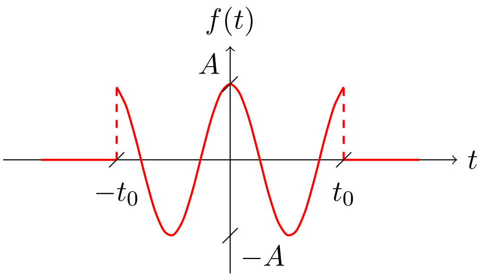
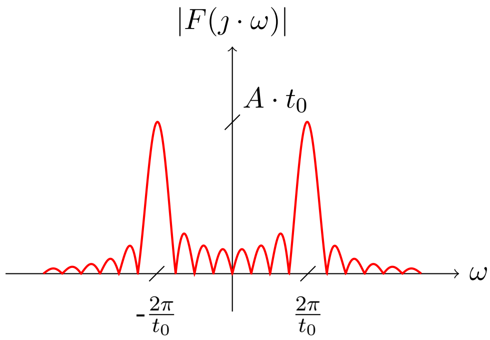
Teoria Sygnałów w zadaniach





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\,(\omega \cdot t_0 + 2\pi) \\ & - Sa\,(\omega \cdot t_0 - 2\pi)] \end{aligned}$$

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Rozdział 1

Podstawowe własności sygnałów

1.1 Podstawowe własności sygnałów

1.1.1 Wartość średnia

1.1.2 Energia sygnału

1.1.3 Moc sygnału

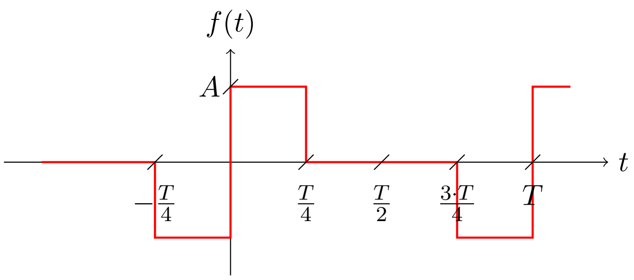
Rozdział 2

Analiza sygnałów okresowych za pomocą szeregów ortogonalnych

2.1 Trygonometryczny szerego Fouriera

2.2 Zespolony szerego Fouriera

Zadanie 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3 \cdot T}{4} + k \cdot T\right) \end{cases} \quad \wedge k \in Z \tag{2.1}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (-\frac{T}{4}; \frac{3 \cdot T}{4})$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot dt + 0\right) = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} dt + A\cdot\int_{0}^{\frac{T}{4}} dt + 0\right) = \\
&= \frac{1}{T}\left(-A\cdot t\Big|_{-\frac{T}{4}}^{0} + A\cdot t\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{1}{T}\left(-A\cdot\left(0-\left(-\frac{T}{4}\right)\right) + A\cdot\left(\frac{T}{4}-0\right)\right) = \\
&= \frac{1}{T}\left(-A\cdot\frac{T}{4} + A\cdot\frac{T}{4}\right) = \\
&= \frac{1}{T}\left(0\right) = \\
&= 0
\end{aligned}
\tag{2.3}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$
F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.4}
$$

For the period $t\in(-\frac{T}{4};\frac{3\cdot T}{4})$, i.e. $k=0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \left\{\begin{array}{ll} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{array}\right\} = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{1}{T}\left(-\frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot dz + \frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot dz\right) = \\
&= \frac{1}{T}\cdot\frac{A}{\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\left(e^{z}\Big|_{-\frac{T}{4}}^{0} - e^{z}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{0} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0} - e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{\jmath \cdot k \cdot 2\pi} \cdot \left(\left(e^0 - e^{\jmath \cdot k \cdot \frac{2\pi}{4}}\right) - \left(e^{-\jmath \cdot k \cdot \frac{2\pi}{4}} - e^0\right)\right) = \\
&= \frac{A}{\jmath \cdot k \cdot 2\pi} \cdot \left(\left(1 - e^{\jmath \cdot k \cdot \frac{\pi}{2}}\right) - \left(e^{-\jmath \cdot k \cdot \frac{\pi}{2}} - 1\right)\right) = \\
&= \frac{A}{\jmath \cdot k \cdot 2\pi} \cdot \left(1 - e^{\jmath \cdot k \cdot \frac{\pi}{2}} - e^{-\jmath \cdot k \cdot \frac{\pi}{2}} + 1\right) = \\
&= \frac{A}{\jmath \cdot k \cdot 2\pi} \cdot \left(2 - \left(e^{\jmath \cdot k \cdot \frac{\pi}{2}} + e^{-\jmath \cdot k \cdot \frac{\pi}{2}}\right)\right) = \\
&= \frac{A}{\jmath \cdot k \cdot 2\pi} \cdot 2 \cdot \left(1 - \frac{e^{\jmath \cdot k \cdot \frac{\pi}{2}} + e^{-\jmath \cdot k \cdot \frac{\pi}{2}}}{2}\right) = \\
&= \frac{A}{\jmath \cdot k \cdot \pi} \cdot \left(1 - \frac{e^{\jmath \cdot k \cdot \frac{\pi}{2}} + e^{-\jmath \cdot k \cdot \frac{\pi}{2}}}{2}\right) = \\
&= \left\{cos(x) \quad = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\right\} = \\
&= \frac{A}{\jmath \cdot k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right) = \\
&= -\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right) = \\
&= \jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
&F_0 = 0 \\
&F_k = \jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1\right)
\end{aligned}
\tag{2.5}
$$

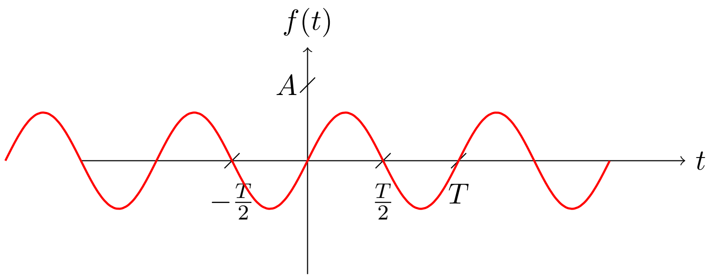
The first several coefficients are equal to:

k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{\pi}$	$\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$	$-\jmath \cdot \frac{A}{3\pi}$
$\lvert F_k \rvert$	$\frac{A}{3\pi}$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{\pi}$	$\frac{A}{\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{\pi}$	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$	$\frac{A}{3\pi}$
$Arg\left\{F_k\right\}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$

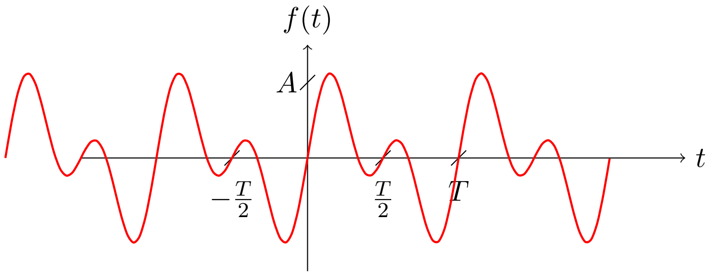
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.6}
$$

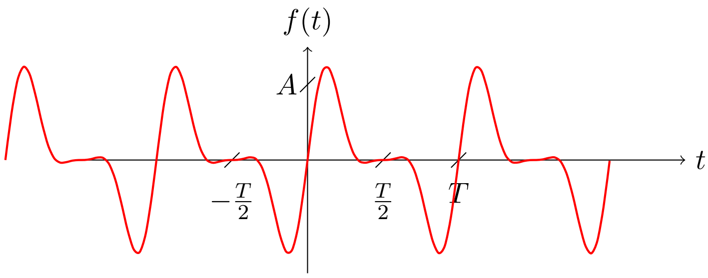
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



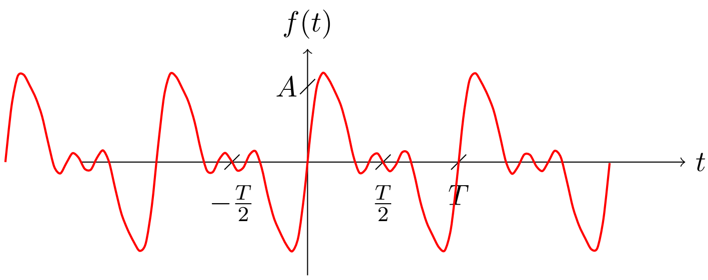
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



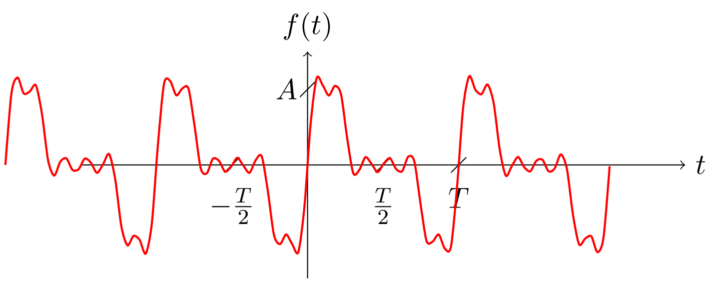
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



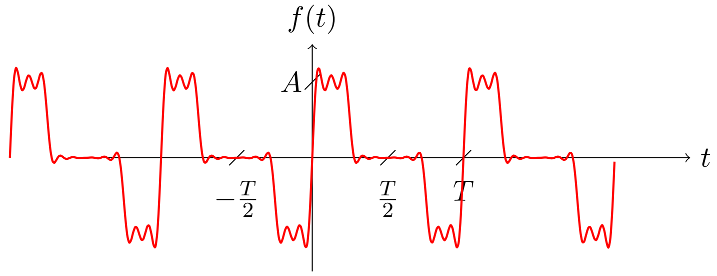
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



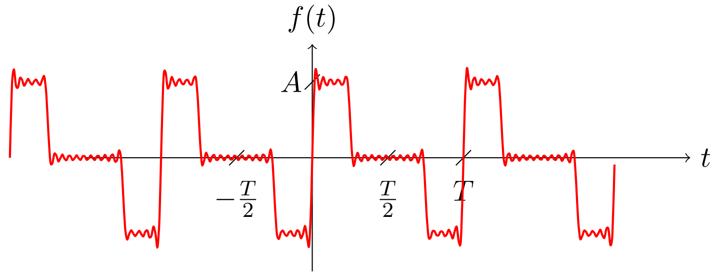
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



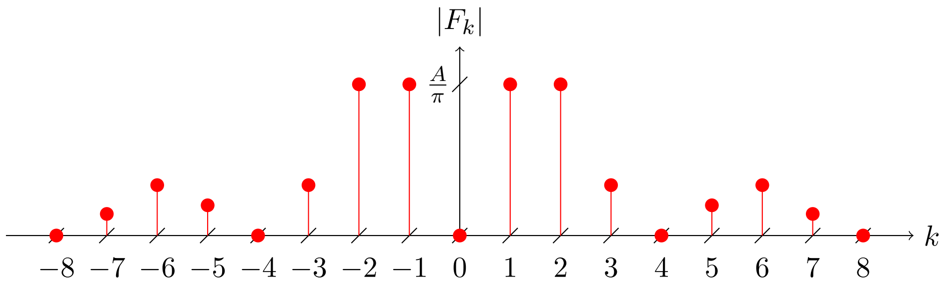
A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



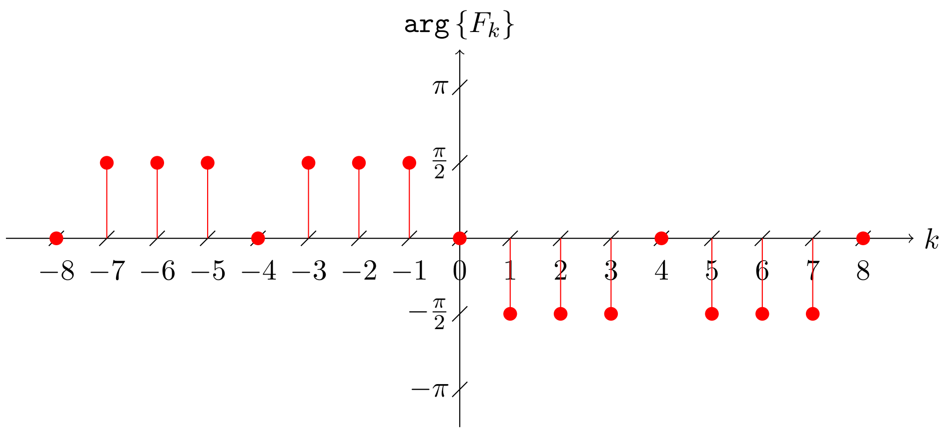
A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.
Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



The magnitude spectrum of a real signal is an even-symmetric function of k.
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



The phase spectrum of a real signal is an odd-symmetric function of k.

2.3 Obliczenia mocy sygnałów - twierdzenie Parsevala

Rozdział 3

Analiza sygnałów nieokresowych. Transformata Fouriera

3.1 Wyznaczanie transformaty Fouriera z definicji

3.2 Wykorzystanie twierdzeń do obliczeń transformaty Fouriera

3.3 Obliczenia energii sygnału za pomocą transformaty Fouriera. Twierdzenie Parsevala

Rozdział 4

Przetwarzanie sygnałów za pomocą układów LTI

4.1 Obliczanie splotu ze wzoru

4.2 Filtry

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