Teoria Sygnałów w zadaniach

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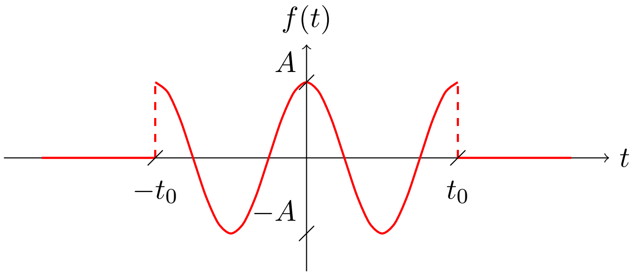
Zadanie 1. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku oraz narysuj jego widmo amplitudowe i fazowe



$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{1}$$

Transformatę Fouriera obliczamy ze wzoru:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{2}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{t_0} A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \\
&= \left\{ cos(x) = \tfrac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2} \right\} \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt + \int_{-t_0}^{t_0} A \cdot \frac{e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t}}{2} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt \\
&= 0 + \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t} \right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + 0 \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \right) \cdot dt \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\frac{2\pi}{t_0}\cdot t - \jmath\cdot\omega\cdot t} + e^{-\jmath\cdot\frac{2\pi}{t_0}\cdot t - \jmath\cdot\omega\cdot t} \right) \cdot dt \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)\cdot t} + e^{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)\cdot t} \right) \cdot dt \\
&= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)\cdot t} \cdot dt + \int_{-t_0}^{t_0} e^{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)\cdot t} \cdot dt \right) \\
&= \left\{ \begin{array}{rlrl} z_1 &= \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t & z_2 &= -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t \\ dz_1 &= \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot dt & dz_2 &= -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot dt \\ dt &= \frac{1}{\jmath\cdot\left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 & dt &= \frac{1}{-\jmath\cdot\left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \end{array} \right\}
\end{aligned}$$

$$= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{z_1} \cdot \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 + \int_{-t_0}^{t_0} e^{z_2} \cdot \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_2} \cdot dz_2 \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{z_1}\Big|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{z_2}\Big|_{-t_0}^{t_0} \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot (-t_0)} \right) + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot (-t_0)} \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right)$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) \cdot \frac{2}{2} + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \cdot \frac{2}{2}$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0}}{2 \cdot \jmath} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0}}{2 \cdot \jmath} \right)$$

$$= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\}$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) \cdot \frac{t_0}{t_0} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \cdot \frac{t_0}{t_0} \right)$$

$$= \frac{A}{2} \cdot \left(\frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + \frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$$

$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} + 2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right)$$

$$= \left\{ Sa(x) = \frac{sin(x)}{x} \right\}$$

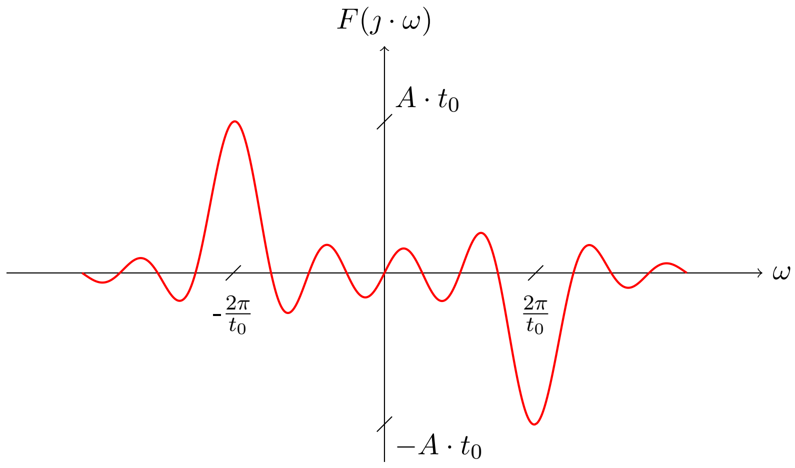
$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + 2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$$

$$= A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$$

Transformata sygnału $f(t)$ to $F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right) \right)$

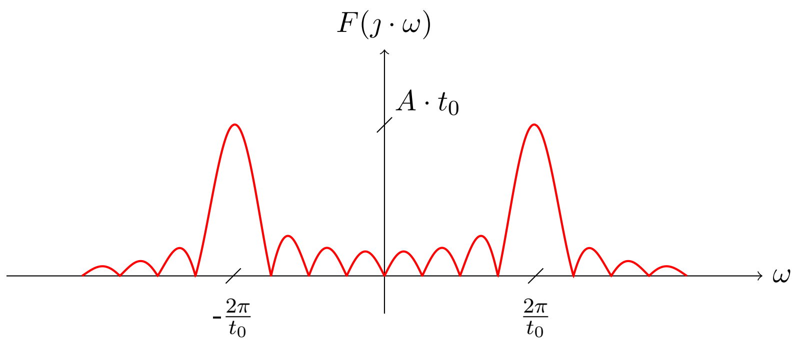
Narysujmy widmo sygnału $f(t)$ czyli:

$$F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)\right) \tag{3}$$



Widmo amplitudowe obliczamy ze wzoru:

$$M(\omega) = |F(j \cdot \omega)| \tag{4}$$



Widmo fazowe obliczamy ze wzoru:

$$\Phi(\omega) = arctg(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}) \tag{5}$$