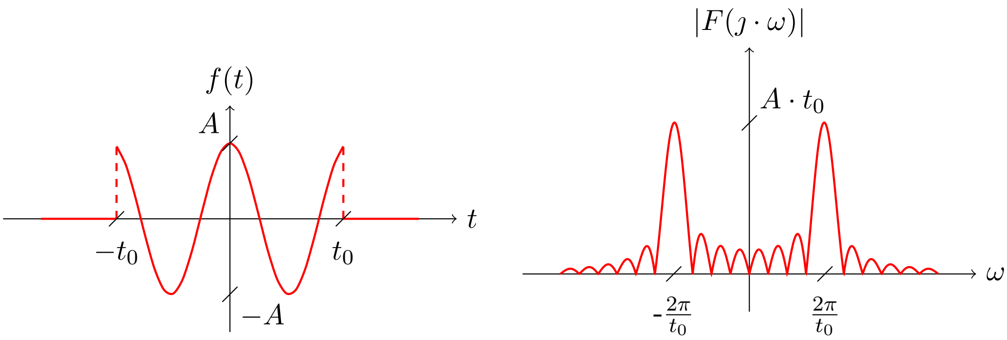
# Signal Theory in practise




$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot [\ Sa\left(\omega \cdot t_0 + 2\pi\right) - Sa\left(\omega \cdot t_0 - 2\pi\right)]$$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# Chapter 1

# Fundamental concepts and measures

## 1.1 Basic signal metrics

### 1.1.1 Mean value of a signal

### 1.1.2 Energy of a signal

### 1.1.3 Power and effective value of a signal

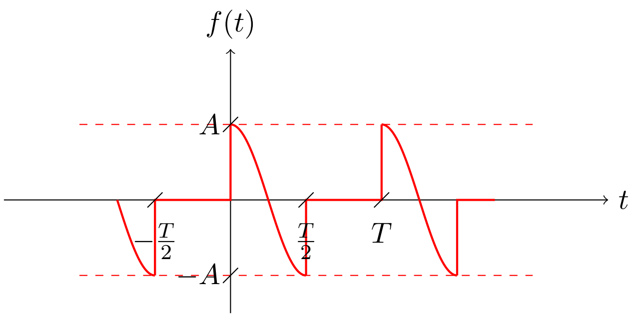
# Chapter 2

# Analysis of periodic signals using orthogonal series

## 2.1 Trigonometric Fourier series

## 2.2 Complex exponential Fourier series

**Task 1.** Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.




Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The $F_0$ coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + 0\right) = \\
&= \frac{A}{T}\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{aligned} z &= \tfrac{2\pi}{T}\cdot t \\ dz &= \tfrac{2\pi}{T}\cdot dt \\ dt &= \tfrac{1}{\frac{2\pi}{T}}\cdot dz \\ dt &= \tfrac{T}{2\pi}\cdot dz \end{aligned}\right\} = \\
&= \frac{A}{T}\cdot\int_0^{\frac{T}{2}} cos\left(z\right)\cdot\frac{T}{2\pi}\cdot dz = \\
&= \frac{A}{T}\cdot\frac{T}{2\pi}\cdot\int_0^{\frac{T}{2}} cos\left(z\right)\cdot dz = \\
&= \frac{A}{T}\cdot\frac{T}{2\pi}\cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi}\cdot\left(sin\left(\frac{2\pi}{T}\cdot\frac{T}{2}\right) - sin\left(\frac{2\pi}{T}\cdot 0\right)\right) = \\
&= \frac{A}{2\pi}\cdot\left(sin\left(pi\right) - sin\left(0\right)\right) = \\
&= \frac{A}{2\pi}\cdot\left(0-0\right) = \\
&= \frac{A}{2\pi}\cdot 0 = \\
&= 0
\end{aligned}
$$

The $F_0$ coefficient equals 0.

The $F_k$ coefficients are defined as:

$$
F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.3}
$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\right)\cdot dt=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\cdot dt\right)=\\
&= \left\{\begin{matrix} z_1=\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t & z_2=-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t\\ dz_1=\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot dt & dz_2=-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot dt\\ dt=\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}\cdot dz_1 & dt=\frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\cdot dz_2\end{matrix}\right\}=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{z_1}\cdot\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}\cdot dz_1+\int_0^{\frac{T}{2}}e^{z_2}\cdot\frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\cdot dz_2\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}\cdot\int_0^{\frac{T}{2}}e^{z_1}\cdot dz_1+\frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\cdot\int_0^{\frac{T}{2}}e^{z_2}\cdot dz_2\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{\jmath\cdot 2\pi\cdot(1-k)}\cdot\int_0^{\frac{T}{2}}e^{z_1}\cdot dz_1-\frac{T}{\jmath\cdot 2\pi\cdot(1+k)}\cdot\int_0^{\frac{T}{2}}e^{z_2}\cdot dz_2\right)=\\
&= \frac{A}{2\cdot T}\cdot\frac{T}{\jmath\cdot 2\pi}\cdot\left(\frac{1}{(1-k)}\cdot\int_0^{\frac{T}{2}}e^{z_1}\cdot dz_1-\frac{1}{(1+k)}\cdot\int_0^{\frac{T}{2}}e^{z_2}\cdot dz_2\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot e^{z_1}\Big|_0^{\frac{T}{2}}-\frac{1}{(1+k)}\cdot e^{z_2}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\Big|_0^{\frac{T}{2}}-\frac{1}{(1+k)}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot\frac{T}{2}}-e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot 0}\right)-\frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot 0}\right)\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)}-e^{0}\right)-\frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)}-1\right)\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)-\frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(-1\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)-\frac{1}{(1+k)}\cdot\left(-1\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(-\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)-\frac{1}{(1+k)}\cdot\left(-\cdot e^{-\jmath\cdot k\cdot\pi}-1\right)\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{\left(-e^{-\jmath\cdot k\cdot\pi}-1\right)\cdot(1+k)}{(1-k)\cdot(1+k)}-\frac{\left(-e^{-\jmath\cdot k\cdot\pi}-1\right)\cdot(1-k)}{(1-k)\cdot(1+k)}\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{-\cdot e^{-\jmath\cdot k\cdot\pi}-1-k\cdot e^{-\jmath\cdot k\cdot\pi}-k}{(1-k)\cdot(1+k)}-\frac{-\cdot e^{-\jmath\cdot k\cdot\pi}-1+k\cdot e^{-\jmath\cdot k\cdot\pi}+k}{(1-k)\cdot(1+k)}\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{-\cdot e^{-\jmath\cdot k\cdot\pi}-1-k\cdot e^{-\jmath\cdot k\cdot\pi}-k+e^{-\jmath\cdot k\cdot\pi}+1-k\cdot e^{-\jmath\cdot k\cdot\pi}-k}{1-k^2}\right)=\\
&= \frac{A}{\jmath\cdot 4\pi}\cdot\left(\frac{-2\cdot k\cdot e^{-\jmath\cdot k\cdot\pi}-2\cdot k}{1-k^2}\right)=\\
&= -\frac{A\cdot k}{\jmath\cdot 2\pi}\cdot\left(\frac{e^{-\jmath\cdot k\cdot\pi}+1}{1-k^2}\right)
\end{aligned}
$$

$$= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1 - k^2} \right)$$

The $F_k$ coefficients equal to $\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate $F_k$ for $k = 1$ directly by definition:

$$\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left( \int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left( A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x\right) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left( A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-1) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+1) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot 0 \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot 2 \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} e^0 \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} 1 \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt = \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \end{matrix} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^z \cdot \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} dt + \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( t\Big|_0^{\frac{T}{2}} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \left( \frac{T}{2} - 0 \right) - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left( \frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left( e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)= \\
&= \frac{A}{2\cdot T}\cdot\frac{T}{2}= \\
&= \frac{A}{4}
\end{aligned}
$$

The $F_1$ coefficients equal to $\frac{A}{4}$.

We have to calculate $F_k$ for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt= \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)= \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot dt\right)= \\
&= \left\{cos\left(x\right)=\tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\}= \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt+0\right)= \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt= \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt= \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt= \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt+\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_0^{\frac{T}{2}} e^{0}\cdot dt\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_0^{\frac{T}{2}} 1\cdot dt\right)= \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_0^{\frac{T}{2}} dt\right)= \\
&= \left\{\begin{matrix} z=\jmath\cdot\frac{4\pi}{T}\cdot t \\ dz=\jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt=\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\}=
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^z\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz+\int_0^{\frac{T}{2}} dt\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^z\cdot dz+\int_0^{\frac{T}{2}} dt\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^z\Big|_0^{\frac{T}{2}}+t\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}+\left(\frac{T}{2}-0\right)\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)+\frac{T}{2}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi}-e^{0}\right)+\frac{T}{2}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)+\frac{T}{2}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0+\frac{T}{2}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(0+\frac{T}{2}\right)=\\
&= \frac{A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{A}{4}
\end{aligned}
$$

The $F_{-1}$ coefficients equal to $\frac{A}{4}$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= \frac{A}{4}\\
F_{-1} &= \frac{A}{4}\\
F_k &= \jmath\cdot\frac{A\cdot k}{2\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)
\end{aligned}
$$

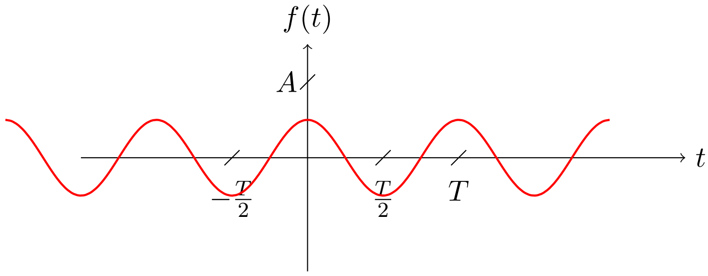
The first several coefficients are equal to:

| $k$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_k$ | $0$ | $\jmath\cdot\frac{4\cdot A}{15\cdot\pi}$ | $0$ | $\jmath\cdot\frac{2\cdot A}{3\cdot\pi}$ | $\frac{A}{4}$ | $0$ | $\frac{A}{4}$ | $-\jmath\cdot\frac{2\cdot A}{3\cdot\pi}$ | $0$ | $-\jmath\cdot\frac{4\cdot A}{15\cdot\pi}$ | $0$ |
| $\|F_k\|$ | $0$ | $\frac{4\cdot A}{15\cdot\pi}$ | $0$ | $\frac{2\cdot A}{3\cdot\pi}$ | $\frac{A}{4}$ | $0$ | $\frac{A}{4}$ | $\frac{2\cdot A}{3\cdot\pi}$ | $0$ | $\frac{4\cdot A}{15\cdot\pi}$ | $0$ |
| $Arg\left\{F_k\right\}$ | $0$ | $\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ | $0$ | $0$ | $0$ | $-\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{2}$ | $0$ |

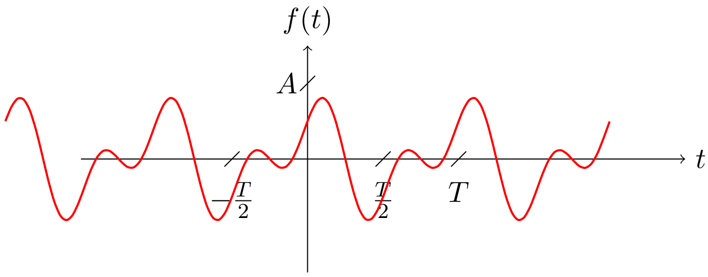
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} + \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k\neq 0 \\ k\neq -1 \wedge k\neq 1}}^{\infty} \left[ \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1-k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{2} \cdot cos\left( \frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k\neq 0 \\ k\neq -1 \wedge k\neq 1}}^{\infty} \left[ \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1-k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.4}
$$

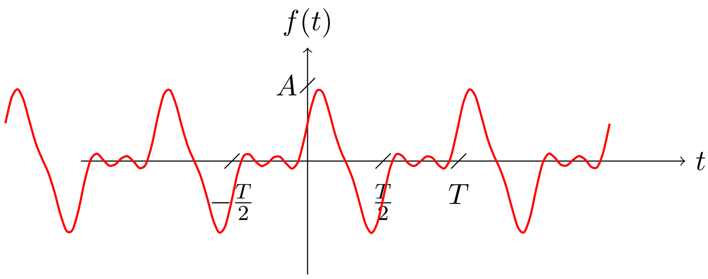
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



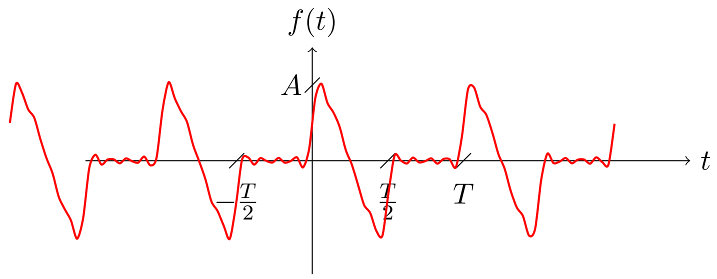

A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



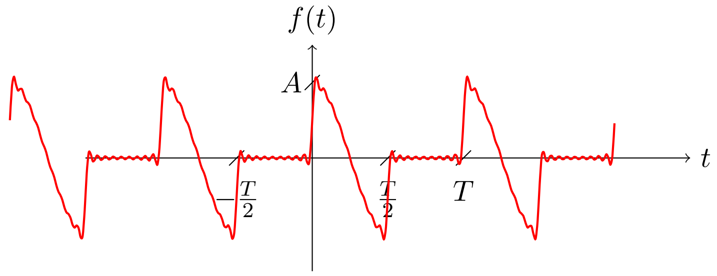

A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



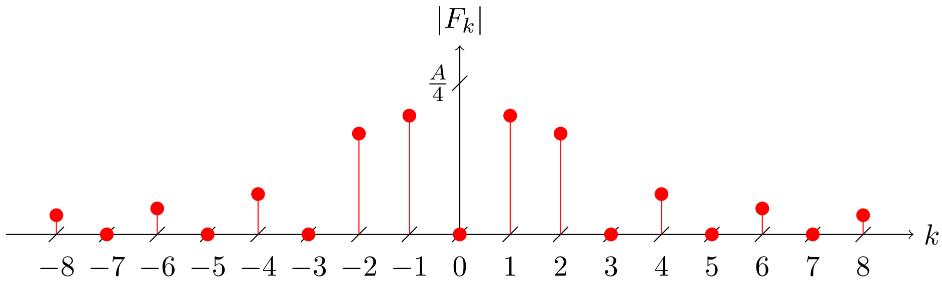

A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:



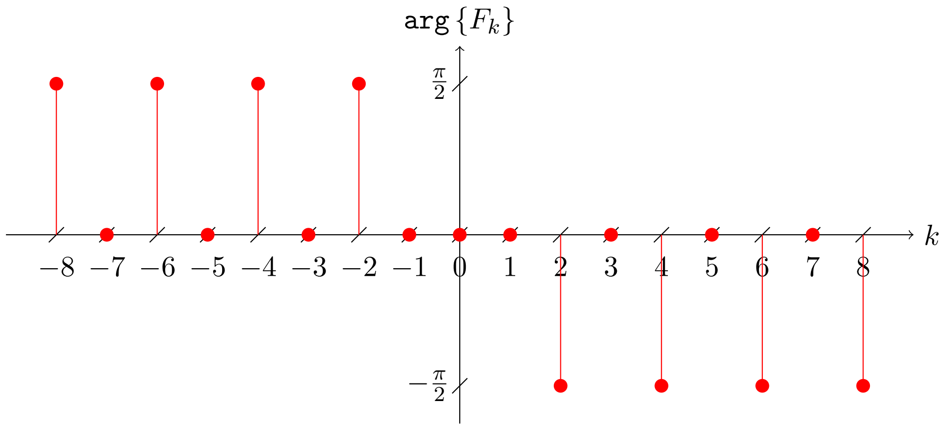

A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:




Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.
Based on coefficents $F_k$ we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

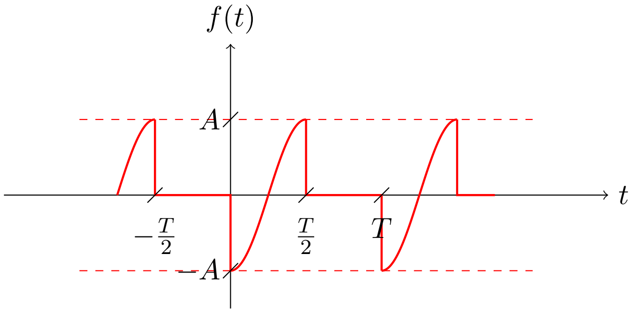



The magnitude spectrum of a real signal is an even-symmetric function of $k$.
Based on coefficents $F_k$ we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.




The phase spectrum of a real signal is an odd-symmetric function of $k$.

**Task 2.** Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.




Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} -A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.5}$$

The $F_0$ coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.6}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left( \int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left( (-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\,(z) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{-A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\,(z) \cdot dz = \\
&= \frac{-A}{2\pi} \cdot sin\,(z)\Big|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot \left( sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{-A}{2\pi} \cdot (0 - 0) = \\
&= \frac{-A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The $F_0$ coefficient equals 0.

The $F_k$ coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.7}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left( \int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left( (-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left( (-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{cc} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{array} \right\} = \\
&= \frac{-A}{2 \cdot T} \cdot \left( \int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left( \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left( \frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left( \frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$\begin{aligned}
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left( e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot \left( e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left( e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot \left( e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left( e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot \left( -1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left( -1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{1}{(1-k)} \cdot \left( -e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left( -e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{\left( -e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left( -e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{-e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left( \frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= \frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left( \frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}$$

The $F_k$ coefficients equal to $-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate $F_k$ for $k=1$ directly by definition:

$$\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left( \int_0^{\frac{T}{2}} (-A) \cdot cos\left( \frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left( (-A) \cdot \int_0^{\frac{T}{2}} cos\left( \frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x\right) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left( (-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left( e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{0}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}1\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{array}{c} z=-\jmath\cdot\frac{4\pi}{T}\cdot t\\ dz=-\jmath\cdot\frac{4\pi}{T}\cdot dt\\ dt=\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\end{array}\right\}=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{z}\cdot\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}}e^{z}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(t\Big|_0^{\frac{T}{2}}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\left(\frac{T}{2}-0\right)-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{-A}{4}
\end{aligned}
$$

The $F_1$ coefficients equal to $\frac{-A}{4}$.
We have to calculate $F_k$ for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}}(-A)\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T}0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=
\end{aligned}
$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) =$$

$$= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} =$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \left\{\begin{array}{c} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{array}\right\} =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\big|_0^{\frac{T}{2}} + t\big|_0^{\frac{T}{2}}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\frac{T}{2} =$$

$$= \frac{-A}{4}$$

The $F_{-1}$ coefficients equal to $\frac{-A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= \frac{-A}{4} \\
F_{-1} &= \frac{-A}{4} \\
F_k &= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}$$

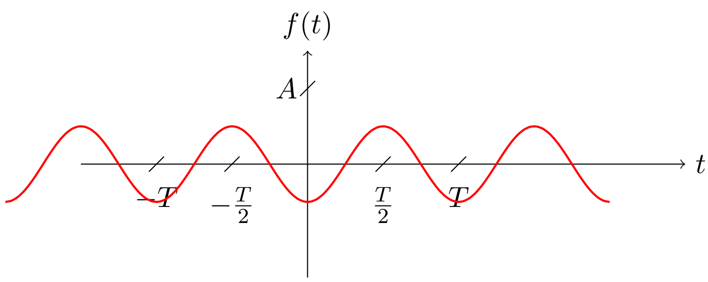
The first several coefficients are equal to:

| $k$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_k$ | $0$ | $-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$ | $0$ | $-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$ | $\frac{-A}{4}$ | $0$ | $\frac{-A}{4}$ | $\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$ | $0$ | $\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$ | $0$ |
| $\lvert F_k \rvert$ | $0$ | $\frac{4 \cdot A}{15 \cdot \pi}$ | $0$ | $\frac{2 \cdot A}{3 \cdot \pi}$ | $\frac{A}{4}$ | $0$ | $\frac{A}{4}$ | $\frac{2 \cdot A}{3 \cdot \pi}$ | $0$ | $\frac{4 \cdot A}{15 \cdot \pi}$ | $0$ |
| $Arg\left\{F_k\right\}$ | $0$ | $-\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{2}$ | $-\pi$ | $0$ | $\pi$ | $\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ | $0$ |

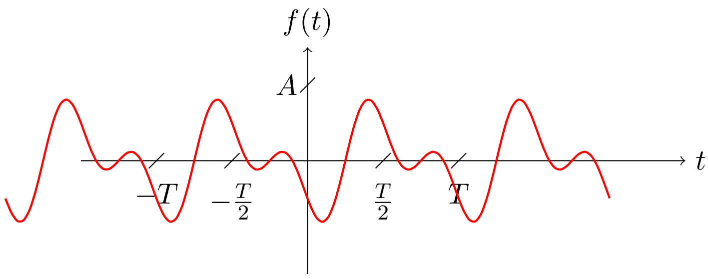
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[ -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{2} \cdot cos\left( \frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[ -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left( \frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.8}$$

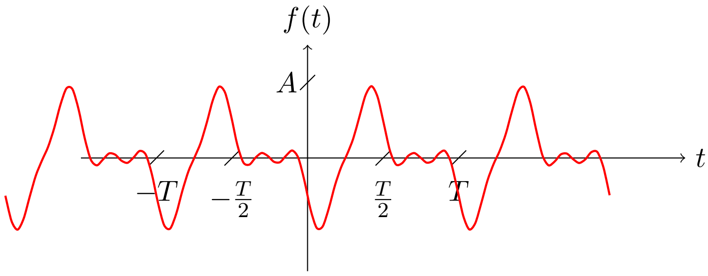
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



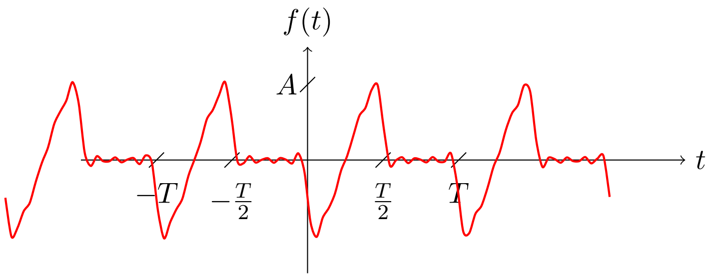

A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



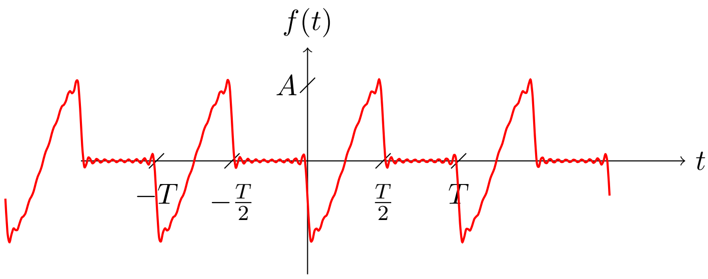

A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



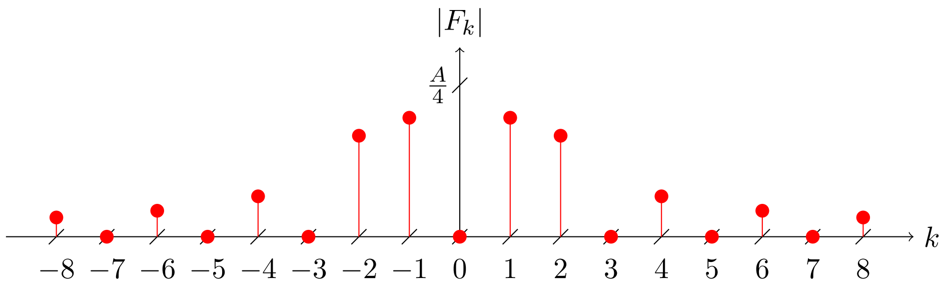

A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:




A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:

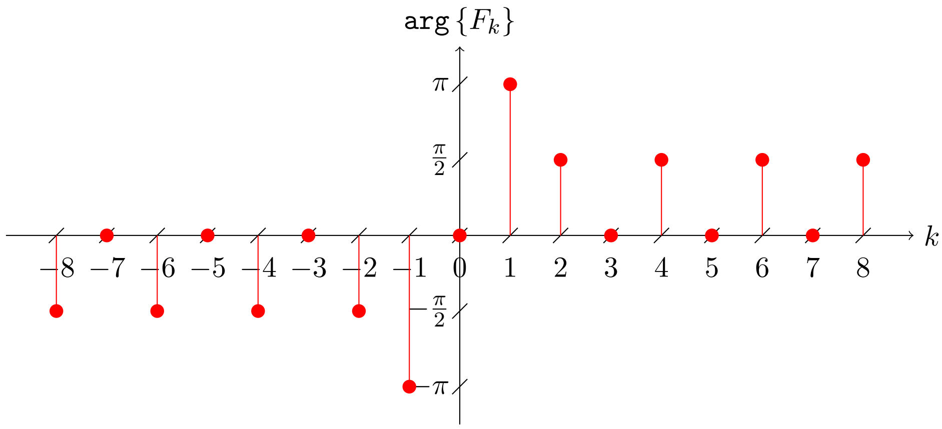



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.
Based on coefficents $F_k$ we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.




The magnitude spectrum of a real signal is an even-symmetric function of $k$.

Based on coefficents $F_k$ we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.




The phase spectrum of a real signal is an odd-symmetric function of $k$.

## 2.3 Computing the power of a signal – the Parseval's theorem

# Chapter 3

# Analysis of non-periodic signals. Fourier Transformation and Transform

## 3.1 Calculation of Fourier Transform by definition

## 3.2 Exploiting properties of the Fourier transform

## 3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

# Chapter 4

# Processing of signals by linear and time invariant (LTI) systems

## 4.1 Linear convolution

## 4.2 Filters

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