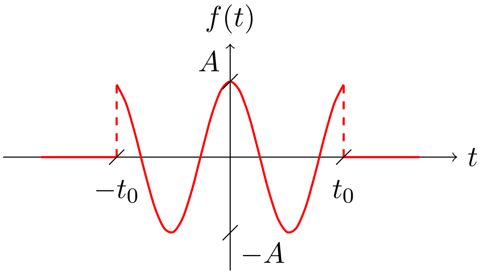
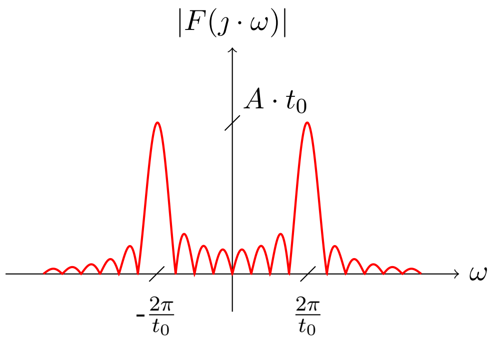
Signal Theory in practise





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot [\ Sa\left(\omega \cdot t_0 + 2\pi\right) - Sa\left(\omega \cdot t_0 - 2\pi\right)]$$

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Chapter 1

Fundamental concepts and measures

1.1 Basic signal metrics

1.1.1 Mean value of a signal

1.1.2 Energy of a signal

1.1.3 Power and effective value of a signal

Chapter 2

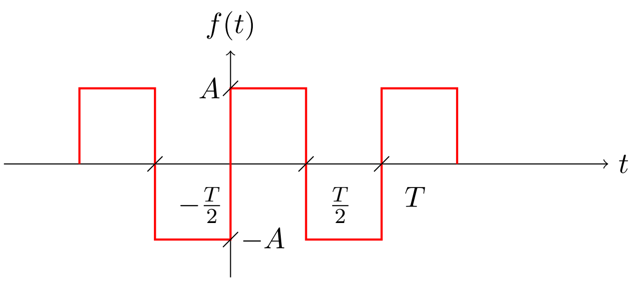
Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

2.2 Complex exponential Fourier series

2.3 Computing the power of a signal – the Parseval's theorem

Task 1. Compute the percentage contribution of the fundamental (first) harmonic in the total power of the periodic square signal shown in the figure below:



$$\frac{P_1}{P} = ? \tag{2.1}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(t) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -A & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.2}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.3}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |-A|^2 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + A^2 \cdot \int_{\frac{T}{2}}^{T} dt \right) = \\
&= \frac{A^2}{T} \cdot \left(t|_0^{\frac{T}{2}} + t|_{\frac{T}{2}}^{T} \right) = \\
&= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 + T - \frac{T}{2} \right) = \\
&= \frac{A^2}{T} \cdot (T) = \\
&= A^2
\end{aligned}$$

The total power of the $f(t)$ signal equals A^2.

Based on Parseval theorem, power of the fundamental harmonic is defined as:

$$P_1 = |F_1|^2 + |F_{-1}|^2 \tag{2.4}$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.5}$$

In order to calculate the P_1, the F_1 coefficient has to be calculated:

$$F_1 = \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.6}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_1 &= \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} -A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt - \int_{\frac{T}{2}}^{T} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z & = -\jmath \cdot \frac{2\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \end{array} \right\} = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} - \int_{\frac{T}{2}}^{T} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{T \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot dz - \int_{\frac{T}{2}}^{T} e^z \cdot dz \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^z|_0^{\frac{T}{2}} - e^z|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot T} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \pi} - e^0 - e^{-\jmath \cdot 2 \cdot \pi} + e^{-\jmath \cdot \pi} \right) = \\
&= \begin{Bmatrix} e^{-\jmath \cdot 2 \cdot \pi} & = cos(2\pi) - \jmath \cdot sin(2\pi) = 1 \\ e^{-\jmath \cdot \pi} & = cos(\pi) - \jmath \cdot sin(\pi) = -1 \end{Bmatrix} = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-1 - 1 - 1 - 1) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-4) = \\
&= \frac{2 \cdot A}{\jmath \cdot \pi} = \\
&= -\jmath \cdot \frac{2 \cdot A}{\pi}
\end{aligned}
$$

The F_1 coefficient equals $-\jmath \cdot \frac{2 \cdot A}{\pi}$.

Thus, the P_1 may be calculated:

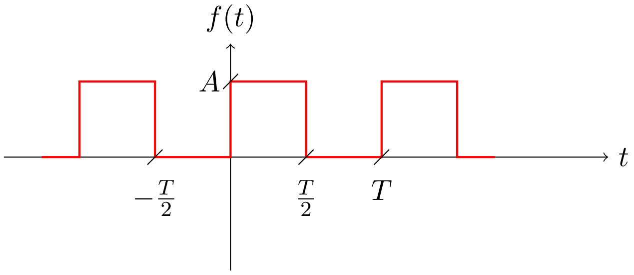
$$
\begin{aligned}
P_1 &= 2 \cdot |F_1|^2 = \\
&= 2 \cdot \left| -\jmath \cdot \frac{2 \cdot A}{\pi} \right|^2 = \\
&= 2 \cdot \left(\frac{2 \cdot A}{\pi} \right)^2 = \\
&= 2 \cdot \frac{4 \cdot A^2}{\pi^2} = \\
&= \frac{8 \cdot A^2}{\pi^2}
\end{aligned}
$$

The power of the fundamental harmonic equals $P_1 = \frac{8 \cdot A^2}{\pi^2}$.

Finally, the percentage contribution of the fundamental harmonic in the total power of the $f(t)$ signal is equal to:

$$
\frac{P_1}{P} = \frac{\frac{8 \cdot A^2}{\pi^2}}{A^2} = \frac{8}{\pi^2} \approx 81\% \tag{2.7}
$$

Task 2. Calculate the percentage contribution of the power of the higher harmonics ($k > 1$) to the total average power of the periodic signal shown below.



$$\frac{P_{>1}}{P} = ? \tag{2.8}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.9}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.10}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ = \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |0|^2 \cdot dt \right) = \\ = \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ = \frac{A^2}{T} \cdot \left(t|_0^{\frac{T}{2}} \right) = \\ = \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ = \frac{A^2}{T} \cdot \left(\frac{T}{2} \right) = \\ = \frac{A^2}{2} \end{aligned}$$

The total power of the $f(t)$ signal equals $\frac{A^2}{2}$.

Based on Parseval theorem, the power of the higher harmonics is defined as:

$$P_{>1} = P - P_0 - P_1 \tag{2.11}$$

where:

$$
\begin{aligned}
P_0 &= |F_0|^2 \\
P_1 &= |F_1|^2 + |F_{-1}|^2
\end{aligned}
$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.12}$$

In order to calculate P_0 and P_1, the F_0 and F_1 coefficients have to be calculated. The F_k coefficients have been calculated in task **??** and are equal to:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big)
\end{aligned}
$$

Now, we may calculate the P_0 and P_1:

$$
\begin{aligned}
P_0 &= |F_0|^2 \\
&= \left| \frac{A}{2} \right|^2 \\
&= \frac{A^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= 2 \cdot |F_1|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1 \Big) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{2\pi} \cdot (-1 - 1) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{A}{2\pi} \cdot (-2) \right|^2 \\
&= 2 \cdot \left| \jmath \cdot \frac{-A}{\pi} \right|^2 \\
&= 2 \cdot \left(\frac{A}{\pi} \right)^2 \\
&= 2 \cdot \frac{A^2}{\pi^2}
\end{aligned}
$$

Finally, the power of the higher harmonics is defined as:

$$
\begin{aligned}
P_{>1} &= P - P_0 - P_1 \\
&= \frac{A^2}{2} - \frac{A^2}{4} - 2 \cdot \frac{A^2}{\pi^2} \\
&= \frac{2 \cdot A^2 \cdot \pi^2}{4\pi^2} - \frac{A^2 \cdot \pi^2}{4\pi^2} - \frac{8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \pi^2 - 8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}
\end{aligned}
$$

The power of the fundamental harmonic equals $P_{>1} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}$.

Finally, the percentage contribution of the higher harmonics in the total power of the $f(t)$ signal is equal to:

$$
\frac{P_{>1}}{P} = \frac{\frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}}{\frac{A^2}{2}} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2} \cdot \frac{2}{A^2} = \frac{\pi^2 - 8}{2\pi^2} = \approx 9\% \tag{2.13}
$$

Task 3. For a certain real-valued periodic signal, its coefficients of expansion to a complex exponential Fourier series are:

$$F_k = \frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2} \wedge k > 0 \tag{2.14}$$

Compute the mean value ($\bar{f}$), knowing that the effective (RMS) value is $U = \frac{A\sqrt{6}}{60}$. During calculation use:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \tag{2.15}$$

From the theoretical considerations we know that:

$$F_0 = \bar{f}$$
$$U = \sqrt{P}$$

In order to calculate $\bar{f}$ we have to calculate F_0. But we know values of the F_k for $k > 0$ only. However, based on Parseval theorem, the power of the signal is defined as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2 \tag{2.16}$$

This equation may be rewritten as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2$$
$$P = \sum_{k=-\infty}^{-1} |F_k|^2 + |F_0|^2 + \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=-\infty}^{-1} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$

Because the $f(t) \in R$, thus $|F_k| = |F_{-k}|$ and we may write:

$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

Now, we can calculate the F_0:

$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

$$
\begin{aligned}
|F_0|^2 &= U^2 - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2 \\
|F_0|^2 &= \left(\frac{A\sqrt{6}}{60}\right)^2 - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot pi^2}\right|^2 \\
|F_0|^2 &= \frac{A^2 \cdot 6}{3600} - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2}\right|^2 \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \left(\frac{A}{k^2 \cdot 4 \cdot \pi^2}\right)^2 \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \frac{A^2}{k^4 \cdot 16 \cdot \pi^4} \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \frac{A^2}{16 \cdot \pi^4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^4} \\
|F_0|^2 &= \frac{A^2}{600} - \frac{A^2}{8 \cdot \pi^4} \cdot \frac{\pi^4}{90} \\
|F_0|^2 &= \frac{A^2}{600} - \frac{A^2}{720} \\
|F_0|^2 &= \frac{720 \cdot A^2}{600 \cdot 720} - \frac{600 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{720 \cdot A^2 - 600 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{120 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{A^2}{5 \cdot 720} \\
|F_0|^2 &= \frac{A^2}{3600} \\
|F_0| &= \sqrt{\frac{A^2}{3600}} \\
|F_0| &= \frac{A}{60} \\
F_0 &= \pm\frac{A}{60}
\end{aligned}
$$

The mean value is equal to $\bar{f} = \pm\frac{A}{60}$.

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

3.2 Exploiting properties of the Fourier transform

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

4.2 Filters

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