UK Complex Statistical Methods, WS 2024/25
Exercise sheet 1

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This was my first LaTeX experience, I hope everything will be rather understandable.

1 Problem 1

Kernel estimator of the s-th derivative $f^{(s)}$ of a density $f \in \mathcal{F}(\beta, L)$, $s < \beta$, with $f(x) \leq f_{\max} < \infty$ for all $x \in \mathbb{R}$, can be defined as follows

$$\widehat{f}^{(s)}(x; h) = \frac{1}{nh^{s+1}} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right),$$

where $h > 0$ is a bandwidth and $K : \mathbb{R} \to \mathbb{R}$ is a bounded kernel with support $[-1, 1]$ satisfying for $\ell = \lfloor \beta \rfloor$

$$\int u^j K(u)\, du = 0, \quad j = 0, 1, \ldots, s-1, s+1, \ldots, \ell,$$

$$\int u^s K(u)\, du = s!.$$

Prove that

(a) the bias of $\widehat{f}^{(s)}(x; h)$ is bounded by $c_1 h^{\beta - s}$ uniformly over the class $\mathcal{F}(\beta, L)$ for appropriate constant c_1 and a given point $x \in \mathbb{R}$; (2 points)

(b) the variance of $\widehat{f}^{(s)}(x; h)$ is bounded by $c_2(nh^{2s+1})^{-1}$ uniformly over the class $\mathcal{F}(\beta, L)$ for appropriate constant c_2 and a given point $x \in \mathbb{R}$; (2 points)

(c) the maximum of the mean squared error of $\widehat{f}^{(s)}(x; h)$ over $\mathcal{F}(\beta, L)$ is of order $O(n^{-2(\beta - s)/(2\beta+1)})$ if the bandwidth $h = h_n$ is chosen optimally. (2 points)

Solutions

I will start with b) as this is the way we've done this in lectures

Proof

b) Let

$$Z_i(x) = K\left(\frac{X_i - x}{h}\right) - \mathbb{E}\left\{K\left(\frac{X_i - x}{h}\right)\right\}.$$

$Z_1(x), \ldots, Z_n(x)$ are i.i.d. random variables with zero mean and variance

$$\begin{aligned}
\operatorname{var}\left\{\hat{f}^{(s)}(x;h)\right\} &= \mathbb{E}\left[\frac{1}{nh^{(s+1)}}\sum_{i=1}^{n} K(\frac{X_i - x}{h})\right] \\
&= \mathbb{E}\left[\frac{1}{nh^{(s+1)}}\sum_{i=1}^{n} K(\frac{X_i - x}{h}) - \frac{1}{nh^{(s+1)}}\sum_{i=1}^{n} \mathbb{E}K(\frac{X_i - x}{h})\right]^2 \qquad (1) \\
&= \frac{1}{nh^{2s+2}}\mathbb{E}\left[\sum_{i=1}^{n} Z_i(x)\right]^2 \\
&= \frac{1}{nh^{2s+2}}\mathbb{E}Z_1^2(x) \text{ (using } i.i.d. \text{ of } Z_i)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{Z_i(x)\}^2 &= \mathbb{E}\left[\left\{K\left(\frac{X_i - x}{h}\right)\right\}^2\right] - \left[\mathbb{E}\left\{K\left(\frac{X_i - x}{h}\right)\right\}\right]^2 \\
&\le \mathbb{E}\left[\left\{K\left(\frac{X_i - x}{h}\right)\right\}^2\right] \text{ (using that second term is } > 0) \\
&= \int \left\{K\left(\frac{u - x}{h}\right)\right\}^2 f(u)\, du \\
&\le f_{\max} h \int \{K(x)\}^2\, dx. \text{ (using } \frac{u-x}{h} = z \text{ and } f(x) \le f_{\max} < \infty) \qquad (2)
\end{aligned}$$

Putting (2) in (1):

$$\begin{aligned}
\operatorname{var}\left\{\hat{f}^{(s)}(x;h)\right\} &\le \frac{h}{nh^{2s+2}} f_{\max} \int \{K(x)\}^2\, dx \\
&= \frac{1}{nh^{2s+1}} f_{\max} \int \{K(x)\}^2\, dx \\
&= \frac{1}{nh^{2s+1}} c_2,
\end{aligned}$$

where

$$c_2 = f_{\max} \int \{K(x)\}^2 \, dx$$

$\blacksquare$

a)

$$\begin{aligned}
\text{bias}\left[\hat{f}^{(s)}(x;h)\right] &= \mathbb{E}\left[\hat{f}^{(s)}(x;h)\right] - f^{(s)}(x) = \\
&= \mathbb{E}\left[\frac{1}{nh^{s+1}} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)\right] - f^{(s)}(x) = \\
&= \frac{1}{h^{s+1}} \int K\left(\frac{u-x}{h}\right) f(u)\, du - f^{(s)}(x) \, (\text{using linearity of}\, \mathbb{E}) = \\
&= \frac{1}{h^s} \int K\left(z\right) f(zh + x)\, dz - f^{(s)}(x) \, (z = \frac{u-x}{h}, du = hdz) = \\
&= \frac{1}{h^s} \int K\left(z\right) \left[f(x) + ... + \frac{(zh)^s}{s!} f^{(s)}(x) + ... + \frac{(zh)^l}{l!} f^{(l)}(x + zh\tau)\right] \, dz \\
-f^{(s)}(x) \, (Taylor) = & \\
&= \frac{1}{h^s} \int K(z) \frac{(zh)^s}{s!} f^{(s)}(x)\, dz + \frac{1}{h^s} \int K(z) \frac{(zh)^l}{l!} f^{(l)}(x + zh\tau)\, dz - f^{(s)}(x) \\
\text{(using conditions on K)} = & \\
&= \frac{1}{h^s} \frac{s!}{s!} h^s f^{(s)}(x) - f^{(s)}(x) + \frac{1}{h^s} \int K(z) \frac{(zh)^l}{l!} f^{(l)}(x + zh\tau)\, dz \\
\text{(using conditions on K)} = & \\
&= \frac{1}{h^s} \int K(z) \frac{(zh)^l}{l!} f^{(l)}(x + zh\tau)\, dz
\end{aligned}$$

Using that:

$$\begin{aligned}
\left|\text{bias}\left[\hat{f}^{(s)}(x;h)\right]\right| &\leq \frac{1}{l!h^s} \int \left|K(z)(zh)^l f^{(l)}(x + zh\tau) - f^{(l)}(x)\right| \, dz \\
&\leq \frac{1}{l!h^s} \int |K(z)| \, |zh|^l L |\tau zh|^{(\beta - l)} \, dz \, (\text{Hölder}) \\
&\leq \frac{L}{l!} h^{(\beta - s)} \int |K(z)||z|^\beta \, dz \\
&= c_1 h^{(\beta - s)}
\end{aligned}$$

where

$$c_1 = \frac{L}{l!} \int |K(z)||z|^\beta \, dz$$

$\blacksquare$

c)

$$\text{MSE}(\widehat{f}^{(s)}(x;h)) = c_1^2 h^{2(\beta-s)} + \frac{c_2}{nh^{2s+1}}$$

$$\begin{aligned}\frac{d}{dh}\text{MSE}(\widehat{f}^{(s)}(x;h)) &= 2(\beta-s)c_1^2 h^{2(\beta-s)-1} - \frac{c_2 n^{-1}(2s+1)}{h^{2(s+1)}} \\ &= \frac{2(\beta-s)c_1^2 h^{2\beta+1} - c_2 n^{-1}(2s+1)}{h^{2(s+1)}}\end{aligned}$$

Which yields:

$$h_{\text{MSE}} = \left[\frac{c_2(2s+1)}{2c_1^2(\beta-s)}\right]^{\frac{1}{2\beta+1}} n^{\frac{-1}{2\beta+1}}$$

And finally:

$$\begin{aligned}&\text{MSE}(\widehat{f}^{(s)}(x;h_{\text{MSE}})) \\ &\qquad = c_2 n^{-1} C n^{\frac{2s+1}{2\beta+1}} + c_1 C n^{\frac{2\beta-2s}{2\beta+1}} \\ &\qquad = \widetilde{C} n^{-\frac{2(\beta-s)}{2\beta+1}} \\ &\qquad = O(n^{-\frac{2(\beta-s)}{2\beta+1}})\end{aligned}$$

$\blacksquare$