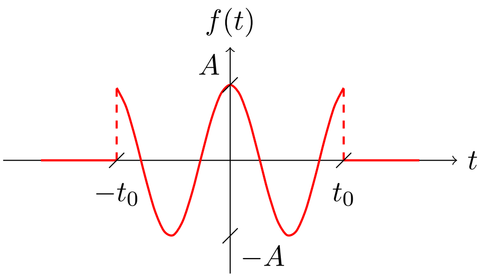
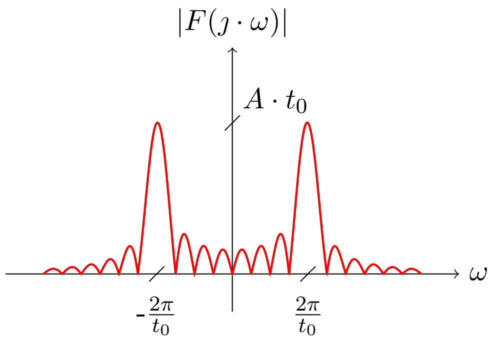
Signal Theory in practise





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\left(\omega \cdot t_0 + 2\pi\right) \\ & - Sa\left(\omega \cdot t_0 - 2\pi\right)] \end{aligned}$$

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Chapter 1

Fundamental concepts and measures

1.1 Basic signal metrics

1.1.1 Mean value of a signal

1.1.2 Energy of a signal

1.1.3 Power and effective value of a signal

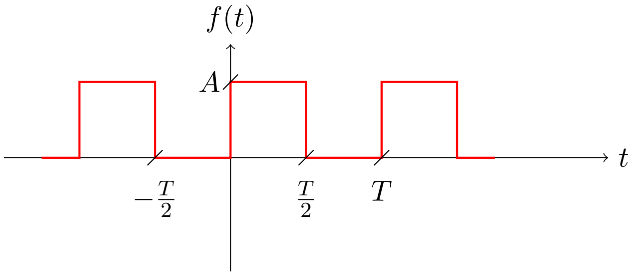
Chapter 2

Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

2.2 Complex exponential Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} dt + 0\right) = \\
&= \frac{1}{T}\left(A\cdot t|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T}\cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}-0\right) = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.3}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.4}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \left\{\begin{aligned} z &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot t \\ dz &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot dt \\ dt &= \tfrac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{aligned}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^z\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^z\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^z|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^0\right) =
\end{aligned}$$

$$= -\frac{A}{\jmath \cdot k \cdot 2\pi}\left(e^{-\jmath \cdot k \cdot \pi} - 1\right) =$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left(e^{-\jmath \cdot k \cdot \pi} - 1\right)$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

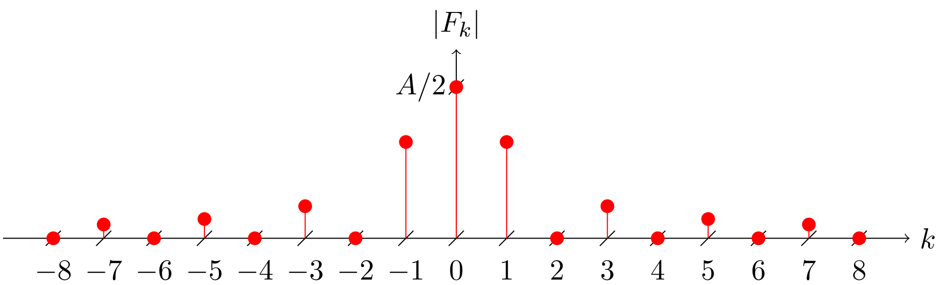
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.5}$$

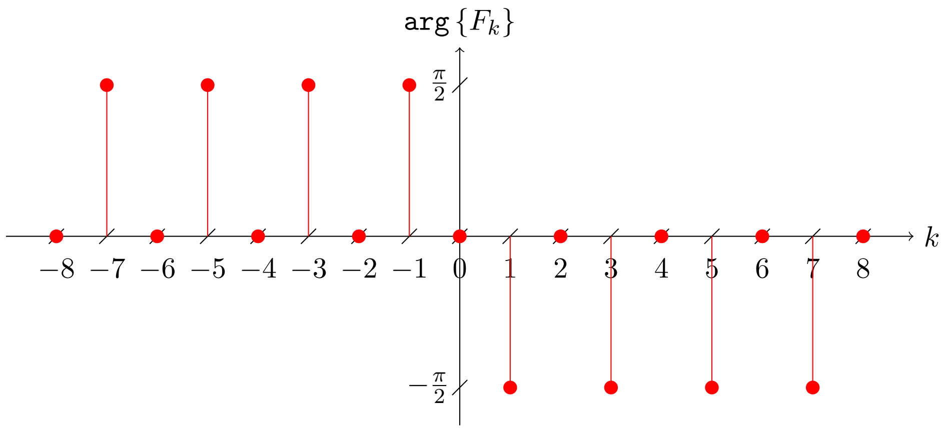
The first several coefficients are equal to:

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	0	$\jmath \cdot \frac{A}{\pi}$	$\frac{A}{2}$	$-\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$
$\lvert F_k \rvert$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{2}$	$\frac{A}{\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$
$Arg\{F_k\}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

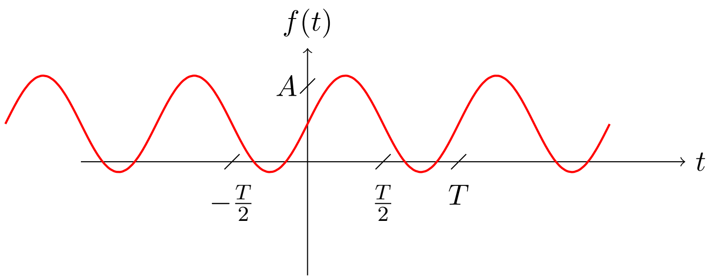


The magnitude spectrum of a real signal is an even-symmetric function of k.
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

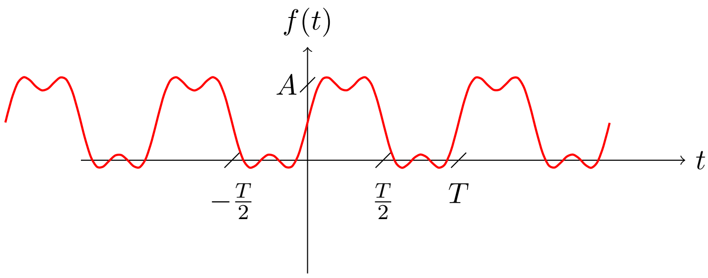


The phase spectrum of a real signal is an odd-symmetric function of k.

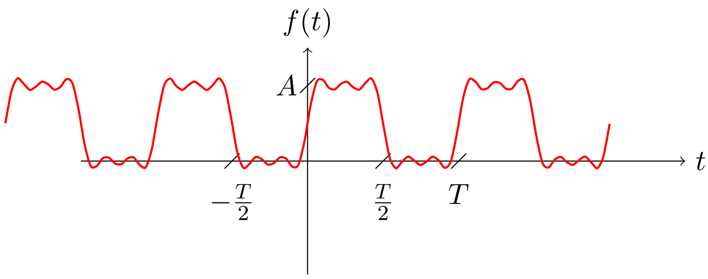
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



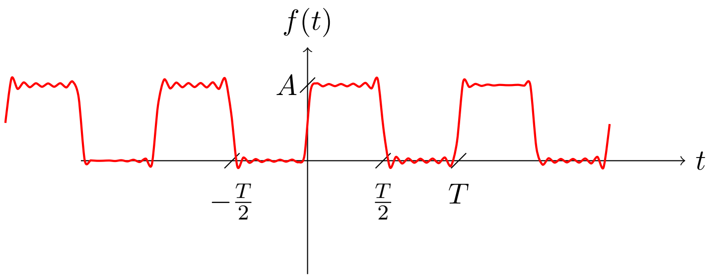
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



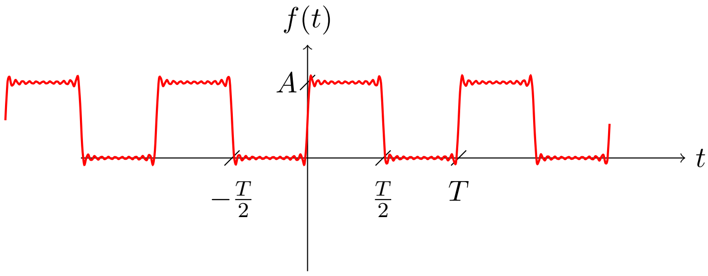
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

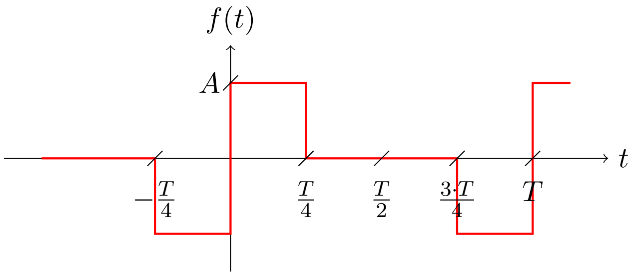


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3\cdot T}{4} + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.6}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.7}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot \int_{-\frac{T}{4}}^{0} dt + A \cdot \int_0^{\frac{T}{4}} dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot t|_{-\frac{T}{4}}^{0} + A \cdot t|_0^{\frac{T}{4}} \right) = \\ &= \frac{1}{T} \left(-A \cdot \left(0 - \left(-\frac{T}{4}\right)\right) + A \cdot \left(\frac{T}{4} - 0\right) \right) = \\ &= \frac{1}{T} \left(-A \cdot \frac{T}{4} + A \cdot \frac{T}{4} \right) = \\ &= \frac{1}{T} (0) = \\ &= 0 \end{aligned} \tag{2.8}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.9}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{1}{T}\left(-\frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot dz + \frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot dz\right) = \\
&= \frac{1}{T}\cdot\frac{A}{\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\left(e^{z}\Big|_{-\frac{T}{4}}^{0} - e^{z}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{0} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0} - e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{0} - e^{\jmath\cdot k\cdot\frac{2\pi}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{4}} - e^{0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 1\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}} + 1\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(2 - \left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot 2\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \left\{ cos(x) \quad = \frac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2} \right\} = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= -\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned} F_0 &= 0 \\ F_k &= \jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1 \right) \end{aligned} \tag{2.10}$$

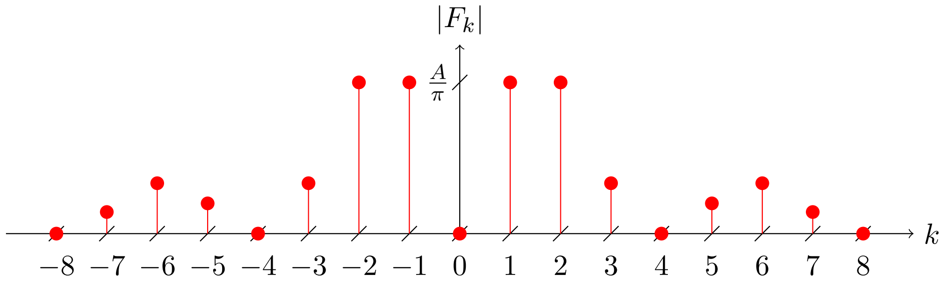
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1 \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.11}$$

The first several coefficients are equal to:

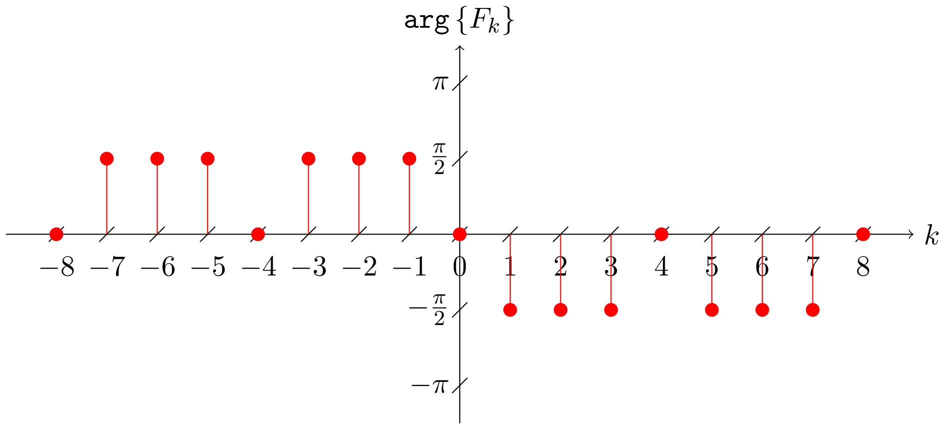
k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{\pi}$	$\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$	$-\jmath \cdot \frac{A}{3\pi}$
$\lvert F_k \rvert$	$\frac{A}{3\pi}$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{\pi}$	$\frac{A}{\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{\pi}$	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$	$\frac{A}{3\pi}$
$Arg\{F_k\}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



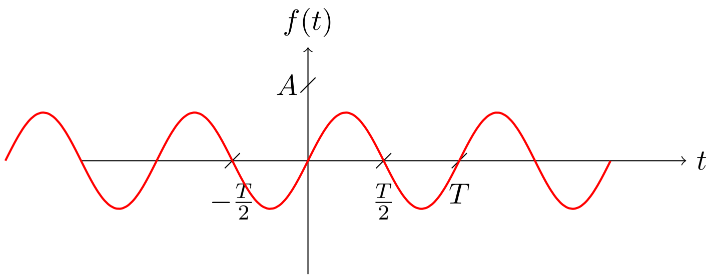
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

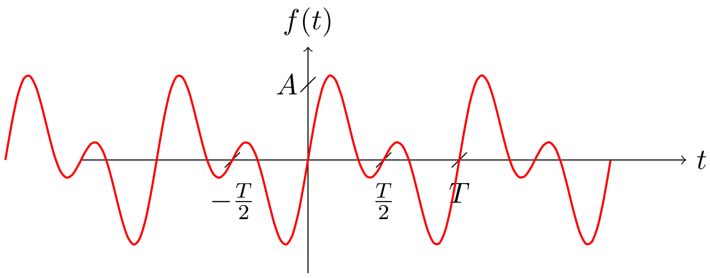


The phase spectrum of a real signal is an odd-symmetric function of k.

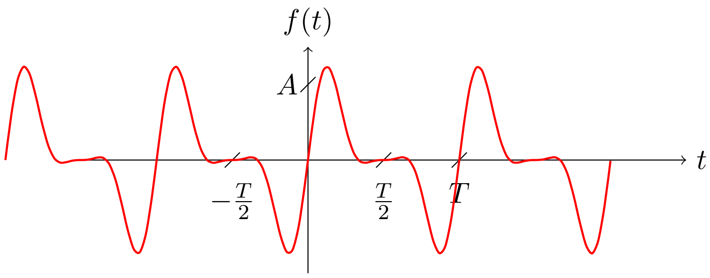
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



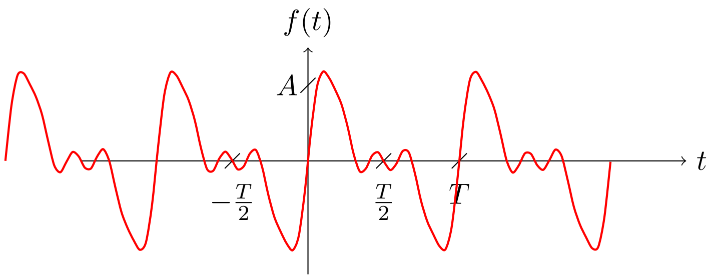
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



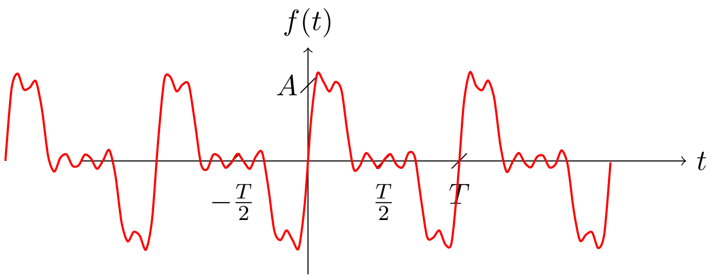
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



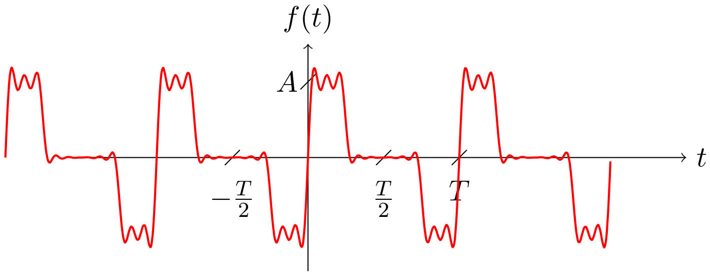
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



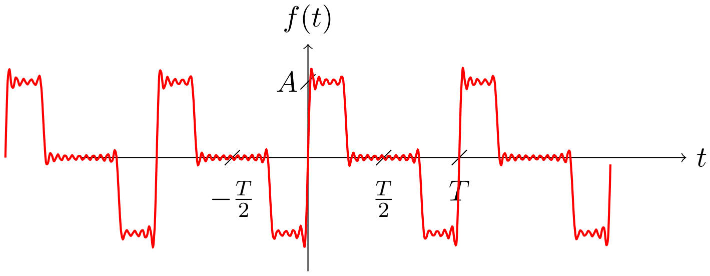
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



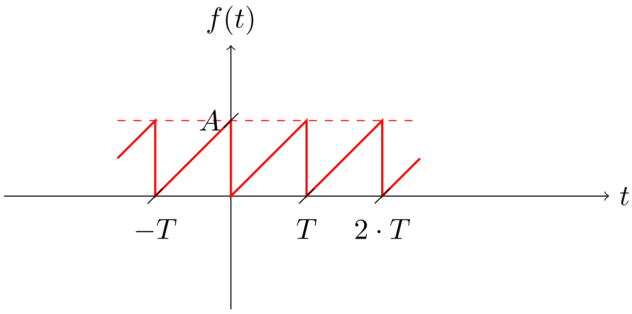
A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 3.

Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.12}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and (T,A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \tfrac{A}{T} \cdot (t - k \cdot T) \quad t \in (0 + k \cdot T; T + k \cdot T) \wedge k \in Z$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.13}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot dt = \\
&= \frac{A}{T^2}\int_0^T t\cdot dt = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot t^2\Big|_0^T = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot\left(T^2 - 0^2\right) = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot T^2 = \\
&= \frac{A}{2}
\end{aligned}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.14}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1\cdot A}{T^2}\int_0^T t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \left\{\begin{matrix} u & = t & dv & = e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \\ du & = dt & v & = \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \end{matrix}\right\} = \\
&= \frac{A}{T^2}\cdot\left(t\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T - \int_0^T \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{T^2}\cdot\left(\left(T\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - 0\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) + \frac{T^2}{\left(-\jmath\cdot k\cdot 2\pi\right)^2}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T\right) = \\
&= \frac{A}{T^2}\cdot\left(\frac{T^2}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} + \frac{T^2}{-\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot 2\pi} - e^0\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot(1-1)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot 0\right) =
\end{aligned}$$

$$
\begin{aligned}
&= A \cdot \left(\frac{1}{-\jmath \cdot k \cdot 2\pi} - 0 \right) = \\
&= \frac{A}{-\jmath \cdot k \cdot 2\pi} = \\
&= \jmath \cdot \frac{A}{k \cdot 2\pi}
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi}
\end{aligned}
$$

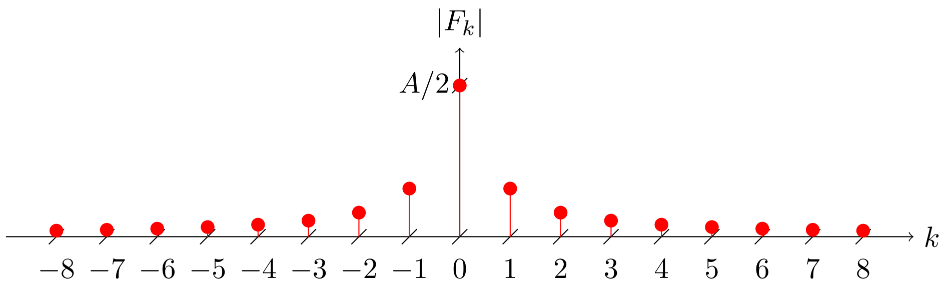
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi} \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.15}
$$

The first several coefficients are equal to:

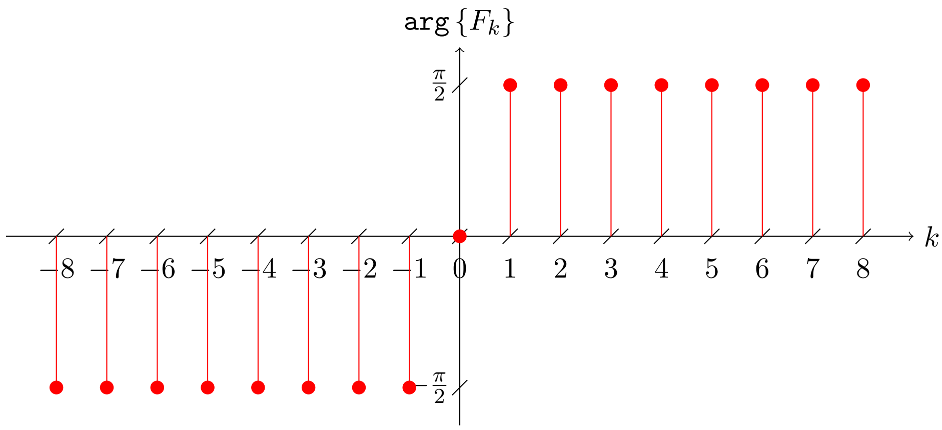
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$-\jmath \cdot \frac{A}{10 \cdot \pi}$	$-\jmath \cdot \frac{A}{8 \cdot \pi}$	$-\jmath \cdot \frac{A}{6 \cdot \pi}$	$-\jmath \cdot \frac{A}{4 \cdot \pi}$	$-\jmath \cdot \frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\jmath \cdot \frac{A}{2 \cdot \pi}$	$\jmath \cdot \frac{A}{4 \cdot \pi}$	$\jmath \cdot \frac{A}{6 \cdot \pi}$	$\jmath \cdot \frac{A}{8 \cdot \pi}$	$\jmath \cdot \frac{A}{10 \cdot \pi}$
$\lvert F_k \rvert$	$\frac{A}{10 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{10 \cdot \pi}$
$Arg\left(F_k\right)$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

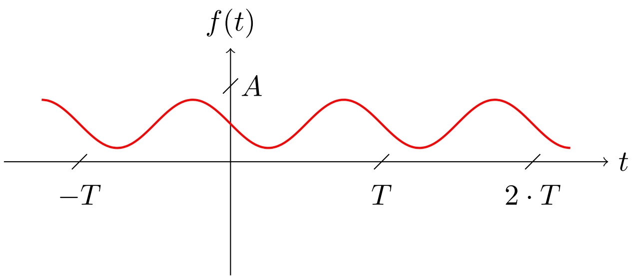


The magnitude spectrum of a real signal is an even-symmetric function of k.

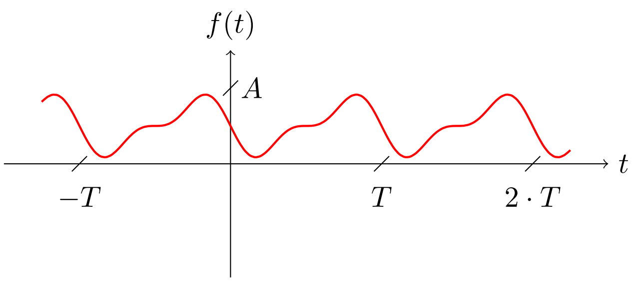
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.



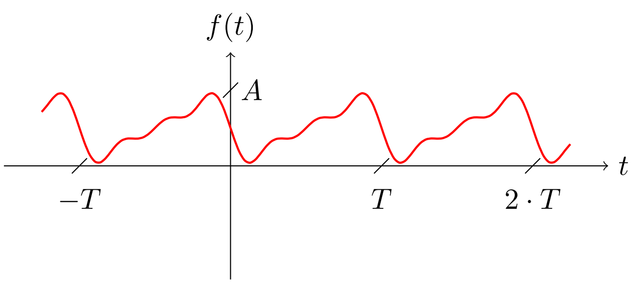
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



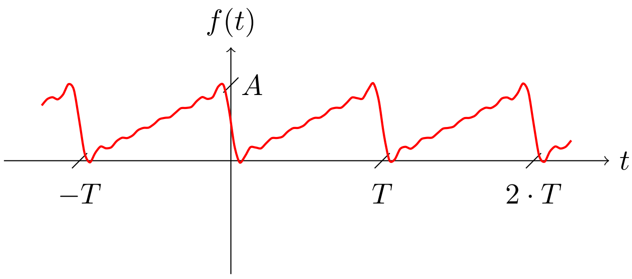
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



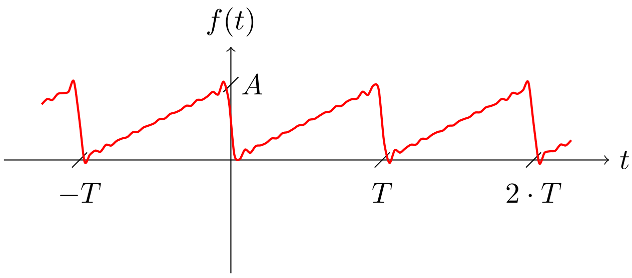
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -7$ to $k_{max} = 7$ results in:



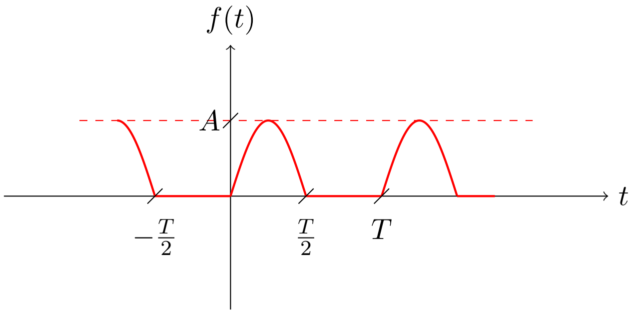
A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 4.

Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.16}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.17}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{matrix} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\,(z) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\,(z) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos\,(z)|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{2\pi}\cdot\left(cos\left(\frac{2\pi}{T}\cdot\frac{T}{2}\right)-cos\left(\frac{2\pi}{T}\cdot 0\right)\right)=\\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\pi\right)-cos\left(0\right)\right)=\\
&= -\frac{A}{2\pi}\cdot\left(-1-1\right)=\\
&= -\frac{A}{2\pi}\cdot\left(-2\right)=\\
&= \frac{A}{\pi}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{\pi}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.18}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot dt\right)=\\
&= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=\\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt+0\right)=\\
&= \frac{1}{T}\cdot\left(\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{1}{T}\cdot\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=\\
&= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-k)}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+k)}\right)\cdot dt=\\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-k)}\cdot dt-\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+k)}\cdot dt\right)=\\
&= \left\{\begin{array}{llll}
z_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t & z_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t\\
dz_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot dt & dz_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot dt\\
dt & = \frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)} & dt & = \frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}
\end{array}\right\}=\\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}-\int_0^{\frac{T}{2}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\right)=\\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}\cdot\int_0^{\frac{T}{2}} e^{z_1}\cdot dz_1-\frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\cdot\int_0^{\frac{T}{2}} e^{z_2}\cdot dz_2\right)=
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T\cdot 2\jmath\cdot\jmath\cdot\frac{2\pi}{T}}\cdot\left(\frac{1}{1-k}\cdot\int_0^{\frac{T}{2}} e^{z_1}\cdot dz_1 + \frac{1}{1+k}\cdot\int_0^{\frac{T}{2}} e^{z_2}\cdot dz_2\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{z_1}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{z_2}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot 0}\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot 0}\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - e^0\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - e^0\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1+k}{(1-k)\cdot(1+k)}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + \frac{1-k}{(1-k)\cdot(1+k)}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right)}{(1-k)\cdot(1+k)} + \frac{(1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + (1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 1 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} - k + e^{-\jmath\cdot\pi\cdot(1+k)} - 1 - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)} + k}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 2 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} + e^{-\jmath\cdot\pi\cdot(1+k)} - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} + e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-1\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k} - 1\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-e^{-\jmath\cdot\pi\cdot k} - 2 - k\cdot e^{-\jmath\cdot\pi\cdot k} - e^{-\jmath\cdot\pi\cdot k} + k\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\pi\cdot k} - 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot\left(\frac{2\cdot e^{-\jmath\cdot\pi\cdot k} + 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot 2\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}\right) = \\
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}\right) \\
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k + 1}{1-k^2}\right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)$ for $k \neq 1 \wedge k \neq -1$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
F_1 = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt =
$$

$$= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$

$$= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \left\{ \begin{array}{ll} z & = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \end{array} \right\} =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(t|_0^{\frac{T}{2}} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) =$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= \frac{A}{4\jmath} = \\
&= -\jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $-\jmath \cdot \frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt\right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0\right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt\right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt\right) = \\
&= \begin{Bmatrix} z & = \jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath \cdot \frac{4\pi}{T}} \end{Bmatrix} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{2} \right) = \\
&= -\frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= -\frac{A}{4\jmath} = \\
&= \jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_{-1} coefficients equal to $\jmath \cdot \frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_{-1} &= \jmath \cdot \frac{A}{4} \\
F_1 &= -\jmath \cdot \frac{A}{4} \\
F_k &= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}
$$

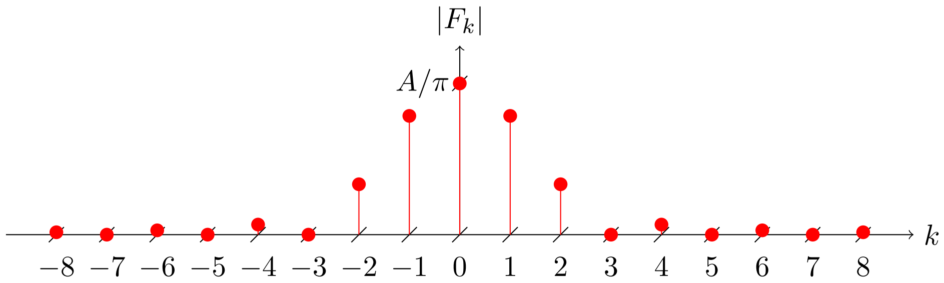
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{\pi} + \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \qquad (2.19)
\end{aligned}
$$

The first several coefficients are equal to:

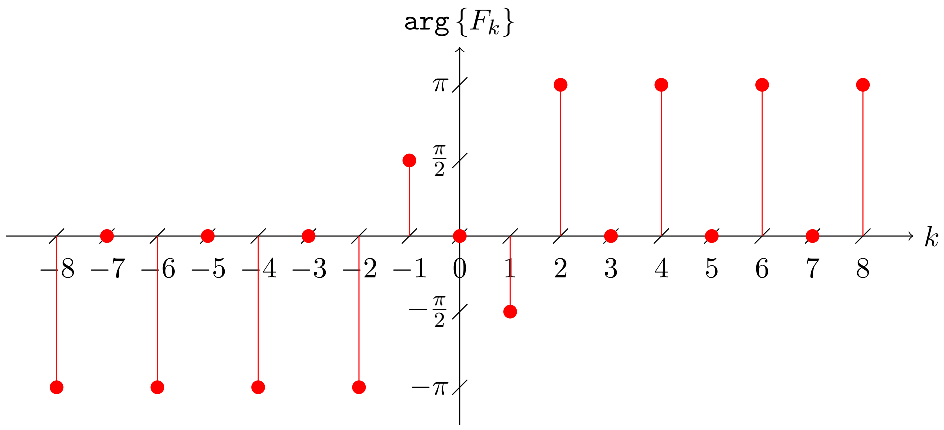
F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\jmath \cdot \frac{A}{4}$	$\frac{A}{\pi}$	$-\jmath \cdot \frac{A}{4}$	$-\frac{A}{3\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{35\pi}$
$\lvert F_k \rvert$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$Arg\{F_k\}$	$-\pi$	0	$-\pi$	0	$-\pi$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	π	0	π	0	π

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



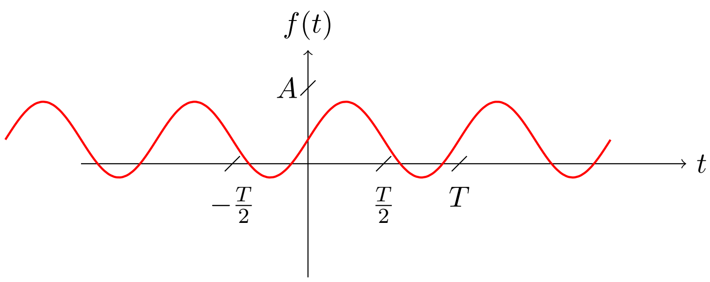
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

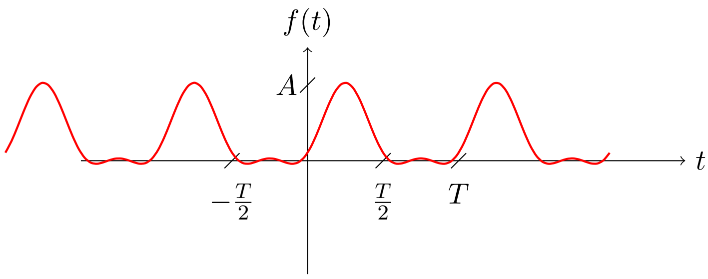


The phase spectrum of a real signal is an odd-symmetric function of k.

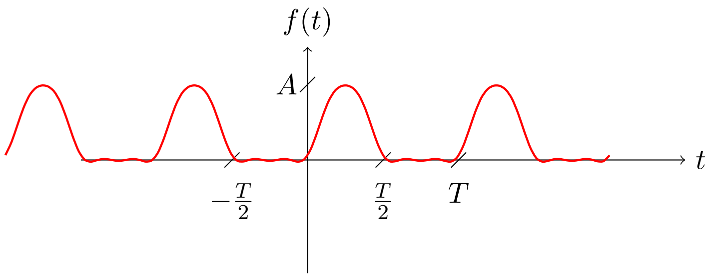
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



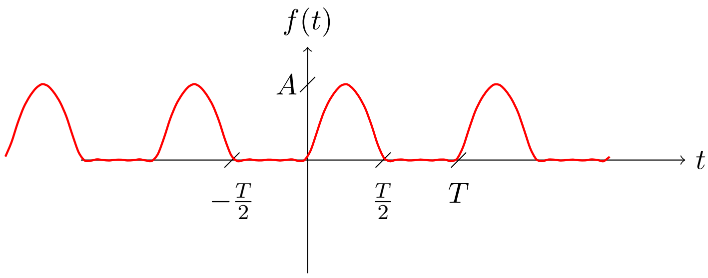
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



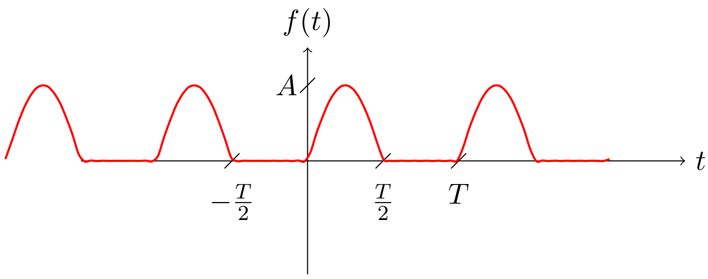
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:

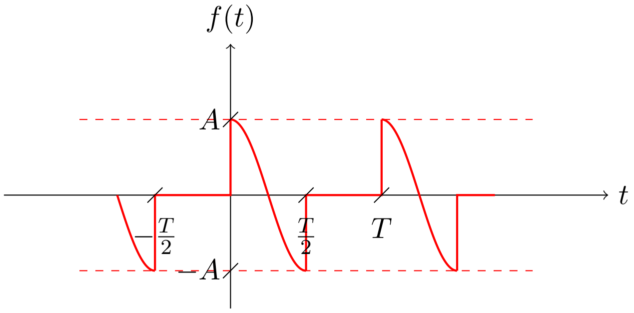


A partial approximation of the $f(t)$ signal from $k_{min} = -12$ to $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 5. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.20}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.21}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = \frac{T}{2\pi} \cdot dz \end{matrix} \right\} = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{A}{2\pi} \cdot (0 - 0) = \\
&= \frac{A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$
F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.22}
$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{matrix} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1} \Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= -\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x \right) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-1) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+1) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot 0 \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot 2 \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt = \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \end{matrix} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{z} \cdot \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(t\Big|_0^{\frac{T}{2}} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\left(\frac{T}{2} - 0 \right) - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1-1) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{2} = \\
&= \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =
\end{aligned}
$$

$$= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) =$$
$$= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} =$$
$$= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =$$
$$= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$
$$= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =$$
$$= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt =$$
$$= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \left\{\begin{matrix} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\} =$$
$$= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) =$$
$$= \frac{A}{2\cdot T}\cdot\frac{T}{2} =$$

$$= \frac{A}{4}$$

The F_{-1} coefficients equal to $\frac{A}{4}$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= \frac{A}{4} \\ F_{-1} &= \frac{A}{4} \\ F_k &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \end{aligned}$$

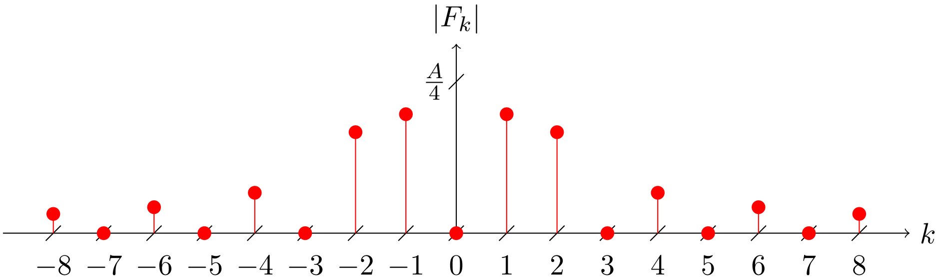
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} + \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\ f(t) &= \frac{A}{2} \cdot cos \left(\frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \end{aligned} \tag{2.23}$$

The first several coefficients are equal to:

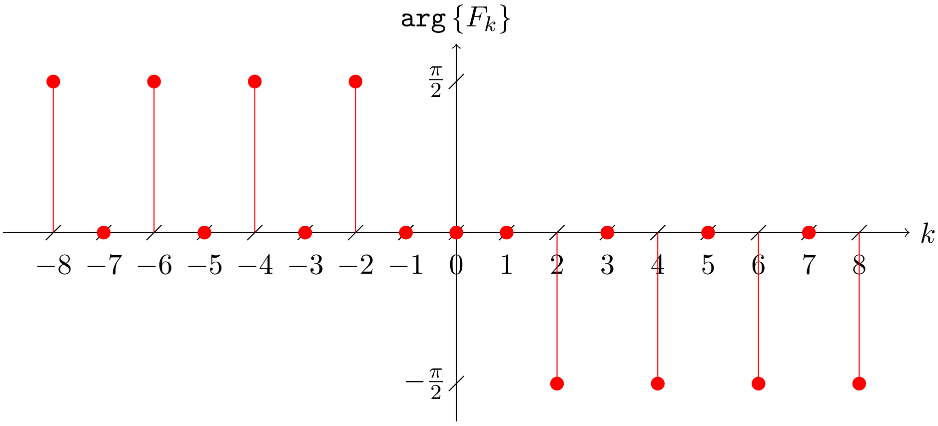
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	0	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

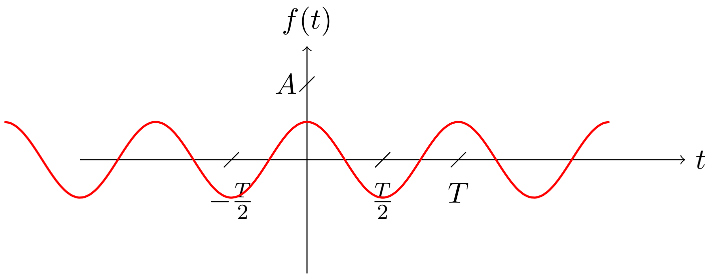


The magnitude spectrum of a real signal is an even-symmetric function of k.

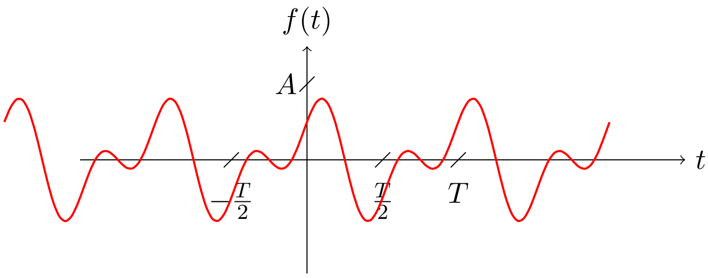
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



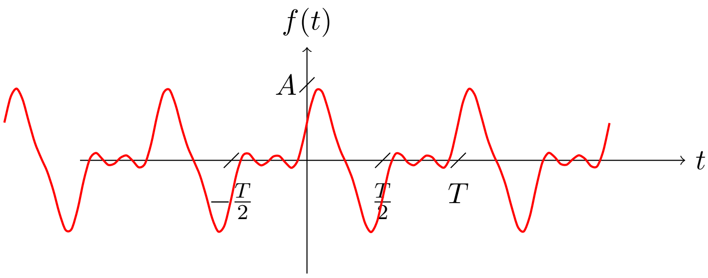
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



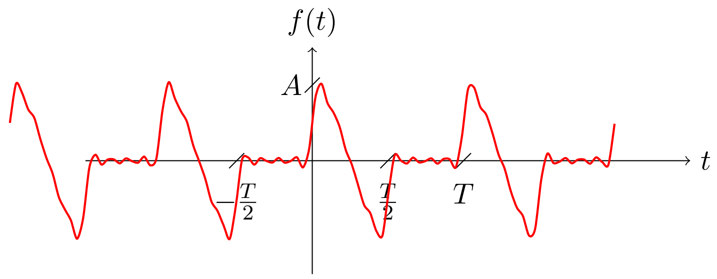
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



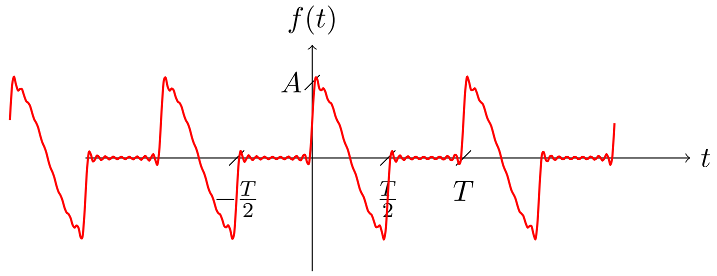
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

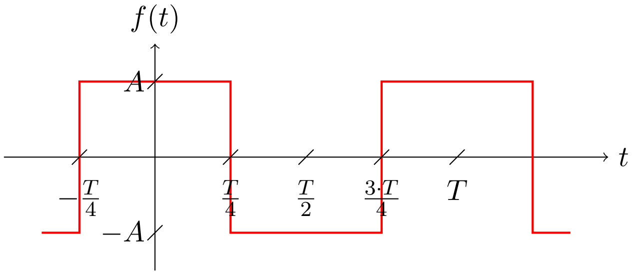


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 6. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function assuming period $t \in \left(\frac{-T}{4}; \frac{3\cdot T}{4}\right)$ is given by:

$$f(x) = \begin{cases} A & t \in \left(-\frac{T}{4} + k\cdot T; \frac{T}{4} + k\cdot T\right) \\ -A & t \in \left(\frac{T}{4} + k\cdot T; \frac{3\cdot T}{4} + k\cdot T\right) \end{cases} \wedge k \in Z \tag{2.24}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T}\int_T f(t)\cdot dt \tag{2.25}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\ &= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} (-A)\cdot dt\right) = \\ &= \frac{1}{T}\left(A\cdot\int_{-\frac{T}{4}}^{\frac{T}{4}} dt - A\cdot\int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} dt\right) = \\ &= \frac{A}{T}\left(t|_{-\frac{T}{4}}^{\frac{T}{4}} - t|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right) = \\ &= \frac{A}{T}\cdot\left[\left(\frac{T}{4} - \left(-\frac{T}{4}\right)\right) - \left(\frac{3\cdot T}{4} - \frac{T}{4}\right)\right] = \\ &= \frac{A}{T}\cdot\left[\frac{T}{4} + \frac{T}{4} - \frac{3\cdot T}{4} + \frac{T}{4}\right] = \\ &= \frac{A}{T}\cdot[0] = \\ &= 0 \end{aligned} \tag{2.26}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.27}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} (-A)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - A\cdot\int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \left\{\begin{aligned} z &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot t \\ dz &= -\jmath\cdot k\cdot\tfrac{2\pi}{T}\cdot dt \\ dt &= \tfrac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{aligned}\right\} = \\
&= \frac{A}{T}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\right] = \\
&= \frac{-A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot dz - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot dz\right] = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{z}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{z}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\left(-\frac{T}{4}\right)} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{3\cdot T}{4}} + e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) = \\
&= \left\{e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} = e^{-\jmath\cdot k\cdot\left(2\pi-\frac{\pi}{2}\right)} = e^{-\jmath\cdot k\cdot 2\pi}\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = 1\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right\} = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) = \\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(2\cdot e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 2\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) = \\
&= \frac{2\cdot A}{\jmath\cdot k\cdot 2\pi}\left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) = \\
&= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= 0 \\
F_k &= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

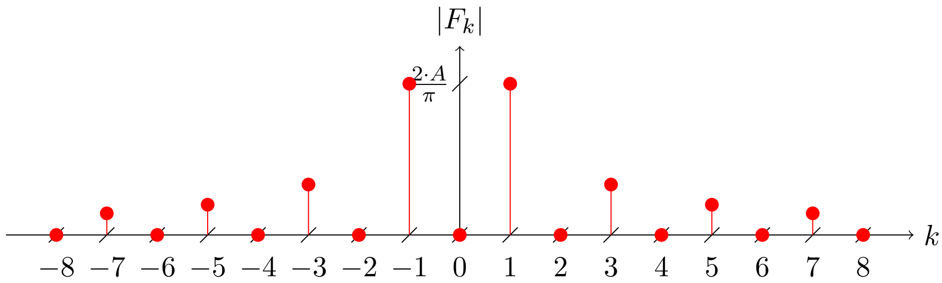
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}$$
$$f(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{2 \cdot A}{k \cdot \pi}\left(sin\left(k \cdot \frac{\pi}{2}\right)\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.28}$$

The first several coefficients are equal to:

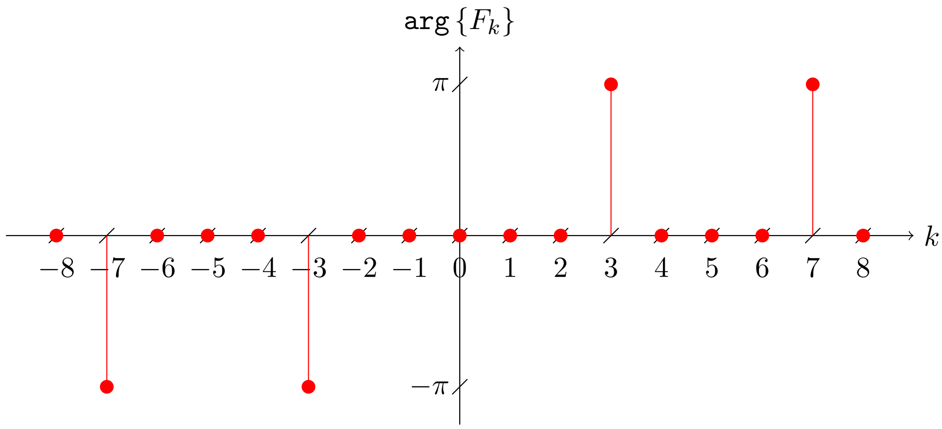
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\frac{2 \cdot A}{5 \cdot \pi}$	0	$-\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$-\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$
$\|F_k\|$	$\frac{2 \cdot A}{5 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$
$Arg\{F_k\}$	0	0	$-\pi$	0	0	0	0	0	π	0	0

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



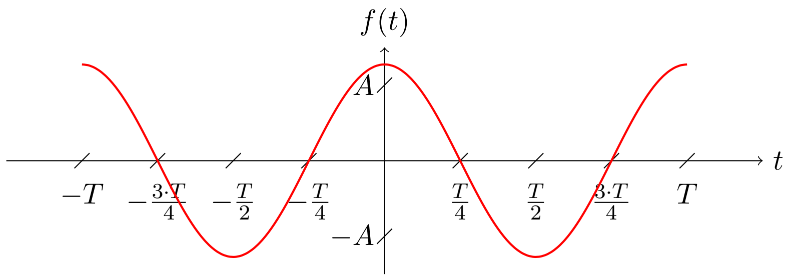
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficents F_k we can plot phase spectrum $\mathtt{arg}\{F_k\}$ of the $f(t)$ signal.

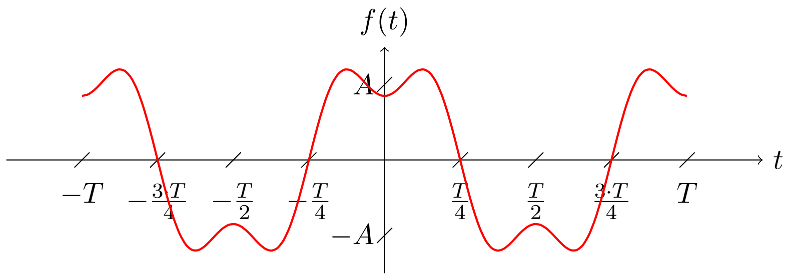


The phase spectrum of a real signal is an odd-symmetric function of k.

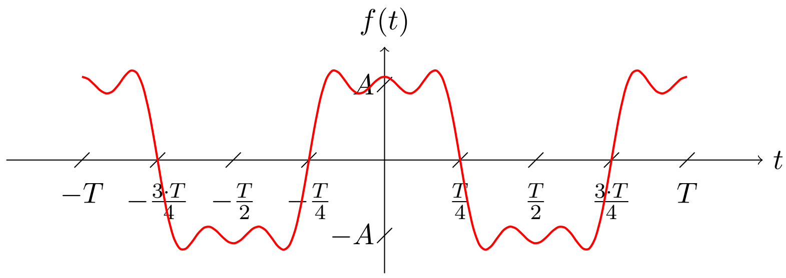
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



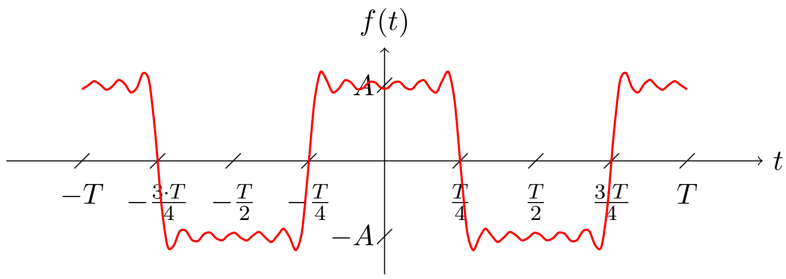
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



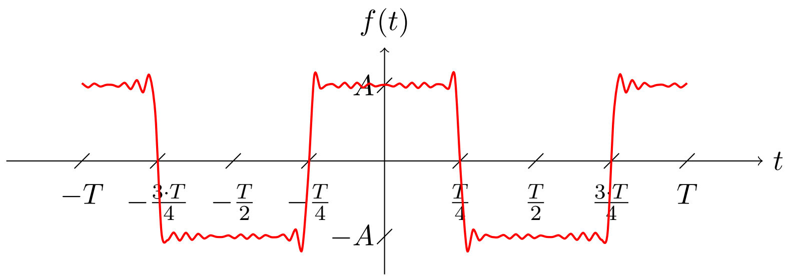
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

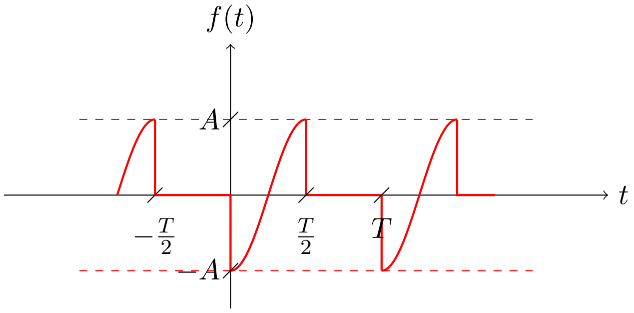


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

Task 7. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} -A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.29}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.30}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{-A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{-A}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{-A}{2\pi} \cdot (0 - 0) = \\
&= \frac{-A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.31}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt\right) = \\
&= \frac{1}{T}\left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\right\} = \\
&= \frac{1}{T}\left((-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0\right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt\right) = \\
&= \left\{\begin{matrix} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{matrix}\right\} = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2\right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2\right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2\right) = \\
&= \frac{-A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{-e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{-A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= \frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x\right) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{0}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}1\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{array}{c} z=-\jmath\cdot\frac{4\pi}{T}\cdot t\\ dz=-\jmath\cdot\frac{4\pi}{T}\cdot dt\\ dt=\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\end{array}\right\}=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{z}\cdot\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}}e^{z}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(t\Big|_0^{\frac{T}{2}}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\left(\frac{T}{2}-0\right)-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{-A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{-A}{4}$.

We have to calculate F_k for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}}(-A)\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T}0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=
\end{aligned}
$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) =$$
$$= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} =$$
$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$
$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =$$
$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \left\{\begin{array}{c} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{array}\right\} =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) =$$
$$= \frac{-A}{2\cdot T}\cdot\frac{T}{2} =$$

$$= \frac{-A}{4}$$

The F_{-1} coefficients equal to $\frac{-A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= \frac{-A}{4} \\
F_{-1} &= \frac{-A}{4} \\
F_k &= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}$$

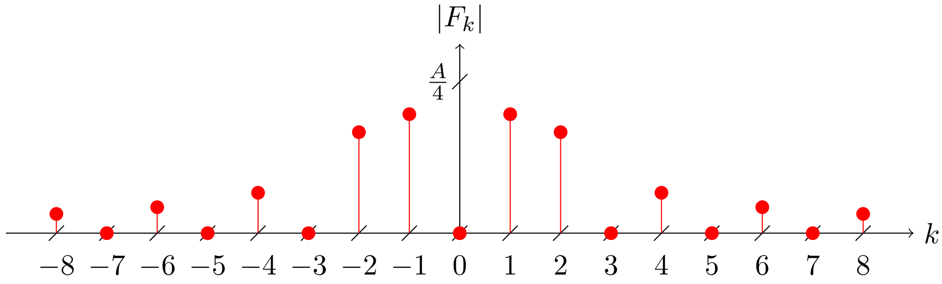
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{2} \cdot cos\left(\frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.32}$$

The first several coefficients are equal to:

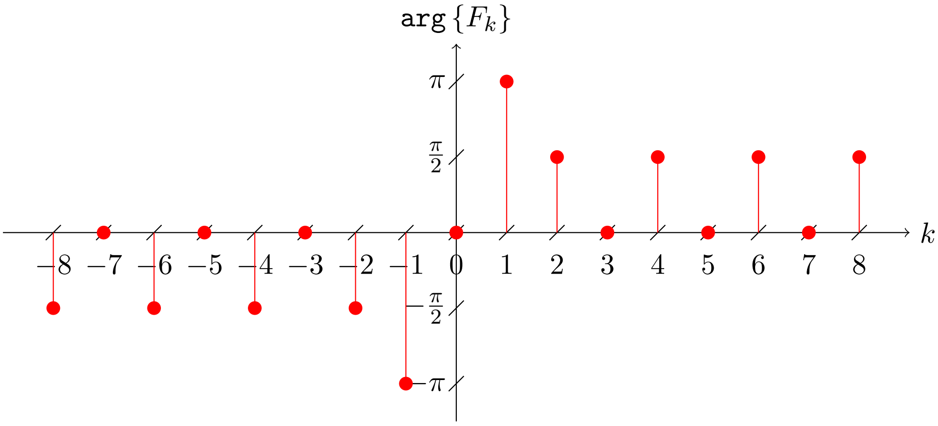
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{-A}{4}$	0	$\frac{-A}{4}$	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\pi$	0	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

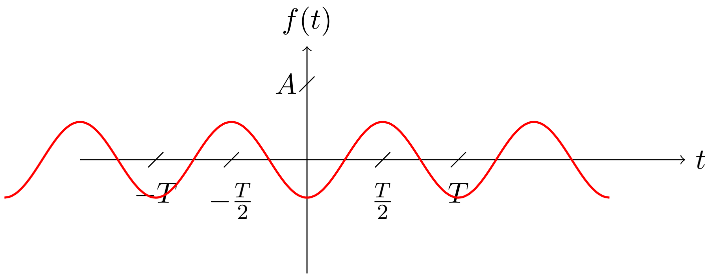


The magnitude spectrum of a real signal is an even-symmetric function of k.

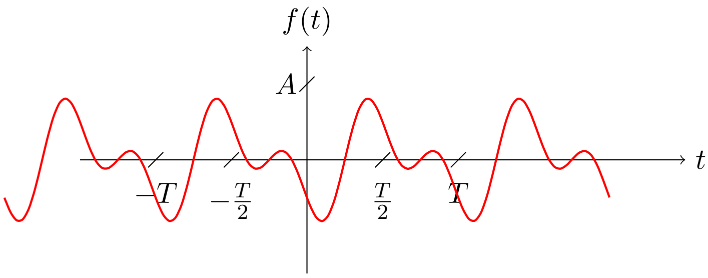
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



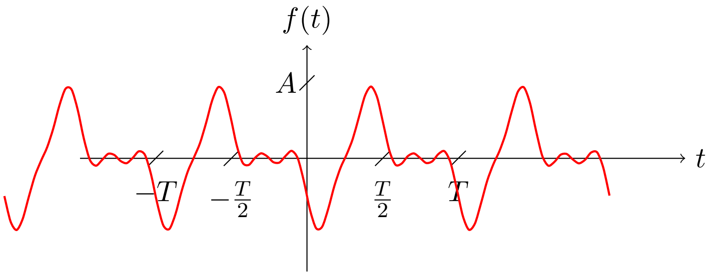
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



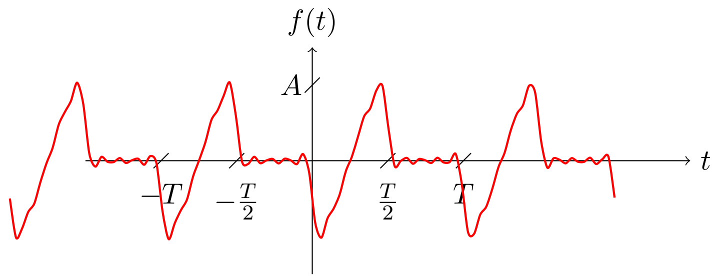
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



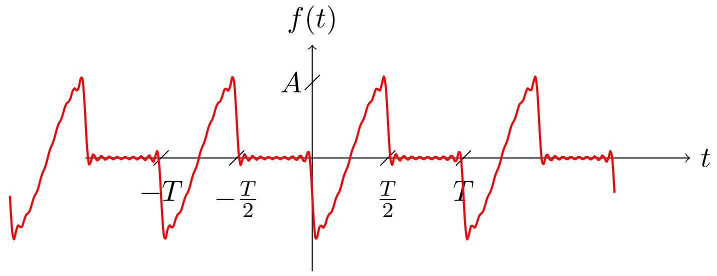
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

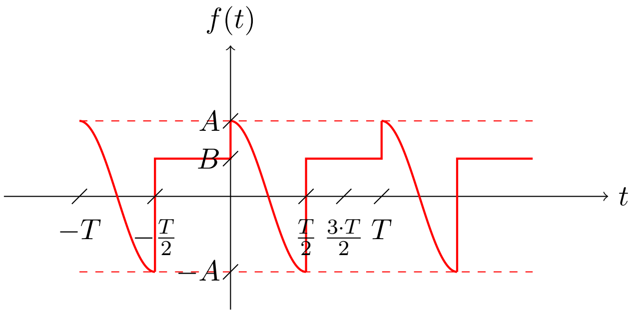


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

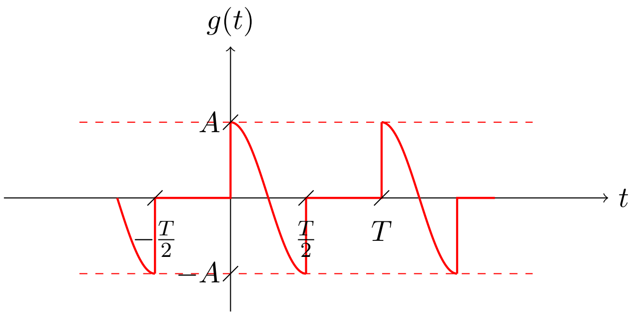
Task 8. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.

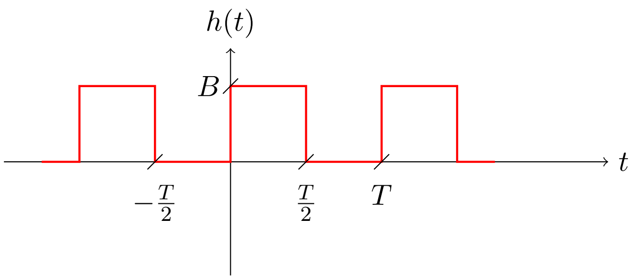


Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ B & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.33}$$

If we look carefully, signal $f(t)$ may be decomposed into two signals $g(t)$ and $h(t)$ for which we have already calculated Fourier series coefficients. The signals are given below:





To be precise, the $f(t)$ signal will be the sum of $g(t)$ and $h(t)$ shifted in time by $\frac{T}{2}$:

$$f(t) = g(t) + h\left(t - \frac{T}{2}\right) \tag{2.34}$$

Based on linearity of complex exponential Fourier series and on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \pi} F_k = G_k + H_k \cdot (-1)^k \tag{2.35}$$

From previous tasks we know, that coefficients for the expansion into a complex exponential Fourier series of $g(t)$ and $h(t)$ signals are equal to:

$$\begin{aligned} G_0 &= 0 \\ G_1 &= \frac{A}{4} \\ G_{-1} &= \frac{A}{4} \\ G_k &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \end{aligned}$$

$$\begin{aligned} H_0 &= \frac{B}{2} \\ H_k &= \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \end{aligned}$$

Right now we know everything to calculate F_k coefficients:

$$F_k = G_k + H_k \cdot (-1)^k$$

$$\begin{aligned} F_0 &= G_0 + H_0 \cdot (-1)^0 = \\ &= 0 + \frac{B}{2} \cdot 1 = \\ &= \frac{B}{2} \end{aligned} \tag{2.36}$$

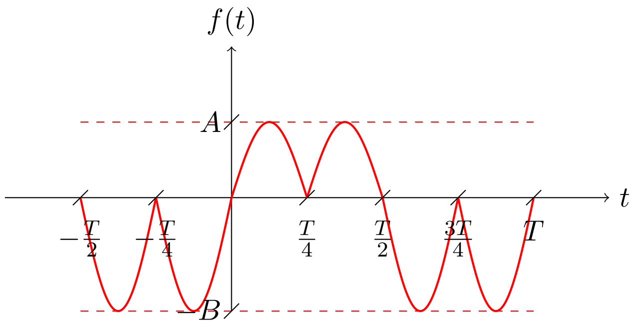
$$\begin{aligned} F_1 &= G_1 + H_1 \cdot (-1)^1 = \\ &= \frac{A}{4} + \jmath \cdot \frac{B}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1 \Big) \cdot (-1) = \\ &= \frac{A}{4} + \jmath \cdot \frac{B}{\pi} \end{aligned} \tag{2.37}$$

$$\begin{aligned} F_{-1} &= G_{-1} + H_{-1} \cdot (-1)^{-1} = \\ &= \frac{A}{4} + \jmath \cdot \frac{B}{(-1) \cdot 2\pi} \cdot \Big((-1)^{-1} - 1 \Big) \cdot (-1) = \\ &= \frac{A}{4} - \jmath \cdot \frac{B}{\pi} \end{aligned} \tag{2.38}$$

$$\begin{aligned} F_k &= G_k + H_k \cdot (-1)^k = \\ &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \cdot (-1)^k = \\ &= \jmath \cdot \left[\frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \frac{B}{k \cdot 2\pi} \cdot \Big(1 - (-1)^k \Big) \right] \end{aligned} \tag{2.39}$$

Task 9.

Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ -A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{4} + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{2} + k \cdot T; \frac{3T}{4} + k \cdot T\right) \\ B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{3T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.40}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.41}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\ &= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\ &= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\ &= \left\{ \begin{aligned} z &= \tfrac{2\pi}{T} \cdot t \\ dz &= \tfrac{2\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{\frac{2\pi}{T}} \end{aligned} \right\} = \\ &= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{2\pi}{T}} = \\ &= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\pi} \cdot \left(-cos\,(z)|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{2\pi}{T} \cdot 0\right) \right) = \\
&= -\frac{A}{2\pi} \cdot (cos\,(\pi) - cos\,(0)) = \\
&= -\frac{A}{2\pi} \cdot (-1-1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{\pi}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.42}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \end{array} \right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)} - \int_0^{\frac{T}{2}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)}\cdot\int_0^{\frac{T}{2}} e^{z_1}\cdot dz_1 - \frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)}\cdot\int_0^{\frac{T}{2}} e^{z_2}\cdot dz_2\right) = \\
&= \frac{A}{T\cdot 2\jmath\cdot\jmath\cdot\frac{2\pi}{T}}\cdot\left(\frac{1}{1-k}\cdot\int_0^{\frac{T}{2}} e^{z_1}\cdot dz_1 + \frac{1}{1+k}\cdot\int_0^{\frac{T}{2}} e^{z_2}\cdot dz_2\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{z_1}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{z_2}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot 0}\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot 0}\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1}{1-k}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - e^0\right) + \frac{1}{1+k}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - e^0\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{1+k}{(1-k)\cdot(1+k)}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + \frac{1-k}{(1-k)\cdot(1+k)}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right)}{(1-k)\cdot(1+k)} + \frac{(1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{(1+k)\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - 1\right) + (1-k)\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 1 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} - k + e^{-\jmath\cdot\pi\cdot(1+k)} - 1 - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)} + k}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi\cdot(1-k)} - 2 + k\cdot e^{\jmath\cdot\pi\cdot(1-k)} + e^{-\jmath\cdot\pi\cdot(1+k)} - k\cdot e^{-\jmath\cdot\pi\cdot(1+k)}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} + e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-1\cdot e^{-\jmath\cdot\pi\cdot k} - 2 + k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k} - 1\cdot e^{-\jmath\cdot\pi\cdot k} - k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-e^{-\jmath\cdot\pi\cdot k} - 2 - k\cdot e^{-\jmath\cdot\pi\cdot k} - e^{-\jmath\cdot\pi\cdot k} + k\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right) = \\
&= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\pi\cdot k} - 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot\left(\frac{2\cdot e^{-\jmath\cdot\pi\cdot k} + 2}{1-k^2}\right) = \\
&= \frac{A}{4\cdot\pi}\cdot 2\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}\right) = \\
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k} + 1}{1-k^2}\right) \\
&= \frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k + 1}{1-k^2}\right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)$ for $k \neq 1 \wedge k \neq -1$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\} = \\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{1}{T}\cdot\left(\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}\cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\cdot dt\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0}\cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}\cdot dt\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{0}\cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} 1\cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \left\{\begin{aligned} &z && = -\jmath\cdot\tfrac{4\pi}{T}\cdot t \\ &dz && = -\jmath\cdot\tfrac{4\pi}{T}\cdot dt \\ &dt && = \tfrac{dz}{-\jmath\cdot\frac{4\pi}{T}} \end{aligned}\right\} = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^{z}\cdot\frac{dz}{-\jmath\cdot\frac{4\pi}{T}}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(t\Big|_0^{\frac{T}{2}} + \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\left(\frac{T}{2}-0\right) + \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\left(\frac{T}{2}-0\right) + \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right) = \\
&= \frac{A}{T\cdot 2\jmath}\cdot\left(\frac{T}{2} + \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1-1)\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0\right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= \frac{A}{4\jmath} = \\
&= -\jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $-\jmath \cdot \frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{\begin{matrix} z & = \jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath \cdot \frac{4\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z \Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \Big|_0^{\frac{T}{2}} - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{2} \right) = \\
&= -\frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= -\frac{A}{4\jmath} = \\
&= \jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_{-1} coefficients equal to $\jmath \cdot \frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_{-1} &= \jmath \cdot \frac{A}{4} \\
F_1 &= -\jmath \cdot \frac{A}{4} \\
F_k &= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}
$$

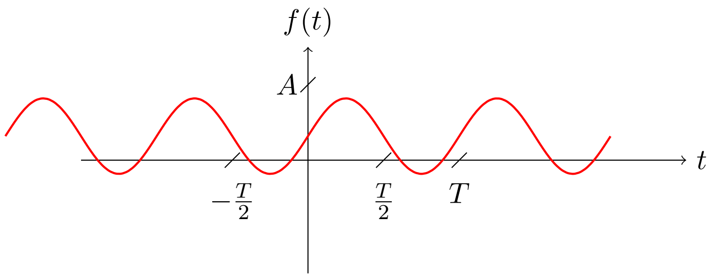
The first several coefficients are equal to:

F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\jmath\cdot\frac{A}{4}$	$\frac{A}{\pi}$	$-\jmath\cdot\frac{A}{4}$	$-\frac{A}{3\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{35\pi}$
$\lvert F_k\rvert$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$Arg\left\{F_k\right\}$	$-\pi$	0	$-\pi$	0	$-\pi$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	π	0	π	0	π

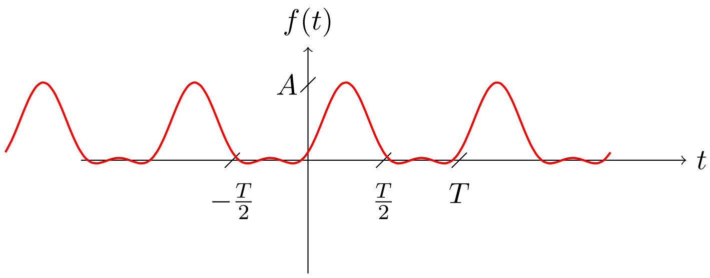
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}$$
$$f(t) = \frac{A}{\pi} + \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t} - \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t} + \sum_{\substack{k=-\infty\\ k\neq 0\\ k\neq -1\wedge k\neq 1}}^{\infty}\left[\frac{A}{2\cdot\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)\right]\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \tag{2.43}$$

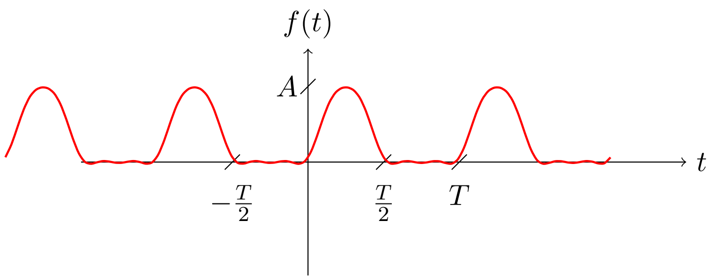
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



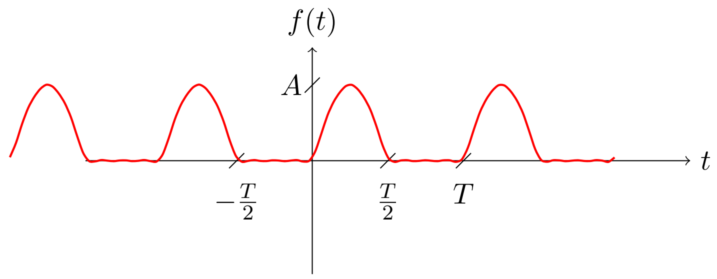
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



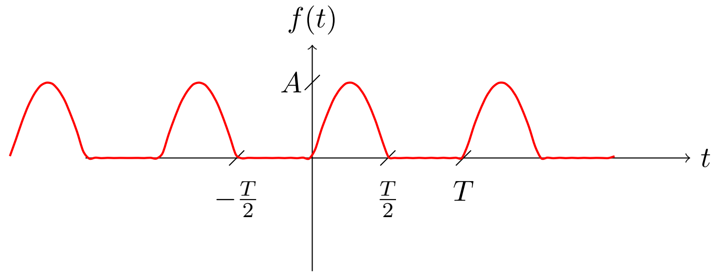
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



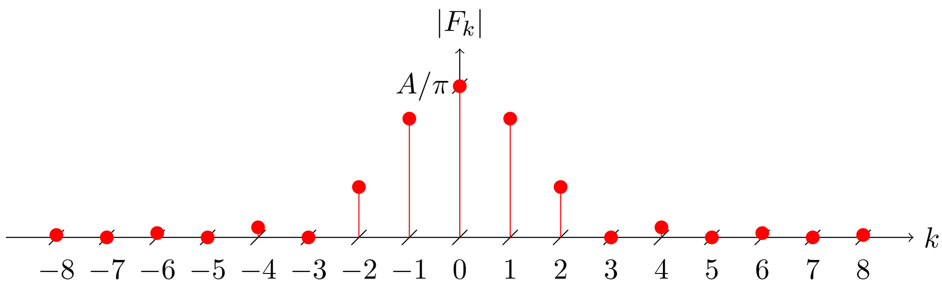
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



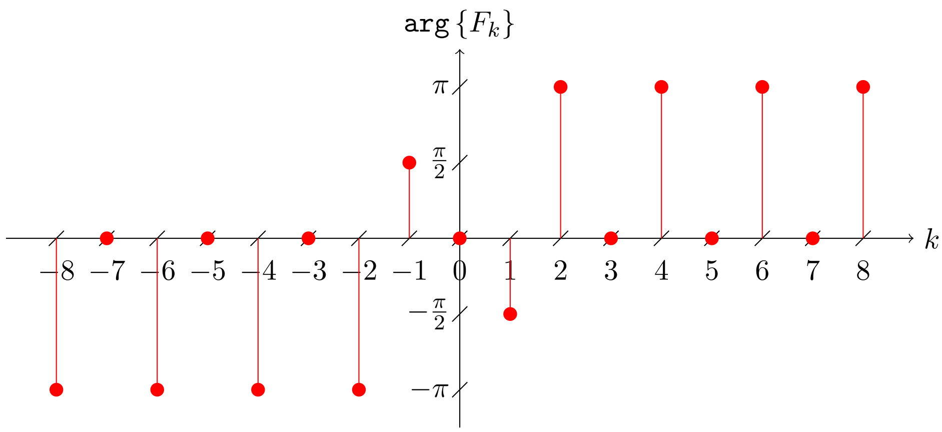
A partial approximation of the $f(t)$ signal from $k_{min} = -12$ to $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal. Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



The magnitude spectrum of a real signal is an even-symmetric function of k.
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



The phase spectrum of a real signal is an odd-symmetric function of k.

2.3 Computing the power of a signal – the Parseval's theorem

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

3.2 Exploiting properties of the Fourier transform

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

4.2 Filters

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