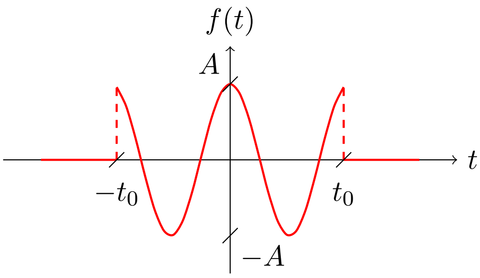
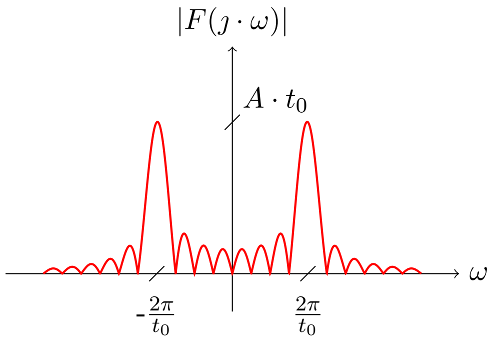
Teoria Sygnałów w zadaniach





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot [\ Sa\left(\omega \cdot t_0 + 2\pi\right) \\ -Sa\left(\omega \cdot t_0 - 2\pi\right)]$$

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Rozdział 1

Podstawowe własności sygnałów

1.1 Podstawowe własności sygnałów

1.1.1 Wartość średnia

1.1.2 Energia sygnału

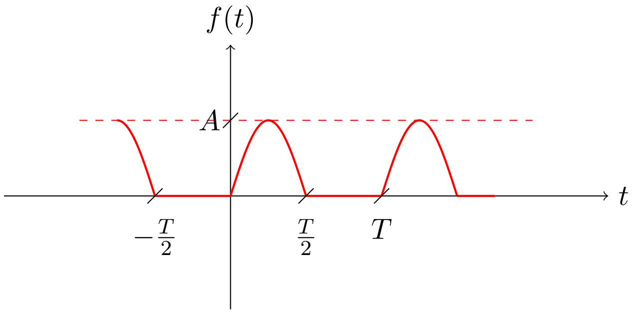
1.1.3 Moc sygnału

Rozdział 2

Analiza sygnałów okresowych za pomocą szeregów ortogonalnych

2.1 Trygonometryczny szerego Fouriera

Zadanie 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



W pierwszej kolejności należy opisać sygnał za pomocą wzoru:
Periodic signal $f(t)$, as a piecewise function assuming period $t \in (0;T)$ is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0;T)$ we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{T}\left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + 0\right) = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{matrix} z & = \frac{2\pi}{T}\cdot t \\ dz & = \frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot dz = \\
&= \frac{A}{2\pi}\cdot\left(-cos\left(z\right)\Big|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\frac{2\pi}{T}\cdot\frac{T}{2}\right) - cos\left(\frac{2\pi}{T}\cdot 0\right)\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{A}{2\pi}\cdot\left(-1-1\right) = \\
&= -\frac{A}{2\pi}\cdot\left(-2\right) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The a_0 coefficient equals $\frac{A}{\pi}$.

The a_k coefficients are defined as:

$$
a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.3}
$$

For the period $t \in (0;T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{\begin{matrix} cos\left(x\right) & = \frac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2} \\ sin\left(x\right) & = \frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath} \end{matrix}\right\} = \\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot\frac{e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot dt + 0\right) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{2\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot\left(e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt\right) =
\end{aligned}
$$

$$= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \right) \cdot dt =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2\jmath} + \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2\jmath} \right) \cdot dt =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k) \right) + sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k) \right) \right) \cdot dt =$$

$$= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k) \right) \cdot dt + \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k) \right) \cdot dt \right) =$$

$$= \left\{ \begin{array}{llll} z_1 & = \frac{2\pi}{T} \cdot t \cdot (1+k) & z_2 & = \frac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 & = \frac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 & = \frac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt & = \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt & = \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{array} \right\} =$$

$$= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} + \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot dz_1 + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot dz_2 \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(-cos\left(z_1\right) \Big|_0^{\frac{T}{2}} \right) + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(-cos\left(z_2\right) \Big|_0^{\frac{T}{2}} \right) \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k) \right) \Bigg|_0^{\frac{T}{2}} \right) + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k) \right) \Bigg|_0^{\frac{T}{2}} \right) \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1+k) \right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1+k) \right) \right) = \right.$$

$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1-k) \right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1-k) \right) \right) \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\pi \cdot (1+k)\right) - cos\left(0\right) \right) = \right.$$

$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\pi \cdot (1-k)\right) - cos\left(0\right) \right) \right) =$$

$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1+k)\right) \right) = \right.$$

$$\left. + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1-k)\right) \right) \right) =$$

$$= \frac{A}{2\pi} \cdot \left(\frac{1}{1+k} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1}{1-k} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$

$$= \frac{A}{2\pi} \cdot \left(\frac{1-k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1+k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$

$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right)}{(1+k)\cdot(1-k)} + \frac{1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k)\cdot(1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right) + 1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k)\cdot(1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \frac{2 - cos\left(\pi \cdot (1+k)\right) + k \cdot cos\left(\pi \cdot (1+k)\right) - cos\left(\pi \cdot (1-k)\right) - k \cdot cos\left(\pi \cdot (1-k)\right)}{1-k^2} =$$
$$= \left\{ \begin{array}{l} cos\left(\pi \cdot (1+k)\right) = cos\left(\pi + k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \\ cos\left(\pi \cdot (1-k)\right) = cos\left(\pi - k \cdot \pi\right) = -cos\left(-k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \end{array} \right\} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + cos\left(k \cdot \pi\right) - k \cdot cos\left(k \cdot \pi\right) + cos\left(k \cdot \pi\right) + k \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + 2 \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{\pi} \cdot \frac{1 + cos\left(k \cdot \pi\right)}{1-k^2}$$

The a_k coefficients equal to $\frac{A}{\pi} \cdot \frac{1+cos(k \cdot \pi)}{1-k^2}$ for $k \neq 1$.

We have to calculate a_k for $k = 1$ directly by definition:

$$a_1 = \frac{2}{T} \int_T f(t) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt =$$
$$= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$
$$= \left\{ \begin{array}{ll} cos\left(x\right) & = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \\ sin\left(x\right) & = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \end{array} \right\} =$$
$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot dt + 0 \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \right) \cdot dt =$$
$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{0} - e^{0} \right) \cdot dt =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}}{2\jmath} \right) \cdot dt =$$
$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot 2\right) \cdot dt =$$

$$
\begin{aligned}
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt = \\
&= \begin{Bmatrix} z & = \frac{4\pi}{T} \cdot t \\ dz & = \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{Bmatrix} = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \\
&= \frac{A}{4\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{4\pi} \cdot \left(-cos\left(\frac{4\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{4\pi}{T} \cdot 0\right)\right) = \\
&= -\frac{A}{4\pi} \cdot \left(cos\left(2\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{A}{4\pi} \cdot \left(1 - 1\right) = \\
&= -\frac{A}{4\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The a_1 coefficient equal to 0.

The b_k coefficients are defined as:

$$
b_k = \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.4}
$$

For the period $t \in (0; T)$ we get:

$$
\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot dt + 0\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =
\end{aligned}
$$

$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot \jmath} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2} - \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2} \right) \cdot dt =$$
$$= -\frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) - cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \right) \cdot dt =$$
$$= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) \cdot dt - \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \cdot dt \right) =$$
$$= \left\{ \begin{array}{llll} z_1 & = \frac{2\pi}{T} \cdot t \cdot (1+k) & z_2 & = \frac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 & = \frac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 & = \frac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt & = \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt & = \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{array} \right\} =$$
$$= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} - \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot dz_1 - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot dz_2 \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(z_1\right)\Big|_0^{\frac{T}{2}} \right) - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(z_2\right)\Big|_0^{\frac{T}{2}} \right) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right)\Big|_0^{\frac{T}{2}} \right) - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right)\Big|_0^{\frac{T}{2}} \right) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1+k)\right) - sin\left(\frac{2\pi}{T} \cdot 0 \cdot (1+k)\right) \right) = \right.$$
$$\left. -\frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1-k)\right) - sin\left(\frac{2\pi}{T} \cdot 0 \cdot (1-k)\right) \right) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\pi \cdot (1+k)\right) - sin\left(0\right) \right) = \right.$$
$$\left. -\frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\pi \cdot (1-k)\right) - sin\left(0\right) \right) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot (0-0) = \right.$$
$$\left. -\frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot (0-0) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot 0 - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot 0 \right) =$$
$$= -\frac{A}{T} \cdot (0-0) =$$
$$= -\frac{A}{T} \cdot 0 =$$
$$= 0$$

The b_k coefficients equal to 0 for $k \neq 1$.

We have to calculate b_k for $k = 1$ directly by definition:

$$
\begin{aligned}
b_1 &= \frac{2}{T}\int_T f(t)\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot sin\left(1\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{ sin\,(x) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath} \right\} = \\
&= \frac{2}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot dt + 0\right) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{2\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt\right) = \\
&= \frac{2}{T}\cdot\frac{A}{2\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}-e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}-e^{0}-e^{0}\right)\cdot dt = \\
&= \frac{A}{T\cdot\jmath\cdot\jmath}\cdot\int_0^{\frac{T}{2}}\left(\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}}{2}-\frac{1+1}{2}\right)\cdot dt = \\
&= -\frac{A}{T}\cdot\int_0^{\frac{T}{2}}\left(cos\left(\frac{2\pi}{T}\cdot t\cdot 2\right)-1\right)\cdot dt = \\
&= -\frac{A}{T}\cdot\left(\int_0^{\frac{T}{2}} cos\left(\frac{4\pi}{T}\cdot t\right)\cdot dt-\int_0^{\frac{T}{2}} dt\right) = \\
&= \left\{\begin{aligned} z &= \tfrac{4\pi}{T}\cdot t \\ dz &= \tfrac{4\pi}{T}\cdot dt \\ dt &= \tfrac{dz}{\frac{4\pi}{T}} \end{aligned}\right\} = \\
&= -\frac{A}{T}\cdot\left(\int_0^{\frac{T}{2}} cos\,(z)\cdot\frac{dz}{\frac{4\pi}{T}}-t|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\int_0^{\frac{T}{2}} cos\,(z)\cdot dz-\left(\frac{T}{2}-0\right)\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\,sin\,(z)|_0^{\frac{T}{2}}-\frac{T}{2}\right) = \\
&= -\frac{A}{T}\cdot\left(\frac{1}{\frac{4\pi}{T}}\,sin\left(\frac{4\pi}{T}\cdot t\right)\bigg|_0^{\frac{T}{2}}-\frac{T}{2}\right) =
\end{aligned}
$$

$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(sin\left(\frac{4\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{4\pi}{T} \cdot 0\right) \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(sin\left(2\pi\right) - sin\left(0\right) \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(0 - 0 \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(-\frac{T}{2} \right) =$$
$$= \frac{A}{2}$$

The b_1 coefficient equal to $\frac{A}{2}$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0 = \frac{A}{\pi}$$
$$a_1 = 0$$
$$a_k = \frac{A}{\pi} \cdot \frac{1 + cos\left(k \cdot \pi\right)}{1 - k^2}$$
$$b_1 = \frac{A}{2}$$
$$b_k = 0$$

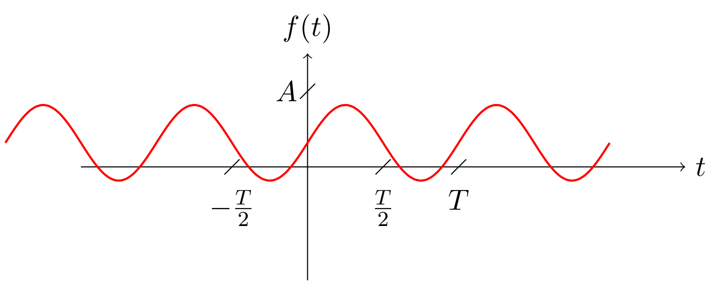
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	$-\frac{2}{3}\frac{A}{\pi}$	0	$-\frac{2}{15}\frac{A}{\pi}$	0	$-\frac{2}{35}\frac{A}{\pi}$
b_k	$\frac{A}{2}$	0	0	0	0	0

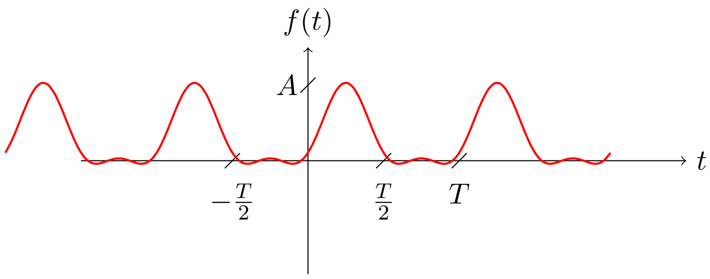
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right] \tag{2.5}$$

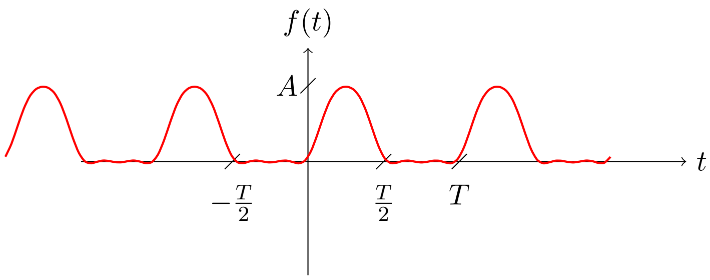
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



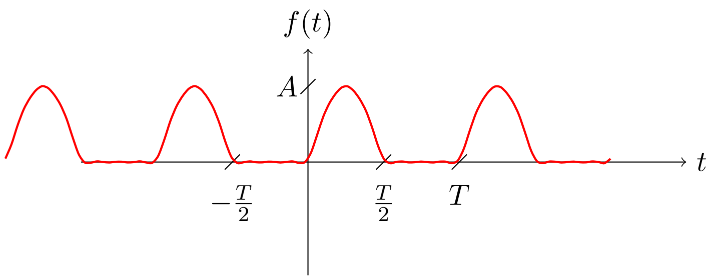
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



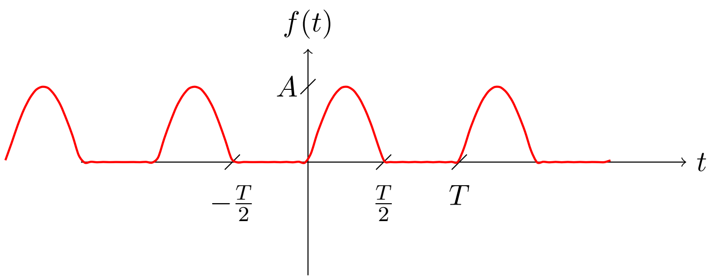
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

2.2 Zespolony szerego Fouriera

2.3 Obliczenia mocy sygnałów - twierdzenie Parsevala

Rozdział 3

Analiza sygnałów nieokresowych. Transformata Fouriera

3.1 Wyznaczanie transformaty Fouriera z definicji

3.2 Wykorzystanie twierdzeń do obliczeń transformaty Fouriera

3.3 Obliczenia energii sygnału za pomocą transformaty Fouriera. Twierdzenie Parsevala

Rozdział 4

Przetwarzanie sygnałów za pomocą układów LTI

4.1 Obliczanie splotu ze wzoru

4.2 Filtry

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