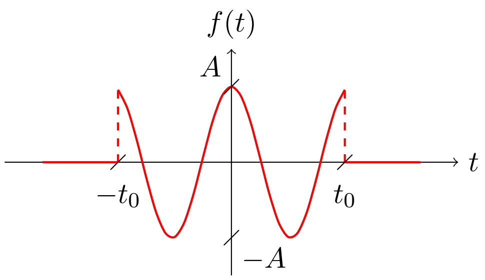
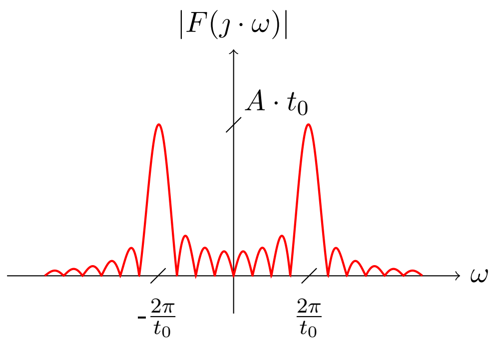
Signal Theory in examples





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$\begin{aligned} F(\jmath\omega) = A \cdot t_0 \cdot [\ & Sa\left(\omega \cdot t_0 + 2\pi\right) \\ & - Sa\left(\omega \cdot t_0 - 2\pi\right)] \end{aligned}$$

Tomasz Grajek, Krzysztof Wegner

April 6, 2020

Poznan University of Technology
Faculty of Computing and Telecommunications
Departament of Multimedia Telecomunications

pl. M. Skłodowskiej-Curie 5
60-965 Poznań

www.et.put.poznan.pl
www.multimedia.edu.pl

ISBN 978-83-939620-3-7
Printed in Poland

Chapter 1

Basic properties of the signals

1.1 Basic properties of the signals

1.1.1 Mean of the signal

1.1.2 Energy of the signal

1.1.3 Power of the signal

Chapter 2

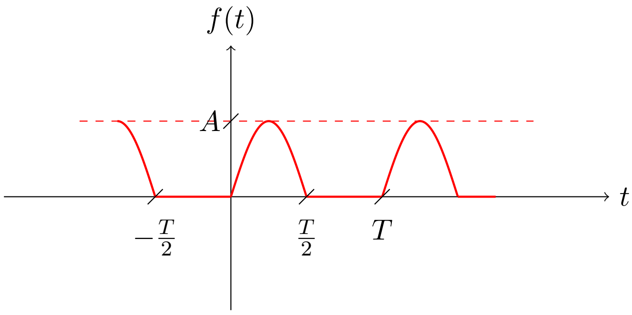
Analisis of the periodic signals by ortogonal series

2.1 Trigonometric Fourier Series

2.2 Complex Fourier Series

Task 1.

Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{A}{T}\left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt + 0\right) = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{matrix} z & = \frac{2\pi}{T}\cdot t \\ dz & = \frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} sin\left(z\right)\cdot dz = \\
&= \frac{A}{2\pi}\cdot\left(-cos\left(z\right)\Big|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\frac{2\pi}{T}\cdot t\right)\Bigg|_0^{\frac{T}{2}}\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\frac{2\pi}{T}\cdot\frac{T}{2}\right) - cos\left(\frac{2\pi}{T}\cdot 0\right)\right) = \\
&= -\frac{A}{2\pi}\cdot\left(cos\left(\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{A}{2\pi}\cdot\left(-1-1\right) = \\
&= -\frac{A}{2\pi}\cdot\left(-2\right) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{\pi}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.3}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{sin\left(x\right) \quad = \frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \end{array} \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} - \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{T \cdot 2\jmath \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\frac{1}{1-k} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{1+k} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{z_1} \Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{z_2} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^0 \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - e^0 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1+k}{(1-k) \cdot (1+k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + \frac{1-k}{(1-k) \cdot (1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right)}{(1-k) \cdot (1+k)} + \frac{(1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + (1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 1 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} - k + e^{-\jmath \cdot \pi \cdot (1+k)} - 1 - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)} + k}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 2 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} + e^{-\jmath \cdot \pi \cdot (1+k)} - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - 2 + k \cdot e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} + e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - k \cdot e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) =
\end{aligned}
$$

$$= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-1\cdot e^{-\jmath\cdot\pi\cdot k}-2+k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k}-1\cdot e^{-\jmath\cdot\pi\cdot k}-k\cdot(-1)\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right)=$$
$$= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-e^{-\jmath\cdot\pi\cdot k}-2-k\cdot e^{-\jmath\cdot\pi\cdot k}-e^{-\jmath\cdot\pi\cdot k}+k\cdot e^{-\jmath\cdot\pi\cdot k}}{1-k^2}\right)=$$
$$= \frac{A}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\pi\cdot k}-2}{1-k^2}\right)=$$
$$= \frac{A}{4\cdot\pi}\cdot\left(\frac{2\cdot e^{-\jmath\cdot\pi\cdot k}+2}{1-k^2}\right)=$$
$$= \frac{A}{4\cdot\pi}\cdot 2\cdot\left(\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2}\right)=$$
$$= \frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2}$$

The F_k coefficients equal to $\frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2}$ for $k\neq 1\wedge k\neq -1$.
We have to calculate F_k for $k=1$ directly by definition:

$$F_1 = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{2}} A\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right)=$$
$$= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=$$
$$= \frac{1}{T}\cdot\left(A\cdot\int_0^{\frac{T}{2}}\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt+0\right)=$$
$$= \frac{1}{T}\cdot\left(\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\frac{A}{2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\right)\cdot dt=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1-1)}\cdot dt-\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(1+1)}\cdot dt\right)=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 0}\cdot dt-\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot 2}\cdot dt\right)=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} e^{0}\cdot dt-\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{A}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{2}} 1\cdot dt-\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=$$

$$
\begin{aligned}
&= \left\{\begin{matrix} z & = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(t\Big|_0^{\frac{T}{2}} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0\right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0\right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= \frac{A}{4\jmath} = \\
&= -\jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $-\jmath \cdot \frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =
\end{aligned}
$$

$$= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt\right) =$$

$$= \left\{\begin{aligned} z &= \jmath \cdot \tfrac{4\pi}{T} \cdot t \\ dz &= \jmath \cdot \tfrac{4\pi}{T} \cdot dt \\ dt &= \tfrac{dz}{\jmath \cdot \frac{4\pi}{T}} \end{aligned}\right\} =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz - \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}}\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - \left(\frac{T}{2} - 0\right)\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0}\right) - \left(\frac{T}{2} - 0\right)\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0}\right) - \frac{T}{2}\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) - \frac{T}{2}\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 - \frac{T}{2}\right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{2}\right) =$$

$$= -\frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} =$$

$$= -\frac{A}{4\jmath} =$$

$$= \jmath \cdot \frac{A}{4}$$

The F_{-1} coefficients equal to $\jmath \cdot \frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_k &= \frac{A}{2\cdot\pi} \cdot \frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2} \\
F_{-1} &= \jmath \cdot \frac{A}{4} \\
F_1 &= -\jmath \cdot \frac{A}{4}
\end{aligned}$$

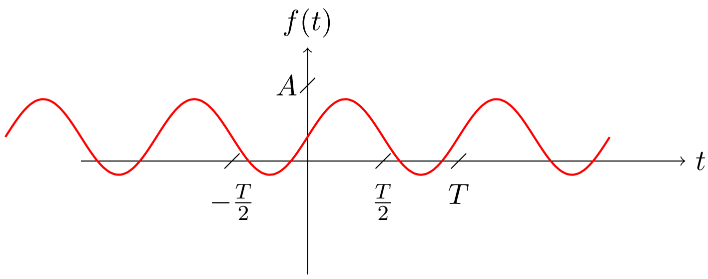
The first several coefficients are equal to:

F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$-\jmath\cdot\frac{A}{4}$	$\frac{A}{\pi}$	$\jmath\cdot\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$\lvert F_k\rvert$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$

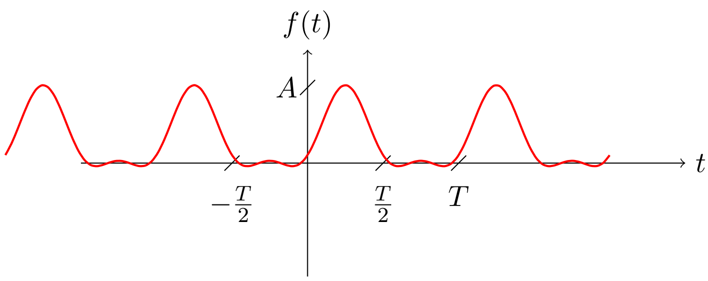
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \\
f(t) &= \frac{A}{\pi} + \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t} - \jmath\cdot\frac{A}{4}\cdot e^{\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t} + \sum_{\substack{k=-\infty \\ k\neq 0 \\ k\neq -1 \wedge k\neq 1}}^{\infty} \left[\frac{A}{2\cdot\pi}\cdot\frac{e^{-\jmath\cdot\pi\cdot k}+1}{1-k^2}\right]\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}
\end{aligned} \tag{2.4}$$

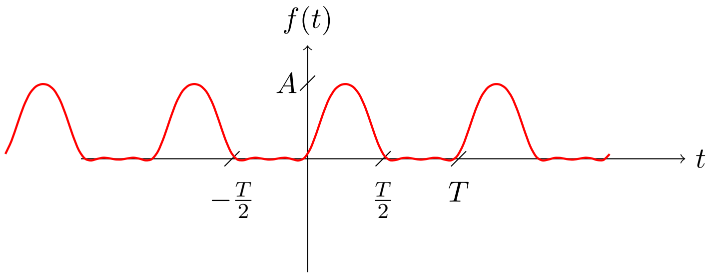
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



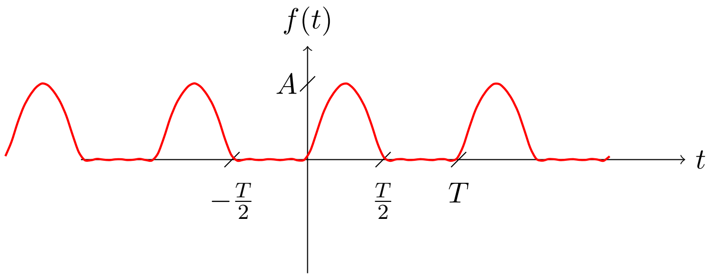
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



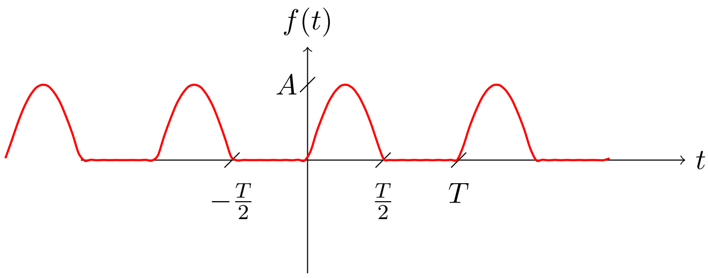
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



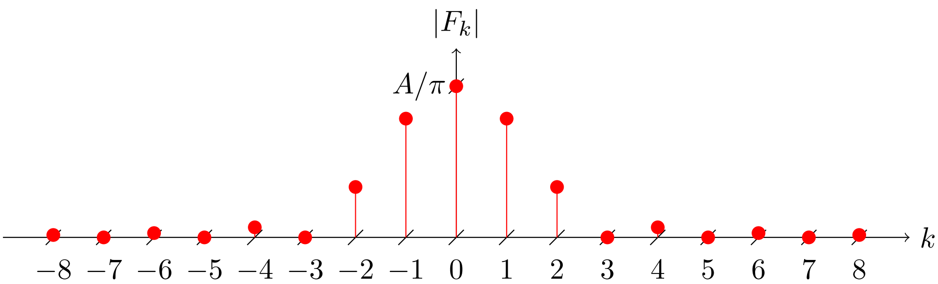
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



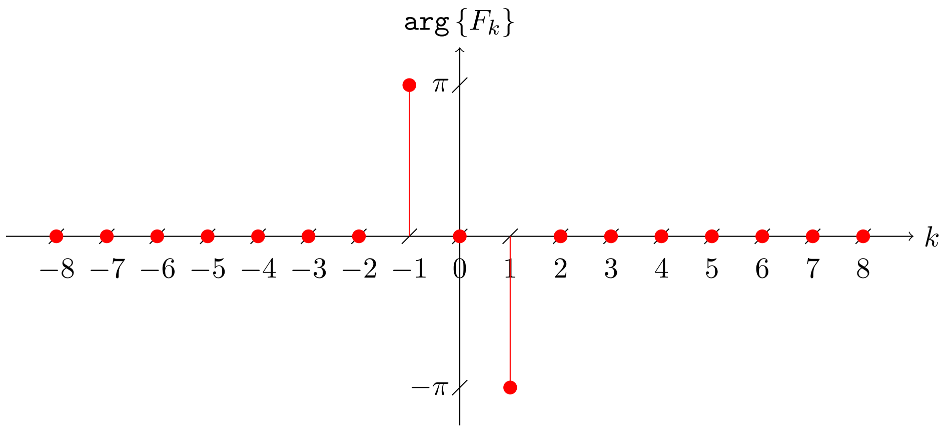
A partial approximation of the $f(t)$ signal from $k_{min} = -12$ to $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in oryginal signal.
Based on coefficents F_k we can plot



Based on coefficents F_k we can plot



arg {F_k}
π
$-\pi$
k
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8

2.3 Calculation of the signals power - Perseval's teorem

Chapter 3

Non-periodic signals analisis - Fourier Transform

3.1 Calculation of Fourier Transform from definition

3.2 Exploiting properties of the Fourier transform

3.3 Signal power calculation based on Fourier transform. Perseval Teorem

Chapter 4

Linear Time Invariant

4.1 Convolutions

4.2 Filters

ISBN 978-83-939620-3-7