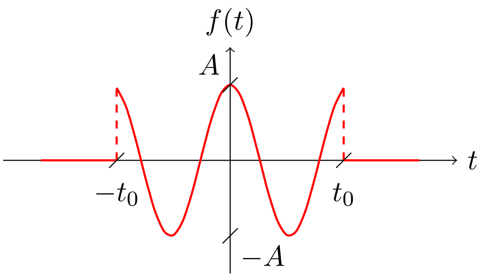
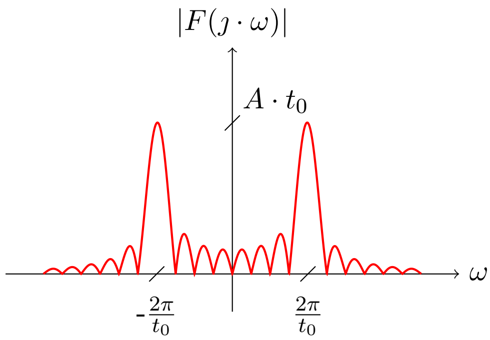
Signal Theory in practise





$$f(t) = A\cdot\Pi\left(\frac{t}{2\cdot t_0}\right)\cdot cos\left(\frac{2\pi}{t_0}\cdot t\right)$$

$$\begin{aligned}F(\jmath\omega) = A\cdot t_0\cdot[\ & Sa\left(\omega\cdot t_0+2\pi\right)\\ & -Sa\left(\omega\cdot t_0-2\pi\right)]\end{aligned}$$

Tomasz Grajek, Krzysztof Wegner

May 8, 2020

Poznan University of Technology
Faculty of Computing and Telecommunications
Institute of Multimedia Telecommunications

pl. M. Skłodowskiej-Curie 5
60-965 Poznań

www.et.put.poznan.pl
www.multimedia.edu.pl

ISBN 978-83-939620-3-7
Printed in Poland

Chapter 1

Fundamental concepts and measures

1.1 Basic signal metrics

1.1.1 Mean value of a signal

1.1.2 Energy of a signal

1.1.3 Power and effective value of a signal

Chapter 2

Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

2.2 Complex exponential Fourier series

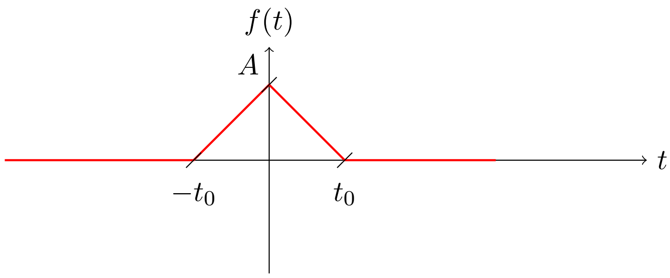
2.3 Computing the power of a signal – the Parseval's theorem

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

Task 1. Compute the Fourier transform of a triangle impulse shown below.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.1}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.2}$$

In order to integrate the $f(t)$ signal, we need to describe it as a piecewise signal.

The simplest form of a linear function is:

$$f(t) = m \cdot t + b \tag{3.3}$$

In the first interval (e.g. $t \in (-t_0; 0)$), linear function crosses two points: $(-t_0, 0)$ and $(0, A)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot (-t_0) + b \\ A = m \cdot 0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot (-t_0) \\ A = b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ A = b \end{cases}$$

$$\begin{cases} A = b \\ \frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{t_0} \cdot t + A$$

In the second interval (e.g. $t \in (0; t_0)$), linear function crosses two points: $(0; A)$ and $(t_0, 0)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot t_0 + b \\ A = m \cdot 0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot t_0 \\ A = b \end{cases}$$

$$\begin{cases} -\frac{b}{t_0} = m \\ A = b \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = -\frac{A}{t_0} \cdot t + A$$

As a result the piecewise linear function is given by:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) = \begin{cases} 0 & \text{for } t \in (-\infty; -t_0) \\ \frac{A}{t_0} \cdot t + A & \text{for } t \in (-t_0; 0) \\ -\frac{A}{t_0} \cdot t + A & \text{for } t \in (0; t_0) \\ 0 & \text{for } t \in (t_0; \infty) \end{cases} \tag{3.4}$$

For the given $f(t)$ signal we get:

$$\begin{aligned} F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\ &= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-t_0}^{0} \left(\frac{A}{t_0} \cdot t + A \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \end{aligned}$$

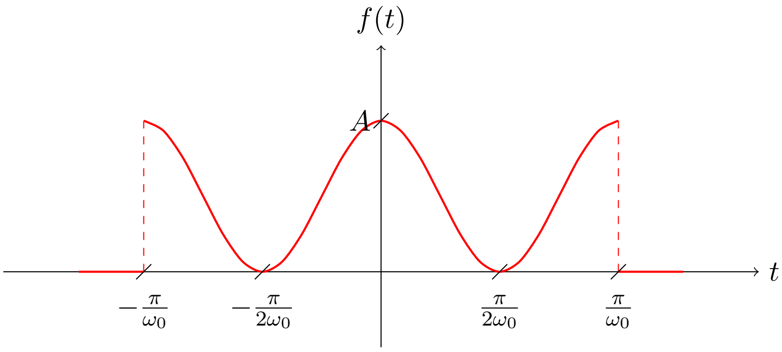
$$+\int_{0}^{t_0}\left(-\frac{A}{t_0}\cdot t+A\right)\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+\int_{t_0}^{\infty}0\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt=$$
$$=\int_{-\infty}^{-t_0}0\cdot dt+\int_{-t_0}^{0}\frac{A}{t_0}\cdot t\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+\int_{-t_0}^{0}A\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt=$$
$$+\int_{0}^{t_0}-\frac{A}{t_0}\cdot t\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+\int_{0}^{t_0}A\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+\int_{t_0}^{\infty}0\cdot dt=$$
$$=0+\frac{A}{t_0}\cdot\int_{-t_0}^{0}t\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+A\cdot\int_{-t_0}^{0}e^{-\jmath\cdot\omega\cdot t}\cdot dt=$$
$$-\frac{A}{t_0}\cdot\int_{0}^{t_0}t\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt+A\cdot\int_{0}^{t_0}e^{-\jmath\cdot\omega\cdot t}\cdot dt+0=$$
$$=\begin{Bmatrix}u & =t & dv & =e^{-\jmath\cdot\omega\cdot t}\cdot dt\\ du & =dt & v & =\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\end{Bmatrix}=$$
$$=\frac{A}{t_0}\cdot\left(t\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}-\int_{-t_0}^{0}\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt\right)=$$
$$+A\cdot\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)=$$
$$-\frac{A}{t_0}\cdot\left(t\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}-\int_{0}^{t_0}\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt\right)=$$
$$+A\cdot\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}\right)=$$
$$=\frac{A}{t_0}\cdot\left(0\cdot e^{-\jmath\cdot\omega\cdot 0}-(-t_0)\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot(-t_0)}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)\right)=$$
$$+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)=$$
$$-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0\cdot e^{-\jmath\cdot\omega\cdot 0}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{0}^{t_0}\right)\right)=$$
$$+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)=$$
$$=\frac{A}{t_0}\cdot\left(0-t_0\cdot\frac{1}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)\right)=$$
$$-\frac{A}{\jmath\cdot\omega}\cdot\left(1-e^{\jmath\cdot\omega\cdot t_0}\right)=$$
$$-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)\right)=$$
$$-\frac{A}{\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-1\right)=$$
$$=-\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{\jmath\cdot\omega}+\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}=$$
$$+\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{-\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}-\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{\jmath\cdot\omega}=$$
$$=-\frac{2\cdot A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=$$
$$=\frac{2\cdot A}{t_0\cdot\omega^2}-\frac{A}{t_0\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=$$
$$=\left\{cos(x)=\frac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\}=$$

$$
\begin{aligned}
&= \frac{2 \cdot A}{t_0 \cdot \omega^2} - \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot cos(\omega \cdot t_0) = \\
&= \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot (1 - cos(\omega \cdot t_0)) = \\
&= \begin{Bmatrix} sin^2(x) = \frac{1}{2} - \frac{1}{2} \cdot cos(2 \cdot x) \\ cos(2 \cdot x) = 1 - 2 \cdot sin^2(x) \end{Bmatrix} = \\
&= \frac{2 \cdot A}{t_0 \cdot \omega^2} \cdot (1 - 1 + 2 \cdot sin^2(\frac{\omega \cdot t_0}{2})) = \\
&= \frac{4 \cdot A}{t_0 \cdot \omega^2} \cdot sin^2(\frac{\omega \cdot t_0}{2}) = \\
&= \frac{A \cdot t_0}{\frac{t_0^2 \cdot \omega^2}{4}} \cdot sin^2(\frac{\omega \cdot t_0}{2}) = \\
&= \left\{ \frac{sin(x)}{x} = Sa(x) \right\} = \\
&= A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})
\end{aligned}
$$

The Fourier transform of the $f(t) = A \cdot \Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega) = A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$.

3.2 Exploiting properties of the Fourier transformation

Task 1. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń, wiedząc że transformata sygnału prostokątnego $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.



Zacznijmy od napisania wzoru sygnału przedstawionego na rysunku

$$f(t) = \begin{cases} 0 & t \in \left(-\infty; -\frac{\pi}{\omega_0}\right) \\ A \cdot cos^2(\omega_0 \cdot t) & t \in \left(-\frac{\pi}{\omega_0}; \frac{\pi}{\omega_0}\right) \\ 0 & t \in \left(\frac{\pi}{\omega_0}; \infty\right) \end{cases}$$

Co możemy zapisać za pomocą sygnałów elementarnych jako

$$f(t) = A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2(\omega_0 \cdot t)$$

Nasz sygnał jest iloczynem pewnej funkcji $h(t)$ oraz cosinusów

$$f(t) = A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2(\omega_0 \cdot t) =$$
$$= A \cdot h(t) \cdot cos^2(\omega_0 \cdot t) =$$

gdzie

$$h(t) = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right)$$

Wyznaczmy transformatę sygnału $h(t)$. Z treści zadania wiemy że transformata sygnału $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.

Postać funkcji $g(t)$ nie jest identyczna z postacią funkcji $h(t)$, funkcja różni się skalą. Wyznaczanym transformaty funkcji przeskalowanej $h(t) = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right)$

Z twierdzenia o zmianie skali mamy

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$h(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

a wiec otrzymujemy

$$\begin{aligned} h(t) = \Pi\left(\frac{t}{T}\right) &= \\ = \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) &= \\ = \Pi\left(t \cdot \frac{\omega_0}{2\pi}\right) &= \\ = g\left(t \cdot \frac{\omega_0}{2\pi}\right) \end{aligned}$$

gdzie

$$\alpha = \frac{\omega_0}{2\pi}$$

a więc

$$\begin{aligned} H(\jmath\omega) &= \frac{1}{\frac{\omega_0}{2\pi}} \cdot G\left(\frac{\jmath\omega}{\frac{\omega_0}{2\pi}}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot G\left(\frac{\jmath\omega \cdot 2\pi}{\omega_0}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot Sa\left(\frac{\frac{\omega \cdot 2\pi}{\omega_0}}{2}\right) = \\ &= \frac{2\pi}{\omega_0} \cdot Sa\left(\omega \cdot \frac{\pi}{\omega_0}\right) \end{aligned}$$

A więc transformata sygnału $h(t)$ jest równa $H(\jmath\omega) = \frac{2\pi}{\omega_0} \cdot Sa\left(\omega \cdot \frac{\pi}{\omega_0}\right)$
Wróćmy do wzoru sygnału i przedstawmy go następująco

$$\begin{aligned} f(t) &= A \cdot \Pi\left(\frac{t \cdot \omega_0}{2\pi}\right) \cdot cos^2\left(\omega_0 \cdot t\right) = \\ &= A \cdot h(t) \cdot cos^2\left(\omega_0 \cdot t\right) = \\ &= A \cdot h(t) \cdot cos\left(\omega_0 \cdot t\right) \cdot cos\left(\omega_0 \cdot t\right) = \\ &= A \cdot k(t) \cdot cos\left(\omega_0 \cdot t\right) = \end{aligned}$$

gdzie

$$k(t) = h(t) \cdot cos\left(\omega_0 \cdot t\right) =$$

$$
\begin{aligned}
&= \left\{ cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2} \right\} = \\
&= h(t) \cdot \frac{e^{\jmath\cdot\omega_0\cdot t} + e^{-\jmath\cdot\omega_0\cdot t}}{2} = \\
&= \frac{1}{2}\left(h(t) \cdot e^{\jmath\cdot\omega_0\cdot t} + h(t)\cdot e^{-\jmath\cdot\omega_0\cdot t} \right)
\end{aligned}
$$

Wyznaczmy transformatę sygnału $k(t)$. Korzystając z twierdzenia o modulacji mamy

$$
\begin{gathered}
h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega) \\
k(t) = h(t)\cdot e^{\jmath\cdot\omega_0\cdot t} \xrightarrow{\mathcal{F}} K(\jmath\omega) = H(\jmath\left(\omega-\omega_0\right))
\end{gathered}
$$

a wiec transformata sygnału $k(t)$ wynosi:

$$
\begin{aligned}
K(\jmath\omega) &= \frac{1}{2}\left(H(\jmath\left(\omega-\omega_0\right)) + H(\jmath\left(\omega+\omega_0\right))\right) = \\
&= \frac{1}{2}\left(\frac{2\pi}{\omega_0}\cdot Sa\left(\left(\omega-\omega_0\right)\cdot\frac{\pi}{\omega_0}\right) + \frac{2\pi}{\omega_0}\cdot Sa\left(\left(\omega+\omega_0\right)\cdot\frac{\pi}{\omega_0}\right)\right) = \\
&= \frac{\pi}{\omega_0}\cdot Sa\left(\left(\omega-\omega_0\right)\cdot\frac{\pi}{\omega_0}\right) + \frac{\pi}{\omega_0}\cdot Sa\left(\left(\omega+\omega_0\right)\cdot\frac{\pi}{\omega_0}\right)
\end{aligned}
$$

Wróćmy do wzoru sygnału $f(t)$

$$
\begin{aligned}
f(t) &= A\cdot\Pi\left(\frac{t\cdot\omega_0}{2\pi}\right)\cdot cos^2\left(\omega_0\cdot t\right) = \\
&= A\cdot h(t)\cdot cos^2\left(\omega_0\cdot t\right) = \\
&= A\cdot h(t)\cdot cos\left(\omega_0\cdot t\right)\cdot cos\left(\omega_0\cdot t\right) = \\
&= A\cdot k(t)\cdot cos\left(\omega_0\cdot t\right) = \\
&= \left\{ cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2} \right\} = \\
&= A\cdot k(t)\cdot\frac{e^{\jmath\cdot\omega_0\cdot t} + e^{-\jmath\cdot\omega_0\cdot t}}{2} = \\
&= \frac{A}{2}\left(k(t)\cdot e^{\jmath\cdot\omega_0\cdot t} + k(t)\cdot e^{-\jmath\cdot\omega_0\cdot t}\right)
\end{aligned}
$$

Znając transformatę sygnału $k(t)$ i korzystając z twierdzenia o modulacji możemy wyznaczyć transformatę sygnału $f(t)$.

$$
\begin{gathered}
k(t) \xrightarrow{\mathcal{F}} K(\jmath\omega) \\
f(t) = k(t)\cdot e^{\jmath\cdot\omega_0\cdot t} \xrightarrow{\mathcal{F}} F(\jmath\omega) = K(\jmath\left(\omega-\omega_0\right))
\end{gathered}
$$

a więc transformata sygnału $f(t)$ wynosi

$$
F(\jmath\omega) = \frac{A}{2}\left(K(\jmath\left(\omega-\omega_0\right)) + K(\jmath\left(\omega+\omega_0\right))\right) =
$$

$$
\begin{aligned}
&= \frac{A}{2}\left(\frac{\pi}{\omega_0}\cdot Sa\left((\omega-\omega_0-\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left((\omega-\omega_0+\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\right.\\
&\left.+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+\omega_0-\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+\omega_0+\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)=\\
&= \frac{A}{2}\left(\frac{\pi}{\omega_0}\cdot Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\right.\\
&\left.+\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)=\\
&= \frac{A}{2}\left(\frac{\pi}{\omega_0}\cdot Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot\frac{\pi}{\omega_0}\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+\right.\\
&\left.+\frac{\pi}{\omega_0}\cdot Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)=\\
&= \frac{A\cdot\pi}{2\cdot\omega_0}\left(Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+Sa\left((\omega+2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ wynosi $F(\jmath\omega) = \frac{A\cdot\pi}{2\cdot\omega_0}\left(Sa\left((\omega-2\cdot\omega_0)\cdot\frac{\pi}{\omega_0}\right)+2\cdot Sa\left(\omega\cdot\frac{\pi}{\omega_0}\right)+Sa\left((\omega+\right.\right.$

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

4.2 Filters

ISBN 978-83-939620-3-7

9 788393 962037