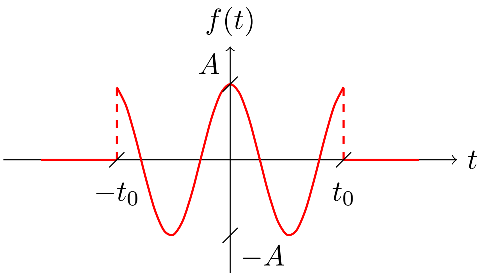
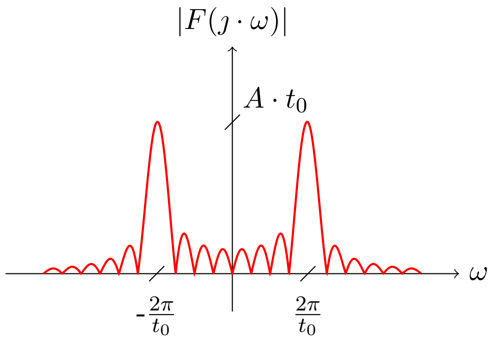
Teoria Sygnałów w zadaniach





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot [\ Sa\left(\omega \cdot t_0 + 2\pi\right) - Sa\left(\omega \cdot t_0 - 2\pi\right)]$$

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Rozdział 1

Podstawowe własności sygnałów

1.1 Podstawowe własności sygnałów

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1.1.2 Energia sygnału

1.1.3 Moc sygnału

Rozdział 2

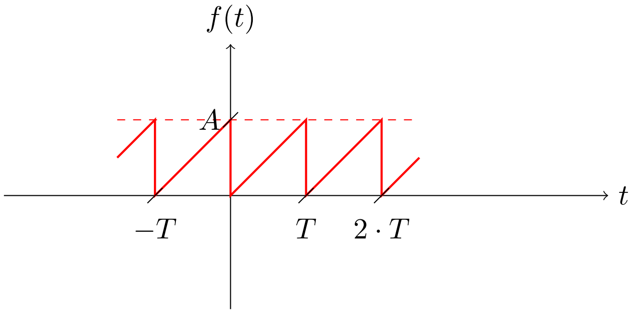
Analiza sygnałów okresowych za pomocą szeregów ortogonalnych

2.1 Trygonometryczny szerego Fouriera

2.2 Zespolony szerego Fouriera

Zadanie 1.

Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.1}$$

In the first period (e.g. $t \in (0; T)$), linear function crosses two points: $(0, 0)$ and (T, A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in C$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, e.g. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \int_0^T \frac{A}{T} \cdot t \cdot dt = \\ &= \frac{A}{T^2} \int_0^T t \cdot dt = \\ &= \frac{A}{T^2} \cdot \frac{1}{2} \cdot t^2 \Big|_0^T = \\ &= \frac{A}{T^2} \cdot \frac{1}{2} \cdot \left(T^2 - 0^2\right) = \\ &= \frac{A}{T^2} \cdot \frac{1}{2} \cdot T^2 = \\ &= \frac{A}{2} \end{aligned}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.3}$$

For the period $t \in (0; T)$, e.g. $k = 0$, we get:

$$\begin{aligned} F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\ &= \frac{1}{T} \int_0^T \frac{A}{T} \cdot t \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \end{aligned}$$

$$
\begin{aligned}
&= \frac{1 \cdot A}{T^2} \int_0^T t \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \\ du & = dt & v & = \frac{T}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{Bmatrix} = \\
&= \frac{A}{T^2} \cdot \left(t \cdot \frac{T}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \Big|_0^T - \int_0^T \frac{T}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T^2} \cdot \left(\left(T \cdot \frac{T}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot T} - 0 \cdot \frac{T}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot 0} \right) + \frac{T^2}{\left(-\jmath \cdot k \cdot 2\pi\right)^2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \Big|_0^T \right) = \\
&= \frac{A}{T^2} \cdot \left(\frac{T^2}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot T} + \frac{T^2}{-\left(k \cdot 2\pi\right)^2} \cdot \left(e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot T} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot 0} \right) \right) = \\
&= A \cdot \left(\frac{1}{-\jmath \cdot k \cdot 2\pi} \cdot e^{-\jmath \cdot k \cdot 2\pi} - \frac{1}{\left(k \cdot 2\pi\right)^2} \cdot \left(e^{-\jmath \cdot k \cdot 2\pi} - e^0 \right) \right) = \\
&= A \cdot \left(\frac{1}{-\jmath \cdot k \cdot 2\pi} - \frac{1}{\left(k \cdot 2\pi\right)^2} \cdot (1 - 1) \right) = \\
&= A \cdot \left(\frac{1}{-\jmath \cdot k \cdot 2\pi} - \frac{1}{\left(k \cdot 2\pi\right)^2} \cdot 0 \right) = \\
&= A \cdot \left(\frac{1}{-\jmath \cdot k \cdot 2\pi} - 0 \right) = \\
&= \frac{A}{-\jmath \cdot k \cdot 2\pi} = \\
&= \jmath \cdot \frac{A}{k \cdot 2\pi}
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi}
\end{aligned}
$$

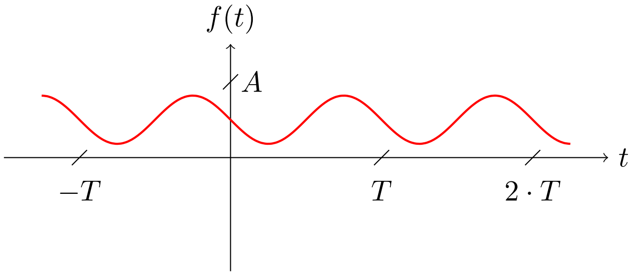
The first several coefficients are equal to:

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$-\jmath \cdot \frac{A}{10 \cdot \pi}$	$-\jmath \cdot \frac{A}{8 \cdot \pi}$	$-\jmath \cdot \frac{A}{6 \cdot \pi}$	$-\jmath \cdot \frac{A}{4 \cdot \pi}$	$-\jmath \cdot \frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\jmath \cdot \frac{A}{2 \cdot \pi}$	$\jmath \cdot \frac{A}{4 \cdot \pi}$	$\jmath \cdot \frac{A}{6 \cdot \pi}$	$\jmath \cdot \frac{A}{8 \cdot \pi}$	$\jmath \cdot \frac{A}{10 \cdot \pi}$
$\lvert F_k \rvert$	$\frac{A}{10 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{10 \cdot \pi}$
$Arg\left(F_k\right)$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

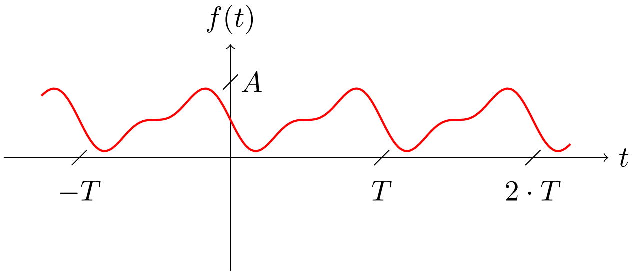
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi}\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.4}
$$

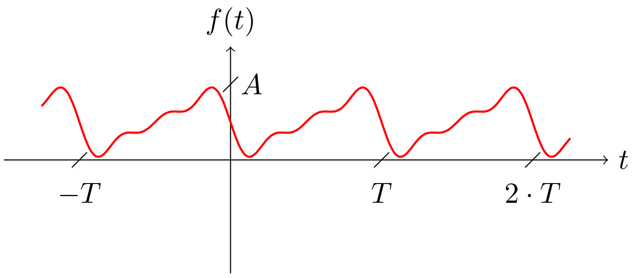
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



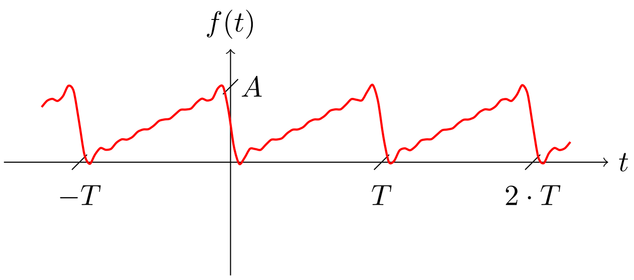
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



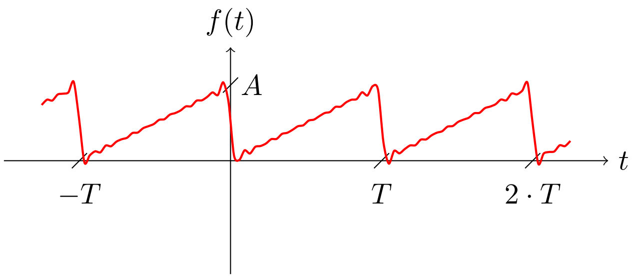
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -7$ to $k_{max} = 7$ results in:

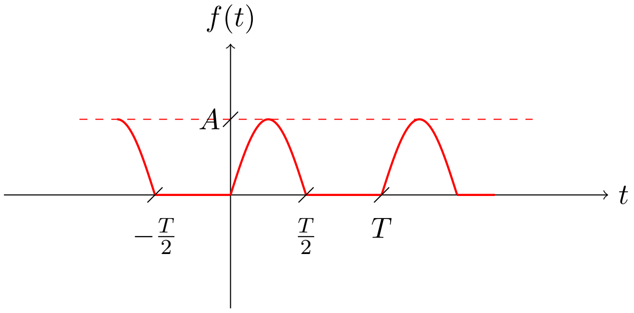


A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = \infty$ to $k_{max} = \infty$ results in oryginal signal.

Zadanie 2. Wyznacz współczynniki zespolonego szeregu fouriera dla okresowego sygnału $f(t)$ przedstawionego na rysunku



W pierwszej kolejności należy opisać sygnał za pomocą wzoru:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in C \tag{2.5}$$

Współczynnik F_0 wyznaczamy ze wzoru

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.6}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{aligned} z &= \tfrac{2\pi}{T} \cdot t \\ dz &= \tfrac{2\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{\frac{2\pi}{T}} \end{aligned} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$\begin{aligned}
&= -\frac{A}{2\pi} \cdot (cos\,(\pi) - cos\,(0)) = \\
&= -\frac{A}{2\pi} \cdot (-1 - 1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}$$

Wartość współczynnika F_0 wynosi $\frac{A}{\pi}$

Współczynnik F_k wyznaczamy ze wzoru

$$F_k = \frac{1}{T}\int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.7}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll}
z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\
dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\
dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)}
\end{array} \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} - \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{T \cdot 2\jmath \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\frac{1}{1-k} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{1+k} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{z_1}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{z_2}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot 0}\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot 0}\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - e^{0}\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - e^{0}\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1+k}{(1-k)\cdot(1+k)}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + \frac{1-k}{(1-k)\cdot(1+k)}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right)}{(1-k)\cdot(1+k)} + \frac{(1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + (1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 1 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} - k + e^{-\jmath\cdot\pi\cdot(1+k)} - 1 - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)} + k}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 2 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} + e^{-\jmath\cdot\pi\cdot(1+k)} - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} + e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-1\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k} - 1\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-e^{-\jmath\cdot\pi\cdot k} - 2 - k\cdot e^{-\jmath\cdot\pi\cdot k} - e^{-\jmath\cdot\pi\cdot k} + k\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\pi\cdot k} - 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot\left(\frac{2\cdot e^{-\jmath\cdot\pi\cdot k} + 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot 2\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}\right) = \\
&= \frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}
\end{aligned}
$$

Wartość współczynnika F_k wynosi $\frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2}$ dla $k \neq 1 \wedge k \neq -1$

F_k dla $k = 1$ musimy wyznaczyć współczynnik raz jeszcze tak wiec wyznaczmy wprost F_1

$$
\begin{aligned}
F_1 &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{sin\left(x\right) \quad = \frac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\jmath}\right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{aligned} z &= -\jmath \cdot \tfrac{4\pi}{T} \cdot t \\ dz &= -\jmath \cdot \tfrac{4\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{-\jmath \cdot \frac{4\pi}{T}} \end{aligned} \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(t\Big|_0^{\frac{T}{2}} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} =
\end{aligned}
$$

$$= \frac{A}{4\jmath} =$$
$$= -\jmath \cdot \frac{A}{4}$$

A wiec wartość współczynnika F_1 wynosi $-\jmath \cdot \frac{A}{4}$
F_k dla $k = -1$ musimy wyznaczyć współczynnik raz jeszcze tak wiec wyznaczmy wprost F_{-1}

$$\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t) \cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t} \cdot dt\right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{ sin\,(x) \quad = \frac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath} \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot dt + 0\right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right) \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot dt\right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t + \jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t + \jmath\cdot\frac{2\pi}{T}\cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt\right) = \\
&= \left\{\begin{matrix} z & = \jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath\cdot\frac{4\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz - \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}}\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} - \left(\frac{T}{2}-0\right)\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot \left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) - \left(\frac{T}{2}-0\right)\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot \left(e^{-\jmath\cdot 2\pi} - e^{0}\right) - \frac{T}{2}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot (1-1) - \frac{T}{2}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 - \frac{T}{2}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(0 - \frac{T}{2}\right) = \\
&= -\frac{A}{T\cdot 2\jmath}\cdot\frac{T}{2} = \\
&= -\frac{A}{4\jmath} = \\
&= \jmath\cdot\frac{A}{4}
\end{aligned}
$$

A wiec wartość współczynnika F_{-1} wynosi $\jmath\cdot\frac{A}{4}$

Ostatecznie współczynniki zespolonego szeregu fouriera dla funkcji przedstawionej na rysunku przyjmują wartości

$$
\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_k &= \frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2} \\
F_{-1} &= \jmath\cdot\frac{A}{4} \\
F_1 &= -\jmath\cdot\frac{A}{4}
\end{aligned}
$$

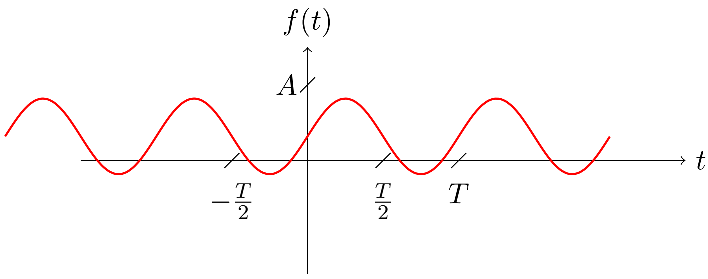
Możemy wyznaczyć kilka wartości współczynników F_k

F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$-\jmath\cdot\frac{A}{4}$	$\frac{A}{\pi}$	$\jmath\cdot\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$\lvert F_k\rvert$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$

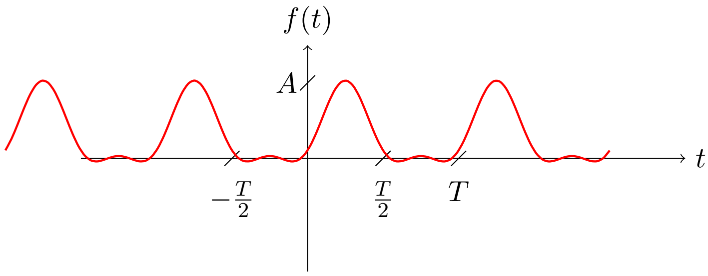
Podstawiając to wzoru aproksymacyjnego funkcje $f(t)$ możemy wyrazić jako

$$f(t) = \sum_{k=-\infty}^{\infty} F_k\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \tag{2.8}$$

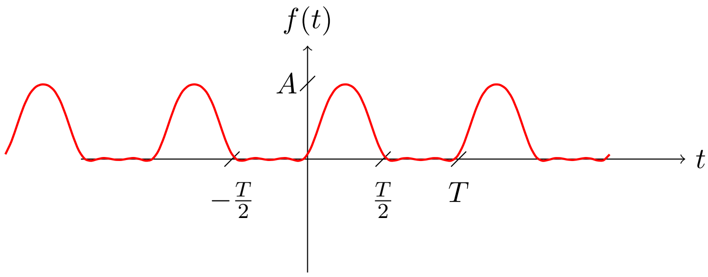
W przypadku sumowania od $k_{min} = -1$ do $k_{max} = 1$ otrzymujemy



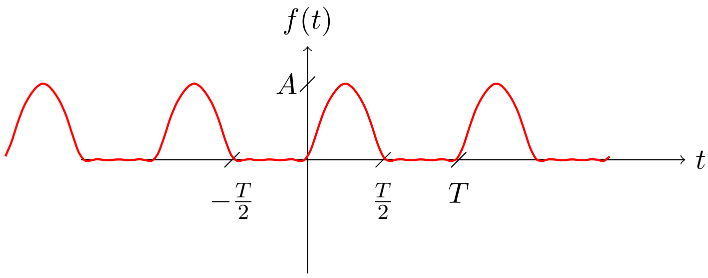
W przypadku sumowania od $k_{min} = -2$ do $k_{max} = 2$ otrzymujemy



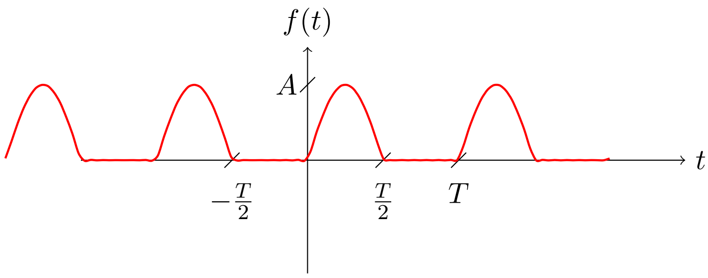
W przypadku sumowania od $k_{min} = -4$ do $k_{max} = 4$ otrzymujemy



W przypadku sumowania od $k_{min} = -6$ do $k_{max} = 6$ otrzymujemy



W przypadku sumowania od $k_{min} = -12$ do $k_{max} = 12$ otrzymujemy



W granicy sumowania od $k_{min} = -\infty$ do $k_{max} = \infty$ otrzymujemy oryginalny sygnał.

2.3 Obliczenia mocy sygnałów - twierdzenie Parsevala

Rozdział 3

Analiza sygnałów nieokresowych. Transformata Fouriera

3.1 Wyznaczanie transformaty Fouriera z definicji

3.2 Wykorzystanie twierdzeń do obliczeń transformaty Fouriera

3.3 Obliczenia energii sygnału za pomocą transformaty Fouriera. Twierdzenie Parsevala

Rozdział 4

Przetwarzanie sygnałów za pomocą układów LTI

4.1 Obliczanie splotu ze wzoru

4.2 Filtry

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