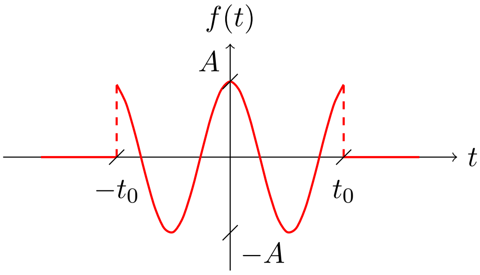
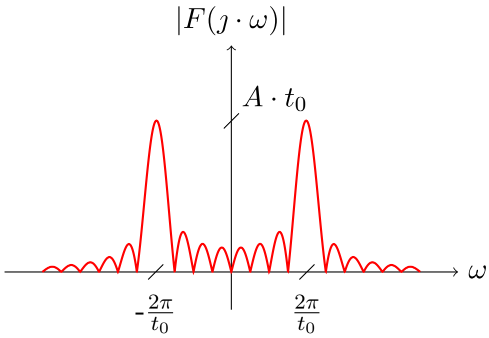
Signal Theory in practise





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\left(\omega \cdot t_0 + 2\pi\right) \\ & -Sa\left(\omega \cdot t_0 - 2\pi\right)] \end{aligned}$$

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Chapter 1

Fundamental concepts and measures

1.1 Basic signal metrics

1.1.1 Mean value of a signal

1.1.2 Energy of a signal

1.1.3 Power and effective value of a signal

Chapter 2

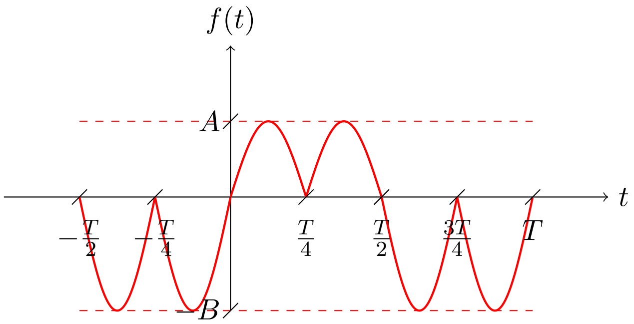
Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

2.2 Complex exponential Fourier series

Task 1.

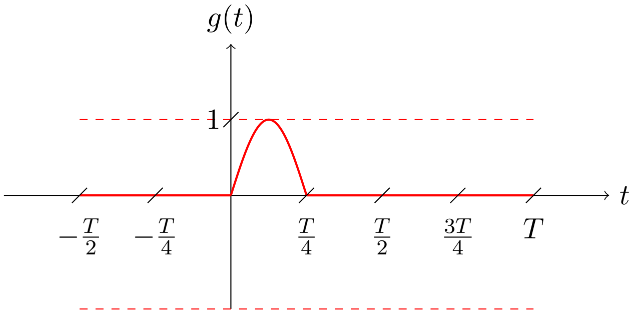
Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(t) = \begin{cases} A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ -A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{4} + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{2} + k \cdot T; \frac{3T}{4} + k \cdot T\right) \\ B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{3T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The F_k coefficients may be calculated directly by definition. However, four integrals have to be solved, each for single interval of one period of the $f(t)$ signal. If we look carefully, signal $f(t)$ may be decomposed into linear combination of shifted in time $g(t)$ signals, for $g(t)$ signal given below:



This is periodic piecewise function, which may be describe as:

$$g(t) = \begin{cases} sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.2}$$

For such a definition of $g(t)$ signal, our $f(t)$ may be described as:

$$g(t) = A \cdot g\left(t\right) + A \cdot g\left(t - \frac{T}{4}\right) - B \cdot g\left(t - \frac{T}{2}\right) - B \cdot g\left(t - \frac{3T}{4}\right) \tag{2.3}$$

Right now, it is enough to calculate G_k - complex exponential Fourier coefficients of $g(t)$ signal. Then, based on linearity and on the effect of signal shift in time on the complex exponential Fourier series, we will be able to derive F_k of $f(t)$ signal.

The G_0 coefficient is defined as:

$$G_0 = \frac{1}{T} \int_T g(t) \cdot dt \tag{2.4}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
G_0 &= \frac{1}{T} \int_T g(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{4}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{1}{T} \int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{aligned} z &= \tfrac{4\pi}{T} \cdot t \\ dz &= \tfrac{4\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{\frac{4\pi}{T}} \end{aligned} \right\} = \\
&= \frac{1}{T} \int_0^{\frac{T}{4}} sin\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} = \\
&= \frac{1}{T \cdot \frac{4\pi}{T}} \int_0^{\frac{T}{4}} sin\left(z\right) \cdot dz =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi} \cdot \left(-cos\,(z)\big|_0^{\frac{T}{4}} \right) = \\
&= -\frac{1}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{4}} \right) = \\
&= -\frac{1}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot \frac{T}{4}\right) - cos\left(\frac{4\pi}{T} \cdot 0\right) \right) = \\
&= -\frac{1}{4\pi} \cdot (cos\,(\pi) - cos\,(0)) = \\
&= -\frac{1}{4\pi} \cdot (-1 - 1) = \\
&= -\frac{1}{4\pi} \cdot (-2) = \\
&= \frac{1}{2\pi}
\end{aligned}
$$

The G_0 coefficient equals $\frac{1}{2\pi}$.

The G_k coefficients are defined as:

$$
G_k = \frac{1}{T} \int_T g(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.5}
$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
G_k &= \frac{1}{T} \int_T g(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{4}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{4}} \frac{e^{\jmath \cdot \frac{4\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{4}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(\frac{1}{2\jmath} \cdot \int_0^{\frac{T}{4}} \left(e^{\jmath \cdot \frac{4\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \frac{1}{2\jmath} \cdot \int_0^{\frac{T}{4}} \left(e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{4}} \left(e^{\jmath \cdot \frac{4\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{4}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (2-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (2+k)} \right) \cdot dt = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (2-k)} \cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (2+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (2-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (2+k) \cdot t \\ dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (2-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (2+k) \cdot dt \\ dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (2-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k)} \end{array} \right\} = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^{z_1} \cdot \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (2-k)} - \int_0^{\frac{T}{4}} e^{z_2} \cdot \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k)} \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (2-k)} \cdot \int_0^{\frac{T}{4}} e^{z_1} \cdot dz_1 - \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k)} \cdot \int_0^{\frac{T}{4}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{1}{T \cdot 2\jmath \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\frac{1}{2-k} \cdot \int_0^{\frac{T}{4}} e^{z_1} \cdot dz_1 + \frac{1}{2+k} \cdot \int_0^{\frac{T}{4}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{1}{2-k} \cdot e^{z_1} \Big|_0^{\frac{T}{4}} + \frac{1}{2+k} \cdot e^{z_2} \Big|_0^{\frac{T}{4}} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{1}{2-k} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (2-k) \cdot t} \Big|_0^{\frac{T}{4}} + \frac{1}{2+k} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k) \cdot t} \Big|_0^{\frac{T}{4}} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{1}{2-k} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (2-k) \cdot \frac{T}{4}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (2-k) \cdot 0} \right) + \frac{1}{2+k} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k) \cdot \frac{T}{4}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (2+k) \cdot 0} \right) \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{1}{2-k} \cdot \left(e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - e^0 \right) + \frac{1}{2+k} \cdot \left(e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - e^0 \right) \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{2+k}{(2-k) \cdot (2+k)} \cdot \left(e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - 1 \right) + \frac{2-k}{(2-k) \cdot (2+k)} \cdot \left(e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - 1 \right) \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{(2+k) \cdot \left(e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - 1 \right)}{(2-k) \cdot (2+k)} + \frac{(2-k) \cdot \left(e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - 1 \right)}{(2-k) \cdot (2+k)} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{(2+k) \cdot \left(e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - 1 \right) + (2-k) \cdot \left(e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - 1 \right)}{(2-k) \cdot (2+k)} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{2 \cdot e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - 2 + k \cdot e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - k + 2 \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - 2 - k \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} + k}{4-k^2} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{2 \cdot e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} - 4 + k \cdot e^{\jmath \cdot \frac{\pi}{2} \cdot (2-k)} + 2 \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)} - k \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot (2+k)}}{4-k^2} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{2 \cdot e^{\jmath \pi} \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - 4 + k \cdot e^{\jmath \pi} \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} + 2 \cdot e^{-\jmath \pi} \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - k \cdot e^{-\jmath \pi} \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{4-k^2} \right) = \\
&= \left\{ \begin{aligned} e^{\jmath \cdot \pi} &= cos\left(\pi\right) + \jmath \cdot sin\left(\pi\right) = -1 \\ e^{-\jmath \cdot \pi} &= cos\left(\pi\right) - \jmath \cdot sin\left(\pi\right) = -1 \end{aligned} \right\} = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{2 \cdot (-1) \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - 4 + k \cdot (-1) \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} + 2 \cdot (-1) \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - k \cdot (-1) \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{4-k^2} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{-2 \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - 4 - k \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - 2 \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} + k \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{4-k^2} \right) = \\
&= \frac{1}{-4 \cdot \pi} \cdot \left(\frac{-4 \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} - 4}{4-k^2} \right) = \\
&= \frac{1}{4 \cdot \pi} \cdot \left(\frac{4 \cdot e^{-\jmath \cdot \frac{k \cdot \pi}{2}} + 4}{4-k^2} \right) = \\
&= \frac{1}{4 \cdot \pi} \cdot 4 \cdot \left(\frac{e^{-\jmath \cdot \frac{k \cdot \pi}{2}} + 1}{4-k^2} \right) = \\
&= \frac{1}{\pi} \cdot \left(\frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{4-k^2} \right) = \\
&= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\left(4-k^2\right)}
\end{aligned}
$$

The G_k coefficients are equal to $\frac{1+e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi(4-k^2)}$ for $k \neq 2 \wedge k \neq -2$.

We have to calculate G_k for $k = 2$ directly by definition:

$$
\begin{aligned}
G_2 &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 0}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 2}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{0}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} 1\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt\right) = \\
&= \left\{\begin{array}{ll} z & = -\jmath\cdot\frac{8\pi}{T}\cdot t \\ dz & = -\jmath\cdot\frac{8\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot\frac{8\pi}{T}} \end{array}\right\} = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} dt - \int_0^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot\frac{8\pi}{T}}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} dt - \frac{1}{-\jmath\cdot\frac{8\pi}{T}}\cdot\int_0^{\frac{T}{4}} e^{z}\cdot dz\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(t\Big|_0^{\frac{T}{4}} + \frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\left(\frac{T}{4}-0\right) + \frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4} + \frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{8\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot\frac{8\pi}{T}\cdot 0}\right)\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \left\{e^{-\jmath\cdot 2\pi}\quad = cos\left(2\pi\right)-\jmath\cdot sin\left(2\pi\right)=1\right\}=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot 0\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+0\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\frac{T}{4}=\\
&= \frac{1}{8\jmath}=\\
&= \frac{-\jmath}{8}
\end{aligned}
$$

The G_2 coefficients equal to $\frac{-\jmath}{8}$.

We have to calculate G_k for $k=-2$ directly by definition:

$$
\begin{aligned}
G_{-2} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot dt\right)=\\
&= \left\{sin\left(x\right)\quad = \frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+0\right)=\\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\right)\cdot dt=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 2}\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 0}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt-\int_0^{\frac{T}{4}} e^{0}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt-\int_0^{\frac{T}{4}} 1\cdot dt\right)=
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{\begin{matrix} z & = \jmath \cdot \frac{8\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{8\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath \cdot \frac{8\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^z \cdot \frac{dz}{\jmath \cdot \frac{8\pi}{T}} - \int_0^{\frac{T}{4}} dt \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot \int_0^{\frac{T}{4}} e^z \cdot dz - \int_0^{\frac{T}{4}} dt \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{4}} - t\Big|_0^{\frac{T}{4}} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot e^{-\jmath \cdot \frac{8\pi}{T} \cdot t}\Big|_0^{\frac{T}{4}} - \left(\frac{T}{4} - 0 \right) \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{8\pi}{T} \cdot \frac{T}{4}} - e^{-\jmath \cdot \frac{8\pi}{T} \cdot 0} \right) - \frac{T}{4} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) - \frac{T}{4} \right) = \\
&= \left\{ e^{-\jmath \cdot 2\pi} \quad = cos\left(2\pi\right) - \jmath \cdot sin\left(2\pi\right) = 1 \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot (1 - 1) - \frac{T}{4} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot 0 - \frac{T}{4} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{4} \right) = \\
&= -\frac{1}{T \cdot 2\jmath} \cdot \frac{T}{4} = \\
&= -\frac{1}{8\jmath} = \\
&= \frac{\jmath}{8}
\end{aligned}
$$

The G_{-2} coefficients equal to $\frac{\jmath}{8}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series for $g(t)$ signal are given by:

$$
\begin{aligned}
G_0 &= \frac{1}{2\pi} \\
G_2 &= \frac{-\jmath}{8} \\
G_{-2} &= \frac{\jmath}{8} \\
G_k &= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\left(4 - k^2\right)}
\end{aligned}
$$

Right now, we may go back to the description of the $f(t)$ signal with shifted in time $g(t)$ signals:

$$f(t) = A \cdot g(t) + A \cdot g\left(t - \frac{T}{4}\right) - B \cdot g\left(t - \frac{T}{2}\right) - B \cdot g\left(t - \frac{3T}{4}\right) \tag{2.6}$$

Recall the linearity and the effect of signal shift in time on the complex exponential Fourier series coefficients:

$$\begin{aligned} n(t) &\rightarrow N_k \\ m(t) &= A \cdot n(t - t_0) \\ M_k &= A \cdot N_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0} \end{aligned}$$

Applying mentioned theorems for $f(t)$ signal, we may write:

$$\begin{aligned} F_k &= A \cdot G_k + A \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{T}{4}} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{T}{2}} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{3T}{4}} = \\ &= A \cdot G_k + A \cdot G_k \cdot e^{-\jmath \cdot \frac{k\pi}{2}} - B \cdot G_k \cdot e^{-\jmath \cdot k \cdot \pi} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{3 \cdot k \cdot \pi}{2}} = \\ &= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left(e^{-\jmath \cdot k\pi} + e^{-\jmath \cdot \frac{3k\pi}{2}}\right) = \\ &= \left\{ \begin{array}{ll} e^{-\jmath \cdot k\pi} & = cos\,(k\pi) + \jmath \cdot sin\,(k\pi) = (-1)^k \\ e^{-\jmath \cdot \frac{3 \cdot k\pi}{2}} & = e^{-\jmath \cdot \left(\frac{2 \cdot k\pi}{2} + \frac{k\pi}{2}\right)} = e^{-\jmath \cdot \frac{2 \cdot k\pi}{2}} \cdot e^{-\jmath \cdot \frac{k\pi}{2}} = (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}} \end{array} \right\} = \\ &= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left((-1)^k + (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\ &= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot (-1)^k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\ &= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) \end{aligned}$$

Now, we may insert G_k coefficients into F_k equation:

$$\begin{aligned} F_k &= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\ &= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\,(4 - k^2)} \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\ &= \frac{\left(1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}\right)^2}{\pi\,(4 - k^2)} \cdot \left(A - B \cdot (-1)^k\right) \end{aligned}$$

Similarly, we may calculate G_0 coefficient:

$$\begin{aligned} F_0 &= G_0 \cdot \left(1 + e^{-\jmath \cdot \frac{0 \cdot \pi}{2}}\right) \cdot \left(A - B \cdot (-1)^0\right) = \\ &= \frac{1}{2\pi} \cdot \left(1 + e^0\right) \cdot (A - B \cdot 1) = \\ &= \frac{1}{2\pi} \cdot (1 + 1) \cdot (A - B) = \\ &= \frac{1}{2\pi} \cdot (2) \cdot (A - B) = \end{aligned}$$

$$= \frac{A-B}{\pi}$$

Similarly, we may calculate G_2 coefficient:

$$\begin{aligned}
F_2 &= G_2 \cdot \left(1 + e^{-\jmath \cdot \frac{2\cdot\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^2\right) = \\
&= \frac{-\jmath}{8} \cdot \left(1 + e^{-\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{e^{-\jmath \cdot \pi} \quad = cos\left(\pi\right) - \jmath \cdot sin\left(\pi\right) = -1\right\} = \\
&= \frac{-\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{-\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}$$

Similarly, we may calculate G_2 coefficient:

$$\begin{aligned}
F_{-2} &= G_{-2} \cdot \left(1 + e^{-\jmath \cdot \frac{(-2)\cdot\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^{-2}\right) = \\
&= \frac{\jmath}{8} \cdot \left(1 + e^{\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{e^{\jmath \cdot \pi} \quad = cos\left(\pi\right) + \jmath \cdot sin\left(\pi\right) = -1\right\} = \\
&= \frac{\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series for $f(t)$ signal are given by:

$$\begin{aligned}
F_0 &= \frac{A-B}{\pi} \\
F_2 &= 0 \\
F_{-2} &= 0 \\
F_k &= \frac{\left(1 + e^{-\jmath \cdot \frac{k\cdot\pi}{2}}\right)^2}{\pi\left(4 - k^2\right)} \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}$$

2.3 Computing the power of a signal – the Parseval's theorem

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

3.2 Exploiting properties of the Fourier transform

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

4.2 Filters

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