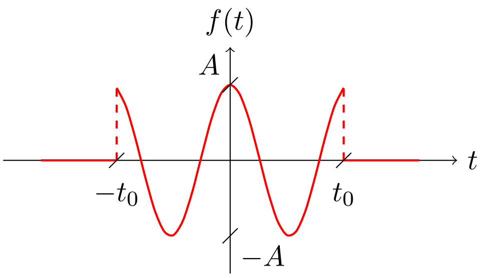
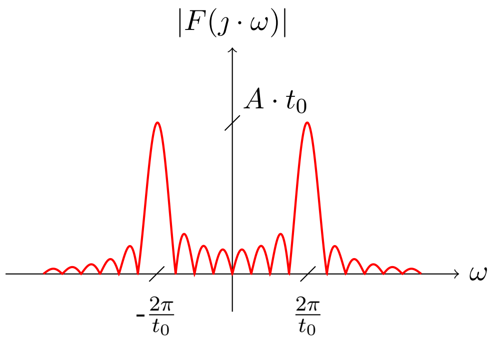
Signal Theory in practise





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\left(\omega \cdot t_0 + 2\pi\right) \\ & -Sa\left(\omega \cdot t_0 - 2\pi\right)] \end{aligned}$$

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Chapter 1

Fundamental concepts and measures

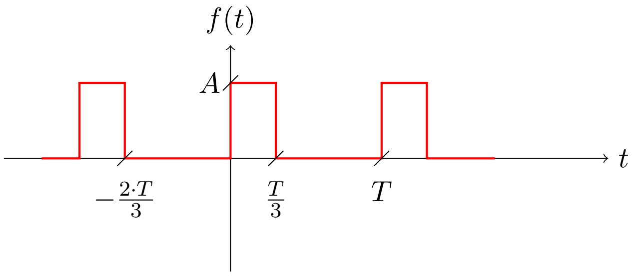
1.1 Basic signal metrics

1.1.1 Mean value of a signal

1.1.2 Energy of a signal

1.1.3 Power and effective value of a signal

Task 1. Compute the average power for the following periodic signal $f(t)$:



Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{3} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{3} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.1}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.2}$$

Compute average power for period $k = 0$

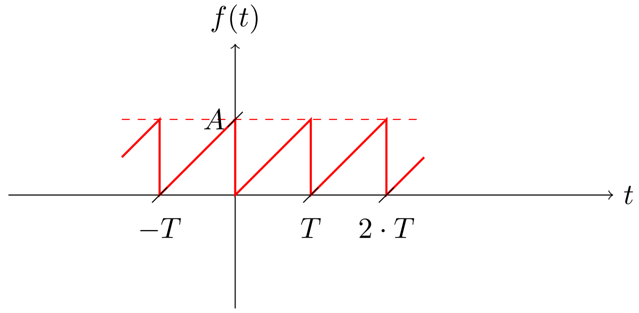
$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} |A|^2 \cdot dt + \int_{\frac{T}{3}}^{T} |0|^2 \cdot dt \right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} A^2 \cdot dt + \int_{\frac{T}{3}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{3}} dt + 0 \right) = \\
&= \frac{A^2}{T} \cdot t|_0^{\frac{T}{3}} = \\
&= \frac{A^2}{T} \cdot \left(\frac{T}{3} - 0 \right) = \\
&= \frac{A^2}{T} \cdot \frac{T}{3} = \\
&= \frac{A^2}{3}
\end{aligned}
$$

Average power equals to $\frac{A^2}{3}$.

Task 2.

Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.3}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and (T,A). So, in order to derive a and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in C$$

The average power for periodic signals is defined by:

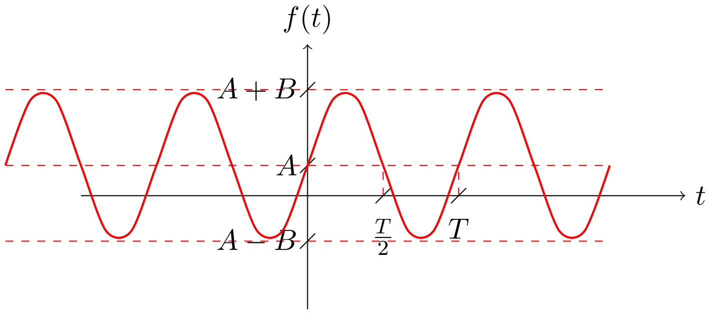
$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.4}$$

In our case we get:

$$
\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| \frac{A}{T} \cdot t \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(\frac{A}{T} \cdot t \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \frac{A^2}{T^2} \cdot t^2 \cdot dt = \\
&= \frac{1}{T} \cdot \frac{A^2}{T^2} \cdot \int_0^T t^2 \cdot dt = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^T \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot 0^3 \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - 0 \right) = \\
&= \frac{A^2}{T^3} \cdot \frac{1}{3} \cdot T^3 = \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}$.

Task 3. Compute the average power for the following periodic signal $f(t) = A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.5}$$

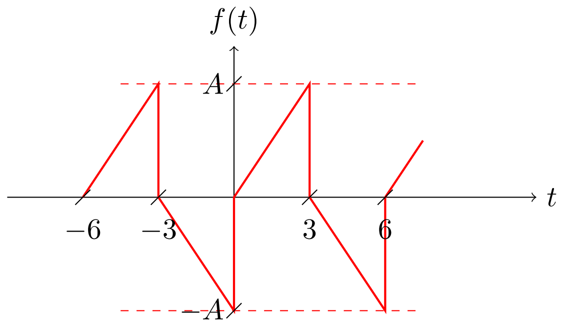
In our case we get:

$$
\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A^2 + 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) + B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \right) \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^T A^2 \cdot dt + \int_0^T 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_0^T B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \frac{A^2}{T} \cdot \int_0^T dt + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{B^2}{T} \cdot \int_0^T sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{l} z = \frac{2\pi}{T} \cdot t \\ dz = \frac{2\pi}{T} \cdot dt \quad dt = \frac{dz}{\frac{2\pi}{T}} = \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{A^2}{T} \cdot t|_0^T + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin(z) \cdot \frac{T}{2\pi} \cdot dz + \frac{B^2}{T} \cdot \int_0^T \frac{1}{2} \cdot \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t \right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot (T - 0) + \frac{2 \cdot A \cdot B}{T} \cdot \frac{T}{2\pi} \cdot \int_0^T sin(z) \cdot dz + \frac{B^2}{T} \cdot \frac{1}{2} \cdot \int_0^T \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t \right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot T + \frac{A \cdot B}{\pi} \cdot \left(-cos(z)|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left(\int_0^T 1 \cdot dt - \int_0^T cos\left(2 \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) = \\
&= \left\{ \begin{array}{l} w = 2 \cdot \frac{2\pi}{T} \cdot t \\ dw = 2 \cdot \frac{2\pi}{T} \cdot dt \quad dt = \frac{dw}{\frac{4\pi}{T}} = \frac{T}{4\pi} \cdot dw \end{array} \right\} = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot t\right) \Big|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left(t|_0^T - \int_0^T cos(w) \cdot \frac{T}{4\pi} \cdot dw \right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot T\right) + cos\left(\frac{2\pi}{T} \cdot 0\right) \right) + \frac{B^2}{2 \cdot T} \cdot \left((T - 0) - \frac{T}{4\pi} \cdot \int_0^T cos(w) \cdot dw \right) =
\end{aligned}
$$

$$= A^2 + \frac{A \cdot B}{\pi} \cdot (-cos\,(2\pi) + cos\,(0)) + \frac{B^2}{2 \cdot T} \cdot \left(T - \frac{T}{4\pi} \cdot -sin\,(w)|_0^T\right) =$$
$$= A^2 + \frac{A \cdot B}{\pi} \cdot (-1 + 1) + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_0^T\right) =$$
$$= A^2 + \frac{A \cdot B}{\pi} \cdot 0 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(2 \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(2 \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot (sin\,(4\pi) - sin\,(0))\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot (0 - 0)\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot (T) =$$
$$= A^2 + \frac{B^2}{2}$$

The average power equals to $A^2 + \frac{B^2}{2}$.

Task 4. Calculate the average power and the effective value (RMS) for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.6}$$

In the first interval of the first period (i.e. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (i.e. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.7}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{6} \cdot \left(\int_0^3 \left| \frac{A}{3} \cdot t \right|^2 \cdot dt + \int_3^6 \left| -\frac{A}{3} \cdot t + A \right|^2 \cdot dt \right) = \\
&= \frac{1}{6} \cdot \int_0^3 \left(\frac{A}{3} \cdot t \right)^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(-\frac{A}{3} \cdot t + A \right)^2 \cdot dt = \\
&= \frac{1}{6} \cdot \int_0^3 \frac{A^2}{9} \cdot t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(\left(-\frac{A}{3} \cdot t \right)^2 - 2 \cdot \frac{A}{3} \cdot t \cdot A + A^2 \right) \cdot dt = \\
&= \frac{A^2}{54} \cdot \int_0^3 t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \frac{A^2}{9} \cdot t^2 \cdot dt - \frac{1}{6} \cdot \int_3^6 \frac{2 \cdot A^2}{3} \cdot t \cdot dt + \frac{1}{6} \cdot \int_3^6 A^2 \cdot dt = \\
&= \frac{A^2}{54} \cdot \left. \frac{t^3}{3} \right|_0^3 + \frac{A^2}{54} \cdot \int_3^6 t^2 \cdot dt - \frac{2 \cdot A^2}{18} \cdot \int_3^6 t^2 \cdot dt + \frac{A^2}{6} \cdot \int_3^6 dt =
\end{aligned}$$

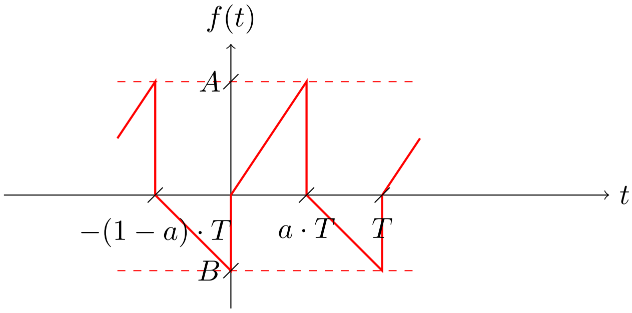
$$\begin{aligned}
&= \frac{A^2}{162} \cdot \left(3^3 - 0^3\right) + \frac{A^2}{54} \cdot \left.\frac{t^3}{3}\right|_3^6 - \frac{2 \cdot A^2}{18} \cdot \left.\frac{t^2}{2}\right|_3^6 + \frac{A^2}{6} \cdot t|_3^6 = \\
&= \frac{A^2}{162} \cdot 27 + \frac{A^2}{162} \cdot \left(6^3 - 3^3\right) - \frac{2 \cdot A^2}{36} \cdot \left(6^2 - 3^2\right) + \frac{A^2}{6} \cdot (6-3) = \\
&= \frac{A^2}{6} + \frac{A^2}{162} \cdot 189 - \frac{2 \cdot A^2}{36} \cdot 27 + \frac{A^2}{6} \cdot 3 = \\
&= \frac{A^2}{6} + \frac{7 \cdot A^2}{6} - \frac{9 \cdot A^2}{6} + \frac{3 \cdot A^2}{6} = \\
&= \frac{2 \cdot A^2}{6} = \\
&= \frac{A^2}{3}
\end{aligned}$$

The average power equals to $\frac{A^2}{3}$.
The effective value (RMS) is defined by:

$$RMS = \sqrt{P} \tag{1.8}$$

Therefore, effective value (RMS) equals to $\frac{A}{\sqrt{3}}$.

Task 5. Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{1.9}$$

In the first interval of the first period (i.e. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a\cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T - a \cdot T} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{T-a\cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{T-a\cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.10}$$

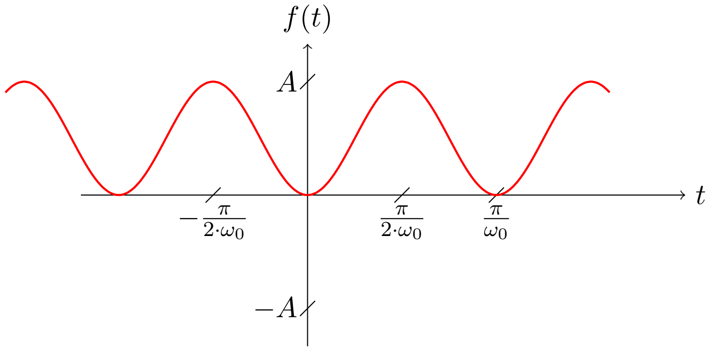
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{a\cdot T} \left| \frac{A}{a \cdot T} \cdot t \right|^2 \cdot dt + \int_{a\cdot T}^{T} \left| \frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right|^2 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \int_0^{a\cdot T} \left(\frac{A}{a \cdot T} \cdot t \right)^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\frac{B}{T - a \cdot T} \cdot t - \frac{B}{1-a} \cdot a \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^{a\cdot T} \frac{A^2}{a^2 \cdot T^2} \cdot t^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\left(\frac{B}{T - a \cdot T} \cdot t \right)^2 - 2 \cdot \frac{B}{T - a \cdot T} \cdot t \cdot \frac{B}{1-a} \cdot a + \left(\frac{B}{1-a} \cdot a \right)^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \int_0^{a\cdot T} t^2 \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \left(\frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 - 2 \cdot \frac{B^2}{T \cdot (1-a)^2} \cdot t \cdot a + \frac{B^2}{(1-a)^2} \cdot a^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^{a\cdot T} \right) + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \frac{B^2}{T^2 \cdot (1-a)^2} \cdot t^2 \cdot dt +
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{T} \cdot \int_{a\cdot T}^{T} 2 \cdot \frac{B^2}{T\cdot(1-a)^2} \cdot t \cdot a \cdot dt + \frac{1}{T} \cdot \int_{a\cdot T}^{T} \frac{B^2}{(1-a)^2} \cdot a^2 \cdot dt = \\
&= \frac{A^2}{a^2\cdot T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^{a\cdot T} \right) + \frac{B^2}{T^3\cdot(1-a)^2} \cdot \int_{a\cdot T}^{T} t^2 \cdot dt + \\
&- \frac{2\cdot B^2}{T^2\cdot(1-a)^2} \cdot a \cdot \int_{a\cdot T}^{T} t \cdot dt + \frac{B^2}{T\cdot(1-a)^2} \cdot a^2 \cdot \int_{a\cdot T}^{T} dt = \\
&= \frac{A^2}{a^2\cdot T^3} \cdot \left(\frac{1}{3} \cdot (a\cdot T)^3 - \frac{1}{3} \cdot 0^3 \right) + \frac{B^2}{T^3\cdot(1-a)^2} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_{a\cdot T}^{T} \right) + \\
&- \frac{2\cdot B^2}{T^2\cdot(1-a)^2} \cdot a \cdot \left(\frac{1}{2} \cdot t^2 \Big|_{a\cdot T}^{T} \right) + \frac{B^2}{T\cdot(1-a)^2} \cdot a^2 \cdot \left(t \big|_{a\cdot T}^{T} \right) = \\
&= \frac{A^2}{a^2\cdot T^3} \cdot \left(\frac{1}{3} \cdot a^3 \cdot T^3 - 0 \right) + \frac{B^2}{T^3\cdot(1-a)^2} \cdot \left(\frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot (a\cdot T)^3 \right) + \\
&- \frac{2\cdot B^2}{T^2\cdot(1-a)^2} \cdot a \cdot \left(\frac{1}{2} \cdot T^2 - \frac{1}{2} \cdot (a\cdot T)^2 \right) + \frac{B^2}{T\cdot(1-a)^2} \cdot a^2 \cdot (T - a\cdot T) = \\
&= \frac{A^2}{a^2\cdot T^3} \cdot \frac{1}{3} \cdot a^3 \cdot T^3 + \frac{B^2}{T^3\cdot(1-a)^2} \cdot \left(\frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot a^3 \cdot T^3 \right) + \\
&- \frac{2\cdot B^2}{T^2\cdot(1-a)^2} \cdot a \cdot \left(\frac{1}{2} \cdot T^2 - \frac{1}{2} \cdot a^2 \cdot T^2 \right) + \frac{B^2}{T\cdot(1-a)^2} \cdot a^2 \cdot (1-a) \cdot T = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{T^3\cdot(1-a)^2} \cdot \left(1 - a^3\right) \cdot \frac{1}{3} \cdot T^3 + \\
&- \frac{2\cdot B^2}{T^2\cdot(1-a)^2} \cdot a \cdot \left(1 - a^2\right) \cdot \frac{1}{2} \cdot T^2 + \frac{B^2}{T\cdot(1-a)^2} \cdot a^2 \cdot (1-a) \cdot T = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{(1-a)^2} \cdot (1-a) \cdot \left(1 + a + a^2\right) \cdot \frac{1}{3} + \\
&- \frac{2\cdot B^2}{(1-a)^2} \cdot a \cdot (1-a) \cdot (1+a) \cdot \frac{1}{2} + \frac{B^2}{1-a} \cdot a^2 = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \left(1 + a + a^2\right) \cdot \frac{1}{3} - \frac{2\cdot B^2}{1-a} \cdot a \cdot (1+a) \cdot \frac{1}{2} + \frac{B^2}{1-a} \cdot a^2 = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \left(\left(1 + a + a^2\right) \cdot \frac{1}{3} - 2 \cdot a \cdot (1+a) \cdot \frac{1}{2} + a^2 \right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \left(\left(1 + a + a^2\right) \cdot \frac{2}{6} - 2 \cdot a \cdot (1+a) \cdot \frac{3}{6} + a^2 \cdot \frac{6}{6} \right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \frac{1}{6} \cdot \left(\left(1 + a + a^2\right) \cdot 2 - 2 \cdot a \cdot (1+a) \cdot 3 + a^2 \cdot 6 \right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \frac{1}{6} \cdot \left(2 + 2\cdot a + 2\cdot a^2 - 6\cdot a - 6\cdot a^2 + 6\cdot a^2\right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \frac{1}{6} \cdot \left(2 - 4\cdot a + 2\cdot a^2\right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \frac{1}{3} \cdot \left(1 - 2\cdot a + a^2\right) = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{1-a} \cdot \frac{1}{3} \cdot (1-a)^2 = \\
&= \frac{A^2}{3} \cdot a + \frac{B^2}{3} \cdot (1-a)
\end{aligned}
$$

The average power equals to $\frac{A^2}{3} \cdot a + \frac{B^2}{3} \cdot (1-a)$.

Task 6. Calculate the average power for the periodic signal $f(t) = A \cdot sin^2(\omega_0 \cdot t)$ given below.



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.11}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| A \cdot sin^2(\omega_0 \cdot t) \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot sin^4(\omega_0 \cdot t) \cdot dt = \\
&= \frac{1}{T} \cdot \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} \right)^4 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{\left(e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \cdot dt = \\
&= \left\{ \begin{array}{llllllllll}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^4 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^0 + 4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^3 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^1 + 6 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^2 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^2}{(2 \cdot \jmath)^4} + \right. \\
&+ \left. \frac{4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^1 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^3 + 1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^0 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-\jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} + \right. \\
&+ \left. \frac{-4 \cdot e^{\jmath \cdot \omega_0 \cdot t} \cdot e^{-3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t - 0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t - \jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t - 2 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{\jmath \cdot \omega_0 \cdot t - 3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t - 4 \cdot \jmath \cdot \omega_0 \cdot t}}{16 \cdot 1} \cdot dt =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} + 6\cdot e^{0\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{16} \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\cdot e^{0}}{16} \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6}{16} \cdot dt = \\
&= \frac{A^2}{16\cdot T} \cdot \int_0^T \left(e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\right) dt = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\int_0^T e^{4\cdot\jmath\cdot\omega_0\cdot t}\cdot dt + \int_0^T e^{-4\cdot\jmath\cdot\omega_0\cdot t}\cdot dt - 4\cdot\int_0^T e^{2\cdot\jmath\cdot\omega_0\cdot t}\cdot dt - 4\cdot\int_0^T e^{-2\cdot\jmath\cdot\omega_0\cdot t}\cdot dt + 6\cdot\int_0^T dt\right) = \\
&= \left\{\begin{matrix} z_1 = 4\cdot\jmath\cdot\omega_0\cdot t & z_2 = -4\cdot\jmath\cdot\omega_0\cdot t & z_3 = 2\cdot\jmath\cdot\omega_0\cdot t & z_4 = -2\cdot\jmath\cdot\omega_0\cdot t = \\ dz_1 = 4\cdot\jmath\cdot\omega_0\cdot dt & dz_2 = -4\cdot\jmath\cdot\omega_0\cdot dt & dz_3 = 2\cdot\jmath\cdot\omega_0\cdot dt & dz_4 = -2\cdot\jmath\cdot\omega_0\cdot dt = \\ dt = \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot dz_1 & dt = \frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot dz_2 & dt = \frac{1}{2\cdot\jmath\cdot\omega_0}\cdot dz_3 & dt = \frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot dz_4 \end{matrix}\right\} = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\int_0^T e^{z_1}\cdot\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot dz_1 + \int_0^T e^{z_2}\cdot\frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot dz_2 + \right. \\
&\left. -\,4\cdot\int_0^T e^{z_3}\cdot\frac{1}{2\cdot\jmath\cdot\omega_0}\cdot dz_3 - 4\cdot\int_0^T e^{z_4}\cdot\frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot dz_4 + 6\cdot\int_0^T dt\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_1}\cdot dz_1 + \frac{1}{-4\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_2}\cdot dz_2 + \right. \\
&\left. -4\cdot\frac{1}{2\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_3}\cdot dz_3 - 4\cdot\frac{1}{-2\cdot\jmath\cdot\omega_0}\cdot\int_0^T e^{z_4}\cdot dz_4 + 6\cdot\int_0^T dt\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{z_1}\Big|_0^T - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{z_2}\Big|_0^T - \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{z_3}\Big|_0^T + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{z_4}\Big|_0^T + \right. \\
&\left. +6\cdot t\Big|_0^T\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \right. \\
&\left. +6\cdot t\Big|_0^T\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{4\cdot\jmath\cdot\omega_0\cdot 0}\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot 0}\right) + \right. \\
&\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{2\cdot\jmath\cdot\omega_0\cdot 0}\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot 0}\right) + 6\cdot(T-0)\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + \right. \\
&\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{0}\right) + 6\cdot T\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) - \frac{1}{4\cdot\jmath\cdot\omega_0}\cdot\left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \right. \\
&\left. -\frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \frac{4}{2\cdot\jmath\cdot\omega_0}\cdot\left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + 6\cdot T\right) = \\
&= \frac{A^2}{16\cdot T} \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} - \frac{1}{4\cdot\jmath\cdot\omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4\cdot\jmath\cdot\omega_0} + \frac{1}{4\cdot\jmath\cdot\omega_0} + \right. \\
&\left. -\frac{4\cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} + \frac{4}{2\cdot\jmath\cdot\omega_0} + \frac{4\cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2\cdot\jmath\cdot\omega_0} - \frac{4}{2\cdot\jmath\cdot\omega_0} + 6\cdot T\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{4 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + \frac{4 \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T \right) = \\
&= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{4 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot T} - 4 \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T \right) = \\
&= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath} - \frac{4}{\omega_0} \cdot \frac{e^{2 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath} + 6 \cdot T \right) = \\
&= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \ cdotx}}{2 \cdot \jmath} \right\} = \\
&= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(4 \cdot \omega_0 \cdot T\right) - \frac{4}{\omega_0} \cdot sin\left(2 \cdot \omega_0 \cdot T\right) + 6 \cdot T \right) = \\
&= \left\{ T = \frac{2\pi}{\omega_0} \right\} = \\
&= \frac{A^2}{16 \cdot \frac{2\pi}{\omega_0}} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(4 \cdot \omega_0 \cdot \frac{2\pi}{\omega_0}\right) - \frac{4}{\omega_0} \cdot sin\left(2 \cdot \omega_0 \cdot \frac{2\pi}{\omega_0}\right) + 6 \cdot \frac{2\pi}{\omega_0} \right) = \\
&= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(8\pi\right) - \frac{4}{\omega_0} \cdot sin\left(4\pi\right) + 6 \cdot \frac{2\pi}{\omega_0} \right) = \\
&= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot 0 - \frac{4}{\omega_0} \cdot 0 + 6 \cdot \frac{2\pi}{\omega_0} \right) = \\
&= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \frac{12\pi}{\omega_0} = \\
&= \frac{3 \cdot A^2}{8}
\end{aligned}
$$

The average power equals to $\frac{3 \cdot A^2}{8}$.

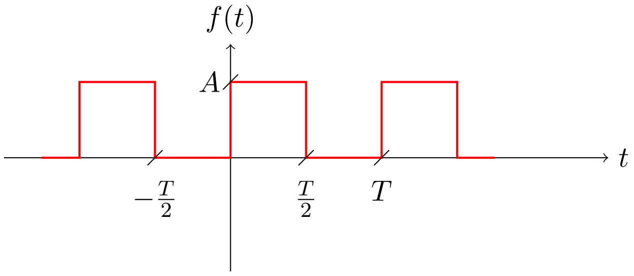
Chapter 2

Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

2.2 Complex exponential Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} dt + 0\right) = \\
&= \frac{1}{T}\left(A\cdot t|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T}\cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}-0\right) = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.3}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.4}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^z\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^z\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^z|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^0\right) =
\end{aligned}$$

$$= -\frac{A}{\jmath \cdot k \cdot 2\pi}\left(e^{-\jmath \cdot k \cdot \pi} - 1\right) =$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left(e^{-\jmath \cdot k \cdot \pi} - 1\right)$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

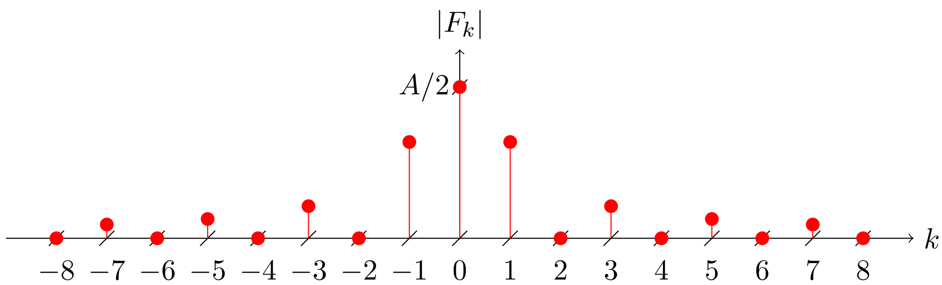
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}$$
$$f(t) = \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.5}$$

The first several coefficients are equal to:

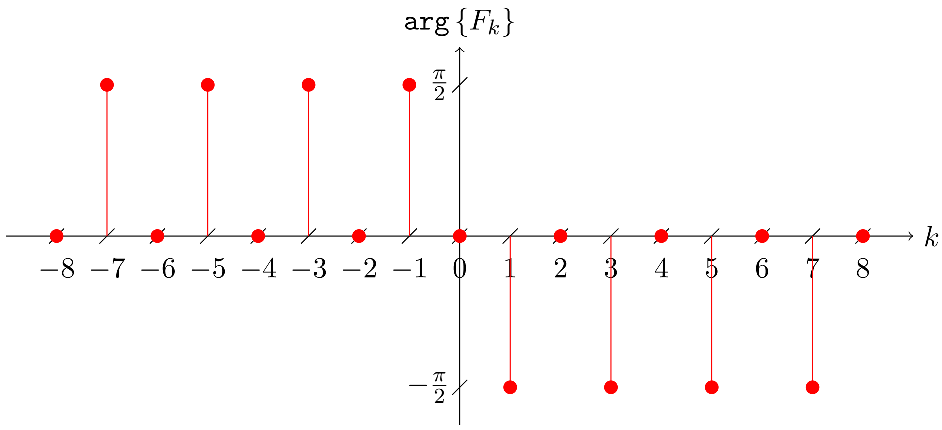
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	0	$\jmath \cdot \frac{A}{\pi}$	$\frac{A}{2}$	$-\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$
$\lvert F_k \rvert$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{2}$	$\frac{A}{\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$
$Arg\{F_k\}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Based on coefficents F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

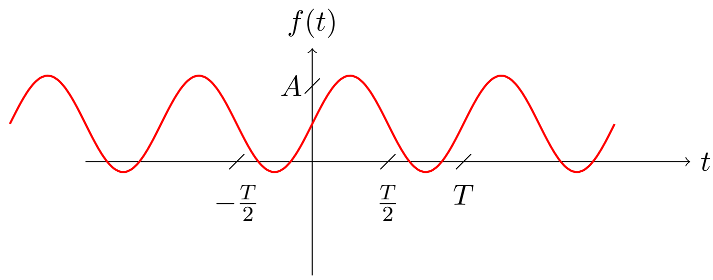


The magnitude spectrum of a real signal is an even-symmetric function of k.

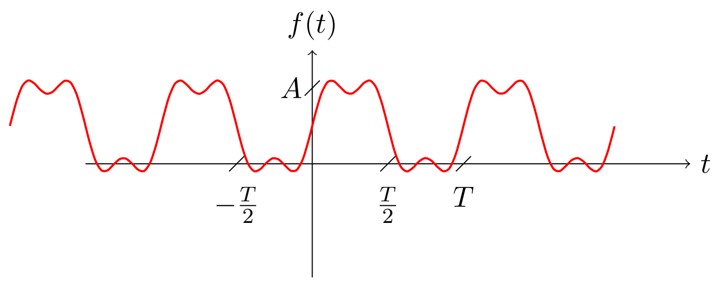
Based on coefficents F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



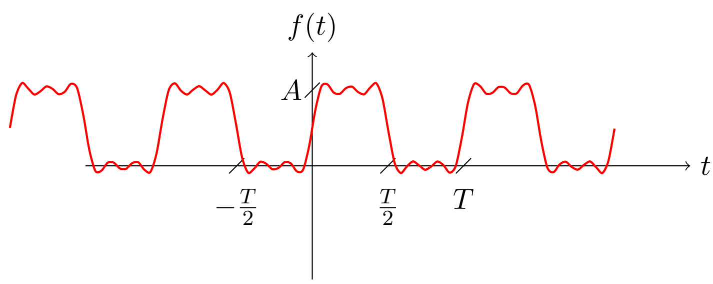
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



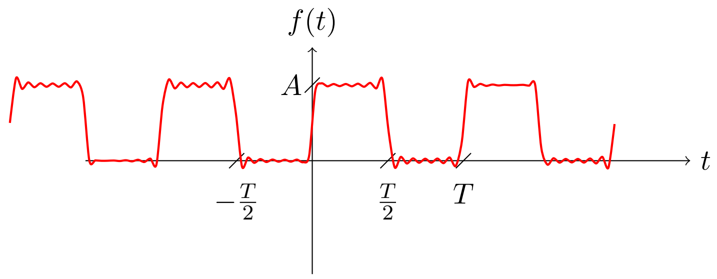
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



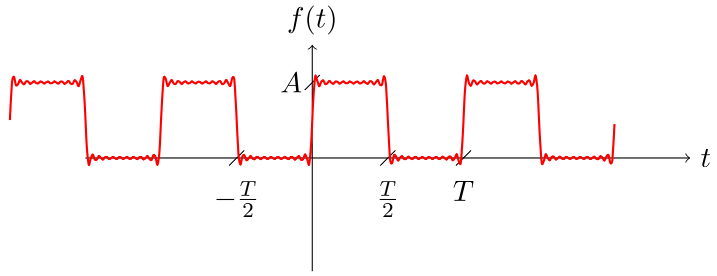
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



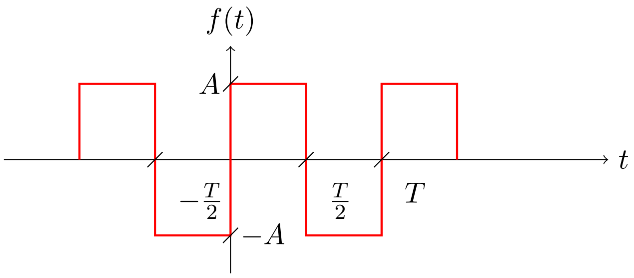
A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.

2.3 Computing the power of a signal – the Parseval's theorem

Task 1. Compute the percentage contribution of the fundamental (first) harmonic in the total power of the periodic square signal shown in the figure below:



$$\frac{P_1}{P} =? \tag{2.6}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -A & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.7}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.8}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |-A|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + A^2 \cdot \int_{\frac{T}{2}}^{T} dt \right) = \\ &= \frac{A^2}{T} \cdot \left(t\Big|_0^{\frac{T}{2}} + t\Big|_{\frac{T}{2}}^{T} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 + T - \frac{T}{2} \right) = \\ &= \frac{A^2}{T} \cdot (T) = \\ &= A^2 \end{aligned}$$

The total power of the $f(t)$ signal equals A^2.

Based on Parseval theorem, power of the fundamental harmonic is defined as:

$$P_1 = |F_1|^2 + |F_{-1}|^2 \tag{2.9}$$

Because the $f(t) \in R$, thus the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.10}$$

In order to calculate P_1, the F_1 coefficient has to be calculated:

$$F_1 = \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.11}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_1 &= \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} -A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt - \int_{\frac{T}{2}}^{T} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z & = -\jmath \cdot \frac{2\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \end{array} \right\} = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} - \int_{\frac{T}{2}}^{T} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \right) = \\
&= -\frac{A}{T \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot dz - \int_{\frac{T}{2}}^{T} e^z \cdot dz \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^z \Big|_0^{\frac{T}{2}} - e^z \Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \Big|_0^{\frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot T} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \pi} - e^0 - e^{-\jmath \cdot 2 \cdot \pi} + e^{-\jmath \cdot \pi} \right) = \\
&= \left\{ \begin{array}{ll} e^{-\jmath \cdot 2 \cdot \pi} & = cos(2\pi) - \jmath \cdot sin(2\pi) = 1 \\ e^{-\jmath \cdot \pi} & = cos(\pi) - \jmath \cdot sin(\pi) = -1 \end{array} \right\} = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-1 - 1 - 1 - 1) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-4) = \\
&= \frac{2 \cdot A}{\jmath \cdot \pi} = \\
&= -\jmath \cdot \frac{2 \cdot A}{\pi}
\end{aligned}$$

The F_1 coefficient equals $-\jmath \cdot \frac{2 \cdot A}{\pi}$.

Thus, P_1 may be calculated:

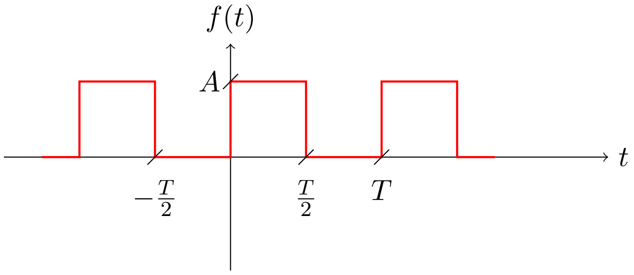
$$\begin{aligned}
P_1 &= 2 \cdot |F_1|^2 = \\
&= 2 \cdot \left| -\jmath \cdot \frac{2 \cdot A}{\pi} \right|^2 = \\
&= 2 \cdot \left(\frac{2 \cdot A}{\pi} \right)^2 = \\
&= 2 \cdot \frac{4 \cdot A^2}{\pi^2} = \\
&= \frac{8 \cdot A^2}{\pi^2}
\end{aligned}$$

The power of the fundamental harmonic equals $P_1 = \frac{8 \cdot A^2}{\pi^2}$.

Finally, the percentage contribution of the fundamental harmonic in the total power of the $f(t)$ signal is equal to:

$$\frac{P_1}{P} = \frac{\frac{8 \cdot A^2}{\pi^2}}{A^2} = \frac{8}{\pi^2} \approx 81\% \tag{2.12}$$

Task 2. Calculate the percentage contribution of the power of higher harmonics ($k > 1$) to the total average power of the periodic signal shown below.



$$\frac{P_{>1}}{P} = ? \tag{2.13}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.14}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.15}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |0|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(t\Big|_0^{\frac{T}{2}} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A^2}{2} \end{aligned}$$

The total power of the $f(t)$ signal equals $\frac{A^2}{2}$.

Based on Parseval theorem, power of the higher harmonics is defined as:

$$P_{>1} = P - P_0 - P_1 \tag{2.16}$$

where:

$$P_0 = |F_0|^2$$
$$P_1 = |F_1|^2 + |F_{-1}|^2$$

Because the $f(t) \in R$, thus the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.17}$$

In order to calculate P_0 and P_1, the F_0 and F_1 coefficients have to be calculated. The F_k coefficients have been calculated in task 1 and are equal to:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \Big((-1)^k - 1\Big)$$

Now, we may calculate the P_0 and P_1:

$$\begin{aligned} P_0 &= |F_0|^2 \\ &= \left|\frac{A}{2}\right|^2 \\ &= \frac{A^2}{4} \end{aligned}$$

$$\begin{aligned} P_1 &= 2 \cdot |F_1|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1\Big)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{2\pi} \cdot (-1 - 1)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{2\pi} \cdot (-2)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{-A}{\pi}\right|^2 \\ &= 2 \cdot \left(\frac{A}{\pi}\right)^2 \\ &= 2 \cdot \frac{A^2}{\pi^2} \end{aligned}$$

Finally, power of the higher harmonics is defined as:

$$\begin{aligned}
P_{>1} &= P - P_0 - P_1 \\
&= \frac{A^2}{2} - \frac{A^2}{4} - 2 \cdot \frac{A^2}{\pi^2} \\
&= \frac{2 \cdot A^2 \cdot \pi^2}{4\pi^2} - \frac{A^2 \cdot \pi^2}{4\pi^2} - \frac{8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \pi^2 - 8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}
\end{aligned}$$

The power of the fundamental harmonic equals $P_{>1} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}$.

Finally, the percentage contribution of the higher harmonics in the total power of the $f(t)$ signal is equal to:

$$\frac{P_{>1}}{P} = \frac{\frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}}{\frac{A^2}{2}} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2} \cdot \frac{2}{A^2} = \frac{\pi^2 - 8}{2\pi^2} = \approx 9\% \tag{2.18}$$

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

3.2 Exploiting properties of the Fourier transform

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

4.2 Filters

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