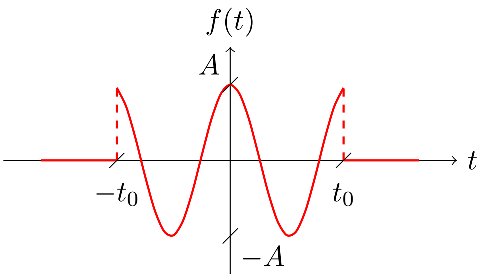
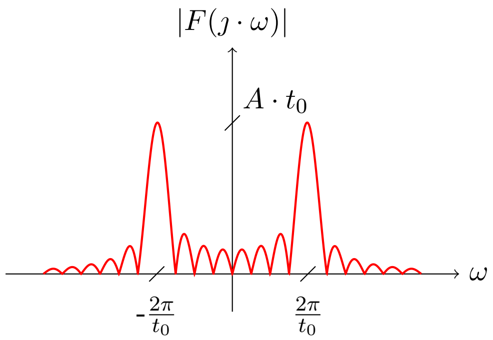
# Signal Theory in practise




$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\left(\omega \cdot t_0 + 2\pi\right) \\ & - Sa\left(\omega \cdot t_0 - 2\pi\right)] \end{aligned}$$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April 18, 2020

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ISBN 978-83-939620-3-7
Printed in Poland

# Chapter 1

# Fundamental concepts and measures

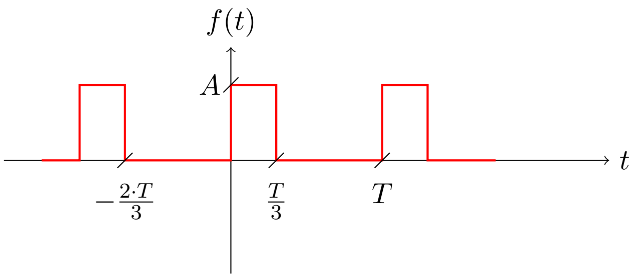
## 1.1 Basic signal metrics

### 1.1.1 Mean value of a signal

### 1.1.2 Energy of a signal

### 1.1.3 Power and effective value of a signal

**Task 1.** Compute the average power for the following periodic signal $f(t)$:




Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{3} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{3} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.1}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.2}$$

Compute average power for period $k = 0$

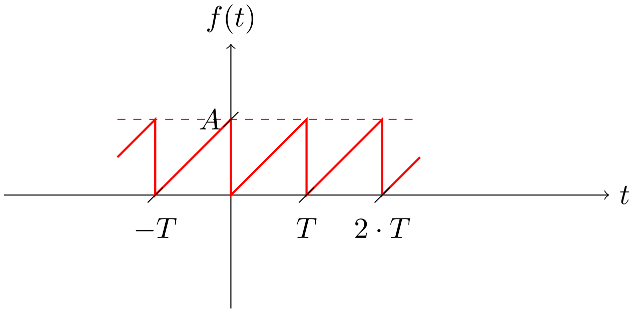
$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left( \int_0^{\frac{T}{3}} |A|^2 \cdot dt + \int_{\frac{T}{3}}^{T} |0|^2 \cdot dt \right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left( \int_0^{\frac{T}{3}} A^2 \cdot dt + \int_{\frac{T}{3}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left( A^2 \cdot \int_0^{\frac{T}{3}} dt + 0 \right) = \\
&= \frac{A^2}{T} \cdot t|_0^{\frac{T}{3}} = \\
&= \frac{A^2}{T} \cdot \left( \frac{T}{3} - 0 \right) = \\
&= \frac{A^2}{T} \cdot \frac{T}{3} = \\
&= \frac{A^2}{3}
\end{aligned}
$$

Average power equals to $\frac{A^2}{3}$.

**Task 2.**

Calculate the average power for the periodic signal $f(t)$ given below:




First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.3}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and $(T,A)$. So, in order to derive $a$ and $b$, the following system of the equations has to be solved:

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in C$$

The average power for periodic signals is defined by:

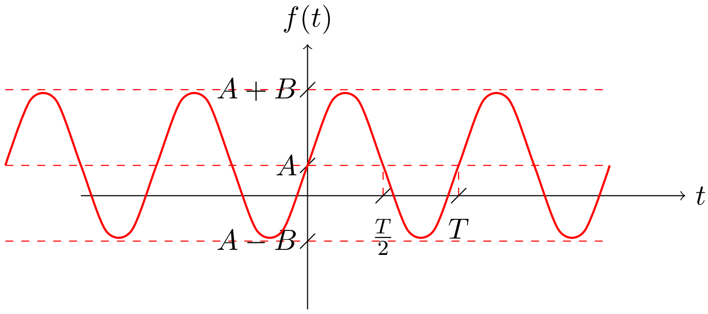
$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.4}$$

In our case we get:

$$
\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| \frac{A}{T} \cdot t \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left( \frac{A}{T} \cdot t \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \frac{A^2}{T^2} \cdot t^2 \cdot dt = \\
&= \frac{1}{T} \cdot \frac{A^2}{T^2} \cdot \int_0^T t^2 \cdot dt = \\
&= \frac{A^2}{T^3} \cdot \left( \frac{1}{3} \cdot t^3 \Big|_0^T \right) = \\
&= \frac{A^2}{T^3} \cdot \left( \frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot 0^3 \right) = \\
&= \frac{A^2}{T^3} \cdot \left( \frac{1}{3} \cdot T^3 - 0 \right) = \\
&= \frac{A^2}{T^3} \cdot \frac{1}{3} \cdot T^3 = \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}$.

**Task 3.** Compute the average power for the following periodic signal $f(t) = A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:




The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.5}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left( A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left( A^2 + 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) + B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \right) \cdot dt = \\
&= \frac{1}{T} \cdot \left( \int_0^T A^2 \cdot dt + \int_0^T 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_0^T B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \frac{A^2}{T} \cdot \int_0^T dt + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{B^2}{T} \cdot \int_0^T sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{l} z = \frac{2\pi}{T} \cdot t \\ dz = \frac{2\pi}{T} \cdot dt \quad dt = \frac{dz}{\frac{2\pi}{T}} = \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{A^2}{T} \cdot t|_0^T + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\left(z\right) \cdot \frac{T}{2\pi} \cdot dz + \frac{B^2}{T} \cdot \int_0^T \frac{1}{2} \cdot \left( 1 - cos\left( 2 \cdot \frac{2\pi}{T} \cdot t \right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot (T - 0) + \frac{2 \cdot A \cdot B}{T} \cdot \frac{T}{2\pi} \cdot \int_0^T sin\left(z\right) \cdot dz + \frac{B^2}{T} \cdot \frac{1}{2} \cdot \int_0^T \left( 1 - cos\left( 2 \cdot \frac{2\pi}{T} \cdot t \right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot T + \frac{A \cdot B}{\pi} \cdot \left( -cos\left(z\right)|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left( \int_0^T 1 \cdot dt - \int_0^T cos\left( 2 \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) = \\
&= \left\{ \begin{array}{l} w = 2 \cdot \frac{2\pi}{T} \cdot t \\ dw = 2 \cdot \frac{2\pi}{T} \cdot dt \quad dt = \frac{dw}{\frac{4\pi}{T}} = \frac{T}{4\pi} \cdot dw \end{array} \right\} = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left( -cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left( t|_0^T - \int_0^T cos\left(w\right) \cdot \frac{T}{4\pi} \cdot dw \right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left( -cos\left(\frac{2\pi}{T} \cdot T\right) + cos\left(\frac{2\pi}{T} \cdot 0\right) \right) + \frac{B^2}{2 \cdot T} \cdot \left( (T - 0) - \frac{T}{4\pi} \cdot \int_0^T cos\left(w\right) \cdot dw \right) =
\end{aligned}$$

$$= A^2 + \frac{A \cdot B}{\pi} \cdot (-cos\,(2\pi) + cos\,(0)) + \frac{B^2}{2 \cdot T} \cdot \left(T - \frac{T}{4\pi} \cdot -sin\,(w)|_0^T\right) =$$

$$= A^2 + \frac{A \cdot B}{\pi} \cdot (-1 + 1) + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_0^T\right) =$$

$$= A^2 + \frac{A \cdot B}{\pi} \cdot 0 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(2 \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(2 \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) =$$

$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot (sin\,(4\pi) - sin\,(0))\right) =$$

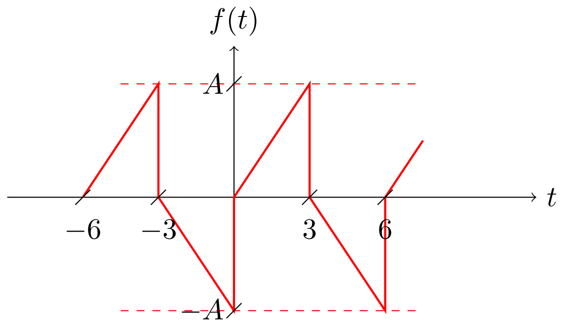
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot (0 - 0)\right) =$$

$$= A^2 + \frac{B^2}{2 \cdot T} \cdot (T) =$$

$$= A^2 + \frac{B^2}{2}$$

The average power equals to $A^2 + \frac{B^2}{2}$.

**Task 4.** Calculate the average power and the effective value (RMS) for the periodic signal $f(t)$ given below:




First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.6}$$

In the first interval of the first period (i.e. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive $a$ and $b$, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (i.e. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive $a$ and $b$, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.7}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{6} \cdot \left( \int_0^3 \left| \frac{A}{3} \cdot t \right|^2 \cdot dt + \int_3^6 \left| -\frac{A}{3} \cdot t + A \right|^2 \cdot dt \right) = \\
&= \frac{1}{6} \cdot \int_0^3 \left( \frac{A}{3} \cdot t \right)^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left( -\frac{A}{3} \cdot t + A \right)^2 \cdot dt = \\
&= \frac{1}{6} \cdot \int_0^3 \frac{A^2}{9} \cdot t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left( \left( -\frac{A}{3} \cdot t \right)^2 - 2 \cdot \frac{A}{3} \cdot t \cdot A + A^2 \right) \cdot dt = \\
&= \frac{A^2}{54} \cdot \int_0^3 t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \frac{A^2}{9} \cdot t^2 \cdot dt - \frac{1}{6} \cdot \int_3^6 \frac{2 \cdot A^2}{3} \cdot t \cdot dt + \frac{1}{6} \cdot \int_3^6 A^2 \cdot dt = \\
&= \frac{A^2}{54} \cdot \frac{t^3}{3} \bigg|_0^3 + \frac{A^2}{54} \cdot \int_3^6 t^2 \cdot dt - \frac{2 \cdot A^2}{18} \cdot \int_3^6 t^2 \cdot dt + \frac{A^2}{6} \cdot \int_3^6 dt =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{162} \cdot \left(3^3 - 0^3\right) + \frac{A^2}{54} \cdot \left.\frac{t^3}{3}\right|_3^6 - \frac{2 \cdot A^2}{18} \cdot \left.\frac{t^2}{2}\right|_3^6 + \frac{A^2}{6} \cdot t|_3^6 = \\
&= \frac{A^2}{162} \cdot 27 + \frac{A^2}{162} \cdot \left(6^3 - 3^3\right) - \frac{2 \cdot A^2}{36} \cdot \left(6^2 - 3^2\right) + \frac{A^2}{6} \cdot (6-3) = \\
&= \frac{A^2}{6} + \frac{A^2}{162} \cdot 189 - \frac{2 \cdot A^2}{36} \cdot 27 + \frac{A^2}{6} \cdot 3 = \\
&= \frac{A^2}{6} + \frac{7 \cdot A^2}{6} - \frac{9 \cdot A^2}{6} + \frac{3 \cdot A^2}{6} = \\
&= \frac{2 \cdot A^2}{6} = \\
&= \frac{A^2}{3}
\end{aligned}
$$

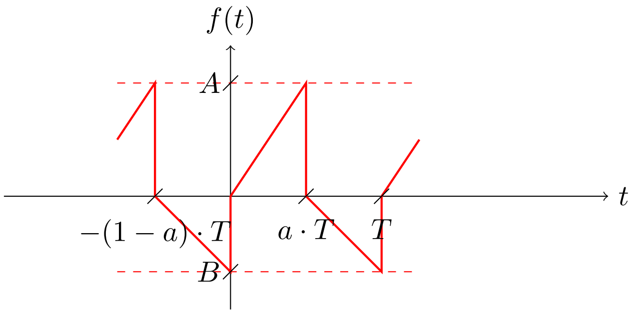
The average power equals to $\frac{A^2}{3}$.

The effective value (RMS) is defined by:

$$RMS = \sqrt{P} \tag{1.8}$$

Therefore, effective value (RMS) equals to $\frac{A}{\sqrt{3}}$.

**Task 5.** Calculate the average power for the periodic signal $f(t)$ given below:




First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{1.9}$$

In the first interval of the first period (i.e. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive $m$ and $b$, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive $m$ and $b$, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a \cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a \cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a \cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a \cdot T} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T - a \cdot T} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a \cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{T-a \cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a \cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{T-a \cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.10}$$

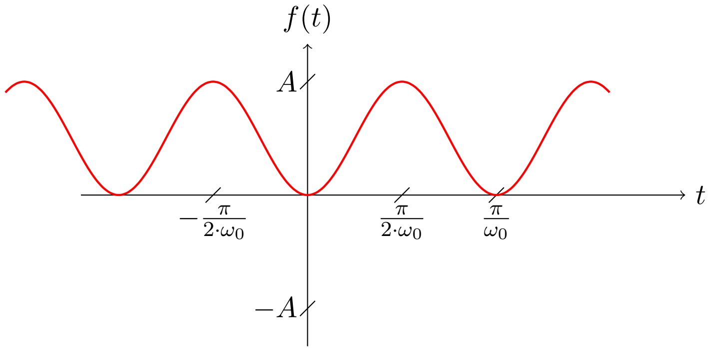
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \left( \int_0^{a \cdot T} \left| \frac{A}{a \cdot T} \cdot t \right|^2 \cdot dt + \int_{a \cdot T}^{T} \left| \frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right|^2 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \int_0^{a \cdot T} \left( \frac{A}{a \cdot T} \cdot t \right)^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left( \frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^{a \cdot T} \frac{A^2}{a^2 \cdot T^2} \cdot t^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left( \left( \frac{B}{T - a \cdot T} \cdot t \right)^2 - 2 \cdot \frac{B}{T - a \cdot T} \cdot t \cdot \frac{B}{1-a} \cdot a + \left( \frac{B}{1-a} \cdot a \right)^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \int_0^{a \cdot T} t^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left( \frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 - 2 \cdot \frac{B^2}{T \cdot (1-a)^2} \cdot t \cdot a + \frac{B^2}{(1-a)^2} \cdot a^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \left( \frac{1}{3} \cdot t^3 \Big|_0^{a \cdot T} \right) + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 \cdot dt +
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{T}\cdot\int_{a\cdot T}^{T}2\cdot\frac{B^2}{T\cdot(1-a)^2}\cdot t\cdot a\cdot dt+\frac{1}{T}\cdot\int_{a\cdot T}^{T}\frac{B^2}{(1-a)^2}\cdot a^2\cdot dt=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot t^3\Big|_0^{a\cdot T}\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\int_{a\cdot T}^{T}t^2\cdot dt+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\int_{a\cdot T}^{T}t\cdot dt+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\int_{a\cdot T}^{T}dt=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot(a\cdot T)^3-\frac{1}{3}\cdot 0^3\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot t^3\Big|_{a\cdot T}^{T}\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot t^2\Big|_{a\cdot T}^{T}\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\left(t\big|_{a\cdot T}^{T}\right)=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot a^3\cdot T^3-0\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot(a\cdot T)^3\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot(a\cdot T)^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(T-a\cdot T)=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\frac{1}{3}\cdot a^3\cdot T^3+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot a^3\cdot T^3\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot a^2\cdot T^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(1-a^3\right)\cdot\frac{1}{3}\cdot T^3+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(1-a^2\right)\cdot\frac{1}{2}\cdot T^2+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{(1-a)^2}\cdot(1-a)\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}+\\
&-\frac{2\cdot B^2}{(1-a)^2}\cdot a\cdot(1-a)\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}-\frac{2\cdot B^2}{1-a}\cdot a\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{1}{3}-2\cdot a\cdot(1+a)\cdot\frac{1}{2}+a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{2}{6}-2\cdot a\cdot(1+a)\cdot\frac{3}{6}+a^2\cdot\frac{6}{6}\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(\left(1+a+a^2\right)\cdot 2-2\cdot a\cdot(1+a)\cdot 3+a^2\cdot 6\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2+2\cdot a+2\cdot a^2-6\cdot a-6\cdot a^2+6\cdot a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2-4\cdot a+2\cdot a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot\left(1-2\cdot a+a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot(1-a)^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)$.

**Task 6.** Calculate the average power for the periodic signal $f(t) = A \cdot sin^2(\omega_0 \cdot t)$ given below.




The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.11}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left|A \cdot sin^2(\omega_0 \cdot t)\right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot sin^4(\omega_0 \cdot t) \cdot dt = \\
&= \frac{1}{T} \cdot \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left( \frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} \right)^4 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{\left(e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \cdot dt = \\
&= \left\{ \begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left( \frac{1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^4 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^0 + 4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^3 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^1 + 6 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^2 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^2}{(2 \cdot \jmath)^4} + \right. \\
&+ \left. \frac{4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^1 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^3 + 1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^0 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left( \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-\jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} + \right. \\
&+ \left. \frac{-4 \cdot e^{\jmath \cdot \omega_0 \cdot t} \cdot e^{-3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t - 0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t - \jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t - 2 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{\jmath \cdot \omega_0 \cdot t - 3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t - 4 \cdot \jmath \cdot \omega_0 \cdot t}}{16 \cdot 1} \cdot dt =
\end{aligned}$$

$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} + 6 \cdot e^{0\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6 \cdot e^{0}}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6}{16} \cdot dt =$$
$$= \frac{A^2}{16 \cdot T} \cdot \int_0^T \left(e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\right) dt =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\int_0^T e^{4\cdot\jmath\cdot\omega_0\cdot t} \cdot dt + \int_0^T e^{-4\cdot\jmath\cdot\omega_0\cdot t} \cdot dt - 4 \cdot \int_0^T e^{2\cdot\jmath\cdot\omega_0\cdot t} \cdot dt - 4 \cdot \int_0^T e^{-2\cdot\jmath\cdot\omega_0\cdot t} \cdot dt + 6 \cdot \int_0^T dt\right) =$$
$$= \left\{\begin{matrix} z_1 = 4 \cdot \jmath \cdot \omega_0 \cdot t & z_2 = -4 \cdot \jmath \cdot \omega_0 \cdot t & z_3 = 2 \cdot \jmath \cdot \omega_0 \cdot t & z_4 = -2 \cdot \jmath \cdot \omega_0 \cdot t = \\ dz_1 = 4 \cdot \jmath \cdot \omega_0 \cdot dt & dz_2 = -4 \cdot \jmath \cdot \omega_0 \cdot dt & dz_3 = 2 \cdot \jmath \cdot \omega_0 \cdot dt & dz_4 = -2 \cdot \jmath \cdot \omega_0 \cdot dt = \\ dt = \frac{1}{4\cdot\jmath\cdot\omega_0} \cdot dz_1 & dt = \frac{1}{-4\cdot\jmath\cdot\omega_0} \cdot dz_2 & dt = \frac{1}{2\cdot\jmath\cdot\omega_0} \cdot dz_3 & dt = \frac{1}{-2\cdot\jmath\cdot\omega_0} \cdot dz_4 \end{matrix}\right\} =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\int_0^T e^{z_1} \cdot \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot dz_1 + \int_0^T e^{z_2} \cdot \frac{1}{-4 \cdot \jmath \cdot \omega_0} \cdot dz_2 +\right.$$
$$\left.- 4 \cdot \int_0^T e^{z_3} \cdot \frac{1}{2 \cdot \jmath \cdot \omega_0} \cdot dz_3 - 4 \cdot \int_0^T e^{z_4} \cdot \frac{1}{-2 \cdot \jmath \cdot \omega_0} \cdot dz_4 + 6 \cdot \int_0^T dt\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_1} \cdot dz_1 + \frac{1}{-4 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_2} \cdot dz_2 +\right.$$
$$\left.-4 \cdot \frac{1}{2 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_3} \cdot dz_3 - 4 \cdot \frac{1}{-2 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_4} \cdot dz_4 + 6 \cdot \int_0^T dt\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{z_1}\Big|_0^T - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{z_2}\Big|_0^T - \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{z_3}\Big|_0^T + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{z_4}\Big|_0^T +\right.$$
$$\left.+6 \cdot t\Big|_0^T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T +\right.$$
$$\left.+6 \cdot t\Big|_0^T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{4\cdot\jmath\cdot\omega_0\cdot 0}\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot 0}\right) +\right.$$
$$\left.-\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{2\cdot\jmath\cdot\omega_0\cdot 0}\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot 0}\right) + 6 \cdot (T - 0)\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) +\right.$$
$$\left.-\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + 6 \cdot T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) +\right.$$
$$\left.-\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + 6 \cdot T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{1}{4 \cdot \jmath \cdot \omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4 \cdot \jmath \cdot \omega_0} + \frac{1}{4 \cdot \jmath \cdot \omega_0} +\right.$$
$$\left.-\frac{4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2 \cdot \jmath \cdot \omega_0} + \frac{4}{2 \cdot \jmath \cdot \omega_0} + \frac{4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2 \cdot \jmath \cdot \omega_0} - \frac{4}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T\right) =$$

$$
\begin{aligned}
&= \frac{A^2}{16\cdot T}\cdot\left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + \frac{4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) = \\
&= \frac{A^2}{16\cdot T}\cdot\left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) = \\
&= \frac{A^2}{16\cdot T}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath} - \frac{4}{\omega_0}\cdot\frac{e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath} + 6\cdot T\right) = \\
&= \left\{sin(x) = \tfrac{e^{\jmath\cdot x} - e^{-\jmath\ cdotx}}{2\cdot\jmath}\right\} = \\
&= \frac{A^2}{16\cdot T}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(4\cdot\omega_0\cdot T\right) - \frac{4}{\omega_0}\cdot sin\left(2\cdot\omega_0\cdot T\right) + 6\cdot T\right) = \\
&= \left\{T = \tfrac{2\pi}{\omega_0}\right\} = \\
&= \frac{A^2}{16\cdot\frac{2\pi}{\omega_0}}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(4\cdot\omega_0\cdot\frac{2\pi}{\omega_0}\right) - \frac{4}{\omega_0}\cdot sin\left(2\cdot\omega_0\cdot\frac{2\pi}{\omega_0}\right) + 6\cdot\frac{2\pi}{\omega_0}\right) = \\
&= \frac{A^2\cdot\omega_0}{32\pi}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(8\pi\right) - \frac{4}{\omega_0}\cdot sin\left(4\pi\right) + 6\cdot\frac{2\pi}{\omega_0}\right) = \\
&= \frac{A^2\cdot\omega_0}{32\pi}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot 0 - \frac{4}{\omega_0}\cdot 0 + 6\cdot\frac{2\pi}{\omega_0}\right) = \\
&= \frac{A^2\cdot\omega_0}{32\pi}\cdot\frac{12\pi}{\omega_0} = \\
&= \frac{3\cdot A^2}{8}
\end{aligned}
$$

The average power equals to $\frac{3\cdot A^2}{8}$.

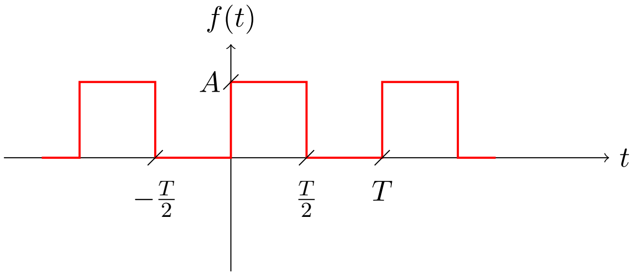
# Chapter 2

# Analysis of periodic signals using orthogonal series

## 2.1 Trigonometric Fourier series

## 2.2 Complex exponential Fourier series

**Task 1.** Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.




Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The $F_0$ coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} dt + 0\right) = \\
&= \frac{1}{T}\left(A\cdot t|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T}\cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}-0\right) = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.3}$$

The $F_0$ coefficient equals $\frac{A}{2}$.

The $F_k$ coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.4}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^z\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^z\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\,e^z|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\,e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^0\right) =
\end{aligned}$$

$$= -\frac{A}{\jmath \cdot k \cdot 2\pi}\left(e^{-\jmath \cdot k \cdot \pi} - 1\right) =$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left(e^{-\jmath \cdot k \cdot \pi} - 1\right)$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

The $F_k$ coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

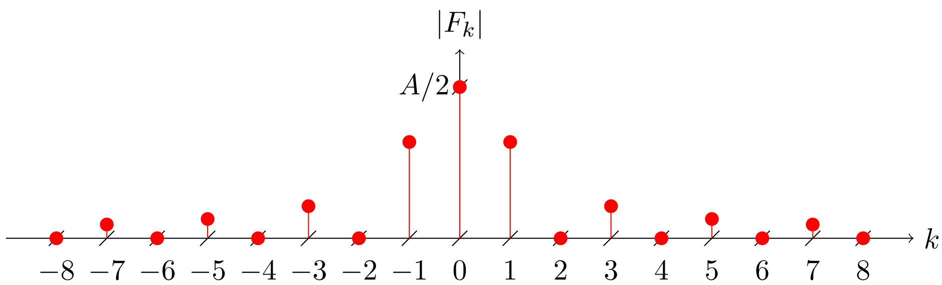
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.5}$$

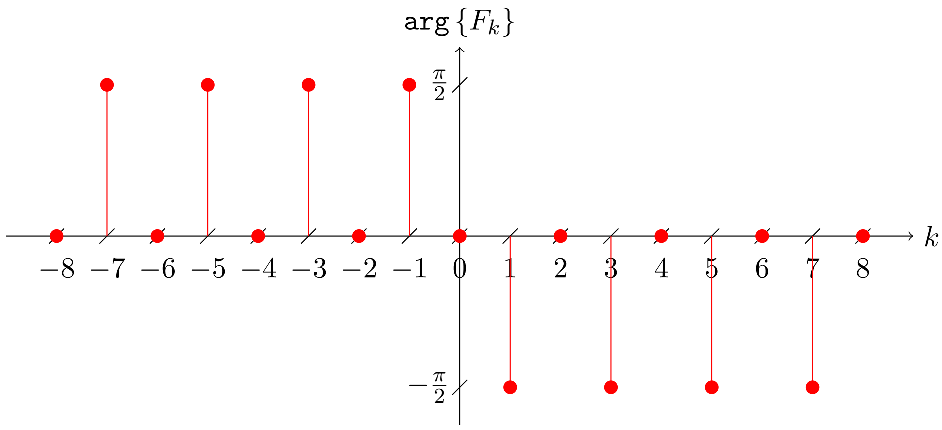
The first several coefficients are equal to:

| $k$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_k$ | $\jmath \cdot \frac{A}{5\pi}$ | $0$ | $\jmath \cdot \frac{A}{3\pi}$ | $0$ | $\jmath \cdot \frac{A}{\pi}$ | $\frac{A}{2}$ | $-\jmath \cdot \frac{A}{\pi}$ | $0$ | $-\jmath \cdot \frac{A}{3\pi}$ | $0$ | $-\jmath \cdot \frac{A}{5\pi}$ |
| $\lvert F_k \rvert$ | $\frac{A}{5\pi}$ | $0$ | $\frac{A}{3\pi}$ | $0$ | $\frac{A}{\pi}$ | $\frac{A}{2}$ | $\frac{A}{\pi}$ | $0$ | $\frac{A}{3\pi}$ | $0$ | $\frac{A}{5\pi}$ |
| $Arg\left\{F_k\right\}$ | $\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{2}$ |

Based on coefficents $F_k$ we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

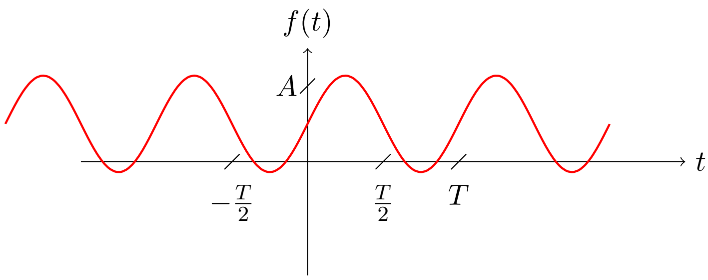



The magnitude spectrum of a real signal is an even-symmetric function of $k$.
Based on coefficents $F_k$ we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

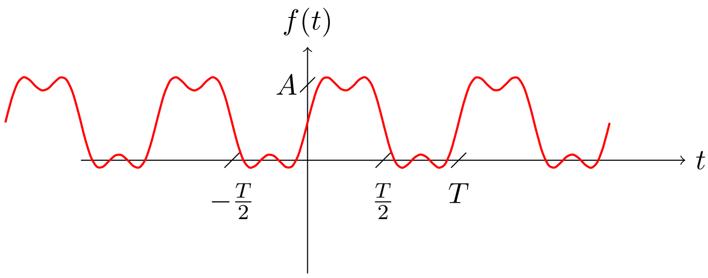



The phase spectrum of a real signal is an odd-symmetric function of $k$.

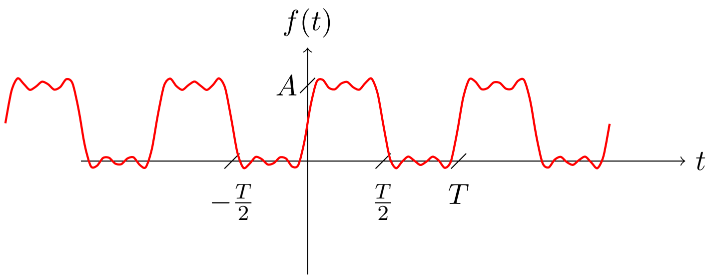
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



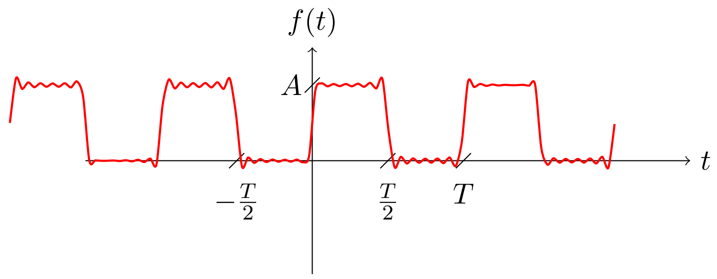

A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



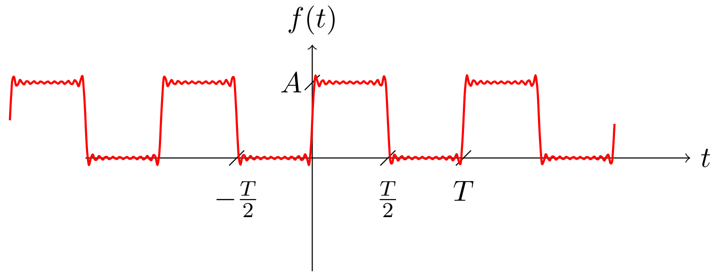

A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:




A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:




A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:




Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

## 2.3 Computing the power of a signal – the Parseval's theorem

**Task 1.** For a certain real-valued periodic signal, its coefficients of expansion to a complex exponential Fourier series are:

$$F_k = \frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2} \wedge k > 0 \tag{2.6}$$

Compute the mean value ($\bar{f}$), knowing that the effective (RMS) value is $U = \frac{A\sqrt{6}}{60}$. During calculation use:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \tag{2.7}$$

From the theoretical considerations we know that:

$$F_0 = \bar{f}$$
$$U = \sqrt{P}$$

In order to calculate $\bar{f}$ we have to calculate $F_0$. But we know values of $F_k$ for $k > 0$ only. However, based on Parseval theorem, power of the signal is defined as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2 \tag{2.8}$$

This equation may be rewritten as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2$$
$$P = \sum_{k=-\infty}^{-1} |F_k|^2 + |F_0|^2 + \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=-\infty}^{-1} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$

Because the $f(t) \in R$, thus $|F_k| = |F_{-k}|$ and we may write:

$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

Now, we can calculate $F_0$:

$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = U^2 - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = \left(\frac{A\sqrt{6}}{60}\right)^2 - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot pi^2}\right|^2$$
$$|F_0|^2 = \frac{A^2 \cdot 6}{3600} - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2}\right|^2$$
$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \left(\frac{A}{k^2 \cdot 4 \cdot \pi^2}\right)^2$$
$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \frac{A^2}{k^4 \cdot 16 \cdot \pi^4}$$
$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \frac{A^2}{16 \cdot \pi^4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^4}$$
$$|F_0|^2 = \frac{A^2}{600} - \frac{A^2}{8 \cdot \pi^4} \cdot \frac{\pi^4}{90}$$
$$|F_0|^2 = \frac{A^2}{600} - \frac{A^2}{720}$$
$$|F_0|^2 = \frac{720 \cdot A^2}{600 \cdot 720} - \frac{600 \cdot A^2}{600 \cdot 720}$$
$$|F_0|^2 = \frac{720 \cdot A^2 - 600 \cdot A^2}{600 \cdot 720}$$
$$|F_0|^2 = \frac{120 \cdot A^2}{600 \cdot 720}$$
$$|F_0|^2 = \frac{A^2}{5 \cdot 720}$$
$$|F_0|^2 = \frac{A^2}{3600}$$
$$|F_0| = \sqrt{\frac{A^2}{3600}}$$
$$|F_0| = \frac{A}{60}$$
$$F_0 = \pm\frac{A}{60}$$

The mean value is equal to $\bar{f} = \pm\frac{A}{60}$.

# Chapter 3

# Analysis of non-periodic signals. Fourier Transformation and Transform

## 3.1 Calculation of Fourier Transform by definition

## 3.2 Exploiting properties of the Fourier transform

## 3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

# Chapter 4

# Processing of signals by linear and time invariant (LTI) systems

## 4.1 Linear convolution

## 4.2 Filters

ISBN 978-83-939620-3-7