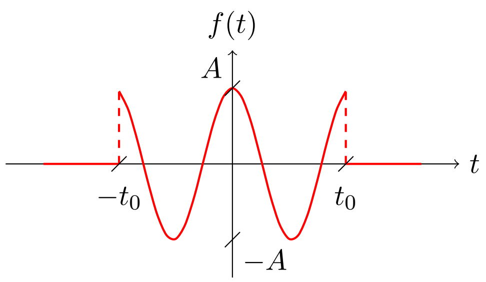
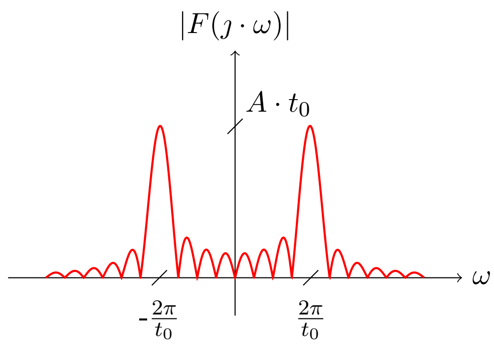
Signal Theory in practise





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot \left[\; Sa\left(\omega \cdot t_0 + 2\pi\right) - Sa\left(\omega \cdot t_0 - 2\pi\right)\right]$$

Tomasz Grajek, Krzysztof Wegner

May 25, 2020

Poznan University of Technology
Faculty of Computing and Telecommunications
Institute of Multimedia Telecommunications

pl. M. Skłodowskiej-Curie 5
60-965 Poznań

www.et.put.poznan.pl
www.multimedia.edu.pl

ISBN 978-83-939620-3-7
Printed in Poland

Chapter 1

Fundamental concepts and measures

Task 1. Expand the following signals into a sum of sine and cosine functions, and a constant, by using the Euler identities.

$$f_1(t) = sin^5(t) - sin^3(t)$$
$$f_2(t) = cos^6(t) - cos^4(t)$$

Euler identities:

$$sin\,(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath}$$
$$cos\,(x) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k \tag{1.1}$$

where $\binom{n}{k}$ are called binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.2}$$

The binomial coefficient $\binom{n}{k}$ appears as the kth entry in the nth row of Pascal's triangle (counting starts at 0). Each entry is the sum of the two above it. Below, example for $n = 6$ is presented:

$$\left\{\begin{array}{lccccccccccccc}
n{=}0: & & & & & & & 1 & & & & & & \\
n{=}1: & & & & & & 1 & & 1 & & & & & \\
n{=}2: & & & & & 1 & & 2 & & 1 & & & & \\
n{=}3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n{=}4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n{=}5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n{=}6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}\right\} \tag{1.3}$$

$$f_1(t) = sin^5(t) - sin^3(t) =$$
$$= \left(\frac{e^{\jmath \cdot t} - e^{-\jmath \cdot t}}{2 \cdot \jmath}\right)^5 - \left(\frac{e^{\jmath \cdot t} - e^{-\jmath \cdot t}}{2 \cdot \jmath}\right)^3 =$$
$$= \frac{\left(e^{\jmath \cdot t} - e^{-\jmath \cdot t}\right)^5}{(2 \cdot \jmath)^5} - \frac{\left(e^{\jmath \cdot t} - e^{-\jmath \cdot t}\right)^3}{(2 \cdot \jmath)^3} =$$

$$
\begin{aligned}
&= \frac{1\cdot(e^{\jmath\cdot t})^5\cdot(-e^{-\jmath\cdot t})^0+5\cdot(e^{\jmath\cdot t})^4\cdot(-e^{-\jmath\cdot t})^1+10\cdot(e^{\jmath\cdot t})^3\cdot(-e^{-\jmath\cdot t})^2}{(2\cdot\jmath)^5}+\\
&+\frac{10\cdot(e^{\jmath\cdot t})^2\cdot(-e^{-\jmath\cdot t})^3+5\cdot(e^{\jmath\cdot t})^1\cdot(-e^{-\jmath\cdot t})^4+1\cdot(e^{\jmath\cdot t})^0\cdot(-e^{-\jmath\cdot t})^5}{(2\cdot\jmath)^5}-\\
&-\left(\frac{1\cdot(e^{\jmath\cdot t})^3\cdot(-e^{-\jmath\cdot t})^0+3\cdot(e^{\jmath\cdot t})^2\cdot(-e^{-\jmath\cdot t})^1+3\cdot(e^{\jmath\cdot t})^1\cdot(-e^{-\jmath\cdot t})^2+1\cdot(e^{\jmath\cdot t})^0\cdot(-e^{-\jmath\cdot t})^3}{(2\cdot\jmath)^3}\right)=\\
&=\frac{1\cdot e^{\jmath\cdot t\cdot 5}\cdot 1+5\cdot e^{\jmath\cdot t\cdot 4}\cdot(-e^{-\jmath\cdot t\cdot 1})+10\cdot e^{\jmath\cdot t\cdot 3}\cdot e^{-\jmath\cdot t\cdot 2}}{(2\cdot\jmath)^5}+\\
&+\frac{10\cdot e^{\jmath\cdot t\cdot 2}\cdot(-e^{-\jmath\cdot t\cdot 3})+5\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 4}+1\cdot 1\cdot(-e^{-\jmath\cdot t\cdot 5})}{(2\cdot\jmath)^5}-\\
&-\left(\frac{1\cdot e^{\jmath\cdot t\cdot 3}\cdot 1+3\cdot e^{\jmath\cdot t\cdot 2}\cdot(-e^{-\jmath\cdot t\cdot 1})+3\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 2}+1\cdot 1\cdot(-e^{-\jmath\cdot t\cdot 3})}{(2\cdot\jmath)^3}\right)=\\
&=\frac{e^{\jmath\cdot t\cdot 5}-5\cdot e^{\jmath\cdot t\cdot 3}+10\cdot e^{\jmath\cdot t}-10\cdot e^{-\jmath\cdot t}+5\cdot e^{-\jmath\cdot t\cdot 3}-e^{-\jmath\cdot t\cdot 5}}{(2\cdot\jmath)^5}-\\
&-\left(\frac{e^{\jmath\cdot t\cdot 3}-3\cdot e^{\jmath\cdot t}+3\cdot e^{-\jmath\cdot t}-e^{-\jmath\cdot t\cdot 3}}{(2\cdot\jmath)^3}\right)=\\
&=\frac{e^{\jmath\cdot t\cdot 5}-e^{-\jmath\cdot t\cdot 5}-5\cdot e^{\jmath\cdot t\cdot 3}+5\cdot e^{-\jmath\cdot t\cdot 3}+10\cdot e^{\jmath\cdot t}-10\cdot e^{-\jmath\cdot t}}{(2\cdot\jmath)^5}-\\
&-\left(\frac{e^{\jmath\cdot t\cdot 3}-e^{-\jmath\cdot t\cdot 3}-3\cdot e^{\jmath\cdot t}+3\cdot e^{-\jmath\cdot t}}{(2\cdot\jmath)^3}\right)=\\
&=\frac{e^{\jmath\cdot t\cdot 5}-e^{-\jmath\cdot t\cdot 5}-5\cdot(e^{\jmath\cdot t\cdot 3}-e^{-\jmath\cdot t\cdot 3})+10\cdot(e^{\jmath\cdot t}-e^{-\jmath\cdot t})}{(2\cdot\jmath)^4\cdot(2\cdot\jmath)}-\\
&-\left(\frac{e^{\jmath\cdot t\cdot 3}-e^{-\jmath\cdot t\cdot 3}-3\cdot(e^{\jmath\cdot t}-\cdot e^{-\jmath\cdot t})}{(2\cdot\jmath)^2\cdot(2\cdot\jmath)}\right)=\\
&=\frac{sin(5\cdot t)-5\cdot sin(3\cdot t)+10\cdot sin(t)}{(2\cdot\jmath)^4}-\\
&-\left(\frac{sin(3\cdot t)-3\cdot sin(t)}{(2\cdot\jmath)^2}\right)=\\
&=\frac{sin(5\cdot t)-5\cdot sin(3\cdot t)+10\cdot sin(t)}{16}+\left(\frac{sin(3\cdot t)-3\cdot sin(t)}{4}\right)=\\
&=\frac{sin(5\cdot t)-5\cdot sin(3\cdot t)+10\cdot sin(t)}{16}+\frac{4\cdot sin(3\cdot t)-12\cdot sin(t)}{16}=\\
&=\frac{sin(5\cdot t)-sin(3\cdot t)-2\cdot sin(t)}{16}
\end{aligned}
$$

To sum up:

$$f_1(t)=sin^5(t)-sin^3(t)=\frac{sin(5\cdot t)-sin(3\cdot t)-2\cdot sin(t)}{16}$$

$$
\begin{aligned}
&f_2(t)=cos^6(t)-cos^4(t)=\\
&=\left(\frac{e^{\jmath\cdot t}+e^{-\jmath\cdot t}}{2}\right)^6-\left(\frac{e^{\jmath\cdot t}+e^{-\jmath\cdot t}}{2}\right)^4=
\end{aligned}
$$

$$= \frac{\left(e^{\jmath \cdot t}+e^{-\jmath \cdot t}\right)^6}{2^6}-\frac{\left(e^{\jmath \cdot t}+e^{-\jmath \cdot t}\right)^4}{2^4}=$$

$$= \frac{1\cdot(e^{\jmath\cdot t})^6\cdot(e^{-\jmath\cdot t})^0+6\cdot(e^{\jmath\cdot t})^5\cdot(e^{-\jmath\cdot t})^1+15\cdot(e^{\jmath\cdot t})^4\cdot(e^{-\jmath\cdot t})^2+20\cdot(e^{\jmath\cdot t})^3\cdot(e^{-\jmath\cdot t})^3}{2^6}+$$

$$+ \frac{15\cdot(e^{\jmath\cdot t})^2\cdot(e^{-\jmath\cdot t})^4+6\cdot(e^{\jmath\cdot t})^1\cdot(e^{-\jmath\cdot t})^5+1\cdot(e^{\jmath\cdot t})^0\cdot(e^{-\jmath\cdot t})^6}{2^6}-$$

$$- \left(\frac{1\cdot(e^{\jmath\cdot t})^4\cdot(e^{-\jmath\cdot t})^0+4\cdot(e^{\jmath\cdot t})^3\cdot(e^{-\jmath\cdot t})^1+6\cdot(e^{\jmath\cdot t})^2\cdot(e^{-\jmath\cdot t})^2+4\cdot(e^{\jmath\cdot t})^1\cdot(e^{-\jmath\cdot t})^3+1\cdot(e^{\jmath\cdot t})^0\cdot(e^{-\jmath\cdot t})^4}{2^4}\right)=$$

$$= \frac{1\cdot e^{\jmath\cdot t\cdot 6}\cdot 1+6\cdot e^{\jmath\cdot t\cdot 5}\cdot e^{-\jmath\cdot t\cdot 1}+15\cdot e^{\jmath\cdot t\cdot 4}\cdot e^{-\jmath\cdot t\cdot 2}+20\cdot e^{\jmath\cdot t\cdot 3}\cdot e^{-\jmath\cdot t\cdot 3}}{2^6}+$$

$$+ \frac{15\cdot e^{\jmath\cdot t\cdot 2}\cdot e^{-\jmath\cdot t\cdot 4}+6\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 5}+1\cdot 1\cdot e^{-\jmath\cdot t\cdot 6}}{2^6}-$$

$$- \left(\frac{1\cdot e^{\jmath\cdot t\cdot 4}\cdot 1+4\cdot e^{\jmath\cdot t\cdot 3}\cdot e^{-\jmath\cdot t\cdot 1}+6\cdot e^{\jmath\cdot t\cdot 2}\cdot e^{-\jmath\cdot t\cdot 2}+4\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 3}+1\cdot 1\cdot e^{-\jmath\cdot t\cdot 4}}{2^4}\right)=$$

$$= \frac{e^{\jmath\cdot t\cdot 6}+6\cdot e^{\jmath\cdot t\cdot 4}+15\cdot e^{\jmath\cdot t\cdot 2}+20\cdot e^{\jmath\cdot t\cdot 0}+15\cdot e^{-\jmath\cdot t\cdot 2}+6\cdot e^{-\jmath\cdot t\cdot 4}+e^{-\jmath\cdot t\cdot 6}}{2^6}-$$

$$- \left(\frac{e^{\jmath\cdot t\cdot 4}+4\cdot e^{\jmath\cdot t\cdot 2}+6\cdot e^{\jmath\cdot t\cdot 0}+4\cdot e^{-\jmath\cdot t\cdot 2}+e^{-\jmath\cdot t\cdot 4}}{2^4}\right)=$$

$$= \frac{e^{\jmath\cdot t\cdot 6}+e^{-\jmath\cdot t\cdot 6}+6\cdot e^{\jmath\cdot t\cdot 4}+6\cdot e^{-\jmath\cdot t\cdot 4}+15\cdot e^{\jmath\cdot t\cdot 2}+15\cdot e^{-\jmath\cdot t\cdot 2}+20}{2^6}-$$

$$- \left(\frac{e^{\jmath\cdot t\cdot 4}+e^{-\jmath\cdot t\cdot 4}+4\cdot e^{\jmath\cdot t\cdot 2}+4\cdot e^{-\jmath\cdot t\cdot 2}+6}{2^4}\right)=$$

$$= \frac{e^{\jmath\cdot t\cdot 6}+e^{-\jmath\cdot t\cdot 6}+6\cdot(e^{\jmath\cdot t\cdot 4}+e^{-\jmath\cdot t\cdot 4})+15\cdot(e^{\jmath\cdot t\cdot 2}+e^{-\jmath\cdot t\cdot 2})+20}{2^5\cdot 2}-$$

$$- \left(\frac{e^{\jmath\cdot t\cdot 4}+e^{-\jmath\cdot t\cdot 4}+4\cdot(e^{\jmath\cdot t\cdot 2}+e^{-\jmath\cdot t\cdot 2})+6}{2^3\cdot 2}\right)=$$

$$= \frac{cos(6\cdot t)+6\cdot cos(4\cdot t)+15\cdot cos(2\cdot t)+10}{2^5}-$$

$$- \left(\frac{cos(4\cdot t)+4\cdot cos(2\cdot t)+3}{2^3}\right)=$$

$$= \frac{cos(6\cdot t)+6\cdot cos(4\cdot t)+15\cdot cos(2\cdot t)+10-4\cdot cos(4\cdot t)-16\cdot cos(2\cdot t)-12}{2^5}=$$

$$= \frac{cos(6\cdot t)+2\cdot cos(4\cdot t)-cos(2\cdot t)-2}{2^5}=$$

$$= \frac{cos(6\cdot t)+2\cdot cos(4\cdot t)-cos(2\cdot t)-2}{32}$$

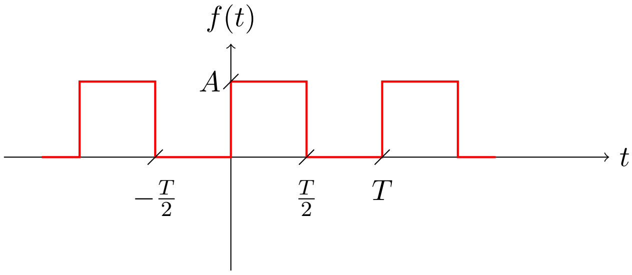
To sum up:

$$f_2(t)=cos^6(t)-cos^4(t)=\frac{cos(6\cdot t)+2\cdot cos(4\cdot t)-cos(2\cdot t)-2}{32}$$

1.1 Basic signal metrics

1.1.1 Mean value of a signal

Task 1. Calculate the mean value of the following periodic signal $f(t)$:



Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.4}$$

The mean value for periodic signals is defined by:

$$\bar{f} = \frac{1}{T} \int_T f(t) \cdot dt \tag{1.5}$$

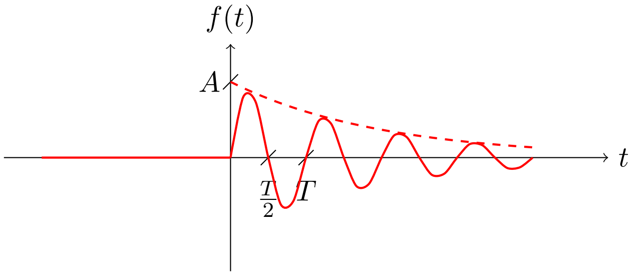
Compute mean value for period $k = 0$:

$$\begin{aligned} \bar{f} &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{1}{T} \left(A \cdot t|_0^{\frac{T}{2}} \right) = \\ &= \frac{A}{T} \cdot t|_0^{\frac{T}{2}} = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A}{2} \end{aligned} \tag{1.6}$$

The mean value equals to $\frac{A}{2}$.

Task 2.

Calculate the mean value of the signal $f(t) = \mathbf{1}(t) \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:



The mean value for non-periodic signals is defined by:

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \cdot dt \tag{1.7}$$

Compute mean value:

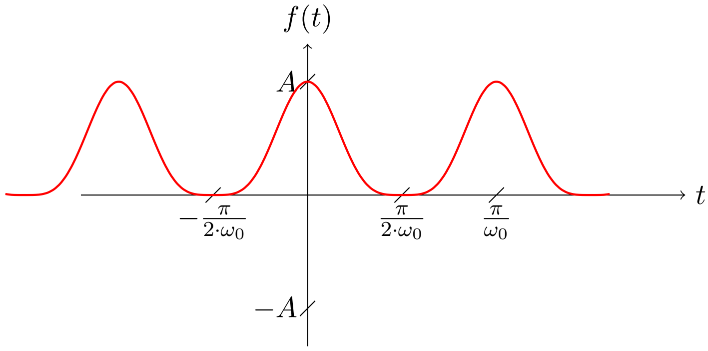
$$\begin{aligned}
\bar{f} &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \cdot dt = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \mathbf{1}(t) \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} 0 \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{0}^{\frac{\tau}{2}} 1 \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} 0 \cdot dt + \int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(0 + \int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \left\{ \begin{matrix} u = sin(\frac{2\pi}{T} \cdot t) & dv = e^{-a \cdot t} \cdot dt \\ du = \frac{2\pi}{T} \cdot cos(\frac{2\pi}{T} \cdot t) \cdot dt & v = -\frac{1}{a} \cdot e^{-a \cdot t} \end{matrix} \right\} = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(-\frac{1}{a} \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \Big|_{0}^{\frac{\tau}{2}} - \int_{0}^{\frac{\tau}{2}} -\frac{1}{a} \cdot e^{-a \cdot t} \cdot \frac{2\pi}{T} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\left(-\frac{1}{a} \cdot e^{-a \cdot \frac{\tau}{2}} \cdot sin\left(\frac{2\pi}{T} \cdot \frac{\tau}{2}\right) + \frac{1}{a} \cdot e^{-a \cdot 0} \cdot sin\left(\frac{2\pi}{T} \cdot 0\right) \right) + \right. \\
&\left. + \frac{1}{a} \cdot \frac{2\pi}{T} \cdot \int_{0}^{\frac{\tau}{2}} \cdot e^{-a \cdot t} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \left\{ \begin{matrix} u = cos(\frac{2\pi}{T} \cdot t) & dv = e^{-a \cdot t} \cdot dt \\ du = -\frac{2\pi}{T} \cdot sin(\frac{2\pi}{T} \cdot t) \cdot dt & v = -\frac{1}{a} \cdot e^{-a \cdot t} \end{matrix} \right\} = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\left(-\frac{1}{a} \cdot e^{-a \cdot \frac{\tau}{2}} \cdot sin\left(\frac{2\pi}{T} \cdot \frac{\tau}{2}\right) + \frac{1}{a} \cdot e^{-a \cdot 0} \cdot sin\left(\frac{2\pi}{T} \cdot 0\right) \right) + \right. \\
&\left. + \frac{1}{a} \cdot \frac{2\pi}{T} \cdot \left(-\frac{1}{a} \cdot e^{-a \cdot t} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \Big|_{0}^{\frac{\tau}{2}} - \int_{0}^{\frac{\tau}{2}} -\frac{1}{a} \cdot e^{-a \cdot t} \cdot \frac{2\pi}{T} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\tau\to\infty} \frac{1}{\tau}\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot 1\cdot 0\right)+\right.\\
&+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot e^{-a\cdot 0}\cdot cos\left(\frac{2\pi}{T}\cdot 0\right)\right)+\right.\\
&\left.\left.+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)\right)=\\
&= \lim_{\tau\to\infty} \frac{1}{\tau}\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+0\right)+\right.\\
&+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot 1\cdot 1\right)+\right.\\
&\left.\left.+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)\right)=\\
&= \lim_{\tau\to\infty} \frac{1}{\tau}\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\right.\\
&-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}+\\
&\left.+\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\left\{\begin{matrix}-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\\ -\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}+\\ +\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt=\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\end{matrix}\right\}=\\
&=\left\{\begin{matrix}-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}=\\ =\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\end{matrix}\right\}=\\
&=\left\{\begin{matrix}-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}=\\ =\left(1-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\right)\cdot\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\end{matrix}\right\}=\\
&=\left\{\begin{matrix}\frac{-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}}{\left(1-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\right)}=\\ =\int_0^{\frac{\tau}{2}} e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\end{matrix}\right\}=\\
&= \lim_{\tau\to\infty} \frac{1}{\tau}\left(\frac{-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}}{\left(1-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\right)}\right)=\\
&=0
\end{aligned}
$$

The mean value equals to 0.

Task 3.

Calculate the mean value of the following periodic signal $f(t) = A \cdot cos^4(\omega_0 \cdot t)$:



The mean value for periodic signals is defined by:

$$\bar{f} = \frac{1}{T}\int_T f(t)\cdot dt \tag{1.8}$$

The period of the given signal has to be identified. In our case: $T = \frac{\pi}{\omega_0}$.
Compute mean value for period $t \in \left(-\frac{\pi}{2\cdot\omega_0}; \frac{\pi}{2\cdot\omega_0}\right)$:

$$\begin{aligned}
\bar{f} &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{\frac{\pi}{\omega_0}}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot cos\left(\omega_0\cdot t\right)^4\cdot dt = \\
&= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot cos\left(\omega_0\cdot t\right)^4\cdot dt = \\
&= \left\{cos(x) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot\left(\frac{e^{\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot\omega_0\cdot t}}{2}\right)^4\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\left(\frac{e^{\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot\omega_0\cdot t}}{2}\right)^2\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{\left(e^{\jmath\cdot\omega_0\cdot t}\right)^2+2\cdot e^{\jmath\cdot\omega_0\cdot t}\cdot e^{-\jmath\cdot\omega_0\cdot t}+\left(e^{-\jmath\cdot\omega_0\cdot t}\right)^2}{2^2}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2\cdot e^{\jmath\cdot\omega_0\cdot t-\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2\cdot e^{0}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}+2}{4}\right)^2\cdot dt =
\end{aligned}$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)\cdot 2 + 2^2}{4^2} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + \left(e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot 2 + 2\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\cdot 2 + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 2\cdot 2\cdot\omega_0\cdot t} + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t - \jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 4\cdot\omega_0\cdot t} + 2\cdot e^{0} + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 4\cdot\omega_0\cdot t} + 2 + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \left(e^{\jmath\cdot 4\cdot\omega_0\cdot t} + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 6\right) \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{\jmath\cdot 4\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{-\jmath\cdot 4\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\cdot dt + \right.$$

$$\left. + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 6\cdot dt\right) =$$

$$= \left\{\begin{matrix} z_1 = \jmath\cdot 4\cdot\omega_0\cdot t & z_2 = -\jmath\cdot 4\cdot\omega_0\cdot t & z_3 = \jmath\cdot 2\cdot\omega_0\cdot t & z_4 = -\jmath\cdot 2\cdot\omega_0\cdot t \\ dz_1 = \jmath\cdot 4\cdot\omega_0\cdot dt & dz_2 = -\jmath\cdot 4\cdot\omega_0\cdot dt & dz_3 = \jmath\cdot 2\cdot\omega_0\cdot dt & dz_4 = -\jmath\cdot 2\cdot\omega_0\cdot dt = \\ dt = \frac{dz_1}{\jmath\cdot 4\cdot\omega_0} & dt = \frac{dz_2}{-\jmath\cdot 4\cdot\omega_0} & dt = \frac{dz_3}{\jmath\cdot 2\cdot\omega_0} & dt = \frac{dz_4}{-\jmath\cdot 2\cdot\omega_0} \end{matrix}\right\} =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot 4\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot 4\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{z_3}\cdot\frac{dz_3}{\jmath\cdot 2\cdot\omega_0} + \right.$$

$$\left. + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{z_4}\cdot\frac{dz_4}{-\jmath\cdot 2\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 6\cdot dt\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_1}\cdot dz_1 + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_2}\cdot dz_2 + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_3}\cdot dz_3 + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_4}\cdot dz_4 + 6\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} dt\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot e^{z_1}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot e^{z_2}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot e^{z_3}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot e^{z_4}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + 6\cdot t\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot e^{\jmath\cdot 4\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot e^{-\jmath\cdot 4\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + 6\cdot t\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot\left(e^{\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{-\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot\left(e^{-\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{+\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \right.$$

$$+ \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot\left(e^{\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{-\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot\left(e^{-\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) +$$

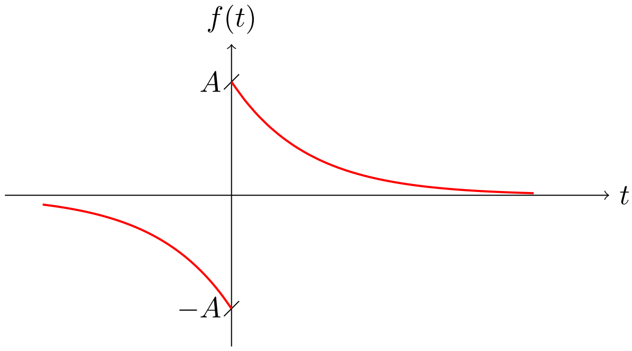
$$\left. + 6\cdot\left(\frac{\pi}{2\cdot\omega_0} + \frac{\pi}{2\cdot\omega_0}\right)\right) =$$

$$
\begin{aligned}
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot \left(e^{\jmath \cdot 2 \cdot \pi} - e^{-\jmath \cdot 2 \cdot \pi} \right) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot \left(e^{-\jmath \cdot 2 \cdot \pi} - e^{\jmath \cdot 2 \cdot \pi} \right) + \right. \\
&\left. + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot (e^{\jmath \cdot \pi} - e^{-\jmath \cdot \pi}) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot (e^{-\jmath \cdot \pi} - e^{\jmath \cdot \pi}) + 6 \cdot \left(\frac{2 \cdot \pi}{2 \cdot \omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot (1 - 1) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot (1 - 1) + \right. \\
&\left. + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot (-1 - (-1)) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot (-1 - (-1)) + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot (0) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot (0) + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot (0) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot (0) + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(0 + 0 + 0 + 0 + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot 6 \cdot \left(\frac{\pi}{\omega_0} \right) = \\
&= \frac{A}{16} \cdot 6 = \\
&= \frac{A}{8} \cdot 3 = \\
&= \frac{3}{8} \cdot A
\end{aligned}
$$

The mean value equals to $\frac{3}{8} \cdot A$.

1.1.2 Energy of a signal

Task 4. Compute energy of $f(t)$ signal given below:



$$f(t) = \begin{cases} -A \cdot e^{a\cdot t} & dla \quad t \in (-\infty; 0) \\ A \cdot e^{-a\cdot t} & dla \quad t \in (0; \infty) \end{cases} \tag{1.9}$$

Total energy of a non-periodic signal (or in short - signal energy) is defined by:

$$E = \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2 \cdot dt \tag{1.10}$$

For the given signal we get

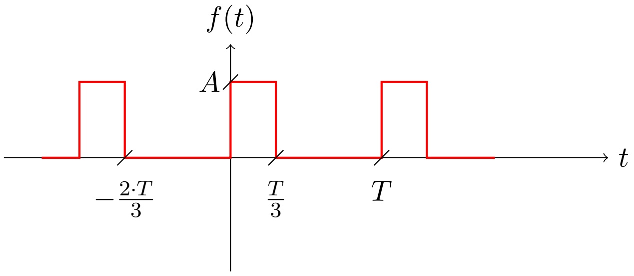
$$\begin{aligned}
E &= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2 \cdot dt = \\
&= \lim_{\tau\to\infty} \left(\int_{-\frac{\tau}{2}}^{0} \left| -A \cdot e^{a\cdot t} \right|^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} \left| A \cdot e^{-a\cdot t} \right|^2 \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \left(\int_{-\frac{\tau}{2}}^{0} \left(-A \cdot e^{a\cdot t} \right)^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} \left(A \cdot e^{-a\cdot t} \right)^2 \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \left(\int_{-\frac{\tau}{2}}^{0} (-A)^2 \cdot \left(e^{a\cdot t} \right)^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} (A)^2 \cdot \left(e^{-a\cdot t} \right)^2 \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \left(\int_{-\frac{\tau}{2}}^{0} A^2 \cdot e^{2\cdot a\cdot t} \cdot dt + \int_{0}^{\frac{\tau}{2}} A^2 \cdot e^{-2\cdot a\cdot t} \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \left(A^2 \cdot \int_{-\frac{\tau}{2}}^{0} e^{2\cdot a\cdot t} \cdot dt + A^2 \cdot \int_{0}^{\frac{\tau}{2}} e^{-2\cdot a\cdot t} \cdot dt \right) = \\
&= \lim_{\tau\to\infty} A^2 \cdot \left(\int_{-\frac{\tau}{2}}^{0} e^{2\cdot a\cdot t} \cdot dt + \int_{0}^{\frac{\tau}{2}} e^{-2\cdot a\cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z = 2 \cdot a \cdot t & w = -2 \cdot a \cdot t \\ dz = 2 \cdot a \cdot dt & dw = -2 \cdot a \cdot dt \\ dt = \frac{dz}{2\cdot a} & dt = \frac{dw}{-2\cdot a} \end{array} \right\} = \\
&= \lim_{\tau\to\infty} A^2 \cdot \left(\int_{-\frac{\tau}{2}}^{0} e^{z} \cdot \frac{dz}{2 \cdot a} + \int_{0}^{\frac{\tau}{2}} e^{w} \cdot \frac{dw}{-2 \cdot a} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(\int_{-\frac{\tau}{2}}^{0} e^{z}\cdot dz - \int_{0}^{\frac{\tau}{2}} e^{w}\cdot dw\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(e^{z}|_{-\frac{\tau}{2}}^{0} - e^{w}|_{0}^{\frac{\tau}{2}}\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(e^{2\cdot a\cdot t}\Big|_{-\frac{\tau}{2}}^{0} - e^{-2\cdot a\cdot dt}\Big|_{0}^{\frac{\tau}{2}}\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(\left(e^{2\cdot a\cdot 0} - e^{-2\cdot a\cdot\frac{\tau}{2}}\right) - \left(e^{-2\cdot a\cdot\frac{\tau}{2}} - e^{-2\cdot a\cdot 0}\right)\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(\left(e^{0} - e^{-a\cdot\tau}\right) - \left(e^{-a\cdot\tau} - e^{0}\right)\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(1 - e^{-a\cdot\tau} - e^{-a\cdot\tau} + 1\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot\left(2 - 2\cdot e^{-a\cdot\tau}\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a}\cdot 2\cdot\left(1 - e^{-a\cdot\tau}\right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{a}\cdot\left(1 - e^{-a\cdot\tau}\right) = \\
&= \frac{A^2}{a}
\end{aligned}
$$

Energy equals to $\frac{A^2}{a}$.

1.1.3 Power and effective value of a signal

Task 5. Compute the average power for the following periodic signal $f(t)$:



Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{3} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{3} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.11}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.12}$$

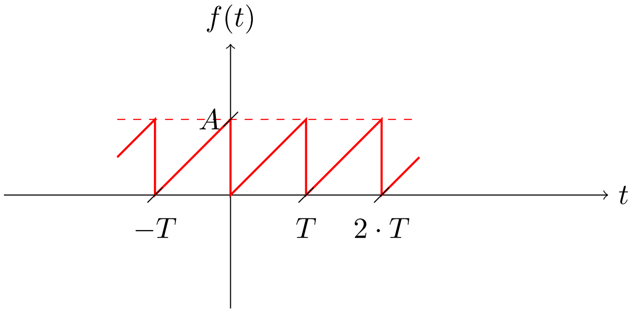
Compute average power for period $k = 0$

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} |A|^2 \cdot dt + \int_{\frac{T}{3}}^{T} |0|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} A^2 \cdot dt + \int_{\frac{T}{3}}^{T} 0 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{3}} dt + 0 \right) = \\ &= \frac{A^2}{T} \cdot t|_0^{\frac{T}{3}} = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{3} - 0 \right) = \\ &= \frac{A^2}{T} \cdot \frac{T}{3} = \\ &= \frac{A^2}{3} \end{aligned}$$

Average power equals to $\frac{A^2}{3}$.

Task 6.

Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.13}$$

In the first period (i.e. $t \in (0; T)$), linear function crosses two points: $(0, 0)$ and (T, A). So, in order to derive a and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in C$$

The average power for periodic signals is defined by:

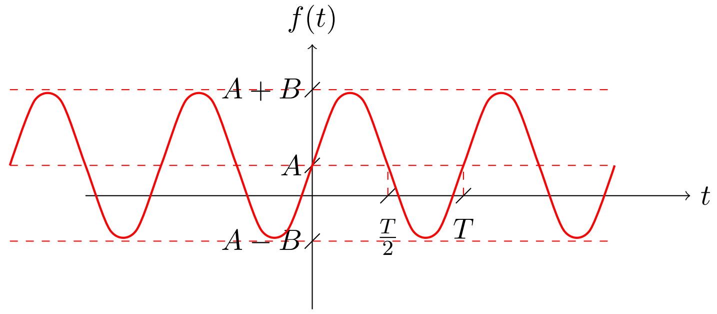
$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.14}$$

In our case we get:

$$
\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| \frac{A}{T} \cdot t \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(\frac{A}{T} \cdot t \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \frac{A^2}{T^2} \cdot t^2 \cdot dt = \\
&= \frac{1}{T} \cdot \frac{A^2}{T^2} \cdot \int_0^T t^2 \cdot dt = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^T \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot 0^3 \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - 0 \right) = \\
&= \frac{A^2}{T^3} \cdot \frac{1}{3} \cdot T^3 = \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}$.

Task 7. Compute the average power for the following periodic signal $f(t) = A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.15}$$

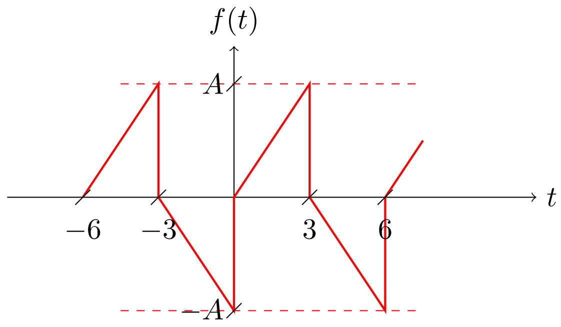
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left|A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A^2 + 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) + B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right)\right) \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^T A^2 \cdot dt + \int_0^T 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_0^T B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \frac{A^2}{T} \cdot \int_0^T dt + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{B^2}{T} \cdot \int_0^T sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \begin{Bmatrix} z = \frac{2\pi}{T} \cdot t \\ dz = \frac{2\pi}{T} \cdot dt \quad dt = \frac{dz}{\frac{2\pi}{T}} = \frac{T}{2\pi} \cdot dz \end{Bmatrix} = \\
&= \frac{A^2}{T} \cdot t|_0^T + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\,(z) \cdot \frac{T}{2\pi} \cdot dz + \frac{B^2}{T} \cdot \int_0^T \frac{1}{2} \cdot \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\right) \cdot dt = \\
&= \frac{A^2}{T} \cdot (T - 0) + \frac{2 \cdot A \cdot B}{T} \cdot \frac{T}{2\pi} \cdot \int_0^T sin\,(z) \cdot dz + \frac{B^2}{T} \cdot \frac{1}{2} \cdot \int_0^T \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\right) \cdot dt = \\
&= \frac{A^2}{T} \cdot T + \frac{A \cdot B}{\pi} \cdot \left(-cos\,(z)|_0^T\right) + \frac{B^2}{2 \cdot T} \cdot \left(\int_0^T 1 \cdot dt - \int_0^T cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \begin{Bmatrix} w = 2 \cdot \frac{2\pi}{T} \cdot t \\ dw = 2 \cdot \frac{2\pi}{T} \cdot dt \quad dt = \frac{dw}{\frac{4\pi}{T}} = \frac{T}{4\pi} \cdot dw \end{Bmatrix} = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^T\right) + \frac{B^2}{2 \cdot T} \cdot \left(t|_0^T - \int_0^T cos\,(w) \cdot \frac{T}{4\pi} \cdot dw\right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot T\right) + cos\left(\frac{2\pi}{T} \cdot 0\right)\right) + \frac{B^2}{2 \cdot T} \cdot \left((T - 0) - \frac{T}{4\pi} \cdot \int_0^T cos\,(w) \cdot dw\right) =
\end{aligned}$$

$$
\begin{aligned}
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(2\pi\right) + cos\left(0\right)\right) + \frac{B^2}{2 \cdot T} \cdot \left(T - \frac{T}{4\pi} \cdot -sin\left(w\right)\Big|_0^T\right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-1 + 1\right) + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_0^T\right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot 0 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(2 \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(2 \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) = \\
&= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(4\pi\right) - sin\left(0\right)\right)\right) = \\
&= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(0 - 0\right)\right) = \\
&= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T\right) = \\
&= A^2 + \frac{B^2}{2}
\end{aligned}
$$

The average power equals to $A^2 + \frac{B^2}{2}$.

Task 8. Calculate the average power and the effective value (RMS) for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.16}$$

In the first interval of the first period (i.e. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (i.e. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.17}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{6} \cdot \left(\int_0^3 \left| \frac{A}{3} \cdot t \right|^2 \cdot dt + \int_3^6 \left| -\frac{A}{3} \cdot t + A \right|^2 \cdot dt \right) = \\
&= \frac{1}{6} \cdot \int_0^3 \left(\frac{A}{3} \cdot t \right)^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(-\frac{A}{3} \cdot t + A \right)^2 \cdot dt = \\
&= \frac{1}{6} \cdot \int_0^3 \frac{A^2}{9} \cdot t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(\left(-\frac{A}{3} \cdot t \right)^2 - 2 \cdot \frac{A}{3} \cdot t \cdot A + A^2 \right) \cdot dt = \\
&= \frac{A^2}{54} \cdot \int_0^3 t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \frac{A^2}{9} \cdot t^2 \cdot dt - \frac{1}{6} \cdot \int_3^6 \frac{2 \cdot A^2}{3} \cdot t \cdot dt + \frac{1}{6} \cdot \int_3^6 A^2 \cdot dt = \\
&= \frac{A^2}{54} \cdot \left. \frac{t^3}{3} \right|_0^3 + \frac{A^2}{54} \cdot \int_3^6 t^2 \cdot dt - \frac{2 \cdot A^2}{18} \cdot \int_3^6 t^2 \cdot dt + \frac{A^2}{6} \cdot \int_3^6 dt =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{162} \cdot \left(3^3 - 0^3\right) + \frac{A^2}{54} \cdot \left.\frac{t^3}{3}\right|_3^6 - \frac{2 \cdot A^2}{18} \cdot \left.\frac{t^2}{2}\right|_3^6 + \frac{A^2}{6} \cdot t|_3^6 = \\
&= \frac{A^2}{162} \cdot 27 + \frac{A^2}{162} \cdot \left(6^3 - 3^3\right) - \frac{2 \cdot A^2}{36} \cdot \left(6^2 - 3^2\right) + \frac{A^2}{6} \cdot (6 - 3) = \\
&= \frac{A^2}{6} + \frac{A^2}{162} \cdot 189 - \frac{2 \cdot A^2}{36} \cdot 27 + \frac{A^2}{6} \cdot 3 = \\
&= \frac{A^2}{6} + \frac{7 \cdot A^2}{6} - \frac{9 \cdot A^2}{6} + \frac{3 \cdot A^2}{6} = \\
&= \frac{2 \cdot A^2}{6} = \\
&= \frac{A^2}{3}
\end{aligned}
$$

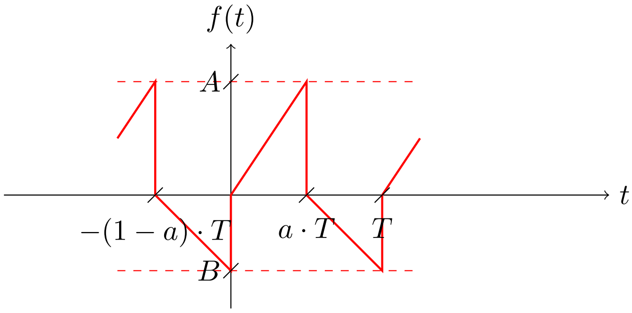
The average power equals to $\frac{A^2}{3}$.

The effective value (RMS) is defined by:

$$RMS = \sqrt{P} \tag{1.18}$$

Therefore, effective value (RMS) equals to $\frac{A}{\sqrt{3}}$.

Task 9. Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{1.19}$$

In the first interval of the first period (i.e. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a\cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T - a \cdot T} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{T-a\cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{T-a\cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.20}$$

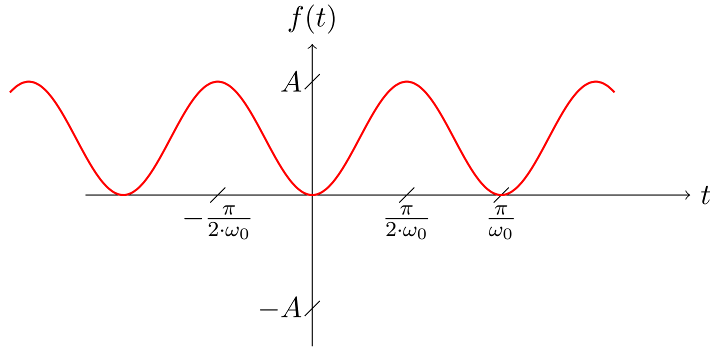
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{a\cdot T} \left| \frac{A}{a \cdot T} \cdot t \right|^2 \cdot dt + \int_{a\cdot T}^{T} \left| \frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right|^2 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \int_0^{a\cdot T} \left(\frac{A}{a \cdot T} \cdot t \right)^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^{a\cdot T} \frac{A^2}{a^2 \cdot T^2} \cdot t^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\left(\frac{B}{T - a \cdot T} \cdot t \right)^2 - 2 \cdot \frac{B}{T - a \cdot T} \cdot t \cdot \frac{B}{1-a} \cdot a + \left(\frac{B}{1-a} \cdot a \right)^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \int_0^{a\cdot T} t^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 - 2 \cdot \frac{B^2}{T \cdot (1-a)^2} \cdot t \cdot a + \frac{B^2}{(1-a)^2} \cdot a^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^{a\cdot T} \right) + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 \cdot dt +
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{T}\cdot\int_{a\cdot T}^{T} 2\cdot\frac{B^2}{T\cdot(1-a)^2}\cdot t\cdot a\cdot dt+\frac{1}{T}\cdot\int_{a\cdot T}^{T}\frac{B^2}{(1-a)^2}\cdot a^2\cdot dt=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot t^3\Big|_0^{a\cdot T}\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\int_{a\cdot T}^{T} t^2\cdot dt+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\int_{a\cdot T}^{T} t\cdot dt+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\int_{a\cdot T}^{T} dt=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot(a\cdot T)^3-\frac{1}{3}\cdot 0^3\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot t^3\Big|_{a\cdot T}^{T}\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot t^2\Big|_{a\cdot T}^{T}\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\left(t|_{a\cdot T}^{T}\right)=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot a^3\cdot T^3-0\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot(a\cdot T)^3\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot(a\cdot T)^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(T-a\cdot T)=\\
&=\frac{A^2}{a^2\cdot T^3}\cdot\frac{1}{3}\cdot a^3\cdot T^3+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot a^3\cdot T^3\right)+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot a^2\cdot T^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(1-a^3\right)\cdot\frac{1}{3}\cdot T^3+\\
&-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(1-a^2\right)\cdot\frac{1}{2}\cdot T^2+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{(1-a)^2}\cdot(1-a)\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}+\\
&-\frac{2\cdot B^2}{(1-a)^2}\cdot a\cdot(1-a)\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}-\frac{2\cdot B^2}{1-a}\cdot a\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{1}{3}-2\cdot a\cdot(1+a)\cdot\frac{1}{2}+a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{2}{6}-2\cdot a\cdot(1+a)\cdot\frac{3}{6}+a^2\cdot\frac{6}{6}\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(\left(1+a+a^2\right)\cdot 2-2\cdot a\cdot(1+a)\cdot 3+a^2\cdot 6\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2+2\cdot a+2\cdot a^2-6\cdot a-6\cdot a^2+6\cdot a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2-4\cdot a+2\cdot a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot\left(1-2\cdot a+a^2\right)=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot(1-a)^2=\\
&=\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)$.

Task 10. Calculate the average power for the periodic signal $f(t) = A \cdot sin^2(\omega_0 \cdot t)$ given below.



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.21}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| A \cdot sin^2(\omega_0 \cdot t) \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot sin^4(\omega_0 \cdot t) \cdot dt = \\
&= \frac{1}{T} \cdot \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} \right)^4 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{\left(e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \cdot dt = \\
&= \left\{ \begin{array}{llllllllll}
n = 0: & & & & & 1 & & & & \\
n = 1: & & & & 1 & & 1 & & & \\
n = 2: & & & 1 & & 2 & & 1 & & \\
n = 3: & & 1 & & 3 & & 3 & & 1 & \\
n = 4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^4 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^0 + 4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^3 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^1 + 6 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^2 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^2}{(2 \cdot \jmath)^4} + \right. \\
&+ \left. \frac{4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^1 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^3 + 1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^0 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-\jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} + \right. \\
&+ \left. \frac{-4 \cdot e^{\jmath \cdot \omega_0 \cdot t} \cdot e^{-3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t - 0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t - \jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t - 2 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{\jmath \cdot \omega_0 \cdot t - 3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t - 4 \cdot \jmath \cdot \omega_0 \cdot t}}{16 \cdot 1} \cdot dt =
\end{aligned}$$

$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} + 6\cdot e^{0\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\cdot e^{0}}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6}{16} \cdot dt =$$
$$= \frac{A^2}{16\cdot T} \cdot \int_0^T \left(e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\right) dt =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\int_0^T e^{4\cdot\jmath\cdot\omega_0\cdot t}\cdot dt + \int_0^T e^{-4\cdot\jmath\cdot\omega_0\cdot t}\cdot dt - 4\cdot\int_0^T e^{2\cdot\jmath\cdot\omega_0\cdot t}\cdot dt - 4\cdot\int_0^T e^{-2\cdot\jmath\cdot\omega_0\cdot t}\cdot dt + 6\cdot\int_0^T dt\right) =$$
$$= \left\{\begin{array}{cccc} z_1 = 4\cdot\jmath\cdot\omega_0\cdot t & z_2 = -4\cdot\jmath\cdot\omega_0\cdot t & z_3 = 2\cdot\jmath\cdot\omega_0\cdot t & z_4 = -2\cdot\jmath\cdot\omega_0\cdot t = \\ dz_1 = 4\cdot\jmath\cdot\omega_0\cdot dt & dz_2 = -4\cdot\jmath\cdot\omega_0\cdot dt & dz_3 = 2\cdot\jmath\cdot\omega_0\cdot dt & dz_4 = -2\cdot\jmath\cdot\omega_0\cdot dt = \\ dt = \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot dz_1 & dt = \frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot dz_2 & dt = \frac{1}{2\cdot\jmath\cdot\omega_0}\cdot dz_3 & dt = \frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot dz_4 \end{array}\right\} =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\int_0^T e^{z_1}\cdot\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot dz_1 + \int_0^T e^{z_2}\cdot\frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot dz_2 + \right.$$
$$\left. - 4\cdot\int_0^T e^{z_3}\cdot\frac{1}{2\cdot\jmath\cdot\omega_0}\cdot dz_3 - 4\cdot\int_0^T e^{z_4}\cdot\frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot dz_4 + 6\cdot\int_0^T dt\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_1}\cdot dz_1 + \frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_2}\cdot dz_2 + \right.$$
$$\left. -4\cdot\frac{1}{2\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_3}\cdot dz_3 - 4\cdot\frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_4}\cdot dz_4 + 6\cdot\int_0^T dt\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{z_1}\big|_0^T - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{z_2}\big|_0^T - \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{z_3}\big|_0^T + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{z_4}\big|_0^T + \right.$$
$$\left. +6\cdot t\big|_0^T\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \right.$$
$$\left. +6\cdot t\big|_0^T\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{4\cdot\jmath\cdot\omega_0\cdot 0}\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot 0}\right) + \right.$$
$$\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{2\cdot\jmath\cdot\omega_0\cdot 0}\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot 0}\right) + 6\cdot(T-0)\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + \right.$$
$$\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + 6\cdot T\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \right.$$
$$\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + 6\cdot T\right) =$$
$$= \frac{A^2}{16\cdot T} \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{1}{4\cdot\jmath\cdot\omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} + \frac{1}{4\cdot\jmath\cdot\omega_0} + \right.$$
$$\left. -\frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + \frac{4}{2\cdot\jmath\cdot\omega_0} + \frac{4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} - \frac{4}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) =$$

$$= \frac{A^2}{16\cdot T}\cdot\left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + \frac{4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) =$$
$$= \frac{A^2}{16\cdot T}\cdot\left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) =$$
$$= \frac{A^2}{16\cdot T}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath} - \frac{4}{\omega_0}\cdot\frac{e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath} + 6\cdot T\right) =$$
$$= \left\{sin(x) = \frac{e^{\jmath\cdot x} - e^{-\jmath\ cdotx}}{2\cdot\jmath}\right\} =$$
$$= \frac{A^2}{16\cdot T}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(4\cdot\omega_0\cdot T\right) - \frac{4}{\omega_0}\cdot sin\left(2\cdot\omega_0\cdot T\right) + 6\cdot T\right) =$$
$$= \left\{T = \frac{2\pi}{\omega_0}\right\} =$$
$$= \frac{A^2}{16\cdot\frac{2\pi}{\omega_0}}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(4\cdot\omega_0\cdot\frac{2\pi}{\omega_0}\right) - \frac{4}{\omega_0}\cdot sin\left(2\cdot\omega_0\cdot\frac{2\pi}{\omega_0}\right) + 6\cdot\frac{2\pi}{\omega_0}\right) =$$
$$= \frac{A^2\cdot\omega_0}{32\pi}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot sin\left(8\pi\right) - \frac{4}{\omega_0}\cdot sin\left(4\pi\right) + 6\cdot\frac{2\pi}{\omega_0}\right) =$$
$$= \frac{A^2\cdot\omega_0}{32\pi}\cdot\left(\frac{1}{2\cdot\omega_0}\cdot 0 - \frac{4}{\omega_0}\cdot 0 + 6\cdot\frac{2\pi}{\omega_0}\right) =$$
$$= \frac{A^2\cdot\omega_0}{32\pi}\cdot\frac{12\pi}{\omega_0} =$$
$$= \frac{3\cdot A^2}{8}$$

The average power equals to $\frac{3\cdot A^2}{8}$.

Task 11. Oblicz moc sygnału $f(t) = A \cdot sin\,(k \cdot t) + B \cdot cos\,(n \cdot t)$.

Pierwszym krokiem jest ustalenie czy sygnał $f(t)$ jest sygnałem okresowym czy nie. Nasz sygnał jest sumą dwóch funkcji okresowych $f_1(t) = A \cdot sin\,(k \cdot t)$ i $f_2(t) = B \cdot cos\,(n \cdot t)$.

Suma funkcji okresowych jest funkcją okresową, wtedy i tylko wtedy gdy stosunek okresów funkcji składowych jest liczbą wymierną

$$\frac{T_1}{T_2} \in \mathcal{W}$$

W naszym przypadku

$$T_1 = \frac{2\pi}{k}$$
$$T_2 = \frac{2\pi}{n}$$
$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{n}} = \frac{n}{k}$$

W ogólności liczby n i k mogą być dowolnymi liczbami rzeczywistymi $n, k \in \mathcal{R}$. Załóżmy jednak iż ułamek $\frac{n}{k}$ jest pewną liczba wymierna $\frac{a}{b}$ gdzie $a, b \in \mathcal{Z}$ są liczbami całkowitymi.

$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{n}} = \frac{n}{k} = \frac{a}{b} \quad a, b \in \mathcal{Z}$$

W takim przypadku okres naszego sygnału jest Najmniejszą Wspólną Wielokrotnością okresów funkcji składowych. Stwórzmy więc tabelę z kolejnymi wielokrotnościami okresów funkcji $f_1(t)$ i $f_2(t)$. Zgodnie z przyjętym przez nas założeniem

Wielokrotność okresu	1	2	3	$\cdots$	a	$\cdots$	b	$\cdots$
T_1	$\frac{2\pi}{k}$	$2 \cdot \frac{2\pi}{k}$	$3 \cdot \frac{2\pi}{k}$	$\cdots$	$a \cdot \frac{2\pi}{k}$	$\cdots$	$\boxed{b \cdot \frac{2\pi}{k}}$	$\cdots$
T_2	$\frac{2\pi}{n}$	$2 \cdot \frac{2\pi}{n}$	$3 \cdot \frac{2\pi}{n}$	$\cdots$	$\boxed{a \cdot \frac{2\pi}{n}}$	$\cdots$	$b \cdot \frac{2\pi}{n}$	$\cdots$

$$\frac{T_1}{T_2} = \frac{a}{b} \Rightarrow b \cdot T_1 = a \cdot T_2$$

a więc a-ta wielokrotność okresu pierwszej funkcji jest równa b-tej wielokrotności okresu drugiej funkcji, a wiec jest ona poszukiwaną przez nas Najmniejszą Wspólną Wielokrotnością. Związku z tym okresem naszego sygnału jest $T = b \cdot T_1 = a \cdot T_2$. Aby obliczyć moc należy wybrać przedział o długości jednego okresu. Przedział może być dowolnie położony, przyjmijmy więc przedział $t \in (0; T)$

Moc sygnału okresowego wyznaczamy ze wzoru

$$P = \frac{1}{T} \int_0^T |f(t)|^2 \cdot dt \tag{1.22}$$

Podstawiamy do wzoru na moc wzór naszej funkcji

$$P = \frac{1}{T} \int_0^T |f(t)|^2 \cdot dt = \\ = \frac{1}{T} \int_0^T |A \cdot sin\,(k \cdot t) + B \cdot cos\,(n \cdot t)|^2 \cdot dt =$$

Ponieważ mamy doczynienia z sygnałem o wartościach rzeczywistych możemy pominąć obliczenie modułu.

$$P = \frac{1}{T} \int_0^T (A \cdot sin\,(k \cdot t) + B \cdot cos\,(n \cdot t))^2 \cdot dt =$$
$$= \frac{1}{T} \int_0^T \Big((A \cdot sin\,(k \cdot t))^2 + 2 \cdot A \cdot sin\,(k \cdot t) \cdot B \cdot cos\,(n \cdot t) + (B \cdot cos\,(n \cdot t))^2\Big) \cdot dt =$$
$$= \frac{1}{T} \int_0^T \Big(A^2 \cdot sin^2\,(k \cdot t) + 2 \cdot A \cdot B \cdot sin\,(k \cdot t) \cdot cos\,(n \cdot t) + B^2 \cdot cos^2\,(n \cdot t)\Big) \cdot dt =$$
$$= \frac{1}{T} \cdot \left(\int_0^T A^2 \cdot sin^2\,(k \cdot t) \cdot dt + \int_0^T 2 \cdot A \cdot B \cdot sin\,(k \cdot t) \cdot cos\,(n \cdot t) \cdot dt + \int_0^T B^2 \cdot cos^2\,(n \cdot t) \cdot dt\right) =$$
$$= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^T sin^2\,(k \cdot t) \cdot dt + 2 \cdot A \cdot B \cdot \int_0^T sin\,(k \cdot t) \cdot cos\,(n \cdot t) \cdot dt + B^2 \cdot \int_0^T cos^2\,(n \cdot t) \cdot dt\right) =$$
$$= \left\{\ sin\,(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \quad cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\ \right\} =$$
$$= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^T \left(\frac{e^{\jmath \cdot k \cdot t} - e^{-\jmath \cdot k \cdot t}}{2 \cdot \jmath}\right)^2 \cdot dt + \right.$$
$$+ 2 \cdot A \cdot B \cdot \int_0^T \frac{e^{\jmath \cdot k \cdot t} - e^{-\jmath \cdot k \cdot t}}{2 \cdot \jmath} \cdot \frac{e^{\jmath \cdot n \cdot t} + e^{-\jmath \cdot n \cdot t}}{2} \cdot dt +$$
$$\left. + B^2 \cdot \int_0^T \left(\frac{e^{\jmath \cdot n \cdot t} + e^{-\jmath \cdot n \cdot t}}{2}\right)^2 \cdot dt\right) =$$
$$= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^T \frac{\left(e^{\jmath \cdot k \cdot t}\right)^2 - 2 \cdot e^{\jmath \cdot k \cdot t} \cdot e^{-\jmath \cdot k \cdot t} + \left(e^{-\jmath \cdot k \cdot t}\right)^2}{(2 \cdot \jmath)^2} \cdot dt + \right.$$
$$+ 2 \cdot A \cdot B \cdot \int_0^T \frac{e^{\jmath \cdot k \cdot t} \cdot e^{\jmath \cdot n \cdot t} + e^{\jmath \cdot k \cdot t} \cdot e^{-\jmath \cdot n \cdot t} - e^{-\jmath \cdot k \cdot t} \cdot e^{\jmath \cdot n \cdot t} - e^{-\jmath \cdot k \cdot t} \cdot e^{-\jmath \cdot n \cdot t}}{2 \cdot \jmath \cdot 2} \cdot dt +$$
$$\left. + B^2 \cdot \int_0^T \frac{\left(e^{\jmath \cdot n \cdot t}\right)^2 + 2 \cdot e^{\jmath \cdot n \cdot t} \cdot e^{-\jmath \cdot n \cdot t} + \left(e^{-\jmath \cdot n \cdot t}\right)^2}{2^2} \cdot dt\right) =$$
$$= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^T \frac{e^{2 \cdot \jmath \cdot k \cdot t} - 2 \cdot e^{\jmath \cdot k \cdot t - \jmath \cdot k \cdot t} + e^{-2 \cdot \jmath \cdot k \cdot t}}{-4} \cdot dt + \right.$$
$$+ 2 \cdot A \cdot B \cdot \int_0^T \frac{e^{\jmath \cdot k \cdot t + \jmath \cdot n \cdot t} + e^{\jmath \cdot k \cdot t - \jmath \cdot n \cdot t} - e^{-\jmath \cdot k \cdot t + \jmath \cdot n \cdot t} - e^{-\jmath \cdot k \cdot t - \jmath \cdot n \cdot t}}{4 \cdot \jmath} \cdot dt +$$
$$\left. + B^2 \cdot \int_0^T \frac{e^{2 \cdot \jmath \cdot n \cdot t} + 2 \cdot e^{\jmath \cdot n \cdot t - \jmath \cdot n \cdot t} + e^{-2 \cdot \jmath \cdot n \cdot t}}{4} \cdot dt\right) =$$
$$= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^T \frac{e^{2 \cdot \jmath \cdot k \cdot t} - 2 \cdot e^0 + e^{-2 \cdot \jmath \cdot k \cdot t}}{-4} \cdot dt + \right.$$
$$+ 2 \cdot A \cdot B \cdot \int_0^T \frac{e^{\jmath \cdot (k+n) \cdot t} + e^{\jmath \cdot (k-n) \cdot t} - e^{-\jmath \cdot (k-n) \cdot t} - e^{-\jmath \cdot (k+n) \cdot t}}{4 \cdot \jmath} \cdot dt +$$
$$\left. + B^2 \cdot \int_0^T \frac{e^{2 \cdot \jmath \cdot n \cdot t} + 2 \cdot e^0 + e^{-2 \cdot \jmath \cdot n \cdot t}}{4} \cdot dt\right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \int_0^T \left(e^{2 \cdot \jmath \cdot k \cdot t} - 2 \cdot 1 + e^{-2 \cdot \jmath \cdot k \cdot t} \right) \cdot dt + \right.$$

$$+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \int_0^T \left(e^{\jmath \cdot (k+n) \cdot t} + e^{\jmath \cdot (k-n) \cdot t} - e^{-\jmath \cdot (k-n) \cdot t} - e^{-\jmath \cdot (k+n) \cdot t} \right) \cdot dt +$$

$$\left. + \frac{B^2}{4} \cdot \int_0^T \left(e^{2 \cdot \jmath \cdot n \cdot t} + 2 \cdot 1 + e^{-2 \cdot \jmath \cdot n \cdot t} \right) \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\int_0^T e^{2 \cdot \jmath \cdot k \cdot t} \cdot dt - \int_0^T 2 \cdot dt + \int_0^T e^{-2 \cdot \jmath \cdot k \cdot t} \cdot dt \right) + \right.$$

$$+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\int_0^T e^{\jmath \cdot (k+n) \cdot t} \cdot dt + \int_0^T e^{\jmath \cdot (k-n) \cdot t} \cdot dt - \int_0^T e^{-\jmath \cdot (k-n) \cdot t} \cdot dt - \int_0^T e^{-\jmath \cdot (k+n) \cdot t} \cdot dt \right) +$$

$$\left. + \frac{B^2}{4} \cdot \left(\int_0^T e^{2 \cdot \jmath \cdot n \cdot t} \cdot dt + \int_0^T 2 \cdot dt + \int_0^T e^{-2 \cdot \jmath \cdot n \cdot t} \cdot dt \right) \right) =$$

$$= \left\{ \begin{array}{llll} z_1 = 2 \cdot \jmath \cdot k \cdot t & z_2 = -2 \cdot \jmath \cdot k \cdot t & z_3 = 2 \cdot \jmath \cdot n \cdot t & z_4 = -2 \cdot \jmath \cdot n \cdot t \\ dz_1 = 2 \cdot \jmath \cdot k \cdot dt & dz_2 = -2 \cdot \jmath \cdot k \cdot dt & dz_3 = 2 \cdot \jmath \cdot n \cdot dt & dz_4 = -2 \cdot \jmath \cdot n \cdot dt \\ dt = \frac{dz_1}{2 \cdot \jmath \cdot k} & dt = \frac{dz_2}{-2 \cdot \jmath \cdot k} & dt = \frac{dz_3}{2 \cdot \jmath \cdot n} & dt = \frac{dz_4}{-2 \cdot \jmath \cdot n} \\ z_5 = 2 \cdot \jmath \cdot (k+n) \cdot t & z_6 = -2 \cdot \jmath \cdot (k+n) \cdot t & z_7 = 2 \cdot \jmath \cdot (k-n) \cdot t & z_8 = -2 \cdot \jmath \cdot (k-n) \cdot t \\ dz_5 = \jmath \cdot (k+n) \cdot dt & dz_6 = -\jmath \cdot (k+n) \cdot dt & dz_7 = \jmath \cdot (k-n) \cdot dt & dz_8 = -\jmath \cdot (k-n) \cdot dt \\ dt = \frac{dz_5}{\jmath \cdot (k+n)} & dt = \frac{dz_6}{-\jmath \cdot (k+n)} & dt = \frac{dz_7}{\jmath \cdot (k-n)} & dt = \frac{dz_8}{-\jmath \cdot (k-n)} \end{array} \right\} =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\int_0^T e^{z_1} \cdot \frac{dz_1}{2 \cdot \jmath \cdot k} - 2 \cdot \int_0^T dt + \int_0^T e^{z_2} \cdot \frac{dz_2}{-2 \cdot \jmath \cdot k} \right) + \right.$$

$$+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\int_0^T e^{z_5} \cdot \frac{dz_5}{\jmath \cdot (k+n)} + \int_0^T e^{z_7} \cdot \frac{dz_7}{\jmath \cdot (k-n)} + \right.$$

$$\left. - \int_0^T e^{z_8} \cdot \frac{dz_8}{-\jmath \cdot (k-n)} - \int_0^T e^{z_6} \cdot \frac{dz_6}{-\jmath \cdot (k+n)} \right) +$$

$$\left. + \frac{B^2}{4} \cdot \left(\int_0^T e^{z_3} \cdot \frac{dz_3}{2 \cdot \jmath \cdot n} + 2 \cdot \int_0^T dt + \int_0^T e^{z_4} \cdot \frac{dz_4}{-2 \cdot \jmath \cdot n} \right) \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \int_0^T e^{z_1} \cdot dz_1 - 2 \cdot \int_0^T dt + \frac{1}{-2 \cdot \jmath \cdot k} \cdot \int_0^T e^{z_2} \cdot dz_2 \right) + \right.$$

$$+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \int_0^T e^{z_5} \cdot dz_5 + \frac{1}{\jmath \cdot (k-n)} \cdot \int_0^T e^{z_7} \cdot dz_7 + \right.$$

$$\left. - \frac{1}{-\jmath \cdot (k-n)} \cdot \int_0^T e^{z_8} \cdot dz_8 - \frac{1}{-\jmath \cdot (k+n)} \cdot \int_0^T e^{z_6} \cdot dz_6 \right) +$$

$$\left. + \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \int_0^T e^{z_3} \cdot dz_3 + 2 \cdot \int_0^T dt + \frac{1}{-2 \cdot \jmath \cdot n} \cdot \int_0^T e^{z_4} \cdot dz_4 \right) \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot e^{z_1}\Big|_0^T - 2 \cdot t\Big|_0^T - \frac{1}{2 \cdot \jmath \cdot k} \cdot e^{z_2}\Big|_0^T \right) + \right.$$

$$+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot e^{z_5}\Big|_0^T + \frac{1}{\jmath \cdot (k-n)} \cdot e^{z_7}\Big|_0^T + \right.$$

$$\left. + \frac{1}{\jmath \cdot (k-n)} \cdot e^{z_8}\Big|_0^T + \frac{1}{\jmath \cdot (k+n)} \cdot e^{z_6}\Big|_0^T \right) +$$

$$\left. + \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot e^{z_3}\Big|_0^T + 2 \cdot t\Big|_0^T - \frac{1}{2 \cdot \jmath \cdot n} \cdot e^{z_4}\Big|_0^T \right) \right) =$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot e^{2 \cdot \jmath \cdot k \cdot t} \Big|_0^T - 2 \cdot t|_0^T - \frac{1}{2 \cdot \jmath \cdot k} \cdot e^{-2 \cdot \jmath \cdot k \cdot t} \Big|_0^T \right) + \right.\\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot e^{\jmath \cdot (k+n) \cdot t} \Big|_0^T + \frac{1}{\jmath \cdot (k-n)} \cdot e^{\jmath \cdot (k-n) \cdot t} \Big|_0^T + \right.\\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot e^{-\jmath \cdot (k-n) \cdot t} \Big|_0^T + \frac{1}{\jmath \cdot (k+n)} \cdot e^{-\jmath \cdot (k+n) \cdot t} \Big|_0^T \right) +\\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot e^{2 \cdot \jmath \cdot n \cdot t} \Big|_0^T + 2 \cdot t|_0^T - \frac{1}{2 \cdot \jmath \cdot n} \cdot e^{-2 \cdot \jmath \cdot n \cdot t} \Big|_0^T \right) \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot k \cdot T} - e^{2 \cdot \jmath \cdot k \cdot 0} \right) - 2 \cdot (T-0) - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot k \cdot T} - e^{-2 \cdot \jmath \cdot k \cdot 0} \right) \right) + \right.\\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot (k+n) \cdot T} - e^{\jmath \cdot (k+n) \cdot 0} \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot (k-n) \cdot T} - e^{\jmath \cdot (k-n) \cdot 0} \right) + \right.\\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot (k-n) \cdot T} - e^{-\jmath \cdot (k-n) \cdot 0} \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot (k+n) \cdot T} - e^{-\jmath \cdot (k+n) \cdot 0} \right) \right) +\\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot n \cdot T} - e^{2 \cdot \jmath \cdot n \cdot 0} \right) + 2 \cdot (T-0) - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot n \cdot T} - e^{-2 \cdot \jmath \cdot n \cdot 0} \right) \right) \right) =\\
&= \left\{ \begin{array}{l} T = b \cdot T_1 = a \cdot T_2 \\ T = b \cdot \frac{2\pi}{k} = a \cdot \frac{2\pi}{n} \end{array} \right\} =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} - e^0 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} - e^0 \right) \right) + \right.\\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot k \cdot T} \cdot e^{\jmath \cdot n \cdot T} - e^0 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot k \cdot T} \cdot e^{-\jmath \cdot n \cdot T} - e^0 \right) + \right.\\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot k \cdot T} \cdot e^{\jmath \cdot n \cdot T} - e^0 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot k \cdot T} \cdot e^{-\jmath \cdot n \cdot T} - e^0 \right) \right) +\\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - e^0 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - e^0 \right) \right) \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) \right) + \right.\\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{-\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \right.\\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{-\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) \right) +\\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) \right) \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) \right) + \right.\\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot b \cdot 2\pi} \cdot e^{\jmath \cdot a \cdot 2\pi} - 1 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot b \cdot 2\pi} \cdot e^{-\jmath \cdot a \cdot 2\pi} - 1 \right) + \right.\\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot b \cdot 2\pi} \cdot e^{\jmath \cdot a \cdot 2\pi} - 1 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot b \cdot 2\pi} \cdot e^{-\jmath \cdot a \cdot 2\pi} - 1 \right) \right) +\\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) \right) \right) =
\end{aligned}
$$

$$= \left\{ \begin{array}{ll} \underset{a\in\mathcal{Z}}{\forall} e^{\jmath\cdot a\cdot 2\pi} = 1 & \underset{a\in\mathcal{Z}}{\forall} e^{2\cdot\jmath\cdot a\cdot 2\pi} = 1 \\ \underset{a\in\mathcal{Z}}{\forall} e^{-\jmath\cdot a\cdot 2\pi} = 1 & \underset{a\in\mathcal{Z}}{\forall} e^{-2\cdot\jmath\cdot a\cdot 2\pi} = 1 \\ \underset{b\in\mathcal{Z}}{\forall} e^{\jmath\cdot b\cdot 2\pi} = 1 & \underset{b\in\mathcal{Z}}{\forall} e^{2\cdot\jmath\cdot b\cdot 2\pi} = 1 \\ \underset{b\in\mathcal{Z}}{\forall} e^{-\jmath\cdot b\cdot 2\pi} = 1 & \underset{b\in\mathcal{Z}}{\forall} e^{-2\cdot\jmath\cdot b\cdot 2\pi} = 1 \end{array} \right\} =$$

$$= \frac{1}{T}\cdot\left(\frac{A^2}{-4}\cdot\left(\frac{1}{2\cdot\jmath\cdot k}\cdot(1-1) - 2\cdot T - \frac{1}{2\cdot\jmath\cdot k}\cdot(1-1)\right) + \right.$$

$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath}\cdot\left(\frac{1}{\jmath\cdot(k+n)}\cdot(1\cdot 1-1) + \frac{1}{\jmath\cdot(k-n)}\cdot(1\cdot 1-1) + \right.$$

$$\left. + \frac{1}{\jmath\cdot(k-n)}\cdot(1\cdot 1-1) + \frac{1}{\jmath\cdot(k+n)}\cdot(1\cdot 1-1)\right) +$$

$$\left. + \frac{B^2}{4}\cdot\left(\frac{1}{2\cdot\jmath\cdot n}\cdot(1-1) + 2\cdot T - \frac{1}{2\cdot\jmath\cdot n}\cdot(1-1)\right)\right) =$$

$$= \frac{1}{T}\cdot\left(\frac{A^2}{-4}\cdot\left(\frac{1}{2\cdot\jmath\cdot k}\cdot 0 - 2\cdot T - \frac{1}{2\cdot\jmath\cdot k}\cdot 0\right) + \right.$$

$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath}\cdot\left(\frac{1}{\jmath\cdot(k+n)}\cdot 0 + \frac{1}{\jmath\cdot(k-n)}\cdot 0 + \right.$$

$$\left. + \frac{1}{\jmath\cdot(k-n)}\cdot 0 + \frac{1}{\jmath\cdot(k+n)}\cdot 0\right) +$$

$$\left. + \frac{B^2}{4}\cdot\left(\frac{1}{2\cdot\jmath\cdot n}\cdot 0 + 2\cdot T - \frac{1}{2\cdot\jmath\cdot n}\cdot 0\right)\right) =$$

$$= \frac{1}{T}\cdot\left(\frac{A^2}{-4}\cdot(0 - 2\cdot T - 0) + \right.$$

$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath}\cdot(0+0+0+0) +$$

$$\left. + \frac{B^2}{4}\cdot(0 + 2\cdot T - 0)\right) =$$

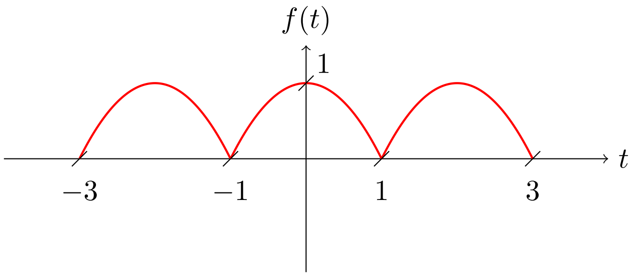
$$= \frac{1}{T}\cdot\left(\frac{A^2}{-4}\cdot(-2\cdot T) + \frac{B^2}{4}\cdot 2\cdot T\right) =$$

$$= \frac{1}{T}\cdot\left(\frac{A^2}{2}\cdot T + \frac{B^2}{2}\cdot T\right) =$$

$$= \frac{A^2}{2} + \frac{B^2}{2}$$

Ostatecznie moc sygnału wynosi $\frac{A^2}{2} + \frac{B^2}{2}$

Task 12. Compute the average power for the following periodic signal $f(t)$:



Signal in the range $t \in (-1; 1)$ is described as:

$$f(t) = 1 - t^2 \tag{1.23}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.24}$$

In this case period T is equal to 2.

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left|1 - t^2\right|^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left(1 - t^2\right)^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left(1 - 2 \cdot t^2 + t^4\right) \cdot dt = \\
&= \frac{1}{2} \cdot \left[\int_{-1}^{1} 1 \cdot dt + \int_{-1}^{1} (-2) \cdot t^2 \cdot dt + \int_{-1}^{1} t^4 \cdot dt\right] = \\
&= \frac{1}{2} \cdot \left[t\Big|_{-1}^{1} - 2 \cdot \frac{t^3}{3}\Big|_{-1}^{1} + \frac{t^5}{5}\Big|_{-1}^{1} \right] = \\
&= \frac{1}{2} \cdot \left[(1 - (-1)) - \frac{2}{3} \cdot (1 - (-1)) + \frac{1}{5} \cdot (1 - (-1))\right] = \\
&= \frac{1}{2} \cdot \left[2 - \frac{4}{3} + \frac{2}{5}\right] = \\
&= \frac{1}{2} \cdot \left[\frac{30}{15} - \frac{20}{15} + \frac{6}{15}\right] = \\
&= \frac{1}{2} \cdot \frac{16}{15} = \\
&= \frac{8}{15}
\end{aligned}$$

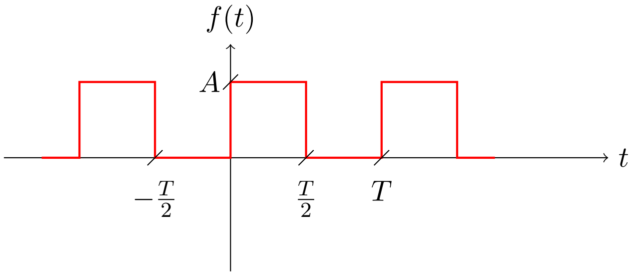
The average power equals to $\frac{8}{15}$.

Chapter 2

Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} dt + 0\right) = \\
&= \frac{1}{T}\left(A\cdot t|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T}\cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}-0\right) = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.3}$$

The a_0 coefficient equals $\frac{A}{2}$.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.4}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\int_0^{\frac{T}{2}} A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{array}{ll} z & = k\cdot\frac{2\pi}{T}\cdot t \\ dz & = k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{k\cdot\frac{2\pi}{T}} \end{array}\right\} = \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} cos\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}} = \\
&= \frac{2\cdot A}{T\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} cos\left(z\right)\cdot dz = \\
&= \frac{A}{k\cdot\pi}\, sin\left(z\right)|_0^{\frac{T}{2}} = \\
&= \frac{A}{k\cdot\pi}\, sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{k\cdot\pi}\left(sin\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right) - sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right) = \\
&= \frac{A}{k\cdot\pi}\left(sin\left(k\cdot\pi\right) - sin\left(0\right)\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{k\cdot\pi}\left(sin\left(k\cdot\pi\right)-0\right)= \\
&= \frac{A}{k\cdot\pi}\cdot sin\left(k\cdot\pi\right)= \\
&= \frac{A}{k\cdot\pi}\cdot 0= \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$
b_k=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.5}
$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt= \\
&= \frac{2}{T}\int_0^{\frac{T}{2}} A\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt= \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt= \\
&= \begin{Bmatrix} z & = k\cdot\frac{2\pi}{T}\cdot t \\ dz & = k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{k\cdot\frac{2\pi}{T}} \end{Bmatrix}= \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}}= \\
&= \frac{2\cdot A}{T\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot dz= \\
&= -\frac{A}{k\cdot\pi}\,cos\left(z\right)\Big|_0^{\frac{T}{2}}= \\
&= -\frac{A}{k\cdot\pi}\,cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}}= \\
&= -\frac{A}{k\cdot\pi}\left(cos\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right)-cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)= \\
&= -\frac{A}{k\cdot\pi}\left(cos\left(k\cdot\pi\right)-cos\left(0\right)\right)= \\
&= -\frac{A}{k\cdot\pi}\left(cos\left(k\cdot\pi\right)-1\right)= \\
&= \frac{A}{k\cdot\pi}\left(1-cos\left(k\cdot\pi\right)\right)
\end{aligned}
$$

The b_k coefficients equal to $\frac{A}{k\cdot\pi}\left(1-cos\left(k\cdot\pi\right)\right)$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$\begin{aligned} a_0 &= \frac{A}{2} \\ a_k &= 0 \\ b_k &= \frac{A}{k \cdot \pi} \left(1 - cos\left(k \cdot \pi\right)\right) \end{aligned} \tag{2.6}$$

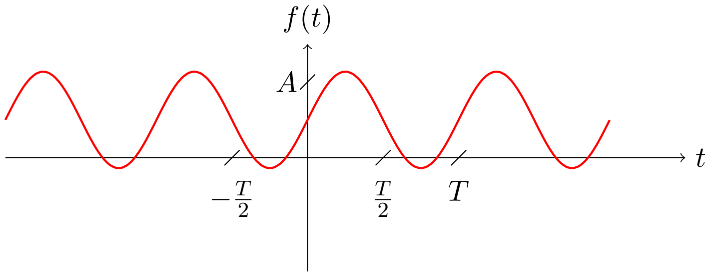
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$	0

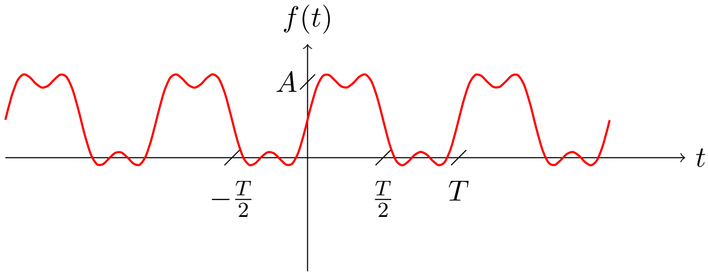
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \\ f(t) &= \frac{A}{2} + \sum_{k=1}^{\infty} \left[\left(\frac{A}{k \cdot \pi}(1 - cos(k \cdot \pi)\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \end{aligned} \tag{2.7}$$

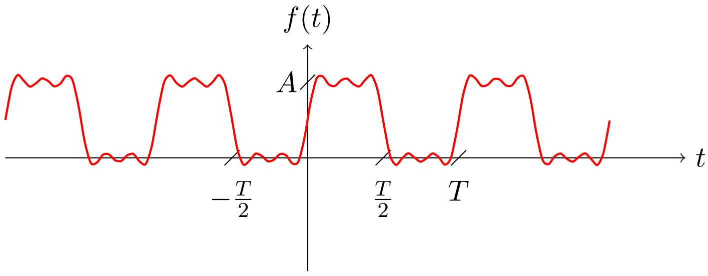
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



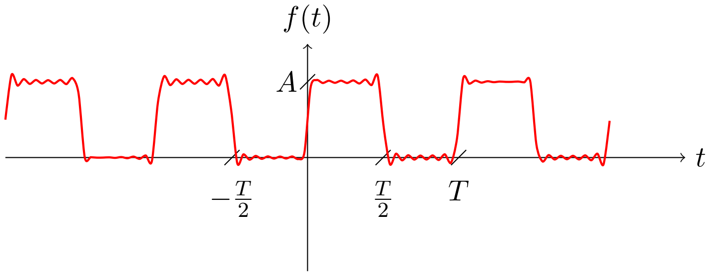
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



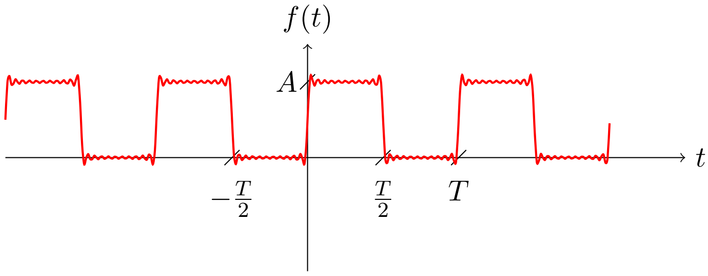
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

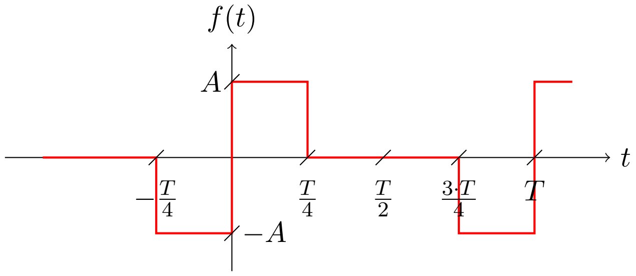


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise linear function assuming period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3 \cdot T}{4} + k \cdot T\right) \end{cases} \quad \wedge k \in Z \tag{2.8}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.9}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_{0}^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3 \cdot T}{4}} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_{0}^{\frac{T}{4}} A \cdot dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot \int_{-\frac{T}{4}}^{0} dt + A \cdot \int_{0}^{\frac{T}{4}} dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot t|_{-\frac{T}{4}}^{0} + A \cdot t|_{0}^{\frac{T}{4}} \right) = \\ &= \frac{1}{T} \left(-A \cdot \left(0 - \left(-\frac{T}{4}\right)\right) + A \cdot \left(\frac{T}{4} - 0\right) \right) = \\ &= \frac{1}{T} \left(-A \cdot \frac{T}{4} + A \cdot \frac{T}{4} \right) = \\ &= \frac{1}{T} (0) = \\ &= 0 \end{aligned}$$

The a_0 coefficient equals 0.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T} \int_T f(t) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.10}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3\cdot T}{4}\right)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_0^{\frac{T}{4}} A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + A\cdot\int_0^{\frac{T}{4}} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \begin{Bmatrix} z & = k\cdot\frac{2\pi}{T}\cdot t \\ dz & = k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dt}{k\cdot\frac{2\pi}{T}} \end{Bmatrix} = \\
&= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} cos\,(z)\cdot\frac{dt}{k\cdot\frac{2\pi}{T}} + A\cdot\int_0^{\frac{T}{4}} cos\,(z)\cdot\frac{dt}{k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{2}{T}\left(-\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} cos\,(z)\cdot dt + \frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_0^{\frac{T}{4}} cos\,(z)\cdot dt\right) = \\
&= \frac{2}{T}\cdot\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\left(-\,sin\,(z)|_{-\frac{T}{4}}^{0} + sin\,(z)|_0^{\frac{T}{4}}\right) = \\
&= \frac{2\cdot A}{k\cdot 2\pi}\cdot\left(-\,sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_{-\frac{T}{4}}^{0} + sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{2\cdot A}{k\cdot 2\pi}\cdot\left(-\left(sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right) - sin\left(-k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right)\right) + \left(sin\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right) - sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-\left(sin\,(0) - sin\left(-k\cdot\frac{2\pi}{4}\right)\right) + \left(sin\left(k\cdot\frac{2\pi}{4}\right) - sin\,(0)\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-\left(0 - sin\left(-k\cdot\frac{\pi}{2}\right)\right) + \left(sin\left(k\cdot\frac{\pi}{2}\right) - 0\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(sin\left(-k\cdot\frac{\pi}{2}\right) + sin\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-sin\left(k\cdot\frac{\pi}{2}\right) + sin\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot(0) = \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.11}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3\cdot T}{4}\right)$ we get:

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_0^{\frac{T}{4}} A\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) =
\end{aligned}
$$

$$= \frac{2}{T}\left(-A \cdot \int_{-\frac{T}{4}}^{0} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + A \cdot \int_{0}^{\frac{T}{4}} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0 \cdot dt\right) =$$

$$= \left\{\begin{aligned} z &= k \cdot \tfrac{2\pi}{T} \cdot t \\ dz &= k \cdot \tfrac{2\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{k \cdot \frac{2\pi}{T}} \end{aligned}\right\} =$$

$$= \frac{2}{T}\left(-A \cdot \int_{-\frac{T}{4}}^{0} sin\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} + A \cdot \int_{0}^{\frac{T}{4}} sin\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} + 0\right) =$$

$$= \frac{2}{T}\left(-\frac{A}{k \cdot \frac{2\pi}{T}} \cdot \int_{-\frac{T}{4}}^{0} sin\left(z\right) \cdot dz + \frac{A}{k \cdot \frac{2\pi}{T}} \cdot \int_{0}^{\frac{T}{4}} sin\left(z\right) \cdot dz\right) =$$

$$= \frac{2}{T} \cdot \frac{A}{k \cdot \frac{2\pi}{T}} \cdot \left(-\int_{-\frac{T}{4}}^{0} sin\left(z\right) \cdot dz + \int_{0}^{\frac{T}{4}} sin\left(z\right) \cdot dz\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(cos\left(z\right)\Big|_{-\frac{T}{4}}^{0} - cos\left(z\right)\Big|_{0}^{\frac{T}{4}}\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_{-\frac{T}{4}}^{0} - cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_{0}^{\frac{T}{4}}\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(\left(cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right) - cos\left(-k \cdot \frac{2\pi}{T} \cdot \frac{T}{4}\right)\right) - \left(cos\left(k \cdot \frac{2\pi}{T} \cdot \frac{T}{4}\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(\left(cos\left(0\right) - cos\left(-k \cdot \frac{\pi}{2}\right)\right) - \left(cos\left(k \cdot \frac{\pi}{2}\right) - cos\left(0\right)\right)\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(cos\left(0\right) - cos\left(-k \cdot \frac{\pi}{2}\right) - cos\left(k \cdot \frac{\pi}{2}\right) + cos\left(0\right)\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right) - cos\left(k \cdot \frac{\pi}{2}\right) + 1\right) =$$

$$= \frac{A}{k \cdot \pi} \cdot \left(2 - 2 \cdot cos\left(k \cdot \frac{\pi}{2}\right)\right) =$$

$$= \frac{2 \cdot A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right)$$

The b_k coefficients equal to $\frac{2 \cdot A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right)$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$\begin{aligned} a_0 &= 0 \\ a_k &= 0 \\ b_k &= \frac{2 \cdot A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right) \end{aligned}$$

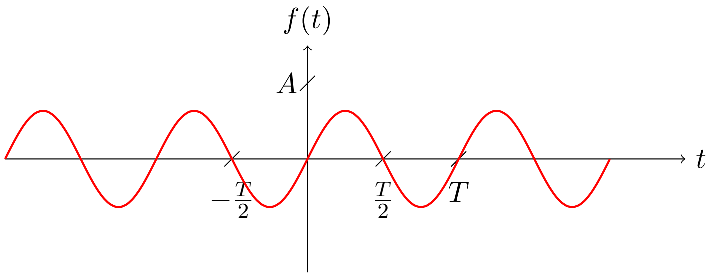
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$\frac{2 \cdot A}{\pi}$	$\frac{2 \cdot A}{\pi}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$	$\frac{2 \cdot A}{3 \cdot \pi}$

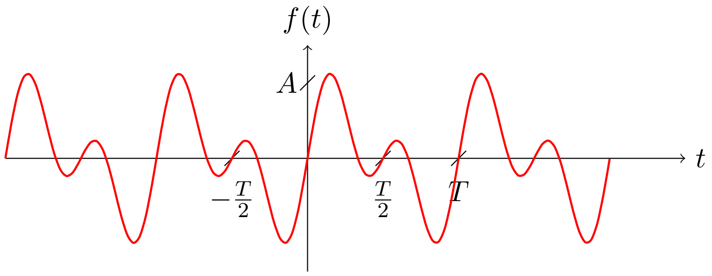
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right]$$
$$f(t) = \sum_{k=1}^{\infty}\left[\frac{2 \cdot A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right]$$

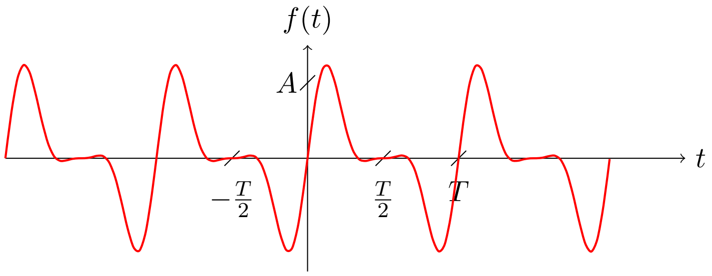
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



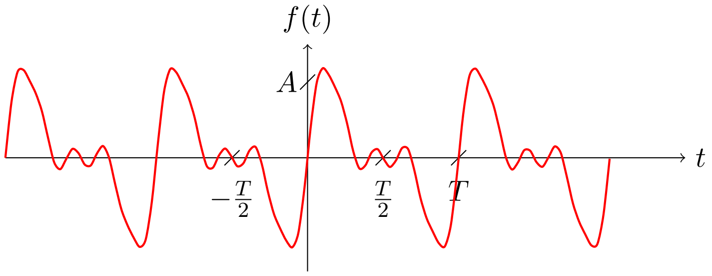
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



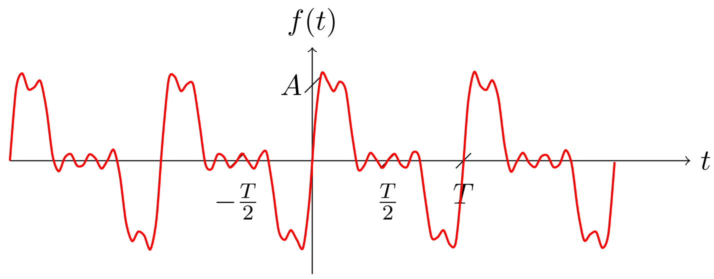
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



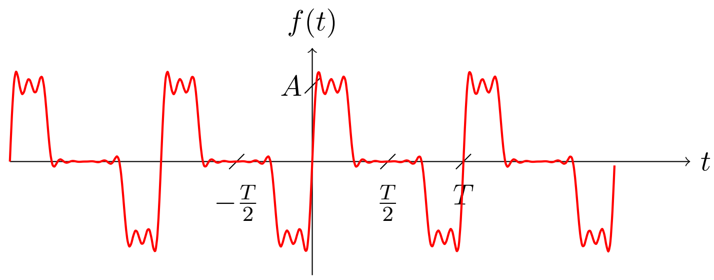
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



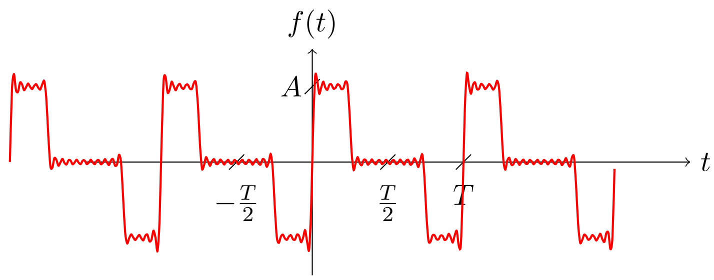
A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

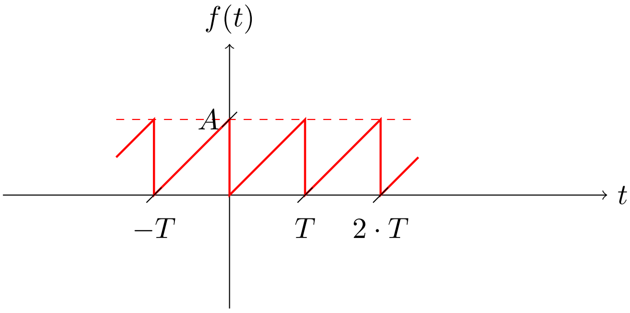


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 3. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.12}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and (T,A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.13}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot dt = \\
&= \frac{A}{T^2}\int_0^T t\cdot dt = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot t^2\Big|_0^T = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot\left(T^2-0^2\right) = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot T^2 = \\
&= \frac{A}{2}
\end{aligned}
$$

The a_0 coefficient equals $\frac{A}{2}$.

The a_k coefficients are defined as:

$$
a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.14}
$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\int_0^T \frac{A}{T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2\cdot A}{T^2}\int_0^T t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{matrix} u & = t & dv & = cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \\ du & = dt & v & = \frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\end{matrix}\right\} = \\
&= \frac{2\cdot A}{T^2}\cdot\left(t\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T - \int_0^T \frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(\left(T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) - 0\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot\left(cos\left(k\cdot\frac{2\pi}{T}\cdot T\right) - cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\right) + \frac{1}{(k\cdot 2\pi)^2}\cdot\left(cos\left(k\cdot 2\pi\right) - cos\left(0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(\frac{1}{k\cdot 2\pi}\cdot 0 + \frac{1}{(k\cdot 2\pi)^2}\cdot(1-1)\right) = \\
&= 2\cdot A\cdot\left(0 + \frac{1}{(k\cdot 2\pi)^2}\cdot 0\right) = \\
&= 2\cdot A\cdot 0 = \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T}\int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.15}$$

For the period $t \in (0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T}\int_0^T \frac{A}{T} \cdot t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2 \cdot A}{T^2}\int_0^T t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du & = dt & v & = -\frac{T}{k \cdot 2\pi} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{Bmatrix} = \\
&= \frac{2 \cdot A}{T^2} \cdot \left(-t \cdot \frac{T}{k \cdot 2\pi} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^T + \int_0^T \frac{T}{k \cdot 2\pi} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \frac{2 \cdot A}{T^2} \cdot \left(-\left(T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - 0 \cdot \frac{T}{k \cdot 2\pi} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right) + \frac{T^2}{(k \cdot 2\pi)^2} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^T \right) = \\
&= \frac{2 \cdot A}{T^2} \cdot \left(-\left(\frac{T^2}{k \cdot 2\pi} \cdot cos\left(k \cdot 2\pi\right)\right) + \frac{T^2}{(k \cdot 2\pi)^2} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) = \\
&= 2 \cdot A \cdot \left(-\left(\frac{1}{k \cdot 2\pi} \cdot 1\right) + \frac{1}{(k \cdot 2\pi)^2} \cdot \left(sin\left(k \cdot 2\pi\right) - sin\left(0\right)\right)\right) = \\
&= 2 \cdot A \cdot \left(-\frac{1}{k \cdot 2\pi} + \frac{1}{(k \cdot 2\pi)^2} \cdot (0-0)\right) = \\
&= 2 \cdot A \cdot \left(-\frac{1}{k \cdot 2\pi} + \frac{1}{(k \cdot 2\pi)^2} \cdot 0\right) = \\
&= 2 \cdot A \cdot \left(-\frac{1}{k \cdot 2\pi}\right) = \\
&= -\frac{2 \cdot A}{k \cdot 2\pi} = \\
&= -\frac{A}{k \cdot \pi}
\end{aligned}$$

The b_k coefficients equal to $-\frac{A}{k \cdot \pi}$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0 = \frac{A}{2}$$

$$a_k = 0$$

$$b_k = -\frac{A}{k \cdot \pi}$$

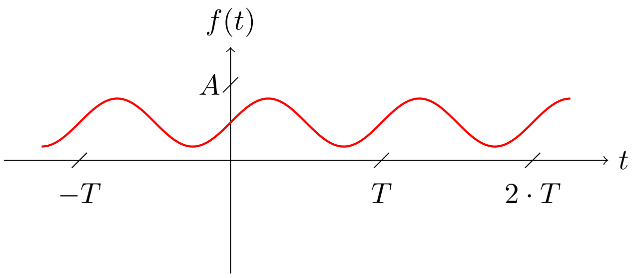
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$-\frac{A}{\pi}$	$-\frac{A}{2\cdot\pi}$	$-\frac{A}{3\cdot\pi}$	$-\frac{A}{4\cdot\pi}$	$-\frac{A}{5\cdot\pi}$	$-\frac{A}{6\cdot\pi}$

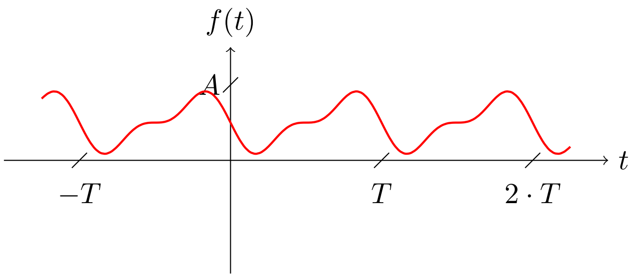
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty}\left[a_k \cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) + b_k \cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \\ f(t) &= \frac{A}{2} + \sum_{k=1}^{\infty}\left[\left(-\frac{A}{k\cdot\pi}\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \end{aligned} \tag{2.16}$$

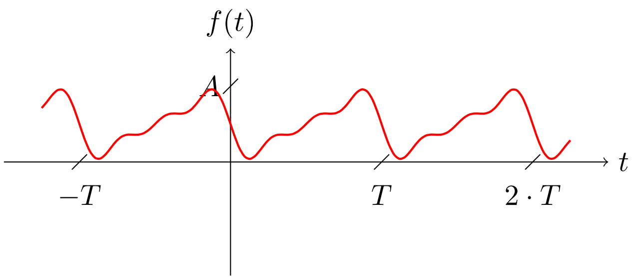
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



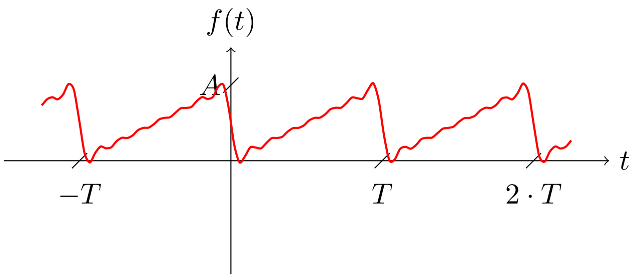
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



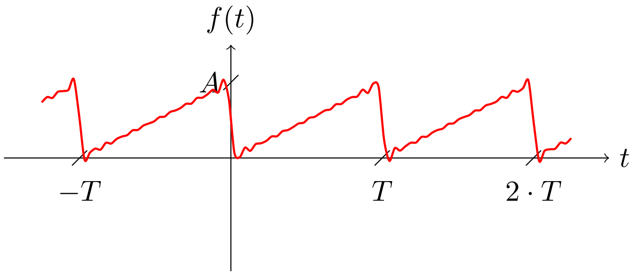
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 7$ results in:

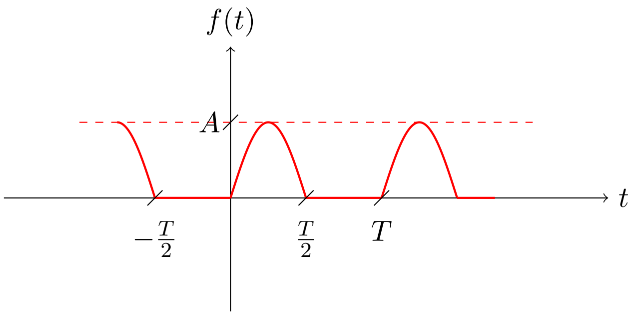


A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 4. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise function assuming period $t \in (0;T)$ is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.17}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.18}$$

For the period $t \in (0;T)$ we get:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{array} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin(z) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin(z) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos(z)|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\bigg|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{2\pi} \cdot (cos\,(\pi) - cos\,(0)) = \\
&= -\frac{A}{2\pi} \cdot (-1-1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The a_0 coefficient equals $\frac{A}{\pi}$.

The a_k coefficients are defined as:

$$
a_k = \frac{2}{T} \int_T f(t) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.19}
$$

For the period $t \in (0; T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T} \int_T f(t) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ \begin{aligned} cos\,(x) &= \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \\ sin\,(x) &= \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \end{aligned} \right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot dt + 0 \right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt \right) = \\
&= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \right) \cdot dt = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2\jmath} + \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2\jmath} \right) \cdot dt = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) + sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \right) \cdot dt = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) \cdot dt + \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \frac{2\pi}{T} \cdot t \cdot (1+k) & z_2 & = \frac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 & = \frac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 & = \frac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt & = \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt & = \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{array} \right\} =
\end{aligned}
$$

$$= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} + \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot dz_1 + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot dz_2 \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(-cos\left(z_1\right)\Big|_0^{\frac{T}{2}} \right) + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(-cos\left(z_2\right)\Big|_0^{\frac{T}{2}} \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k) \right)\Big|_0^{\frac{T}{2}} \right) + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k) \right)\Big|_0^{\frac{T}{2}} \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1+k) \right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1+k) \right) \right) = \right.$$
$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1-k) \right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1-k) \right) \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\pi \cdot (1+k)\right) - cos\left(0\right) \right) + \right.$$
$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\pi \cdot (1-k)\right) - cos\left(0\right) \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1+k)\right) \right) + \right.$$
$$\left. + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1-k)\right) \right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1}{1+k} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1}{1-k} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1-k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1+k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right)}{(1+k) \cdot (1-k)} + \frac{1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k) \cdot (1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right) + 1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k) \cdot (1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \frac{2 - cos\left(\pi \cdot (1+k)\right) + k \cdot cos\left(\pi \cdot (1+k)\right) - cos\left(\pi \cdot (1-k)\right) - k \cdot cos\left(\pi \cdot (1-k)\right)}{1-k^2} =$$
$$= \left\{ \begin{matrix} cos\left(\pi \cdot (1+k)\right) = cos\left(\pi + k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \\ cos\left(\pi \cdot (1-k)\right) = cos\left(\pi - k \cdot \pi\right) = -cos\left(-k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \end{matrix} \right\} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + cos\left(k \cdot \pi\right) - k \cdot cos\left(k \cdot \pi\right) + cos\left(k \cdot \pi\right) + k \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + 2 \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{\pi} \cdot \frac{1 + cos\left(k \cdot \pi\right)}{1-k^2}$$

The a_k coefficients equal to $\frac{A}{\pi} \cdot \frac{1+cos(k\cdot\pi)}{1-k^2}$ for $k \neq 1$.
We have to calculate a_k for $k = 1$ directly by definition:

$$a_1 = \frac{2}{T} \int_T f(t) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt =$$

$$= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$
$$= \left\{ \begin{matrix} cos\left(x\right) & = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \\ sin\left(x\right) & = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \end{matrix} \right\} =$$
$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot dt + 0 \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)}\right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}\right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{0} - e^{0}\right) \cdot dt =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}}{2\jmath}\right) \cdot dt =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot 2\right) \cdot dt =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt =$$
$$= \left\{ \begin{matrix} z & = \frac{4\pi}{T} \cdot t \\ dz & = \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{matrix} \right\} =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} =$$
$$= \frac{A}{T \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz =$$
$$= \frac{A}{4\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}} \right) =$$
$$= \frac{A}{4\pi} \cdot \left(-cos\left(\frac{4\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} \right) =$$
$$= -\frac{A}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{4\pi}{T} \cdot 0\right) \right) =$$
$$= -\frac{A}{4\pi} \cdot \left(cos\left(2\pi\right) - cos\left(0\right)\right) =$$
$$= -\frac{A}{4\pi} \cdot \left(1 - 1\right) =$$

$$
\begin{aligned}
&= -\frac{A}{4\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The a_1 coefficient equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.20}$$

For the period $t \in (0; T)$ we get:

$$
\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot dt + 0\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot \jmath} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2} - \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2}\right) \cdot dt = \\
&= -\frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) - cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right)\right) \cdot dt = \\
&= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) \cdot dt - \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \cdot dt\right) = \\
&= \left\{ \begin{matrix} z_1 & = \frac{2\pi}{T} \cdot t \cdot (1+k) & z_2 & = \frac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 & = \frac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 & = \frac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt & = \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt & = \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{matrix} \right\} = \\
&= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} - \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)}\right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot dz_1 - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot dz_2\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot\left(sin\left(z_1\right)\Big|_0^{\frac{T}{2}}\right)-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot\left(sin\left(z_2\right)\Big|_0^{\frac{T}{2}}\right)\right)=\\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot\left(sin\left(\frac{2\pi}{T}\cdot t\cdot(1+k)\right)\Bigg|_0^{\frac{T}{2}}\right)-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot\left(sin\left(\frac{2\pi}{T}\cdot t\cdot(1-k)\right)\Bigg|_0^{\frac{T}{2}}\right)\right)=\\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot\left(sin\left(\frac{2\pi}{T}\cdot\frac{T}{2}\cdot(1+k)\right)-sin\left(\frac{2\pi}{T}\cdot 0\cdot(1+k)\right)\right)+\right.\\
&\left.-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot\left(sin\left(\frac{2\pi}{T}\cdot\frac{T}{2}\cdot(1-k)\right)-sin\left(\frac{2\pi}{T}\cdot 0\cdot(1-k)\right)\right)\right)=\\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot\left(sin\left(\pi\cdot(1+k)\right)-sin\left(0\right)\right)+\right.\\
&\left.-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot\left(sin\left(\pi\cdot(1-k)\right)-sin\left(0\right)\right)\right)=\\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot(0-0)\right. =\\
&\left.-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot(0-0)\right)=\\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{2\pi}{T}\cdot(1+k)}\cdot 0-\frac{1}{\frac{2\pi}{T}\cdot(1-k)}\cdot 0\right)=\\
&= -\frac{A}{T}\cdot(0-0)=\\
&= -\frac{A}{T}\cdot 0=\\
&= 0
\end{aligned}
$$

The b_k coefficients equal to 0 for $k \neq 1$.

We have to calculate b_k for $k = 1$ directly by definition:

$$
\begin{aligned}
b_1 &= \frac{2}{T}\int_T f(t)\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt=\\
&= \frac{2}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot dt\right)=\\
&= \left\{sin\left(x\right)\quad=\frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=\\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}}\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot dt+0\right)=\\
&= \frac{2}{T}\cdot\left(\frac{A}{2\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\frac{A}{2\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)} - e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} - e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} - e^{0} - e^{0}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot\jmath}\cdot\int_0^{\frac{T}{2}}\left(\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}}{2} - \frac{1+1}{2}\right)\cdot dt = \\
&= -\frac{A}{T}\cdot\int_0^{\frac{T}{2}}\left(cos\left(\frac{2\pi}{T}\cdot t\cdot 2\right) - 1\right)\cdot dt = \\
&= -\frac{A}{T}\cdot\left(\int_0^{\frac{T}{2}} cos\left(\frac{4\pi}{T}\cdot t\right)\cdot dt - \int_0^{\frac{T}{2}} dt\right) = \\
&= \left\{\begin{matrix} z & = \frac{4\pi}{T}\cdot t \\ dz & = \frac{4\pi}{T}\cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{matrix}\right\} = \\
&= -\frac{A}{T}\cdot\left(\int_0^{\frac{T}{2}} cos\left(z\right)\cdot\frac{dz}{\frac{4\pi}{T}} - t|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\int_0^{\frac{T}{2}} cos\left(z\right)\cdot dz - \left(\frac{T}{2} - 0\right)\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\, sin\left(z\right)|_0^{\frac{T}{2}} - \frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\, sin\left(\frac{4\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}} - \frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\left(sin\left(\frac{4\pi}{T}\cdot\frac{T}{2}\right) - sin\left(\frac{4\pi}{T}\cdot 0\right)\right) - \frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\left(sin\left(2\pi\right) - sin\left(0\right)\right) - \frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\left(0 - 0\right) - \frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(-\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned}
$$

The b_1 coefficient equal to $\frac{A}{2}$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$
\begin{aligned}
a_0 &= \frac{A}{\pi} \\
a_1 &= 0 \\
a_k &= \frac{A}{\pi}\cdot\frac{1+cos\left(k\cdot\pi\right)}{1-k^2} \\
b_1 &= \frac{A}{2}
\end{aligned}
$$

$$b_k = 0$$

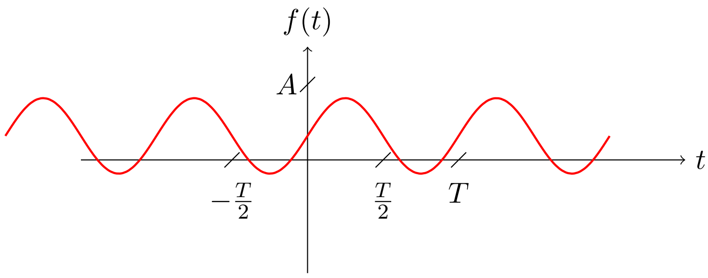
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	$-\frac{2}{3}\frac{A}{\pi}$	0	$-\frac{2}{15}\frac{A}{\pi}$	0	$-\frac{2}{35}\frac{A}{\pi}$
b_k	$\frac{A}{2}$	0	0	0	0	0

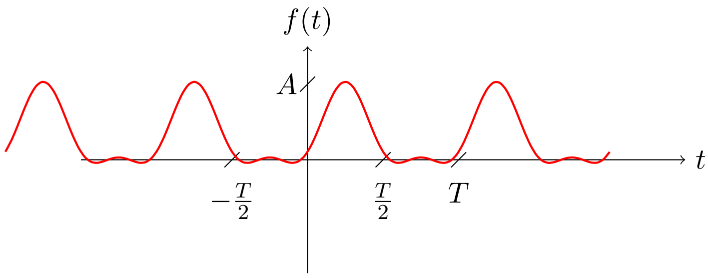
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \tag{2.21}$$

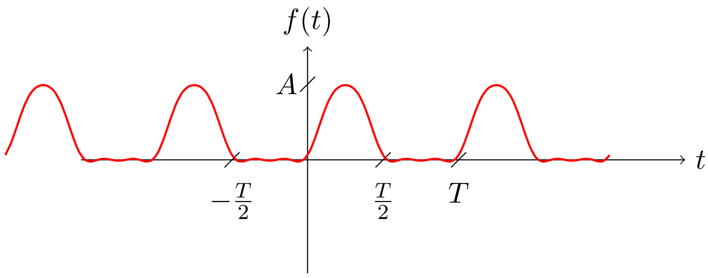
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



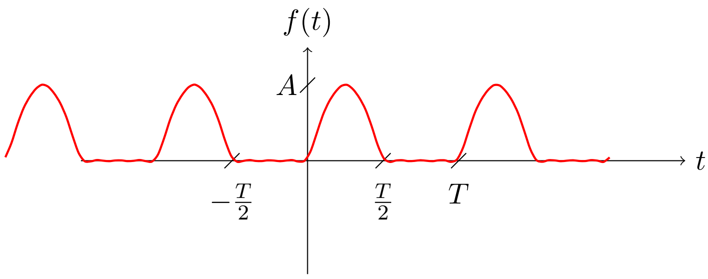
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



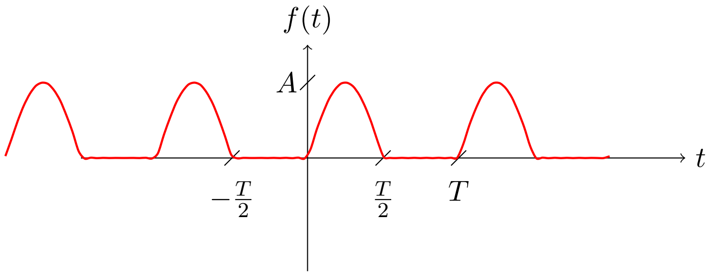
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:

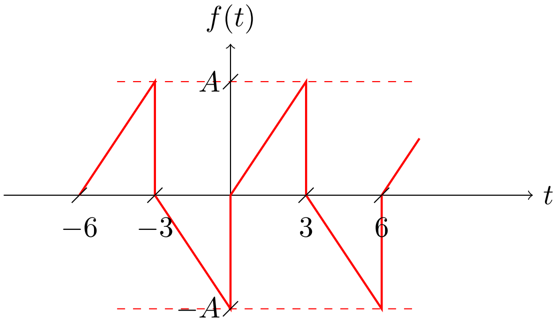


A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 5. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of a linear function is:

$$f(t) = a \cdot t + b \tag{2.22}$$

In the first interval of the first period (e.g. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3, A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (e.g. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3\cdot a = b \\ -A = 3\cdot a \end{cases}$$
$$\begin{cases} -3\cdot a = b \\ -\frac{A}{3} = a \end{cases}$$
$$\begin{cases} -3\cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$
$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3}\cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3}\cdot t & for \quad t \in (0;3) \\ -\frac{A}{3}\cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3}\cdot (t-k\cdot 6) & for \quad t \in (0+k\cdot 6; 3+k\cdot 6) \\ -\frac{A}{3}\cdot (t-k\cdot 6) + A & for \quad t \in (3+k\cdot 6; 6+k\cdot 6) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T}\int_T f(t)\cdot dt \tag{2.23}$$

For the period $t \in (0;6)$, i.e. $k = 0$, we get:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_T f(t)\cdot dt \\ &= \frac{1}{6}\cdot\left[\int_0^3 \frac{A}{3}\cdot t\cdot dt + \int_3^6 \left(-\frac{A}{3}\cdot t + A\right)\cdot dt\right] = \\ &= \frac{A}{18}\cdot\int_0^3 t\cdot dt - \frac{A}{18}\cdot\int_3^6 t\cdot dt + \frac{A}{6}\cdot\int_3^6 dt = \\ &= \frac{A}{18}\cdot\left.\frac{t^2}{2}\right|_0^3 - \frac{A}{18}\cdot\left.\frac{t^2}{2}\right|_3^6 + \frac{A}{6}\cdot t|_3^6 = \\ &= \frac{A}{36}\cdot\left(3^2-0^2\right) - \frac{A}{36}\cdot\left(6^2-3^2\right) + \frac{A}{6}\cdot(6-3) = \\ &= \frac{A}{36}\cdot 9 - \frac{A}{36}\cdot 27 + \frac{A}{6}\cdot 3 = \\ &= \frac{9\cdot A}{36} - \frac{27\cdot A}{36} + \frac{18\cdot A}{36} = \end{aligned}$$

$$= 0$$

The a_0 coefficient equals 0.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.24}$$

For the period $t \in (0; 6)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\\
&= \frac{2}{6}\cdot\left[\int_0^3 \frac{A}{3}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt + \int_3^6\left(-\frac{A}{3}\cdot t + A\right)\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt\right] =\\
&= \frac{A}{9}\cdot\int_0^3 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt - \frac{A}{9}\cdot\int_3^6 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt + \frac{A}{3}\cdot\int_3^6 cos\left(k\cdot\frac{\pi}{3}\cdot t\right)dt =\\
&= \begin{Bmatrix} u & = t & dv & = cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\\ du & = dt & v & = \frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\end{Bmatrix} =\\
&= \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3 - \int_0^3\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] -\\
&- \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 - \int_3^6\frac{3}{k\cdot 3\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] +\\
&+ \frac{A}{3}\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 =\\
&= \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) - 0\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 0\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3\right] -\\
&- \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - 3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - sin\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\pi\right) - 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 0\right)\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot 2\pi\right) - 3\cdot sin\left(k\pi\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 6\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot 2\pi\right) - sin\left(k\cdot\pi\right)\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right) - \frac{3}{k\cdot\pi}\cdot cos\left(0\right)\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot 0 - 3\cdot 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[0 - 0\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right) - \frac{3}{k\cdot\pi}\cdot 1\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot 1 - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right] =\\
&= \frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right) - \frac{A}{k^2\cdot\pi^2} - \frac{A}{k^2\cdot\pi^2} + \frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot (cos\,(k \cdot \pi) - 1) = \\
&= \frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \Big((-1)^k - 1\Big)
\end{aligned}
$$

The a_k coefficients equal to $\frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \Big((-1)^k - 1\Big)$.
The b_k coefficients are defined as:

$$
b_k = \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.25}
$$

For the period $t \in (0; 6)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\
&= \frac{2}{6} \cdot \left[\int_0^3 \frac{A}{3} \cdot t \cdot sin\left(k \cdot \frac{2\pi}{6} \cdot t\right) \cdot dt + \int_3^6 \left(-\frac{A}{3} \cdot t + A\right) \cdot sin\left(k \cdot \frac{2\pi}{6} \cdot t\right) \cdot dt\right] = \\
&= \frac{A}{9} \cdot \int_0^3 t \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt - \frac{A}{9} \cdot \int_3^6 t \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt + \frac{A}{3} \cdot \int_3^6 sin\left(k \cdot \frac{\pi}{3} \cdot t\right) dt = \\
&= \left\{\begin{matrix} u & = t & dv & = sin\,(k \cdot \frac{\pi}{3} \cdot t) \cdot dt \\ du & = dt & v & = -\frac{3}{k \cdot \pi} \cdot cos\,(k \cdot \frac{\pi}{3} \cdot t) \end{matrix}\right\} = \\
&= \frac{A}{9} \cdot \left[t \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_0^3 - \int_0^3 \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt\right] - \\
&- \frac{A}{9} \cdot \left[t \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6 - \int_3^6 \frac{-3}{k \cdot 3\pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt\right] + \\
&+ \frac{A}{3} \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6 = \\
&= \frac{-3 \cdot A}{9 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 3\right) - 0 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 0\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_0^3\right] + \\
&+ \frac{3 \cdot A}{9 \cdot k \cdot \pi} \cdot \left[6 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 6\right) - 3 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 3\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6\right] - \\
&- \frac{A}{k \cdot \pi} \cdot \left[cos\left(k \cdot \frac{\pi}{3} \cdot 6\right) - cos\left(k \cdot \frac{\pi}{3} \cdot 3\right)\right] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\,(k \cdot \pi) - 0 - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 3\right) + \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 0\right)\right] + \\
&+ \frac{A}{3 \cdot k \cdot \pi} \cdot \left[6 \cdot cos\,(k \cdot 2\pi) - 3 \cdot cos\,(k \cdot \pi) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 6\right) + \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 3\right)\right] - \\
&- \frac{A}{k \cdot \pi} \cdot [cos\,(k \cdot 2\pi) - cos\,(k \cdot \pi)] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\,(k \cdot \pi) - \frac{3}{k \cdot \pi} \cdot sin\,(k \cdot \pi) + \frac{3}{k \cdot \pi} \cdot sin\,(0)\right] + \\
&+ \frac{A}{3 \cdot k \cdot \pi} \cdot \left[6 \cdot 1 - 3 \cdot cos\,(k \cdot \pi) - \frac{3}{k \cdot \pi} \cdot sin\,(k \cdot 2\pi) + \frac{3}{k \cdot \pi} \cdot sin\,(k \cdot \pi)\right] - \\
&- \frac{A}{k \cdot \pi} \cdot [1 - cos\,(k \cdot \pi)] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\,(k \cdot \pi) - \frac{3}{k \cdot \pi} \cdot 0 + \frac{3}{k \cdot \pi} \cdot 0\right] +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot 0+\frac{3}{k\cdot\pi}\cdot 0\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)\right]+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)\right]-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{2\cdot A}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)-\frac{A}{k\cdot\pi}+\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{A}{k\cdot\pi}=\\
&=\frac{-A}{k\cdot\pi}\cdot(-1)^k+\frac{A}{k\cdot\pi}=\\
&=\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)
\end{aligned}
$$

The b_k coefficients equal to $\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$
\begin{aligned}
a_0&=0\\
a_k&=\frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left((-1)^k-1\right)\\
b_k&=\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)
\end{aligned}
$$

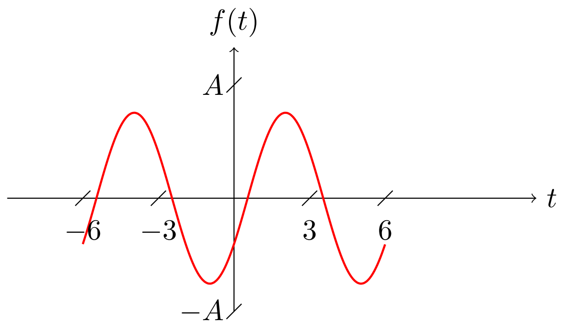
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	$\frac{-4\cdot A}{\pi^2}$	0	$\frac{-4\cdot A}{9\cdot\pi^2}$	0	$\frac{-4\cdot A}{25\cdot\pi^2}$	0
b_k	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$	0

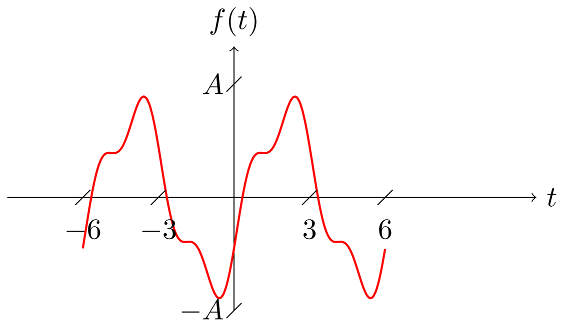
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t)&=a_0+\sum_{k=1}^{\infty}\left[a_k\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+b_k\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right]\\
f(t)&=\sum_{k=1}^{\infty}\left[\frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left((-1)^k-1\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+\left(\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right]
\end{aligned}
\tag{2.26}
$$

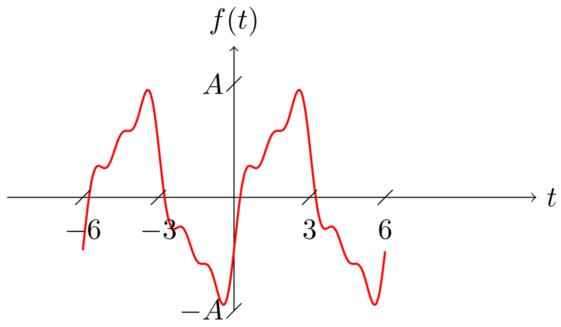
A partial approximation of the $f(t)$ signal for $k_{max}=1$ results in:



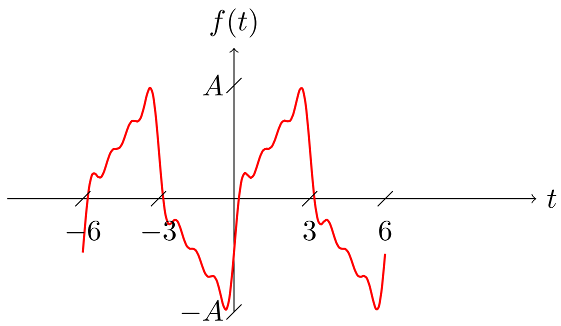
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



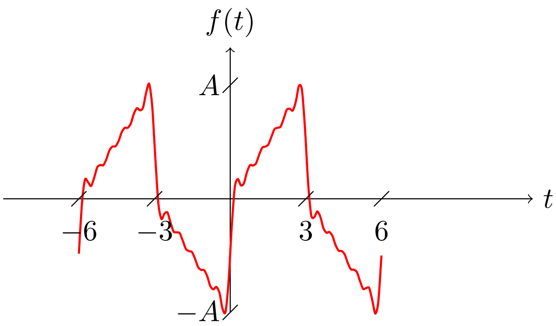
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



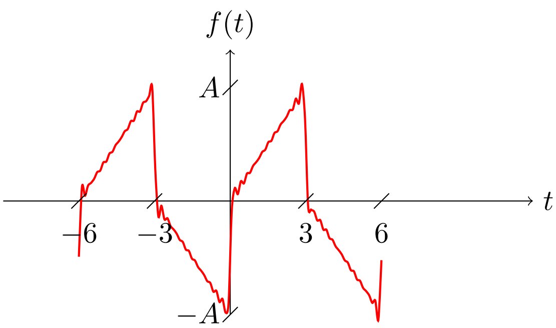
A partial approximation of the $f(t)$ signal for $k_{max} = 7$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

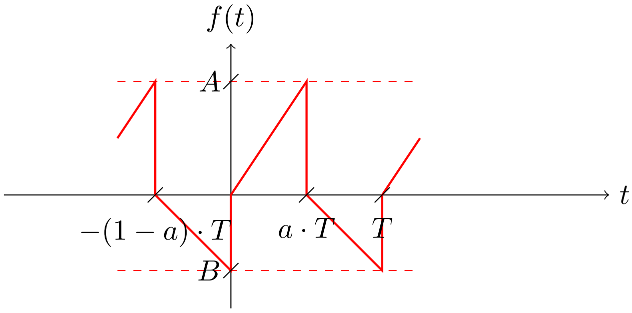


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 6. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{2.27}$$

In the first interval of the first period (e.g. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a\cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T\cdot(1-a)} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T \cdot (1-a)} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.28}$$

For the period $t \in (0; T)$ we get:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^T \left(-\frac{B}{(1-a) \cdot T} \cdot t + \frac{B}{1-a} \cdot a \right) \cdot dt \right) = \\
&= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^T -\frac{B}{(1-a) \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^T \frac{B}{1-a} \cdot a \cdot dt \right) = \\
&= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \int_0^{a\cdot T} t \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a\cdot T}^T t \cdot dt + \frac{B}{1-a} \cdot a \cdot \int_{a\cdot T}^T dt \right) = \\
&= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \left. \frac{t^2}{2} \right|_0^{a\cdot T} - \frac{B}{(1-a) \cdot T} \cdot \left. \frac{t^2}{2} \right|_{a\cdot T}^T + \frac{B}{1-a} \cdot a \cdot t|_{a\cdot T}^T \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{(a\cdot T)^2}{2}-\frac{0^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{(a\cdot T)^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot(T-a\cdot T)\right)= \\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}-\frac{0}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{a^2\cdot T^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot T\cdot(1-a)\right)= \\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot T^2\cdot\left(\frac{1}{2}-\frac{a^2}{2}\right)+B\cdot a\cdot T\right)= \\
&= \frac{1}{T}\left(A\cdot\left(\frac{a\cdot T}{2}\right)-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot\left(1-a^2\right)+B\cdot a\cdot T\right)= \\
&= \frac{1}{T}\left(A\cdot\frac{a\cdot T}{2}-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot(1-a)\cdot(1+a)+B\cdot a\cdot T\right)= \\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot(1+a)+B\cdot a\cdot T\right)= \\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot a+B\cdot a\cdot T\right)= \\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}+B\cdot T\cdot\frac{1}{2}\cdot a\right)= \\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}+B\cdot\frac{1}{2}\cdot a= \\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}(1-a)= \\
&= \frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)
\end{aligned}
$$

The a_0 coefficient equals $\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$.

The a_k coefficients are defined as:

$$a_k=\frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.29}$$

For the period $t\in(0;T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt= \\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}\left(-\frac{B}{(1-a)\cdot T}\cdot t+\frac{B}{1-a}\cdot a\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)= \\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}-\frac{B}{(1-a)\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right. \\
&\left.+\int_{a\cdot T}^{T}\frac{B}{1-a}\cdot a\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)= \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right. \\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)= \\
&= \left\{\begin{array}{l} z=k\cdot\frac{2\pi}{T}\cdot t \\ dz=k\cdot\frac{2\pi}{T}\cdot dt \\ \frac{dz}{k\cdot\frac{2\pi}{T}}=dt \end{array}\right\}=
\end{aligned}
$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \int_{a \cdot T}^{T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \int_{a \cdot T}^{T} cos\,(z) \cdot dz \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\, sin\,(z)|_{a \cdot T}^{T} \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\, sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$
$$= \left\{ \begin{array}{ll} u = t & dv = cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array} \right\} =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \int_0^{a \cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$
$$- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} - \int_{a \cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$
$$- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a \cdot T}^{T} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$
$$= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) + \right.$$

$$
\begin{aligned}
&- \frac{B}{(1-a)\cdot T}\cdot\left(t\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Bigg|_{a\cdot T}^{T} - \frac{1}{k\cdot\frac{2\pi}{T}}\cdot\int_{a\cdot T}^{T} sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}}\right) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(sin\left(k\cdot 2\pi\right) - sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\left(\left(a\cdot T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right) - 0\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot -\right)\right) + \right.\right.\\
&- \frac{1}{\left(k\cdot\frac{2\pi}{T}\right)^2}\cdot\int_0^{a\cdot T} sin\left(z\right)\cdot dz\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\left(\left(T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) - a\cdot T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right)\right) + \right.\\
&- \frac{1}{\left(k\cdot\frac{2\pi}{T}\right)^2}\cdot\int_{a\cdot T}^{T} sin\left(z\right)\cdot dz\Bigg) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(0 - sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\left(\left(a\cdot T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) - 0\right) + \right.\right.\\
&- \frac{1}{k^2\cdot\frac{4\pi^2}{T^2}}\cdot\left(-cos\left(z\right)\right)\Big|_0^{a\cdot T}\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\left(\left(T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\right) - a\cdot T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\right) + \right.\\
&- \frac{1}{k^2\cdot\frac{4\pi^2}{T^2}}\cdot\left(-cos\left(z\right)\right)\Big|_{a\cdot T}^{T}\Bigg) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(-sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\left(a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \right.\right.\\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right)\Bigg|_0^{a\cdot T}\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{k\cdot 2\pi}\cdot 0 - a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \right.\\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right)\Bigg|_{a\cdot T}^{T}\Bigg) + \\
&- \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\Bigg) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\left(a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \right.\right.\\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right) + cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\left(0 - a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) + cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi \cdot a\right) + cos\left(0\right)\right)\right) + \right.\\
&- \frac{B}{(1-a) \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi\right) + cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&\left.- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot T^2 \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi \cdot a\right) + 1\right)\right) + \right.\\
&- \frac{B}{(1-a) \cdot T} \cdot T^2 \left(-a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(-1 + cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&\left.- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a} \cdot T \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \right.\\
&+ \frac{B}{1-a} \cdot T \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&\left.- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2 \cdot A}{a} \left(\frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&+ \frac{2 \cdot B}{1-a} \left(\frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&- \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) = \\
&= \frac{2 \cdot A}{a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{2 \cdot A}{a} \cdot \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&+ \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{2 \cdot B}{1-a} \cdot \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) = \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{A}{a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot \frac{1}{2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot \frac{1}{2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) \\
&= \left\{ \ \tfrac{1}{2} \cdot \left(1 - cos\left(x\right)\right) = sin^2\left(\tfrac{x}{2}\right) \ \right\} = \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) +
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{A}{a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)+\\
&-\frac{B}{1-a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)\\
&=\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)
\end{aligned}
$$

The a_k coefficients equal to $\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)$.
The b_k coefficients are defined as:

$$
b_k=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.30}
$$

For the period $t\in(0;T)$ we get:

$$
\begin{aligned}
b_k&=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt=\\
&=\frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}\left(-\frac{B}{(1-a)\cdot T}\cdot t+\frac{B}{1-a}\cdot a\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}-\frac{B}{(1-a)\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\int_{a\cdot T}^{T}\frac{B}{1-a}\cdot a\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\left\{\begin{array}{l} z=k\cdot\frac{2\pi}{T}\cdot t\\ dz=k\cdot\frac{2\pi}{T}\cdot dt\\ \frac{dz}{k\cdot\frac{2\pi}{T}}=dt\end{array}\right\}=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}}\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\int_{a\cdot T}^{T}sin\left(z\right)\cdot dz\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.-\frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\,cos\left(z\right)\Big|_{a\cdot T}^{T}\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.
\end{aligned}
$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T}\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{ll} u = t & dv = sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = -\frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} + \int_0^{a \cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} + \int_{a \cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a \cdot T}^{T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}}\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a \cdot T}^{T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}}\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\left(cos\,(k \cdot 2\pi) - cos\,(k \cdot 2\pi \cdot a)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) + 0 \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right) + \right.\right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_0^{a \cdot T} cos\,(z) \cdot dz\right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) + a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right) + \right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_{a \cdot T}^{T} cos\,(z) \cdot dz\right) +$$

$$
\begin{aligned}
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + 0 \right) \right.\right. + \\
&+ \left.\left. \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\,(z)\right)\Big|_0^{a \cdot T} \right) \right. + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi) + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right)\right. + \\
&+ \left. \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\,(z)\right)\Big|_{a \cdot T}^{T} \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right.\right. + \\
&+ \left. \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Big|_0^{a \cdot T} \right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot 1 + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right)\right. + \\
&+ \left. \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Big|_{a \cdot T}^{T} \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right.\right. + \\
&+ \left.\frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(T \cdot \frac{T}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) \right. + \\
&+ \left.\frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right.\right. + \\
&+ \left. \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi \cdot a) - sin\,(0)) \right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) \right. + \\
&+ \left. \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi) - sin\,(k \cdot 2\pi \cdot a)) \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right.\right. +
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(sin\left(k \cdot 2\pi \cdot a\right) - 0\right)\Bigg) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \Bigg(\frac{T^2}{k \cdot 2\pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&+ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(0 - sin\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T} \cdot \Bigg(\frac{A}{a \cdot T} \cdot \Bigg(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \Bigg(\frac{T^2}{k \cdot 2\pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T} \cdot \Bigg(\frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T} \cdot \frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{2}{T} \cdot \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{2}{T} \cdot \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{A}{a} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= -\frac{A}{a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot \left(-1 + a \cdot cos\left(k \cdot 2\pi \cdot a\right)\right) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= -\frac{A}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&+ \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} - \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot a \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) = \\
&= -\frac{A}{k \cdot \pi} \cdot cos\left(k \cdot 2\pi \cdot a\right) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&+ \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\left(k \cdot 2\pi \cdot a\right) +
\end{aligned}
$$

$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}-\frac{B}{1-a}\cdot a\cdot\frac{1}{k\cdot\pi}=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+$$
$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}\cdot(1-a)=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+\frac{B}{k\cdot\pi}$$

The b_k coefficients equal to $\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0=\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$$
$$a_k=\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)$$
$$b_k=\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$$

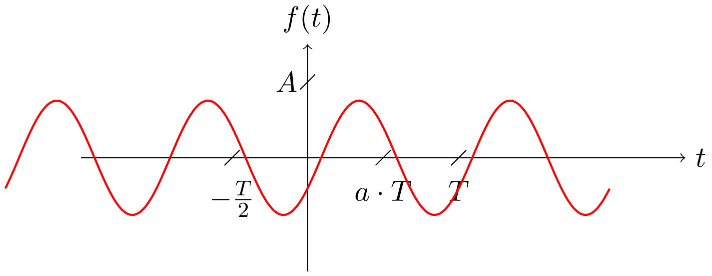
The first six coefficients are equal to:

k	a_k	b_k
1	$\frac{A}{\pi}\cdot sin\left(2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{\pi^2}\cdot sin^2\left(\pi\cdot a\right)$	$\frac{B}{\pi}-\frac{A}{\pi}\cdot cos\left(2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{2\pi^2}\cdot sin\left(2\pi\cdot a\right)$
2	$\frac{A}{2\pi}\cdot sin\left(4\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{4\pi^2}\cdot sin^2\left(2\pi\cdot a\right)$	$\frac{B}{2\pi}-\frac{A}{2\pi}\cdot cos\left(4\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{8\pi^2}\cdot sin\left(4\pi\cdot a\right)$
3	$\frac{A}{3\pi}\cdot sin\left(6\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{9\pi^2}\cdot sin^2\left(3\pi\cdot a\right)$	$\frac{B}{3\pi}-\frac{A}{3\pi}\cdot cos\left(6\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{18\pi^2}\cdot sin\left(6\pi\cdot a\right)$
4	$\frac{A}{4\pi}\cdot sin\left(8\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{16\pi^2}\cdot sin^2\left(4\pi\cdot a\right)$	$\frac{B}{4\pi}-\frac{A}{4\pi}\cdot cos\left(8\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{32\pi^2}\cdot sin\left(8\pi\cdot a\right)$
5	$\frac{A}{5\pi}\cdot sin\left(10\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{25\pi^2}\cdot sin^2\left(5\pi\cdot a\right)$	$\frac{B}{5\pi}-\frac{A}{5\pi}\cdot cos\left(10\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{50\pi^2}\cdot sin\left(10\pi\cdot a\right)$
6	$\frac{A}{6\pi}\cdot sin\left(12\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{36\pi^2}\cdot sin^2\left(6\pi\cdot a\right)$	$\frac{B}{6\pi}-\frac{A}{6\pi}\cdot cos\left(12\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{72\pi^2}\cdot sin\left(12\pi\cdot a\right)$

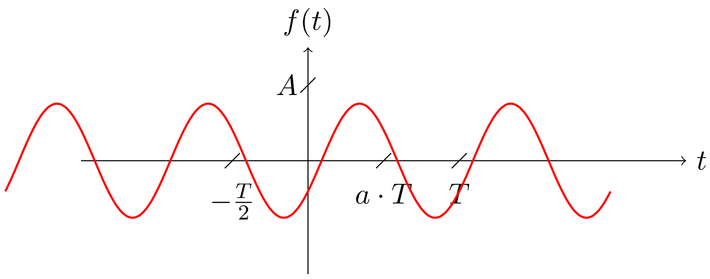
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t)=a_0+\sum_{k=1}^{\infty}\left[a_k\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+b_k\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \tag{2.31}$$

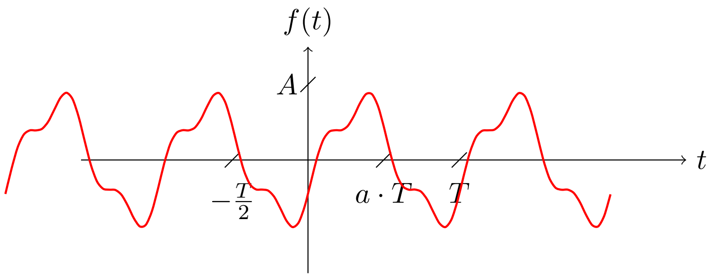
A partial approximation of the $f(t)$ signal for $k_{max}=1$ results in:



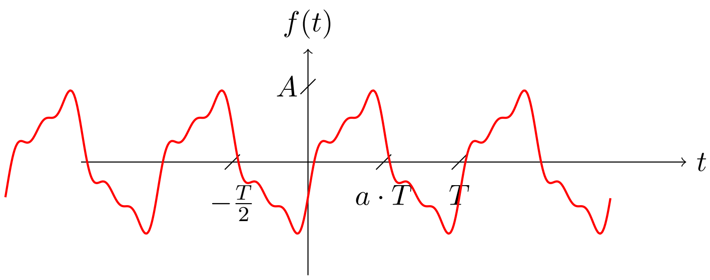
A partial approximation of the $f(t)$ signal for $k_{max}=2$ results in:



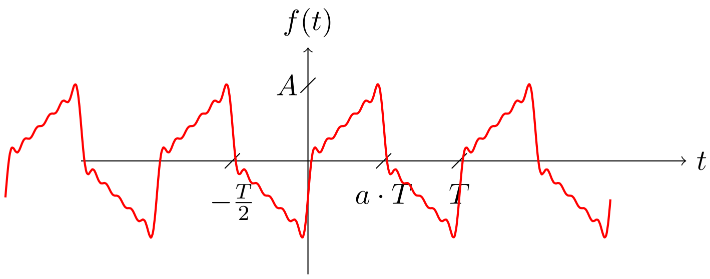
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



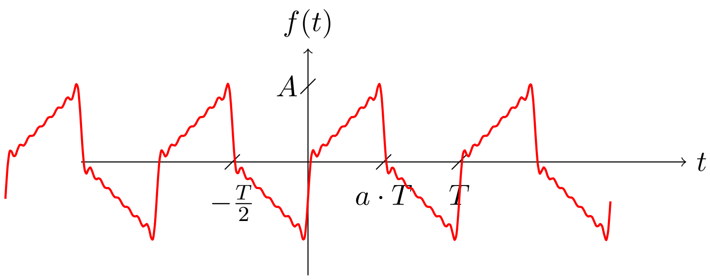
A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



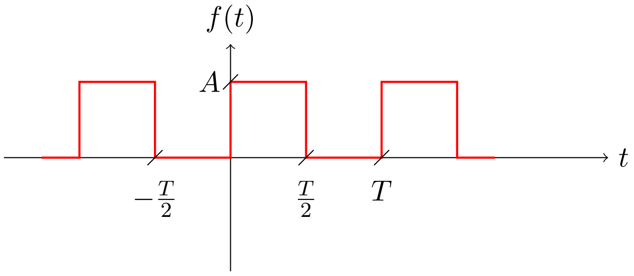
A partial approximation of the $f(t)$ signal for $k_{max} = 16$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

2.2 Complex exponential Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.32}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.33}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot dt + \int_{\frac{T}{2}}^T 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{1}{T} \left(A \cdot t|_0^{\frac{T}{2}} \right) = \\ &= \frac{A}{T} \cdot t|_0^{\frac{T}{2}} = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A}{2} \end{aligned} \tag{2.34}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.35}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \begin{Bmatrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{Bmatrix} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^z\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^z\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\; e^z\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\; e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^0\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - 1\right) = \\
&= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\pi} - 1\right) \\
&= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)
\end{aligned}
$$

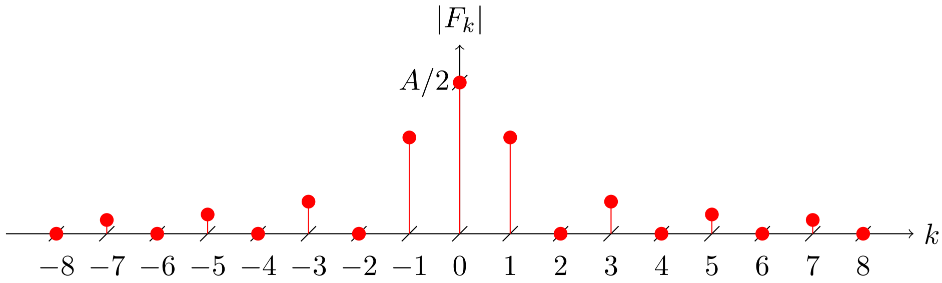
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \\
f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[\jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)\right]\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}
\end{aligned}
\tag{2.36}
$$

The first several coefficients are equal to:

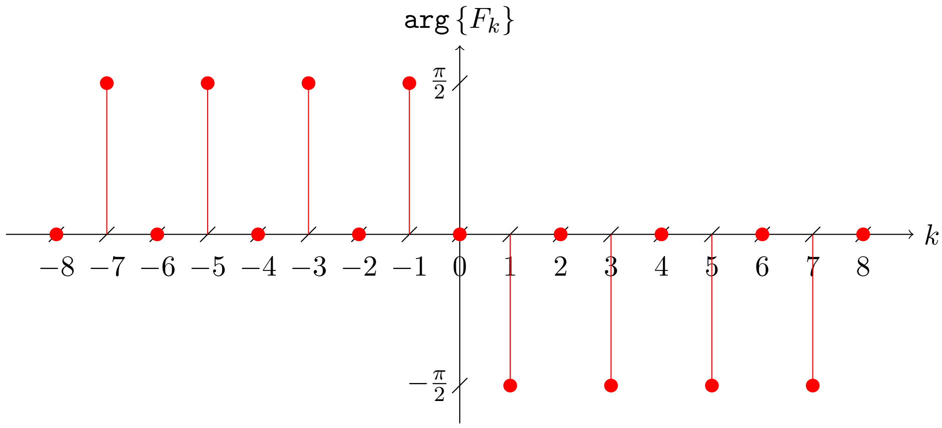
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	0	$\jmath \cdot \frac{A}{\pi}$	$\frac{A}{2}$	$-\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$
$\lvert F_k \rvert$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{2}$	$\frac{A}{\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$
$Arg\left\{F_k\right\}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



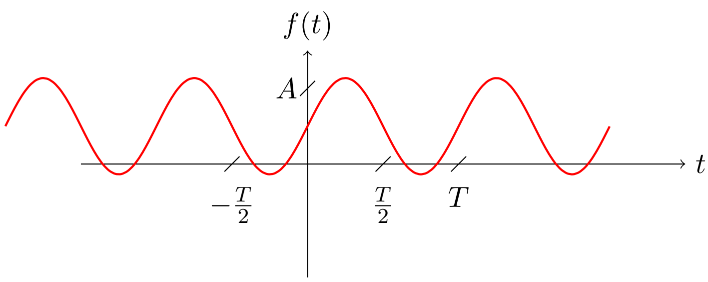
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

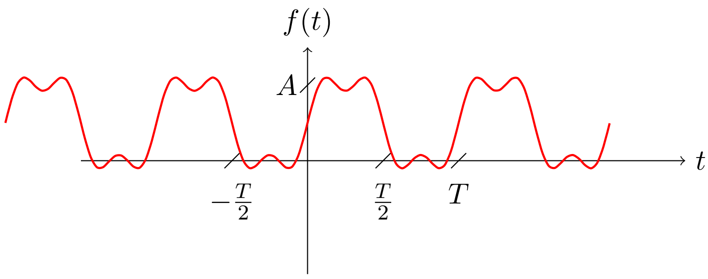


The phase spectrum of a real signal is an odd-symmetric function of k.

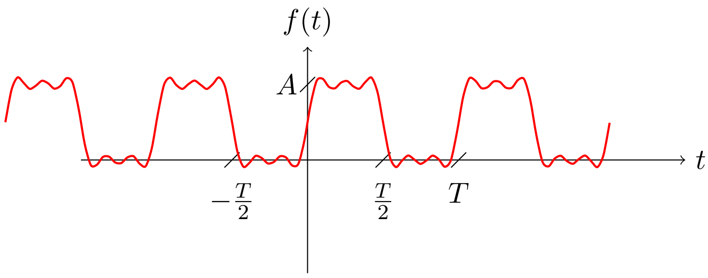
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



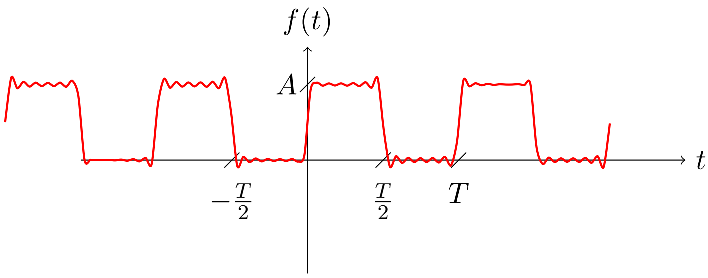
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



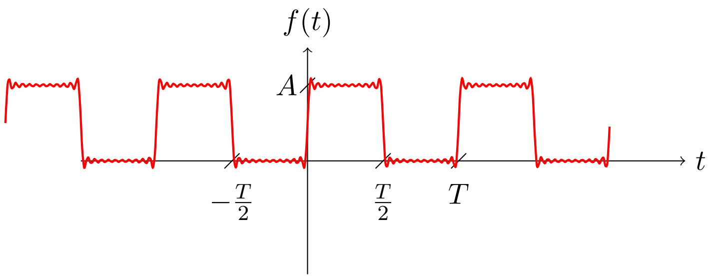
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

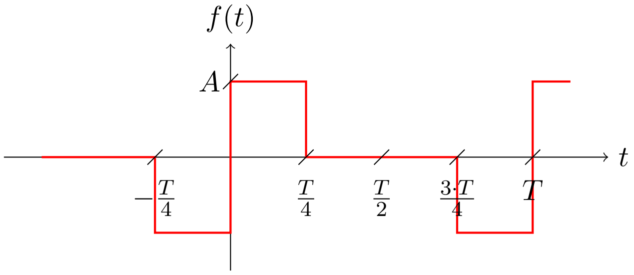


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3 \cdot T}{4} + k \cdot T\right) \end{cases} \quad \wedge k \in Z \tag{2.37}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.38}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot \int_{-\frac{T}{4}}^{0} dt + A \cdot \int_0^{\frac{T}{4}} dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot t|_{-\frac{T}{4}}^{0} + A \cdot t|_0^{\frac{T}{4}} \right) = \\ &= \frac{1}{T} \left(-A \cdot \left(0 - \left(-\frac{T}{4}\right)\right) + A \cdot \left(\frac{T}{4} - 0\right) \right) = \\ &= \frac{1}{T} \left(-A \cdot \frac{T}{4} + A \cdot \frac{T}{4} \right) = \\ &= \frac{1}{T} (0) = \\ &= 0 \end{aligned} \tag{2.39}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.40}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \left\{\begin{aligned} z &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot t \\ dz &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot dt \\ dt &= \tfrac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{aligned}\right\} = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{1}{T}\left(-\frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot dz + \frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot dz\right) = \\
&= \frac{1}{T}\cdot\frac{A}{\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\left(e^{z}\big|_{-\frac{T}{4}}^{0} - e^{z}\big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{0} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0} - e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{0} - e^{\jmath\cdot k\cdot\frac{2\pi}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{4}} - e^{0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 1\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}} + 1\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(2 - \left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot 2\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \left\{cos(x) \quad = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= -\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= 0 \\
F_k &= \jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos \left(k \cdot \frac{\pi}{2} \right) - 1 \right)
\end{aligned}
\tag{2.41}
$$

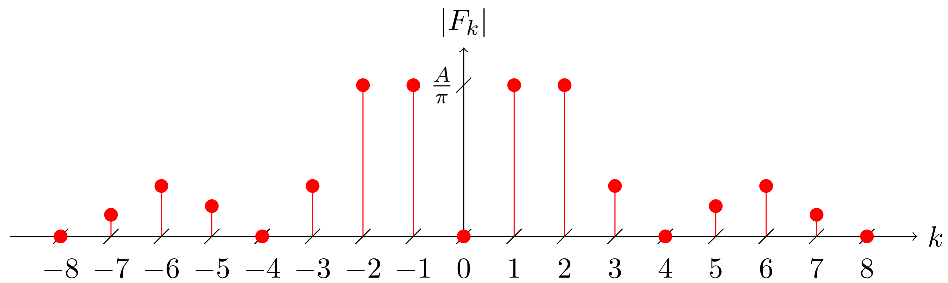
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos \left(k \cdot \frac{\pi}{2} \right) - 1 \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.42}
$$

The first several coefficients are equal to:

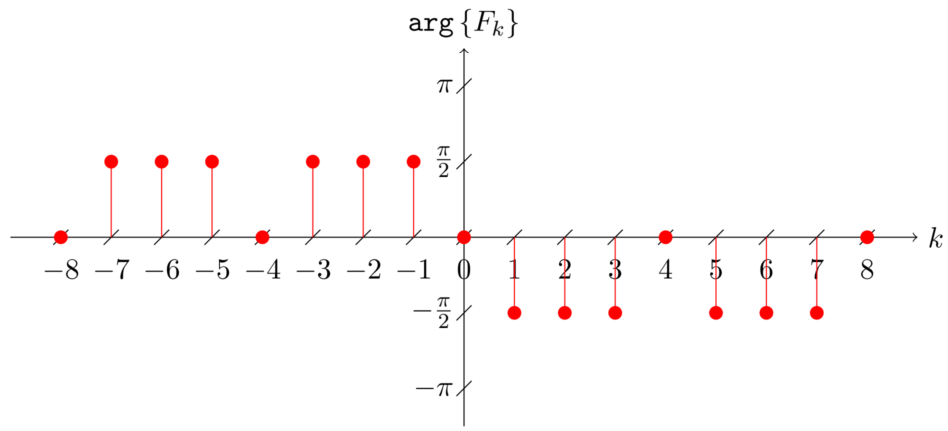
k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{\pi}$	$\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$	$-\jmath \cdot \frac{A}{3\pi}$
$\lvert F_k \rvert$	$\frac{A}{3\pi}$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{\pi}$	$\frac{A}{\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{\pi}$	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$	$\frac{A}{3\pi}$
$Arg\{F_k\}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



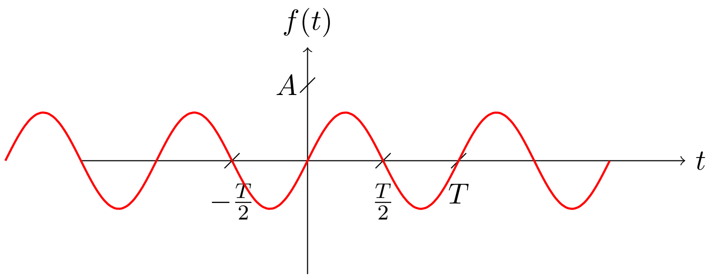
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

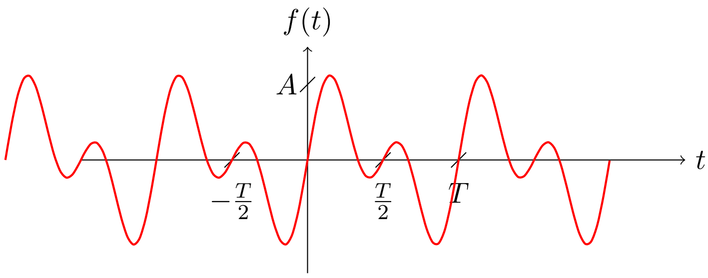


The phase spectrum of a real signal is an odd-symmetric function of k.

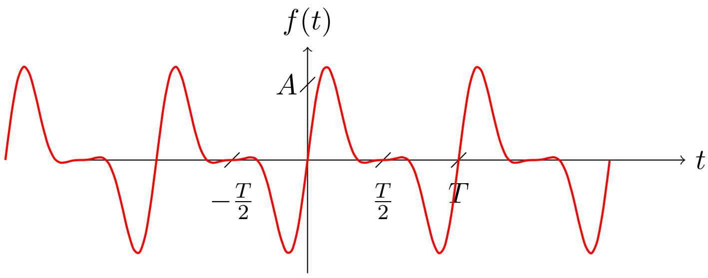
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



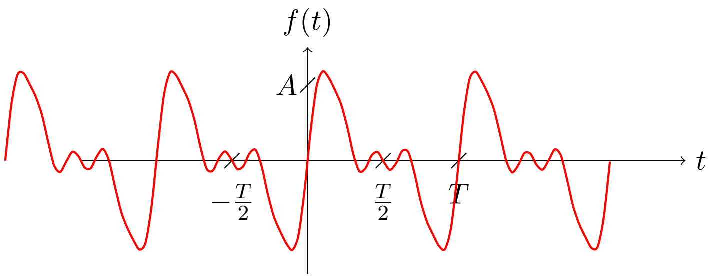
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



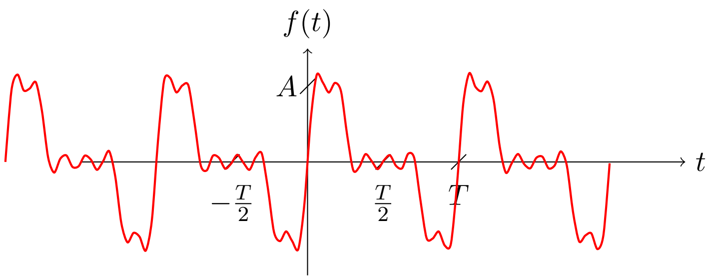
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



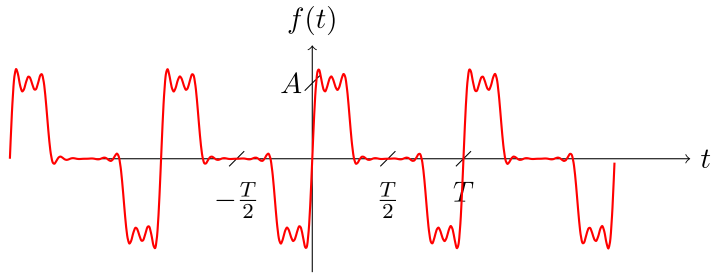
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



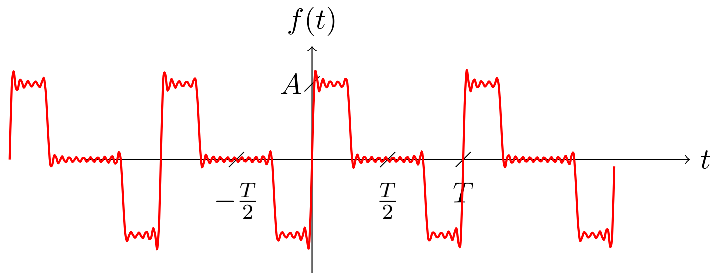
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

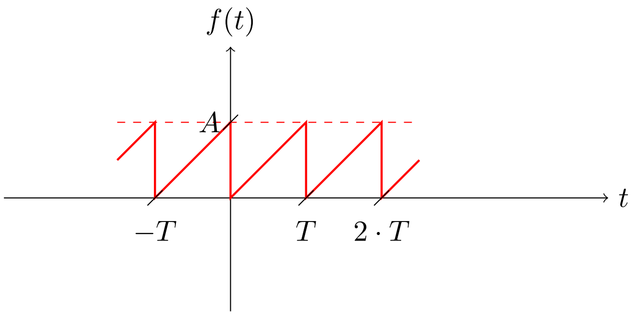


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 3. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.43}$$

In the first period (i.e. $t \in (0; T)$), linear function crosses two points: $(0, 0)$ and (T, A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \tfrac{A}{T} \cdot (t - k \cdot T) \quad t \in (0 + k \cdot T; T + k \cdot T) \wedge k \in Z$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.44}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot dt = \\
&= \frac{A}{T^2}\int_0^T t\cdot dt = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot t^2\Big|_0^T = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot\left(T^2-0^2\right) = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot T^2 = \\
&= \frac{A}{2}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$
F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.45}
$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1\cdot A}{T^2}\int_0^T t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \\ du & = dt & v & = \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \end{Bmatrix} = \\
&= \frac{A}{T^2}\cdot\left(t\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T - \int_0^T \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{T^2}\cdot\left(\left(T\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - 0\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) + \frac{T^2}{\left(-\jmath\cdot k\cdot 2\pi\right)^2}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T\right) = \\
&= \frac{A}{T^2}\cdot\left(\frac{T^2}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} + \frac{T^2}{-\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot 2\pi} - e^0\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot(1-1)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot 0\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - 0\right) =
\end{aligned}
$$

$$= \frac{A}{-\jmath \cdot k \cdot 2\pi} =$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi}$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi}$$

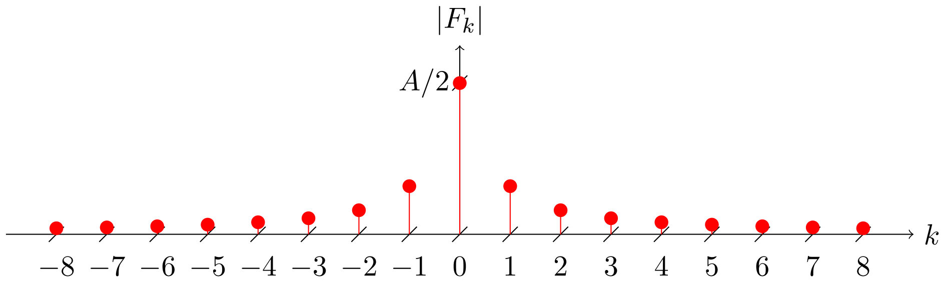
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi} \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.46}$$

The first several coefficients are equal to:

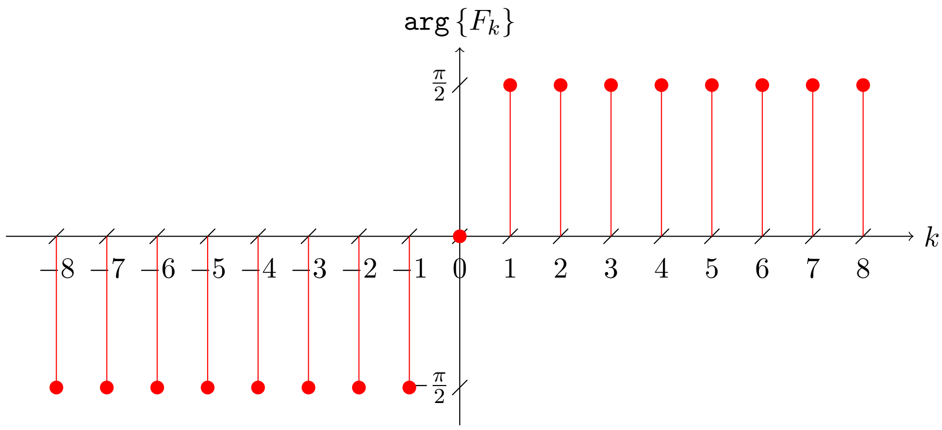
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$-\jmath \cdot \frac{A}{10 \cdot \pi}$	$-\jmath \cdot \frac{A}{8 \cdot \pi}$	$-\jmath \cdot \frac{A}{6 \cdot \pi}$	$-\jmath \cdot \frac{A}{4 \cdot \pi}$	$-\jmath \cdot \frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\jmath \cdot \frac{A}{2 \cdot \pi}$	$\jmath \cdot \frac{A}{4 \cdot \pi}$	$\jmath \cdot \frac{A}{6 \cdot \pi}$	$\jmath \cdot \frac{A}{8 \cdot \pi}$	$\jmath \cdot \frac{A}{10 \cdot \pi}$
$\lvert F_k \rvert$	$\frac{A}{10 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{10 \cdot \pi}$
$Arg\left(F_k\right)$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



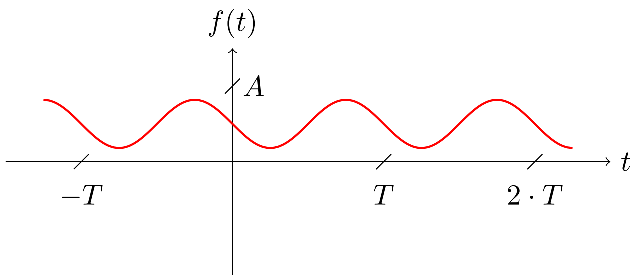
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

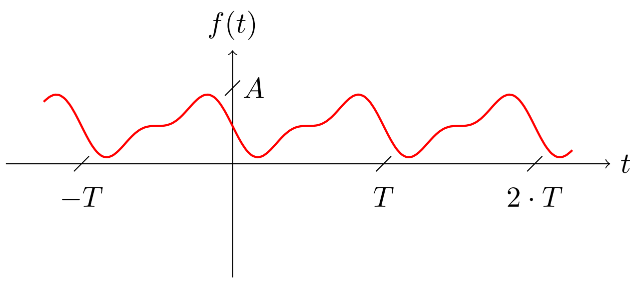


The phase spectrum of a real signal is an odd-symmetric function of k.

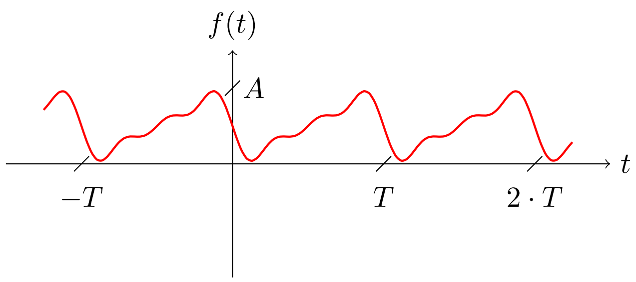
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



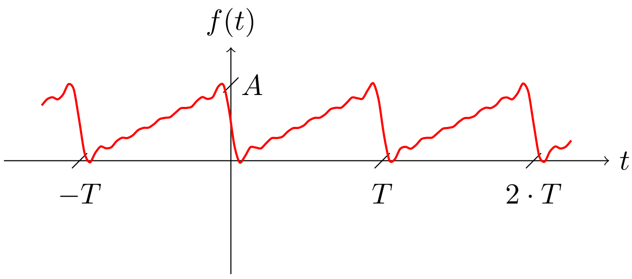
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



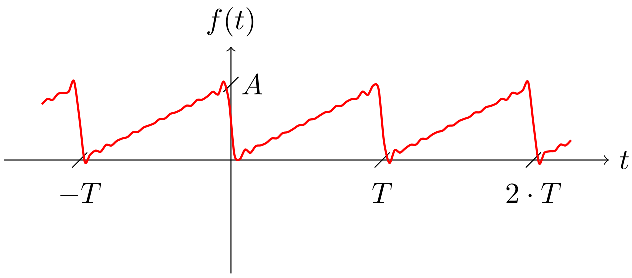
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -7$ to $k_{max} = 7$ results in:

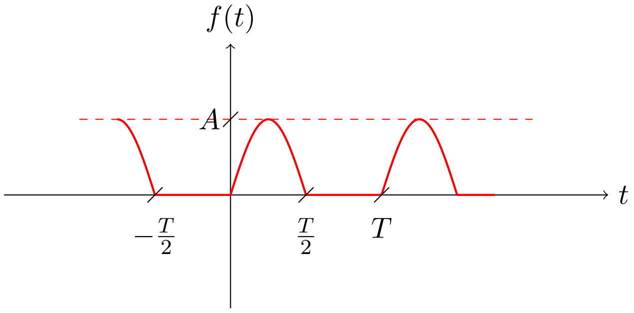


A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 4. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.47}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.48}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{aligned} z &= \tfrac{2\pi}{T} \cdot t \\ dz &= \tfrac{2\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{\frac{2\pi}{T}} \end{aligned} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{2\pi} \cdot \left(cos \left(\frac{2\pi}{T} \cdot \frac{T}{2} \right) - cos \left(\frac{2\pi}{T} \cdot 0 \right) \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\pi\right) - cos\left(0\right) \right) = \\
&= -\frac{A}{2\pi} \cdot (-1-1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{\pi}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.49}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\left(x\right) \quad = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \end{array} \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} - \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\frac{1}{1-k} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{1+k} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{z_1} \Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{z_2} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^0 \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - e^0 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1+k}{(1-k) \cdot (1+k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + \frac{1-k}{(1-k) \cdot (1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right)}{(1-k) \cdot (1+k)} + \frac{(1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + (1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 1 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} - k + e^{-\jmath \cdot \pi \cdot (1+k)} - 1 - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)} + k}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 2 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} + e^{-\jmath \cdot \pi \cdot (1+k)} - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - 2 + k \cdot e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} + e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - k \cdot e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-1 \cdot e^{-\jmath \cdot \pi \cdot k} - 2 + k \cdot (-1) \cdot e^{-\jmath \cdot \pi \cdot k} - 1 \cdot e^{-\jmath \cdot \pi \cdot k} - k \cdot (-1) \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-e^{-\jmath \cdot \pi \cdot k} - 2 - k \cdot e^{-\jmath \cdot \pi \cdot k} - e^{-\jmath \cdot \pi \cdot k} + k \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-2 \cdot e^{-\jmath \cdot \pi \cdot k} - 2}{1-k^2} \right) = \\
&= \frac{A}{4 \cdot \pi} \cdot \left(\frac{2 \cdot e^{-\jmath \cdot \pi \cdot k} + 2}{1-k^2} \right) = \\
&= \frac{A}{4 \cdot \pi} \cdot 2 \cdot \left(\frac{e^{-\jmath \cdot \pi \cdot k} + 1}{1-k^2} \right) = \\
&= \frac{A}{2 \cdot \pi} \cdot \left(\frac{e^{-\jmath \cdot \pi \cdot k} + 1}{1-k^2} \right) \\
&= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)$ for $k \neq 1 \wedge k \neq -1$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
F_1 = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt =
$$

$$= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$

$$= \left\{ sin\,(x) \quad = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \left\{ \begin{aligned} z \quad &= -\jmath \cdot \tfrac{4\pi}{T} \cdot t \\ dz \quad &= -\jmath \cdot \tfrac{4\pi}{T} \cdot dt \\ dt \quad &= \tfrac{dz}{-\jmath \cdot \frac{4\pi}{T}} \end{aligned} \right\} =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(t\Big|_0^{\frac{T}{2}} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) =$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= \frac{A}{4\jmath} = \\
&= -\jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $-\jmath \cdot \frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt \right) = \\
&= \begin{Bmatrix} z & = \jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath \cdot \frac{4\pi}{T}} \end{Bmatrix} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{2} \right) = \\
&= -\frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= -\frac{A}{4\jmath} = \\
&= \jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_{-1} coefficients equal to $\jmath \cdot \frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_{-1} &= \jmath \cdot \frac{A}{4} \\
F_1 &= -\jmath \cdot \frac{A}{4} \\
F_k &= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}
$$

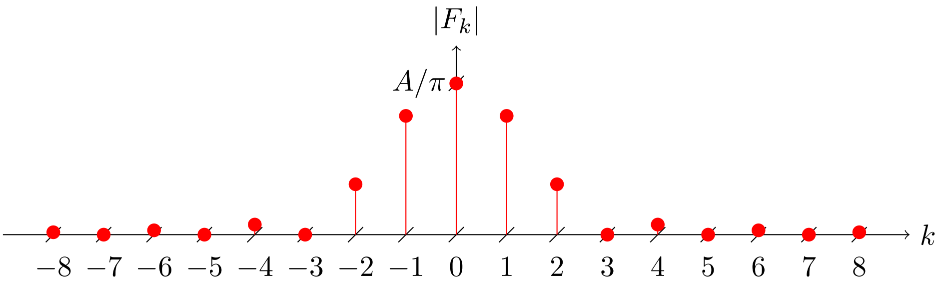
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{\pi} + \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \qquad (2.50)
\end{aligned}
$$

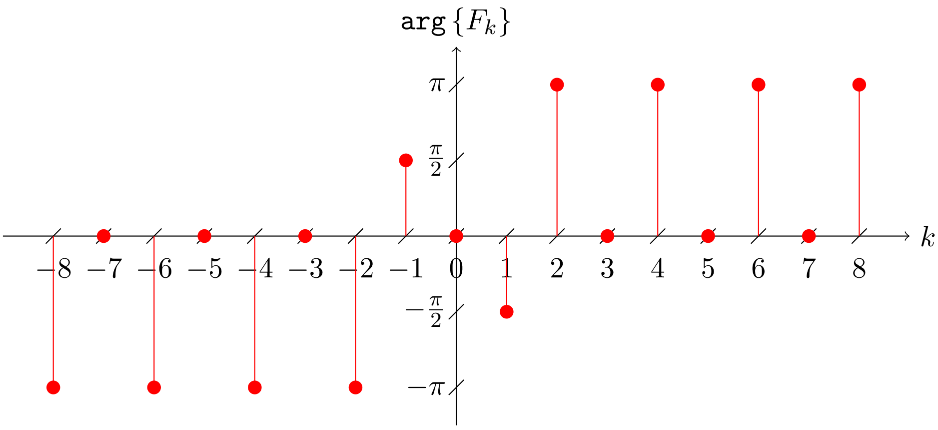
The first several coefficients are equal to:

F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\jmath \cdot \frac{A}{4}$	$\frac{A}{\pi}$	$-\jmath \cdot \frac{A}{4}$	$-\frac{A}{3\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{35\pi}$
$\|F_k\|$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$Arg\left\{F_k\right\}$	$-\pi$	0	$-\pi$	0	$-\pi$	$\frac{\pi}{2}$	0	-$\frac{\pi}{2}$	π	0	π	0	π

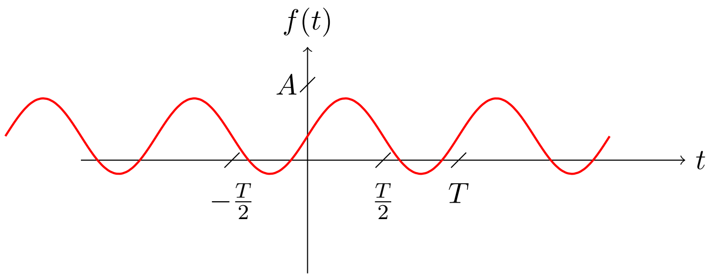
Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



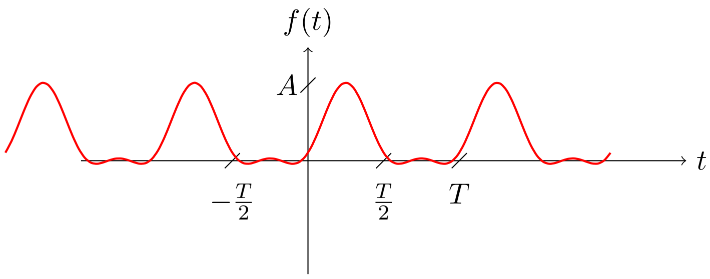
The magnitude spectrum of a real signal is an even-symmetric function of k.
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.



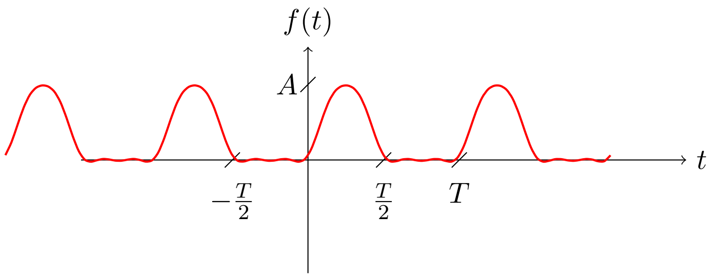
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



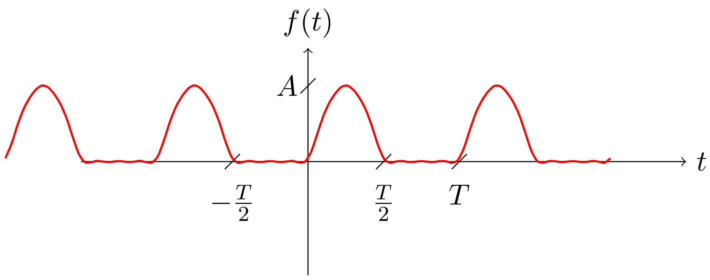
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



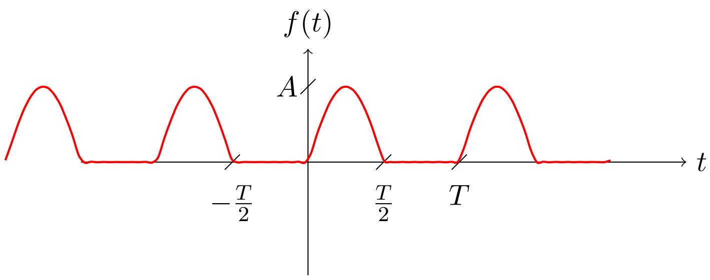
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:

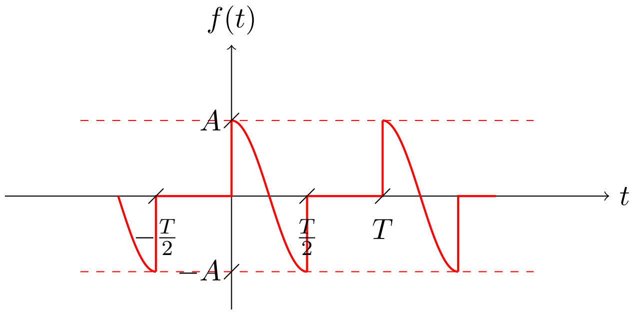


A partial approximation of the $f(t)$ signal from $k_{min} = -12$ to $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 5. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.51}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.52}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = \frac{T}{2\pi} \cdot dz \end{matrix} \right\} = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{A}{2\pi} \cdot (0 - 0) = \\
&= \frac{A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.53}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{matrix} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^0 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= -\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x\right) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{2\cdot T}\cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} e^{0}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} 1\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \left\{\begin{matrix} z = -\jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = -\jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\} = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{z}\cdot \frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\int_0^{\frac{T}{2}} dt + \frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(t\Big|_0^{\frac{T}{2}} - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\left(\frac{T}{2}-0\right) - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\frac{T}{2} - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\frac{T}{2} - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi} - e^{0}\right)\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\frac{T}{2} - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\frac{T}{2} - \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right) = \\
&= \frac{A}{2\cdot T}\cdot \left(\frac{T}{2} - 0\right) = \\
&= \frac{A}{2\cdot T}\cdot \frac{T}{2} = \\
&= \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{A}{4}$.

We have to calculate F_k for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) = \\
&= \left\{\begin{matrix} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\} = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\frac{T}{2} =
\end{aligned}
$$

$$= \frac{A}{4}$$

The F_{-1} coefficients equal to $\frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = 0$$
$$F_1 = \frac{A}{4}$$
$$F_{-1} = \frac{A}{4}$$
$$F_k = \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)$$

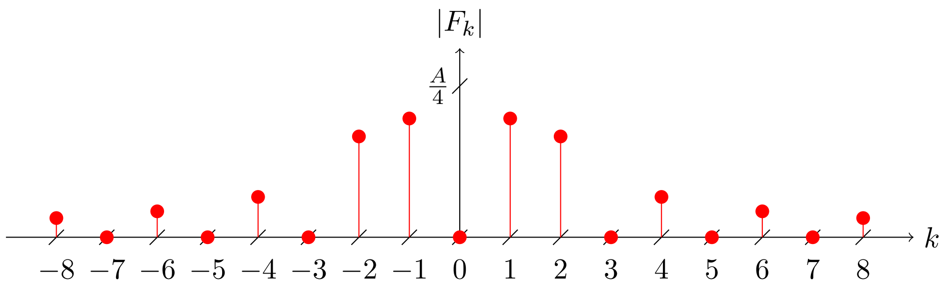
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} + \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{2} \cdot cos\left(\frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.54}$$

The first several coefficients are equal to:

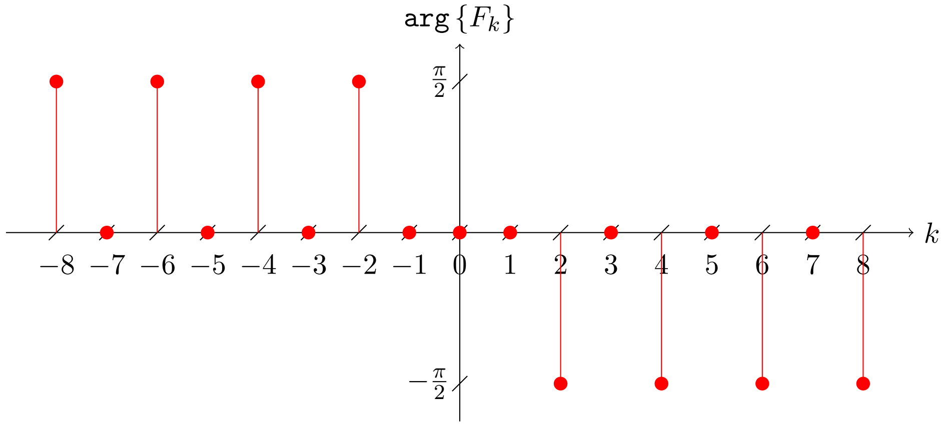
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	0	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

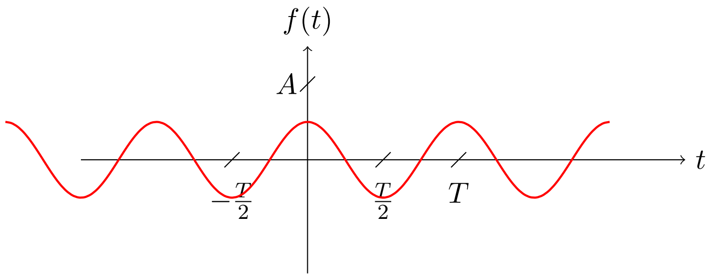


The magnitude spectrum of a real signal is an even-symmetric function of k.

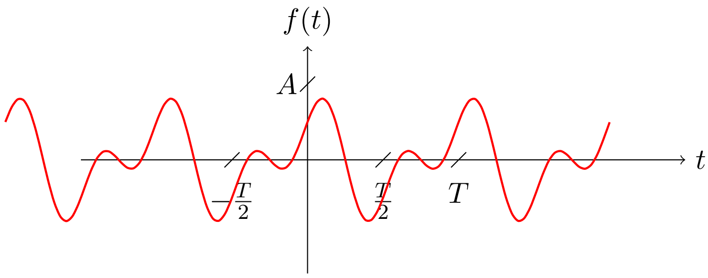
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



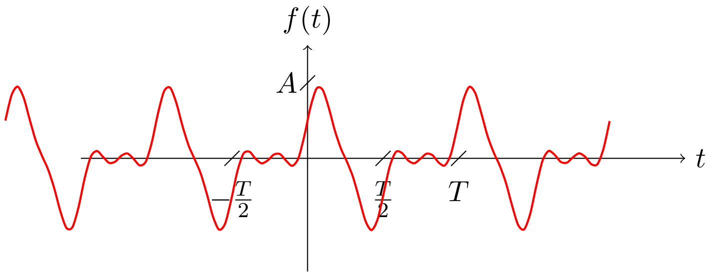
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



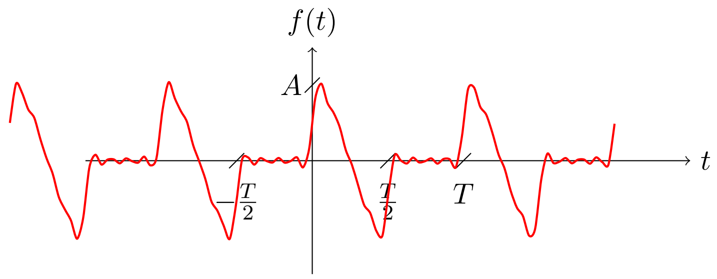
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



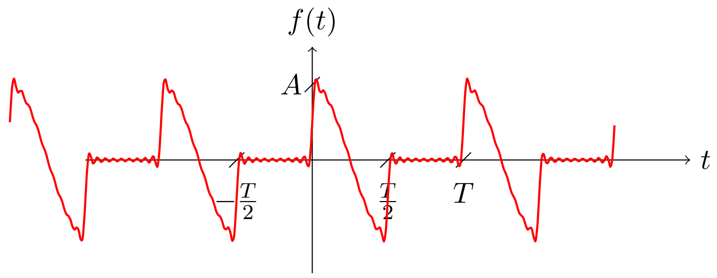
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

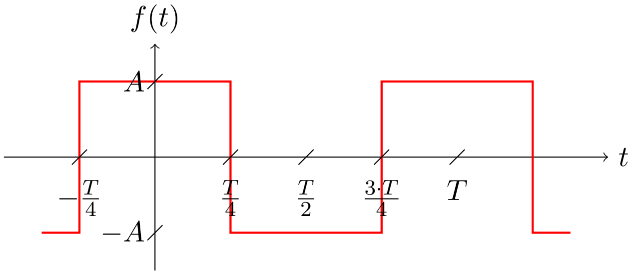


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 6. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function assuming period $t \in \left(\frac{-T}{4}; \frac{3\cdot T}{4}\right)$ is given by:

$$f(x) = \begin{cases} A & t \in \left(-\frac{T}{4} + k\cdot T; \frac{T}{4} + k\cdot T\right) \\ -A & t \in \left(\frac{T}{4} + k\cdot T; \frac{3\cdot T}{4} + k\cdot T\right) \end{cases} \wedge k \in Z \tag{2.55}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T}\int_T f(t)\cdot dt \tag{2.56}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\ &= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} (-A)\cdot dt\right) = \\ &= \frac{1}{T}\left(A\cdot\int_{-\frac{T}{4}}^{\frac{T}{4}} dt - A\cdot\int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} dt\right) = \\ &= \frac{A}{T}\left(t|_{-\frac{T}{4}}^{\frac{T}{4}} - t|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right) = \\ &= \frac{A}{T}\cdot\left[\left(\frac{T}{4} - \left(-\frac{T}{4}\right)\right) - \left(\frac{3\cdot T}{4} - \frac{T}{4}\right)\right] = \\ &= \frac{A}{T}\cdot\left[\frac{T}{4} + \frac{T}{4} - \frac{3\cdot T}{4} + \frac{T}{4}\right] = \\ &= \frac{A}{T}\cdot[0] = \\ &= 0 \end{aligned} \tag{2.57}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.58}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} (-A)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \frac{1}{T}\left(A\cdot\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - A\cdot\int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \frac{A}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t\\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt\\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\end{matrix}\right\} =\\
&= \frac{A}{T}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\right] =\\
&= \frac{-A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot dz - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot dz\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{z}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{z}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\left(-\frac{T}{4}\right)} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{3\cdot T}{4}} + e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \left\{e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} = e^{-\jmath\cdot k\cdot\left(2\pi-\frac{\pi}{2}\right)} = e^{-\jmath\cdot k\cdot 2\pi}\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = 1\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right\} =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(2\cdot e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 2\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{2\cdot A}{\jmath\cdot k\cdot 2\pi}\left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= 0\\
F_k &= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

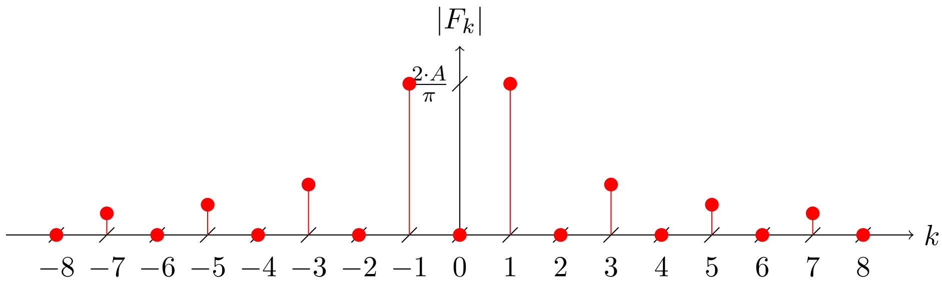
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}$$
$$f(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{2 \cdot A}{k \cdot \pi}\left(sin\left(k \cdot \frac{\pi}{2}\right)\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.59}$$

The first several coefficients are equal to:

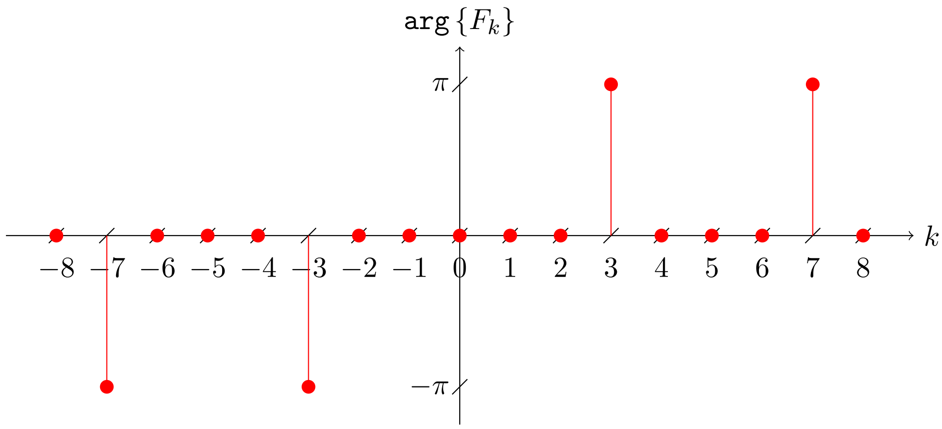
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\frac{2\cdot A}{5\cdot\pi}$	0	$-\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{\pi}$	0	$-\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$
$\lvert F_k\rvert$	$\frac{2\cdot A}{5\cdot\pi}$	0	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$
$Arg\left\{F_k\right\}$	0	0	$-\pi$	0	0	0	0	0	π	0	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



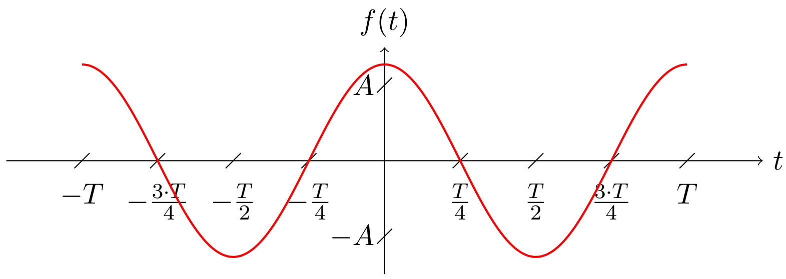
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

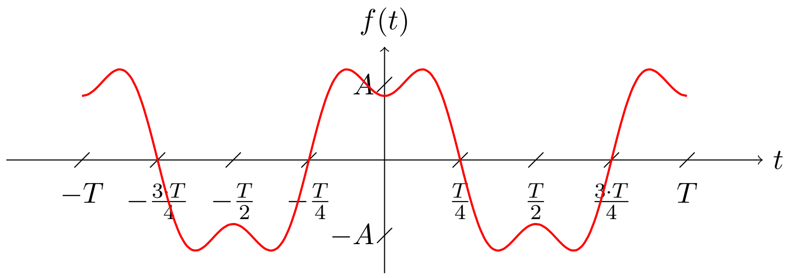


The phase spectrum of a real signal is an odd-symmetric function of k.

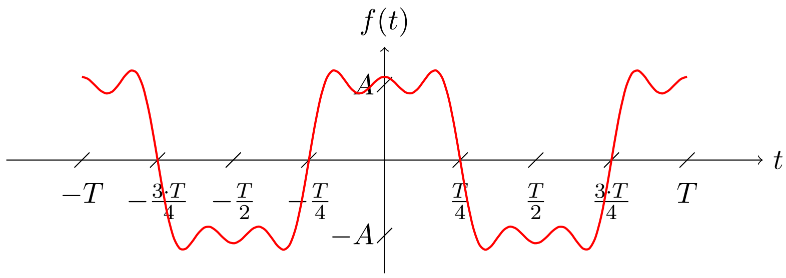
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



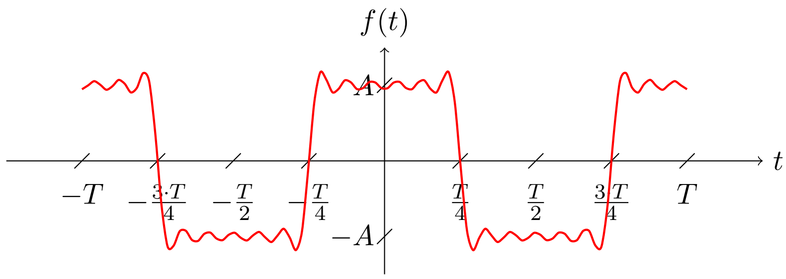
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



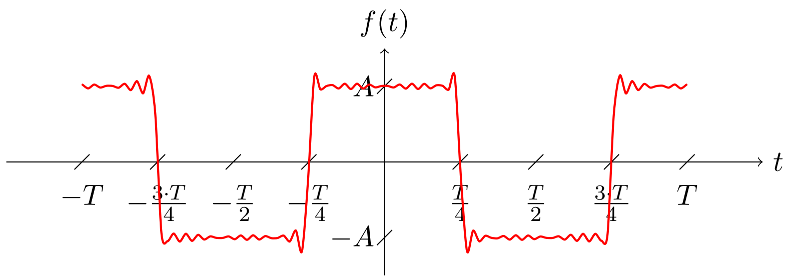
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

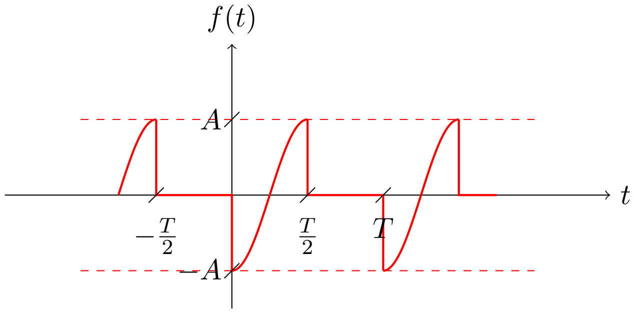


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 7. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} -A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.60}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T}\int_T f(t) \cdot dt \tag{2.61}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T}\int_T f(t) \cdot dt = \\ &= \frac{1}{T}\left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\ &= \frac{1}{T}\left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0\right) = \\ &= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\ &= \left\{\begin{array}{lll} z & = & \frac{2\pi}{T} \cdot t \\ dz & = & \frac{2\pi}{T} \cdot dt \\ dt & = & \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = & \frac{T}{2\pi} \cdot dz \end{array}\right\} = \\ &= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\ &= \frac{-A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\ &= \frac{-A}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\ &= \frac{-A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\ &= \frac{-A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right)\right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{-A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{-A}{2\pi} \cdot (0 - 0) = \\
&= \frac{-A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.62}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{matrix} \right\} = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{-e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= \frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{0}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}1\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{matrix} z=-\jmath\cdot\frac{4\pi}{T}\cdot t\\ dz=-\jmath\cdot\frac{4\pi}{T}\cdot dt\\ dt=\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\end{matrix}\right\}=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{z}\cdot\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}}e^{z}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(t|_0^{\frac{T}{2}}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\left(\frac{T}{2}-0\right)-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{-A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{-A}{4}$.
We have to calculate F_k for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}}(-A)\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T}0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=
\end{aligned}
$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) =$$

$$= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} =$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \left\{\begin{array}{c} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{array}\right\} =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\frac{T}{2} =$$

$$= \frac{-A}{4}$$

The F_{-1} coefficients equal to $\frac{-A}{4}$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= 0\\
F_1 &= \frac{-A}{4}\\
F_{-1} &= \frac{-A}{4}\\
F_k &= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2}\right)
\end{aligned}$$

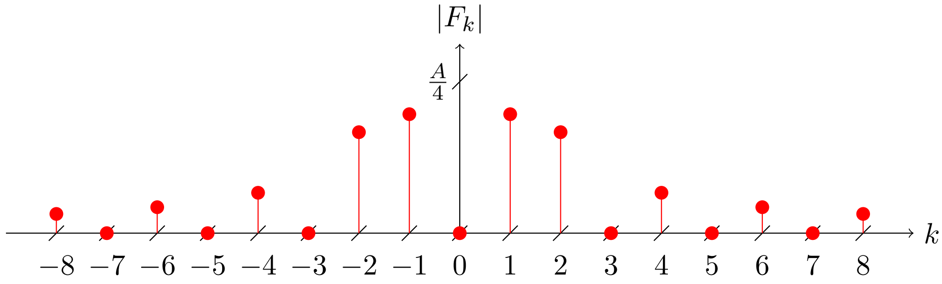
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\\
f(t) &= -\frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty\\ k\neq 0\\ k\neq -1 \wedge k\neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2}\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\\
f(t) &= -\frac{A}{2} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) + \sum_{\substack{k=-\infty\\ k\neq 0\\ k\neq -1 \wedge k\neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2}\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.63}$$

The first several coefficients are equal to:

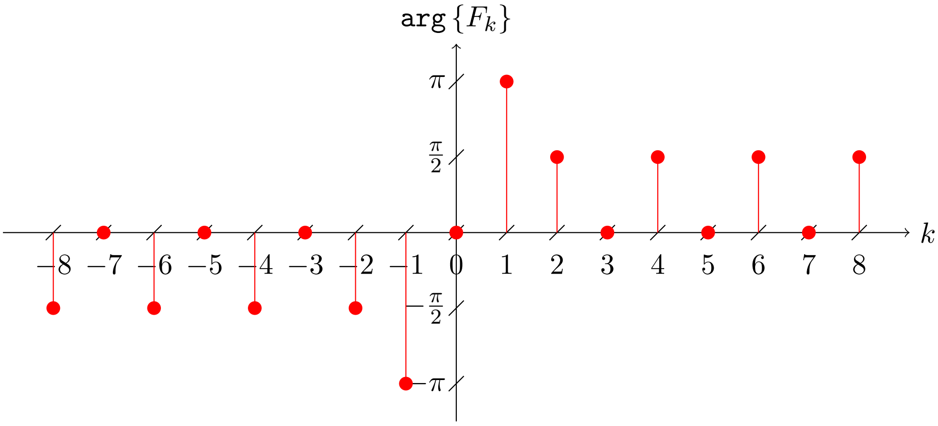
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{-A}{4}$	0	$\frac{-A}{4}$	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\pi$	0	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

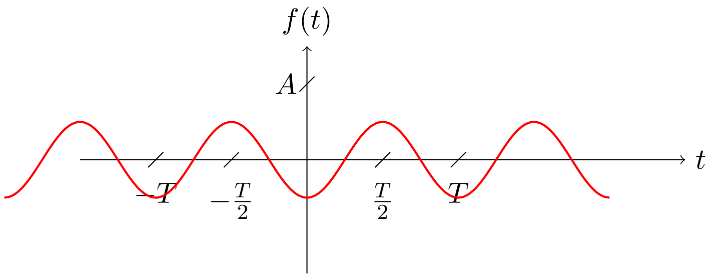


The magnitude spectrum of a real signal is an even-symmetric function of k.

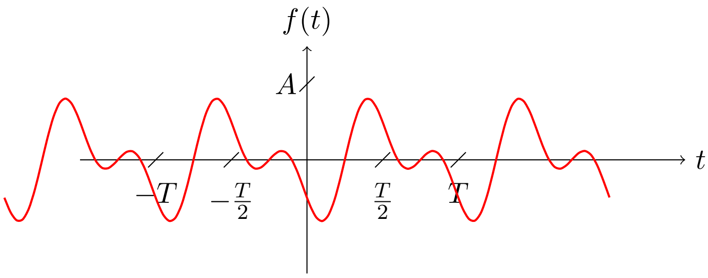
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



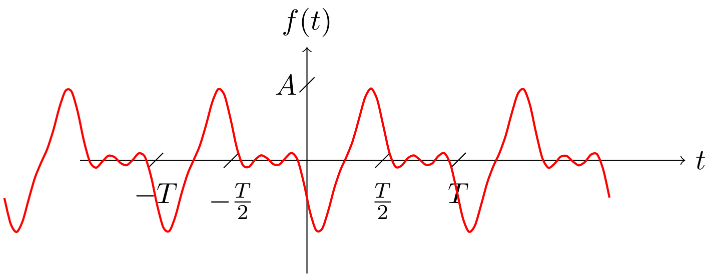
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



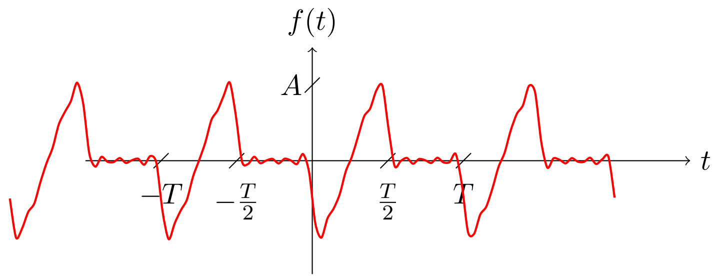
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



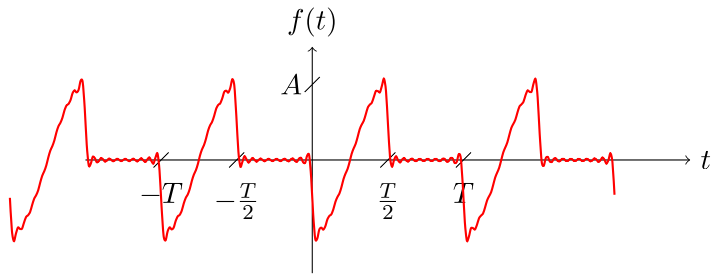
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

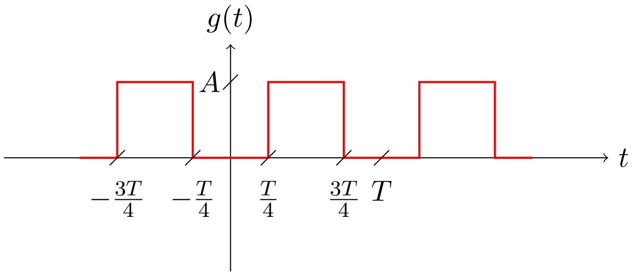


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 8. Calculate coefficients of the periodic signal $g(t)$ shown below for the expansion into a complex exponential Fourier series. Use properties of the complex series and coefficients calculated in task 1.



Periodic signal $g(t)$, as a piecewise linear function, is given by:

$$g(x) = \begin{cases} 0 & t \in \left(0 + k\cdot T; \frac{T}{4} + k\cdot T\right) \\ A & t \in \left(\frac{T}{4} + k\cdot T; \frac{3T}{4} + k\cdot T\right) \wedge k \in Z \\ 0 & t \in \left(\frac{3T}{4} + k\cdot T; T + k\cdot T\right) \end{cases} \tag{2.64}$$

You may notice, that the $g(t)$ is the $f(t)$ from task 1 shifted in time by $\frac{T}{4}$:

$$g(t) = f\left(t - \frac{T}{4}\right)$$

The coefficients for the expansion into a complex exponential Fourier series of $f(t)$ signal from task 1 are equal to:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k\cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

Based on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$g(t) = f(t - t_0)$$
$$G_k = F_k \cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot t_0}$$

Right now the G_k coefficients are equal to:

$$\begin{aligned} G_k &= F_k \cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot t_0} = \\ &= \left\{\ t_0 = \tfrac{T}{4}\ \right\} = \\ &= \jmath \cdot \frac{A}{k\cdot 2\pi} \cdot \left((-1)^k - 1\right) \cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot\frac{T}{4}} \\ &= \jmath \cdot \frac{A}{k\cdot 2\pi} \cdot \left((-1)^k - 1\right) \cdot e^{-\jmath\cdot\pi\cdot k\cdot\frac{1}{2}} \end{aligned}$$

$$= \jmath \cdot \frac{A \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot k}}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big)$$

Similarly, we have to calculate G_0 coefficient:

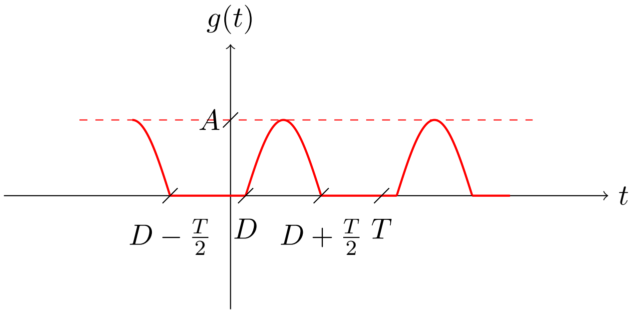
$$\begin{aligned} G_0 &= F_0 \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0 \cdot t_0} = \\ &= F_0 \cdot e^0 = \\ &= F_0 \cdot 1 = \\ &= F_0 = \\ &= \frac{A}{2} \end{aligned}$$

Please note that $G_0 = F_0$. It is obvious, if you recall, that complex exponential Fourier series coefficient for $k = 0$ is equal to mean value of the signal. The mean value does not change with shifting of the signal it time.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned} G_0 &= \frac{A}{2} \\ G_k &= \jmath \cdot \frac{A \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot k}}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \end{aligned}$$

Task 9. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use the knowledge about properties of complex exponential Fourier series and data from Task 4.



Periodic signal $g(t)$, as a piecewise linear function, is given by:

$$g(t) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot (t-D)\right) & t \in \left(D + k \cdot T; D + \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(D + \frac{T}{2} + k \cdot T; D + T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.65}$$

Note, that the $g(t)$ signal is the same as $f(t)$ signal from the Task $\frac{T}{2}$, but shifted in time:

$$g(t) = f\,(t - D)$$

From Task 4 we know, that F_k coefficients for the expansion into a complex exponential Fourier series of $f(t)$ signal are equal to:

$$F_0 = \frac{A}{\pi}$$
$$F_{-1} = \jmath \cdot \frac{A}{4}$$
$$F_1 = -\jmath \cdot \frac{A}{4}$$
$$F_k = \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2}\right)$$

Based on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$g(t) = f(t - t_0)$$
$$G_k = F_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0}$$

Right now we know everything to calculate G_k coefficients:

$$G_k = F_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0} =$$

$$
\begin{aligned}
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot t_0} = \\
&= \left\{\ t_0 = D\ \right\} = \\
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot D}
\end{aligned}
$$

Also G_1 and G_{-1} have to be calculated separately:

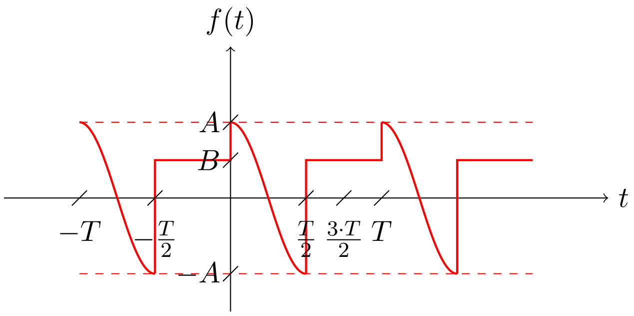
$$
\begin{aligned}
G_1 &= F_1\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot 1\cdot t_0} = \\
&= -\jmath\cdot\frac{A}{4}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t_0} = \\
&= \left\{\ t_0 = D\ \right\} = \\
&= -\jmath\cdot\frac{A}{4}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot D}
\end{aligned}
$$

$$
\begin{aligned}
G_{-1} &= F_{-1}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1)\cdot t_0} = \\
&= \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t_0} = \\
&= \left\{\ t_0 = D\ \right\} = \\
&= \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot D}
\end{aligned}
$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
G_0 &= \frac{A}{\pi} \\
G_{-1} &= \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot D} \\
G_1 &= -\jmath\cdot\frac{A}{4}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot D} \\
G_k &= \frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot D}
\end{aligned}
$$

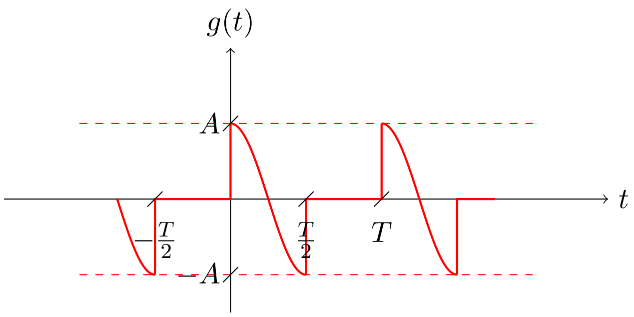
Task 10. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.

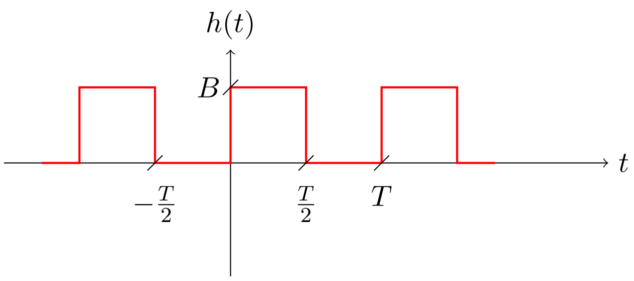


Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(t) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ B & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.66}$$

If we look carefully, signal $f(t)$ may be decomposed into two signals $g(t)$ and $h(t)$ for which we have already calculated Fourier series coefficients. The signals are given below:





To be precise, the $f(t)$ signal will be the sum of $g(t)$ and $h(t)$ shifted in time by $\frac{T}{2}$:

$$f(t) = g(t) + h\left(t - \frac{T}{2}\right) \tag{2.67}$$

Based on linearity of complex exponential Fourier series and on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot \frac{T}{2}}$$

$$F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \pi}$$
$$F_k = G_k + H_k \cdot (-1)^k$$

From previous tasks we know, that coefficients for the expansion into a complex exponential Fourier series of $g(t)$ and $h(t)$ signals are equal to:

$$\begin{aligned} G_0 &= 0 \\ G_1 &= \frac{A}{4} \\ G_{-1} &= \frac{A}{4} \\ G_k &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \end{aligned}$$

$$\begin{aligned} H_0 &= \frac{B}{2} \\ H_k &= \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \end{aligned}$$

Right now we know everything to calculate F_k coefficients:

$$\begin{aligned} F_k &= G_k + H_k \cdot (-1)^k = \\ &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \cdot (-1)^k = \\ &= \jmath \cdot \left[\frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \frac{B}{k \cdot 2\pi} \cdot \Big(1 - (-1)^k \Big) \right] \end{aligned}$$

Similarly, we have to calculate F_0:

$$\begin{aligned} F_0 &= G_0 + H_0 \cdot (-1)^0 = \\ &= 0 + \frac{B}{2} \cdot 1 = \\ &= \frac{B}{2} \end{aligned}$$

Also F_1 and F_{-1} have to be calculated separately:

$$F_1 = G_1 + H_1 \cdot (-1)^1 =$$

$$
\begin{aligned}
&= \frac{A}{4} + \jmath \cdot \frac{B}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1\Big) \cdot (-1) = \\
&= \frac{A}{4} + \jmath \cdot \frac{B}{\pi}
\end{aligned}
$$

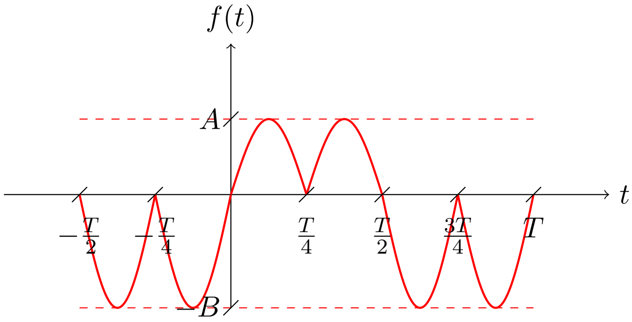
$$
\begin{aligned}
F_{-1} &= G_{-1} + H_{-1} \cdot (-1)^{-1} = \\
&= \frac{A}{4} + \jmath \cdot \frac{B}{(-1) \cdot 2\pi} \cdot \Big((-1)^{-1} - 1\Big) \cdot (-1) = \\
&= \frac{A}{4} - \jmath \cdot \frac{B}{\pi}
\end{aligned}
$$

To sum up, complex exponential Fourier coefficients for $f(t)$ are equal to:

$$
\begin{aligned}
F_0 &= \frac{B}{2} \\
F_1 &= \frac{A}{4} + \jmath \cdot \frac{B}{\pi} \\
F_{-1} &= \frac{A}{4} - \jmath \cdot \frac{B}{\pi} \\
F_k &= \jmath \cdot \left[\frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2}\right) + \frac{B}{k \cdot 2\pi} \cdot \Big(1 - (-1)^k\Big)\right]
\end{aligned}
$$

Task 11.

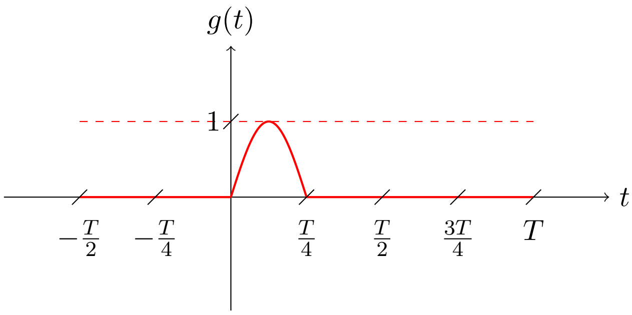
Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(t) = \begin{cases} A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ -A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{4} + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{2} + k \cdot T; \frac{3T}{4} + k \cdot T\right) \\ B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{3T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.68}$$

The F_k coefficients may be calculated directly by definition. However, four integrals have to be solved, each for single interval of one period of the $f(t)$ signal. If we look carefully, signal $f(t)$ may be decomposed into linear combination of shifted in time $g(t)$ signals, for $g(t)$ signal given below:



This is periodic piecewise function, which may be describe as:

$$g(t) = \begin{cases} sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.69}$$

For such a definition of $g(t)$ signal, our $f(t)$ may be described as:

$$g(t) = A \cdot g\left(t\right) + A \cdot g\left(t - \frac{T}{4}\right) - B \cdot g\left(t - \frac{T}{2}\right) - B \cdot g\left(t - \frac{3T}{4}\right) \tag{2.70}$$

Right now, it is enough to calculate G_k - complex exponential Fourier coefficients of $g(t)$ signal. Then, based on linearity and on the effect of signal shift in time on the complex exponential Fourier series, we will be able to derive F_k of $f(t)$ signal.

The G_0 coefficient is defined as:

$$G_0 = \frac{1}{T}\int_T g(t)\cdot dt \tag{2.71}$$

For the period $t \in (0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
G_0 &= \frac{1}{T}\int_T g(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot dt + \frac{1}{T}\int_{\frac{T}{4}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot dt + 0\right) = \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{matrix} z & = \frac{4\pi}{T}\cdot t \\ dz & = \frac{4\pi}{T}\cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} sin\left(z\right)\cdot \frac{dz}{\frac{4\pi}{T}} = \\
&= \frac{1}{T\cdot\frac{4\pi}{T}}\int_0^{\frac{T}{4}} sin\left(z\right)\cdot dz = \\
&= \frac{1}{4\pi}\cdot\left(-cos\left(z\right)\Big|_0^{\frac{T}{4}}\right) = \\
&= -\frac{1}{4\pi}\cdot\left(cos\left(\frac{4\pi}{T}\cdot t\right)\Bigg|_0^{\frac{T}{4}}\right) = \\
&= -\frac{1}{4\pi}\cdot\left(cos\left(\frac{4\pi}{T}\cdot\frac{T}{4}\right) - cos\left(\frac{4\pi}{T}\cdot 0\right)\right) = \\
&= -\frac{1}{4\pi}\cdot\left(cos\left(\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{1}{4\pi}\cdot\left(-1-1\right) = \\
&= -\frac{1}{4\pi}\cdot\left(-2\right) = \\
&= \frac{1}{2\pi}
\end{aligned}$$

The G_0 coefficient equals $\frac{1}{2\pi}$.

The G_k coefficients are defined as:

$$G_k = \frac{1}{T}\int_T g(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.72}$$

For the period $t \in (0;T)$, i.e. $k=0$, we get:

$$
\begin{aligned}
G_k &= \frac{1}{T}\int_T g(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt =\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \left\{ sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath} \right\} =\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot dt\right) =\\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =\\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =\\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =\\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2-k)} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2+k)}\right)\cdot dt =\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2-k)}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2+k)}\cdot dt\right) =\\
&= \left\{\begin{array}{llll} z_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot t & z_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot t\\ dz_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot dt & dz_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot dt\\ dt & = \frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)} & dt & = \frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)} \end{array}\right\} =\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)} - \int_0^{\frac{T}{4}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)}\right) =\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)}\cdot\int_0^{\frac{T}{4}} e^{z_1}\cdot dz_1 - \frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)}\cdot\int_0^{\frac{T}{4}} e^{z_2}\cdot dz_2\right) =\\
&= \frac{1}{T\cdot 2\jmath\cdot\jmath\cdot\frac{2\pi}{T}}\cdot\left(\frac{1}{2-k}\cdot\int_0^{\frac{T}{4}} e^{z_1}\cdot dz_1 + \frac{1}{2+k}\cdot\int_0^{\frac{T}{4}} e^{z_2}\cdot dz_2\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot e^{z_1}\Big|_0^{\frac{T}{4}} + \frac{1}{2+k}\cdot e^{z_2}\Big|_0^{\frac{T}{4}}\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot t}\Big|_0^{\frac{T}{4}} + \frac{1}{2+k}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot t}\Big|_0^{\frac{T}{4}}\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot\frac{T}{4}} - e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot 0}\right) + \frac{1}{2+k}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot\frac{T}{4}} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot 0}\right)\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)} - e^{0}\right) + \frac{1}{2+k}\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)} - e^{0}\right)\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2+k}{(2-k)\cdot(2+k)}\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)} - 1\right) + \frac{2-k}{(2-k)\cdot(2+k)}\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)} - 1\right)\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{(2+k)\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)} - 1\right)}{(2-k)\cdot(2+k)} + \frac{(2-k)\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)} - 1\right)}{(2-k)\cdot(2+k)}\right) =\\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{(2+k)\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)} - 1\right) + (2-k)\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)} - 1\right)}{(2-k)\cdot(2+k)}\right) =
\end{aligned}
$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-2+k\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-k+2\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-2-k\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}+k}{4-k^2}\right)=$$
$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-4+k\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}+2\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-k\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}}{4-k^2}\right)=$$
$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4+k\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+2\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-k\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$
$$= \begin{Bmatrix} e^{\jmath\cdot\pi} & = cos\left(\pi\right)+\jmath\cdot sin\left(\pi\right)=-1 \\ e^{-\jmath\cdot\pi} & = cos\left(\pi\right)-\jmath\cdot sin\left(\pi\right)=-1 \end{Bmatrix}=$$
$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4+k\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+2\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-k\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$
$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4-k\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-2\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+k\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$
$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{-4\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4}{4-k^2}\right)=$$
$$= \frac{1}{4\cdot\pi}\cdot\left(\frac{4\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+4}{4-k^2}\right)=$$
$$= \frac{1}{4\cdot\pi}\cdot 4\cdot\left(\frac{e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+1}{4-k^2}\right)=$$
$$= \frac{1}{\pi}\cdot\left(\frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$
$$= \frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{\pi\left(4-k^2\right)}$$

The G_k coefficients are equal to $\frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{\pi(4-k^2)}$ for $k\neq 2\wedge k\neq -2$.
We have to calculate G_k for $k=2$ directly by definition:

$$G_2 = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt=$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot dt\right)=$$
$$= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+0\right)=$$
$$= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=$$
$$= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=$$

$$
\begin{aligned}
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 0}\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 2}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{0}\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} 1\cdot dt-\int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{matrix} z &= -\jmath\cdot\frac{8\pi}{T}\cdot t\\ dz &= -\jmath\cdot\frac{8\pi}{T}\cdot dt\\ dt &= \frac{dz}{-\jmath\cdot\frac{8\pi}{T}}\end{matrix}\right\}=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} dt-\int_0^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot\frac{8\pi}{T}}\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} dt-\frac{1}{-\jmath\cdot\frac{8\pi}{T}}\cdot\int_0^{\frac{T}{4}} e^{z}\cdot dz\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(t\Big|_0^{\frac{T}{4}}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{4}}\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\left(\frac{T}{4}-0\right)+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\Big|_0^{\frac{T}{4}}\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{8\pi}{T}\cdot\frac{T}{4}}-e^{-\jmath\cdot\frac{8\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \left\{e^{-\jmath\cdot 2\pi}\quad = cos\left(2\pi\right)-\jmath\cdot sin\left(2\pi\right)=1\right\}=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot 0\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{T}{4}+0\right)=\\
&= \frac{1}{T\cdot 2\jmath}\cdot\frac{T}{4}=\\
&= \frac{1}{8\jmath}=\\
&= \frac{-\jmath}{8}
\end{aligned}
$$

The G_2 coefficients equal to $\frac{-\jmath}{8}$.

We have to calculate G_k for $k=-2$ directly by definition:

$$
\begin{aligned}
G_{-2} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot dt\right) = \\
&= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\} = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 2}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 0}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt - \int_0^{\frac{T}{4}} e^{0}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt - \int_0^{\frac{T}{4}} 1\cdot dt\right) = \\
&= \left\{\begin{matrix} z & = \jmath\cdot\frac{8\pi}{T}\cdot t \\ dz & = \jmath\cdot\frac{8\pi}{T}\cdot dt \\ dt & = \frac{dz}{\jmath\cdot\frac{8\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{z}\cdot\frac{dz}{\jmath\cdot\frac{8\pi}{T}} - \int_0^{\frac{T}{4}} dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\int_0^{\frac{T}{4}} e^{z}\cdot dz - \int_0^{\frac{T}{4}} dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{4}} - t\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\Big|_0^{\frac{T}{4}} - \left(\frac{T}{4}-0\right)\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{8\pi}{T}\cdot\frac{T}{4}}-e^{-\jmath\cdot\frac{8\pi}{T}\cdot 0}\right) - \frac{T}{4}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right) - \frac{T}{4}\right) =
\end{aligned}
$$

$$\begin{aligned}
&= \left\{e^{-\jmath \cdot 2\pi} \quad = cos\left(2\pi\right) - \jmath \cdot sin\left(2\pi\right) = 1\right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot (1-1) - \frac{T}{4}\right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot 0 - \frac{T}{4}\right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{4}\right) = \\
&= -\frac{1}{T \cdot 2\jmath} \cdot \frac{T}{4} = \\
&= -\frac{1}{8\jmath} = \\
&= \frac{\jmath}{8}
\end{aligned}$$

The G_{-2} coefficients equal to $\frac{\jmath}{8}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series for $g(t)$ signal are given by:

$$\begin{aligned}
G_0 &= \frac{1}{2\pi} \\
G_2 &= \frac{-\jmath}{8} \\
G_{-2} &= \frac{\jmath}{8} \\
G_k &= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\left(4 - k^2\right)}
\end{aligned}$$

Right now, we may go back to the description of the $f(t)$ signal with shifted in time $g(t)$ signals:

$$f(t) = A \cdot g\left(t\right) + A \cdot g\left(t - \frac{T}{4}\right) - B \cdot g\left(t - \frac{T}{2}\right) - B \cdot g\left(t - \frac{3T}{4}\right) \tag{2.73}$$

Recall the linearity and the effect of signal shift in time on the complex exponential Fourier series coefficients:

$$\begin{aligned}
n(t) &\rightarrow N_k \\
m(t) &= A \cdot n(t - t_0) \\
M_k &= A \cdot N_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0}
\end{aligned}$$

Applying mentioned theorems for $f(t)$ signal, we may write:

$$\begin{aligned}
F_k &= A \cdot G_k + A \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{T}{4}} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{T}{2}} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{3T}{4}} = \\
&= A \cdot G_k + A \cdot G_k \cdot e^{-\jmath \cdot \frac{k\pi}{2}} - B \cdot G_k \cdot e^{-\jmath \cdot k \cdot \pi} - B \cdot G_k \cdot e^{-\jmath \cdot \frac{3 \cdot k \cdot \pi}{2}} =
\end{aligned}$$

$$
\begin{aligned}
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left(e^{-\jmath \cdot k\pi} + e^{-\jmath \cdot \frac{3k\pi}{2}}\right) = \\
&= \left\{\begin{matrix} e^{-\jmath \cdot k\pi} & = cos\left(k\pi\right) + \jmath \cdot sin\left(k\pi\right) = (-1)^k \\ e^{-\jmath \cdot \frac{3 \cdot k\pi}{2}} & = e^{-\jmath \cdot \left(\frac{2 \cdot k\pi}{2} + \frac{k\pi}{2}\right)} = e^{-\jmath \cdot \frac{2 \cdot k\pi}{2}} \cdot e^{-\jmath \cdot \frac{k\pi}{2}} = (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}} \end{matrix}\right\} = \\
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left((-1)^k + (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot (-1)^k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\
&= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}
$$

Now, we may insert G_k coefficients into F_k equation:

$$
\begin{aligned}
F_k &= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\
&= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\left(4 - k^2\right)} \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\
&= \frac{\left(1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}\right)^2}{\pi\left(4 - k^2\right)} \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}
$$

Similarly, we may calculate F_0 coefficient:

$$
\begin{aligned}
F_0 &= G_0 \cdot \left(1 + e^{-\jmath \cdot \frac{0 \cdot \pi}{2}}\right) \cdot \left(A - B \cdot (-1)^0\right) = \\
&= \frac{1}{2\pi} \cdot \left(1 + e^0\right) \cdot (A - B \cdot 1) = \\
&= \frac{1}{2\pi} \cdot (1 + 1) \cdot (A - B) = \\
&= \frac{1}{2\pi} \cdot (2) \cdot (A - B) = \\
&= \frac{A - B}{\pi}
\end{aligned}
$$

Similarly, we may calculate F_2 coefficient:

$$
\begin{aligned}
F_2 &= G_2 \cdot \left(1 + e^{-\jmath \cdot \frac{2 \cdot \pi}{2}}\right) \cdot \left(A - B \cdot (-1)^2\right) = \\
&= \frac{-\jmath}{8} \cdot \left(1 + e^{-\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{e^{-\jmath \cdot \pi} \quad = cos\left(\pi\right) - \jmath \cdot sin\left(\pi\right) = -1\right\} = \\
&= \frac{-\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{-\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}
$$

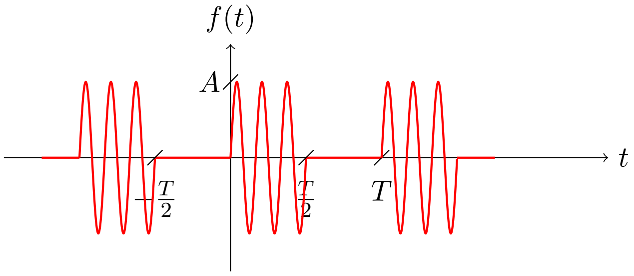
Similarly, we may calculate F_{-2} coefficient:

$$\begin{aligned}
F_{-2} &= G_{-2} \cdot \left(1 + e^{-\jmath \cdot \frac{(-2)\cdot\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^{-2}\right) = \\
&= \frac{\jmath}{8} \cdot \left(1 + e^{\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{ e^{\jmath \cdot \pi} \quad = cos\,(\pi) + \jmath \cdot sin\,(\pi) = -1 \right\} = \\
&= \frac{\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series for $f(t)$ signal are given by:

$$\begin{aligned}
F_0 &= \frac{A - B}{\pi} \\
F_2 &= 0 \\
F_{-2} &= 0 \\
F_k &= \frac{\left(1 + e^{-\jmath \cdot \frac{k\cdot\pi}{2}}\right)^2}{\pi\,(4 - k^2)} \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}$$

Task 12. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{12\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.74}$$

$$\begin{aligned} g(t) &= f\left(t\right) \cdot sin\left(\frac{12\pi}{T} \cdot t\right) \\ &= \left\{\ sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath}\ \right\} = \\ &= f\left(t\right) \cdot \frac{e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}}{2 \cdot \jmath} \\ &= \frac{1}{2 \cdot \jmath}\left(f\left(t\right) \cdot e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - f\left(t\right) \cdot e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}\right) \end{aligned}$$

$$\begin{aligned} F_0 &= \frac{A}{2} \\ F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \end{aligned}$$

$$\begin{aligned} g^1(t) &= f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0 \cdot t} \\ G^1_k &= F_{k-k_0} \end{aligned}$$

$$\begin{aligned} g(t) &= \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k^1_0 \cdot t} - \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k^2_0 \cdot t} \\ g(t) &= g^1(t) - g^2(t) \\ G_k &= G^1_k - G^2_k \\ G_k &= \frac{1}{2 \cdot \jmath}\left(F_{k-k^1_0} - F_{k-k^2_0}\right) \end{aligned}$$

$$e^{\jmath \cdot \frac{12\pi}{T} \cdot t} = e^{\jmath \cdot \frac{2\ cdot 6\pi}{T} \cdot t}$$

$$= e^{\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t} \Rightarrow k_0^1 = 6$$

$$\begin{aligned}
e^{-\jmath\cdot\frac{12\pi}{T}\cdot t} &= e^{-\jmath\cdot\frac{2\ cdot6\pi}{T}\cdot t}\\
&= e^{-\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t}\\
&= e^{\jmath\cdot\frac{2\pi}{T}\cdot(-6)\cdot t} \Rightarrow k_0^2 = -6
\end{aligned}$$

$$\begin{aligned}
G_k &= \frac{1}{2\cdot\jmath}\left(F_{k-k_0^1} - F_{k-k_0^2}\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-k_0^1)\cdot 2\pi}\cdot\left((-1)^{k-k_0^1}-1\right) - \jmath\cdot\frac{A}{(k-k_0^2)\cdot 2\pi}\cdot\left((-1)^{k-k_0^2}-1\right)\right) =\\
&= \left\{\ k_0^1 = 6 \quad k_0^2 = -6\ \right\} =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k-(-6))\cdot 2\pi}\cdot\left((-1)^{k-(-6)}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k+6}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{6}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi} - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{1}{2\cdot\jmath}\cdot\jmath\cdot\frac{A}{2\pi}\left(\frac{1}{k-6} - \frac{1}{k+6}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{k+6}{(k-6)\cdot(k+6)} - \frac{k-6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{k+6-k+6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{12}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{\pi}\left(\frac{3}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{3\cdot A}{\pi\cdot(k^2-36)}\cdot\left((-1)^{k}-1\right)
\end{aligned}$$

$$\begin{aligned}
G_6 &= \frac{1}{2\cdot\jmath}\left(F_{6-k_0^1} - F_{6-k_0^2}\right) =\\
&= \left\{\ k_0^1 = 6 \quad k_0^2 = -6\ \right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2 \cdot \jmath} \left(F_{6-6} - F_{6-(-6)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{0} - F_{6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{0} - F_{12} \right)
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{12)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\frac{A}{2} - \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \left((-1)^{12} - 1 \right) \right) = \\
&= \frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} - \frac{1}{2 \cdot \jmath} \cdot \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \left((-1)^{12} - 1 \right) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (1 - 1) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (0) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot 0 = \\
&= \frac{A}{4 \cdot \jmath} - 0 = \\
&= \frac{A}{4 \cdot \jmath}
\end{aligned}
$$

$$
\begin{aligned}
G_{-6} &= \frac{1}{2 \cdot \jmath} \left(F_{-6-k_0^1} - F_{-6-k_0^2} \right) = \\
&= \left\{ \; k_0^1 = 6 \quad k_0^2 = -6 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-6-6} - F_{-6-(-6)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_{-6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_{0} \right)
\end{aligned}
$$

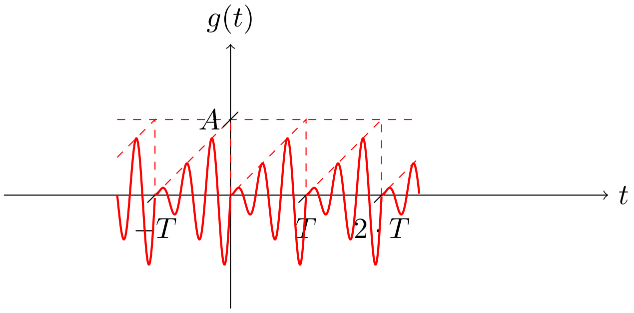
$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_{0)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{-12 \cdot 2\pi} \cdot \left((-1)^{-12} - 1 \right) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (1 - 1) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (0) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(0 - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(- \frac{A}{2} \right) =
\end{aligned}
$$

$$= -\frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} =$$
$$= -\frac{A}{4 \cdot \jmath}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$G_{-6} = -\frac{A}{4 \cdot \jmath}$$
$$G_6 = \frac{A}{4 \cdot \jmath}$$
$$G_k = \frac{3 \cdot A}{\pi \cdot (k^2 - 36)} \cdot \Big((-1)^k - 1\Big)$$

Task 13. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$g(x) = \frac{A}{T} \cdot t \cdot sin\left(\frac{6\pi}{T} \cdot t\right) \tag{2.75}$$

$$\begin{aligned} g(t) &= f\left(t\right) \cdot sin\left(\frac{6\pi}{T} \cdot t\right) \\ &= \left\{ \; sin\left(x\right) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \; \right\} = \\ &= f\left(t\right) \cdot \frac{e^{\jmath \cdot \frac{6\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{6\pi}{T} \cdot t}}{2 \cdot \jmath} \\ &= \frac{1}{2 \cdot \jmath} \left(f\left(t\right) \cdot e^{\jmath \cdot \frac{6\pi}{T} \cdot t} - f\left(t\right) \cdot e^{-\jmath \cdot \frac{6\pi}{T} \cdot t}\right) \end{aligned}$$

$$\begin{aligned} a_0 &= \frac{A}{2} \\ a_k &= 0 \\ b_k &= -\frac{A}{k \cdot \pi} \end{aligned}$$

xx

$$\begin{aligned} F_0 &= a_0 \\ F_k &= a_k - \jmath \cdot b_k \end{aligned}$$

xx

$$\begin{aligned} F_0 &= a_0 = \frac{A}{2} \\ F_k &= a_k - \jmath \cdot b_k = \\ &= 0 - \jmath \cdot \left(-\frac{A}{k \cdot \pi}\right) = \end{aligned}$$

$$= \jmath \cdot \frac{A}{k \cdot \pi}$$

$$g^1(t) = f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0 \cdot t}$$
$$G_k^1 = F_{k-k_0}$$

$$\begin{aligned}
g(t) &= \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^1 \cdot t} - \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^2 \cdot t} \\
g(t) &= g^1(t) - g^2(t) \\
G_k &= G_k^1 - G_k^2 \\
G_k &= \frac{1}{2 \cdot \jmath} \left(F_{k-k_0^1} - F_{k-k_0^2} \right)
\end{aligned}$$

$$\begin{aligned}
e^{\jmath \cdot \frac{6\pi}{T} \cdot t} &= e^{\jmath \cdot \frac{2\ cdot 3\pi}{T} \cdot t} \\
&= e^{\jmath \cdot \frac{2\pi}{T} \cdot 3 \cdot t} \Rightarrow k_0^1 = 3
\end{aligned}$$

$$\begin{aligned}
e^{-\jmath \cdot \frac{6\pi}{T} \cdot t} &= e^{-\jmath \cdot \frac{2\ cdot 3\pi}{T} \cdot t} \\
&= e^{-\jmath \cdot \frac{2\pi}{T} \cdot 3 \cdot t} \\
&= e^{\jmath \cdot \frac{2\pi}{T} \cdot (-3) \cdot t} \Rightarrow k_0^2 = -3
\end{aligned}$$

$$\begin{aligned}
G_k &= \frac{1}{2 \cdot \jmath} \left(F_{k-k_0^1} - F_{k-k_0^2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{(k - k_0^1) \cdot \pi} - \jmath \cdot \frac{A}{(k - k_0^2) \cdot \pi} \right) = \\
&= \left\{ \; k_0^1 = 3 \quad k_0^2 = -3 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{(k - 3) \cdot \pi} - \jmath \cdot \frac{A}{(k - (-3)) \cdot \pi} \right) = \\
&= \frac{1}{2 \cdot \jmath} \cdot \jmath \cdot \frac{A}{\pi} \left(\frac{1}{k-3} - \frac{1}{k+3} \right) = \\
&= \frac{1}{2} \cdot \frac{A}{\pi} \left(\frac{k+3}{(k-3) \cdot (k+3)} - \frac{k-3}{(k+3) \cdot (k-1)} \right) = \\
&= \frac{1}{2} \cdot \frac{A}{\pi} \cdot \frac{k+3-(k-3)}{(k-3) \cdot (k+3)} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \frac{A}{\pi} \cdot \frac{k+3-k+3}{k^2-9} = \\
&= \frac{1}{2} \cdot \frac{A}{\pi} \cdot \frac{6}{k^2-9} = \\
&= \frac{A}{\pi} \cdot \frac{3}{k^2-9}
\end{aligned}
$$

$$
\begin{aligned}
G_3 &= \frac{1}{2 \cdot \jmath} \left(F_{3-k_0^1} - F_{3-k_0^2} \right) = \\
&= \left\{ \; k_0^1 = 3 \quad k_0^2 = -3 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{3-3} - F_{3-(-3)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{3+3} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_6 \right)
\end{aligned}
$$

$$
\begin{aligned}
G_3 &= \frac{1}{2 \cdot \jmath} \left(F_0 - F_6 \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\frac{A}{2} - \jmath \cdot \frac{A}{6 \cdot \pi} \right) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{2 \cdot 6 \cdot \pi} = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot \pi} = \\
&= -\frac{A}{12 \cdot \pi} + \frac{A}{4 \cdot \jmath}
\end{aligned}
$$

$$
\begin{aligned}
G_{-3} &= \frac{1}{2 \cdot \jmath} \left(F_{-3-k_0^1} - F_{-3-k_0^2} \right) = \\
&= \left\{ \; k_0^1 = 3 \quad k_0^2 = -3 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-3-3} - F_{-3-(-3)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-6} - F_{-3+3} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-6} - F_0 \right)
\end{aligned}
$$

$$
\begin{aligned}
G_{-3} &= \frac{1}{2 \cdot \jmath} \left(F_{-6} - F_{0)} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{-6 \cdot \pi} - \frac{A}{2} \right) =
\end{aligned}
$$

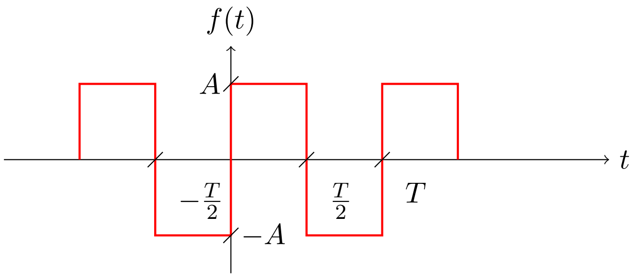
$$= \frac{A}{-12 \cdot \pi} - \frac{A}{4 \cdot \jmath} =$$
$$= -\frac{A}{12 \cdot \pi} - \frac{A}{4 \cdot \jmath}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$G_{-3} = -\frac{A}{12 \cdot \pi} - \frac{A}{4 \cdot \jmath}$$
$$G_{3} = -\frac{A}{12 \cdot \pi} + \frac{A}{4 \cdot \jmath}$$
$$G_{k} = \frac{A}{\pi} \cdot \frac{3}{k^2 - 9}$$

2.3 Computing the power of a signal – the Parseval's theorem

Task 1. Compute the percentage contribution of the fundamental (first) harmonic in the total power of the periodic square signal shown in the figure below:



$$\frac{P_1}{P} = ? \tag{2.76}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(t) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -A & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.77}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.78}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |-A|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + A^2 \cdot \int_{\frac{T}{2}}^{T} dt \right) = \\ &= \frac{A^2}{T} \cdot \left(t|_0^{\frac{T}{2}} + t|_{\frac{T}{2}}^{T} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 + T - \frac{T}{2} \right) = \\ &= \frac{A^2}{T} \cdot (T) = \\ &= A^2 \end{aligned}$$

The total power of the $f(t)$ signal equals A^2.

Based on Parseval theorem, power of the fundamental harmonic is defined as:

$$P_1 = |F_1|^2 + |F_{-1}|^2 \tag{2.79}$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.80}$$

In order to calculate the P_1, the F_1 coefficient has to be calculated:

$$F_1 = \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.81}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_1 &= \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} -A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt - \int_{\frac{T}{2}}^{T} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z & = -\jmath \cdot \frac{2\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \end{matrix} \right\} = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} - \int_{\frac{T}{2}}^{T} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \right) = \\
&= -\frac{A}{T \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot dz - \int_{\frac{T}{2}}^{T} e^z \cdot dz \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^z\Big|_0^{\frac{T}{2}} - e^z\Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot T} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \pi} - e^0 - e^{-\jmath \cdot 2 \cdot \pi} + e^{-\jmath \cdot \pi} \right) = \\
&= \left\{ \begin{matrix} e^{-\jmath \cdot 2 \cdot \pi} & = cos(2\pi) - \jmath \cdot sin(2\pi) = 1 \\ e^{-\jmath \cdot \pi} & = cos(\pi) - \jmath \cdot sin(\pi) = -1 \end{matrix} \right\} = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-1 - 1 - 1 - 1) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-4) = \\
&= \frac{2 \cdot A}{\jmath \cdot \pi} = \\
&= -\jmath \cdot \frac{2 \cdot A}{\pi}
\end{aligned}$$

The F_1 coefficient equals $-\jmath \cdot \frac{2 \cdot A}{\pi}$.

Thus, the P_1 may be calculated:

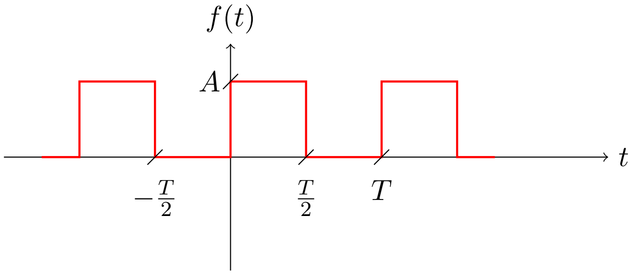
$$\begin{aligned} P_1 &= 2 \cdot |F_1|^2 = \\ &= 2 \cdot \left| -\jmath \cdot \frac{2 \cdot A}{\pi} \right|^2 = \\ &= 2 \cdot \left(\frac{2 \cdot A}{\pi} \right)^2 = \\ &= 2 \cdot \frac{4 \cdot A^2}{\pi^2} = \\ &= \frac{8 \cdot A^2}{\pi^2} \end{aligned}$$

The power of the fundamental harmonic equals $P_1 = \frac{8 \cdot A^2}{\pi^2}$.

Finally, the percentage contribution of the fundamental harmonic in the total power of the $f(t)$ signal is equal to:

$$\frac{P_1}{P} = \frac{\frac{8 \cdot A^2}{\pi^2}}{A^2} = \frac{8}{\pi^2} \approx 81\% \tag{2.82}$$

Task 2. Calculate the percentage contribution of the power of the higher harmonics ($k > 1$) to the total average power of the periodic signal shown below.



$$\frac{P_{>1}}{P} = ? \tag{2.83}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(t) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.84}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.85}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |0|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(t \Big|_0^{\frac{T}{2}} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A^2}{2} \end{aligned}$$

The total power of the $f(t)$ signal equals $\frac{A^2}{2}$.

Based on Parseval theorem, the power of the higher harmonics is defined as:

$$P_{>1} = P - P_0 - P_1 \tag{2.86}$$

where:

$$
\begin{aligned}
P_0 &= |F_0|^2 \\
P_1 &= |F_1|^2 + |F_{-1}|^2
\end{aligned}
$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.87}$$

In order to calculate P_0 and P_1, the F_0 and F_1 coefficients have to be calculated. The F_k coefficients have been calculated in task 1 and are equal to:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big)
\end{aligned}
$$

Now, we may calculate the P_0 and P_1:

$$
\begin{aligned}
P_0 &= |F_0|^2 \\
&= \left| \frac{A}{2} \right|^2 \\
&= \frac{A^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= 2 \cdot |F_1|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1 \Big) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{2\pi} \cdot (-1 - 1) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{2\pi} \cdot (-2) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{-A}{\pi} \right|^2 \\
&= 2 \cdot \left(\frac{A}{\pi} \right)^2 \\
&= 2 \cdot \frac{A^2}{\pi^2}
\end{aligned}
$$

Finally, the power of the higher harmonics is defined as:

$$
\begin{aligned}
P_{>1} &= P - P_0 - P_1 \\
&= \frac{A^2}{2} - \frac{A^2}{4} - 2 \cdot \frac{A^2}{\pi^2} \\
&= \frac{2 \cdot A^2 \cdot \pi^2}{4\pi^2} - \frac{A^2 \cdot \pi^2}{4\pi^2} - \frac{8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \pi^2 - 8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}
\end{aligned}
$$

The power of the fundamental harmonic equals $P_{>1} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}$.

Finally, the percentage contribution of the higher harmonics in the total power of the $f(t)$ signal is equal to:

$$
\frac{P_{>1}}{P} = \frac{\frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}}{\frac{A^2}{2}} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2} \cdot \frac{2}{A^2} = \frac{\pi^2 - 8}{2\pi^2} = \approx 9\% \tag{2.88}
$$

Task 3. For a certain real-valued periodic signal, its coefficients of expansion to a complex exponential Fourier series are:

$$F_k = \frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2} \wedge k > 0 \tag{2.89}$$

Compute the mean value ($\bar{f}$), knowing that the effective (RMS) value is $U = \frac{A\sqrt{6}}{60}$. During calculation use:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \tag{2.90}$$

From the theoretical considerations we know that:

$$F_0 = \bar{f}$$
$$U = \sqrt{P}$$

In order to calculate $\bar{f}$ we have to calculate F_0. But we know values of the F_k for $k > 0$ only. However, based on Parseval theorem, the power of the signal is defined as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2 \tag{2.91}$$

This equation may be rewritten as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2$$
$$P = \sum_{k=-\infty}^{-1} |F_k|^2 + |F_0|^2 + \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=-\infty}^{-1} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$

Because the $f(t) \in R$, thus $|F_k| = |F_{-k}|$ and we may write:

$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

Now, we can calculate the F_0:

$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

$$|F_0|^2 = U^2 - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

$$|F_0|^2 = \left(\frac{A\sqrt{6}}{60}\right)^2 - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot pi^2}\right|^2$$

$$|F_0|^2 = \frac{A^2 \cdot 6}{3600} - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2}\right|^2$$

$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \left(\frac{A}{k^2 \cdot 4 \cdot \pi^2}\right)^2$$

$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \frac{A^2}{k^4 \cdot 16 \cdot \pi^4}$$

$$|F_0|^2 = \frac{A^2}{600} - 2 \cdot \frac{A^2}{16 \cdot \pi^4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^4}$$

$$|F_0|^2 = \frac{A^2}{600} - \frac{A^2}{8 \cdot \pi^4} \cdot \frac{\pi^4}{90}$$

$$|F_0|^2 = \frac{A^2}{600} - \frac{A^2}{720}$$

$$|F_0|^2 = \frac{720 \cdot A^2}{600 \cdot 720} - \frac{600 \cdot A^2}{600 \cdot 720}$$

$$|F_0|^2 = \frac{720 \cdot A^2 - 600 \cdot A^2}{600 \cdot 720}$$

$$|F_0|^2 = \frac{120 \cdot A^2}{600 \cdot 720}$$

$$|F_0|^2 = \frac{A^2}{5 \cdot 720}$$

$$|F_0|^2 = \frac{A^2}{3600}$$

$$|F_0| = \sqrt{\frac{A^2}{3600}}$$

$$|F_0| = \frac{A}{60}$$

$$F_0 = \pm\frac{A}{60}$$

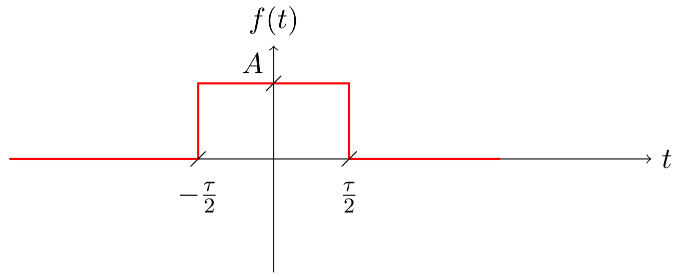
The mean value is equal to $\bar{f} = \pm\frac{A}{60}$.

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

Task 1. Compute the Fourier transform of a rectangular impulse shown below. Compute and draw magnitude and phase spectra.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \Pi(\frac{t}{\tau}) \tag{3.1}$$

Which can be expresed as:

$$f(t) = \begin{cases} 0 & \text{for } t \in (-\infty; -\frac{\tau}{2}) \\ A & \text{for } t \in (-\frac{\tau}{2}; \frac{\tau}{2}) \\ 0 & \text{for } t \in (\frac{\tau}{2}; \infty) \end{cases} \tag{3.2}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.3}$$

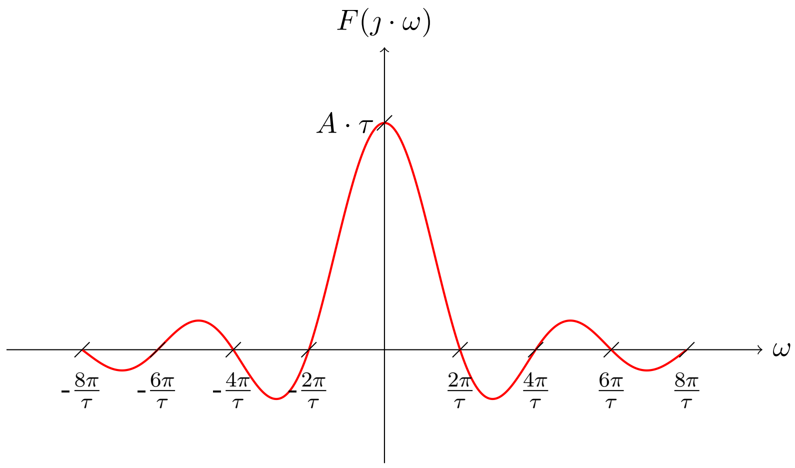
For the given $f(t)$ signal we get:

$$
\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} A \cdot \Pi(\frac{t}{\tau}) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-\frac{\tau}{2}} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{\frac{\tau}{2}}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-\frac{\tau}{2}} 0 \cdot dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{\frac{\tau}{2}}^{\infty} 0 \cdot dt = \\
&= 0 + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + 0 = \\
&= A \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = -\jmath \cdot \omega \cdot t \\ dz & = -\jmath \cdot \omega \cdot dt \\ dt & = \frac{1}{-\jmath\cdot\omega} \cdot dz \end{array} \right\} = \\
&= A \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{z} \cdot \frac{1}{-\jmath \cdot \omega} \cdot dz = \\
&= A \cdot \frac{1}{-\jmath \cdot \omega} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{z} \cdot dz = \\
&= A \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{z}|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \\
&= A \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \\
&= \frac{A}{-\jmath \cdot \omega} \cdot \left(e^{-\jmath\cdot\omega\cdot\frac{\tau}{2}} - e^{-\jmath\cdot\omega\cdot(-\frac{\tau}{2})}\right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(e^{\jmath\cdot\omega\cdot\frac{\tau}{2}} - e^{-\jmath\cdot\omega\cdot\frac{\tau}{2}}\right) = \\
&= \left\{ sin(x) = \frac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath} \right\} = \\
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot \frac{\tau}{2}\right) = \\
&= \left\{ \frac{sin(x)}{x} = Sa(x) \right\} = \\
&= A \cdot \tau \cdot Sa\left(\omega \cdot \frac{\tau}{2}\right)
\end{aligned}
$$

The Fourier transform of the $f(t) = A \cdot \Pi(\frac{t}{\tau})$ is equal to $F(\jmath\omega) = A \cdot \tau \cdot Sa\left(\omega \cdot \frac{\tau}{2}\right)$.

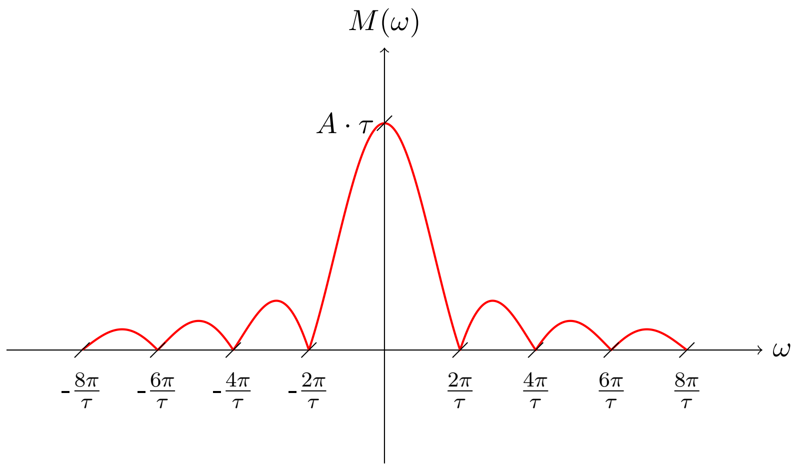
Draw complex spectrum of the $f(t) = A \cdot \Pi(\frac{t}{\tau})$:

$$
F(\jmath\omega) = A \cdot \tau \cdot Sa\left(\omega \cdot \frac{\tau}{2}\right) \tag{3.4}
$$



The magnitude spectrum is defined as:

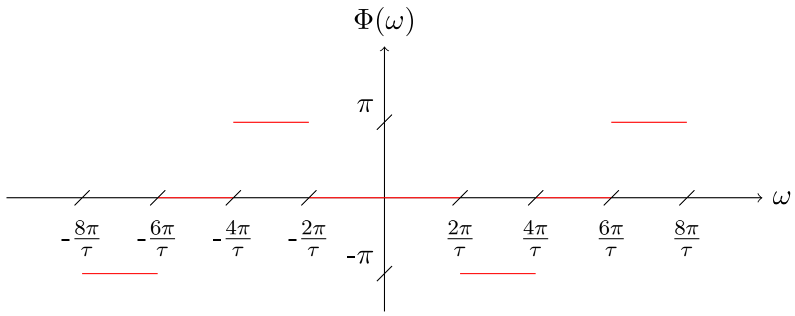
$$M(\omega) = |F(j \cdot \omega)| \tag{3.5}$$



The magnitude spectrum of a real signal is an even-symmetric function of w.

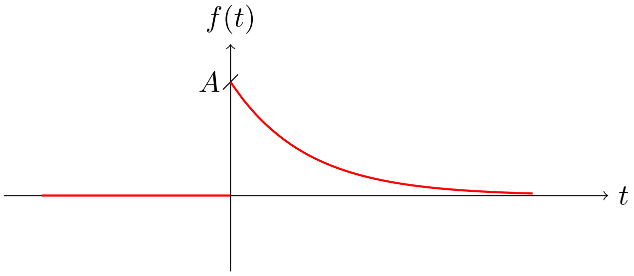
The phase spectrum is defined as:

$$\Phi(\omega) = arctan2\left(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}\right) \tag{3.6}$$



The phase spectrum of a real signal is an odd-symmetric function of w.

Task 2. Compute the Fourier transform of a impulse $f(t) = A \cdot \mathbb{1}(t) \cdot e^{-a \cdot t}$ shown below. Compute and draw magnitude and phase spectra.



The signal $f(t)$, as a piecewise function, is given by:

$$f(t) = \begin{cases} 0 & \text{for } t \in (-\infty; 0) \\ A \cdot e^{-a \cdot t} & \text{for } t \in (0; \infty) \end{cases} \tag{3.7}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.8}$$

For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 0 + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{0}^{\infty} e^{-(a + \jmath \cdot \omega) \cdot t} \cdot dt = \\
&= \lim_{\tau \to \infty} A \cdot \int_{0}^{\tau} e^{-(a + \jmath \cdot \omega) \cdot t} \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = -(a + \jmath \cdot \omega) \cdot t \\ dz & = -(a + \jmath \cdot \omega) \cdot dt \\ dt & = \frac{1}{-(a + \jmath \cdot \omega)} \cdot dz \end{array} \right\} = \\
&= \lim_{\tau \to \infty} A \cdot \int_{0}^{\tau} e^{z} \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot dz = \\
&= A \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} \int_{0}^{\tau} e^{z} \cdot dz = \\
&= A \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} e^{z} \big|_{0}^{\tau} = \\
&= \frac{A}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} e^{-(a + \jmath \cdot \omega) \cdot t} \Big|_{0}^{\tau} =
\end{aligned}$$

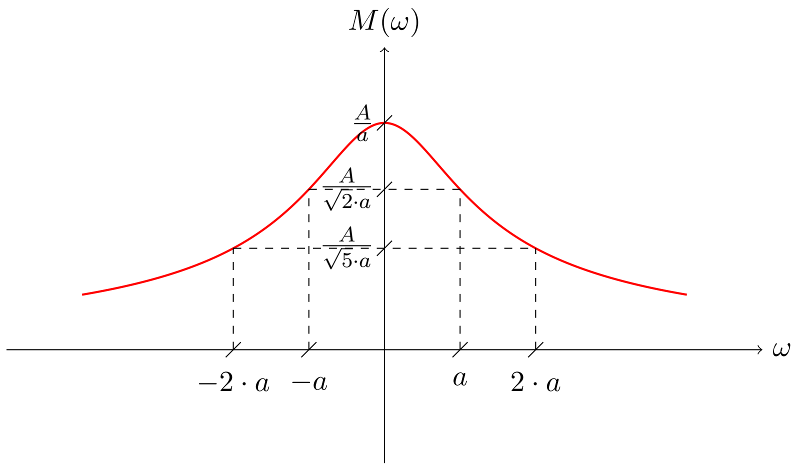
$$
\begin{aligned}
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau}-e^{-(a+\jmath\cdot\omega)\cdot 0}\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau}-e^{0}\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}e^{-(a+\jmath\cdot\omega)\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}e^{-a\cdot\tau+\jmath\cdot\omega\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}e^{-a\cdot\tau}\cdot e^{\jmath\cdot\omega\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}e^{-a\cdot\tau}\cdot\lim_{\tau\to\infty}e^{\jmath\cdot\omega\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(0\cdot\lim_{\tau\to\infty}e^{\jmath\cdot\omega\cdot\tau}-1\right)=\\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot(0-1)=\\
&= \frac{A}{a+\jmath\cdot\omega}
\end{aligned}
$$

The Fourier transform of the $f(t)$ signal is equal to $F(\jmath\omega)=\frac{A}{a+\jmath\cdot\omega}$.
Let's explicitly determine the real and imaginary part:

$$
\begin{aligned}
F(\jmath\omega) &= \frac{A}{(a+\jmath\cdot\omega)}=\\
&= \frac{A}{(a+\jmath\cdot\omega)}\cdot\frac{(a-\jmath\cdot\omega)}{(a-\jmath\cdot\omega)}=\\
&= \frac{A\cdot(a-\jmath\cdot\omega)}{(a^2+\omega^2)}=\\
&= \frac{A\cdot a}{(a^2+\omega^2)}-\jmath\cdot\frac{A\cdot\omega}{(a^2+\omega^2)}
\end{aligned}
$$

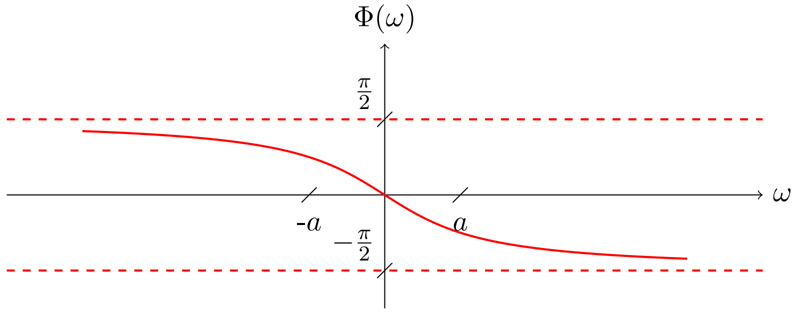
The magnitude spectrum is defined as:

$$
\begin{aligned}
M(\omega) &= |F(j\omega)| =\\
&= \sqrt{\left(\frac{A\cdot a}{(a^2+\omega^2)}\right)^2+\left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\right)^2}=\\
&= \sqrt{\frac{A^2\cdot(a^2+\omega^2)}{(a^2+\omega^2)^2}}=\\
&= \sqrt{\frac{A^2}{(a^2+\omega^2)}}=\\
&= \frac{A}{\sqrt{a^2+\omega^2}}
\end{aligned}
$$



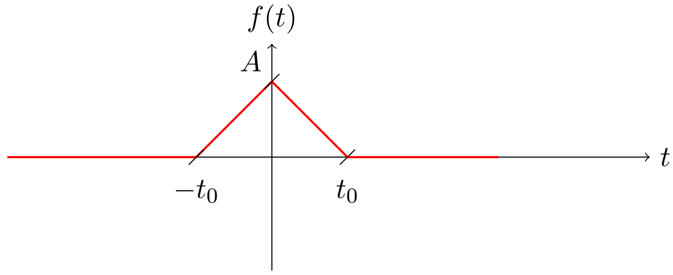
The magnitude spectrum of a real signal is an even-symmetric function of w.
The phase spectrum is defined as:

$$\begin{aligned}
\Phi(\omega) &= arctan2(\frac{Im\{F(\jmath\omega)\}}{Re\{F(\jmath\omega)\}}) = \\
&= arctan2\left(\frac{\left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\right)}{\left(\frac{A\cdot a}{(a^2+\omega^2)}\right)}\right) = \\
&= arctan2\left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\cdot\frac{(a^2+\omega^2)}{A\cdot a}\right) = \\
&= arctan2\left(-\frac{\omega}{a}\right)
\end{aligned}$$



The phase spectrum of a real signal is an odd-symmetric function of w.

Task 3. Compute the Fourier transform of a triangle impulse shown below.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.9}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.10}$$

In order to integrate the $f(t)$ signal, we need to describe it as a piecewise signal.

The simplest form of a linear function is:

$$f(t) = m \cdot t + b \tag{3.11}$$

In the first interval (e.g. $t \in (-t_0; 0)$), linear function crosses two points: $(-t_0, 0)$ and $(0, A)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot (-t_0) + b \\ A = m \cdot 0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot (-t_0) \\ A = b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ A = b \end{cases}$$

$$\begin{cases} A = b \\ \frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{t_0} \cdot t + A$$

In the second interval (e.g. $t \in (0; t_0)$), linear function crosses two points: $(0; A)$ and $(t_0, 0)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot t_0 + b \\ A = m \cdot 0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot t_0 \\ A = b \end{cases}$$

$$\begin{cases} -\frac{b}{t_0} = m \\ A = b \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = -\frac{A}{t_0} \cdot t + A$$

As a result the piecewise linear function is given by:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) = \begin{cases} 0 & \text{for } t \in (-\infty; -t_0) \\ \frac{A}{t_0} \cdot t + A & \text{for } t \in (-t_0; 0) \\ -\frac{A}{t_0} \cdot t + A & \text{for } t \in (0; t_0) \\ 0 & \text{for } t \in (t_0; \infty) \end{cases} \tag{3.12}$$

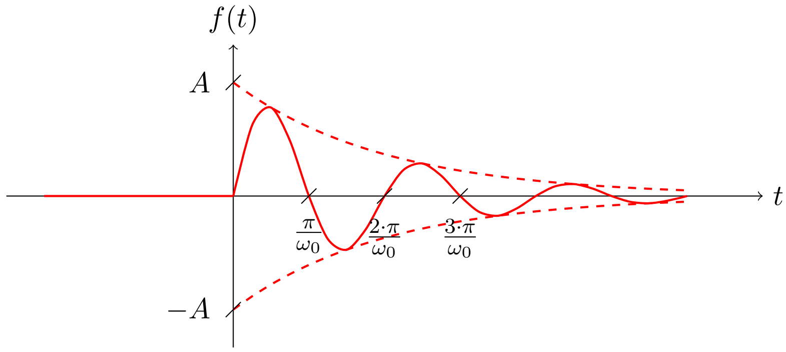
For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} \left(\frac{A}{t_0} \cdot t + A\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \left(-\frac{A}{t_0} \cdot t + A\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt + \int_{-t_0}^{0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} -\frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{t_0} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 + \frac{A}{t_0} \cdot \int_{-t_0}^{0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + A \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&- \frac{A}{t_0} \cdot \int_{0}^{t_0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + A \cdot \int_{0}^{t_0} e^{-\jmath\cdot\omega\cdot t} \cdot dt + 0 = \\
&= \left\{ \begin{matrix} u & = t & dv & = e^{-\jmath\cdot\omega\cdot t} \cdot dt \\ du & = dt & v & = \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \end{matrix} \right\} = \\
&= \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t} \Big|_{-t_0}^{0} - \int_{-t_0}^{0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) =
\end{aligned}$$

$$
\begin{aligned}
&+A\cdot\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)=\\
&-\frac{A}{t_0}\cdot\left(t\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}-\int_0^{t_0}\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt\right)=\\
&+A\cdot\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}\right)=\\
&=\frac{A}{t_0}\cdot\left(0\cdot e^{-\jmath\cdot\omega\cdot 0}-(-t_0)\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot(-t_0)}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)\right)=\\
&+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)=\\
&-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0\cdot e^{-\jmath\cdot\omega\cdot 0}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}\right)\right)=\\
&+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)=\\
&=\frac{A}{t_0}\cdot\left(0-t_0\cdot\frac{1}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)\right)=\\
&-\frac{A}{\jmath\cdot\omega}\cdot\left(1-e^{\jmath\cdot\omega\cdot t_0}\right)=\\
&-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)\right)=\\
&-\frac{A}{\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-1\right)=\\
&=-\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{\jmath\cdot\omega}+\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}=\\
&+\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{-\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}-\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{\jmath\cdot\omega}=\\
&=-\frac{2\cdot A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}-\frac{A}{t_0\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=\\
&=\left\{cos(x)=\tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\}=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}-\frac{2\cdot A}{t_0\cdot\omega^2}\cdot cos(\omega\cdot t_0)=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}\cdot(1-cos(\omega\cdot t_0))=\\
&=\begin{Bmatrix}sin^2(x)=\frac{1}{2}-\frac{1}{2}\cdot cos(2\cdot x)\\ cos(2\cdot x)=1-2\cdot sin^2(x)\end{Bmatrix}=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}\cdot(1-1+2\cdot sin^2(\frac{\omega\cdot t_0}{2}))=\\
&=\frac{4\cdot A}{t_0\cdot\omega^2}\cdot sin^2(\frac{\omega\cdot t_0}{2})=\\
&=\frac{A\cdot t_0}{\frac{t_0^2\cdot\omega^2}{4}}\cdot sin^2(\frac{\omega\cdot t_0}{2})=\\
&=\left\{\tfrac{sin(x)}{x}=Sa(x)\right\}=\\
&=A\cdot t_0\cdot Sa^2(\frac{\omega\cdot t_0}{2})
\end{aligned}
$$

The Fourier transform of the $f(t) = A \cdot \Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega) = A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$.

Task 4. Compute the Fourier transform of a signal shown below. Compute and draw magnitude and phase spectra.



The $f(t)$ signal, as a piecewise function is given by:

$$f(t) = \begin{cases} 0 & \text{for } t \in (-\infty; 0) \\ e^{-a\cdot t} \cdot sin(\omega_0 \cdot t) & \text{for } t \in (0; \infty) \end{cases} \tag{3.13}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.14}$$

For the given $f(t)$ signal we get:

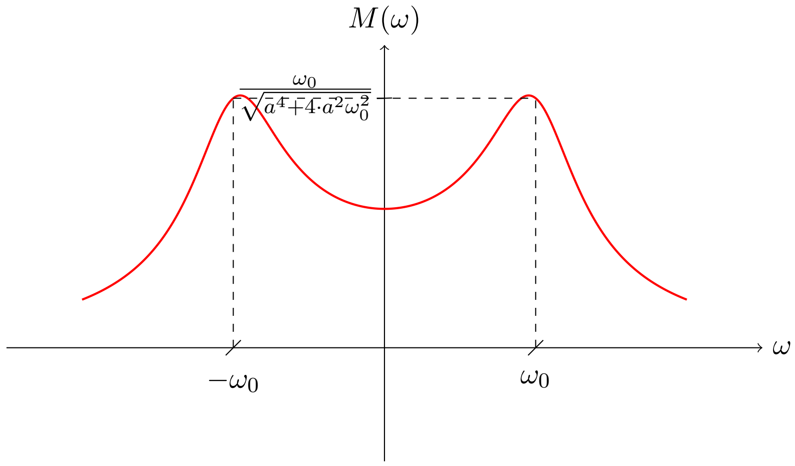
$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{\infty} e^{-a\cdot t} \cdot sin(\omega_0 \cdot t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ sin(x) = \tfrac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath} \right\} = \\
&= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} e^{-a\cdot t} \cdot \left(\frac{e^{\jmath\cdot\omega_0\cdot t} - e^{-\jmath\cdot\omega_0\cdot t}}{2\cdot\jmath} \right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= 0 + \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \left(\int_{0}^{\tau} e^{-a\cdot t} \cdot e^{\jmath\cdot\omega_0\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt - \int_{0}^{\tau} e^{-a\cdot t} \cdot e^{-\jmath\cdot\omega_0\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \left(\int_{0}^{\tau} e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t} \cdot dt - \int_{0}^{\tau} e^{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z & = & (-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t & w & = & (-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t \\ dz & = & (-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot dt & dw & = & (-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot dt \\ dt & = & \frac{1}{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot dz & dt & = & \frac{1}{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot dw \end{matrix} \right\} = \\
&= \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \int_{0}^{\tau} e^{z} \cdot \frac{dz}{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} - \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \int_{0}^{\tau} e^{w} \cdot \frac{dw}{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \int_{0}^{\tau} e^{z} \cdot dz - \frac{1}{2\cdot\jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \int_{0}^{\tau} e^{w} \cdot dw = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{z}\Big|_{0}^{\tau} - \frac{1}{2\cdot\jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{w}\Big|_{0}^{\tau} = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t}\Big|_{0}^{\tau} +
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty} e^{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t}\Big|_0^{\tau}\\
&= \frac{1}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot 0}\right)+\\
&- \frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-e^{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot 0}\right)=\\
&= \frac{1}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}\left(e^{-a\cdot\tau}\cdot e^{(\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}\right)-\lim_{\tau\to\infty}1\right)+\\
&- \frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}\left(e^{-a\cdot\tau}\cdot e^{(-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}\right)-\lim_{\tau\to\infty}1\right)=\\
&= \frac{1}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}(e^{-a\cdot\tau})\cdot\lim_{\tau\to\infty}e^{(\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-1\right)+\\
&- \frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty}(e^{-a\cdot\tau})\cdot\lim_{\tau\to\infty}e^{(-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-1\right)=\\
&= \frac{1}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(0\cdot\lim_{\tau\to\infty}e^{(\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-1\right)+\\
&- \frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot\left(0\cdot\lim_{\tau\to\infty}e^{(-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot\tau}-1\right)=\\
&= \frac{-1}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}+\frac{1}{2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}=\\
&= \frac{-(2\cdot\jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega))+2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}{2\cdot \jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot 2\cdot \jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}=\\
&= \frac{2\cdot\jmath\cdot a+2\cdot\jmath^2\cdot\omega_0+2\cdot\jmath^2\cdot\omega-2\cdot\jmath\cdot a+2\cdot\jmath^2\cdot\omega_0-2\cdot\jmath^2\cdot\omega}{4\cdot\jmath^2\cdot(a^2+a\cdot\jmath\cdot\omega_0+a\cdot\jmath\cdot\omega-a\cdot\jmath\cdot\omega_0-\jmath^2\cdot\omega_0^2-\jmath^2\cdot\omega_0\cdot\omega+a\cdot\jmath\cdot\omega+\jmath^2\cdot\omega_0\cdot\omega+\jmath^2\cdot\omega^2)}=\\
&= \frac{4\cdot\jmath^2\cdot\omega_0}{4\cdot\jmath^2\cdot(a^2+2\cdot a\cdot\jmath\cdot\omega-\jmath^2\cdot\omega_0^2+\jmath^2\cdot\omega^2)}=\\
&= \frac{\omega_0}{a^2+2\cdot a\cdot\jmath\cdot\omega+\omega_0^2-\omega^2}=\\
&= \frac{\omega_0}{\omega_0^2+(a^2+2\cdot a\cdot\jmath\cdot\omega-\omega^2)}=\\
&= \frac{\omega_0}{\omega_0^2+(a+\jmath\cdot\omega)^2}
\end{aligned}
$$

The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega)=\frac{\omega_0}{\omega_0^2+(a+\jmath\cdot\omega)^2}$.
The magnitude spectrum is defined as:

$$
\begin{aligned}
M(\omega)=|F(j\omega)|&=\\
&=\left|\frac{\omega_0}{\omega_0^2+(a+\jmath\cdot\omega)^2}\right|=\\
&=\left|\frac{\omega_0}{\omega_0^2+a^2+2\cdot a\cdot\jmath\cdot\omega+(\jmath\cdot\omega)^2}\right|=\\
&=\left|\frac{\omega_0}{\omega_0^2+a^2+2\cdot a\cdot\jmath\cdot\omega-\omega^2}\right|=\\
&=\left|\frac{\omega_0}{\omega_0^2-\omega^2+a^2+\jmath\cdot 2\cdot a\cdot\omega}\right|=\\
&=\left\{\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right\}=\\
&=\frac{|\omega_0|}{|\omega_0^2-\omega^2+a^2+\jmath\cdot 2\cdot a\cdot\omega|}=
\end{aligned}
$$

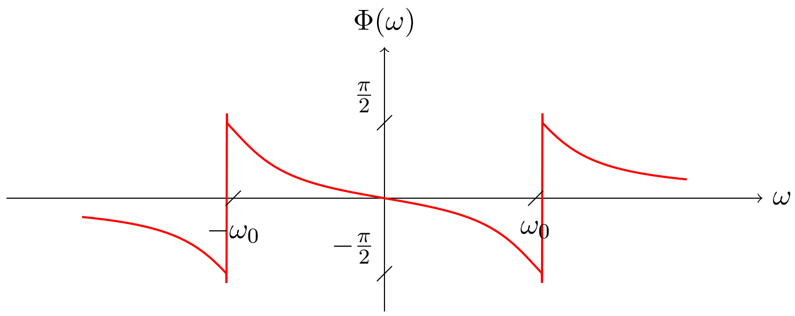
$$= \left\{ |a + \jmath \cdot b| = \sqrt{a^2 + b^2} \right\} =$$
$$= \frac{\omega_0}{\sqrt{\left(\omega_0^2 - \omega^2 + a^2\right)^2 + \left(2 \cdot a \cdot \omega\right)^2}}$$



The magnitude spectrum of a real signal is an even-symmetric function of w.

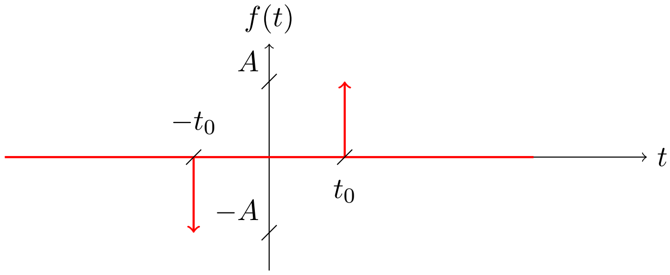
The phase spectrum is defined as:

$$\begin{aligned}
\Phi(\omega) &= arg\left(\frac{\omega_0}{\omega_0^2 + (a + \jmath \cdot \omega)^2}\right) = \\
&= arg\left(\frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega + (\jmath \cdot \omega)^2}\right) = \\
&= arg\left(\frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega - \omega^2}\right) = \\
&= arg\left(\frac{\omega_0}{\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega}\right) = \\
&= \left\{ arg\left(\tfrac{z_1}{z_2}\right) = arg\left(z_1\right) - arg\left(z_2\right) \right\} = \\
&= arg\left(\omega_0\right) - arg\left(\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega\right) = \\
&= \left\{ arg\left(a + \jmath \cdot b\right) = arctan(\tfrac{b}{a}) \right\} = \\
&= arctan\left(\frac{0}{\omega_0}\right) - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) = \\
&= arctan\left(0\right) - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) = \\
&= 0 - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) = \\
&= -arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right)
\end{aligned}$$



The phase spectrum of a real signal is an odd-symmetric function of w.

Task 5. Compute the Fourier transform of a signal shown below. Compute and draw magnitude and phase spectra.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0) \tag{3.15}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.16}$$

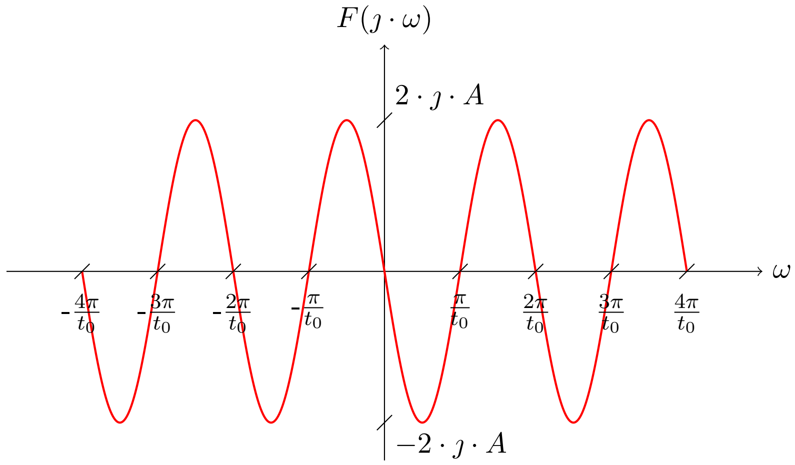
For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} A \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \int_{-\infty}^{\infty} A \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - A \cdot \int_{-\infty}^{\infty} \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot t_0} - A \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot t_0} - A \cdot e^{\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(e^{-\jmath \cdot \omega \cdot t_0} - e^{\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(-e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= -A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= -A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= -2 \cdot \jmath \cdot A \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0)
\end{aligned}$$

The Fourier transform of the $f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)$ is equal to $F(\jmath\omega) = -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0)$.

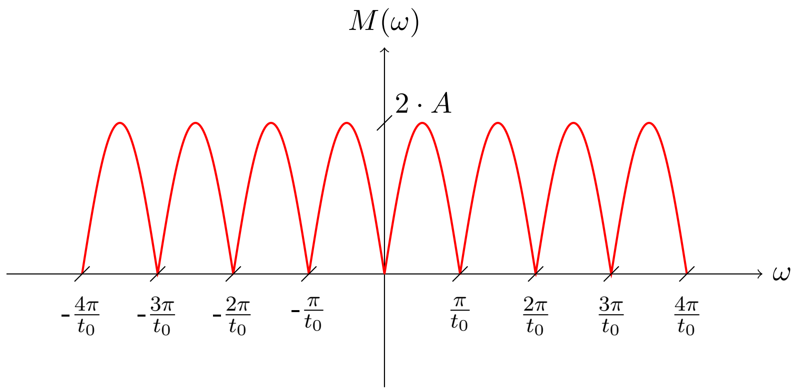
Draw complex spectrum of the $f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)$:

$$F(\jmath\omega) = -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0) \tag{3.17}$$



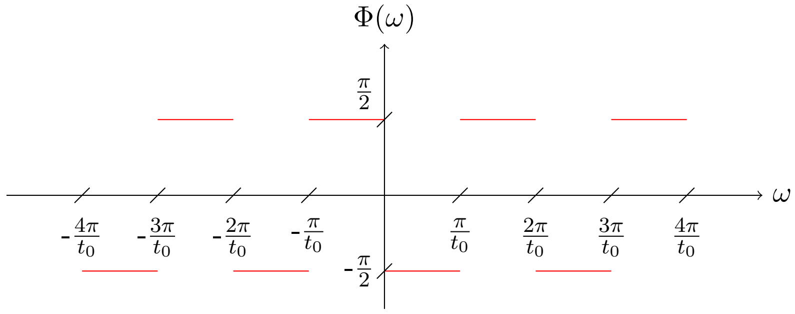
The magnitude spectrum is defined as:

$$\begin{aligned} M(\omega) &= |F(\jmath \cdot \omega)| \\ &= \sqrt{(-2 \cdot A \cdot sin(\omega \cdot t_0))^2} \\ &= \sqrt{4 \cdot A^2 \cdot sin^2(\omega \cdot t_0)} \\ &= 2 \cdot A \cdot |sin(\omega \cdot t_0)| \end{aligned}$$

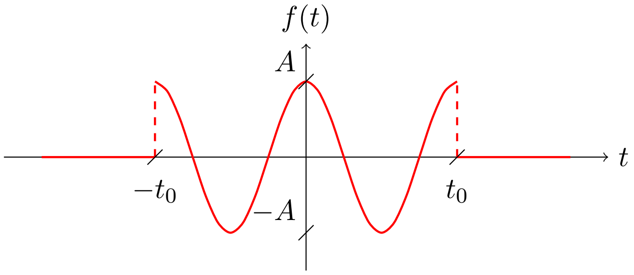


The phase spectrum is defined as:

$$\Phi(\omega) = arctan2\left(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}\right) \tag{3.18}$$


$\Phi(\omega)$
$\frac{\pi}{2}$
ω
-$\frac{4\pi}{t_0}$
-$\frac{3\pi}{t_0}$
-$\frac{2\pi}{t_0}$
-$\frac{\pi}{t_0}$
$\frac{\pi}{t_0}$
$\frac{2\pi}{t_0}$
$\frac{3\pi}{t_0}$
$\frac{4\pi}{t_0}$
-$\frac{\pi}{2}$

Task 6. Compute the Fourier transform of a signal shown below. Compute and draw magnitude and phase spectra.



$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.19}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.20}$$

For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{t_0} A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ cos(x) = \tfrac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2} \right\} = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt + \int_{-t_0}^{t_0} A \cdot \frac{e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t}}{2} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 + \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t} \right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + 0 = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t - \jmath\cdot\omega\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t - \jmath\cdot\omega\cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)\cdot t} + e^{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)\cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)\cdot t} \cdot dt + \int_{-t_0}^{t_0} e^{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)\cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z_1 = \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t & z_2 = -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t \\ dz_1 = \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot dt & dz_2 = -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot dt \\ dt = \frac{1}{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 & dt = \frac{1}{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \end{matrix} \right\} =
\end{aligned}$$

$$= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{z_1} \cdot \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 + \int_{-t_0}^{t_0} e^{z_2} \cdot \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_2} \cdot dz_2 \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{z_1}\Big|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{z_2}\Big|_{-t_0}^{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot (-t_0)} \right) + \right.$$

$$\left. + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot (-t_0)} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \right.$$

$$\left. + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \right.$$

$$\left. + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) \cdot \frac{2}{2} + \right.$$

$$\left. + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \cdot \frac{2}{2} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0}}{2 \cdot \jmath} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0}}{2 \cdot \jmath} \right) =$$

$$= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} =$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \right) =$$

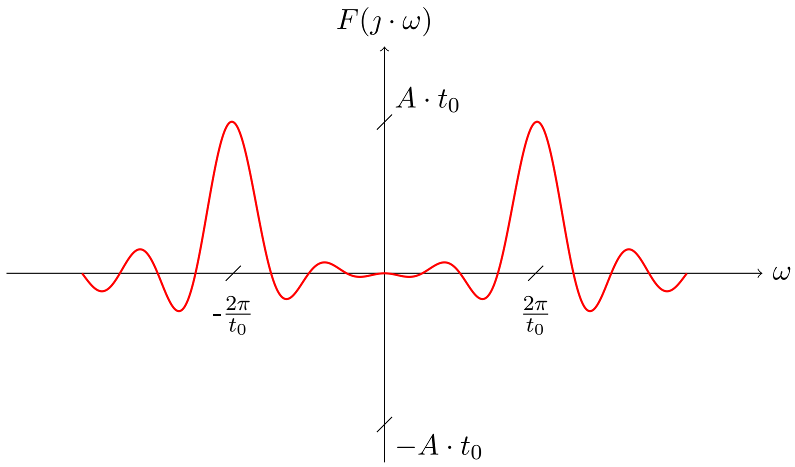
$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) \cdot \frac{t_0}{t_0} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \cdot \frac{t_0}{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) + \frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \right) =$$

$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} + 2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) =$$
$$= \left\{ Sa(x) = \frac{sin(x)}{x} \right\} =$$
$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + 2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right) =$$
$$= A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$$

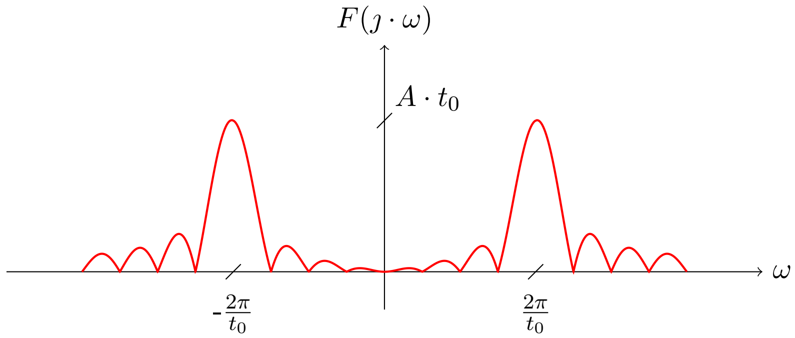
The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$.
Draw complex spectrum of the $f(t)$:

$$F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right) \tag{3.21}$$



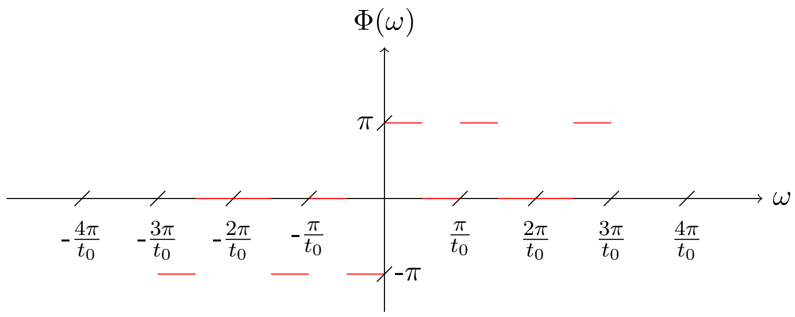
The magnitude spectrum is defined as:

$$M(\omega) = |F(j \cdot \omega)| \tag{3.22}$$



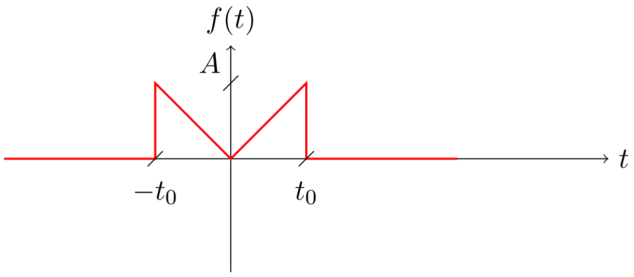
The magnitude spectrum of a real signal is an even-symmetric function of w.
The phase spectrum is defined as:

$$\Phi(\omega) = arctan2\left(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}\right) \tag{3.23}$$



The phase spectrum of a real signal is an odd-symmetric function of w.

Task 7. Compute the Fourier transform of a the $f(t)$ signal shown below.



The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.24}$$

In order to integrate the $f(t)$ signal, we need to describe it as a piecewise linear signal.

The simplest form of a linear function is:

$$f(t) = m \cdot t + b \tag{3.25}$$

In the first interval (e.g. $t \in (-t_0; 0)$), linear function crosses two points: $(-t_0, A)$ and $(0, 0)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} A = m \cdot (-t_0) + b \\ 0 = m \cdot 0 + b \end{cases}$$

$$\begin{cases} A = m \cdot (-t_0) + b \\ 0 = b \end{cases}$$

$$\begin{cases} A = m \cdot (-t_0) + 0 \\ 0 = b \end{cases}$$

$$\begin{cases} -\frac{A}{t_0} = m \\ 0 = b \end{cases}$$

As a result we get:

$$f(t) = -\frac{A}{t_0} \cdot t$$

In the second interval (e.g. $t \in (0; t_0)$), linear function crosses two points: $(0; 0)$ and (t_0, A). So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} A = m \cdot t_0 + b \\ 0 = m \cdot 0 + b \end{cases}$$

$$\begin{cases} A = m \cdot t_0 + b \\ 0 = b \end{cases}$$

$$\begin{cases} A = m \cdot t_0 + 0 \\ 0 = b \end{cases}$$

$$\begin{cases} \frac{A}{t_0} = m \\ 0 = b \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{t_0} \cdot t$$

As a result the piecewise linear function is given by:

$$f(t) = \begin{cases} 0 & \text{for} \quad t \in (-\infty; -t_0) \\ -\frac{A}{t_0} \cdot t & \text{for} \quad t \in (-t_0; 0) \\ \frac{A}{t_0} \cdot t & \text{for} \quad t \in (0; t_0) \\ 0 & \text{for} \quad t \in (t_0; \infty) \end{cases} \tag{3.26}$$

For the given $f(t)$ signal we get:

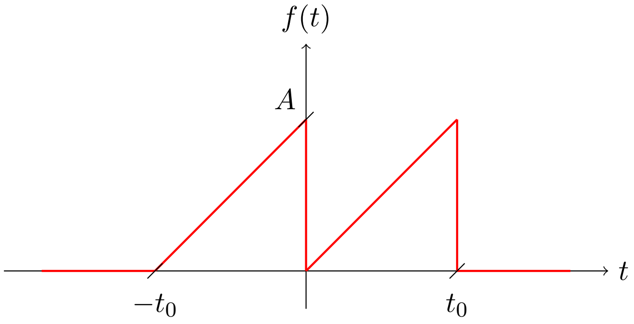
$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-t_0}^{0} \left(-\frac{A}{t_0} \cdot t \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt - \int_{-t_0}^{0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 - \frac{A}{t_0} \cdot \int_{-t_0}^{0} t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \frac{A}{t_0} \cdot \int_{0}^{t_0} t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + 0 = \\
&= \left\{ \begin{matrix} u & = t & dv & = e^{-\jmath \cdot \omega \cdot t} \cdot dt \\ du & = dt & v & = \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \end{matrix} \right\} = \\
&= -\frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{-t_0}^{0} - \int_{-t_0}^{0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&+ \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{0}^{t_0} - \int_{0}^{t_0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= -\frac{A}{t_0} \cdot \left(0 \cdot e^{-\jmath \cdot \omega \cdot 0} - (-t_0) \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} + \frac{1}{\jmath \cdot \omega} \left(\frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{-t_0}^{0} \right) \right) + \\
&+ \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - 0 \cdot e^{-\jmath \cdot \omega \cdot 0} + \frac{1}{\jmath \cdot \omega} \left(\frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{0}^{t_0} \right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{t_0} \cdot \left(0 - t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot 0} - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&+ \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - 0 - \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot 0}\right)\right) = \\
&= \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{0} - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&+ \frac{A}{t_0} \cdot \left(-t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot t_0} - e^{0}\right)\right) = \\
&= \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2} \left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&- \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right)\right) = \\
&= \frac{A}{t_0} \cdot t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right) + \\
&- \frac{A}{t_0} \cdot t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right) - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}\right) + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} = \\
&= \frac{2 \cdot A}{2 \cdot \jmath \cdot \omega} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}\right) + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ 2 \cdot \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - \frac{2}{2} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}\right)\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - 2 \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}}{2}\right) = \\
&= \left\{ \begin{array}{l} sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \\ cos\left(x\right) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \end{array} \right\} = \\
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - 2 \cdot cos\left(\omega \cdot t_0\right)\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot 4 \cdot \left(\frac{1}{2} - \frac{1}{2} \cdot cos\left(\omega \cdot t_0\right)\right) = \\
&= \left\{ sin^2\left(x\right) = \tfrac{1}{2} - \tfrac{1}{2} \cdot cos\left(2 \cdot x\right) \right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{4}{\jmath^2 \cdot \omega^2} \cdot sin^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{t_0}{t_0} sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{4}{\jmath^2 \cdot \omega^2} \cdot \frac{t_0}{t_0} \cdot sin^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right) = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\left(\omega \cdot t_0\right)}{t_0 \cdot \omega} + A \cdot t_0 \cdot \frac{4}{-1 \cdot \omega^2 \cdot t_0^2} \cdot sin^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right) = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\left(\omega \cdot t_0\right)}{t_0 \cdot \omega} + A \cdot t_0 \cdot \frac{-1}{\frac{\omega^2 \cdot t_0^2}{4}} \cdot sin^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right) = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\left(\omega \cdot t_0\right)}{t_0 \cdot \omega} - A \cdot t_0 \cdot \frac{sin^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right)}{\left(\frac{\omega \cdot t_0}{2}\right)^2} = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\left(\omega \cdot t_0\right)}{t_0 \cdot \omega} - A \cdot t_0 \cdot \left(\frac{sin\left(\frac{1}{2} \cdot \omega \cdot t_0\right)}{\frac{1}{2} \cdot \omega \cdot t_0}\right)^2 = \\
&= \left\{Sa\left(x\right) = \frac{sin(x)}{x}\right\} = \\
&= 2 \cdot A \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - A \cdot t_0 \cdot Sa^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right)
\end{aligned}
$$

The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega) = 2 \cdot A \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - A \cdot t_0 \cdot Sa^2\left(\frac{1}{2} \cdot \omega \cdot t_0\right)$.

Task 8. Compute the Fourier transform of a signal shown below. Compute and draw magnitude and phase spectra.



$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{t_0} \cdot t + A & t \in (-t_0; 0) \\ \frac{A}{t_0} \cdot t & t \in (0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.27}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.28}$$

For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} \left(\frac{A}{t_0} \cdot t + A\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0dt + \int_{-t_0}^{0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= \frac{A}{t_0} \cdot \int_{-t_0}^{0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + A \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt + \frac{A}{t_0} \cdot \int_{0}^{t_0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ \begin{matrix} u & = t & dv & = e^{-\jmath\cdot\omega\cdot t} \cdot dt \\ du & = dt & v & \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \end{matrix} \right\} = \\
&= \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \Big|_{-t_0}^{0} + \int_{-t_0}^{0} \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) + \\
&+ A \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt + \\
&+ \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \Big|_{0}^{t_0} + \int_{0}^{t_0} \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) = \\
&= \frac{A}{t_0} \cdot \left(\left(0 \cdot \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot 0} + t_0 \cdot \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot(-t_0)} \right) + \frac{1}{-\jmath\cdot\omega} \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) + \\
&+ A \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt +
\end{aligned}$$

$$+\frac{A}{t_0}\cdot\left(\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot 0}\right)+\frac{1}{-\jmath\cdot\omega}\cdot\int_0^{t_0}e^{-\jmath\cdot\omega\cdot t}\cdot dt\right)=$$

$$=\left\{\begin{matrix} z & =-\jmath\cdot\omega\cdot t\\ dz & =-\jmath\cdot\omega\cdot dt\\ dt & =\frac{dz}{-\jmath\cdot\omega}\end{matrix}\right\}=$$

$$=\frac{A}{t_0}\cdot\left(\left(0+t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}\right)+\frac{1}{-\jmath\cdot\omega}\cdot\int_{-t_0}^{0}e^{z}\cdot\frac{dz}{-\jmath\cdot\omega}\right)+$$

$$+A\cdot\int_{-t_0}^{0}e^{z}\cdot\frac{dz}{-\jmath\cdot\omega}+$$

$$+\frac{A}{t_0}\cdot\left(\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0\right)+\frac{1}{-\jmath\cdot\omega}\cdot\int_0^{t_0}e^{z}\cdot\frac{dz}{-\jmath\cdot\omega}\right)=$$

$$=\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{1}{-\jmath\cdot\omega}\cdot\frac{1}{-\jmath\cdot\omega}\cdot\int_{-t_0}^{0}e^{z}\cdot dz\right)+$$

$$+A\cdot\frac{1}{-\jmath\cdot\omega}\cdot\int_{-t_0}^{0}e^{z}\cdot dz+$$

$$+\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{1}{-\jmath\cdot\omega}\cdot\frac{1}{-\jmath\cdot\omega}\cdot\int_0^{t_0}e^{z}\cdot dz\right)=$$

$$=\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot e^{z}\big|_{-t_0}^{0}\right)+$$

$$-A\cdot\frac{1}{\jmath\cdot\omega}\cdot e^{z}\big|_{-t_0}^{0}+$$

$$+\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot e^{z}\big|_{0}^{t_0}\right)=$$

$$=\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)+$$

$$-A\cdot\frac{1}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}+$$

$$+\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}\right)=$$

$$=\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)\right)+$$

$$-A\cdot\frac{1}{\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)+$$

$$+\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)\right)=$$

$$=\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot\left(e^{0}-e^{\jmath\cdot\omega\cdot t_0}\right)\right)+$$

$$-A\cdot\frac{1}{\jmath\cdot\omega}\cdot\left(e^{0}-e^{\jmath\cdot\omega\cdot t_0}\right)+$$

$$+\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{1}{-1\cdot\omega^2}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{0}\right)\right)=$$

$$=\frac{A}{t_0}\cdot t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}+\frac{A}{t_0}\cdot\frac{1}{-1\cdot\omega^2}\cdot\left(1-e^{\jmath\cdot\omega\cdot t_0}\right)+$$

$$-A\cdot\frac{1}{\jmath\cdot\omega}\cdot\left(1-e^{\jmath\cdot\omega\cdot t_0}\right)+$$

$$+\frac{A}{t_0}\cdot t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{t_0}\cdot\frac{1}{-1\cdot\omega^2}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-1\right)=$$

$$
\begin{aligned}
&= -\frac{A}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0 \cdot \omega^2} \cdot \left(1 - e^{\jmath \cdot \omega \cdot t_0}\right) + \\
&- \frac{A}{\jmath \cdot \omega} + \frac{A}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \\
&- \frac{A}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0 \cdot \omega^2} \cdot \left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right) = \\
&= -\frac{A}{t_0 \cdot \omega^2} + \frac{A}{t_0 \cdot \omega^2} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{\jmath \cdot \omega} + \\
&- \frac{A}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0 \cdot \omega^2} = \\
&= \frac{A}{t_0 \cdot \omega^2} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{A}{\jmath \cdot \omega} - \frac{A}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{A}{t_0 \cdot \omega^2} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}\right) - \frac{A}{\jmath \cdot \omega} \left(1 + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \frac{A \cdot 2 \cdot \jmath}{t_0 \cdot \omega^2} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} - \frac{A}{\jmath \cdot \omega} \left(1 + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \left\{ sin\left(x\right) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega} \left(1 + e^{-\jmath \cdot \omega \cdot t_0}\right)
\end{aligned}
$$

The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega) = \jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega}\left(1 + e^{-\jmath \cdot \omega \cdot t_0}\right)$.
Draw complex spectrum of the $f(t)$:

$$
M\left(\omega\right) = \left|F(\jmath\omega)\right| \tag{3.29}
$$

.

$$
\begin{aligned}
M\left(\omega\right) &= \left|F(\jmath\omega)\right| = \\
&= \left|\jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega} \left(1 + e^{-\jmath \cdot \omega \cdot t_0}\right)\right| = \\
&= \left\{ e^{\jmath \cdot x} = cos\left(x\right) + \jmath \cdot sin\left(x\right) \right\} = \\
&= \left|\jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega} \left(1 + cos\left(-\omega \cdot t_0\right) + \jmath \cdot sin\left(-\omega \cdot t_0\right)\right)\right| = \\
&= \left|\jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega} \left(1 + cos\left(\omega \cdot t_0\right) - \jmath \cdot sin\left(\omega \cdot t_0\right)\right)\right| = \\
&= \left|\jmath \cdot \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \frac{A}{\omega} + \jmath \cdot \frac{A}{\omega} \cdot cos\left(\omega \cdot t_0\right) - \jmath \cdot \frac{A}{\omega} \cdot \jmath \cdot sin\left(\omega \cdot t_0\right)\right| = \\
&= \left|\jmath \cdot \left(\frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{\omega} + \frac{A}{\omega} \cdot cos\left(\omega \cdot t_0\right)\right) + \frac{A}{\omega} \cdot sin\left(\omega \cdot t_0\right)\right| = \\
&= \left|\frac{A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \jmath \cdot \left(\frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{\omega} \cdot \left(1 + cos\left(\omega \cdot t_0\right)\right)\right)\right| = \\
&= \left|\frac{A}{\omega} \cdot \left(sin\left(\omega \cdot t_0\right) + \jmath \cdot \left(\frac{2}{t_0 \cdot \omega} \cdot sin\left(\omega \cdot t_0\right) + 1 + cos\left(\omega \cdot t_0\right)\right)\right)\right| = \\
&= \left|\frac{A}{\omega}\right| \cdot \left|sin\left(\omega \cdot t_0\right) + \jmath \cdot \left(\frac{2}{t_0 \cdot \omega} \cdot sin\left(\omega \cdot t_0\right) + 1 + cos\left(\omega \cdot t_0\right)\right)\right| = \\
&= \frac{A}{|\omega|} \cdot \sqrt{sin^2\left(\omega \cdot t_0\right) + \left(\frac{2}{t_0 \cdot \omega} \cdot sin\left(\omega \cdot t_0\right) + 1 + cos\left(\omega \cdot t_0\right)\right)^2}
\end{aligned}
$$

$$M\left(w\right)=\frac{A}{|\omega|}\cdot\sqrt{sin^{2}\left(\omega\cdot t_{0}\right)+\left(\frac{2}{t_{0}\cdot\omega}\cdot sin\left(\omega\cdot t_{0}\right)+1+cos\left(\omega\cdot t_{0}\right)\right)^{2}}=0\ \|\cdot\frac{|\omega|}{A}$$
$$\sqrt{sin^{2}\left(\omega\cdot t_{0}\right)+\left(\frac{2}{t_{0}\cdot\omega}\cdot sin\left(\omega\cdot t_{0}\right)+1+cos\left(\omega\cdot t_{0}\right)\right)^{2}}=0\ \|\left(\cdot\right)^{2}$$
$$sin^{2}\left(\omega\cdot t_{0}\right)+\left(\frac{2}{t_{0}\cdot\omega}\cdot sin\left(\omega\cdot t_{0}\right)+1+cos\left(\omega\cdot t_{0}\right)\right)^{2}=0$$

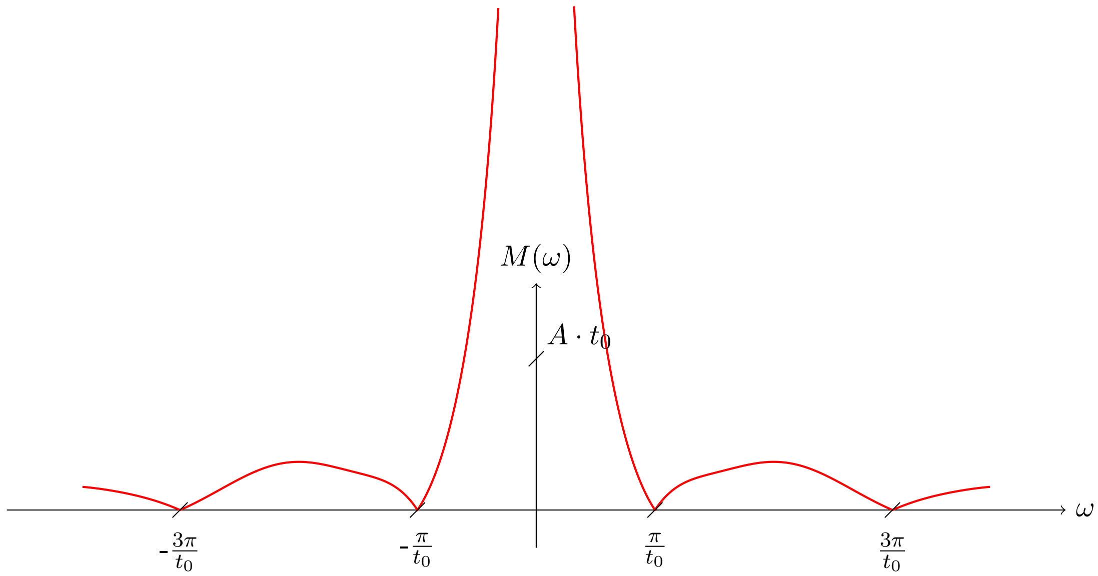
$$sin^{2}\left(\omega\cdot t_{0}\right)=0$$
$$sin\left(\omega\cdot t_{0}\right)=0$$
$$sin\left(\omega\cdot t_{0}\right)=sin\left(\pi\cdot k\right)$$
$$\omega\cdot t_{0}=\pi\cdot k$$
$$\omega=\frac{\pi}{t_{0}}\cdot k$$

$$\left(\frac{2}{t_{0}\cdot\omega}\cdot sin\left(\omega\cdot t_{0}\right)+1+cos\left(\omega\cdot t_{0}\right)\right)^{2}=0$$
$$\frac{2}{t_{0}\cdot\omega}\cdot sin\left(\omega\cdot t_{0}\right)+1+cos\left(\omega\cdot t_{0}\right)=0$$
$$\begin{Bmatrix}1+cos(x)=2\cdot cos^{2}\left(\frac{x}{2}\right)\\ Sa\left(x\right)=\frac{sin(x)}{x}\end{Bmatrix}$$
$$2\cdot Sa\left(\omega\cdot t_{0}\right)+2\cdot cos^{2}\left(\frac{\omega\cdot t_{0}}{2}\right)=0$$

$$2\cdot Sa\left(\omega\cdot t_{0}\right)=0$$
$$Sa\left(\omega\cdot t_{0}\right)=0$$
$$Sa\left(\omega\cdot t_{0}\right)=Sa\left(\pi\cdot k\right)$$
$$\omega\cdot t_{0}=\pi\cdot k$$
$$\omega=\frac{\pi}{t_{0}}\cdot k$$

$$2\cdot cos^{2}\left(\frac{\omega\cdot t_{0}}{2}\right)=0$$

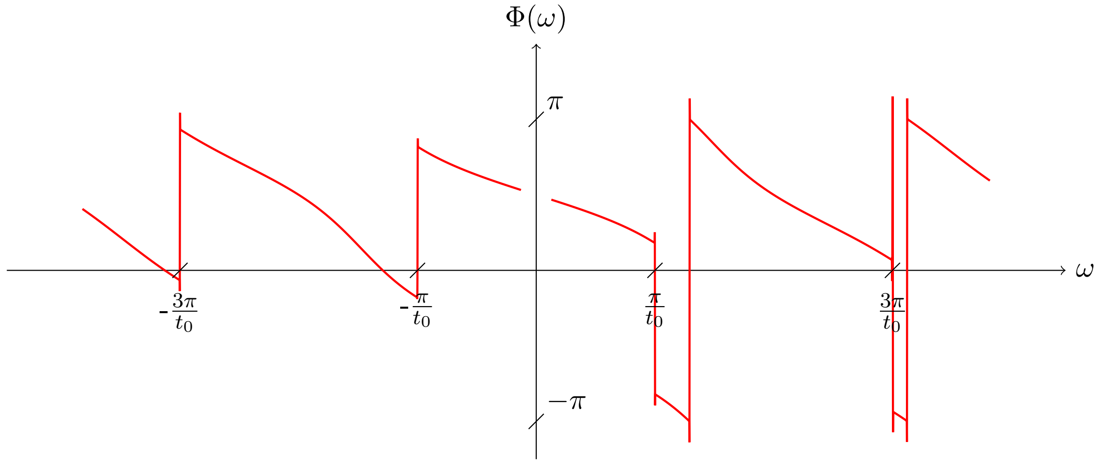
$$cos^2\left(\frac{\omega \cdot t_0}{2}\right) = 0$$
$$cos\left(\frac{\omega \cdot t_0}{2}\right) = 0$$
$$cos\left(\frac{\omega \cdot t_0}{2}\right) = cos\left(\frac{\pi}{2} + \pi \cdot k\right)$$
$$\frac{\omega \cdot t_0}{2} = \frac{\pi}{2} + \pi \cdot k$$
$$\omega \cdot t_0 = \pi + 2 \cdot \pi \cdot k$$
$$\omega = \frac{\pi + 2 \cdot \pi \cdot k}{t_0}$$



The magnitude spectrum of a real signal is an even-symmetric function of w.

The phase spectrum is defined as:

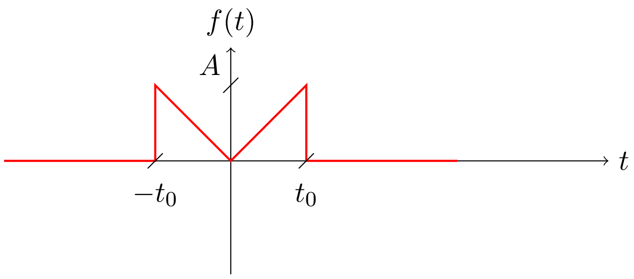
$$\Phi(\omega) = arctan2\left(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}\right) \tag{3.30}$$



The phase spectrum of a real signal is an odd-symmetric function of w.

3.2 Exploiting properties of the Fourier transformation

Task 1. Compute the Fourier transform of the $f(t)$ signal shown below using theorems describing the properties of Fourier transformation. Exploit the following transforms $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$ and $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \left(\Pi\left(\frac{t}{2 \cdot t_0}\right) - \Lambda\left(\frac{t}{t_0}\right)\right) \tag{3.31}$$

Based on linearity of the Fourier transformation, we can calculate transforms for elementary signals separately:

$$f(t) = A \cdot (f_1(t) - f_2(t)) \tag{3.32}$$

where:

$$f_1(t) = \Pi\left(\frac{t}{2 \cdot t_0}\right)$$
$$f_2(t) = \Lambda\left(\frac{t}{t_0}\right)$$

Calculate the Fourier transform $F_1(\jmath\omega)$ for the first signal $f_1(t)$.

We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

Based on the scaling theorem:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

we get:

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{2 \cdot t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{2 \cdot t_0}}}{2}\right)$$
$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} 2 \cdot t_0 \cdot Sa\left(\frac{\omega \cdot 2 \cdot t_0}{2}\right)$$

$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} 2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right)$$

The Fourier transform of $f_1(t)$ signal is equal to:

$$F_1(\jmath\omega) = \mathcal{F}\{f_1(t)\} = 2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) \tag{3.33}$$

Now, we calculate the Fourier transform $F_2(\jmath\omega)$ for the second signal $f_2(t)$.
We know that: $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.
Based on the scaling theorem:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

we get:

$$\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right)$$
$$\Lambda(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa^2\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right)$$
$$\Lambda(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)$$

The Fourier transform of $f_2(t)$ signal is equal to:

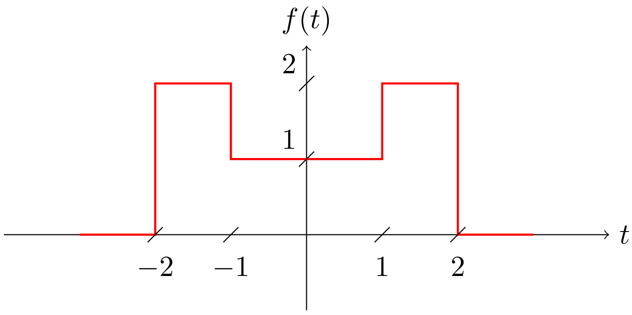
$$F_2(\jmath\omega) = \mathcal{F}\{f_2(t)\} = t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right) \tag{3.34}$$

Finally, the Fourier transform for $f(t)$ signal is equal to:

$$F(\jmath\omega) = \mathcal{F}\{f(t)\} = A \cdot \left(2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)\right)$$

The Fourier transform of $f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) - A \cdot \Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega) = 2 \cdot A \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - A \cdot t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)$.

Task 2. Compute the Fourier transform of the $f(t)$ signal shown below using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$. Give solutions for at least two different descriptions of the $f(t)$ signal.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = 2 \cdot \Pi\left(\frac{t}{4}\right) - \Pi\left(\frac{t}{2}\right) \tag{3.35}$$

Based on linearity of the Fourier transformation, we can calculate transforms for elementary signals separately:

$$f(t) = 2 \cdot f_1(t) - f_2(t) \tag{3.36}$$

where:

$$f_1(t) = \Pi\left(\frac{t}{4}\right)$$
$$f_2(t) = \Pi\left(\frac{t}{2}\right)$$

Calculate the Fourier transform $F_1(\jmath\omega)$ for the first signal $f_1(t)$.

We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

Based on the scaling theorem:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f_1(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

we get:

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{4}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{4}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{4}}}{2}\right)$$
$$\Pi(\frac{t}{4}) \xrightarrow{\mathcal{F}} 4 \cdot Sa\left(\frac{\omega \cdot 4}{2}\right)$$
$$\Pi(\frac{t}{4}) \xrightarrow{\mathcal{F}} 4 \cdot Sa\left(2 \cdot \omega\right)$$

The Fourier transform of $f_1(t)$ signal is equal to:

$$F_1(\jmath\omega) = \mathcal{F}\{f_1(t)\} = 4 \cdot Sa\left(2 \cdot \omega\right) \tag{3.37}$$

Now, we calculate the Fourier transform $F_2(\jmath\omega)$ for the second signal $f_2(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.
Based on the scaling theorem:

$$\begin{aligned} g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\ f_2(t) = g(\alpha \cdot t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha}) \end{aligned}$$

we get:

$$\begin{aligned} \Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \\ \Pi(\frac{t}{2}) &\xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{2}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{2}}}{2}\right) \\ \Pi(\frac{t}{2}) &\xrightarrow{\mathcal{F}} 2 \cdot Sa\left(\frac{\omega \cdot 2}{2}\right) \\ \Pi(\frac{t}{2}) &\xrightarrow{\mathcal{F}} 2 \cdot Sa\left(\omega\right) \end{aligned}$$

The Fourier transform of $f_2(t)$ signal is equal to:

$$F_2(\jmath\omega) = \mathcal{F}\{f_2(t)\} = 2 \cdot Sa\left(\omega\right) \tag{3.38}$$

Finally, the Fourier transform for $f(t)$ signal is equal to:

$$F(\jmath\omega) = \mathcal{F}\{f(t)\} = 2 \cdot (4 \cdot Sa\left(2 \cdot \omega\right)) - 2 \cdot Sa\left(\omega\right) = 8 \cdot Sa\left(2 \cdot \omega\right) - 2 \cdot Sa\left(\omega\right)$$

The Fourier transform of $f(t) = 2 \cdot \Pi\left(\frac{t}{4}\right) - \Pi\left(\frac{t}{2}\right)$ is equal to $F(\jmath\omega) = 8 \cdot Sa\left(2 \cdot \omega\right) - 2 \cdot Sa\left(\omega\right)$.
Other ways to describe the $f(t)$ signal using elementary signals:

$$\begin{aligned} f(t) &= \Pi\left(\frac{t}{4}\right) + \Pi\left(t - \frac{3}{2}\right) + \Pi\left(t + \frac{3}{2}\right) \\ f(t) &= \Pi\left(\frac{t}{2}\right) + 2 \cdot \Pi\left(t - \frac{3}{2}\right) + 2 \cdot \Pi\left(t + \frac{3}{2}\right) \end{aligned}$$

SOLUTION II:

Let's consider the following description of the $f(t)$ signal:

$$f(t) = \Pi\left(\frac{t}{4}\right) + \Pi\left(t - \frac{3}{2}\right) + \Pi\left(t + \frac{3}{2}\right)$$

Based on linearity of the Fourier transformation, we can calculate transforms for elementary signals separately:

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

where:

$$\begin{aligned}
f_1(t) &= \Pi\left(\frac{t}{4}\right) \\
f_2(t) &= \Pi\left(t - \frac{3}{2}\right) \\
f_3(t) &= \Pi\left(t + \frac{3}{2}\right)
\end{aligned}$$

Calculate the Fourier transform $F_1(\jmath\omega)$ for the first signal $f_1(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.
Based on the scaling theorem:

$$\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f_1(t) = g(\alpha \cdot t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})
\end{aligned}$$

we get:

$$\begin{aligned}
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \\
\Pi(\frac{t}{4}) &\xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{4}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{4}}}{2}\right) \\
\Pi(\frac{t}{4}) &\xrightarrow{\mathcal{F}} 4 \cdot Sa\left(\frac{\omega \cdot 4}{2}\right) \\
\Pi(\frac{t}{4}) &\xrightarrow{\mathcal{F}} 4 \cdot Sa\left(2 \cdot \omega\right)
\end{aligned}$$

The Fourier transform of the $f_1(t)$ signal is equal to:

$$F_1(\jmath\omega) = \mathcal{F}\{f_1(t)\} = 4 \cdot Sa\left(2 \cdot \omega\right) \tag{3.39}$$

Calculate the Fourier transform $F_2(\jmath\omega)$ for the second signal $f_2(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

Based on the shifting in time theorem:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f_2(t) = g(t - t_0) \xrightarrow{\mathcal{F}} F_2(\jmath\omega) = G(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0}$$

we get:

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi\left(t - \frac{3}{2}\right) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{3}{2}}$$

The Fourier transform of the $f_2(t)$ signal is equal to:

$$F_2(\jmath\omega) = \mathcal{F}\{f_2(t)\} = Sa\left(\frac{\omega}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{3}{2}} \tag{3.40}$$

Calculate the Fourier transform $F_3(\jmath\omega)$ for the third signal $f_3(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.
Based on the shifting in time theorem:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f_3(t) = g(t - t_0) \xrightarrow{\mathcal{F}} F_3(\jmath\omega) = G(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0}$$

we get:

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi\left(t + \frac{3}{2}\right) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \left(-\frac{3}{2}\right)}$$
$$\Pi\left(t + \frac{3}{2}\right) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{3}{2}}$$

The Fourier transform of the $f_3(t)$ signal is equal to:

$$F_3(\jmath\omega) = \mathcal{F}\{f_3(t)\} = Sa\left(\frac{\omega}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{3}{2}} \tag{3.41}$$

Finally, the Fourier transform of the $f(t)$ signal is equal to:

$$\begin{aligned} F(\jmath\omega) = \mathcal{F}\{f(t)\} &= 4 \cdot Sa\left(2 \cdot \omega\right) + Sa\left(\frac{\omega}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{3}{2}} + Sa\left(\frac{\omega}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{3}{2}} = \\ &= 4 \cdot Sa\left(2 \cdot \omega\right) + Sa\left(\frac{\omega}{2}\right) \cdot \left(e^{-\jmath \cdot \omega \cdot \frac{3}{2}} + e^{\jmath \cdot \omega \cdot \frac{3}{2}}\right) = \\ &= 4 \cdot Sa\left(2 \cdot \omega\right) + Sa\left(\frac{\omega}{2}\right) \cdot 2 \cdot \frac{e^{-\jmath \cdot \omega \cdot \frac{3}{2}} + e^{\jmath \cdot \omega \cdot \frac{3}{2}}}{2} = \end{aligned}$$

$$
\begin{aligned}
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= 4\cdot Sa\left(2\cdot\omega\right) + Sa\left(\frac{\omega}{2}\right)\cdot 2\cdot cos\left(\omega\cdot\frac{3}{2}\right) = \\
&= 4\cdot Sa\left(2\cdot\omega\right) + 2\cdot Sa\left(\frac{\omega}{2}\right)\cdot cos\left(\omega\cdot\frac{3}{2}\right)
\end{aligned}
$$

The Fourier transform of the $f(t) = \Pi\left(\frac{t}{4}\right) + \Pi\left(t-\frac{3}{2}\right) + \Pi\left(t+\frac{3}{2}\right)$ signal is equal to $F(\jmath\omega) = 4\cdot Sa\left(2\cdot\omega\right) + 2\cdot Sa\left(\frac{\omega}{2}\right)\cdot cos\left(\omega\cdot\frac{3}{2}\right)$.

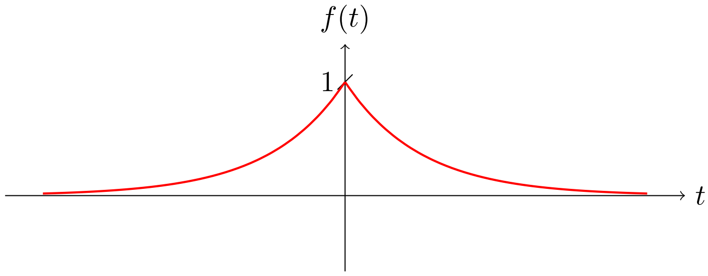
Final remarks:

Results derived for Solution I and II are equal to:

$$
\begin{aligned}
F(\jmath\omega) &= 8\cdot Sa\left(2\cdot\omega\right) - 2\cdot Sa\left(\omega\right) \\
F(\jmath\omega) &= 4\cdot Sa\left(2\cdot\omega\right) + 2\cdot Sa\left(\frac{\omega}{2}\right)\cdot cos\left(\omega\cdot\frac{3}{2}\right)
\end{aligned}
$$

It can be shown that this is the same transform (although the obtained equations describing the transform are different).

Task 3. Compute the Fourier transform of the $f(t) = e^{-|t|}$ signal shown below using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{A \cdot \mathbb{1}(t) \cdot e^{-a \cdot t}\} = \frac{A}{a + \jmath \cdot \omega}$.

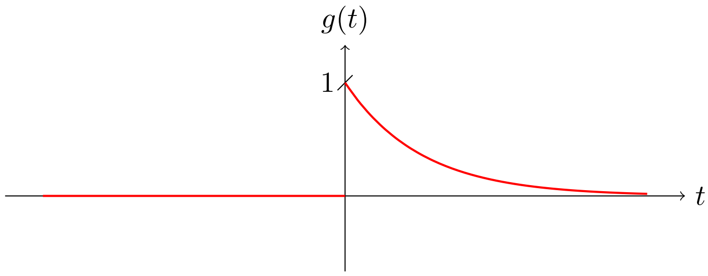


The signal $f(t)$, as a piecewise function, is given by:

$$f(t) = \begin{cases} e^{t} & \text{for } t \in (-\infty; 0) \\ e^{-t} & \text{for } t \in (0; \infty) \end{cases} \tag{3.42}$$

Let's denote $\mathbb{1}(t) \cdot e^{-t}$ signal as $g(t)$:

$$g(t) = \mathbb{1}(t) \cdot e^{-t} \tag{3.43}$$



Now, the $f(t)$ signal may be expressed as a linear combination of $g(t)$ and $g(-t)$ signals:

$$f(t) = g(t) + g(-t) \tag{3.44}$$

Based on linearity of the Fourier transformation, we can calculate transforms for elementary signals separately.

Let's calculate the Fourier transform $G(\jmath\omega)$ for the first signal $g(t)$.

We know that:

$$\mathcal{F}\{A \cdot \mathbb{1}(t) \cdot e^{-a \cdot t}\} = \frac{A}{a + \jmath \cdot \omega} \tag{3.45}$$

Substituting $A = 1$ and $a = 1$ we directly get the Fourier transform $G(\jmath\omega)$ for $g(t)$ signal:

$$A \cdot \mathbb{1}(t) \cdot e^{-a \cdot t} \xrightarrow{\mathcal{F}} \frac{A}{a + \jmath \cdot \omega}$$
$$\left\{ \begin{matrix} A = 1 \\ a = 1 \end{matrix} \right\}$$

$$g(t) = \mathbb{1}(t) \cdot e^{-t} \xrightarrow{\mathcal{F}} \frac{1}{1 + \jmath \cdot \omega} = G(\jmath\omega)$$

Based on the scaling theorem, we can derive the Fourier transform of the $g(-t)$ signal:

$$\begin{aligned} g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\ f(t) = g(\alpha \cdot t) &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha}) \end{aligned}$$

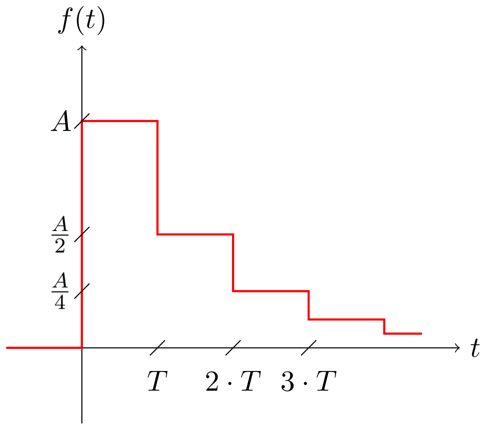
$$\begin{aligned} g(t) &\xrightarrow{\mathcal{F}} \frac{1}{1 + \jmath \cdot \omega} \\ g(-1 \cdot t) &\xrightarrow{\mathcal{F}} \frac{1}{|-1|} \cdot \frac{1}{1 + \jmath \cdot \frac{\omega}{-1}} \\ g(-t) &\xrightarrow{\mathcal{F}} \frac{1}{1} \cdot \frac{1}{1 - \jmath \cdot \omega} \\ g(-t) &\xrightarrow{\mathcal{F}} \frac{1}{1 - \jmath \cdot \omega} \end{aligned}$$

Finally, the Fourier transform for the $f(t)$ signal is equal to:

$$\begin{aligned} F(\jmath\omega) = \mathcal{F}\{f(t)\} &= \frac{1}{1 + \jmath \cdot \omega} + \frac{1}{1 - \jmath \cdot \omega} \\ F(\jmath\omega) &= \frac{1 - \jmath \cdot \omega + 1 + \jmath \cdot \omega}{(1 + \jmath \cdot \omega) \cdot (1 - \jmath \cdot \omega)} \\ F(\jmath\omega) &= \frac{2}{1 - \jmath^2 \cdot \omega^2} \\ F(\jmath\omega) &= \frac{2}{1 + \omega^2} \end{aligned}$$

The Fourier transform of the $f(t) = e^{-|t|}$ is equal to $F(\jmath\omega) = \frac{2}{1+\omega^2}$.

Task 4. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń, wiedząc że transformata sygnału prostokątnego $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.



Sygnał zbudowany jest z ciągu poprzesuwanych sygnałów prostokątnych o wykładniczo malejącej amplitudzie.

$$f(t) = \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot \Pi\left(\frac{t - \frac{T}{2} - n \cdot T}{T}\right)$$

Nasz sygnał jest nieskończoną sumą funkcji prostokątnych. Korzystając z liniowość transformaty fouriera

$$f_1(t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega)$$
$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$
$$f(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot F_1(\jmath\omega) + \beta \cdot F_2(\jmath\omega)$$

możemy napisać że:

$$F(\jmath\omega) = \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot H_n(\jmath\omega)$$

gdzie $H_n(\jmath\omega)$ jest transformatą Fouriera odpowiednio przesuniętego sygnału prostokątnego $h_n(t) = \Pi\left(\frac{t-\frac{T}{2}-n\cdot T}{T}\right)$.

Transformata sygnału $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa(\frac{\omega}{2})$. Postać funkcji $g(t)$ nie jest identyczna z postacią funkcji $h_n(t)$, funkcja różni się skalą i przesunięciem. Zacznijmy od skali.

Wyznaczanym transformaty funkcji przeskalowanej $h(t) = \Pi\left(\frac{t}{T}\right)$

Z twierdzenia o zmianie skali mamy

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$h(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

a wiec otrzymujemy

$$\begin{aligned} h(t) &= \Pi\left(\frac{t}{T}\right) = \\ &= \Pi\left(\frac{1}{T}\cdot t\right) = \\ &= g\left(\frac{1}{T}\cdot t\right) \end{aligned}$$

$$\alpha = \frac{1}{T}$$

$$\begin{aligned} H(\jmath\omega) &= \frac{1}{\frac{1}{T}}\cdot G\left(\frac{\jmath\omega}{\frac{1}{T}}\right) = \\ &= \frac{1}{\frac{1}{T}}\cdot Sa\left(\frac{\frac{\omega}{\frac{1}{T}}}{2}\right) = \\ &= T\cdot Sa\left(\frac{\omega\cdot T}{2}\right) \end{aligned}$$

Dalej wyznaczanym transformaty funkcji przeskalowanej i przesuniętej $h_n(t) = \Pi\left(\frac{t-\frac{T}{2}-n\cdot T}{T}\right)$ Korzystając z twierdzenia o przesunięciu w dziedzinie czasu

$$\begin{aligned} h_n(t) &\xrightarrow{\mathcal{F}} H_n(\jmath\omega) \\ h(t) = h_n(t-t_0) &\xrightarrow{\mathcal{F}} H(\jmath\omega) = H_n(\jmath\omega)\cdot e^{-\jmath\cdot\omega\cdot t_0} \end{aligned}$$

możemy napisać że:

$$\begin{aligned} H_n(\jmath\omega) &= H(\jmath\omega)\cdot e^{-\jmath\cdot\omega\cdot t_0} = \\ &= T\cdot Sa\left(\frac{\omega\cdot T}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\left(\frac{T}{2}+n\cdot T\right)} \end{aligned}$$

Ostatecznie wzór na transformatę sygnału $f(t)$ jest równy

$$\begin{aligned} F(\jmath\omega) &= \sum_{n=0}^{\infty}\frac{A}{2^n}\cdot H_n(\jmath\omega) = \\ &= \sum_{n=0}^{\infty}\frac{A}{2^n}\cdot T\cdot Sa\left(\frac{\omega\cdot T}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\left(\frac{T}{2}+n\cdot T\right)} = \\ &= \sum_{n=0}^{\infty}\frac{A}{2^n}\cdot T\cdot Sa\left(\frac{\omega\cdot T}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\frac{T}{2}}\cdot e^{-\jmath\cdot\omega\cdot n\cdot T} = \\ &= \sum_{n=0}^{\infty} T\cdot Sa\left(\frac{\omega\cdot T}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\frac{T}{2}}\cdot\frac{A}{2^n}\cdot e^{-\jmath\cdot\omega\cdot n\cdot T} = \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot A \cdot \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^{n} = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^{n}
\end{aligned}
$$

Można zauważyć że suma w rozwiązaniu to szereg geometryczny. Z wzoru na sumę szeregu geometrycznego mamy

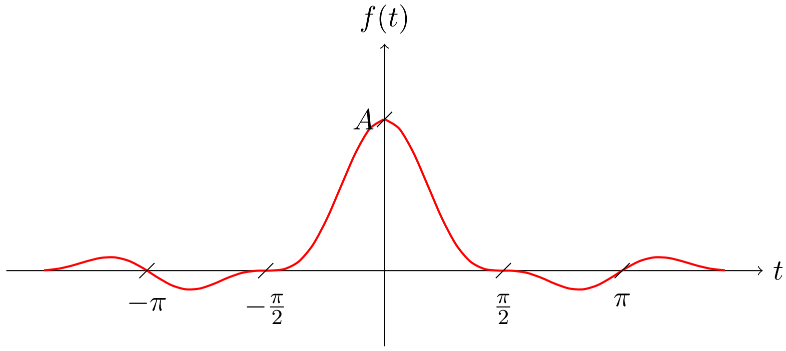
$$
\begin{aligned}
F(\jmath\omega) &= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^{n} = \\
&= \left\{\textstyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}\right\} = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ równa się:

$$
F(\jmath\omega) = A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}
$$

Task 5. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń, wiedząc że transformata sygnału prostokątnego $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$..

$$f(t) = A \cdot \frac{cos^2(\omega_0 \cdot t)}{\omega_0 \cdot t} \cdot sin(2 \cdot \omega_0 \cdot t)$$



Przepismy wzór naszej funkcji następująco

$$\begin{aligned}
f(t) &= A \cdot \frac{cos^2(\omega_0 \cdot t)}{\omega_0 \cdot t} \cdot sin(2 \cdot \omega_0 \cdot t) = \\
&= A \cdot cos^2(\omega_0 \cdot t) \frac{sin(2 \cdot \omega_0 \cdot t)}{\omega_0 \cdot t} = \\
&= A \cdot cos^2(\omega_0 \cdot t) \frac{2 \cdot sin(2 \cdot \omega_0 \cdot t)}{2 \cdot \omega_0 \cdot t} = \\
&= \left\{ Sa(x) = \frac{sin(x)}{x} \right\} = \\
&= A \cdot cos^2(\omega_0 \cdot t)\, 2 \cdot Sa(2 \cdot \omega_0 \cdot t) = \\
&= 2 \cdot A \cdot Sa(2 \cdot \omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)
\end{aligned}$$

Nasz sygnał jest iloczynem pewnej funkcji $h(t)$ oraz cosinusów

$$\begin{aligned}
f(t) &= 2 \cdot A \cdot Sa(2 \cdot \omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) = \\
f(t) &= 2 \cdot A \cdot h(t) \cdot cos(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)
\end{aligned}$$

gdzie

$$h(t) = Sa(2 \cdot \omega_0 \cdot t)$$

Wyznaczmy transformatę sygnału $h(t)$. Z treści zadania wiemy że transformata sygnału $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.

Postać funkcji $g(t)$ nie jest identyczna z postacią funkcji $h(t)$, funkcja różni się skalą. Wyznaczanym transformaty funkcji przeskalowanej $h(t) = \Pi\left(\frac{t}{T}\right)$

Z twierdzenia o zmianie skali mamy

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$h(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

a wiec otrzymujemy

$$\begin{aligned} h(t) &= \Pi\left(\frac{t}{T}\right) = \\ &= \Pi\left(\frac{1}{T} \cdot t\right) = \\ &= g\left(\frac{1}{T} \cdot t\right) \end{aligned}$$

$$\alpha = \frac{1}{T}$$

$$\begin{aligned} H(\jmath\omega) &= \frac{1}{\frac{1}{T}} \cdot G\left(\frac{\jmath\omega}{\frac{1}{T}}\right) = \\ &= \frac{1}{\frac{1}{T}} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{T}}}{2}\right) = \\ &= T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \end{aligned}$$

Dalej wyznaczanym transformaty funkcji przeskalowanej i przesuniętej $h_n(t) = \Pi\left(\frac{t - \frac{T}{2} - n \cdot T}{T}\right)$ Korzystając z twierdzenia o przesunięciu w dziedzinie czasu

$$h_n(t) \xrightarrow{\mathcal{F}} H_n(\jmath\omega)$$
$$h(t) = h_n(t - t_0) \xrightarrow{\mathcal{F}} H(\jmath\omega) = H_n(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0}$$

możemy napisać że:

$$\begin{aligned} H_n(\jmath\omega) &= H(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\ &= T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \left(\frac{T}{2} + n \cdot T\right)} \end{aligned}$$

Ostatecznie wzór na transformatę sygnału $f(t)$ jest równy

$$\begin{aligned} F(\jmath\omega) &= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot H_n(\jmath\omega) = \\ &= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \left(\frac{T}{2} + n \cdot T\right)} = \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot e^{-\jmath \cdot \omega \cdot n \cdot T} = \\
&= \sum_{n=0}^{\infty} T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{A}{2^n} \cdot e^{-\jmath \cdot \omega \cdot n \cdot T} = \\
&= \sum_{n=0}^{\infty} T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot A \cdot \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n
\end{aligned}
$$

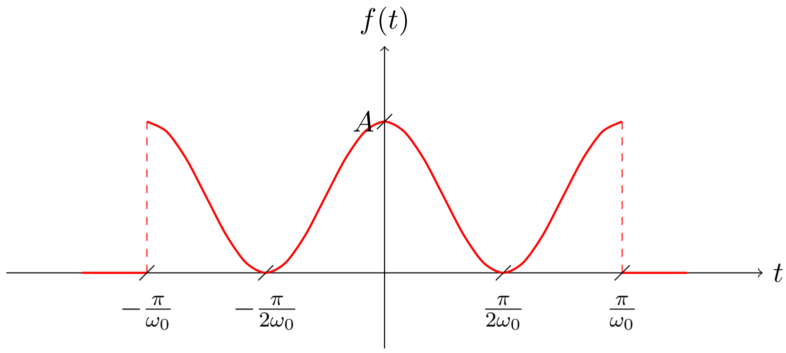
Można zauważyć że suma w rozwiązaniu to szereg geometryczny. Z wzoru na sumę szeregu geometrycznego mamy

$$
\begin{aligned}
F(\jmath\omega) &= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n = \\
&= \left\{\sum_{n=0}^{\infty} q^n = \tfrac{1}{1-q}\right\} = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ równa się:

$$
F(\jmath\omega) = A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}
$$

Task 6. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń, wiedząc że transformata sygnału prostokątnego $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.



Zacznijmy od napisania wzoru sygnału przedstawionego na rysunku

$$f(t) = \begin{cases} 0 & t \in \left(-\infty; -\frac{\pi}{\omega_0}\right) \\ A \cdot cos^2\left(\omega_0 \cdot t\right) & t \in \left(-\frac{\pi}{\omega_0}; \frac{\pi}{\omega_0}\right) \\ 0 & t \in \left(\frac{\pi}{\omega_0}; \infty\right) \end{cases}$$

Co możemy zapisać za pomocą sygnałów elementarnych jako

$$f(t) = A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2\left(\omega_0 \cdot t\right)$$

Nasz sygnał jest iloczynem pewnej funkcji $h(t)$ oraz cosinusów

$$\begin{aligned} f(t) &= A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2\left(\omega_0 \cdot t\right) = \\ &= A \cdot h(t) \cdot cos^2\left(\omega_0 \cdot t\right) = \end{aligned}$$

gdzie

$$h(t) = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right)$$

Wyznaczmy transformatę sygnału $h(t)$. Z treści zadania wiemy że transformata sygnału $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.

Postać funkcji $g(t)$ nie jest identyczna z postacią funkcji $h(t)$, funkcja różni się skalą. Wyznaczanym transformaty funkcji przeskalowanej $h(t) = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right)$

Z twierdzenia o zmianie skali mamy

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$h(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

a wiec otrzymujemy

$$\begin{aligned} h(t) = \Pi\left(\frac{t}{T}\right) = \\ = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) = \\ = \Pi\left(t \cdot \frac{\omega_0}{2\pi}\right) = \\ = g\left(t \cdot \frac{\omega_0}{2\pi}\right) \end{aligned}$$

gdzie

$$\alpha = \frac{\omega_0}{2\pi}$$

a więc

$$\begin{aligned} H(\jmath\omega) &= \frac{1}{\frac{\omega_0}{2\pi}} \cdot G\left(\frac{\jmath\omega}{\frac{\omega_0}{2\pi}}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot G\left(\frac{\jmath\omega \cdot 2\pi}{\omega_0}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot Sa\left(\frac{\frac{\omega \cdot 2\pi}{\omega_0}}{2}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot Sa\left(\omega \cdot \frac{\pi}{\omega_0}\right) \end{aligned}$$

A więc transformata sygnału $h(t)$ jest równa $H(\jmath\omega) = \frac{2\pi}{\omega_0} \cdot Sa\left(\omega \cdot \frac{\pi}{\omega_0}\right)$
Wróćmy do wzoru sygnału i przedstawmy go następująco

$$\begin{aligned} f(t) &= A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2\left(\omega_0 \cdot t\right) = \\ &= A \cdot h(t) \cdot cos^2\left(\omega_0 \cdot t\right) = \\ &= A \cdot h(t) \cdot cos\left(\omega_0 \cdot t\right) \cdot cos\left(\omega_0 \cdot t\right) = \\ &= A \cdot k(t) \cdot cos\left(\omega_0 \cdot t\right) = \end{aligned}$$

gdzie

$$\begin{aligned} k(t) &= h(t) \cdot cos\left(\omega_0 \cdot t\right) = \\ &= \left\{cos\left(x\right) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\right\} = \end{aligned}$$

$$
\begin{aligned}
&= h(t) \cdot \frac{e^{\jmath \cdot \omega_0 \cdot t} + e^{-\jmath \cdot \omega_0 \cdot t}}{2} = \\
&= \frac{1}{2} \left(h(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} + h(t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \right)
\end{aligned}
$$

Wyznaczmy transformatę sygnału $k(t)$. Korzystając z twierdzenia o modulacji mamy

$$
\begin{gathered}
h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega) \\
k(t) = h(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} K(\jmath\omega) = H(\jmath(\omega - \omega_0))
\end{gathered}
$$

a wiec transformata sygnału $k(t)$ wynosi:

$$
\begin{aligned}
K(\jmath\omega) &= \frac{1}{2} \left(H(\jmath(\omega - \omega_0)) + H(\jmath(\omega + \omega_0)) \right) = \\
&= \frac{1}{2} \left(\frac{2\pi}{\omega_0} \cdot Sa\left((\omega - \omega_0) \cdot \frac{\pi}{\omega_0} \right) + \frac{2\pi}{\omega_0} \cdot Sa\left((\omega + \omega_0) \cdot \frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\pi}{\omega_0} \cdot Sa\left((\omega - \omega_0) \cdot \frac{\pi}{\omega_0} \right) + \frac{\pi}{\omega_0} \cdot Sa\left((\omega + \omega_0) \cdot \frac{\pi}{\omega_0} \right)
\end{aligned}
$$

Wróćmy do wzoru sygnału $f(t)$

$$
\begin{aligned}
f(t) &= A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi} \right) \cdot cos^2(\omega_0 \cdot t) = \\
&= A \cdot h(t) \cdot cos^2(\omega_0 \cdot t) = \\
&= A \cdot h(t) \cdot cos(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) = \\
&= A \cdot k(t) \cdot cos(\omega_0 \cdot t) = \\
&= \left\{ cos(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= A \cdot k(t) \cdot \frac{e^{\jmath \cdot \omega_0 \cdot t} + e^{-\jmath \cdot \omega_0 \cdot t}}{2} = \\
&= \frac{A}{2} \left(k(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} + k(t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \right)
\end{aligned}
$$

Znając transformatę sygnału $k(t)$ i korzystając z twierdzenia o modulacji możemy wyznaczyć transformatę sygnału $f(t)$.

$$
\begin{gathered}
k(t) \xrightarrow{\mathcal{F}} K(\jmath\omega) \\
f(t) = k(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F(\jmath\omega) = K(\jmath(\omega - \omega_0))
\end{gathered}
$$

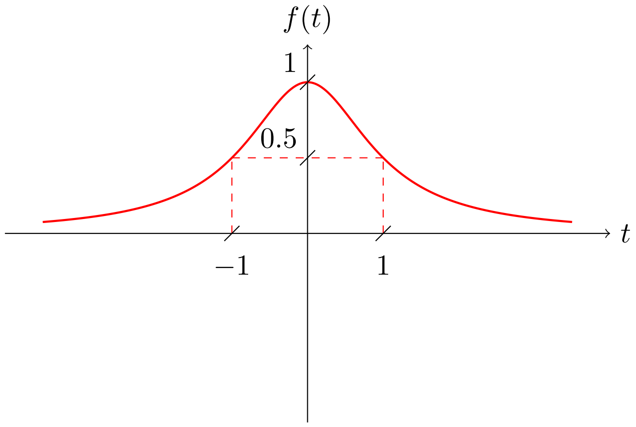
a więc transformata sygnału $f(t)$ wynosi

$$
\begin{aligned}
F(\jmath\omega) &= \frac{A}{2} \left(K(\jmath(\omega - \omega_0)) + K(\jmath(\omega + \omega_0)) \right) = \\
&= \frac{A}{2} \left(\frac{\pi}{\omega_0} \cdot Sa\left((\omega - \omega_0 - \omega_0) \cdot \frac{\pi}{\omega_0} \right) + \frac{\pi}{\omega_0} \cdot Sa\left((\omega - \omega_0 + \omega_0) \cdot \frac{\pi}{\omega_0} \right) + \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+\omega_0-\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+\omega_0+\omega_0)\cdot\frac{\pi}{\omega_0}\right)\Bigg)=\\
&=\frac{A}{2}\left(\frac{\pi}{\omega_0}\cdot Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\right.\\
&\left.+\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)=\\
&=\frac{A}{2}\left(\frac{\pi}{\omega_0}\cdot Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\right.\\
&\left.+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)=\\
&=\frac{A\cdot\pi}{2\cdot\omega_0}\left(Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ wynosi $F(\jmath\omega)=\frac{A\cdot\pi}{2\cdot\omega_0}\left(Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+Sa\left((\omega+\right.\right.$

Task 7. Compute the Fourier transform of the $f(t) = \frac{1}{1+t^2}$ signal shown below using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2}$.



We know that:

$$\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2} \tag{3.46}$$

Let's denote $g(t) = e^{-|t|}$ and $G(\jmath\omega) = \frac{2}{1+\omega^2}$.
Based on duality theorem we can derive the following transform:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$h(t) = G(t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = 2\pi \cdot g(-\omega)$$

$$\begin{aligned} H(\jmath\omega) &= 2\pi \cdot g(-\omega) = \\ &= 2\pi \cdot e^{-|-\omega|} = \\ &= 2\pi \cdot e^{-|\omega|} \end{aligned}$$

Unfortunately, the $h(t) = \frac{2}{1+t^2}$ signal in not equal to $f(t) = \frac{1}{1+t^2}$ signal. But:

$$f(t) = \frac{1}{2} \cdot h(t) \tag{3.47}$$

Based on linearity theorem we can calculate:

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$
$$f(t) = \alpha \cdot h(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot H(\jmath\omega)$$

Finally, we get:

$$f(t) = \frac{1}{2} \cdot \frac{2}{1+t^2}$$

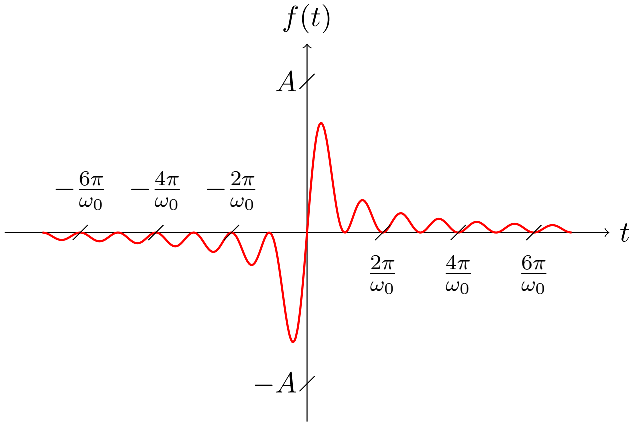
$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{2} \cdot 2\pi \cdot e^{-|\omega|} = \\
&= \pi \cdot e^{-|\omega|}
\end{aligned}
$$

The Fourier transform of the $f(t) = \frac{1}{1+t^2}$ signal is equal to $F(\jmath\omega) = \pi \cdot e^{-|\omega|}$.

Task 8. Compute the Fourier transform of the $f(t) = Sa(\omega_0 \cdot t) \cdot sin(\omega_0 \cdot t)$ signal using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa(\omega_0 \cdot t) \cdot sin(\omega_0 \cdot t) \tag{3.48}$$

$$\Pi(t) \xrightarrow{F} Sa\left(\frac{\omega}{2}\right) \tag{3.49}$$



Firstly, note that using Euler's identity we can write:

$$\begin{aligned}
f(t) &= Sa(\omega_0 \cdot t) \cdot sin(\omega_0 \cdot t) = \\
&= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= Sa(\omega_0 \cdot t) \cdot \frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} = \\
&= \frac{1}{2 \cdot \jmath} \cdot \left(Sa(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} - Sa(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \right) = \\
&= \left\{ \begin{aligned} f_1(t) &= Sa(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\ f_2(t) &= Sa(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \end{aligned} \right\} = \\
&= \frac{1}{2 \cdot \jmath} \cdot (f_1(t) - f_2(t))
\end{aligned}$$

Please note, that $f_1(t)$ and $f_2(t)$ signals are product of Sa and complex exponential functions:

$$\begin{aligned}
f_1(t) &= Sa(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\
f_2(t) &= Sa(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{-\jmath \cdot \omega_0 \cdot t}
\end{aligned}$$

Knowing the Fourier transform of $g(t) = Sa(\omega_0 \cdot t)$ we can use the modulation theorem to derive the Fourier transform of $f_x(t)$:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$f(t) = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F(\jmath\omega) = G(\jmath\,(\omega - \omega_0))$$

We know that $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$. Based on time-frequency duality theorem we can derive transform for $g(t)$ signal:

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$
$$g(t) = H(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = 2\pi \cdot h(-\omega)$$

Let's start with deriving Fourier transform for Sa signal using $h(t) = \Pi(t)$:

$$h(t) = \Pi(t) \xrightarrow{F} H(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$$
$$g_1(t) = H(t) = Sa\left(\frac{t}{2}\right) \xrightarrow{F} G_1(\jmath\omega) = 2\pi \cdot h(-\jmath\omega) = 2\pi \cdot \Pi\left(-\omega\right) = 2\pi \cdot \Pi\left(\omega\right)$$

We derived Fourier transform for $g_1(t)$ signal. However, this is not exact the same function as $g(t)$.

$$\begin{aligned} g(t) &= Sa\left(\omega_0 \cdot t\right) = \\ &= Sa\left(\omega_0 \cdot t \cdot \frac{2}{2}\right) = \\ &= Sa\left(2 \cdot \omega_0 \cdot \frac{t}{2}\right) = \\ &= Sa\left(\frac{2 \cdot \omega_0 \cdot t}{2}\right) = \\ &= \left\{a = 2 \cdot \omega_0\right\} = \\ &= Sa\left(\frac{a \cdot t}{2}\right) = \\ &= g_1(a \cdot t) \end{aligned}$$

Now, we can calculate the Fourier transform for $g(t) = g_1(a \cdot t)$ signal using the scaling theorem:

$$g_1(t) \xrightarrow{\mathcal{F}} G_1(\jmath\omega)$$
$$g(t) = g_1(\alpha \cdot t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha})$$

$$\begin{aligned} G(\jmath\omega) &= \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha}) = \\ &= \left\{\alpha = 2 \cdot \omega_0\right\} = \\ &= \frac{1}{|2 \cdot \omega_0|} \cdot G_1(\frac{\omega}{2 \cdot \omega_0}) = \\ &= \left\{G_1(\jmath\omega) = 2\pi \cdot \Pi\left(\omega\right)\right\} = \end{aligned}$$

$$\begin{aligned}&= \frac{1}{2\cdot\omega_0}\cdot 2\pi\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right) = \\ &= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)\end{aligned}$$

As a result the Fourier transform for $g(t) = Sa\left(\omega_0 \cdot t\right)$ signal is equal to $G(\jmath\omega) = \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)$.
Now, we are able to derive transforms for:

$$\begin{aligned}f_1(t) &= Sa\left(\omega_0\cdot t\right)\cdot e^{\jmath\cdot\omega_0\cdot t}\\ f_2(t) &= Sa\left(\omega_0\cdot t\right)\cdot e^{-\jmath\cdot\omega_0\cdot t}\end{aligned}$$

Based on the modulation theorem:

$$\begin{aligned}g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega)\\ f_1(t) = g(t)\cdot e^{\jmath\cdot\omega_0\cdot t} &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) = G(\jmath\left(\omega-\omega_0\right))\end{aligned}$$

we get:

$$\begin{aligned}F_1(\jmath\omega) &= G\left(\jmath\left(\omega-\omega_0\right)\right) = \\ &= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right)\end{aligned}$$

$$\begin{aligned}F_2(\jmath\omega) &= G\left(\jmath\left(\omega+\omega_0\right)\right) = \\ &= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\end{aligned}$$

Finally, based on the linearity theorem:

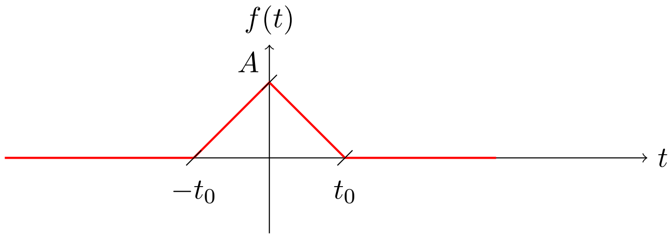
$$\begin{aligned}f_1(t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega)\\ f_2(t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega)\\ f(t) = \alpha\cdot f_1(t) + \beta\cdot f_2(t) &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha\cdot F_1(\jmath\omega) + \beta\cdot F_2(\jmath\omega)\end{aligned}$$

we get:

$$\begin{aligned}F(\jmath\omega) &= F_1(\jmath\omega) - F_2(\jmath\omega) = \\ &= \frac{1}{2\cdot\jmath}\cdot\left(\frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) - \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)\end{aligned}$$

The Fourier transform of $f(t) = Sa\left(\omega_0\cdot t\right)\cdot sin\left(\omega_0\cdot t\right)$ is equal to $F(\jmath\omega) = \frac{1}{2\cdot\jmath}\cdot\left(\frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) - \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)$.

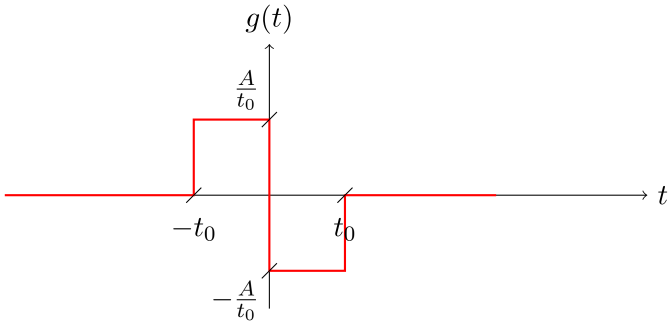
Task 9. Compute the Fourier transform of the $f(t)$ signal given below using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.



First of all, describe the $f(t)$ signal using the elementary signals:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.50}$$

Let's derive derivative of the $f(t)$ signal as $g(t) = \frac{\partial}{\partial t} f(t)$:



Using the elementary signals we can write:

$$g(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) - \frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0}) \tag{3.51}$$

You can check that by integrating the $g(t)$ signal we'll get the $f(t)$ signal:

$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.52}$$

Therefore, the Fourier transform of the $f(t)$ signal can be determined from the integration theorem. In this case we will integrate the $g(t)$ signal:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.53}$$

In order to derive $F(\jmath\omega)$ we have to calculate the $G(\jmath\omega)$ transform of the $g(t)$ signal:

$$g(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) - \frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0}) \tag{3.54}$$

Based on linearity of the Fourier transformation, we can calculate transforms for gate signals separately:

$$g(t) = g_1(t) - g_2(t) \tag{3.55}$$

where:

$$g_1(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0})$$
$$g_2(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0})$$

Let's calculate the Fourier transform $G_1(\jmath\omega)$ for the first signal $g_1(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot(-\frac{t_0}{2})} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} \frac{A}{t_0} \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} A \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$

The Fourier transform of $g_1(t)$ signal is equal to:

$$G_1(\jmath\omega) = \mathcal{F}\{g_1(t)\} = A \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \tag{3.56}$$

Now, we calculate the Fourier transform $G_2(\jmath\omega)$ for the second signal $g_2(t)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot(\frac{t_0}{2})} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} \frac{A}{t_0} \cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} A \cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$

The Fourier transform of $g_2(t)$ signal is equal to:

$$G_2(\jmath\omega) = \mathcal{F}\{g_2(t)\} = A \cdot e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \tag{3.57}$$

Finally, the Fourier transform for $g(t)$ signal is equal to:

$$\begin{aligned}
G(\jmath\omega) = \mathcal{F}\{g(t)\} = A \cdot e^{\jmath \cdot \omega \cdot \frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) - A \cdot e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \\
G(\jmath\omega) = A \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot \left(e^{\jmath \cdot \omega \cdot \frac{t_0}{2}} - e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}}\right) \\
G(\jmath\omega) = A \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot \left(e^{\jmath \cdot \omega \cdot \frac{t_0}{2}} - e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}}\right) \\
\left\{ sin(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} \\
G(\jmath\omega) = A \cdot 2 \cdot \jmath \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot sin\left(\frac{\omega \cdot t_0}{2}\right)
\end{aligned}$$

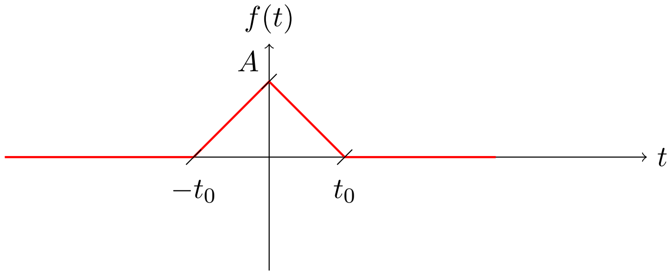
We derived the $G(\jmath\omega)$ transform. Now, based on the integration theorem, we can calculate the Fourier transform of the $f(t)$ signal.

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.58}$$

$$\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot A \cdot 2 \cdot \jmath \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot sin\left(\frac{\omega \cdot t_0}{2}\right) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{array}{l} G(0) = A \cdot 2 \cdot \jmath \cdot Sa\left(\frac{0 \cdot t_0}{2}\right) \cdot sin\left(\frac{0 \cdot t_0}{2}\right) \\ G(0) = A \cdot 2 \cdot \jmath \cdot Sa(0) \cdot sin(0) \\ G(0) = A \cdot 2 \cdot \jmath \cdot 1 \cdot 0 \\ G(0) = 0 \end{array} \right\} = \\
&= \frac{1}{\jmath \cdot \omega} \cdot A \cdot 2 \cdot \jmath \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot sin\left(\frac{\omega \cdot t_0}{2}\right) = \\
&= \left\{ \tfrac{sin(x)}{x} = Sa(x) \right\} = \\
&= \frac{A \cdot 2 \cdot t_0}{\omega \cdot t_0} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot sin\left(\frac{\omega \cdot t_0}{2}\right) = \\
&= A \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) = \\
&= A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})
\end{aligned}$$

The Fourier transform of $f(t) = A \cdot \Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega) = A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$.

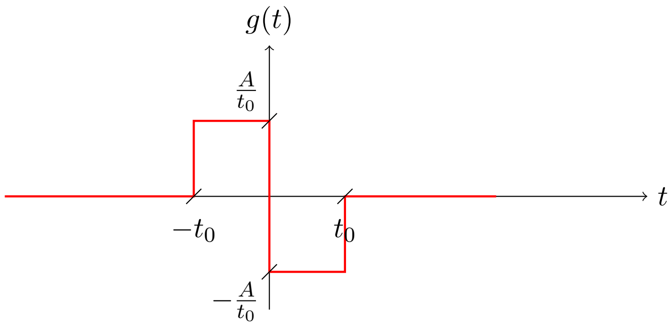
Task 10. Compute the Fourier transform of the $f(t)$ signal given below using theorems describing the properties of Fourier transformation. Exploit the sampling property of Dirac delta.



First of all, describe the $f(t)$ signal using the elementary signals:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.59}$$

Let's derive derivative of the $f(t)$ signal as $g(t) = \frac{\partial}{\partial t} f(t)$:



You can check that by integrating the $g(t)$ signal we'll get the $f(t)$ signal:

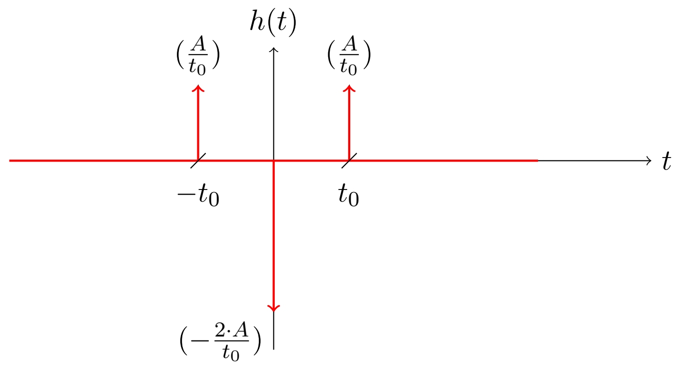
$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.60}$$

Therefore, the Fourier transform of the $f(t)$ signal can be determined from the integration theorem. In this case we will integrate the $g(t)$ signal:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.61}$$

Is it possible to calculate derivative again to simply the analysed signal? Let's calculate the derivative of the $g(t)$ signal, so the second derivative of the $f(t)$ signal:

$$h(t) = \frac{\partial}{\partial t} g(t) = \frac{\partial^2}{\partial t^2} f(t) \tag{3.62}$$



Using the elementary signals we can write:

$$h(t) = \frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0)) \tag{3.63}$$

You can check that by integrating the $h(t)$ signal we'll get the $g(t)$ signal:

$$g(t) = \int_{-\infty}^{t} h(x) \cdot dx \tag{3.64}$$

Therefore, the Fourier transform of the $g(t)$ signal can be determined from the integration theorem. In this case we will integrate the $h(t)$ signal:

$$G(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) \tag{3.65}$$

In order to derive $G(\jmath\omega)$ we have to calculate the $H(\jmath\omega)$ transform of the $h(t)$ signal:

$$h(t) = \frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0)) \tag{3.66}$$

Based on the linearity theorem, we can derive transforms separately for each Dirac delta signal:

$$\begin{aligned}
H(\jmath\omega) &= \mathcal{F}\{h(t)\} = \\
&= \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0))\right\} = \\
&= \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (-t_0))\right\} - \mathcal{F}\left\{\frac{2 \cdot A}{t_0} \cdot \delta(t)\right\} + \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (t_0))\right\} = \\
&= \frac{A}{t_0} \cdot \mathcal{F}\left\{\delta(t - (-t_0))\right\} - \frac{2 \cdot A}{t_0} \cdot \mathcal{F}\left\{\delta(t)\right\} + \frac{A}{t_0} \cdot \mathcal{F}\left\{\delta(t - (t_0))\right\} = \\
&= \left\{\begin{matrix} \delta(t) \xrightarrow{\mathcal{F}} 1 \\ \delta(t - (-t_0)) \xrightarrow{\mathcal{F}} 1 \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} \\ \delta(t - (t_0)) \xrightarrow{\mathcal{F}} 1 \cdot e^{-\jmath \cdot \omega \cdot t_0} \end{matrix}\right\} = \\
&= \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - \frac{2 \cdot A}{t_0} \cdot 1 + \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{A}{t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - 2 + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \left\{cos(x) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{t_0} \cdot (2 \cdot cos(\omega \cdot t_0) - 2) = \\
&= \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1)
\end{aligned}
$$

The Fourier transform of the $h(t)$ signal is equal to:

$$
H(\jmath\omega) = \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1) \tag{3.67}
$$

We derived the $H(\jmath\omega)$ transform. Now, based on the integration theorem, we can calculate the $G(\jmath\omega)$ Fourier transform.

$$
\begin{aligned}
G(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \left\{ \begin{array}{l} H(0) = \frac{2 \cdot A}{t_0} \cdot (cos(0 \cdot t_0) - 1) \\ H(0) = \frac{2 \cdot A}{t_0} \cdot (cos(0) - 1) \\ H(0) = \frac{2 \cdot A}{t_0} \cdot (1 - 1) \\ H(0) = 0 \end{array} \right\} = \\
&= \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1)
\end{aligned}
$$

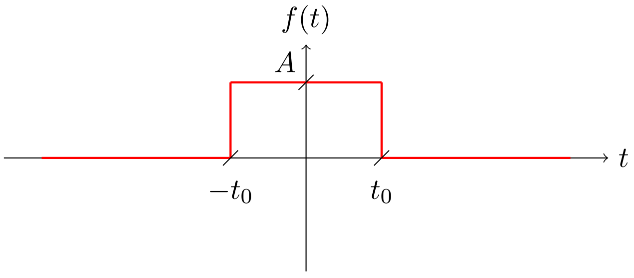
We derived the $G(\jmath\omega)$ transform. Using the integration theorem once again, we can calculate the $F(\jmath\omega)$ Fourier transform.

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{array}{l} G(0) = \frac{2 \cdot A}{\jmath \cdot 0 \cdot t_0} \cdot (cos(0 \cdot t_0) - 1) \\ G(0) = \frac{0}{0}!!! \\ G(0) = \int_{-\infty}^{\infty} g(t) \cdot dt = \int_{-t_0}^{0} \frac{A}{t_0} \cdot dt + \int_{0}^{t_0} (-\frac{A}{t_0}) \cdot dt \\ G(0) = \frac{A}{t_0} \cdot (0 - (-t_0)) - \frac{A}{t_0} \cdot (t_0 - 0) = A - A \\ G(0) = 0 \end{array} \right\} = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) = \\
&= \frac{2 \cdot A}{\jmath^2 \cdot \omega^2 \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) = \\
&= \frac{2 \cdot A}{\omega^2 \cdot t_0} \cdot (1 - cos(\omega \cdot t_0)) = \\
&= \left\{ \begin{array}{l} sin^2(x) = \frac{1}{2} - \frac{1}{2} \cdot cos(2 \cdot x) \\ cos(2 \cdot x) = 1 - 2 \cdot sin^2(x) \end{array} \right\} = \\
&= \frac{2 \cdot A}{\omega^2 \cdot t_0} \cdot \left(1 - 1 + 2 \cdot sin^2 \left(\frac{\omega \cdot t_0}{2} \right) \right) =
\end{aligned}
$$

$$= \frac{4 \cdot A}{\omega^2 \cdot t_0} \cdot sin^2\left(\frac{\omega \cdot t_0}{2}\right) =$$
$$= \left\{\frac{sin(x)}{x} = Sa(x)\right\} =$$
$$= A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$$

The Fourier transform of $f(t) = A \cdot \Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega) = A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$.

Task 11. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



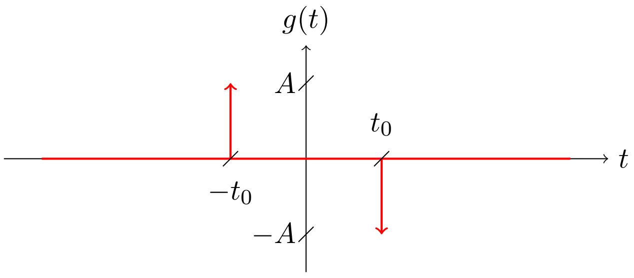
$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ A & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.68}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0) \tag{3.69}$$

czyli po prostu

$$g(t) = f'(t) = A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0) \tag{3.70}$$



Wyznaczanie transformaty sygnału $g(t)$ złożonego z delt diracka jest znacznie prostsze.

$$G(\jmath\omega) = \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.71}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned} G(\jmath\omega) &= \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\ &= \int_{-\infty}^{\infty} (A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\ &= \int_{-\infty}^{\infty} \left(A \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} - A \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t}\right) \cdot dt = \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} A \cdot \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \int_{-\infty}^{\infty} A \cdot \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - A \cdot \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot e^{\jmath \cdot \omega \cdot t_0} - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= A \cdot 2 \cdot \jmath \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= A \cdot 2 \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0)
\end{aligned}
$$

Transformata sygnału $g(t)$ to $G(\jmath\omega) = \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $f(t)$ na podstawie transformaty sygnału $g(t) = f'(t)$

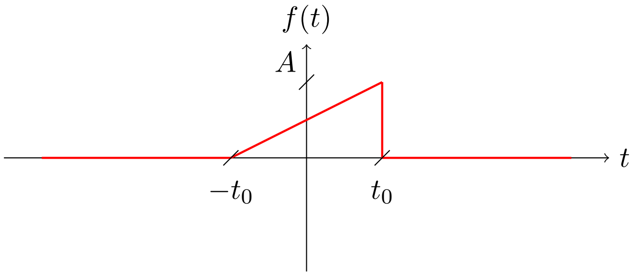
$$
g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)
$$
$$
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
$$

Podstawiając obliczona wcześniej transformatę $G(\jmath\omega)$ sygnału $g(t)$ otrzymujemy transformatę $F(\jmath\omega)$ sygnału $f(t)$

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(0) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot sin\,(0 \cdot t_0) = \\
&= \frac{1}{\omega} \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot sin\,(0) = \\
&= \frac{1}{\omega} \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) \cdot \frac{t_0}{t_0} + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot 0 = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\,(\omega \cdot t_0)}{\omega \cdot t_0} + 0 = \\
&= \left\{ Sa\,(x) = \tfrac{sin(x)}{x} \right\} = \\
&= 2 \cdot A \cdot t_0 \cdot Sa\,(\omega \cdot t_0)
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = 2 \cdot A \cdot t_0 \cdot Sa\,(\omega \cdot t_0)$.

Task 12. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



W pierwszej kolejności trzeba wyznaczyć jawną postać równań opisujących funkcję $f(t)$.

W tym celu wyznaczamy równanie prostej na odcinku $(-t_0, t_0)$

Ogólne równanie prostej to:

$$f(t) = m \cdot t + b \tag{3.72}$$

Dla rozważanego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(-t_0, 0)$ oraz (t_0, A). Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} 0 = m \cdot (-t_0) + b \\ A = m \cdot t_0 + b \end{cases}$$
$$\begin{cases} -b = m \cdot (-t_0) \\ A = m \cdot t_0 + b \end{cases}$$
$$\begin{cases} \frac{b}{t_0} = m \\ A = \frac{b}{t_0} \cdot t_0 + b \end{cases}$$
$$\begin{cases} \frac{b}{t_0} = m \\ A = b + b \end{cases}$$
$$\begin{cases} \frac{b}{t_0} = m \\ A = 2 \cdot b \end{cases}$$
$$\begin{cases} \frac{b}{t_0} = m \\ \frac{A}{2} = b \end{cases}$$
$$\begin{cases} \frac{A}{2 \cdot t_0} = m \\ \frac{A}{2} = b \end{cases}$$

Równianie prostej dla t z zakresu $(-t_0, t_0)$ to:

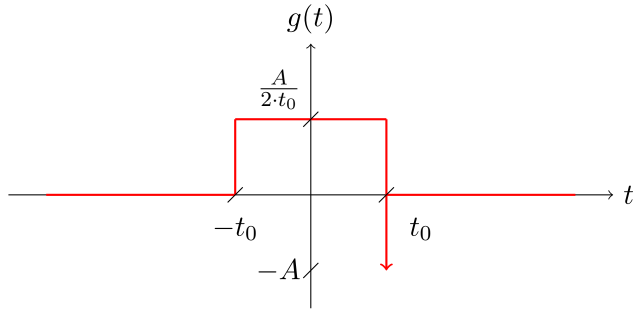
$$f(t) = \frac{A}{2 \cdot t_0} \cdot t + \frac{A}{2}$$

Podsumowując, sygnal $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} \cdot t + \frac{A}{2} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.73}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} - A \cdot \delta(t - t_0) \tag{3.74}$$

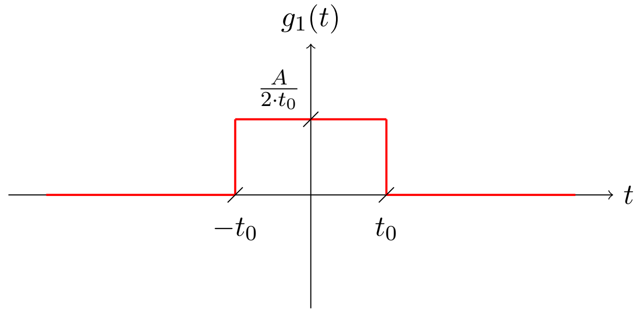


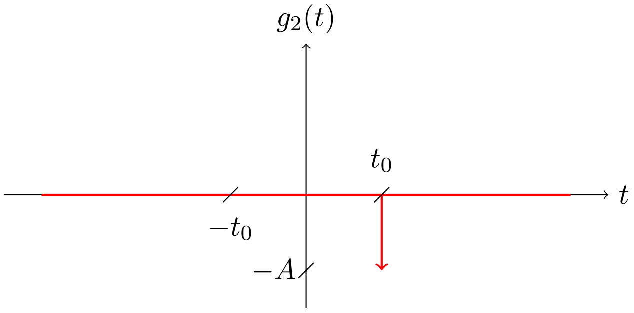
Funkcja $g(t)$ składa się z dwóch sygnałów $g_1(t)$ i $g_2(t)$

$$g(t) = g_1(t) + g_2(t) \tag{3.75}$$

$$g_1(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.76}$$

$$g_2(t) = -A \cdot \delta(t - t_0) \tag{3.77}$$





Wyznaczenie transformaty sygnału $g_2(t)$ złożonego z delty diracka jest znacznie prostsze.

$$G_2(\jmath\omega) = \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.78}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

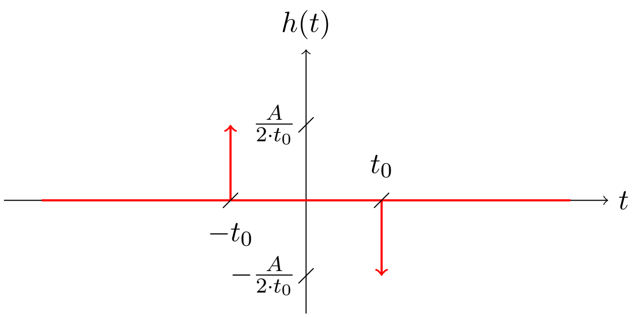
$$\begin{aligned}
G_2(\jmath\omega) &= \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (-A \cdot \delta(t - t_0)) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= -A \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= -A \cdot e^{-\jmath\cdot\omega\cdot t_0}
\end{aligned}$$

Transformata sygnału $g_2(t)$ to $G_2(\jmath\omega) = -A \cdot e^{-\jmath\cdot\omega\cdot t_0}$
Funkcja $g_1(t)$ jest jeszcze zbyt złożona tak wiec wyznaczamy pochodną raz jeszcze

$$h(t) = g_1'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + \frac{A}{2 \cdot t_0}\delta(t + t_0) - \frac{A}{2 \cdot t_0}\delta(t - t_0) \tag{3.79}$$

czyli po prostu

$$h(t) = g_1'(t) = \frac{A}{2 \cdot t_0}\delta(t + t_0) - \frac{A}{2 \cdot t_0}\delta(t - t_0) \tag{3.80}$$



Wyznaczanie transformaty sygnału $h(t)$ złożonego z delt diracka jest znacznie prostsze.

$$H(\jmath\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.81}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
H(\jmath\omega) &= \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(\frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) - \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(\frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} - \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \right) \cdot dt = \\
&= \int_{-\infty}^{\infty} \frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \int_{-\infty}^{\infty} \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot t_0} \cdot \int_{-\infty}^{\infty} \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \frac{A}{2 \cdot t_0} \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= \frac{A}{2 \cdot t_0} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - \frac{A}{2 \cdot t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{A}{2 \cdot t_0} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{2 \cdot t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{A}{2 \cdot t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= \frac{A}{2 \cdot t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{\jmath}{\jmath} = \\
&= \frac{A}{t_0} \cdot \jmath \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \frac{A}{t_0} \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= \jmath \cdot \frac{A}{t_0} \cdot sin\,(\omega \cdot t_0)
\end{aligned}$$

Transformata sygnału $h(t)$ to $H(\jmath\omega) = \jmath \cdot \frac{A}{t_0} \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $g_1(t)$ na podstawie transformaty sygnału $h(t) = g_1'(t)$

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$

$$g_1(t) = \int_{-\infty}^{t} h(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0)$$

Podstawiając obliczona wcześniej transformatę $H(\jmath\omega)$ sygnału $h(t)$ otrzymujemy transformatę $G_1(\jmath\omega)$ sygnału $g_1(t)$

$$G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(0) \cdot H(0) =$$

$$
\begin{aligned}
&= \frac{1}{\jmath \cdot \omega} \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(\omega \cdot t_0\right) + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(0 \cdot t_0\right) = \\
&= \frac{1}{\omega} \cdot \frac{A}{t_0} \cdot sin\left(\omega \cdot t_0\right) + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(0\right) = \\
&= A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot 0 = \\
&= A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} + 0 = \\
&= \left\{ Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\
&= A \cdot Sa\left(\omega \cdot t_0\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $g_1(t)$ jest równa $G_1(\jmath\omega) = A \cdot Sa\left(\omega \cdot t_0\right)$.

Korzystając z jednorodności transformaty Fouriera

$$
\begin{aligned}
g_1(t) &\xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\
g_2(t) &\xrightarrow{\mathcal{F}} G_2(\jmath\omega) \\
g(t) = \alpha \cdot g_1(t) + \beta \cdot g_2(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) = \alpha \cdot G_1(\jmath\omega) + \beta \cdot G_2(\jmath\omega)
\end{aligned}
$$

można wyznaczyć transformatę Fouriera $G(\jmath\omega)$ funkcji $g(t)$

$$
\begin{aligned}
G(\jmath\omega) &= G_1(\jmath\omega) + G_2(\jmath\omega) = \\
&= A \cdot Sa\left(\omega \cdot t_0\right) - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)
\end{aligned}
$$

Znając transformatę $G(\jmath\omega)$ i korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $F(\jmath\omega)$ funkcji $f(t)$

$$
\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{aligned}
$$

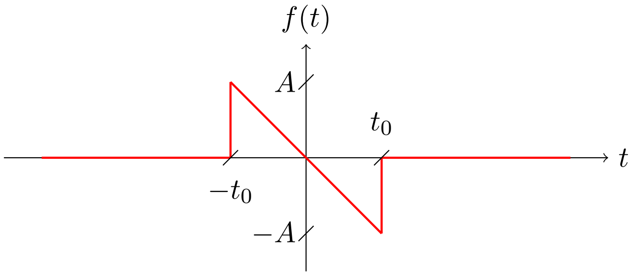
Podstawiając otrzymujemy

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(0) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot A \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot \left(Sa\left(0 \cdot t_0\right) - e^{-\jmath \cdot 0 \cdot t_0} \right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot \left(Sa\left(0\right) - e^{0} \right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot (1 - 1) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot 0 =
\end{aligned}
$$

$$= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + 0 =$$
$$= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)$.

Task 13. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



W pierwszej kolejności trzeba wyznaczyć jawną postać równań opisujących funkcję $f(t)$.

W tym celu wyznaczamy równanie prostej na odcinku $(-t_0, t_0)$

Ogólne równanie prostej to:

$$f(t) = m \cdot t + b \tag{3.82}$$

Dla rozważanego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(-t_0, A)$ oraz $(t_0, -A)$. Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} A = m \cdot (-t_0) + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} A = -m \cdot t_0 + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} A - A = -m \cdot t_0 + b + m \cdot t_0 + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = 2 \cdot b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 \end{cases}$$

$$\begin{cases} 0 = b \\ -\frac{A}{t_0} = m \end{cases}$$

Równianie prostej dla t z zakresu $(-t_0, t_0)$ to:

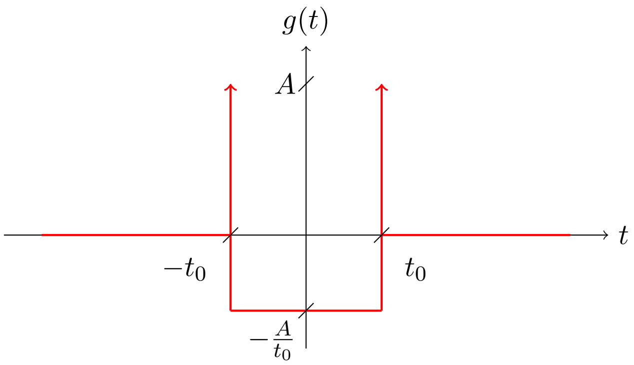
$$f(t) = -\frac{A}{t_0} \cdot t$$

Podsumowując, sygnal $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} \cdot t & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.83}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + A \cdot \delta(t + t_0) + A \cdot \delta(t - t_0) \tag{3.84}$$

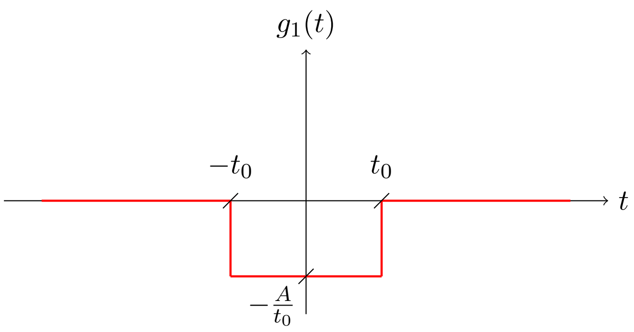


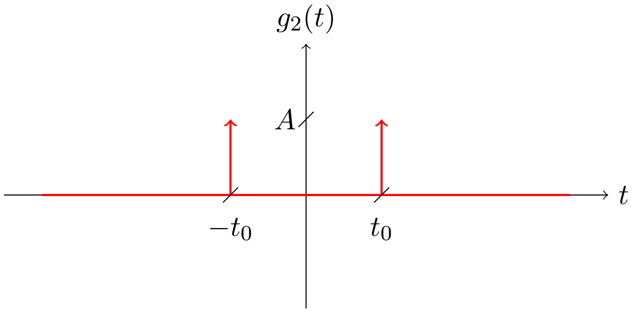
Funkcja $g(t)$ składa się z dwóch sygnałów $g_1(t)$ i $g_2(t)$

$$g(t) = g_1(t) + g_2(t) \tag{3.85}$$

$$g_1(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.86}$$

$$g_2(t) = A \cdot \delta(t + t_0) + A \cdot \delta(t - t_0) \tag{3.87}$$





Wyznaczenie transformaty sygnału $g_2(t)$ złożonego z delt diracka jest znacznie prostsze.

$$G_2(\jmath\omega) = \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.88}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
G_2(\jmath\omega) &= \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (A \cdot \delta(t+t_0) + A \cdot \delta(t-t_0)) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} (\delta(t+t_0) + \delta(t-t_0)) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= A \cdot \left(\int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot \left(e^{-\jmath\cdot\omega\cdot(-t_0)} + e^{-\jmath\cdot\omega\cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath\cdot\omega\cdot t_0} + e^{-\jmath\cdot\omega\cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath\cdot\omega\cdot t_0} + e^{-\jmath\cdot\omega\cdot t_0} \right) \cdot \frac{2}{2} = \\
&= 2 \cdot A \cdot \frac{e^{\jmath\cdot\omega\cdot t_0} + e^{-\jmath\cdot\omega\cdot t_0}}{2} = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2} \right\} = \\
&= 2 \cdot A \cdot cos\,(\omega \cdot t_0)
\end{aligned}$$

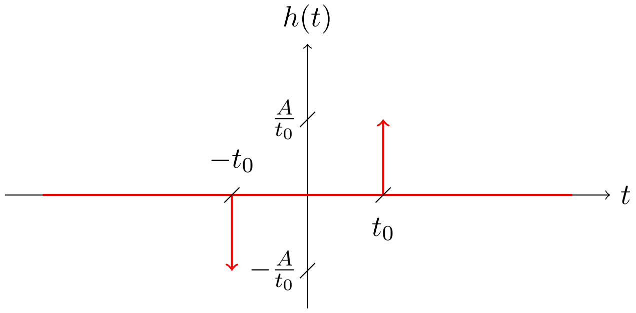
Transformata sygnału $g_2(t)$ to $G_2(\jmath\omega) = 2 \cdot A \cdot cos\,(\omega \cdot t_0)$

Funkcja $g_1(t)$ jest jeszcze zbyt złożona tak wiec wyznaczamy pochodną raz jeszcze

$$h(t) = g_1'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} - \frac{A}{t_0}\delta(t+t_0) + \frac{A}{t_0}\delta(t-t_0) \tag{3.89}$$

czyli po prostu

$$h(t) = g_1'(t) = -\frac{A}{t_0}\delta(t+t_0) + \frac{A}{t_0}\delta(t-t_0) \tag{3.90}$$



Wyznaczanie transformaty sygnału $h(t)$ złożonego z delt diracka jest znacznie prostsze.

$$H(\jmath\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.91}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
H(\jmath\omega) &= \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(-\frac{A}{t_0} \cdot \delta(t+t_0) + \frac{A}{t_0} \cdot \delta(t-t_0) \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(-\frac{A}{t_0} \cdot \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} + \frac{A}{t_0} \cdot \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \right) \cdot dt = \\
&= -\int_{-\infty}^{\infty} \frac{A}{t_0} \cdot \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-\infty}^{\infty} \frac{A}{t_0} \cdot \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= -\frac{A}{t_0} \cdot \int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \frac{A}{t_0} \cdot \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= -\frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} + \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= -\frac{A}{t_0} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= -\frac{A}{t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= -\frac{A}{t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= -\frac{2 \cdot A}{t_0} \cdot \jmath \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= -\frac{2 \cdot A}{t_0} \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= -\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\,(\omega \cdot t_0)
\end{aligned}$$

Transformata sygnału $h(t)$ to $H(\jmath\omega) = -\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $g_1(t)$ na podstawie transformaty sygnału $h(t) = g_1'(t)$

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$

$$g_1(t) = \int_{-\infty}^{t} h(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0)$$

Podstawiając obliczona wcześniej transformatę $H(\jmath\omega)$ sygnału $h(t)$ otrzymujemy transformatę $G_1(\jmath\omega)$ sygnału $g_1(t)$

$$\begin{aligned}
G_1(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \left(-\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(\omega \cdot t_0\right)\right) + \pi \cdot \delta(\omega) \cdot \left(-\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(0 \cdot t_0\right)\right) = \\
&= -\frac{1}{\omega} \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(\omega \cdot t_0\right) - \pi \cdot \delta(\omega) \cdot \jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(0\right) = \\
&= -2 \cdot A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} - \pi \cdot \delta(\omega) \cdot \jmath \cdot \frac{2 \cdot A}{t_0} \cdot 0 = \\
&= -2 \cdot A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} - 0 = \\
&= \left\{ Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\
&= -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right)
\end{aligned}$$

Ostatecznie transformata sygnału $g_1(t)$ jest równa $G_1(\jmath\omega) = -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right)$.

Korzystając z jednorodności transformaty Fouriera

$$\begin{gathered}
g_1(t) \xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\
g_2(t) \xrightarrow{\mathcal{F}} G_2(\jmath\omega) \\
g(t) = \alpha \cdot g_1(t) + \beta \cdot g_2(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \alpha \cdot G_1(\jmath\omega) + \beta \cdot G_2(\jmath\omega)
\end{gathered}$$

można wyznaczyć transformatę Fouriera $G(\jmath\omega)$ funkcji $g(t)$

$$\begin{aligned}
G(\jmath\omega) = G_1(\jmath\omega) + G_2(\jmath\omega) &= \\
&= -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right) + 2 \cdot A \cdot cos\left(\omega \cdot t_0\right) = \\
&= -2 \cdot A \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right)
\end{aligned}$$

Znając transformatę $G(\jmath\omega)$ i korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $F(\jmath\omega)$ funkcji $f(t)$

$$\begin{gathered}
g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{gathered}$$

Podstawiając otrzymujemy

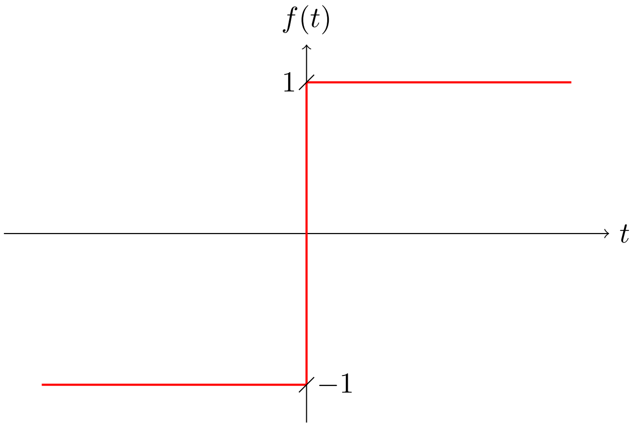
$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) =$$

$$= \frac{1}{\jmath \cdot \omega} \cdot (-2 \cdot A \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0))) + \pi \cdot \delta(\omega) \cdot (-2 \cdot A \cdot (Sa\,(0 \cdot t_0) - cos\,(0 \cdot t_0))) =$$
$$= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0)) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot (Sa\,(0) - cos\,(0)) =$$
$$= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0)) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot (1 - 1) =$$
$$= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0)) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot 0 =$$
$$= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0)) - 0 =$$
$$= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0))$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot (Sa\,(\omega \cdot t_0) - cos\,(\omega \cdot t_0))$.

Task 14.

Compute the Fourier transform of the $f(t) = sgn(t)$ signal using theorems describing the properties of Fourier transformation.

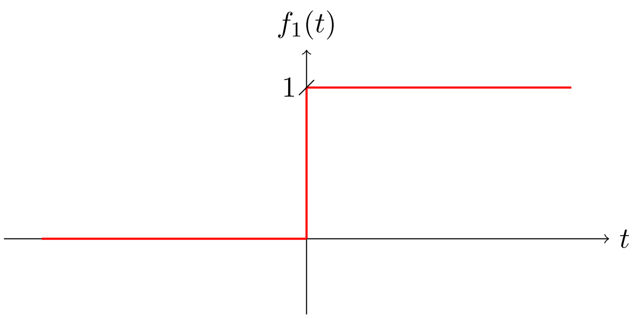


Let's describe the $f(t)$ signal using the elementary signals - step signals:

$$\begin{aligned} f(t) &= sgn(t) = \\ &= \mathbb{1}(t) - \mathbb{1}(-t) = \\ &= f_1(t) - f_2(t) \end{aligned}$$

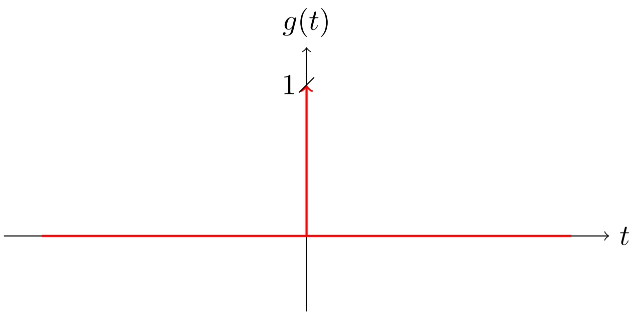
where:

$$f_1(t) = \mathbb{1}(t)$$
$$f_2(t) = \mathbb{1}(-t)$$



The transform of the $f_1(t) = \mathbb{1}(t)$ signal cannot be derived by classical Fourier transformation. However, the derivative $f_1'(t)$ can be easily derived:

$$g(t) = f_1'(t) = \delta(t)$$



The Fourier transform for the $g(t) = \delta(t)$ signal can be calculated as:

$$\begin{aligned} G(\jmath\omega) &= \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath\cdot\omega\cdot t} = \\ &= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-\jmath\cdot\omega\cdot t} = \\ &= \left\{ \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\ &= e^{-\jmath\cdot\omega\cdot 0} = \\ &= e^0 = \\ &= 1 \end{aligned}$$

The Fourier transform of the $g(t) = \delta(t)$ signal is equal to $G(\jmath\omega) = 1$.
Based on the integration theorem we can calculate the Fourier transform of the $f_1(t)$ signal:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f_1(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)$$

$$\begin{aligned} F_1(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\ &= \frac{1}{\jmath \cdot \omega} \cdot 1 + \pi \cdot \delta(\omega) \cdot 1 = \\ &= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega) \end{aligned}$$

The Fourier transform of the step signal $f_1(t) = \mathbb{1}(t)$ is equal to $F_1(\jmath\omega) = \frac{1}{\jmath\cdot\omega} + \pi \cdot \delta(\omega)$.
The $f_2(t)$ signal can be described as:

$$\begin{aligned} f_2(t) &= \mathbb{1}(-t) = \\ &= \mathbb{1}(-1 \cdot t) = \\ &= f_1(-1 \cdot t) \end{aligned}$$

Therefore, the Fourier transform of the $f_2(t)$ signal can be derived using the scaling theorem:

$$f_1(t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega)$$
$$f_2(t) = f_1(\alpha \cdot t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega) = \frac{1}{|\alpha|} \cdot F_1(\jmath\frac{\omega}{\alpha})$$

$$\begin{aligned}
F_2(\jmath\omega) &= \frac{1}{|a|} \cdot F_1(\jmath\frac{\omega}{a}) = \\
&= \Big\{a = -1\Big\} = \\
&= \frac{1}{|-1|} \cdot \frac{1}{\jmath \cdot \frac{\omega}{-1}} + \pi \cdot \delta(\frac{\omega}{-1}) = \\
&= \frac{1}{1} \cdot \frac{1}{-\jmath \cdot \omega} + \pi \cdot \delta(-\omega) = \\
&= -\frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)
\end{aligned}$$

The Fourier transform of the $f_2(t)$ signal is equal to $F_2(\jmath\omega) = -\frac{1}{\jmath\cdot\omega} + \pi \cdot \delta(\omega)$.
Finally, the Fourier transform of the $f(t)$ signal can be derived using the linearity theorem:

$$f_1(t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega)$$
$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$
$$f(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot F_1(\jmath\omega) + \beta \cdot F_2(\jmath\omega)$$

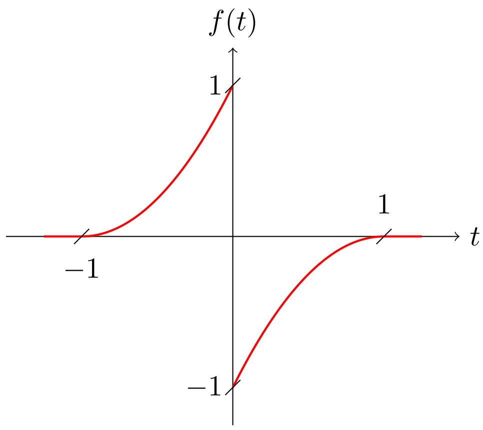
$$\begin{aligned}
F(\jmath\omega) &= F_1(\jmath\omega) - F_2(\jmath\omega) = \\
&= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega) - \left(-\frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)\right) = \\
&= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega) + \frac{1}{\jmath \cdot \omega} - \pi \cdot \delta(\omega) = \\
&= \frac{2}{\jmath \cdot \omega}
\end{aligned}$$

The Fourier transform of the $f(t) = sgn(t)$ is equal to $F(\jmath\omega) = \frac{2}{\jmath\cdot\omega}$.

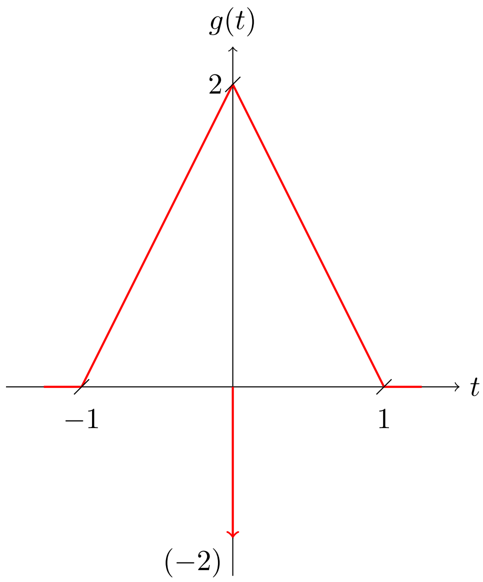
Task 15. Compute the Fourier transform of the $f(t)$ signal given below using theorems describing the properties of Fourier transformation. Exploit the following transform $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.

$$f(t) = \begin{cases} 0 & \text{for } t \in (-\infty; -1) \\ (t+1)^2 & \text{for } t \in (-1; 0) \\ -(t-1)^2 & \text{for } t \in (0; 1) \\ 0 & \text{for } t \in (1; \infty) \end{cases} \tag{3.92}$$

$$\Lambda(t) \xrightarrow{F} Sa^2\left(\frac{\omega}{2}\right) \tag{3.93}$$



Let's derive derivative of the $f(t)$ signal as $g(t) = \frac{\partial}{\partial t} f(t)$:



Using the elementary signals we can write:

$$g(t) = 2 \cdot \Lambda(t) - 2 \cdot \delta(t) \tag{3.94}$$

You can check that by integrating the $g(t)$ signal we'll get the $f(t)$ signal:

$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.95}$$

Therefore, the Fourier transform of the $f(t)$ signal can be determined from the integration theorem. In this case we will integrate the $g(t)$ signal:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.96}$$

In order to derive $F(\jmath\omega)$ we have to calculate the $G(\jmath\omega)$ transform of the $g(t)$ signal.

Based on linearity of the Fourier transformation, we can calculate transforms for elementary signals separately:

$$g(t) = 2 \cdot (g_1(t) - g_2(t)) \tag{3.97}$$

where:

$$\begin{aligned} g_1(t) &= \Lambda(t) \\ g_2(t) &= \delta(t) \end{aligned}$$

Calculate the Fourier transform $G_1(\jmath\omega)$ for the first signal $g_1(t)$.

We know that: $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.

The Fourier transform of $g_1(t)$ signal is equal to:

$$G_1(\jmath\omega) = \mathcal{F}\{g_1(t)\} = Sa^2\left(\frac{\omega}{2}\right) \tag{3.98}$$

Now, let's calculate the Fourier transform $G_2(\jmath\omega)$ for the second signal $g_2(t)$. The Fourier transform for the $g_2(t) = \delta(t)$ signal can be calculated as:

$$\begin{aligned} G_2(\jmath\omega) &= \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} = \\ &= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-\jmath \cdot \omega \cdot t} = \\ &= \left\{ \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\ &= e^{-\jmath \cdot \omega \cdot 0} = \\ &= e^0 = \\ &= 1 \end{aligned}$$

The Fourier transform of the $g_2(t) = \delta(t)$ signal is equal to $G_2(\jmath\omega) = 1$.

Finally, the Fourier transform of the $g(t)$ signal can be derived using the linearity theorem:

$$\begin{aligned} g_1(t) &\xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\ g_2(t) &\xrightarrow{\mathcal{F}} G_2(\jmath\omega) \end{aligned}$$

$$g(t) = \alpha \cdot g_1(t) + \beta \cdot g_2(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \alpha \cdot G_1(\jmath\omega) + \beta \cdot G_2(\jmath\omega)$$

$$\begin{aligned}
g(t) &= 2 \cdot (g_1(t) - g_2(t)) \\
G(\jmath\omega) &= 2 \cdot (G_1(\jmath\omega) - G_2(\jmath\omega)) = \\
&= 2 \cdot \left(Sa^2\left(\frac{\omega}{2}\right) - 1 \right)
\end{aligned}$$

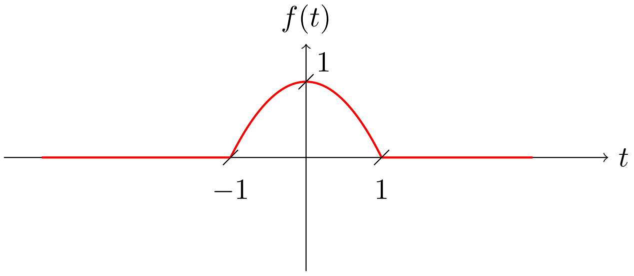
We derived the $G(\jmath\omega)$ transform. Now, based on the integration theorem, we can calculate the $F(\jmath\omega)$ Fourier transform.

$$\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot 2 \cdot \left(Sa^2\left(\frac{\omega}{2}\right) - 1 \right) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{matrix} G(0) = 2 \cdot \left(Sa^2\left(\frac{0}{2}\right) - 1 \right) \\ G(0) = 2 \cdot \left(Sa^2\left(0\right) - 1 \right) \\ G(0) = 2 \cdot (1 - 1) \\ G(0) = 0 \end{matrix} \right\} = \\
&= \frac{2}{\jmath \cdot \omega} \cdot \left(Sa^2\left(\frac{\omega}{2}\right) - 1 \right)
\end{aligned}$$

The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega) = \frac{2}{\jmath \cdot \omega} \cdot \left(Sa^2\left(\frac{\omega}{2}\right) - 1 \right)$.

Task 16.

Compute the Fourier transform of the $f(t)$ signal given below using theorems describing the properties of the Fourier transformation. Exploit the following transform $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

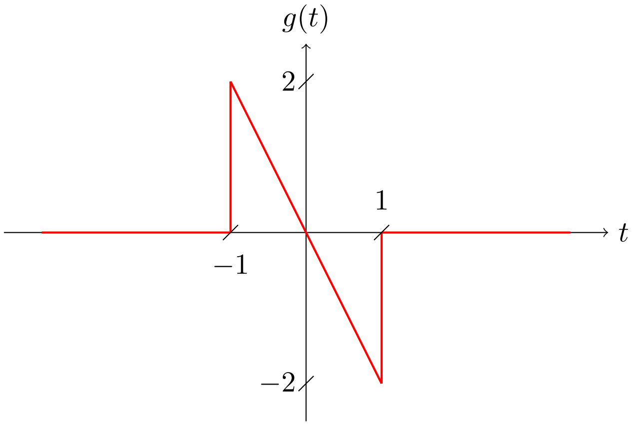


The $f(t)$ signal, as a piecewise function, may be described as:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ 1 - t^2 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} \tag{3.99}$$

First of all, let's calculate the $f(t)$ signal derivative:

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ -2 \cdot t & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} \tag{3.100}$$



You can check that by integrating the $g(t)$ signal we'll get the $f(t)$ signal:

$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.101}$$

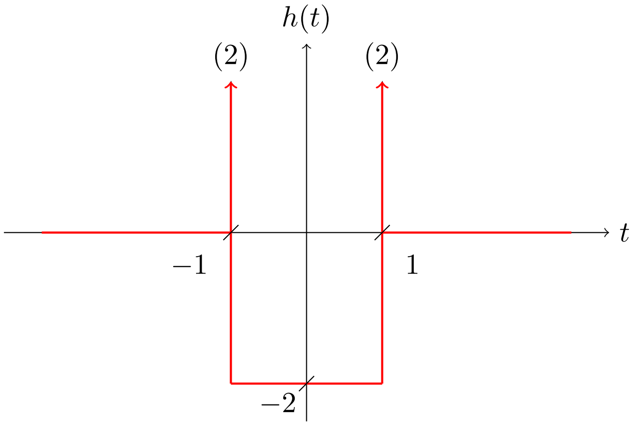
Therefore, the Fourier transform of the $f(t)$ signal can be determined from the integration theorem. In this case we will integrate the $g(t)$ signal:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.102}$$

Is it possible to calculate derivative again to simplify the analysed signal? Let's calculate the derivative of the $g(t)$ signal, so the second derivative of the $f(t)$ signal:

$$h(t) = \frac{\partial}{\partial t} g(t) = \frac{\partial^2}{\partial t^2} f(t) \tag{3.103}$$

$$h(t) = g'(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ -2 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} + 2 \cdot \delta(t+1) + 2 \cdot \delta(t-1) \tag{3.104}$$



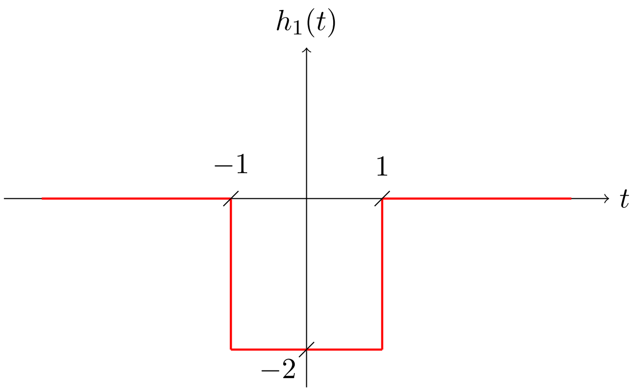
The $h(t)$ signal is a linear combination of signals $h_1(t)$ i $h_2(t)$:

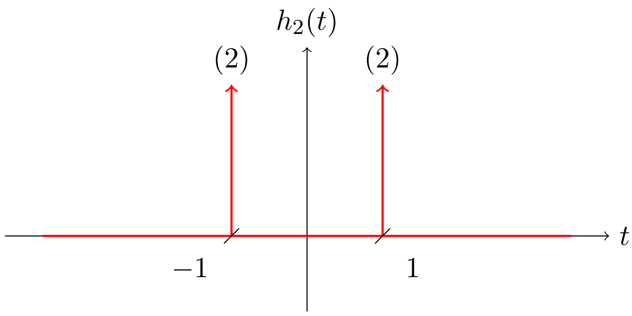
$$h(t) = h_1(t) + h_2(t) \tag{3.105}$$

where:

$$h_1(t) = -2 \cdot \Pi(\frac{t}{2}) \tag{3.106}$$

$$h_2(t) = 2 \cdot \delta(t+1) + 2 \cdot \delta(t-1) \tag{3.107}$$





Let's calculate the Fourier transform $H_1(\jmath\omega)$ for the first signal $h_1(t)$.
We know that: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.
Based on the scaling theorem:

$$x(t) \xrightarrow{\mathcal{F}} X(\jmath\omega)$$
$$h_1(t) = x(\alpha \cdot t) \xrightarrow{\mathcal{F}} H_1(\jmath\omega) = \frac{1}{|\alpha|} \cdot X(\jmath\frac{\omega}{\alpha})$$

we get:

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{1}{2} \cdot t) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{2}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{2}}}{2}\right)$$
$$\Pi(\frac{t}{2}) \xrightarrow{\mathcal{F}} 2 \cdot Sa\left(\frac{\omega \cdot 2}{2}\right)$$
$$\Pi(\frac{t}{2}) \xrightarrow{\mathcal{F}} 2 \cdot Sa\left(\omega\right)$$
$$-2 \cdot \Pi(\frac{t}{2}) \xrightarrow{\mathcal{F}} -2 \cdot 2 \cdot Sa\left(\omega\right)$$
$$-2 \cdot \Pi(\frac{t}{2}) \xrightarrow{\mathcal{F}} -4 \cdot Sa\left(\omega\right)$$

The Fourier transform of the $h_1(t)$ signal is equal to:

$$H_1(\jmath\omega) = \mathcal{F}\{H_1(t)\} = -4 \cdot Sa\left(\omega\right) \tag{3.108}$$

Let's calculate the Fourier transform of the $h_2(t)$ signal using the sampling property of the Dirac impulse.

$$H_2(\jmath\omega) = \int_{-\infty}^{\infty} h_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.109}$$

$$H_2(\jmath\omega) = \int_{-\infty}^{\infty} h_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt =$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} (2 \cdot \delta(t+1) + 2 \cdot \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \int_{-\infty}^{\infty} (\delta(t+1) + \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \left(\int_{-\infty}^{\infty} \delta(t+1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-\infty}^{\infty} \delta(t-1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= 2 \cdot \left(e^{-\jmath \cdot \omega \cdot (-1)} + e^{-\jmath \cdot \omega \cdot 1} \right) = \\
&= 2 \cdot \left(e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega} \right) = \\
&= 2 \cdot \left(e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega} \right) \cdot \frac{2}{2} = \\
&= 4 \cdot \frac{e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega}}{2} = \\
&= \left\{ cos\,(x) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= 4 \cdot cos\,(\omega)
\end{aligned}
$$

The Fourier transform of the $h_2(t)$ signal is equal to $H_2(\jmath\omega) = 4 \cdot cos\,(\omega)$.

Based on the linearity theorem:

$$
\begin{aligned}
h_1(t) &\xrightarrow{\mathcal{F}} H_1(\jmath\omega) \\
h_2(t) &\xrightarrow{\mathcal{F}} H_2(\jmath\omega) \\
h(t) = \alpha \cdot h_1(t) + \beta \cdot h_2(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) = \alpha \cdot H_1(\jmath\omega) + \beta \cdot H_2(\jmath\omega)
\end{aligned}
$$

The Fourier transform $H(\jmath\omega)$ of the $h(t)$ signal can be derived as:

$$
\begin{aligned}
H(\jmath\omega) &= H_1(\jmath\omega) + H_2(\jmath\omega) = \\
&= -4 \cdot Sa\,(\omega) + 4 \cdot cos\,(\omega) = \\
&= 4 \cdot (cos\,(\omega) - Sa\,(\omega))
\end{aligned}
$$

We derived the $H(\jmath\omega)$ transform. Now, based on the integration theorem, we can calculate the $G(\jmath\omega)$ Fourier transform.

$$
\begin{aligned}
h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) \\
g(t) = \int_{-\infty}^{t} h(\tau) \cdot d\tau &\xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0)
\end{aligned}
$$

$$
\begin{aligned}
G(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \left\{ \begin{array}{l} H(0) = 4 \cdot (cos\,(0) - Sa\,(0)) \\ H(0) = 4 \cdot (1-1) \\ H(0) = 4 \cdot 0 \\ H(0) = 0 \end{array} \right\} =
\end{aligned}
$$

$$= \frac{1}{\jmath \cdot \omega} \cdot (4 \cdot (cos\,(\omega) - Sa\,(\omega))) + 0 =$$
$$= \frac{4}{\jmath \cdot \omega} \cdot (cos\,(\omega) - Sa\,(\omega))$$

The Fourier transform of the $g(t)$ signal is equal to $G(\jmath\omega) = \frac{4}{\jmath\cdot\omega} \cdot (cos\,(\omega) - Sa\,(\omega))$.

We derived the $G(\jmath\omega)$ transform. Using the integration theorem once again, we can calculate the $F(\jmath\omega)$ Fourier transform.

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)$$

$$\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{array}{l} G(0) = \frac{4}{\jmath\cdot 0} \cdot (cos\,(0) - Sa\,(0)) \\ G(0) = \frac{0}{0}!!! \\ G(0) = \int_{-\infty}^{\infty} g(t) \cdot dt = \int_{-1}^{1} (-2) \cdot t \cdot dt = (-2) \cdot \left.\frac{t^2}{2}\right|_{-1}^{1} \\ G(0) = (-2) \cdot \left(\frac{1}{2} - \frac{1}{2}\right) = (-2) \cdot 0 \\ G(0) = 0 \end{array} \right\} = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{4}{\jmath \cdot \omega} \cdot (cos\,(\omega) - Sa\,(\omega)) + 0 = \\
&= \frac{4}{\jmath^2 \cdot \omega^2} \cdot (cos\,(\omega) - Sa\,(\omega)) = \\
&= \frac{4}{(-1) \cdot \omega^2} \cdot (cos\,(\omega) - Sa\,(\omega)) = \\
&= \frac{4}{\omega^2} \cdot (Sa\,(\omega) - cos\,(\omega))
\end{aligned}$$

The Fourier transform of the $f(t)$ signal is equal to $F(\jmath\omega) = \frac{4}{\omega^2} \cdot (Sa\,(\omega) - cos\,(\omega))$.

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Task 1. Compute the energy of the $f(t) = Sa\left(\omega_0 \cdot t\right)$ signal. Exploit the following transform $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa\left(\omega_0 \cdot t\right) \tag{3.110}$$

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.111}$$

The energy of the non-periodic signal may be derived as:

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \cdot dt \tag{3.112}$$

For the given $f(t)$ signal we get:

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} |f(t)|^2 \cdot dt = \\
&= \int_{-\infty}^{\infty} \left|Sa\left(\omega_0 \cdot t\right)\right|^2 \cdot dt = \\
&= \left\{ \quad Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\
&= \int_{-\infty}^{\infty} \left|\frac{sin\left(\omega_0 \cdot t\right)}{\left(\omega_0 \cdot t\right)}\right|^2 \cdot dt = \\
&= \int_{-\infty}^{\infty} \frac{sin^2\left(\omega_0 \cdot t\right)}{\left(\omega_0 \cdot t\right)^2} \cdot dt = \\
&= ....
\end{aligned}$$

In this approach, we have to integrate by parts even several times. Maybe there is an easier approach?

Let's consider the Parseval's theorem given below:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.113}$$

The Fourier transform $F(\jmath\omega)$ of the $f(t)$ signal has to be derived.

We know that:

$$g(t) = \Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.114}$$

Based on the time-frequency duality theorem:

$$\begin{gathered}
g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = G(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = 2\pi \cdot g(-\omega)
\end{gathered}$$

we get:

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega)$$
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$

Now, the $Sa\left(\frac{t}{2}\right)$ signal has to be scaled in time in order to get the $Sa\left(\omega_0 \cdot t\right)$ signal. Let's exploit the scaling in time theorem with $\alpha = 2 \cdot \omega_0$:

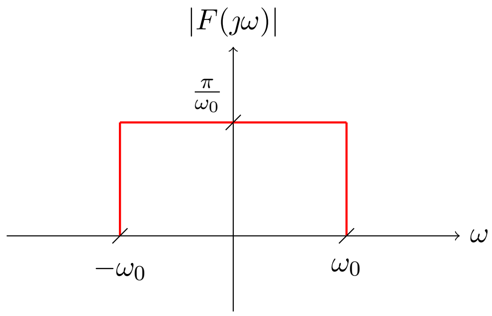
$$f(t) \xrightarrow{\mathcal{F}} F(\jmath\omega)$$
$$g(t) = f(\alpha \cdot t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{|\alpha|} \cdot F(\jmath\frac{\omega}{\alpha})$$

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$
$$Sa\left(2 \cdot \omega_0 \cdot \frac{t}{2}\right) \xrightarrow{\mathcal{F}} \frac{1}{2 \cdot \omega_0} \cdot 2\pi \cdot \Pi(\frac{\omega}{2 \cdot \omega_0})$$
$$Sa\left(\omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0} \cdot \Pi(\frac{\omega}{2 \cdot \omega_0})$$

The energy will be computed using Parseval's theorem:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.115}$$

Let's draw the magnitude spectrum of the $f(t)$ signal - $|F(\jmath\omega)|$:



$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -\omega_0) \\ \frac{\pi}{\omega_0} & \omega \in (-\omega_0; \omega_0) \\ 0 & \omega \in (\omega_0; \infty) \end{cases}$$

Let's derive $|F(\jmath\omega)|^2$ also:

$$|F(\jmath\omega)|^2 = \begin{cases} 0 & \omega \in (-\infty; -\omega_0) \\ \left(\frac{\pi}{\omega_0}\right)^2 & \omega \in (-\omega_0; \omega_0) \\ 0 & \omega \in (\omega_0; \infty) \end{cases}$$

Finally, let's calculate the energy using Parseval's theorem:

$$\begin{aligned}
E &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\int_{-\infty}^{-\omega_0} 0 \cdot d\omega + \int_{-\omega_0}^{\omega_0} \left(\frac{\pi}{\omega_0}\right)^2 \cdot d\omega + \int_{\omega_0}^{\infty} 0 \cdot d\omega \right) = \\
&= \frac{1}{2\pi} \cdot \left(0 + \left(\frac{\pi}{\omega_0}\right)^2 \cdot \int_{-\omega_0}^{\omega_0} d\omega + 0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{\pi}{\omega_0}\right)^2 \cdot \left(\omega|_{-\omega_0}^{\omega_0} \right) = \\
&= \frac{\pi}{2 \cdot \omega_0^2} \cdot (\omega_0 - (-\omega_0)) = \\
&= \frac{\pi}{2 \cdot \omega_0^2} \cdot (2 \cdot \omega_0) = \\
&= \frac{\pi}{\omega_0}
\end{aligned}$$

The energy of the $f(t) = Sa\left(\omega_0 \cdot t\right)$ signal is equal to $E = \frac{\pi}{\omega_0}$.

Task 2. Oblicz, jaka część energi sygnału $f(t) = A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right)$ przypada na wartości pulsacji $|\omega| < 2 \cdot \omega_0$. Wykorzystaj informację, że transformata sygnału $\Pi(t)$ jest równa $Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right) \tag{3.116}$$

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.117}$$

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} =? \tag{3.118}$$

Ponieważ musimy obliczyć energię tylko dla pewnego zakresu pulsacji, to wykorzystamy twierdzenie Parsevala:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.119}$$

W tym podejściu musimy obliczyć transformatę Fouriera sygnału $f(t)$, czyli $F(\jmath\omega)$.

Ponieważ możemy korzystać tylko ze znanych twierdzeń oraz wiedzy o transformacie sygnału $\Pi(t)$, to spróbujmy przekształcić sygnał $f(t)$ do postaci, w której wprost możemy zastosować twierdzenia. Zauważmy, że:

$$\begin{aligned}
f(t) &= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right) = \\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2}\right\} = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{2\cdot\jmath\cdot\omega_0\cdot t} + e^{-2\cdot\jmath\cdot\omega_0\cdot t}}{2}\right)^2 = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{(e^{2\cdot\jmath\cdot\omega_0\cdot t})^2 + 2 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + (e^{-2\cdot\jmath\cdot\omega_0\cdot t})^2}{4}\right) = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + 2 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t - 2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{4}\right) = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + 2 \cdot e^{0} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{4}\right) = \\
&= \frac{A}{4} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot e^{4\cdot\jmath\cdot\omega_0\cdot t} + \frac{A}{2} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) + \frac{A}{4} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t} = \\
&= f_1(t) + f_2(t) + f_3(t)
\end{aligned}$$

Korzystając z liniowości przekształcenia Fouriera możemy niezależnie obliczyć transformaty dla sygnałów $f_1(t)$, $f_2(t)$ i $f_3(t)$, a następnie zsumować te transformaty. Zacznijmy od sygnału $f_2(t)$:

Skoro wiemy, że:

$$g(t) = \Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.120}$$

to, na podstawie twierdzenia o symetrii przekształcenia Fouriera:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$f_2(t) = G(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega) = 2\pi \cdot g(-\omega)$$

otrzymujemy:

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega)$$
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$

Teraz musimy przeskalować $Sa\left(\frac{t}{2}\right)$ tak, aby otrzymać $Sa\left(2 \cdot \omega_0 \cdot t\right)$. W tym celu skorzystamy z twierdzenia o zmianie skali podstawiając $\alpha = 4 \cdot \omega_0$:

$$f(t) \xrightarrow{\mathcal{F}} F(\jmath\omega)$$
$$f_1(t) = f(\alpha \cdot t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{|\alpha|} \cdot F(\jmath\frac{\omega}{\alpha})$$

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$
$$Sa\left(4 \cdot \omega_0 \cdot \frac{t}{2}\right) \xrightarrow{\mathcal{F}} \frac{1}{4 \cdot \omega_0} \cdot 2\pi \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$Sa\left(2 \cdot \omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{\pi}{2 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$f_2(t) = \frac{A}{2} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) = F_2(\jmath\omega)$$

Podsumowując, transformata sygnału $f_2(t)$ to $F_2(\jmath\omega) = \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$.

Zauważmy, że $f_1(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t}$, czyli $f_1(t)$ to zmodulowany sygnał $f_2(t)$. Stosując twierdzenie o modulacji:

$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$
$$f_1(t) = f_2(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = F_2(\jmath\left(\omega - \omega_0\right))$$

otrzymujemy:

$$f_2(t) \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$$
$$f_1(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right) = F_1(\jmath\omega)$$

Podsumowując, transformata sygnału $f_1(t)$ to $F_1(\jmath\omega) = \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$.

Podobnie, zauważmy, że $f_3(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}$, czyli $f_3(t)$ to zmodulowany sygnał $f_2(t)$. Stosując twierdzenie o modulacji:

$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$

$$f_3(t) = f_2(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F_3(\jmath\omega) = F_2(\jmath(\omega - \omega_0))$$

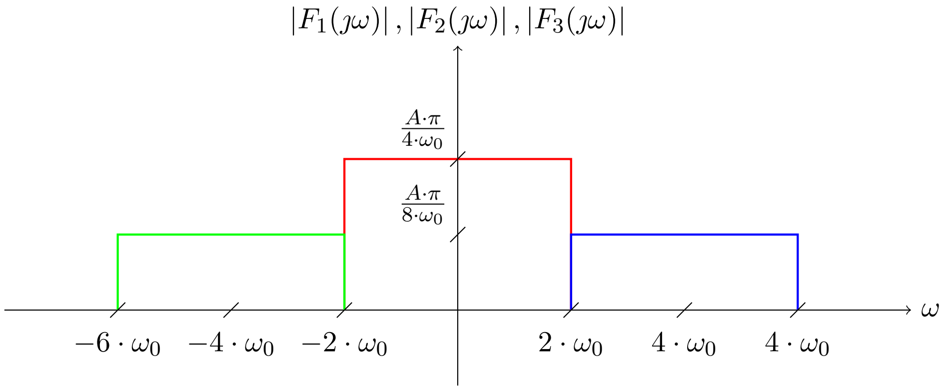
otrzymujemy:

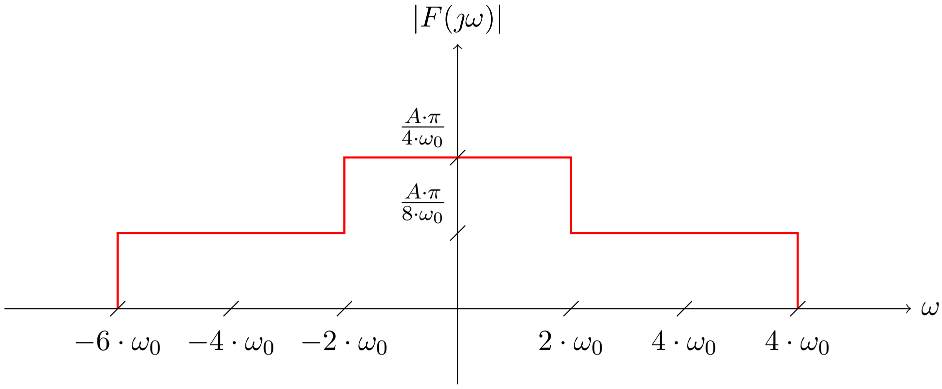
$$\begin{aligned} f_2(t) &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) \\ f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t} &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) \\ f_3(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t} &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) = F_3(\jmath\omega) \end{aligned}$$

Podsumowując, transformata sygnału $f_3(t)$ to $F_3(\jmath\omega) = \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$.
Teraz możemy podać transformatę sygnału $f(t) = f_1(t) + f_2(t) + f_3(t)$,

$$\begin{aligned} F(\jmath\omega) &= F_1(\jmath\omega) + F_2(\jmath\omega) + F_3(\jmath\omega) = \\ &= \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right) + \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) + \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) \end{aligned}$$

Narysujmy widmo amplitudowe sygnału $f(t)$, czyli $|F(\jmath\omega)|$.





$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -6\cdot\omega_0) \\ \frac{A\cdot\pi}{8\cdot\omega_0} & \omega \in (-6\cdot\omega_0; -2\cdot\omega_0) \\ \frac{A\cdot\pi}{4\cdot\omega_0} & \omega \in (-2\cdot\omega_0; 2\cdot\omega_0) \\ \frac{A\cdot\pi}{8\cdot\omega_0} & \omega \in (2\cdot\omega_0; 6\cdot\omega_0) \\ 0 & \omega \in (6\cdot\omega_0; \infty) \end{cases}$$

Ponieważ energię wyznaczamy ze wzoru:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.121}$$

to wyznaczmy $|F(\jmath\omega)|^2$:

$$|F(\jmath\omega)|^2 = \begin{cases} 0 & \omega \in (-\infty; -6\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 & \omega \in (-6\cdot\omega_0; -2\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{4\cdot\omega_0}\right)^2 & \omega \in (-2\cdot\omega_0; 2\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 & \omega \in (2\cdot\omega_0; 6\cdot\omega_0) \\ 0 & \omega \in (6\cdot\omega_0; \infty) \end{cases}$$

$$\begin{aligned}
E &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\int_{-\infty}^{-6\cdot\omega_0} 0 \cdot d\omega + \int_{-6\cdot\omega_0}^{-2\cdot\omega_0} \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 \cdot d\omega + \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left(\frac{A\cdot\pi}{4\cdot\omega_0}\right)^2 \cdot d\omega + \int_{2\cdot\omega_0}^{6\cdot\omega_0} \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 \cdot d\omega + \int_{6\cdot\omega_0}^{\infty} 0 \cdot d\omega \right) = \\
&= \frac{1}{2\pi} \cdot \left(0 + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \int_{-6\cdot\omega_0}^{-2\cdot\omega_0} d\omega + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} d\omega + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \int_{2\cdot\omega_0}^{6\cdot\omega_0} d\omega + 0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \omega|_{-6\cdot\omega_0}^{-2\cdot\omega_0} + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot \omega|_{-2\cdot\omega_0}^{2\cdot\omega_0} + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \omega|_{2\cdot\omega_0}^{6\cdot\omega_0} \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot (-2\cdot\omega_0 - (-6\cdot\omega_0)) + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot (2\cdot\omega_0 - (-2\cdot\omega_0)) + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot (6\cdot\omega_0 - 2\cdot\omega_0) \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot 4\cdot\omega_0 + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot 4\cdot\omega_0 + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot 4\cdot\omega_0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{16\cdot\omega_0} + \frac{A^2\cdot\pi^2}{4\cdot\omega_0} + \frac{A^2\cdot\pi^2}{16\cdot\omega_0} \right) = \\
&= \frac{1}{2\pi} \cdot \frac{A^2\cdot\pi^2}{4\cdot\omega_0} \cdot \left(\frac{1}{4} + 1 + \frac{1}{4} \right) = \\
&= \frac{A^2\cdot\pi}{8\cdot\omega_0} \cdot \left(\frac{2}{4} + 1 \right) =
\end{aligned}$$

$$= \frac{A^2 \cdot \pi}{8 \cdot \omega_0} \cdot \left(\frac{3}{2}\right) =$$
$$= \frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}$$

Energia sygnału $f(t) = A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right)$ równa się $E = \frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}$.

Energię sygnału dla pewnego zakresu pulsacji, także można wyznaczyć z twierdzenia Parsevala, ale zmieniając granice w całce zgodnie z oczekiwanym zakresem pulsacji, czyli dla pulsacji $|\omega| < 2 \cdot \omega_0$ otrzymamy wzór:

$$E_{|\omega|<2\cdot\omega_0} = \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.122}$$

Podstawiając dane dla naszego sygnału otrzymamy:

$$\begin{aligned}
E_{|\omega|<2\cdot\omega_0} &= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left|\frac{A \cdot \pi}{4 \cdot \omega_0}\right|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left(\frac{A \cdot \pi}{4 \cdot \omega_0}\right)^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A \cdot \pi}{4 \cdot \omega_0}\right)^2 \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2 \cdot \pi^2}{16 \cdot \omega_0^2}\right) \cdot \omega\Big|_{-2\cdot\omega_0}^{2\cdot\omega_0} = \\
&= \frac{A^2 \cdot \pi}{32 \cdot \omega_0^2} \cdot (2 \cdot \omega_0 - (-2 \cdot \omega_0)) = \\
&= \frac{A^2 \cdot \pi}{32 \cdot \omega_0^2} \cdot (4 \cdot \omega_0) = \\
&= \frac{A^2 \cdot \pi}{8 \cdot \omega_0}
\end{aligned}$$

Podsumowując $E_{|\omega|<\omega_0} = \frac{A^2 \cdot \pi}{8 \cdot \omega_0}$.

Teraz możemy obliczyć:

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} =? \tag{3.123}$$

Podstawiając nasze wcześniejsze wyniki otrzymujemy:

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} = \frac{\frac{A^2 \cdot \pi}{8 \cdot \omega_0}}{\frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}} = \frac{A^2 \cdot \pi}{8 \cdot \omega_0} \cdot \frac{16 \cdot \omega_0}{3 \cdot A^2 \cdot \pi} = \frac{2}{3} \approx 66\%$$

Na pulsacje z zakresu $|\omega| < 2 \cdot \omega_0$ przypada około 66% energii sygnału.

Task 3.

Oblicz, jaka część energi sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ przypada na wartości pulsacji $|\omega| < \omega_0$. Wykorzystaj informację, że transformata sygnału $\Lambda(t)$ jest równa $Sa^2\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) \tag{3.124}$$

$$\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right) \tag{3.125}$$

$$\frac{E_{|\omega|<\omega_0}}{E} =? \tag{3.126}$$

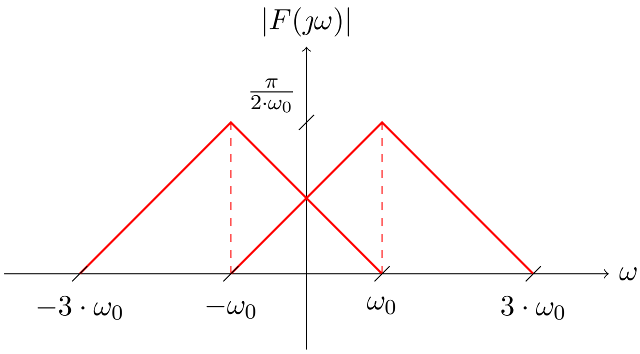
Całkowitą energię sygnału można wyznaczyc z twierdzenia Parsevala:

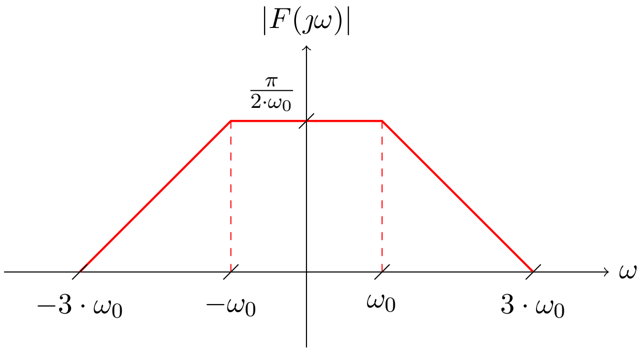
$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.127}$$

W tym celu musimy wyznaczyc transformatę sygnału $f(t)$.

W jednym z wcześniejszych zadań obliczyliśmy, że transformata Fouriera sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ jest równa $F(\jmath\omega) = \frac{1}{2} \cdot \left(\frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) + \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)$.

Narysujmy widmo amplitudowe sygnału $f(t)$, czyli $|F(\jmath\omega)|$.





$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -3\cdot\omega_0\rangle \\ \frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0} & \omega \in (-3\cdot\omega_0; -\omega_0\rangle \\ \frac{\pi}{2\cdot\omega_0} & \omega \in (-\omega_0; \omega_0\rangle \\ -\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0} & \omega \in (\omega_0; 3\cdot\omega_0\rangle \\ 0 & \omega \in (3\cdot\omega_0; \infty) \end{cases}$$

Podstawiając do wzoru na energię całkowitą, otrzymujemy:

$$E = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}|F(\jmath\omega)|^2\cdot d\omega =$$

$$= \frac{1}{2\pi}\cdot\left[\int_{-\infty}^{-3\cdot\omega_0}|0|^2\cdot d\omega + \int_{-3\cdot\omega_0}^{-\omega_0}\left|\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0}\right|^2\cdot d\omega + \int_{-\omega_0}^{\omega_0}\left|\frac{\pi}{2\cdot\omega_0}\right|^2\cdot d\omega + \right.$$

$$\left. + \int_{\omega_0}^{3\cdot\omega_0}\left|-\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0}\right|^2\cdot d\omega + \int_{3\cdot\omega_0}^{\infty}|0|^2\cdot d\omega\right] =$$

$$= \frac{1}{2\pi}\cdot\left[0 + \int_{-3\cdot\omega_0}^{-\omega_0}\left(\left(\frac{\pi}{4\cdot\omega_0^2}\right)^2\cdot\omega^2 + 2\cdot\frac{\pi}{4\cdot\omega_0^2}\cdot\frac{3\cdot\pi}{4\cdot\omega_0}\cdot\omega + \left(\frac{3\cdot\pi}{4\cdot\omega_0}\right)^2\right)\cdot d\omega + \frac{\pi^2}{4\cdot\omega_0^2}\cdot\int_{-\omega_0}^{\omega_0}d\omega + \right.$$

$$\left. + \int_{\omega_0}^{3\cdot\omega_0}\left(\left(-\frac{\pi}{4\cdot\omega_0^2}\right)^2\cdot\omega^2 - 2\cdot\frac{\pi}{4\cdot\omega_0^2}\cdot\frac{3\cdot\pi}{4\cdot\omega_0}\cdot\omega + \left(\frac{3\cdot\pi}{4\cdot\omega_0}\right)^2\right)\cdot d\omega + 0\right] =$$

$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}\omega^2\cdot d\omega + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}\omega\cdot d\omega + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}d\omega + \frac{\pi^2}{4\cdot\omega_0^2}\cdot\omega|_{-\omega_0}^{\omega_0} + \right.$$

$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\int_{\omega_0}^{3\cdot\omega_0}\omega^2\cdot d\omega - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\int_{\omega_0}^{3\cdot\omega_0}\omega\cdot d\omega + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\int_{\omega_0}^{3\cdot\omega_0}d\omega\right] =$$

$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\left.\frac{\omega^3}{3}\right|_{-3\cdot\omega_0}^{-\omega_0} + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left.\frac{\omega^2}{2}\right|_{-3\cdot\omega_0}^{-\omega_0} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\omega|_{-3\cdot\omega_0}^{-\omega_0} + \frac{\pi^2}{4\cdot\omega_0^2}\cdot(\omega_0 - (-\omega_0)) + \right.$$

$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\left.\frac{\omega^3}{3}\right|_{\omega_0}^{3\cdot\omega_0} - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left.\frac{\omega^2}{2}\right|_{\omega_0}^{3\cdot\omega_0} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\omega|_{\omega_0}^{3\cdot\omega_0}\right] =$$

$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\left(-\frac{\omega_0^3}{3} - \left(-\frac{27\cdot\omega_0^3}{3}\right)\right) + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(\frac{\omega_0^2}{2} - \frac{9\cdot\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot(-\omega_0 - (-3\cdot\omega_0)) + \right.$$

$$\left. + \frac{\pi^2}{4\cdot\omega_0^2}\cdot 2\cdot\omega_0 + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\left(\frac{27\cdot\omega_0^3}{3} - \frac{\omega_0^3}{3}\right) - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(\frac{9\cdot\omega_0^2}{2} - \frac{\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot(3\cdot\omega_0 - \omega_0)\right] =$$

$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\frac{26\cdot\omega_0^3}{3} + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(-\frac{8\cdot\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot 2\cdot\omega_0 + \frac{\pi^2}{2\cdot\omega_0} + \right.$$

$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\frac{26\cdot\omega_0^3}{3} - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\frac{8\cdot\omega_0^2}{2} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot 2\cdot\omega_0\right] =$$

$$= \frac{1}{2\pi}\cdot\left[\frac{26\cdot\pi^2}{48\cdot\omega_0} - \frac{48\cdot\pi^2}{32\cdot\omega_0} + \frac{18\cdot\pi^2}{16\cdot\omega_0} + \frac{\pi^2}{2\cdot\omega_0} + \frac{26\cdot\pi^2}{48\cdot\omega_0} - \frac{48\cdot\pi^2}{32\cdot\omega_0} + \frac{18\cdot\pi^2}{16\cdot\omega_0}\right] =$$

$$= \frac{1}{2\pi}\cdot\frac{\pi^2}{2\cdot\omega_0}\cdot\left[\frac{26}{24} - \frac{48}{16} + \frac{18}{8} + 1 + \frac{26}{24} - \frac{48}{16} + \frac{18}{8}\right] =$$

$$= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{52}{24} - \frac{96}{16} + \frac{36}{8} + 1\right] =$$

$$
\begin{aligned}
&= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{52}{24} - 6 + \frac{108}{24} + 1\right] = \\
&= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{160}{24} - 5\right] = \\
&= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{160}{24} - \frac{120}{24}\right] = \\
&= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{40}{24}\right] = \\
&= \frac{5\cdot\pi}{12\cdot\omega_0}
\end{aligned}
$$

Podsumowując, całkowita energia sygnału $f(t) = Sa^2(\omega_0\cdot t)\cdot cos(\omega_0\cdot t)$ to $E = \frac{5\cdot\pi}{12\cdot\omega_0}$.

Energię sygnału dla pewnego zakresu pulsacji, także można wyznaczyc z twierdzenia Parsevala, ale zmieniając granice w całce zgodnie z oczekiwanym zakresem pulsacji, czyli dla pulsacji $|\omega| < \omega_0$ otrzymamy wzór:

$$
E_{|\omega|<\omega_0} = \frac{1}{2\pi}\cdot\int_{-\omega_0}^{\omega_0}|F(\jmath\omega)|^2\cdot d\omega \tag{3.128}
$$

Podstawiając dane dla naszego sygnału otrzymamy:

$$
\begin{aligned}
E_{|\omega|<\omega_0} &= \frac{1}{2\pi}\cdot\int_{-\omega_0}^{\omega_0}|F(\jmath\omega)|^2\cdot d\omega = \\
&= \frac{1}{2\pi}\cdot\int_{-\omega_0}^{\omega_0}\left|\frac{\pi}{2\cdot\omega_0}\right|^2\cdot d\omega = \\
&= \frac{1}{2\pi}\cdot\int_{-\omega_0}^{\omega_0}\left(\frac{\pi}{2\cdot\omega_0}\right)^2\cdot d\omega = \\
&= \frac{1}{2\pi}\cdot\left(\frac{\pi}{2\cdot\omega_0}\right)^2\cdot\int_{-\omega_0}^{\omega_0} d\omega = \\
&= \frac{1}{2\pi}\cdot\left(\frac{\pi^2}{4\cdot\omega_0^2}\right)\cdot\omega|_{-\omega_0}^{\omega_0} = \\
&= \frac{\pi}{8\cdot\omega_0^2}\cdot(\omega_0 - (-\omega_0)) = \\
&= \frac{\pi}{8\cdot\omega_0^2}\cdot(2\cdot\omega_0) = \\
&= \frac{\pi}{4\cdot\omega_0}
\end{aligned}
$$

Podsumowując $E_{|\omega|<\omega_0} = \frac{\pi}{4\cdot\omega_0}$.

Teraz możemy obliczyć:

$$
\frac{E_{|\omega|<\omega_0}}{E} =? \tag{3.129}
$$

Podstawiając nasze wcześniejsze wyniki otrzymujemy:

$$
\frac{E_{|\omega|<\omega_0}}{E} = \frac{\frac{\pi}{4\cdot\omega_0}}{\frac{5\cdot\pi}{12\cdot\omega_0}} = \frac{\pi}{4\cdot\omega_0}\cdot\frac{12\cdot\omega_0}{5\cdot\pi} = \frac{12}{20} = \frac{6}{10} = 60\%
$$

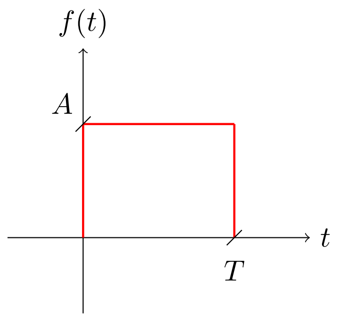
Na pulsacje z zakresu $|\omega| < \omega_0$ przypada 60% energi sygnału.

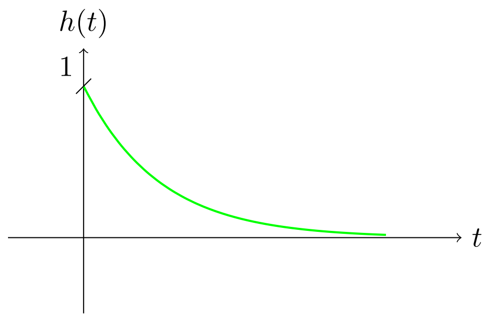
Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

Task 1. Oblicz splot sygnałów $f(t) = A \cdot \Pi\left(\frac{t-T}{T}\right)$ i $h(t) = \mathbb{1}(t) \cdot e^{-a \cdot t}$





Wzór na slot sygnałów

$$y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) \cdot d\tau \tag{4.1}$$

Wzory sygnałów pod całką

$$f(\tau) = A \cdot \Pi\left(\frac{\tau}{T}\right)$$

$$h(t-\tau) = \mathbb{1}(t) \cdot e^{-a \cdot (t-\tau)}$$

$$f(\tau) = \begin{cases} 0 & \tau \in (-\infty; 0) \\ A & \tau \in (0; T) \\ 0 & \tau \in (T; \infty) \end{cases}$$

$$h(t-\tau) = \begin{cases} e^{-a \cdot (t-\tau)} & \tau \in (-\infty; t) \\ 0 & \tau \in (t; \infty) \end{cases}$$

Wykresy obu funkcji w dziedzinie τ dla różnych wartości t:

Po wymnożeniu obu funkcji, dla przykładowych wartości t, otrzymujemy (ciągła, czerwona linia):

Z wykresu widać, że dla różnych wartości t otrzymujemy różny kształt funkcji podcałkowej $f(\tau) \cdot h(t-\tau)$. W związku z tym, wyznaczymy splot oddzielnie dla posczególnych przedziałów wartości t

Przedział 1 Dla wartości t spełniających warunek $t < 0$ otrzymujemy:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} 0 \cdot d\tau = \\ &= 0 \end{aligned}$$

Przedział 2 Dla wartości t spełniających warunki $t \geq 0$ i $t < T$ otrzymujemy

$$f(\tau)\cdot h(t-\tau)=\begin{cases}0 & \tau\in(-\infty;0)\\ A\cdot e^{-a\cdot(t-\tau)} & \tau\in(0;t)\\ 0 & \tau\in(t;\infty)\end{cases}$$

Wartość splotu $y(t)$ wyznaczamy ze wzoru:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} f(\tau)\cdot h(t-\tau)\cdot d\tau = \\
&= \int_{-\infty}^{0} 0\cdot d\tau + \int_{0}^{t}\left(A\cdot e^{-a\cdot(t-\tau)}\right)\cdot d\tau + \int_{t}^{\infty} 0\cdot d\tau = \\
&= 0 + A\cdot\int_{0}^{t}\left(e^{-a\cdot t}\cdot e^{a\cdot\tau}\right)\cdot d\tau + 0 = \\
&= A\cdot e^{-a\cdot t}\cdot\int_{0}^{t}\left(e^{a\cdot\tau}\right)\cdot d\tau = \\
&= A\cdot e^{-a\cdot t}\cdot\frac{1}{a}\cdot e^{a\cdot\tau}\Big|_0^t = \\
&= \frac{A}{a}\cdot e^{-a\cdot t}\cdot\left(e^{a\cdot t}-e^{a\cdot 0}\right) = \\
&= \frac{A}{a}\cdot e^{-a\cdot t}\cdot\left(e^{a\cdot t}-1\right) = \\
&= \frac{A}{a}\cdot\left(e^{a\cdot t}\cdot e^{-a\cdot t}-1\cdot e^{-a\cdot t}\right) = \\
&= \frac{A}{a}\cdot\left(e^{a\cdot t-a\cdot t}-e^{-a\cdot t}\right) = \\
&= \frac{A}{a}\cdot\left(e^{0}-e^{-a\cdot t}\right) = \\
&= \frac{A}{a}\cdot\left(1-e^{-a\cdot t}\right)
\end{aligned}$$

Przedział 3 Dla wartości t spełniających warunki $t\geq T$ otrzymujemy

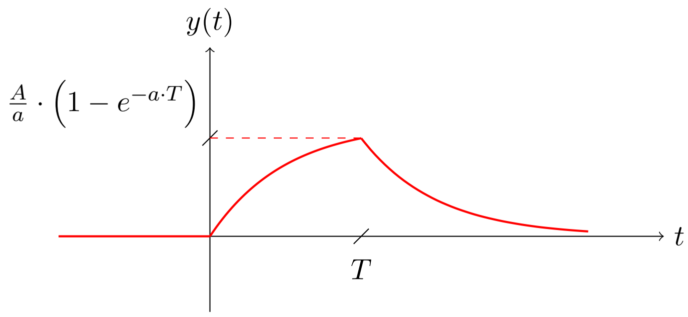
$$f(\tau) \cdot h(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; 0) \\ A \cdot e^{-a \cdot (t-\tau)} & \tau \in (0; T) \\ 0 & \tau \in (T; \infty) \end{cases}$$

Wartość splotu $y(t)$ wyznaczamy ze wzoru:

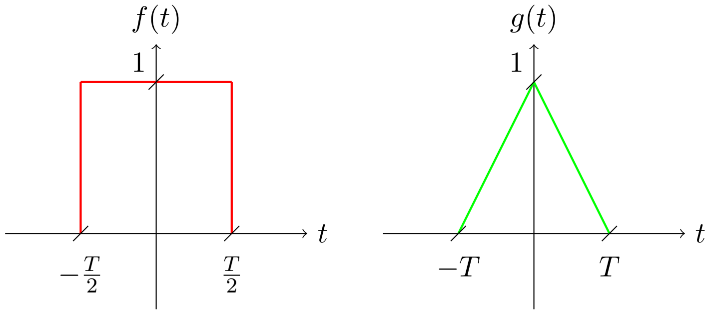
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) \cdot d\tau = \\ &= \int_{-\infty}^{0} 0 \cdot d\tau + \int_{0}^{T} \left(A \cdot e^{-a \cdot (t-\tau)} \right) \cdot d\tau + \int_{T}^{\infty} 0 \cdot d\tau = \\ &= 0 + A \cdot \int_{0}^{T} \left(e^{-a \cdot t} \cdot e^{a \cdot \tau} \right) \cdot d\tau + 0 = \\ &= A \cdot e^{-a \cdot t} \cdot \int_{0}^{T} \left(e^{a \cdot \tau} \right) \cdot d\tau = \\ &= A \cdot e^{-a \cdot t} \cdot \frac{1}{a} \cdot e^{a \cdot \tau} \Big|_{0}^{T} = \\ &= \frac{A}{a} \cdot e^{-a \cdot t} \cdot \left(e^{a \cdot T} - e^{a \cdot 0} \right) = \\ &= \frac{A}{a} \cdot e^{-a \cdot t} \cdot \left(e^{a \cdot T} - 1 \right) \end{aligned}$$

Podsumowując:

$$y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) \cdot d\tau = \begin{cases} 0 & t \in (-\infty; 0) \\ \frac{A}{a} \cdot (1 - e^{-a \cdot t}) & t \in (0; T) \\ \frac{A}{a} \cdot e^{-a \cdot t} \cdot \left(e^{a \cdot T} - 1 \right) & t \in (T; \infty) \end{cases}$$


$y(t)$
$\frac{A}{a} \cdot \left(1 - e^{-a \cdot T}\right)$
t
T

Task 2. Oblicz splot sygnałów $f(t) = \Pi\left(\frac{t}{T}\right)$ i $g(t) = \Lambda\left(\frac{t}{T}\right)$



Wzór na slot sygnałów

$$h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau \tag{4.2}$$

Wzory sygnałów pod całką

$$f(\tau) = \Pi\left(\frac{\tau}{T}\right)$$
$$g(t-\tau) = \Lambda\left(\frac{t-\tau}{T}\right)$$

$$f(\tau) = \begin{cases} 0 & \tau \in \left(-\infty; -\frac{T}{2}\right) \\ A & \tau \in \left(-\frac{T}{2}; \frac{T}{2}\right) \\ 0 & \tau \in \left(\frac{T}{2}; \infty\right) \end{cases}$$

$$g(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; t-T); \\ \frac{1}{T} \cdot \tau - \frac{t-T}{T} & \tau \in (t-T; t) \\ -\frac{1}{T} \cdot \tau - \frac{-t-T}{T} & \tau \in (t; t+T) \\ 0 & \tau \in (t+T; \infty); \end{cases}$$

Wykresy obu funkcji dla różnych wartości t

Po wymnożeniu obu funkcji dla przykładowych wartości t otrzymujemy

Jak widać dla różnych wartości t otrzymujemy różny kształt funkcji podcałkowej $f(\tau) \cdot g(t-\tau)$.

Przedział 1 .

Dla wartości t spełniających warunek $t + T < -\frac{T}{2}$

$$\begin{aligned} t + T &< -\frac{T}{2} \\ t &< -\frac{T}{2} - T \\ t &< -\frac{3}{2} \cdot T \end{aligned}$$

w wyniku mnożenia otrzymyjemy 0 a więc wartość splotu jest także równa 0

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} 0 \cdot d\tau \\ &= 0 \end{aligned}$$

Przedział 2 .

Dla wartości t spełniających warunki $t+T \geq -\frac{T}{2}$ i $t < -\frac{T}{2}$

$$\begin{aligned} t+T \geq -\frac{T}{2} &\qquad \wedge \qquad & t < -\frac{T}{2} \\ t \geq -\frac{T}{2} - T &\qquad \wedge \qquad & t < -\frac{T}{2} \\ t \geq -\frac{3}{2} \cdot T &\qquad \wedge \qquad & t < -\frac{T}{2} \end{aligned}$$

a więc $t \in \left\langle -\frac{3}{2} \cdot T, -\frac{T}{2} \right)$
w wyniku mnożenia otrzymujemy prostą zdefiniowaną na odcinku $t \in \left(-\frac{T}{2}, t+T \right)$.

$$f(\tau) \cdot g(t-\tau) = \begin{cases} 0 & \tau \in \left(-\infty; -\frac{T}{2} \right) \\ -\frac{1}{T} \cdot \tau - \frac{-t-T}{T} & \tau \in \left(-\frac{T}{2}; t+T \right) \\ 0 & \tau \in (t+T; \infty) \end{cases}$$

wartość splotu wyznaczamy z ze wzoru

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau = \\ &= \int_{-\infty}^{\frac{T}{2}} 0 \cdot d\tau + \int_{-\frac{T}{2}}^{t+T} \left(-\frac{1}{T} \cdot \tau - \frac{-t-T}{T} \right) \cdot d\tau + \int_{t+T}^{\infty} 0 \cdot d\tau = \\ &= 0 - \int_{-\frac{T}{2}}^{t+T} \frac{1}{T} \cdot \tau d\tau + \int_{-\frac{T}{2}}^{t+T} \frac{t+T}{T} \cdot d\tau + 0 = \\ &= -\frac{1}{T} \cdot \int_{-\frac{T}{2}}^{t+T} \tau \cdot d\tau + \frac{t+T}{T} \cdot \int_{-\frac{T}{2}}^{t+T} d\tau = \\ &= -\frac{1}{T} \cdot \left(\frac{1}{2} \cdot \tau^2 \right)_{-\frac{T}{2}}^{t+T} + \frac{t+T}{T} \cdot (\tau)_{-\frac{T}{2}}^{t+T} = \\ &= -\frac{1}{T} \cdot \frac{1}{2} \cdot \left((t+T)^2 - \left(-\frac{T}{2} \right)^2 \right) + \frac{t+T}{T} \cdot \left(t+T - \left(-\frac{T}{2} \right) \right) = \\ &= -\frac{1}{2 \cdot T} \cdot \left(t^2 + 2 \cdot t \cdot T + T^2 - \frac{T^2}{4} \right) + \frac{t+T}{T} \cdot \left(t+T+\frac{T}{2} \right) = \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{t+T}{T}\cdot\left(t+\frac{3}{2}\cdot T\right)=\\
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{1}{T}\cdot\left(t^2+\frac{3}{2}\cdot t\cdot T+t\cdot T+\frac{3}{2}\cdot T^2\right)=\\
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{2}{2\cdot T}\cdot\left(t^2+\frac{5}{2}\cdot t\cdot T+\frac{3}{2}\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(-t^2-2\cdot t\cdot T-\frac{3}{4}\cdot T^2\right)+\frac{1}{2\cdot T}\cdot\left(2\cdot t^2+5\cdot t\cdot T+3\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(-t^2-2\cdot t\cdot T-\frac{3}{4}\cdot T^2+2\cdot t^2+5\cdot t\cdot T+3\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(t^2+3\cdot t\cdot T+2\frac{1}{4}\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot t^2+\frac{1}{2\cdot T}\cdot 3\cdot t\cdot T+\frac{1}{2\cdot T}\cdot\frac{9}{4}\cdot T^2=\\
&= \frac{1}{2\cdot T}\cdot t^2+\frac{3}{2}\cdot t+\frac{9}{8}\cdot T
\end{aligned}
$$

Przedział 3 .

Dla wartości t spełniających warunki $t \geq -\frac{T}{2}$ i $t < \frac{T}{2}$

$$
t \geq -\frac{T}{2} \qquad\qquad \wedge \qquad\qquad t < \frac{T}{2}
$$

a więc $t \in \left\langle -\frac{1}{2}\cdot T, \frac{1}{2}\cdot T\right)$

w wyniku mnożenia otrzymujemy dwie proste zdefiniowaną na odcinkach $t \in \left(-\frac{T}{2}, t\right)$ oraz $t \in \left(t, \frac{T}{2}\right)$.

$$
f(\tau)\cdot g(t-\tau)=\begin{cases}
0 & \tau\in\left(-\infty;-\frac{T}{2}\right)\\
\frac{1}{T}\cdot\tau-\frac{t-T}{T} & \tau\in\left(-\frac{T}{2};t\right)\\
-\frac{1}{T}\cdot\tau-\frac{-t-T}{T} & \tau\in\left(t;\frac{T}{2}\right)\\
0 & \tau\in\left(\frac{T}{2};\infty\right)
\end{cases}
$$

wartość splotu wyznaczamy z ze wzoru

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\cdot d\tau = \\
&= \int_{-\infty}^{-\frac{T}{2}} 0\cdot d\tau + \int_{-\frac{T}{2}}^{t} \left(\frac{1}{T}\cdot\tau - \frac{t-T}{T}\right)\cdot d\tau + \int_{t}^{\frac{T}{2}} \left(-\frac{1}{T}\cdot\tau - \frac{-t-T}{T}\right)\cdot d\tau + \int_{\frac{T}{2}}^{\infty} 0\cdot d\tau = \\
&= 0 + \int_{-\frac{T}{2}}^{t} \frac{1}{T}\cdot\tau\cdot d\tau - \int_{-\frac{T}{2}}^{t} \frac{t-T}{T}\cdot d\tau + \int_{t}^{\frac{T}{2}} \left(-\frac{1}{T}\cdot\tau\right)\cdot d\tau - \int_{t}^{\frac{T}{2}} \frac{-t-T}{T}\cdot d\tau + 0 = \\
&= \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{t} \tau\cdot d\tau - \frac{t-T}{T}\cdot\int_{-\frac{T}{2}}^{t} d\tau - \frac{1}{T}\cdot\int_{t}^{\frac{T}{2}} \tau\cdot d\tau + \frac{t+T}{T}\cdot\int_{t}^{\frac{T}{2}} d\tau = \\
&= \frac{1}{T}\cdot\frac{1}{2}\cdot\tau^2\Big|_{-\frac{T}{2}}^{t} - \frac{t-T}{T}\cdot\tau|_{-\frac{T}{2}}^{t} - \frac{1}{T}\cdot\frac{1}{2}\cdot\tau^2\Big|_{t}^{\frac{T}{2}} + \frac{t+T}{T}\cdot\tau|_{t}^{\frac{T}{2}} = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 - \left(-\frac{T}{2}\right)^2\right) - \frac{t-T}{T}\cdot\left(t - \left(-\frac{T}{2}\right)\right) - \frac{1}{2\cdot T}\cdot\left(\left(\frac{T}{2}\right)^2 - t^2\right) + \frac{t+T}{T}\cdot\left(\frac{T}{2} - t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 + \frac{1}{4}\cdot T^2\right) - \frac{t-T}{T}\cdot\left(t + \frac{T}{2}\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2 - t^2\right) + \frac{t+T}{T}\cdot\left(\frac{T}{2} - t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 + \frac{1}{4}\cdot T^2\right) - \frac{2}{2\cdot T}\cdot(t-T)\cdot\left(t + \frac{T}{2}\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2 - t^2\right) + \frac{2}{2\cdot T}\cdot(t+T)\cdot\left(\frac{T}{2} - t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 + \frac{1}{4}\cdot T^2\right) - \frac{2}{2\cdot T}\cdot\left(t^2 + \frac{1}{2}\cdot t\cdot T - t\cdot T - \frac{1}{2}\cdot T^2\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2 - t^2\right) + \\
&+ \frac{2}{2\cdot T}\cdot\left(\frac{1}{2}\cdot t\cdot T - t^2 + \frac{1}{2}\cdot T^2 - t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 + \frac{1}{4}\cdot T^2\right) + \frac{1}{2\cdot T}\cdot\left(-2\cdot t^2 - t\cdot T + 2\cdot t\cdot T + T^2\right) + \frac{1}{2\cdot T}\cdot\left(-\frac{1}{4}\cdot T^2 + t^2\right) + \\
&+ \frac{1}{2\cdot T}\cdot\left(t\cdot T - 2\cdot t^2 + T^2 - 2\cdot t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2 + \frac{1}{4}\cdot T^2 - 2\cdot t^2 - t\cdot T + 2\cdot t\cdot T + T^2 - \frac{1}{4}\cdot T^2 + t^2 + t\cdot T - 2\cdot t^2 + T^2 - 2\cdot t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(-2\cdot t^2 + 2\cdot T^2\right) = \\
&= \frac{1}{T}\cdot\left(-t^2 + T^2\right) = \\
&= -\frac{1}{T}\cdot t^2 + T
\end{aligned}
$$

Przedział 4 .

Dla wartości t spełniających warunki $t - T \geq -\frac{T}{2}$ i $t - T < \frac{T}{2}$

$$
\begin{aligned}
t - T &\geq -\frac{T}{2} & \wedge && t - T &< \frac{T}{2} \\
t &\geq -\frac{T}{2} + T & \wedge && t &< \frac{T}{2} + T \\
t &\geq \frac{1}{2} \cdot T & \wedge && t &< \frac{3}{2} \cdot T
\end{aligned}
$$

a więc $t \in \left\langle \frac{1}{2} \cdot T, \frac{3}{2} \cdot T \right)$
w wyniku mnożenia otrzymujemy prostą zdefiniowaną na odcinku $t \in \left(t - T, \frac{T}{2}\right)$.

$$
f(\tau) \cdot g(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; t-T) \\ \frac{1}{T} \cdot \tau - \frac{t-T}{T} & \tau \in \left(t-T; \frac{T}{2}\right) \\ 0 & \tau \in \left(\frac{T}{2}; \infty\right) \end{cases}
$$

wartość splotu wyznaczamy z ze wzoru

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau = \\
&= \int_{-\infty}^{t-T} 0 \cdot d\tau + \int_{t-T}^{\frac{T}{2}} \left(\frac{1}{T} \cdot \tau - \frac{t-T}{T}\right) \cdot d\tau + \int_{\frac{T}{2}}^{\infty} 0 \cdot d\tau = \\
&= 0 + \int_{t-T}^{\frac{T}{2}} frac1T \cdot \tau \cdot d\tau - \int_{t-T}^{\frac{T}{2}} \frac{t-T}{T} \cdot d\tau + 0 = \\
&= \frac{1}{T} \cdot \int_{t-T}^{\frac{T}{2}} \tau \cdot d\tau - \frac{t-T}{T} \cdot \int_{t-T}^{\frac{T}{2}} d\tau = \\
&= \frac{1}{T} \cdot \frac{1}{2} \cdot \tau^2 \Big|_{t-T}^{\frac{T}{2}} - \frac{t-T}{T} \cdot \tau|_{t-T}^{\frac{T}{2}} = \\
&= \frac{1}{T} \cdot \frac{1}{2} \cdot \left(\left(\frac{T}{2}\right)^2 - (t-T)^2\right) - \frac{t-T}{T} \cdot \left(\frac{T}{2} - (t-T)\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{1}{4} \cdot T^2 - \left(t^2 - 2 \cdot t \cdot T + T^2\right)\right) - \frac{t-T}{T} \cdot \left(\frac{T}{2} - t + T\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{1}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T - T^2\right) - \frac{1}{T} \cdot (t-T) \cdot \left(\frac{3}{2} \cdot T - t\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T\right) - \frac{2}{2 \cdot T} \cdot \left(\frac{3}{2} \cdot t \cdot T - t^2 - \frac{3}{2} \cdot T^2 + t \cdot T\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T\right) - \frac{1}{2 \cdot T} \cdot \left(\frac{6}{2} \cdot t \cdot T - 2 \cdot t^2 - \frac{6}{2} \cdot T^2 + 2 \cdot t \cdot T\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T - \frac{6}{2} \cdot t \cdot T + 2 \cdot t^2 + \frac{6}{2} \cdot T^2 - 2 \cdot t \cdot T\right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{9}{4} \cdot T^2 - 3 \cdot t \cdot T + t^2\right) = \\
&= \frac{1}{2 \cdot T} \cdot \frac{9}{4} \cdot T^2 - \frac{1}{2 \cdot T} \cdot 3 \cdot t \cdot T + \frac{1}{2 \cdot T} \cdot t^2 = \\
&= \frac{9}{8} \cdot T - \frac{3}{2} \cdot t + \frac{1}{2 \cdot T} \cdot t^2
\end{aligned}
$$

Przedział 5 .

Dla wartości t spełniających warunek $t - T \geq \frac{T}{2}$.

$$\begin{aligned} t - T &\geq \frac{T}{2} \\ t &\geq \frac{T}{2} + T \\ t &\geq \frac{3}{2} \cdot T \end{aligned}$$

a więc $t \in \left\langle \frac{3}{2} \cdot T, \infty \right)$
w wyniku mnożenia otrzymujemy sygnał zerowy

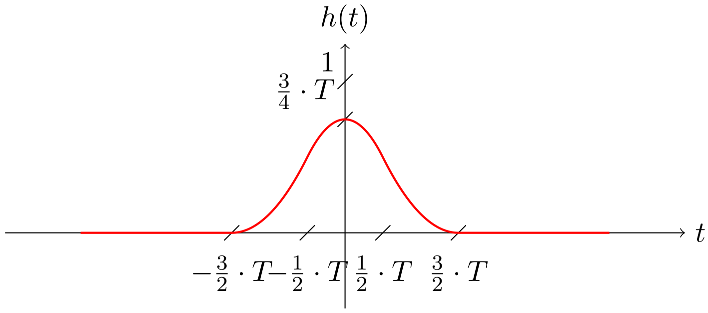
$$f(\tau) \cdot g(t - \tau) = 0$$

a więc wartość splotu wyznaczona ze wzoru

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) \cdot d\tau = \\ &= \int_{-\infty}^{\infty} 0 \cdot d\tau = \\ &= 0 \end{aligned}$$

Podsumowanie Zbierając wyniki, wynik splotu wyrażony jest jako funkcja o pięciu przedziałach

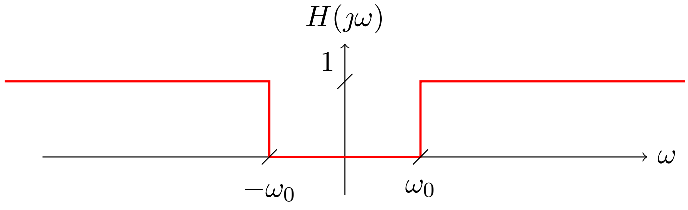
$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) \cdot d\tau = \\ &= \begin{cases} 0 & \tau \in \left(-\infty; -\frac{3}{2} \cdot T\right); \\ \frac{1}{2 \cdot T} \cdot t^2 + \frac{3}{2} \cdot t + \frac{9}{8} \cdot T & \tau \in \left(-\frac{3}{2} \cdot T; -\frac{1}{2} \cdot T\right); \\ -\frac{1}{T} \cdot t^2 + \frac{3}{4} \cdot T & \tau \in \left(-\frac{1}{2} \cdot T; \frac{1}{2} \cdot T\right); \\ \frac{9}{8} \cdot T - \frac{3}{2} \cdot t + \frac{1}{2 \cdot T} \cdot t^2 & \tau \in \left(\frac{1}{2} \cdot T; \frac{3}{2} \cdot T\right); \\ 0 & \tau \in \left(\frac{3}{2} \cdot T; \infty\right); \end{cases} \end{aligned}$$


$h(t)$
1
$\frac{3}{4} \cdot T$
t
$-\frac{3}{2} \cdot T$
$-\frac{1}{2} \cdot T$
$\frac{1}{2} \cdot T$
$\frac{3}{2} \cdot T$

4.2 Filters

Task 1.

Na układ LTI o transmitancji podanej poniżej, podano sygnał $u(t) = A \cdot Sa\left(3 \cdot \omega_0 \cdot t\right)$. Wyznacz odpowiedź układu $y(t)$ wiedząc, że $\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$.



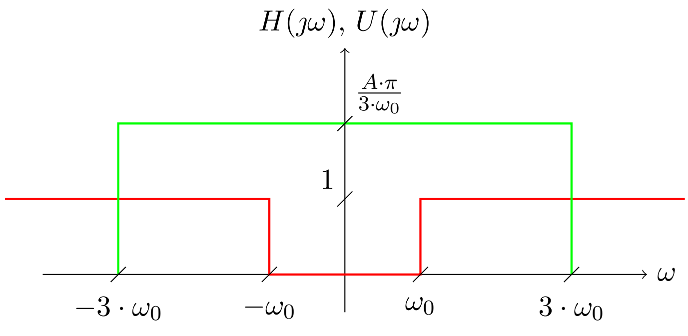
Wiemy, że odpowiedź układu LTI można obliczyć z zależności $y(t) = u(t) * h(t)$, gdzie $h(t)$ jest odpowiedzią impulsową układu. Wiemy także, że transformatę odpowiedzi układu można wyznaczyć ze wzoru $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$.

Ponieważ wyznaczenie splotu liniowego sygnałów jest bardziej skomplikowane niż operacja mnożenia, dlatego spróbujemy skorzystać z tej drugie zależności, czyli mnożenia transformat. W tym celu musimy wyznaczyć transformatę sygnału wejściowego $u(t)$, czyli $U(\jmath\omega)$.

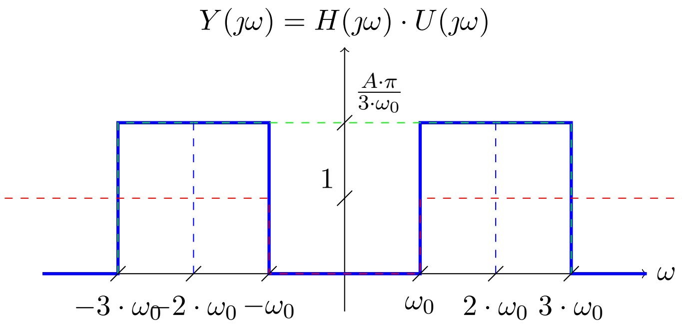
$$
\begin{aligned}
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \\
Sa\left(\frac{t}{2}\right) &\xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega) \\
Sa\left(6 \cdot \omega_0 \cdot \frac{t}{2}\right) &\xrightarrow{\mathcal{F}} \frac{1}{|6 \cdot \omega_0|} \cdot 2\pi \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right) \\
A \cdot Sa\left(3 \cdot \omega_0 \cdot t\right) &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{3 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right)
\end{aligned}
$$

Transformata sygnału wejściowego $u(t)$ to $U(\jmath\omega) = \frac{A \cdot \pi}{3 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right)$.

Transformatę sygnału wyjściowego, czyli $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$ wyznaczymy graficznie, W tym celu na wykresie transmitancji $H(\jmath\omega)$ dodamy transformatę $U(\jmath\omega)$:



Teraz dokonujmy operacji mnożenia transformat $U(\jmath\omega)$ przez $H(\jmath\omega)$



$$Y(\jmath\omega) = \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega-2\cdot\omega_0}{2\cdot\omega_0}\right) + \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega+2\cdot\omega_0}{2\cdot\omega_0}\right) \tag{4.3}$$

Skoro zanmy transformatę sygnału wyjściowego, to spróbujmy wyznaczyć sygnał wyjściowy w dziedzinie czasu wykorzystując wcześniejsze obliczenia. Transformata $Y(\jmath\omega)$ to suma dwóch przeskalowanych prostokątów, przesuniętych na osi pulsacji. W takim razie można wnioskować, że sygnał w dziedzinie czasiu to będzie suma dwóch zmodulowanych i przeskalowanych funkcji $Sa(t)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi\cdot\Pi(-\omega)$$
$$Sa\left(2\cdot\omega_0\cdot\frac{t}{2}\right) \xrightarrow{\mathcal{F}} \frac{1}{|2\cdot\omega_0|}\cdot 2\pi\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)$$
$$Sa(\omega_0\cdot t) \xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)$$
$$e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) \xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-2\cdot\omega_0}{2\cdot\omega_0}\right)$$
$$\frac{A}{3}\cdot e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) \xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega-2\cdot\omega_0}{2\cdot\omega_0}\right)$$
$$\frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) \xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega-(-2\cdot\omega_0)}{2\cdot\omega_0}\right)$$
$$\frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) \xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega+2\cdot\omega_0}{2\cdot\omega_0}\right)$$

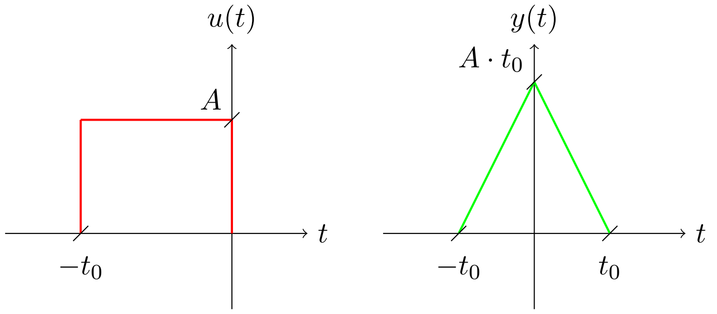
Podsumowując sygnał wyjściowy $y(t)$:

$$\begin{aligned} y(t) &= \frac{A}{3}\cdot e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) + \frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) = \\ &= \frac{A}{3}\cdot Sa(\omega_0\cdot t)\cdot\left(e^{(\jmath\cdot 2\cdot\omega_0\cdot t)} + e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\right) = \\ &= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\ &= \frac{2\cdot A}{3}\cdot Sa(\omega_0\cdot t)\cdot cos\left(2\cdot\omega_0\cdot t\right)) \end{aligned}$$

Odpowiedź układu to $y(t) = \frac{2\cdot A}{3}\cdot Sa(\omega_0\cdot t)\cdot cos\left(2\cdot\omega_0\cdot t\right))$.

Task 2.

Wyznacz odpowiedź implusową $h(t)$ układu LTI, wiedząc, że sygnały $u(t)$ oraz $y(t)$ wygladają jak na poniższych wykresach. Wykorzystaj informacje o transformatach sygnałów: $\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$ oraz $\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right)$.



Wiemy, że transformatę odpowiedzi układu można wyznaczyć ze wzoru $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$ oraz że $h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$. W związku z tym $H(\jmath\omega) = \frac{Y(\jmath\omega)}{U(\jmath\omega)}$ oraz $h(t) \xrightarrow{\mathcal{F}^{-1}} H(\jmath\omega)$.

W pierwszym kroku wyznaczmy transformaty sygnałów $u(t)$ oraz $y(t)$:

$$
\begin{aligned}
u(t) &= A \cdot \Pi\left(\frac{t + \frac{t_0}{2}}{t_0}\right) \\
U(\jmath\omega) &= \mathcal{F}\{u(t)\} \\
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \\
\Pi\left(\frac{1}{t_0} \cdot t\right) &\xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \\
\Pi\left(\frac{t + \frac{t_0}{2}}{t_0}\right) &\xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{t_0}{2}} \\
A \cdot \Pi\left(\frac{t + \frac{t_0}{2}}{t_0}\right) &\xrightarrow{\mathcal{F}} A \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{t_0}{2}}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= A \cdot t_0 \cdot \Lambda\left(\frac{t}{t_0}\right) \\
Y(\jmath\omega) &= \mathcal{F}\{y(t)\} \\
\Lambda(t) &\xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right) \\
\Lambda\left(\frac{1}{t_0} \cdot t\right) &\xrightarrow{\mathcal{F}} t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right) \\
A \cdot t_0 \cdot \Lambda\left(\frac{t}{t_0}\right) &\xrightarrow{\mathcal{F}} A \cdot t_0^2 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)
\end{aligned}
$$

Skoro zanmy transformaty sygnałów wejściowego i wyjściowego, to możemy wyznaczyc transmitancję układu, czyli $H(\jmath\omega)$.

$$
\begin{aligned}
H(\jmath\omega) &= \frac{Y(\jmath\omega)}{U(\jmath\omega)} = \\
&= \frac{A \cdot t_0^2 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)}{A \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{\jmath \cdot \omega \cdot \frac{t_0}{2}}} = \\
&= t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}}
\end{aligned}
$$

Teraz możemy wyznaczć odpowiedź implusową układu $h(t)$:

$$
\begin{aligned}
h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) \\
? &\xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{t_0}{2}} \\
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)
\end{aligned}
$$

$$\Pi\left(\frac{1}{t_0}\cdot t\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$
$$\Pi\left(\frac{t-\frac{t_0}{2}}{t_0}\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}$$

Odpowiedź implusowa układu to $h(t) = \Pi\left(\frac{t-\frac{t_0}{2}}{t_0}\right)$.

ISBN 978-83-939620-3-7