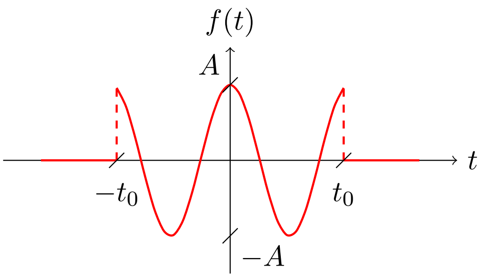
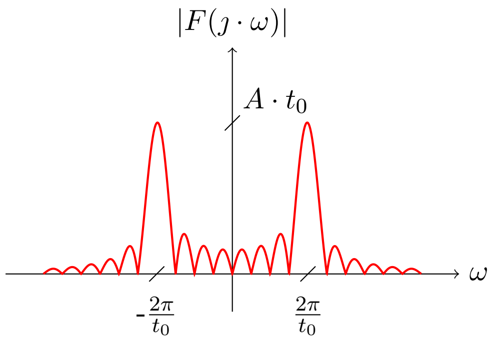
Signal Theory in practise





$$f(t) = A\cdot\Pi\left(\frac{t}{2\cdot t_0}\right)\cdot cos\left(\frac{2\pi}{t_0}\cdot t\right)$$

$$F(\jmath\omega) = A\cdot t_0\cdot\left[\ Sa\left(\omega\cdot t_0+2\pi\right) - Sa\left(\omega\cdot t_0-2\pi\right)\right]$$

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Chapter 1

Fundamental concepts and measures

Task 1. Expand the following signals into a sum of sine and cosine functions, and a constant, by using the Euler identities.

$$f_1(t) = sin^5(t) - sin^3(t)$$
$$f_2(t) = cos^6(t) - cos^4(t)$$

Euler identities:

$$sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath}$$
$$cos\left(x\right) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k \tag{1.1}$$

where $\binom{n}{k}$ are called binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.2}$$

The binomial coefficient $\binom{n}{k}$ appears as the kth entry in the nth row of Pascal's triangle (counting starts at 0). Each entry is the sum of the two above it. Below, example for $n = 6$ is presented:

$$\left\{\begin{array}{lccccccccccccc}
n{=}0: & & & & & & & 1 & & & & & & \\
n{=}1: & & & & & & 1 & & 1 & & & & & \\
n{=}2: & & & & & 1 & & 2 & & 1 & & & & \\
n{=}3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n{=}4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n{=}5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n{=}6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}\right\} \tag{1.3}$$

$$\begin{aligned}
f_1(t) &= sin^5(t) - sin^3(t) = \\
&= \left(\frac{e^{\jmath \cdot t} - e^{-\jmath \cdot t}}{2 \cdot \jmath}\right)^5 - \left(\frac{e^{\jmath \cdot t} - e^{-\jmath \cdot t}}{2 \cdot \jmath}\right)^3 = \\
&= \frac{\left(e^{\jmath \cdot t} - e^{-\jmath \cdot t}\right)^5}{(2 \cdot \jmath)^5} - \frac{\left(e^{\jmath \cdot t} - e^{-\jmath \cdot t}\right)^3}{(2 \cdot \jmath)^3} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1 \cdot (e^{\jmath \cdot t})^5 \cdot (-e^{-\jmath \cdot t})^0 + 5 \cdot (e^{\jmath \cdot t})^4 \cdot (-e^{-\jmath \cdot t})^1 + 10 \cdot (e^{\jmath \cdot t})^3 \cdot (-e^{-\jmath \cdot t})^2}{(2 \cdot \jmath)^5} + \\
&+ \frac{10 \cdot (e^{\jmath \cdot t})^2 \cdot (-e^{-\jmath \cdot t})^3 + 5 \cdot (e^{\jmath \cdot t})^1 \cdot (-e^{-\jmath \cdot t})^4 + 1 \cdot (e^{\jmath \cdot t})^0 \cdot (-e^{-\jmath \cdot t})^5}{(2 \cdot \jmath)^5} - \\
&- \left(\frac{1 \cdot (e^{\jmath \cdot t})^3 \cdot (-e^{-\jmath \cdot t})^0 + 3 \cdot (e^{\jmath \cdot t})^2 \cdot (-e^{-\jmath \cdot t})^1 + 3 \cdot (e^{\jmath \cdot t})^1 \cdot (-e^{-\jmath \cdot t})^2 + 1 \cdot (e^{\jmath \cdot t})^0 \cdot (-e^{-\jmath \cdot t})^3}{(2 \cdot \jmath)^3} \right) = \\
&= \frac{1 \cdot e^{\jmath \cdot t \cdot 5} \cdot 1 + 5 \cdot e^{\jmath \cdot t \cdot 4} \cdot (-e^{-\jmath \cdot t \cdot 1}) + 10 \cdot e^{\jmath \cdot t \cdot 3} \cdot e^{-\jmath \cdot t \cdot 2}}{(2 \cdot \jmath)^5} + \\
&+ \frac{10 \cdot e^{\jmath \cdot t \cdot 2} \cdot (-e^{-\jmath \cdot t \cdot 3}) + 5 \cdot e^{\jmath \cdot t \cdot 1} \cdot e^{-\jmath \cdot t \cdot 4} + 1 \cdot 1 \cdot (-e^{-\jmath \cdot t \cdot 5})}{(2 \cdot \jmath)^5} - \\
&- \left(\frac{1 \cdot e^{\jmath \cdot t \cdot 3} \cdot 1 + 3 \cdot e^{\jmath \cdot t \cdot 2} \cdot (-e^{-\jmath \cdot t \cdot 1}) + 3 \cdot e^{\jmath \cdot t \cdot 1} \cdot e^{-\jmath \cdot t \cdot 2} + 1 \cdot 1 \cdot (-e^{-\jmath \cdot t \cdot 3})}{(2 \cdot \jmath)^3} \right) = \\
&= \frac{e^{\jmath \cdot t \cdot 5} - 5 \cdot e^{\jmath \cdot t \cdot 3} + 10 \cdot e^{\jmath \cdot t} - 10 \cdot e^{-\jmath \cdot t} + 5 \cdot e^{-\jmath \cdot t \cdot 3} - e^{-\jmath \cdot t \cdot 5}}{(2 \cdot \jmath)^5} - \\
&- \left(\frac{e^{\jmath \cdot t \cdot 3} - 3 \cdot e^{\jmath \cdot t} + 3 \cdot e^{-\jmath \cdot t} - e^{-\jmath \cdot t \cdot 3}}{(2 \cdot \jmath)^3} \right) = \\
&= \frac{e^{\jmath \cdot t \cdot 5} - e^{-\jmath \cdot t \cdot 5} - 5 \cdot e^{\jmath \cdot t \cdot 3} + 5 \cdot e^{-\jmath \cdot t \cdot 3} + 10 \cdot e^{\jmath \cdot t} - 10 \cdot e^{-\jmath \cdot t}}{(2 \cdot \jmath)^5} - \\
&- \left(\frac{e^{\jmath \cdot t \cdot 3} - e^{-\jmath \cdot t \cdot 3} - 3 \cdot e^{\jmath \cdot t} + 3 \cdot e^{-\jmath \cdot t}}{(2 \cdot \jmath)^3} \right) = \\
&= \frac{e^{\jmath \cdot t \cdot 5} - e^{-\jmath \cdot t \cdot 5} - 5 \cdot (e^{\jmath \cdot t \cdot 3} - e^{-\jmath \cdot t \cdot 3}) + 10 \cdot (e^{\jmath \cdot t} - e^{-\jmath \cdot t})}{(2 \cdot \jmath)^4 \cdot (2 \cdot \jmath)} - \\
&- \left(\frac{e^{\jmath \cdot t \cdot 3} - e^{-\jmath \cdot t \cdot 3} - 3 \cdot (e^{\jmath \cdot t} - \cdot e^{-\jmath \cdot t})}{(2 \cdot \jmath)^2 \cdot (2 \cdot \jmath)} \right) = \\
&= \frac{sin(5 \cdot t) - 5 \cdot sin(3 \cdot t) + 10 \cdot sin(t)}{(2 \cdot \jmath)^4} - \\
&- \left(\frac{sin(3 \cdot t) - 3 \cdot sin(t)}{(2 \cdot \jmath)^2} \right) = \\
&= \frac{sin(5 \cdot t) - 5 \cdot sin(3 \cdot t) + 10 \cdot sin(t)}{16} + \left(\frac{sin(3 \cdot t) - 3 \cdot sin(t)}{4} \right) = \\
&= \frac{sin(5 \cdot t) - 5 \cdot sin(3 \cdot t) + 10 \cdot sin(t)}{16} + \frac{4 \cdot sin(3 \cdot t) - 12 \cdot sin(t)}{16} = \\
&= \frac{sin(5 \cdot t) - sin(3 \cdot t) - 2 \cdot sin(t)}{16}
\end{aligned}
$$

To sum up:

$$f_1(t) = sin^5(t) - sin^3(t) = \frac{sin(5 \cdot t) - sin(3 \cdot t) - 2 \cdot sin(t)}{16}$$

$$
\begin{aligned}
&f_2(t) = cos^6(t) - cos^4(t) = \\
&\quad = \left(\frac{e^{\jmath \cdot t} + e^{-\jmath \cdot t}}{2} \right)^6 - \left(\frac{e^{\jmath \cdot t} + e^{-\jmath \cdot t}}{2} \right)^4 =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(e^{\jmath\cdot t}+e^{-\jmath\cdot t}\right)^6}{2^6} - \frac{\left(e^{\jmath\cdot t}+e^{-\jmath\cdot t}\right)^4}{2^4} = \\
&= \frac{1\cdot(e^{\jmath\cdot t})^6\cdot(e^{-\jmath\cdot t})^0 + 6\cdot(e^{\jmath\cdot t})^5\cdot(e^{-\jmath\cdot t})^1 + 15\cdot(e^{\jmath\cdot t})^4\cdot(e^{-\jmath\cdot t})^2 + 20\cdot(e^{\jmath\cdot t})^3\cdot(e^{-\jmath\cdot t})^3}{2^6} + \\
&+ \frac{15\cdot(e^{\jmath\cdot t})^2\cdot(e^{-\jmath\cdot t})^4 + 6\cdot(e^{\jmath\cdot t})^1\cdot(e^{-\jmath\cdot t})^5 + 1\cdot(e^{\jmath\cdot t})^0\cdot(e^{-\jmath\cdot t})^6}{2^6} - \\
&- \left(\frac{1\cdot(e^{\jmath\cdot t})^4\cdot(e^{-\jmath\cdot t})^0 + 4\cdot(e^{\jmath\cdot t})^3\cdot(e^{-\jmath\cdot t})^1 + 6\cdot(e^{\jmath\cdot t})^2\cdot(e^{-\jmath\cdot t})^2 + 4\cdot(e^{\jmath\cdot t})^1\cdot(e^{-\jmath\cdot t})^3 + 1\cdot(e^{\jmath\cdot t})^0\cdot(e^{-\jmath\cdot t})^4}{2^4}\right) = \\
&= \frac{1\cdot e^{\jmath\cdot t\cdot 6}\cdot 1 + 6\cdot e^{\jmath\cdot t\cdot 5}\cdot e^{-\jmath\cdot t\cdot 1} + 15\cdot e^{\jmath\cdot t\cdot 4}\cdot e^{-\jmath\cdot t\cdot 2} + 20\cdot e^{\jmath\cdot t\cdot 3}\cdot e^{-\jmath\cdot t\cdot 3}}{2^6} + \\
&+ \frac{15\cdot e^{\jmath\cdot t\cdot 2}\cdot e^{-\jmath\cdot t\cdot 4} + 6\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 5} + 1\cdot 1\cdot e^{-\jmath\cdot t\cdot 6}}{2^6} - \\
&- \left(\frac{1\cdot e^{\jmath\cdot t\cdot 4}\cdot 1 + 4\cdot e^{\jmath\cdot t\cdot 3}\cdot e^{-\jmath\cdot t\cdot 1} + 6\cdot e^{\jmath\cdot t\cdot 2}\cdot e^{-\jmath\cdot t\cdot 2} + 4\cdot e^{\jmath\cdot t\cdot 1}\cdot e^{-\jmath\cdot t\cdot 3} + 1\cdot 1\cdot e^{-\jmath\cdot t\cdot 4}}{2^4}\right) = \\
&= \frac{e^{\jmath\cdot t\cdot 6} + 6\cdot e^{\jmath\cdot t\cdot 4} + 15\cdot e^{\jmath\cdot t\cdot 2} + 20\cdot e^{\jmath\cdot t\cdot 0} + 15\cdot e^{-\jmath\cdot t\cdot 2} + 6\cdot e^{-\jmath\cdot t\cdot 4} + e^{-\jmath\cdot t\cdot 6}}{2^6} - \\
&- \left(\frac{e^{\jmath\cdot t\cdot 4} + 4\cdot e^{\jmath\cdot t\cdot 2} + 6\cdot e^{\jmath\cdot t\cdot 0} + 4\cdot e^{-\jmath\cdot t\cdot 2} + e^{-\jmath\cdot t\cdot 4}}{2^4}\right) = \\
&= \frac{e^{\jmath\cdot t\cdot 6} + e^{-\jmath\cdot t\cdot 6} + 6\cdot e^{\jmath\cdot t\cdot 4} + 6\cdot e^{-\jmath\cdot t\cdot 4} + 15\cdot e^{\jmath\cdot t\cdot 2} + 15\cdot e^{-\jmath\cdot t\cdot 2} + 20}{2^6} - \\
&- \left(\frac{e^{\jmath\cdot t\cdot 4} + e^{-\jmath\cdot t\cdot 4} + 4\cdot e^{\jmath\cdot t\cdot 2} + 4\cdot e^{-\jmath\cdot t\cdot 2} + 6}{2^4}\right) = \\
&= \frac{e^{\jmath\cdot t\cdot 6} + e^{-\jmath\cdot t\cdot 6} + 6\cdot(e^{\jmath\cdot t\cdot 4} + e^{-\jmath\cdot t\cdot 4}) + 15\cdot(e^{\jmath\cdot t\cdot 2} + e^{-\jmath\cdot t\cdot 2}) + 20}{2^5\cdot 2} - \\
&- \left(\frac{e^{\jmath\cdot t\cdot 4} + e^{-\jmath\cdot t\cdot 4} + 4\cdot(e^{\jmath\cdot t\cdot 2} + e^{-\jmath\cdot t\cdot 2}) + 6}{2^3\cdot 2}\right) = \\
&= \frac{cos(6\cdot t) + 6\cdot cos(4\cdot t) + 15\cdot cos(2\cdot t) + 10}{2^5} - \\
&- \left(\frac{cos(4\cdot t) + 4\cdot cos(2\cdot t) + 3}{2^3}\right) = \\
&= \frac{cos(6\cdot t) + 6\cdot cos(4\cdot t) + 15\cdot cos(2\cdot t) + 10 - 4\cdot cos(4\cdot t) - 16\cdot cos(2\cdot t) - 12}{2^5} = \\
&= \frac{cos(6\cdot t) + 2\cdot cos(4\cdot t) - cos(2\cdot t) - 2}{2^5} = \\
&= \frac{cos(6\cdot t) + 2\cdot cos(4\cdot t) - cos(2\cdot t) - 2}{32}
\end{aligned}
$$

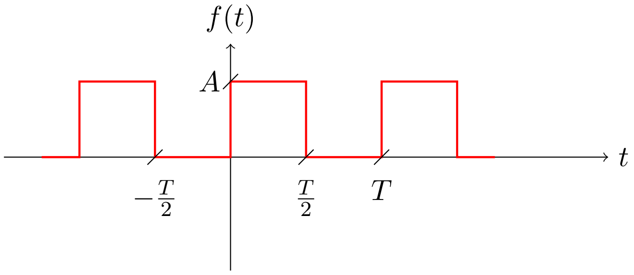
To sum up:

$$f_2(t) = cos^6(t) - cos^4(t) = \frac{cos(6\cdot t) + 2\cdot cos(4\cdot t) - cos(2\cdot t) - 2}{32}$$

1.1 Basic signal metrics

1.1.1 Mean value of a signal

Task 1. Calculate the mean value of the following periodic signal $f(t)$:



Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.4}$$

The mean value for periodic signals is defined by:

$$\bar{f} = \frac{1}{T} \int_T f(t) \cdot dt \tag{1.5}$$

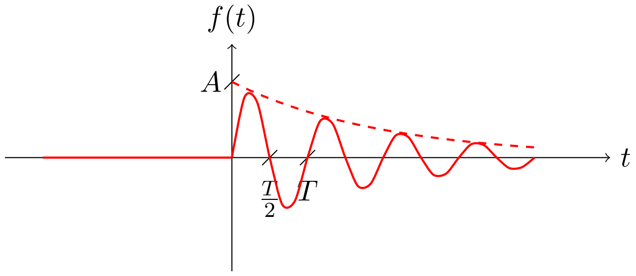
Compute mean value for period $k = 0$:

$$\begin{aligned} \bar{f} &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{1}{T} \left(A \cdot t|_0^{\frac{T}{2}} \right) = \\ &= \frac{A}{T} \cdot t|_0^{\frac{T}{2}} = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A}{2} \end{aligned} \tag{1.6}$$

The mean value equals to $\frac{A}{2}$.

Task 2.

Calculate the mean value of the signal $f(t) = \mathbf{1}(t) \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:



The mean value for non-periodic signals is defined by:

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \cdot dt \tag{1.7}$$

Compute mean value:

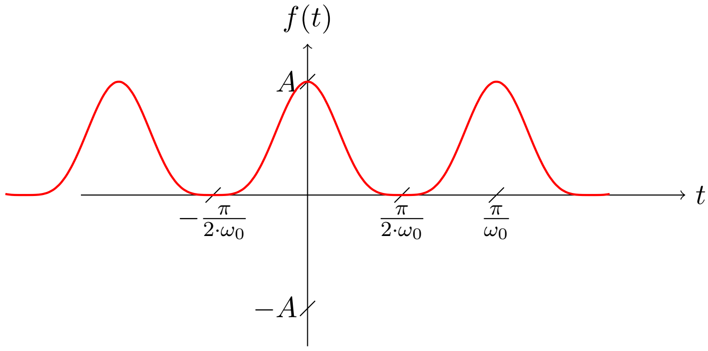
$$\begin{aligned}
\bar{f} &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \cdot dt = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \mathbf{1}(t) \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} 0 \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{0}^{\frac{\tau}{2}} 1 \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} 0 \cdot dt + \int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(0 + \int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\int_{0}^{\frac{\tau}{2}} e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \left\{ \begin{matrix} u = sin(\frac{2\pi}{T} \cdot t) & dv = e^{-a \cdot t} \cdot dt \\ du = \frac{2\pi}{T} \cdot cos(\frac{2\pi}{T} \cdot t) \cdot dt & v = -\frac{1}{a} \cdot e^{-a \cdot t} \end{matrix} \right\} = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(-\frac{1}{a} \cdot e^{-a \cdot t} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \Big|_{0}^{\frac{\tau}{2}} - \int_{0}^{\frac{\tau}{2}} -\frac{1}{a} \cdot e^{-a \cdot t} \cdot \frac{2\pi}{T} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\left(-\frac{1}{a} \cdot e^{-a \cdot \frac{\tau}{2}} \cdot sin\left(\frac{2\pi}{T} \cdot \frac{\tau}{2}\right) + \frac{1}{a} \cdot e^{-a \cdot 0} \cdot sin\left(\frac{2\pi}{T} \cdot 0\right) \right) + \right. \\
&\left. + \frac{1}{a} \cdot \frac{2\pi}{T} \cdot \int_{0}^{\frac{\tau}{2}} \cdot e^{-a \cdot t} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \left\{ \begin{matrix} u = cos(\frac{2\pi}{T} \cdot t) & dv = e^{-a \cdot t} \cdot dt \\ du = -\frac{2\pi}{T} \cdot sin(\frac{2\pi}{T} \cdot t) \cdot dt & v = -\frac{1}{a} \cdot e^{-a \cdot t} \end{matrix} \right\} = \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \left(\left(-\frac{1}{a} \cdot e^{-a \cdot \frac{\tau}{2}} \cdot sin\left(\frac{2\pi}{T} \cdot \frac{\tau}{2}\right) + \frac{1}{a} \cdot e^{-a \cdot 0} \cdot sin\left(\frac{2\pi}{T} \cdot 0\right) \right) + \right. \\
&\left. + \frac{1}{a} \cdot \frac{2\pi}{T} \cdot \left(-\frac{1}{a} \cdot e^{-a \cdot t} \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \Big|_{0}^{\frac{\tau}{2}} - \int_{0}^{\frac{\tau}{2}} -\frac{1}{a} \cdot e^{-a \cdot t} \cdot \frac{2\pi}{T} \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\tau\to\infty}\frac{1}{\tau}\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot 1\cdot 0\right)+\right.\\
&+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot e^{-a\cdot 0}\cdot cos\left(\frac{2\pi}{T}\cdot 0\right)\right)+\right.\\
&\left.\left.+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)\right)=\\
&= \lim_{\tau\to\infty}\frac{1}{\tau}\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+0\right)+\right.\\
&+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\left(\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a}\cdot 1\cdot 1\right)+\right.\\
&\left.\left.+\frac{1}{a}\cdot\frac{2\pi}{T}\cdot\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)\right)=\\
&= \lim_{\tau\to\infty}\frac{1}{\tau}\left(-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\right.\\
&-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}+\\
&\left.+\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\cdot\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\left\{\begin{aligned}&-\tfrac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)+\\&-\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)+\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}+\\&+\tfrac{1}{a^2}\cdot\tfrac{T^2}{4\pi^2}\cdot\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt=\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt\end{aligned}\right\}=\\
&=\left\{\begin{aligned}&-\tfrac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)-\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)+\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}=\\&=\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt-\tfrac{1}{a^2}\cdot\tfrac{T^2}{4\pi^2}\cdot\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt\end{aligned}\right\}=\\
&=\left\{\begin{aligned}&-\tfrac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)-\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\tfrac{2\pi}{T}\cdot\tfrac{\tau}{2}\right)+\tfrac{1}{a^2}\cdot\tfrac{T}{2\pi}=\\&=\left(1-\tfrac{1}{a^2}\cdot\tfrac{T^2}{4\pi^2}\right)\cdot\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt\end{aligned}\right\}=\\
&=\left\{\begin{aligned}&\tfrac{-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}}{\left(1-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\right)}=\\&=\textstyle\int_0^{\frac{\tau}{2}}e^{-a\cdot t}\cdot sin\left(\tfrac{2\pi}{T}\cdot t\right)\cdot dt\end{aligned}\right\}=\\
&=\lim_{\tau\to\infty}\frac{1}{\tau}\left(\frac{-\frac{1}{a}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot sin\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)-\frac{1}{a^2}\cdot\frac{T}{2\pi}\cdot e^{-a\cdot\frac{\tau}{2}}\cdot cos\left(\frac{2\pi}{T}\cdot\frac{\tau}{2}\right)+\frac{1}{a^2}\cdot\frac{T}{2\pi}}{\left(1-\frac{1}{a^2}\cdot\frac{T^2}{4\pi^2}\right)}\right)=\\
&=0
\end{aligned}
$$

The mean value equals to 0.

Task 3.

Calculate the mean value of the following periodic signal $f(t) = A \cdot cos^4(\omega_0 \cdot t)$:



The mean value for periodic signals is defined by:

$$\bar{f} = \frac{1}{T}\int_T f(t)\cdot dt \tag{1.8}$$

The period of the given signal has to be identified. In our case: $T = \frac{\pi}{\omega_0}$.
Compute mean value for period $t \in \left(-\frac{\pi}{2\cdot\omega_0}; \frac{\pi}{2\cdot\omega_0}\right)$:

$$\begin{aligned}
\bar{f} &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{\frac{\pi}{\omega_0}}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot cos\left(\omega_0\cdot t\right)^4\cdot dt = \\
&= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot cos\left(\omega_0\cdot t\right)^4\cdot dt = \\
&= \left\{cos(x) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} A\cdot\left(\frac{e^{\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot\omega_0\cdot t}}{2}\right)^4\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\left(\frac{e^{\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot\omega_0\cdot t}}{2}\right)^2\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{\left(e^{\jmath\cdot\omega_0\cdot t}\right)^2+2\cdot e^{\jmath\cdot\omega_0\cdot t}\cdot e^{-\jmath\cdot\omega_0\cdot t}+\left(e^{-\jmath\cdot\omega_0\cdot t}\right)^2}{2^2}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2\cdot e^{\jmath\cdot\omega_0\cdot t-\jmath\cdot\omega_0\cdot t}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2\cdot e^{0}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+2+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}}{4}\right)^2\cdot dt = \\
&= \frac{\omega_0}{\pi}\cdot A\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\left(\frac{e^{\jmath\cdot 2\cdot\omega_0\cdot t}+e^{-\jmath\cdot 2\cdot\omega_0\cdot t}+2}{4}\right)^2\cdot dt =
\end{aligned}$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)\cdot 2 + 2^2}{4^2} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{\left(e^{\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + \left(e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\right)^2 + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot 2 + 2\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\cdot 2 + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 2\cdot 2\cdot\omega_0\cdot t} + 2\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t - \jmath\cdot 2\cdot\omega_0\cdot t} + e^{-\jmath\cdot 2\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 4\cdot\omega_0\cdot t} + 2\cdot e^{0} + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot A \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \frac{e^{\jmath\cdot 4\cdot\omega_0\cdot t} + 2 + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 4}{16} \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} \left(e^{\jmath\cdot 4\cdot\omega_0\cdot t} + e^{-\jmath\cdot 4\cdot\omega_0\cdot t} + 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t} + 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t} + 6\right) \cdot dt =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{\jmath\cdot 4\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{-\jmath\cdot 4\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\cdot dt + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\cdot dt + \right.$$

$$\left. + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 6\cdot dt\right) =$$

$$= \left\{\begin{matrix} z_1 = \jmath\cdot 4\cdot\omega_0\cdot t & z_2 = -\jmath\cdot 4\cdot\omega_0\cdot t & z_3 = \jmath\cdot 2\cdot\omega_0\cdot t & z_4 = -\jmath\cdot 2\cdot\omega_0\cdot t \\ dz_1 = \jmath\cdot 4\cdot\omega_0\cdot dt & dz_2 = -\jmath\cdot 4\cdot\omega_0\cdot dt & dz_3 = \jmath\cdot 2\cdot\omega_0\cdot dt & dz_4 = -\jmath\cdot 2\cdot\omega_0\cdot dt = \\ dt = \frac{dz_1}{\jmath\cdot 4\cdot\omega_0} & dt = \frac{dz_2}{-\jmath\cdot 4\cdot\omega_0} & dt = \frac{dz_3}{\jmath\cdot 2\cdot\omega_0} & dt = \frac{dz_4}{-\jmath\cdot 2\cdot\omega_0} \end{matrix}\right\} =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot 4\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot 4\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{z_3}\cdot\frac{dz_3}{\jmath\cdot 2\cdot\omega_0} + \right.$$

$$\left. + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 4\cdot e^{z_4}\cdot\frac{dz_4}{-\jmath\cdot 2\cdot\omega_0} + \int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} 6\cdot dt\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_1}\cdot dz_1 + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_2}\cdot dz_2 + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_3}\cdot dz_3 + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} e^{z_4}\cdot dz_4 + 6\cdot\int_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} dt\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot e^{z_1}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot e^{z_2}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot e^{z_3}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot e^{z_4}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + 6\cdot t\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot e^{\jmath\cdot 4\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot e^{-\jmath\cdot 4\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot e^{\jmath\cdot 2\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + \right.$$

$$\left. + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot e^{-\jmath\cdot 2\cdot\omega_0\cdot t}\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}} + 6\cdot t\Big|_{-\frac{\pi}{2\cdot\omega_0}}^{\frac{\pi}{2\cdot\omega_0}}\right) =$$

$$= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath\cdot 4\cdot\omega_0}\cdot\left(e^{\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{-\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \frac{1}{-\jmath\cdot 4\cdot\omega_0}\cdot\left(e^{-\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{+\jmath\cdot 4\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \right.$$

$$+ \frac{4}{\jmath\cdot 2\cdot\omega_0}\cdot\left(e^{\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{-\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) + \frac{4}{-\jmath\cdot 2\cdot\omega_0}\cdot\left(e^{-\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}} - e^{\jmath\cdot 2\cdot\omega_0\cdot\frac{\pi}{2\cdot\omega_0}}\right) +$$

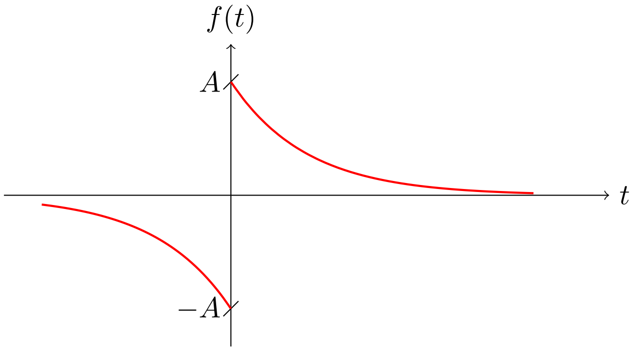
$$\left. + 6\cdot\left(\frac{\pi}{2\cdot\omega_0} + \frac{\pi}{2\cdot\omega_0}\right)\right) =$$

$$
\begin{aligned}
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot \left(e^{\jmath \cdot 2 \cdot \pi} - e^{-\jmath \cdot 2 \cdot \pi} \right) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot \left(e^{-\jmath \cdot 2 \cdot \pi} - e^{\jmath \cdot 2 \cdot \pi} \right) + \right. \\
&\left. + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot \left(e^{\jmath \cdot \pi} - e^{-\jmath \cdot \pi} \right) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot \left(e^{-\jmath \cdot \pi} - e^{\jmath \cdot \pi} \right) + 6 \cdot \left(\frac{2 \cdot \pi}{2 \cdot \omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot (1 - 1) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot (1 - 1) + \right. \\
&\left. + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot (-1 - (-1)) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot (-1 - (-1)) + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(\frac{1}{\jmath \cdot 4 \cdot \omega_0} \cdot (0) + \frac{1}{-\jmath \cdot 4 \cdot \omega_0} \cdot (0) + \frac{4}{\jmath \cdot 2 \cdot \omega_0} \cdot (0) + \frac{4}{-\jmath \cdot 2 \cdot \omega_0} \cdot (0) + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot \left(0 + 0 + 0 + 0 + 6 \cdot \left(\frac{\pi}{\omega_0} \right) \right) = \\
&= \frac{\omega_0}{\pi} \cdot \frac{A}{16} \cdot 6 \cdot \left(\frac{\pi}{\omega_0} \right) = \\
&= \frac{A}{16} \cdot 6 = \\
&= \frac{A}{8} \cdot 3 = \\
&= \frac{3}{8} \cdot A
\end{aligned}
$$

The mean value equals to $\frac{3}{8} \cdot A$.

1.1.2 Energy of a signal

Task 4. Compute energy of $f(t)$ signal given below:



$$f(t) = \begin{cases} -A \cdot e^{a \cdot t} & dla \quad t \in (-\infty; 0) \\ A \cdot e^{-a \cdot t} & dla \quad t \in (0; \infty) \end{cases} \tag{1.9}$$

Total energy of a non-periodic signal (or in short - signal energy) is defined by:

$$E = \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2 \cdot dt \tag{1.10}$$

For the given signal we get

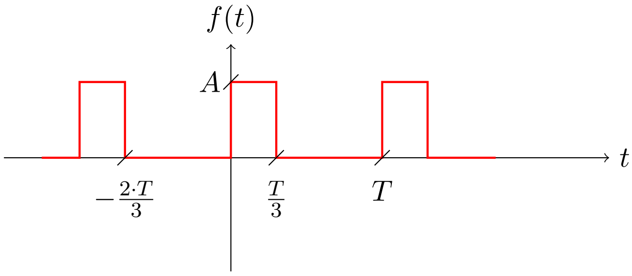
$$\begin{aligned}
E &= \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2 \cdot dt = \\
&= \lim_{\tau \to \infty} \left(\int_{-\frac{\tau}{2}}^{0} \left| -A \cdot e^{a \cdot t} \right|^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} \left| A \cdot e^{-a \cdot t} \right|^2 \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \left(\int_{-\frac{\tau}{2}}^{0} \left(-A \cdot e^{a \cdot t} \right)^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} \left(A \cdot e^{-a \cdot t} \right)^2 \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \left(\int_{-\frac{\tau}{2}}^{0} (-A)^2 \cdot \left(e^{a \cdot t} \right)^2 \cdot dt + \int_{0}^{\frac{\tau}{2}} (A)^2 \cdot \left(e^{-a \cdot t} \right)^2 \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \left(\int_{-\frac{\tau}{2}}^{0} A^2 \cdot e^{2 \cdot a \cdot t} \cdot dt + \int_{0}^{\frac{\tau}{2}} A^2 \cdot e^{-2 \cdot a \cdot t} \cdot dt \right) = \\
&= \lim_{\tau \to \infty} \left(A^2 \cdot \int_{-\frac{\tau}{2}}^{0} e^{2 \cdot a \cdot t} \cdot dt + A^2 \cdot \int_{0}^{\frac{\tau}{2}} e^{-2 \cdot a \cdot t} \cdot dt \right) = \\
&= \lim_{\tau \to \infty} A^2 \cdot \left(\int_{-\frac{\tau}{2}}^{0} e^{2 \cdot a \cdot t} \cdot dt + \int_{0}^{\frac{\tau}{2}} e^{-2 \cdot a \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z = 2 \cdot a \cdot t & w = -2 \cdot a \cdot t \\ dz = 2 \cdot a \cdot dt & dw = -2 \cdot a \cdot dt \\ dt = \frac{dz}{2 \cdot a} & dt = \frac{dw}{-2 \cdot a} \end{array} \right\} = \\
&= \lim_{\tau \to \infty} A^2 \cdot \left(\int_{-\frac{\tau}{2}}^{0} e^{z} \cdot \frac{dz}{2 \cdot a} + \int_{0}^{\frac{\tau}{2}} e^{w} \cdot \frac{dw}{-2 \cdot a} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(\int_{-\frac{\tau}{2}}^{0} e^{z} \cdot dz - \int_{0}^{\frac{\tau}{2}} e^{w} \cdot dw \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(e^{z}\big|_{-\frac{\tau}{2}}^{0} - e^{w}\big|_{0}^{\frac{\tau}{2}} \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(e^{2\cdot a\cdot t}\Big|_{-\frac{\tau}{2}}^{0} - e^{-2\cdot a\cdot dt}\Big|_{0}^{\frac{\tau}{2}} \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(\left(e^{2\cdot a\cdot 0} - e^{-2\cdot a\cdot \frac{\tau}{2}} \right) - \left(e^{-2\cdot a\cdot \frac{\tau}{2}} - e^{-2\cdot a\cdot 0} \right) \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(\left(e^{0} - e^{-a\cdot\tau} \right) - \left(e^{-a\cdot\tau} - e^{0} \right) \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(1 - e^{-a\cdot\tau} - e^{-a\cdot\tau} + 1 \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot \left(2 - 2\cdot e^{-a\cdot\tau} \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{2\cdot a} \cdot 2 \cdot \left(1 - e^{-a\cdot\tau} \right) = \\
&= \lim_{\tau\to\infty} \frac{A^2}{a} \cdot \left(1 - e^{-a\cdot\tau} \right) = \\
&= \frac{A^2}{a}
\end{aligned}
$$

Energy equals to $\frac{A^2}{a}$.

1.1.3 Power and effective value of a signal

Task 5. Compute the average power for the following periodic signal $f(t)$:



Signal $f(t)$ can de described as:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{3} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{3} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{1.11}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.12}$$

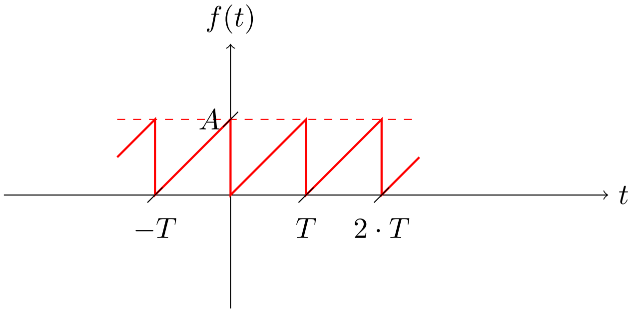
Compute average power for period $k = 0$

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} |A|^2 \cdot dt + \int_{\frac{T}{3}}^{T} |0|^2 \cdot dt\right) = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{3}} A^2 \cdot dt + \int_{\frac{T}{3}}^{T} 0 \cdot dt\right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{3}} dt + 0\right) = \\ &= \frac{A^2}{T} \cdot t|_0^{\frac{T}{3}} = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{3} - 0\right) = \\ &= \frac{A^2}{T} \cdot \frac{T}{3} = \\ &= \frac{A^2}{3} \end{aligned}$$

Average power equals to $\frac{A^2}{3}$.

Task 6.

Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.13}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and (T,A). So, in order to derive a and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in C$$

The average power for periodic signals is defined by:

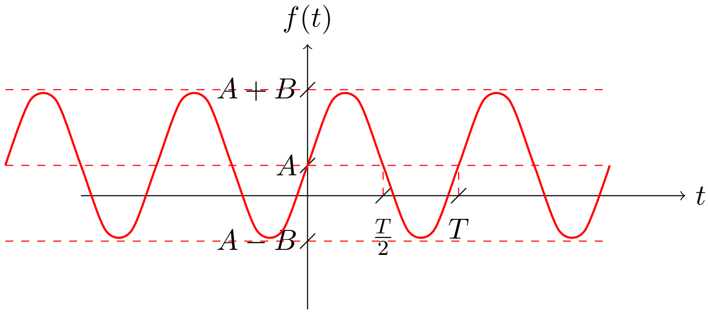
$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.14}$$

In our case we get:

$$
\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| \frac{A}{T} \cdot t \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(\frac{A}{T} \cdot t \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \frac{A^2}{T^2} \cdot t^2 \cdot dt = \\
&= \frac{1}{T} \cdot \frac{A^2}{T^2} \cdot \int_0^T t^2 \cdot dt = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^T \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - \frac{1}{3} \cdot 0^3 \right) = \\
&= \frac{A^2}{T^3} \cdot \left(\frac{1}{3} \cdot T^3 - 0 \right) = \\
&= \frac{A^2}{T^3} \cdot \frac{1}{3} \cdot T^3 = \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}$.

Task 7. Compute the average power for the following periodic signal $f(t) = A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right)$ given below:



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.15}$$

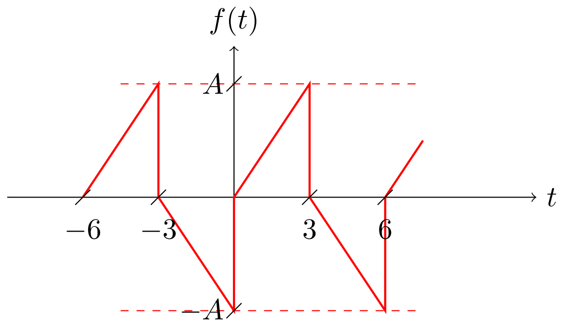
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left| A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A + B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left(A^2 + 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) + B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \right) \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^T A^2 \cdot dt + \int_0^T 2 \cdot A \cdot B \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_0^T B^2 \cdot sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \frac{A^2}{T} \cdot \int_0^T dt + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{B^2}{T} \cdot \int_0^T sin^2\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{l} z = \frac{2\pi}{T} \cdot t \\ dz = \frac{2\pi}{T} \cdot dt \quad dt = \frac{dz}{\frac{2\pi}{T}} = \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{A^2}{T} \cdot t|_0^T + \frac{2 \cdot A \cdot B}{T} \cdot \int_0^T sin\,(z) \cdot \frac{T}{2\pi} \cdot dz + \frac{B^2}{T} \cdot \int_0^T \frac{1}{2} \cdot \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot (T - 0) + \frac{2 \cdot A \cdot B}{T} \cdot \frac{T}{2\pi} \cdot \int_0^T sin\,(z) \cdot dz + \frac{B^2}{T} \cdot \frac{1}{2} \cdot \int_0^T \left(1 - cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) \right) \cdot dt = \\
&= \frac{A^2}{T} \cdot T + \frac{A \cdot B}{\pi} \cdot \left(-cos\,(z)|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left(\int_0^T 1 \cdot dt - \int_0^T cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) = \\
&= \left\{ \begin{array}{l} w = 2 \cdot \frac{2\pi}{T} \cdot t \\ dw = 2 \cdot \frac{2\pi}{T} \cdot dt \quad dt = \frac{dw}{\frac{4\pi}{T}} = \frac{T}{4\pi} \cdot dw \end{array} \right\} = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot t\right) \Big|_0^T \right) + \frac{B^2}{2 \cdot T} \cdot \left(t|_0^T - \int_0^T cos\,(w) \cdot \frac{T}{4\pi} \cdot dw \right) = \\
&= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(\frac{2\pi}{T} \cdot T\right) + cos\left(\frac{2\pi}{T} \cdot 0\right) \right) + \frac{B^2}{2 \cdot T} \cdot \left((T - 0) - \frac{T}{4\pi} \cdot \int_0^T cos\,(w) \cdot dw \right) =
\end{aligned}$$

$$= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-cos\left(2\pi\right) + cos\left(0\right)\right) + \frac{B^2}{2 \cdot T} \cdot \left(T - \frac{T}{4\pi} \cdot -sin\left(w\right)\Big|_0^T\right) =$$
$$= A^2 + \frac{A \cdot B}{\pi} \cdot \left(-1 + 1\right) + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_0^T\right) =$$
$$= A^2 + \frac{A \cdot B}{\pi} \cdot 0 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(2 \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(2 \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(sin\left(4\pi\right) - sin\left(0\right)\right)\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T + \frac{T}{4\pi} \cdot \left(0 - 0\right)\right) =$$
$$= A^2 + \frac{B^2}{2 \cdot T} \cdot \left(T\right) =$$
$$= A^2 + \frac{B^2}{2}$$

The average power equals to $A^2 + \frac{B^2}{2}$.

Task 8. Calculate the average power and the effective value (RMS) for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{1.16}$$

In the first interval of the first period (i.e. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (i.e. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.17}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{6} \cdot \left(\int_0^3 \left| \frac{A}{3} \cdot t \right|^2 \cdot dt + \int_3^6 \left| -\frac{A}{3} \cdot t + A \right|^2 \cdot dt \right) = \\
&= \frac{1}{6} \cdot \int_0^3 \left(\frac{A}{3} \cdot t \right)^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(-\frac{A}{3} \cdot t + A \right)^2 \cdot dt = \\
&= \frac{1}{6} \cdot \int_0^3 \frac{A^2}{9} \cdot t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \left(\left(-\frac{A}{3} \cdot t \right)^2 - 2 \cdot \frac{A}{3} \cdot t \cdot A + A^2 \right) \cdot dt = \\
&= \frac{A^2}{54} \cdot \int_0^3 t^2 \cdot dt + \frac{1}{6} \cdot \int_3^6 \frac{A^2}{9} \cdot t^2 \cdot dt - \frac{1}{6} \cdot \int_3^6 \frac{2 \cdot A^2}{3} \cdot t \cdot dt + \frac{1}{6} \cdot \int_3^6 A^2 \cdot dt = \\
&= \frac{A^2}{54} \cdot \left. \frac{t^3}{3} \right|_0^3 + \frac{A^2}{54} \cdot \int_3^6 t^2 \cdot dt - \frac{2 \cdot A^2}{18} \cdot \int_3^6 t^2 \cdot dt + \frac{A^2}{6} \cdot \int_3^6 dt =
\end{aligned}$$

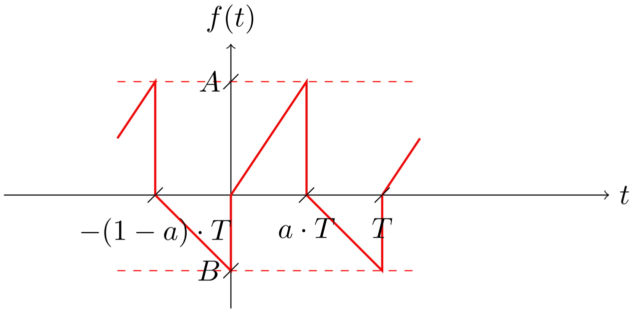
$$
\begin{aligned}
&= \frac{A^2}{162} \cdot \left(3^3 - 0^3\right) + \frac{A^2}{54} \cdot \left.\frac{t^3}{3}\right|_3^6 - \frac{2 \cdot A^2}{18} \cdot \left.\frac{t^2}{2}\right|_3^6 + \frac{A^2}{6} \cdot t|_3^6 = \\
&= \frac{A^2}{162} \cdot 27 + \frac{A^2}{162} \cdot \left(6^3 - 3^3\right) - \frac{2 \cdot A^2}{36} \cdot \left(6^2 - 3^2\right) + \frac{A^2}{6} \cdot (6 - 3) = \\
&= \frac{A^2}{6} + \frac{A^2}{162} \cdot 189 - \frac{2 \cdot A^2}{36} \cdot 27 + \frac{A^2}{6} \cdot 3 = \\
&= \frac{A^2}{6} + \frac{7 \cdot A^2}{6} - \frac{9 \cdot A^2}{6} + \frac{3 \cdot A^2}{6} = \\
&= \frac{2 \cdot A^2}{6} = \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power equals to $\frac{A^2}{3}$.
The effective value (RMS) is defined by:

$$
RMS = \sqrt{P} \tag{1.18}
$$

Therefore, effective value (RMS) equals to $\frac{A}{\sqrt{3}}$.

Task 9. Calculate the average power for the periodic signal $f(t)$ given below:



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{1.19}$$

In the first interval of the first period (i.e. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T - a \cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T - a \cdot T} \cdot a \cdot T = b \\ -\frac{B}{T - a \cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1 - a} \cdot a = b \\ -\frac{B}{T - a \cdot T} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T - a \cdot T} \cdot t + \frac{B}{1 - a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a \cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{T - a \cdot T} \cdot t + \frac{B}{1 - a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a \cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{T - a \cdot T} \cdot (t - k \cdot T) + \frac{B}{1 - a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.20}$$

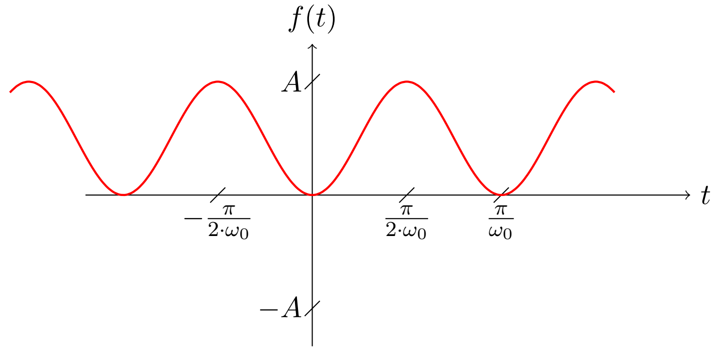
In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{a \cdot T} \left| \frac{A}{a \cdot T} \cdot t \right|^2 \cdot dt + \int_{a \cdot T}^{T} \left| \frac{B}{T - a \cdot T} \cdot t - \frac{B}{1 - a} \cdot a \right|^2 \cdot dt \right) = \\
&= \frac{1}{T} \cdot \int_0^{a \cdot T} \left(\frac{A}{a \cdot T} \cdot t \right)^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left(\frac{B}{T - a \cdot T} \cdot t - \frac{B}{1 - a} \cdot a \right)^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^{a \cdot T} \frac{A^2}{a^2 \cdot T^2} \cdot t^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left(\left(\frac{B}{T - a \cdot T} \cdot t \right)^2 - 2 \cdot \frac{B}{T - a \cdot T} \cdot t \cdot \frac{B}{1 - a} \cdot a + \left(\frac{B}{1 - a} \cdot a \right)^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \int_0^{a \cdot T} t^2 \cdot dt + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \left(\frac{B^2}{T^2 \cdot (1 - a)^2} \cdot t^2 - 2 \cdot \frac{B^2}{T \cdot (1 - a)^2} \cdot t \cdot a + \frac{B^2}{(1 - a)^2} \cdot a^2 \right) \cdot dt = \\
&= \frac{A^2}{a^2 \cdot T^3} \cdot \left(\frac{1}{3} \cdot t^3 \Big|_0^{a \cdot T} \right) + \frac{1}{T} \cdot \int_{a \cdot T}^{T} \frac{B^2}{T^2 \cdot (1 - a)^2} \cdot t^2 \cdot dt +
\end{aligned}$$

$$-\frac{1}{T}\cdot\int_{a\cdot T}^{T}2\cdot\frac{B^2}{T\cdot(1-a)^2}\cdot t\cdot a\cdot dt+\frac{1}{T}\cdot\int_{a\cdot T}^{T}\frac{B^2}{(1-a)^2}\cdot a^2\cdot dt=$$
$$=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot t^3\Big|_0^{a\cdot T}\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\int_{a\cdot T}^{T}t^2\cdot dt+$$
$$-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\int_{a\cdot T}^{T}t\cdot dt+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\int_{a\cdot T}^{T}dt=$$
$$=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot(a\cdot T)^3-\frac{1}{3}\cdot 0^3\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot t^3\Big|_{a\cdot T}^{T}\right)+$$
$$-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot t^2\Big|_{a\cdot T}^{T}\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot\left(t\Big|_{a\cdot T}^{T}\right)=$$
$$=\frac{A^2}{a^2\cdot T^3}\cdot\left(\frac{1}{3}\cdot a^3\cdot T^3-0\right)+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot(a\cdot T)^3\right)+$$
$$-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot(a\cdot T)^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(T-a\cdot T)=$$
$$=\frac{A^2}{a^2\cdot T^3}\cdot\frac{1}{3}\cdot a^3\cdot T^3+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(\frac{1}{3}\cdot T^3-\frac{1}{3}\cdot a^3\cdot T^3\right)+$$
$$-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(\frac{1}{2}\cdot T^2-\frac{1}{2}\cdot a^2\cdot T^2\right)+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{T^3\cdot(1-a)^2}\cdot\left(1-a^3\right)\cdot\frac{1}{3}\cdot T^3+$$
$$-\frac{2\cdot B^2}{T^2\cdot(1-a)^2}\cdot a\cdot\left(1-a^2\right)\cdot\frac{1}{2}\cdot T^2+\frac{B^2}{T\cdot(1-a)^2}\cdot a^2\cdot(1-a)\cdot T=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{(1-a)^2}\cdot(1-a)\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}+$$
$$-\frac{2\cdot B^2}{(1-a)^2}\cdot a\cdot(1-a)\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(1+a+a^2\right)\cdot\frac{1}{3}-\frac{2\cdot B^2}{1-a}\cdot a\cdot(1+a)\cdot\frac{1}{2}+\frac{B^2}{1-a}\cdot a^2=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{1}{3}-2\cdot a\cdot(1+a)\cdot\frac{1}{2}+a^2\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\left(\left(1+a+a^2\right)\cdot\frac{2}{6}-2\cdot a\cdot(1+a)\cdot\frac{3}{6}+a^2\cdot\frac{6}{6}\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(\left(1+a+a^2\right)\cdot 2-2\cdot a\cdot(1+a)\cdot 3+a^2\cdot 6\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2+2\cdot a+2\cdot a^2-6\cdot a-6\cdot a^2+6\cdot a^2\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{6}\cdot\left(2-4\cdot a+2\cdot a^2\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot\left(1-2\cdot a+a^2\right)=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{1-a}\cdot\frac{1}{3}\cdot(1-a)^2=$$
$$=\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)$$

The average power equals to $\frac{A^2}{3}\cdot a+\frac{B^2}{3}\cdot(1-a)$.

Task 10. Calculate the average power for the periodic signal $f(t) = A \cdot sin^2(\omega_0 \cdot t)$ given below.



The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.21}$$

In our case we get:

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T \left|A \cdot sin^2(\omega_0 \cdot t)\right|^2 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot sin^4(\omega_0 \cdot t) \cdot dt = \\
&= \frac{1}{T} \cdot \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} \right)^4 \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{\left(e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \cdot dt = \\
&= \left\{ \begin{array}{lccccccccc}
n = 0: & & & & & 1 & & & & \\
n = 1: & & & & 1 & & 1 & & & \\
n = 2: & & & 1 & & 2 & & 1 & & \\
n = 3: & & 1 & & 3 & & 3 & & 1 & \\
n = 4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \right\} = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^4 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^0 + 4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^3 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^1 + 6 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^2 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^2}{(2 \cdot \jmath)^4} + \right. \\
&\left. + \frac{4 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^1 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^3 + 1 \cdot \left(e^{\jmath \cdot \omega_0 \cdot t}\right)^0 \cdot \left(-e^{-\jmath \cdot \omega_0 \cdot t}\right)^4}{(2 \cdot \jmath)^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-\jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} + \right. \\
&\left. + \frac{-4 \cdot e^{\jmath \cdot \omega_0 \cdot t} \cdot e^{-3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t} \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}}{2^4 \cdot \jmath^4} \right) \cdot dt = \\
&= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot t - 0 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{3 \cdot \jmath \cdot \omega_0 \cdot t - \jmath \cdot \omega_0 \cdot t} + 6 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot t - 2 \cdot \jmath \cdot \omega_0 \cdot t} - 4 \cdot e^{\jmath \cdot \omega_0 \cdot t - 3 \cdot \jmath \cdot \omega_0 \cdot t} + e^{0 \cdot \jmath \cdot \omega_0 \cdot t - 4 \cdot \jmath \cdot \omega_0 \cdot t}}{16 \cdot 1} \cdot dt =
\end{aligned}$$

$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} + 6 \cdot e^{0\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6 \cdot e^0}{16} \cdot dt =$$
$$= \frac{1}{T} \cdot \int_0^T A^2 \cdot \frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6}{16} \cdot dt =$$
$$= \frac{A^2}{16 \cdot T} \cdot \int_0^T \left(e^{4\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} - 4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + 6\right) dt =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\int_0^T e^{4\cdot\jmath\cdot\omega_0\cdot t} \cdot dt + \int_0^T e^{-4\cdot\jmath\cdot\omega_0\cdot t} \cdot dt - 4 \cdot \int_0^T e^{2\cdot\jmath\cdot\omega_0\cdot t} \cdot dt - 4 \cdot \int_0^T e^{-2\cdot\jmath\cdot\omega_0\cdot t} \cdot dt + 6 \cdot \int_0^T dt\right) =$$
$$= \left\{\begin{matrix} z_1 = 4 \cdot \jmath \cdot \omega_0 \cdot t & z_2 = -4 \cdot \jmath \cdot \omega_0 \cdot t & z_3 = 2 \cdot \jmath \cdot \omega_0 \cdot t & z_4 = -2 \cdot \jmath \cdot \omega_0 \cdot t = \\ dz_1 = 4 \cdot \jmath \cdot \omega_0 \cdot dt & dz_2 = -4 \cdot \jmath \cdot \omega_0 \cdot dt & dz_3 = 2 \cdot \jmath \cdot \omega_0 \cdot dt & dz_4 = -2 \cdot \jmath \cdot \omega_0 \cdot dt = \\ dt = \frac{1}{4\cdot\jmath\cdot\omega_0} \cdot dz_1 & dt = \frac{1}{-4\cdot\jmath\cdot\omega_0} \cdot dz_2 & dt = \frac{1}{2\cdot\jmath\cdot\omega_0} \cdot dz_3 & dt = \frac{1}{-2\cdot\jmath\cdot\omega_0} \cdot dz_4 \end{matrix}\right\} =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\int_0^T e^{z_1} \cdot \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot dz_1 + \int_0^T e^{z_2} \cdot \frac{1}{-4 \cdot \jmath \cdot \omega_0} \cdot dz_2 + \right.$$
$$\left. - 4 \cdot \int_0^T e^{z_3} \cdot \frac{1}{2 \cdot \jmath \cdot \omega_0} \cdot dz_3 - 4 \cdot \int_0^T e^{z_4} \cdot \frac{1}{-2 \cdot \jmath \cdot \omega_0} \cdot dz_4 + 6 \cdot \int_0^T dt\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_1} \cdot dz_1 + \frac{1}{-4 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_2} \cdot dz_2 + \right.$$
$$\left. -4 \cdot \frac{1}{2 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_3} \cdot dz_3 - 4 \cdot \frac{1}{-2 \cdot \jmath \cdot \omega_0} \cdot \int_0^T e^{z_4} \cdot dz_4 + 6 \cdot \int_0^T dt\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{z_1}\Big|_0^T - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{z_2}\Big|_0^T - \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{z_3}\Big|_0^T + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{z_4}\Big|_0^T + \right.$$
$$\left. +6 \cdot t\Big|_0^T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T - \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t}\Big|_0^T + \right.$$
$$\left. +6 \cdot t\Big|_0^T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^{4\cdot\jmath\cdot\omega_0\cdot 0}\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^{-4\cdot\jmath\cdot\omega_0\cdot 0}\right) + \right.$$
$$\left. -\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^{2\cdot\jmath\cdot\omega_0\cdot 0}\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^{-2\cdot\jmath\cdot\omega_0\cdot 0}\right) + 6 \cdot (T - 0)\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - e^0\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - e^0\right) + \right.$$
$$\left. -\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - e^0\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - e^0\right) + 6 \cdot T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) - \frac{1}{4 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-4\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \right.$$
$$\left. -\frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + \frac{4}{2 \cdot \jmath \cdot \omega_0} \cdot \left(e^{-2\cdot\jmath\cdot\omega_0\cdot T} - 1\right) + 6 \cdot T\right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{1}{4 \cdot \jmath \cdot \omega_0} - \frac{e^{-4\cdot\jmath\cdot\omega_0\cdot T}}{4 \cdot \jmath \cdot \omega_0} + \frac{1}{4 \cdot \jmath \cdot \omega_0} + \right.$$
$$\left. -\frac{4 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot T}}{2 \cdot \jmath \cdot \omega_0} + \frac{4}{2 \cdot \jmath \cdot \omega_0} + \frac{4 \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot T}}{2 \cdot \jmath \cdot \omega_0} - \frac{4}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T\right) =$$

$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{4 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + \frac{4 \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T \right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{4 \cdot \jmath \cdot \omega_0} - \frac{4 \cdot e^{2 \cdot \jmath \cdot \omega_0 \cdot T} - 4 \cdot e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath \cdot \omega_0} + 6 \cdot T \right) =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot \frac{e^{4 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-4 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath} - \frac{4}{\omega_0} \cdot \frac{e^{2 \cdot \jmath \cdot \omega_0 \cdot T} - e^{-2 \cdot \jmath \cdot \omega_0 \cdot T}}{2 \cdot \jmath} + 6 \cdot T \right) =$$
$$= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \ cdotx}}{2 \cdot \jmath} \right\} =$$
$$= \frac{A^2}{16 \cdot T} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(4 \cdot \omega_0 \cdot T\right) - \frac{4}{\omega_0} \cdot sin\left(2 \cdot \omega_0 \cdot T\right) + 6 \cdot T \right) =$$
$$= \left\{ T = \frac{2\pi}{\omega_0} \right\} =$$
$$= \frac{A^2}{16 \cdot \frac{2\pi}{\omega_0}} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(4 \cdot \omega_0 \cdot \frac{2\pi}{\omega_0}\right) - \frac{4}{\omega_0} \cdot sin\left(2 \cdot \omega_0 \cdot \frac{2\pi}{\omega_0}\right) + 6 \cdot \frac{2\pi}{\omega_0} \right) =$$
$$= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot sin\left(8\pi\right) - \frac{4}{\omega_0} \cdot sin\left(4\pi\right) + 6 \cdot \frac{2\pi}{\omega_0} \right) =$$
$$= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \left(\frac{1}{2 \cdot \omega_0} \cdot 0 - \frac{4}{\omega_0} \cdot 0 + 6 \cdot \frac{2\pi}{\omega_0} \right) =$$
$$= \frac{A^2 \cdot \omega_0}{32\pi} \cdot \frac{12\pi}{\omega_0} =$$
$$= \frac{3 \cdot A^2}{8}$$

The average power equals to $\frac{3 \cdot A^2}{8}$.

Task 11. Oblicz moc sygnału $f(t) = A \cdot sin\left(k \cdot t\right) + B \cdot cos\left(n \cdot t\right)$.

Pierwszym krokiem jest ustalenie czy sygnał $f(t)$ jest sygnałem okresowym czy nie. Nasz sygnał jest sumą dwóch funkcji okresowych $f_1(t) = A \cdot sin\left(k \cdot t\right)$ i $f_2(t) = B \cdot cos\left(n \cdot t\right)$.

Suma funkcji okresowych jest funkcją okresową, wtedy i tylko wtedy gdy stosunek okresów funkcji składowych jest liczbą wymierną

$$\frac{T_1}{T_2} \in \mathcal{W}$$

W naszym przypadku

$$T_1 = \frac{2\pi}{k}$$
$$T_2 = \frac{2\pi}{n}$$
$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{n}} = \frac{n}{k}$$

W ogólności liczby n i k mogą być dowolnymi liczbami rzeczywistymi $n, k \in \mathcal{R}$. Załóżmy jednak iż ułamek $\frac{n}{k}$ jest pewną liczba wymierna $\frac{a}{b}$ gdzie $a, b \in \mathcal{Z}$ są liczbami całkowitymi.

$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{n}} = \frac{n}{k} = \frac{a}{b} \quad a, b \in \mathcal{Z}$$

W takim przypadku okres naszego sygnału jest Najmniejszą Wspólną Wielokrotnością okresów funkcji składowych. Stwórzmy więc tabelę z kolejnymi wielokrotnościami okresów funkcji $f_1(t)$ i $f_2(t)$. Zgodnie z przyjętym przez nas założeniem

Wielokrotność okresu	1	2	3	$\cdots$	a	$\cdots$	b	$\cdots$
T_1	$\frac{2\pi}{k}$	$2 \cdot \frac{2\pi}{k}$	$3 \cdot \frac{2\pi}{k}$	$\cdots$	$a \cdot \frac{2\pi}{k}$	$\cdots$	$\boxed{b \cdot \frac{2\pi}{k}}$	$\cdots$
T_2	$\frac{2\pi}{n}$	$2 \cdot \frac{2\pi}{n}$	$3 \cdot \frac{2\pi}{n}$	$\cdots$	$\boxed{a \cdot \frac{2\pi}{n}}$	$\cdots$	$b \cdot \frac{2\pi}{n}$	$\cdots$

$$\frac{T_1}{T_2} = \frac{a}{b} \Rightarrow b \cdot T_1 = a \cdot T_2$$

a więc a-ta wielokrotność okresu pierwszej funkcji jest równa b-tej wielokrotności okresu drugiej funkcji, a wiec jest ona poszukiwaną przez nas Najmniejszą Wspólną Wielokrotnością. Związku z tym okresem naszego sygnału jest $T = b \cdot T_1 = a \cdot T_2$. Aby obliczyć moc należy wybrać przedział o długości jednego okresu. Przedział może być dowolnie położony, przyjmijmy więc przedział $t \in (0; T)$

Moc sygnału okresowego wyznaczamy ze wzoru

$$P = \frac{1}{T} \int_0^T |f(t)|^2 \cdot dt \tag{1.22}$$

Podstawiamy do wzoru na moc wzór naszej funkcji

$$\begin{aligned} P &= \frac{1}{T}\int_0^T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T}\int_0^T |A\cdot sin\,(k\cdot t) + B\cdot cos\,(n\cdot t)|^2 \cdot dt = \end{aligned}$$

Ponieważ mamy doczynienia z sygnałem o wartościach rzeczywistych możemy pominąć obliczenie modułu.

$$\begin{aligned}
P &= \frac{1}{T}\int_0^T (A\cdot sin\,(k\cdot t) + B\cdot cos\,(n\cdot t))^2 \cdot dt = \\
&= \frac{1}{T}\int_0^T \Big((A\cdot sin\,(k\cdot t))^2 + 2\cdot A\cdot sin\,(k\cdot t)\cdot B\cdot cos\,(n\cdot t) + (B\cdot cos\,(n\cdot t))^2\Big)\cdot dt = \\
&= \frac{1}{T}\int_0^T \Big(A^2\cdot sin^2\,(k\cdot t) + 2\cdot A\cdot B\cdot sin\,(k\cdot t)\cdot cos\,(n\cdot t) + B^2\cdot cos^2\,(n\cdot t)\Big)\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^T A^2\cdot sin^2\,(k\cdot t)\cdot dt + \int_0^T 2\cdot A\cdot B\cdot sin\,(k\cdot t)\cdot cos\,(n\cdot t)\cdot dt + \int_0^T B^2\cdot cos^2\,(n\cdot t)\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A^2\cdot\int_0^T sin^2\,(k\cdot t)\cdot dt + 2\cdot A\cdot B\cdot\int_0^T sin\,(k\cdot t)\cdot cos\,(n\cdot t)\cdot dt + B^2\cdot\int_0^T cos^2\,(n\cdot t)\cdot dt\right) = \\
&= \left\{\ sin\,(x) = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\cdot\jmath} \quad cos\,(x) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\ \right\} = \\
&= \frac{1}{T}\cdot\left(A^2\cdot\int_0^T \left(\frac{e^{\jmath\cdot k\cdot t}-e^{-\jmath\cdot k\cdot t}}{2\cdot\jmath}\right)^2\cdot dt + \right.\\
&+ 2\cdot A\cdot B\cdot\int_0^T \frac{e^{\jmath\cdot k\cdot t}-e^{-\jmath\cdot k\cdot t}}{2\cdot\jmath}\cdot\frac{e^{\jmath\cdot n\cdot t}+e^{-\jmath\cdot n\cdot t}}{2}\cdot dt + \\
&+ \left. B^2\cdot\int_0^T \left(\frac{e^{\jmath\cdot n\cdot t}+e^{-\jmath\cdot n\cdot t}}{2}\right)^2\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A^2\cdot\int_0^T \frac{\left(e^{\jmath\cdot k\cdot t}\right)^2 - 2\cdot e^{\jmath\cdot k\cdot t}\cdot e^{-\jmath\cdot k\cdot t} + \left(e^{-\jmath\cdot k\cdot t}\right)^2}{(2\cdot\jmath)^2}\cdot dt + \right.\\
&+ 2\cdot A\cdot B\cdot\int_0^T \frac{e^{\jmath\cdot k\cdot t}\cdot e^{\jmath\cdot n\cdot t} + e^{\jmath\cdot k\cdot t}\cdot e^{-\jmath\cdot n\cdot t} - e^{-\jmath\cdot k\cdot t}\cdot e^{\jmath\cdot n\cdot t} - e^{-\jmath\cdot k\cdot t}\cdot e^{-\jmath\cdot n\cdot t}}{2\cdot\jmath\cdot 2}\cdot dt + \\
&+ \left. B^2\cdot\int_0^T \frac{\left(e^{\jmath\cdot n\cdot t}\right)^2 + 2\cdot e^{\jmath\cdot n\cdot t}\cdot e^{-\jmath\cdot n\cdot t} + \left(e^{-\jmath\cdot n\cdot t}\right)^2}{2^2}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A^2\cdot\int_0^T \frac{e^{2\cdot\jmath\cdot k\cdot t} - 2\cdot e^{\jmath\cdot k\cdot t-\jmath\cdot k\cdot t} + e^{-2\cdot\jmath\cdot k\cdot t}}{-4}\cdot dt + \right.\\
&+ 2\cdot A\cdot B\cdot\int_0^T \frac{e^{\jmath\cdot k\cdot t+\jmath\cdot n\cdot t} + e^{\jmath\cdot k\cdot t-\jmath\cdot n\cdot t} - e^{-\jmath\cdot k\cdot t+\jmath\cdot n\cdot t} - e^{-\jmath\cdot k\cdot t-\jmath\cdot n\cdot t}}{4\cdot\jmath}\cdot dt + \\
&+ \left. B^2\cdot\int_0^T \frac{e^{2\cdot\jmath\cdot n\cdot t} + 2\cdot e^{\jmath\cdot n\cdot t-\jmath\cdot n\cdot t} + e^{-2\cdot\jmath\cdot n\cdot t}}{4}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(A^2\cdot\int_0^T \frac{e^{2\cdot\jmath\cdot k\cdot t} - 2\cdot e^{0} + e^{-2\cdot\jmath\cdot k\cdot t}}{-4}\cdot dt + \right.\\
&+ 2\cdot A\cdot B\cdot\int_0^T \frac{e^{\jmath\cdot(k+n)\cdot t} + e^{\jmath\cdot(k-n)\cdot t} - e^{-\jmath\cdot(k-n)\cdot t} - e^{-\jmath\cdot(k+n)\cdot t}}{4\cdot\jmath}\cdot dt + \\
&+ \left. B^2\cdot\int_0^T \frac{e^{2\cdot\jmath\cdot n\cdot t} + 2\cdot e^{0} + e^{-2\cdot\jmath\cdot n\cdot t}}{4}\cdot dt\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \int_0^T \left(e^{2\cdot\jmath\cdot k\cdot t} - 2\cdot 1 + e^{-2\cdot\jmath\cdot k\cdot t} \right) \cdot dt + \right.\\
&+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \int_0^T \left(e^{\jmath\cdot(k+n)\cdot t} + e^{\jmath\cdot(k-n)\cdot t} - e^{-\jmath\cdot(k-n)\cdot t} - e^{-\jmath\cdot(k+n)\cdot t} \right) \cdot dt +\\
&\left. + \frac{B^2}{4} \cdot \int_0^T \left(e^{2\cdot\jmath\cdot n\cdot t} + 2\cdot 1 + e^{-2\cdot\jmath\cdot n\cdot t} \right) \cdot dt \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\int_0^T e^{2\cdot\jmath\cdot k\cdot t} \cdot dt - \int_0^T 2\cdot dt + \int_0^T e^{-2\cdot\jmath\cdot k\cdot t} \cdot dt \right) + \right.\\
&+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\int_0^T e^{\jmath\cdot(k+n)\cdot t} \cdot dt + \int_0^T e^{\jmath\cdot(k-n)\cdot t} \cdot dt - \int_0^T e^{-\jmath\cdot(k-n)\cdot t} \cdot dt - \int_0^T e^{-\jmath\cdot(k+n)\cdot t} \cdot dt \right) +\\
&\left. + \frac{B^2}{4} \cdot \left(\int_0^T e^{2\cdot\jmath\cdot n\cdot t} \cdot dt + \int_0^T 2\cdot dt + \int_0^T e^{-2\cdot\jmath\cdot n\cdot t} \cdot dt \right) \right) =\\
&= \left\{ \begin{array}{llll}
z_1 = 2\cdot\jmath\cdot k\cdot t & z_2 = -2\cdot\jmath\cdot k\cdot t & z_3 = 2\cdot\jmath\cdot n\cdot t & z_4 = -2\cdot\jmath\cdot n\cdot t\\
dz_1 = 2\cdot\jmath\cdot k\cdot dt & dz_2 = -2\cdot\jmath\cdot k\cdot dt & dz_3 = 2\cdot\jmath\cdot n\cdot dt & dz_4 = -2\cdot\jmath\cdot n\cdot dt\\
dt = \frac{dz_1}{2\cdot\jmath\cdot k} & dt = \frac{dz_2}{-2\cdot\jmath\cdot k} & dt = \frac{dz_3}{2\cdot\jmath\cdot n} & dt = \frac{dz_4}{-2\cdot\jmath\cdot n}\\
z_5 = 2\cdot\jmath\cdot(k+n)\cdot t & z_6 = -2\cdot\jmath\cdot(k+n)\cdot t & z_7 = 2\cdot\jmath\cdot(k-n)\cdot t & z_8 = -2\cdot\jmath\cdot(k-n)\cdot t\\
dz_5 = \jmath\cdot(k+n)\cdot dt & dz_6 = -\jmath\cdot(k+n)\cdot dt & dz_7 = \jmath\cdot(k-n)\cdot dt & dz_8 = -\jmath\cdot(k-n)\cdot dt\\
dt = \frac{dz_5}{\jmath\cdot(k+n)} & dt = \frac{dz_6}{-\jmath\cdot(k+n)} & dt = \frac{dz_7}{\jmath\cdot(k-n)} & dt = \frac{dz_8}{-\jmath\cdot(k-n)}
\end{array} \right\} =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\int_0^T e^{z_1} \cdot \frac{dz_1}{2\cdot\jmath\cdot k} - 2\cdot \int_0^T dt + \int_0^T e^{z_2} \cdot \frac{dz_2}{-2\cdot\jmath\cdot k} \right) + \right.\\
&+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\int_0^T e^{z_5} \cdot \frac{dz_5}{\jmath\cdot(k+n)} + \int_0^T e^{z_7} \cdot \frac{dz_7}{\jmath\cdot(k-n)} + \right.\\
&\left. - \int_0^T e^{z_8} \cdot \frac{dz_8}{-\jmath\cdot(k-n)} - \int_0^T e^{z_6} \cdot \frac{dz_6}{-\jmath\cdot(k+n)} \right) +\\
&\left. + \frac{B^2}{4} \cdot \left(\int_0^T e^{z_3} \cdot \frac{dz_3}{2\cdot\jmath\cdot n} + 2\cdot \int_0^T dt + \int_0^T e^{z_4} \cdot \frac{dz_4}{-2\cdot\jmath\cdot n} \right) \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2\cdot\jmath\cdot k} \cdot \int_0^T e^{z_1} \cdot dz_1 - 2\cdot \int_0^T dt + \frac{1}{-2\cdot\jmath\cdot k} \cdot \int_0^T e^{z_2} \cdot dz_2 \right) + \right.\\
&+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\frac{1}{\jmath\cdot(k+n)} \cdot \int_0^T e^{z_5} \cdot dz_5 + \frac{1}{\jmath\cdot(k-n)} \cdot \int_0^T e^{z_7} \cdot dz_7 + \right.\\
&\left. - \frac{1}{-\jmath\cdot(k-n)} \cdot \int_0^T e^{z_8} \cdot dz_8 - \frac{1}{-\jmath\cdot(k+n)} \cdot \int_0^T e^{z_6} \cdot dz_6 \right) +\\
&\left. + \frac{B^2}{4} \cdot \left(\frac{1}{2\cdot\jmath\cdot n} \cdot \int_0^T e^{z_3} \cdot dz_3 + 2\cdot \int_0^T dt + \frac{1}{-2\cdot\jmath\cdot n} \cdot \int_0^T e^{z_4} \cdot dz_4 \right) \right) =\\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2\cdot\jmath\cdot k} \cdot e^{z_1}\Big|_0^T - 2\cdot t\Big|_0^T - \frac{1}{2\cdot\jmath\cdot k} \cdot e^{z_2}\Big|_0^T \right) + \right.\\
&+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\frac{1}{\jmath\cdot(k+n)} \cdot e^{z_5}\Big|_0^T + \frac{1}{\jmath\cdot(k-n)} \cdot e^{z_7}\Big|_0^T + \right.\\
&\left. + \frac{1}{\jmath\cdot(k-n)} \cdot e^{z_8}\Big|_0^T + \frac{1}{\jmath\cdot(k+n)} \cdot e^{z_6}\Big|_0^T \right) +\\
&\left. + \frac{B^2}{4} \cdot \left(\frac{1}{2\cdot\jmath\cdot n} \cdot e^{z_3}\Big|_0^T + 2\cdot t\Big|_0^T - \frac{1}{2\cdot\jmath\cdot n} \cdot e^{z_4}\Big|_0^T \right) \right) =
\end{aligned}
$$

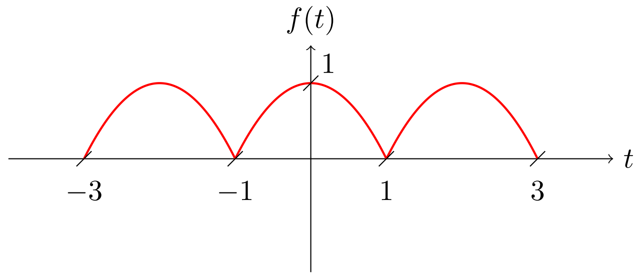
$$
\begin{aligned}
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot e^{2 \cdot \jmath \cdot k \cdot t} \Big|_0^T - 2 \cdot t|_0^T - \frac{1}{2 \cdot \jmath \cdot k} \cdot e^{-2 \cdot \jmath \cdot k \cdot t} \Big|_0^T \right) + \right. \\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot e^{\jmath \cdot (k+n) \cdot t} \Big|_0^T + \frac{1}{\jmath \cdot (k-n)} \cdot e^{\jmath \cdot (k-n) \cdot t} \Big|_0^T + \right. \\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot e^{-\jmath \cdot (k-n) \cdot t} \Big|_0^T + \frac{1}{\jmath \cdot (k+n)} \cdot e^{-\jmath \cdot (k+n) \cdot t} \Big|_0^T \right) + \\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot e^{2 \cdot \jmath \cdot n \cdot t} \Big|_0^T + 2 \cdot t|_0^T - \frac{1}{2 \cdot \jmath \cdot n} \cdot e^{-2 \cdot \jmath \cdot n \cdot t} \Big|_0^T \right) \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot k \cdot T} - e^{2 \cdot \jmath \cdot k \cdot 0} \right) - 2 \cdot (T - 0) - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot k \cdot T} - e^{-2 \cdot \jmath \cdot k \cdot 0} \right) \right) + \right. \\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot (k+n) \cdot T} - e^{\jmath \cdot (k+n) \cdot 0} \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot (k-n) \cdot T} - e^{\jmath \cdot (k-n) \cdot 0} \right) + \right. \\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot (k-n) \cdot T} - e^{-\jmath \cdot (k-n) \cdot 0} \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot (k+n) \cdot T} - e^{-\jmath \cdot (k+n) \cdot 0} \right) \right) + \\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot n \cdot T} - e^{2 \cdot \jmath \cdot n \cdot 0} \right) + 2 \cdot (T - 0) - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot n \cdot T} - e^{-2 \cdot \jmath \cdot n \cdot 0} \right) \right) \right) = \\
&= \left\{ \begin{array}{l} T = b \cdot T_1 = a \cdot T_2 \\ T = b \cdot \frac{2\pi}{k} = a \cdot \frac{2\pi}{n} \end{array} \right\} = \\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} - e^0 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} - e^0 \right) \right) + \right. \\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot k \cdot T} \cdot e^{\jmath \cdot n \cdot T} - e^0 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot k \cdot T} \cdot e^{-\jmath \cdot n \cdot T} - e^0 \right) + \right. \\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot k \cdot T} \cdot e^{\jmath \cdot n \cdot T} - e^0 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot k \cdot T} \cdot e^{-\jmath \cdot n \cdot T} - e^0 \right) \right) + \\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - e^0 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - e^0 \right) \right) \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) \right) + \right. \\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{-\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \right. \\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot k \cdot b \cdot \frac{2\pi}{k}} \cdot e^{-\jmath \cdot n \cdot a \cdot \frac{2\pi}{n}} - 1 \right) \right) + \\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) \right) \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) - 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot k} \cdot \left(e^{-2 \cdot \jmath \cdot b \cdot 2\pi} - 1 \right) \right) + \right. \\
&+ \frac{2 \cdot A \cdot B}{4 \cdot \jmath} \cdot \left(\frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{\jmath \cdot b \cdot 2\pi} \cdot e^{\jmath \cdot a \cdot 2\pi} - 1 \right) + \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{\jmath \cdot b \cdot 2\pi} \cdot e^{-\jmath \cdot a \cdot 2\pi} - 1 \right) + \right. \\
&+ \left. \frac{1}{\jmath \cdot (k-n)} \cdot \left(e^{-\jmath \cdot b \cdot 2\pi} \cdot e^{\jmath \cdot a \cdot 2\pi} - 1 \right) + \frac{1}{\jmath \cdot (k+n)} \cdot \left(e^{-\jmath \cdot b \cdot 2\pi} \cdot e^{-\jmath \cdot a \cdot 2\pi} - 1 \right) \right) + \\
&+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) + 2 \cdot T - \frac{1}{2 \cdot \jmath \cdot n} \cdot \left(e^{-2 \cdot \jmath \cdot a \cdot 2\pi} - 1 \right) \right) \right) =
\end{aligned}
$$

$$= \left\{ \begin{array}{ll} \underset{a\in\mathcal{Z}}{\forall} e^{\jmath\cdot a\cdot 2\pi} = 1 & \underset{a\in\mathcal{Z}}{\forall} e^{2\cdot\jmath\cdot a\cdot 2\pi} = 1 \\ \underset{a\in\mathcal{Z}}{\forall} e^{-\jmath\cdot a\cdot 2\pi} = 1 & \underset{a\in\mathcal{Z}}{\forall} e^{-2\cdot\jmath\cdot a\cdot 2\pi} = 1 \\ \underset{b\in\mathcal{Z}}{\forall} e^{\jmath\cdot b\cdot 2\pi} = 1 & \underset{b\in\mathcal{Z}}{\forall} e^{2\cdot\jmath\cdot b\cdot 2\pi} = 1 \\ \underset{b\in\mathcal{Z}}{\forall} e^{-\jmath\cdot b\cdot 2\pi} = 1 & \underset{b\in\mathcal{Z}}{\forall} e^{-2\cdot\jmath\cdot b\cdot 2\pi} = 1 \end{array} \right\} =$$

$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2\cdot\jmath\cdot k} \cdot (1-1) - 2\cdot T - \frac{1}{2\cdot\jmath\cdot k} \cdot (1-1) \right) + \right.$$
$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\frac{1}{\jmath\cdot(k+n)} \cdot (1\cdot 1-1) + \frac{1}{\jmath\cdot(k-n)} \cdot (1\cdot 1-1) + \right.$$
$$+ \left. \frac{1}{\jmath\cdot(k-n)} \cdot (1\cdot 1-1) + \frac{1}{\jmath\cdot(k+n)} \cdot (1\cdot 1-1) \right) +$$
$$+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2\cdot\jmath\cdot n} \cdot (1-1) + 2\cdot T - \frac{1}{2\cdot\jmath\cdot n} \cdot (1-1) \right) \right) =$$
$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot \left(\frac{1}{2\cdot\jmath\cdot k} \cdot 0 - 2\cdot T - \frac{1}{2\cdot\jmath\cdot k} \cdot 0 \right) + \right.$$
$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot \left(\frac{1}{\jmath\cdot(k+n)} \cdot 0 + \frac{1}{\jmath\cdot(k-n)} \cdot 0 + \right.$$
$$+ \left. \frac{1}{\jmath\cdot(k-n)} \cdot 0 + \frac{1}{\jmath\cdot(k+n)} \cdot 0 \right) +$$
$$+ \left. \frac{B^2}{4} \cdot \left(\frac{1}{2\cdot\jmath\cdot n} \cdot 0 + 2\cdot T - \frac{1}{2\cdot\jmath\cdot n} \cdot 0 \right) \right) =$$
$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot (0 - 2\cdot T - 0) + \right.$$
$$+ \frac{2\cdot A\cdot B}{4\cdot\jmath} \cdot (0+0+0+0) +$$
$$+ \left. \frac{B^2}{4} \cdot (0 + 2\cdot T - 0) \right) =$$
$$= \frac{1}{T} \cdot \left(\frac{A^2}{-4} \cdot (-2\cdot T) + \frac{B^2}{4} \cdot 2\cdot T \right) =$$
$$= \frac{1}{T} \cdot \left(\frac{A^2}{2} \cdot T + \frac{B^2}{2} \cdot T \right) =$$
$$= \frac{A^2}{2} + \frac{B^2}{2}$$

Ostatecznie moc sygnału wynosi $\frac{A^2}{2} + \frac{B^2}{2}$

Task 12. Compute the average power for the following periodic signal $f(t)$:



Signal in the range $t \in (-1; 1)$ is described as:

$$f(t) = 1 - t^2 \tag{1.23}$$

The average power for periodic signals is defined by:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{1.24}$$

In this case period T is equal to 2.

$$\begin{aligned}
P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left|1 - t^2\right|^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left(1 - t^2\right)^2 \cdot dt = \\
&= \frac{1}{2} \cdot \int_{-1}^{1} \left(1 - 2 \cdot t^2 + t^4\right) \cdot dt = \\
&= \frac{1}{2} \cdot \left[\int_{-1}^{1} 1 \cdot dt + \int_{-1}^{1} (-2) \cdot t^2 \cdot dt + \int_{-1}^{1} t^4 \cdot dt\right] = \\
&= \frac{1}{2} \cdot \left[t\Big|_{-1}^{1} - 2 \cdot \frac{t^3}{3}\Big|_{-1}^{1} + \frac{t^5}{5}\Big|_{-1}^{1} \right] = \\
&= \frac{1}{2} \cdot \left[(1 - (-1)) - \frac{2}{3} \cdot (1 - (-1)) + \frac{1}{5} \cdot (1 - (-1))\right] = \\
&= \frac{1}{2} \cdot \left[2 - \frac{4}{3} + \frac{2}{5}\right] = \\
&= \frac{1}{2} \cdot \left[\frac{30}{15} - \frac{20}{15} + \frac{6}{15}\right] = \\
&= \frac{1}{2} \cdot \frac{16}{15} = \\
&= \frac{8}{15}
\end{aligned}$$

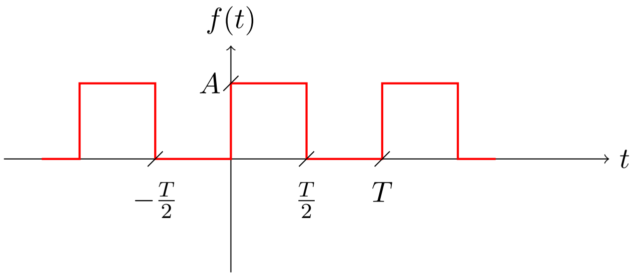
The average power equals to $\frac{8}{15}$.

Chapter 2

Analysis of periodic signals using orthogonal series

2.1 Trigonometric Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.1}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.2}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} dt + 0\right) = \\
&= \frac{1}{T}\left(A\cdot t|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{T}\cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}-0\right) = \\
&= \frac{A}{T}\cdot\left(\frac{T}{2}\right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.3}$$

The a_0 coefficient equals $\frac{A}{2}$.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.4}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\int_0^{\frac{T}{2}} A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \left\{\begin{array}{ll} z & = k\cdot\frac{2\pi}{T}\cdot t \\ dz & = k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{k\cdot\frac{2\pi}{T}} \end{array}\right\} = \\
&= \frac{2\cdot A}{T}\int_0^{\frac{T}{2}} cos\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}} = \\
&= \frac{2\cdot A}{T\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} cos\left(z\right)\cdot dz = \\
&= \frac{A}{k\cdot\pi}\, sin\left(z\right)|_0^{\frac{T}{2}} = \\
&= \frac{A}{k\cdot\pi}\, sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{k\cdot\pi}\left(sin\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right) - sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right) = \\
&= \frac{A}{k\cdot\pi}\left(sin\left(k\cdot\pi\right) - sin\left(0\right)\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{k \cdot \pi} \left(sin \left(k \cdot \pi \right) - 0 \right) = \\
&= \frac{A}{k \cdot \pi} \cdot sin \left(k \cdot \pi \right) = \\
&= \frac{A}{k \cdot \pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$
b_k = \frac{2}{T} \int_T f(t) \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \tag{2.5}
$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt = \\
&= \frac{2}{T} \int_0^{\frac{T}{2}} A \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt = \\
&= \frac{2 \cdot A}{T} \int_0^{\frac{T}{2}} sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = k \cdot \frac{2\pi}{T} \cdot t \\ dz & = k \cdot \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{k \cdot \frac{2\pi}{T}} \end{matrix} \right\} = \\
&= \frac{2 \cdot A}{T} \int_0^{\frac{T}{2}} sin \left(z \right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} = \\
&= \frac{2 \cdot A}{T \cdot k \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin \left(z \right) \cdot dz = \\
&= -\frac{A}{k \cdot \pi} \left. cos \left(z \right) \right|_0^{\frac{T}{2}} = \\
&= -\frac{A}{k \cdot \pi} \left. cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \right|_0^{\frac{T}{2}} = \\
&= -\frac{A}{k \cdot \pi} \left(cos \left(k \cdot \frac{2\pi}{T} \cdot \frac{T}{2} \right) - cos \left(k \cdot \frac{2\pi}{T} \cdot 0 \right) \right) = \\
&= -\frac{A}{k \cdot \pi} \left(cos \left(k \cdot \pi \right) - cos \left(0 \right) \right) = \\
&= -\frac{A}{k \cdot \pi} \left(cos \left(k \cdot \pi \right) - 1 \right) = \\
&= \frac{A}{k \cdot \pi} \left(1 - cos \left(k \cdot \pi \right) \right)
\end{aligned}
$$

The b_k coefficients equal to $\frac{A}{k \cdot \pi} \left(1 - cos \left(k \cdot \pi \right) \right)$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$\begin{aligned} a_0 &= \frac{A}{2} \\ a_k &= 0 \\ b_k &= \frac{A}{k \cdot \pi} \left(1 - cos\left(k \cdot \pi\right)\right) \end{aligned} \tag{2.6}$$

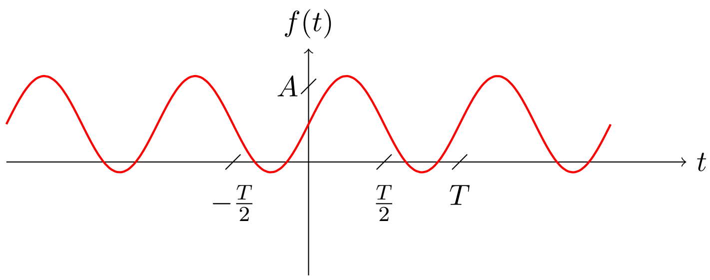
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$	0

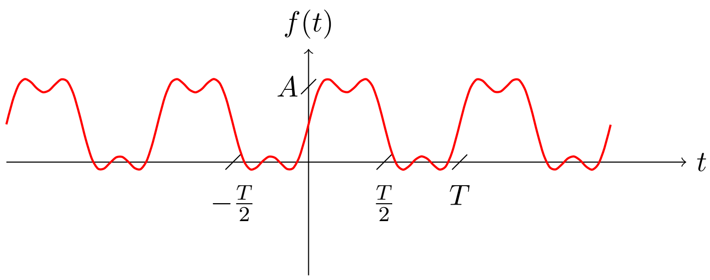
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \\ f(t) &= \frac{A}{2} + \sum_{k=1}^{\infty} \left[\left(\frac{A}{k \cdot \pi}(1 - cos(k \cdot \pi)\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \end{aligned} \tag{2.7}$$

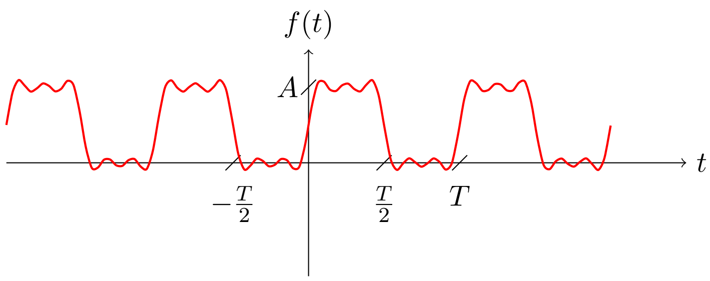
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



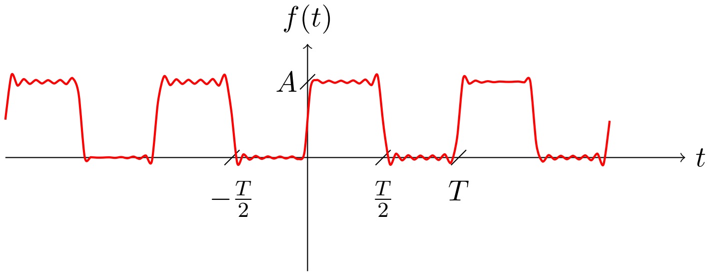
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



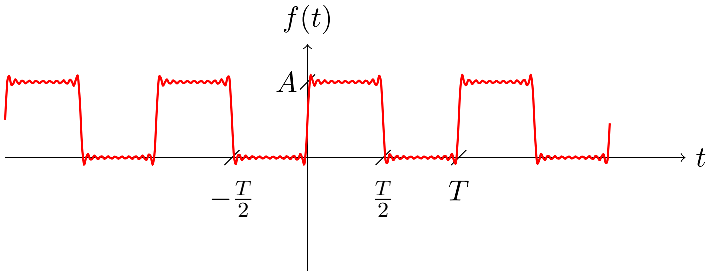
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

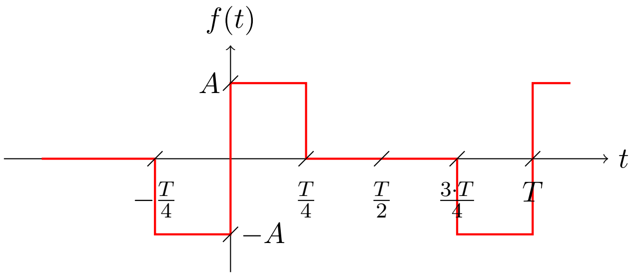


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise linear function assuming period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3 \cdot T}{4} + k \cdot T\right) \end{cases} \quad \wedge k \in Z \tag{2.8}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.9}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_{0}^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3 \cdot T}{4}} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_{0}^{\frac{T}{4}} A \cdot dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot \int_{-\frac{T}{4}}^{0} dt + A \cdot \int_{0}^{\frac{T}{4}} dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot t|_{-\frac{T}{4}}^{0} + A \cdot t|_{0}^{\frac{T}{4}} \right) = \\ &= \frac{1}{T} \left(-A \cdot \left(0 - \left(-\frac{T}{4}\right)\right) + A \cdot \left(\frac{T}{4} - 0\right) \right) = \\ &= \frac{1}{T} \left(-A \cdot \frac{T}{4} + A \cdot \frac{T}{4} \right) = \\ &= \frac{1}{T} (0) = \\ &= 0 \end{aligned}$$

The a_0 coefficient equals 0.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T} \int_T f(t) \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.10}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \left\{\begin{array}{ll} z & = k\cdot\frac{2\pi}{T}\cdot t \\ dz & = k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dt}{k\cdot\frac{2\pi}{T}} \end{array}\right\} = \\
&= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} cos\left(z\right)\cdot\frac{dt}{k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} cos\left(z\right)\cdot\frac{dt}{k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{2}{T}\left(-\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} cos\left(z\right)\cdot dt + \frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} cos\left(z\right)\cdot dt\right) = \\
&= \frac{2}{T}\cdot\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\left(-\,sin\left(z\right)|_{-\frac{T}{4}}^{0} + sin\left(z\right)|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{2\cdot A}{k\cdot 2\pi}\cdot\left(-\,sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_{-\frac{T}{4}}^{0} + sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{2\cdot A}{k\cdot 2\pi}\cdot\left(-\left(sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right) - sin\left(-k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right)\right) + \left(sin\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right) - sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-\left(sin\left(0\right) - sin\left(-k\cdot\frac{2\pi}{4}\right)\right) + \left(sin\left(k\cdot\frac{2\pi}{4}\right) - sin\left(0\right)\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-\left(0 - sin\left(-k\cdot\frac{\pi}{2}\right)\right) + \left(sin\left(k\cdot\frac{\pi}{2}\right) - 0\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(sin\left(-k\cdot\frac{\pi}{2}\right) + sin\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot\left(-sin\left(k\cdot\frac{\pi}{2}\right) + sin\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \frac{A}{k\cdot\pi}\cdot(0) = \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.11}$$

For the period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ we get:

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) =
\end{aligned}
$$

$$= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) =$$
$$= \left\{\begin{aligned} z &= k\cdot\tfrac{2\pi}{T}\cdot t\\ dz &= k\cdot\tfrac{2\pi}{T}\cdot dt\\ dt &= \tfrac{dz}{k\cdot\frac{2\pi}{T}}\end{aligned}\right\} =$$
$$= \frac{2}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}} + 0\right) =$$
$$= \frac{2}{T}\left(-\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} sin\left(z\right)\cdot dz + \frac{A}{k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} sin\left(z\right)\cdot dz\right) =$$
$$= \frac{2}{T}\cdot\frac{A}{k\cdot\frac{2\pi}{T}}\cdot\left(-\int_{-\frac{T}{4}}^{0} sin\left(z\right)\cdot dz + \int_{0}^{\frac{T}{4}} sin\left(z\right)\cdot dz\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(cos\left(z\right)\Big|_{-\frac{T}{4}}^{0} - cos\left(z\right)\Big|_{0}^{\frac{T}{4}}\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_{-\frac{T}{4}}^{0} - cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_{0}^{\frac{T}{4}}\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(\left(cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right) - cos\left(-k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right)\right) - \left(cos\left(k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right) - cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(\left(cos\left(0\right) - cos\left(-k\cdot\frac{\pi}{2}\right)\right) - \left(cos\left(k\cdot\frac{\pi}{2}\right) - cos\left(0\right)\right)\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(cos\left(0\right) - cos\left(-k\cdot\frac{\pi}{2}\right) - cos\left(k\cdot\frac{\pi}{2}\right) + cos\left(0\right)\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right) - cos\left(k\cdot\frac{\pi}{2}\right) + 1\right) =$$
$$= \frac{A}{k\cdot\pi}\cdot\left(2 - 2\cdot cos\left(k\cdot\frac{\pi}{2}\right)\right) =$$
$$= \frac{2\cdot A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right)$$

The b_k coefficients equal to $\frac{2\cdot A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right)$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0 = 0$$
$$a_k = 0$$
$$b_k = \frac{2\cdot A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right)$$

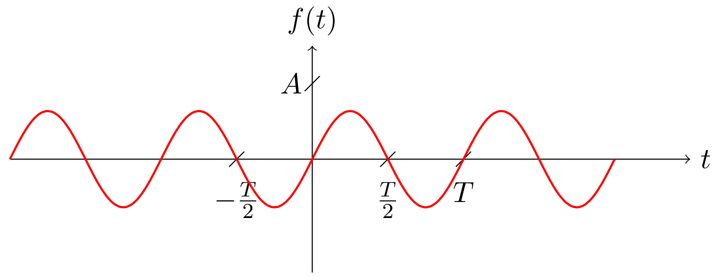
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$\frac{2\cdot A}{\pi}$	$\frac{2\cdot A}{\pi}$	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$	$\frac{2\cdot A}{3\cdot\pi}$

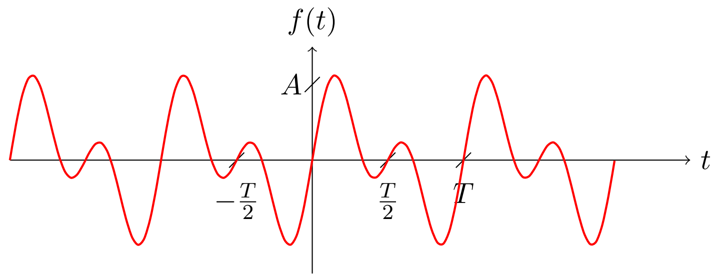
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series:

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right]$$
$$f(t) = \sum_{k=1}^{\infty} \left[\frac{2 \cdot A}{k \cdot \pi} \cdot \left(1 - cos\left(k \cdot \frac{\pi}{2}\right)\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right]$$

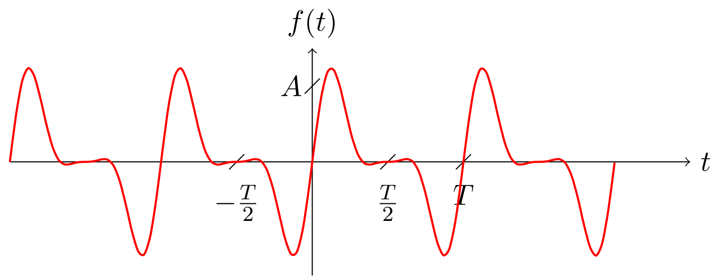
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



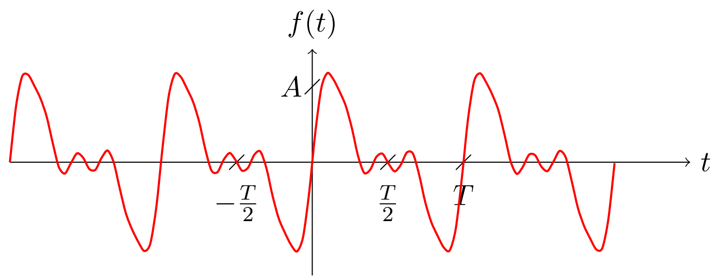
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



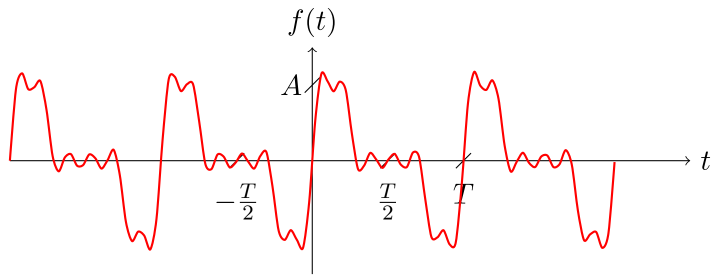
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



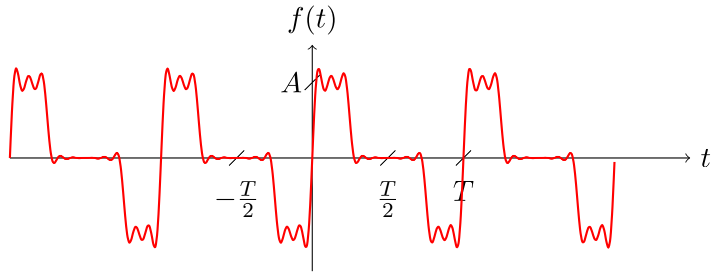
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



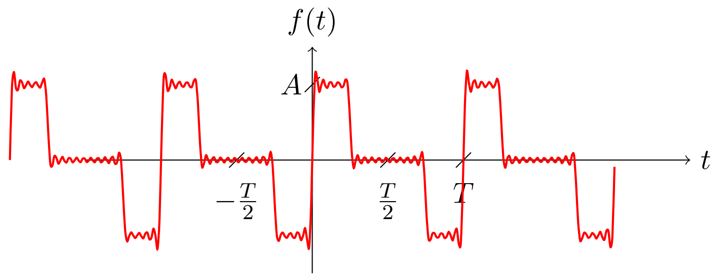
A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

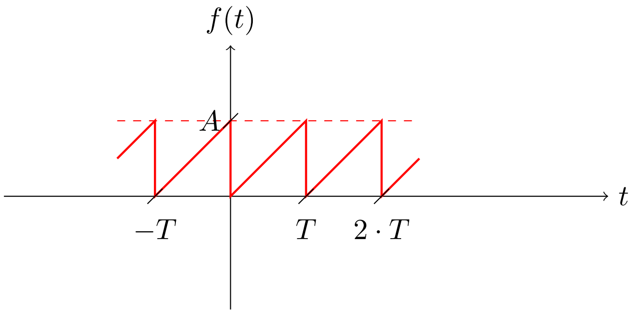


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 3. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.12}$$

In the first period (i.e. $t \in (0;T)$), linear function crosses two points: $(0,0)$ and (T,A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \frac{A}{T} \cdot (t - k \cdot T) \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.13}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot dt = \\
&= \frac{A}{T^2}\int_0^T t\cdot dt = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot t^2\Big|_0^T = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot\left(T^2-0^2\right) = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot T^2 = \\
&= \frac{A}{2}
\end{aligned}
$$

The a_0 coefficient equals $\frac{A}{2}$.

The a_k coefficients are defined as:

$$
a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.14}
$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\int_0^T \frac{A}{T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2\cdot A}{T^2}\int_0^T t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \\ du & = dt & v & = \frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{Bmatrix} = \\
&= \frac{2\cdot A}{T^2}\cdot\left(t\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T - \int_0^T \frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(\left(T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) - 0\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot\left(cos\left(k\cdot\frac{2\pi}{T}\cdot T\right) - cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\right) + \frac{1}{(k\cdot 2\pi)^2}\cdot\left(cos\left(k\cdot 2\pi\right) - cos\left(0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(\frac{1}{k\cdot 2\pi}\cdot 0 + \frac{1}{(k\cdot 2\pi)^2}\cdot(1-1)\right) = \\
&= 2\cdot A\cdot\left(0 + \frac{1}{(k\cdot 2\pi)^2}\cdot 0\right) = \\
&= 2\cdot A\cdot 0 = \\
&= 0
\end{aligned}
$$

The a_k coefficients equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.15}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2}{T}\int_0^T \frac{A}{T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \frac{2\cdot A}{T^2}\int_0^T t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \\ du & = dt & v & = -\frac{T}{k\cdot 2\pi}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{Bmatrix} = \\
&= \frac{2\cdot A}{T^2}\cdot\left(-t\cdot\frac{T}{k\cdot 2\pi}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T + \int_0^T\frac{T}{k\cdot 2\pi}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(-\left(T\cdot\frac{T}{k\cdot 2\pi}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot T\right) - 0\cdot\frac{T}{k\cdot 2\pi}\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\Big|_0^T\right) = \\
&= \frac{2\cdot A}{T^2}\cdot\left(-\left(\frac{T^2}{k\cdot 2\pi}\cdot cos\left(k\cdot 2\pi\right)\right) + \frac{T^2}{(k\cdot 2\pi)^2}\cdot\left(sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) - sin\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(-\left(\frac{1}{k\cdot 2\pi}\cdot 1\right) + \frac{1}{(k\cdot 2\pi)^2}\cdot\left(sin\left(k\cdot 2\pi\right) - sin\left(0\right)\right)\right) = \\
&= 2\cdot A\cdot\left(-\frac{1}{k\cdot 2\pi} + \frac{1}{(k\cdot 2\pi)^2}\cdot(0-0)\right) = \\
&= 2\cdot A\cdot\left(-\frac{1}{k\cdot 2\pi} + \frac{1}{(k\cdot 2\pi)^2}\cdot 0\right) = \\
&= 2\cdot A\cdot\left(-\frac{1}{k\cdot 2\pi}\right) = \\
&= -\frac{2\cdot A}{k\cdot 2\pi} = \\
&= -\frac{A}{k\cdot\pi}
\end{aligned}$$

The b_k coefficients equal to $-\frac{A}{k\cdot\pi}$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$\begin{aligned}
a_0 &= \frac{A}{2} \\
a_k &= 0 \\
b_k &= -\frac{A}{k\cdot\pi}
\end{aligned}$$

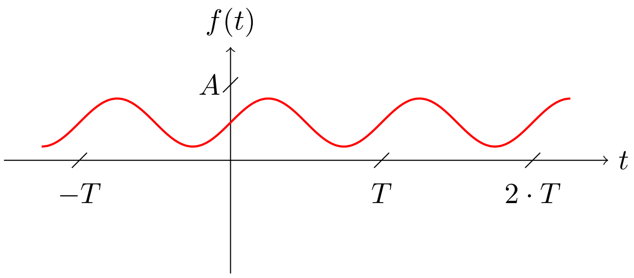
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	0	0	0	0	0
b_k	$-\frac{A}{\pi}$	$-\frac{A}{2\cdot\pi}$	$-\frac{A}{3\cdot\pi}$	$-\frac{A}{4\cdot\pi}$	$-\frac{A}{5\cdot\pi}$	$-\frac{A}{6\cdot\pi}$

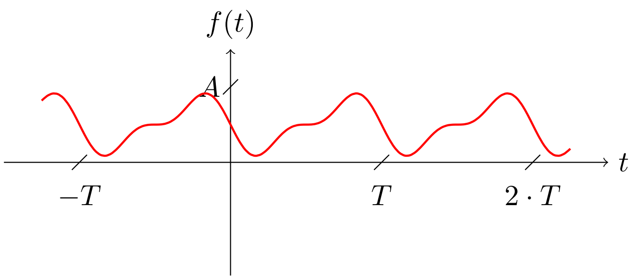
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty}\left[a_k \cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) + b_k \cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \\ f(t) &= \frac{A}{2} + \sum_{k=1}^{\infty}\left[\left(-\frac{A}{k\cdot\pi}\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \end{aligned} \tag{2.16}$$

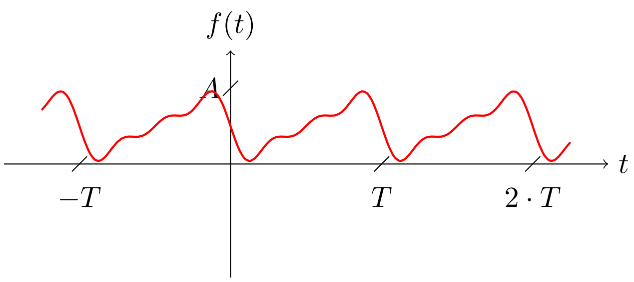
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



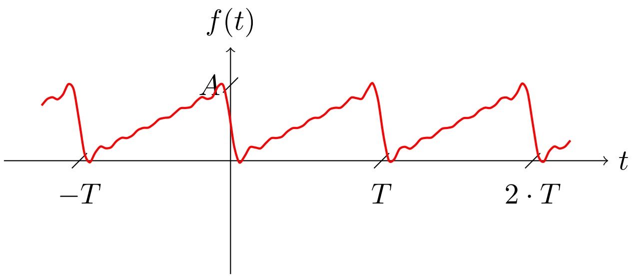
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



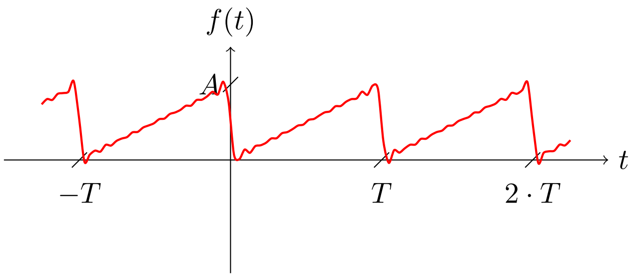
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 7$ results in:

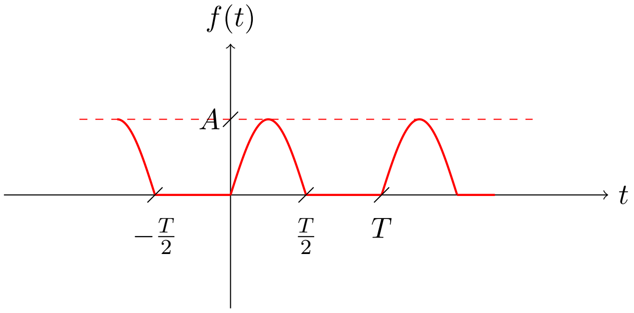


A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 4. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



Periodic signal $f(t)$, as a piecewise function assuming period $t \in (0;T)$ is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.17}$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.18}$$

For the period $t \in (0;T)$ we get:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{matrix} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{2\pi} \cdot \left(cos\left(\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{A}{2\pi} \cdot (-1-1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The a_0 coefficient equals $\frac{A}{\pi}$.

The a_k coefficients are defined as:

$$
a_k = \frac{2}{T} \int_T f(t) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.19}
$$

For the period $t \in (0; T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T} \int_T f(t) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{\begin{aligned} cos\left(x\right) &= \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \\ sin\left(x\right) &= \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \end{aligned}\right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot dt + 0\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}\right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}\right) \cdot dt = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2\jmath} + \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2\jmath}\right) \cdot dt = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) + sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right)\right) \cdot dt = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) \cdot dt + \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \cdot dt\right) = \\
&= \left\{\begin{aligned} z_1 &= \tfrac{2\pi}{T} \cdot t \cdot (1+k) & z_2 &= \tfrac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 &= \tfrac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 &= \tfrac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt &= \tfrac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt &= \tfrac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{aligned}\right\} =
\end{aligned}
$$

$$= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} + \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_1\right) \cdot dz_1 + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} sin\left(z_2\right) \cdot dz_2 \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(-cos\left(z_1\right)\Big|_0^{\frac{T}{2}} \right) + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(-cos\left(z_2\right)\Big|_0^{\frac{T}{2}} \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right)\Bigg|_0^{\frac{T}{2}} \right) + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right)\Bigg|_0^{\frac{T}{2}} \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1+k)\right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1+k)\right) \right) = \right.$$
$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1-k)\right) - cos\left(\frac{2\pi}{T} \cdot 0 \cdot (1-k)\right) \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{-1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(\pi \cdot (1+k)\right) - cos\left(0\right) \right) + \right.$$
$$\left. + \frac{-1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(\pi \cdot (1-k)\right) - cos\left(0\right) \right) \right) =$$
$$= \frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1+k)\right) \right) + \right.$$
$$\left. + \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(cos\left(0\right) - cos\left(\pi \cdot (1-k)\right) \right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1}{1+k} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1}{1-k} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1-k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1+k)\right)\right) + \frac{1+k}{(1+k) \cdot (1-k)} \cdot \left(1 - cos\left(\pi \cdot (1-k)\right)\right) \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right)}{(1+k) \cdot (1-k)} + \frac{1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k) \cdot (1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \left(\frac{1 - cos\left(\pi \cdot (1+k)\right) - k + k \cdot cos\left(\pi \cdot (1+k)\right) + 1 - cos\left(\pi \cdot (1-k)\right) + k - k \cdot cos\left(\pi \cdot (1-k)\right)}{(1+k) \cdot (1-k)} \right) =$$
$$= \frac{A}{2\pi} \cdot \frac{2 - cos\left(\pi \cdot (1+k)\right) + k \cdot cos\left(\pi \cdot (1+k)\right) - cos\left(\pi \cdot (1-k)\right) - k \cdot cos\left(\pi \cdot (1-k)\right)}{1-k^2} =$$
$$= \left\{ \begin{array}{l} cos\left(\pi \cdot (1+k)\right) = cos\left(\pi + k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \\ cos\left(\pi \cdot (1-k)\right) = cos\left(\pi - k \cdot \pi\right) = -cos\left(-k \cdot \pi\right) = -cos\left(k \cdot \pi\right) \end{array} \right\} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + cos\left(k \cdot \pi\right) - k \cdot cos\left(k \cdot \pi\right) + cos\left(k \cdot \pi\right) + k \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{2\pi} \cdot \frac{2 + 2 \cdot cos\left(k \cdot \pi\right)}{1-k^2} =$$
$$= \frac{A}{\pi} \cdot \frac{1 + cos\left(k \cdot \pi\right)}{1-k^2}$$

The a_k coefficients equal to $\frac{A}{\pi} \cdot \frac{1+cos(k\cdot\pi)}{1-k^2}$ for $k \neq 1$.
We have to calculate a_k for $k = 1$ directly by definition:

$$a_1 = \frac{2}{T} \int_T f(t) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt =$$

$$= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$

$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$

$$= \left\{ \begin{matrix} cos\left(x\right) & = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \\ sin\left(x\right) & = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \end{matrix} \right\} =$$

$$= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot dt + 0 \right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt \right) =$$

$$= \frac{2}{T} \cdot \frac{A}{2 \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}\right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^0 - e^0\right) \cdot dt =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}}{2\jmath} \right) \cdot dt =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \cdot 2\right) \cdot dt =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt =$$

$$= \left\{ \begin{matrix} z & = \frac{4\pi}{T} \cdot t \\ dz & = \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{matrix} \right\} =$$

$$= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} =$$

$$= \frac{A}{T \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz =$$

$$= \frac{A}{4\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{4\pi} \cdot \left(-cos\left(\frac{4\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} \right) =$$

$$= -\frac{A}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{4\pi}{T} \cdot 0\right) \right) =$$

$$= -\frac{A}{4\pi} \cdot \left(cos\left(2\pi\right) - cos\left(0\right)\right) =$$

$$= -\frac{A}{4\pi} \cdot \left(1 - 1\right) =$$

$$= -\frac{A}{4\pi} \cdot 0 =$$
$$= 0$$

The a_1 coefficient equal to 0.

The b_k coefficients are defined as:

$$b_k = \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.20}$$

For the period $t \in (0; T)$ we get:

$$\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot dt + 0\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}\right) \cdot \left(e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt\right) = \\
&= \frac{2}{T} \cdot \frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}\right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot \jmath} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)}}{2} - \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)}}{2}\right) \cdot dt = \\
&= -\frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) - cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right)\right) \cdot dt = \\
&= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1+k)\right) \cdot dt - \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \cdot (1-k)\right) \cdot dt\right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \frac{2\pi}{T} \cdot t \cdot (1+k) & z_2 & = \frac{2\pi}{T} \cdot t \cdot (1-k) \\ dz_1 & = \frac{2\pi}{T} \cdot (1+k) \cdot dt & z_2 & = \frac{2\pi}{T} \cdot (1-k) \cdot dt \\ dt & = \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} \wedge k \neq -1 & dt & = \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)} \wedge k \neq 1 \end{array} \right\} = \\
&= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot \frac{dz_1}{\frac{2\pi}{T} \cdot (1+k)} - \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot \frac{dz_2}{\frac{2\pi}{T} \cdot (1-k)}\right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_1\right) \cdot dz_1 - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} cos\left(z_2\right) \cdot dz_2\right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(z_1\right)\Big|_0^{\frac{T}{2}} \right) - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(z_2\right)\Big|_0^{\frac{T}{2}} \right) \right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1+k) \right)\Bigg|_0^{\frac{T}{2}} \right) - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot t \cdot (1-k) \right)\Bigg|_0^{\frac{T}{2}} \right) \right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1+k) \right) - sin\left(\frac{2\pi}{T} \cdot 0 \cdot (1+k) \right) \right) + \right. \\
&\left. - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2} \cdot (1-k) \right) - sin\left(\frac{2\pi}{T} \cdot 0 \cdot (1-k) \right) \right) \right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot \left(sin\left(\pi \cdot (1+k) \right) - sin\left(0\right) \right) + \right. \\
&\left. - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot \left(sin\left(\pi \cdot (1-k) \right) - sin\left(0\right) \right) \right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot (0-0) = \right. \\
&\left. - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot (0-0) \right) = \\
&= -\frac{A}{T} \cdot \left(\frac{1}{\frac{2\pi}{T} \cdot (1+k)} \cdot 0 - \frac{1}{\frac{2\pi}{T} \cdot (1-k)} \cdot 0 \right) = \\
&= -\frac{A}{T} \cdot (0-0) = \\
&= -\frac{A}{T} \cdot 0 = \\
&= 0
\end{aligned}
$$

The b_k coefficients equal to 0 for $k \neq 1$.

We have to calculate b_k for $k = 1$ directly by definition:

$$
\begin{aligned}
b_1 &= \frac{2}{T} \int_T f(t) \cdot sin\left(1 \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt = \\
&= \frac{2}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t \right) \cdot sin\left(1 \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot sin\left(1 \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \right) \cdot sin\left(\frac{2\pi}{T} \cdot t \right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{2}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot dt + 0 \right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt \right) = \\
&= \frac{2}{T} \cdot \frac{A}{2\jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} - e^{0} - e^{0} \right) \cdot dt =$$
$$= \frac{A}{T \cdot \jmath \cdot \jmath} \cdot \int_0^{\frac{T}{2}} \left(\frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2}}{2} - \frac{1+1}{2} \right) \cdot dt =$$
$$= -\frac{A}{T} \cdot \int_0^{\frac{T}{2}} \left(cos\left(\frac{2\pi}{T} \cdot t \cdot 2 \right) - 1 \right) \cdot dt =$$
$$= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(\frac{4\pi}{T} \cdot t \right) \cdot dt - \int_0^{\frac{T}{2}} dt \right) =$$
$$= \left\{ \begin{array}{ll} z & = \frac{4\pi}{T} \cdot t \\ dz & = \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{array} \right\} =$$
$$= -\frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} - t|_0^{\frac{T}{2}} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz - \left(\frac{T}{2} - 0 \right) \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \, sin\left(z\right)|_0^{\frac{T}{2}} - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \, sin\left(\frac{4\pi}{T} \cdot t \right) \Big|_0^{\frac{T}{2}} - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(sin\left(\frac{4\pi}{T} \cdot \frac{T}{2} \right) - sin\left(\frac{4\pi}{T} \cdot 0 \right) \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(sin\left(2\pi\right) - sin\left(0\right) \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(\frac{1}{\frac{4\pi}{T}} \left(0 - 0 \right) - \frac{T}{2} \right) =$$
$$= -\frac{A}{T} \cdot \left(-\frac{T}{2} \right) =$$
$$= \frac{A}{2}$$

The b_1 coefficient equal to $\frac{A}{2}$.

To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0 = \frac{A}{\pi}$$
$$a_1 = 0$$
$$a_k = \frac{A}{\pi} \cdot \frac{1 + cos\left(k \cdot \pi\right)}{1 - k^2}$$
$$b_1 = \frac{A}{2}$$

$$b_k = 0$$

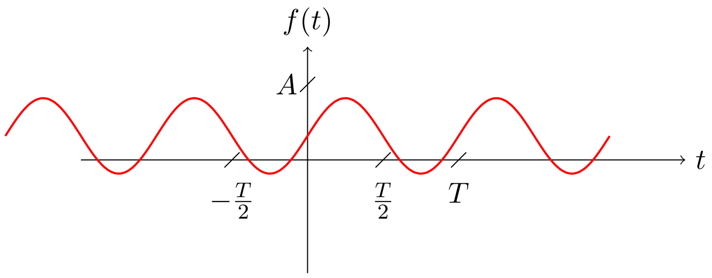
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	0	$-\frac{2}{3}\frac{A}{\pi}$	0	$-\frac{2}{15}\frac{A}{\pi}$	0	$-\frac{2}{35}\frac{A}{\pi}$
b_k	$\frac{A}{2}$	0	0	0	0	0

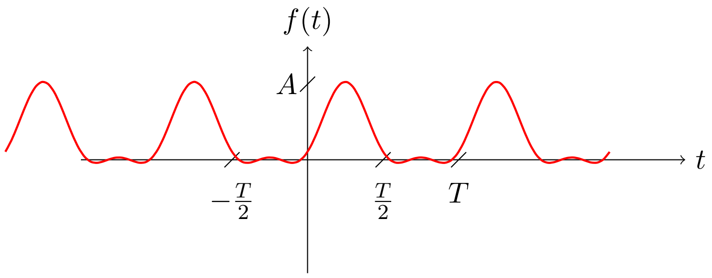
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \tag{2.21}$$

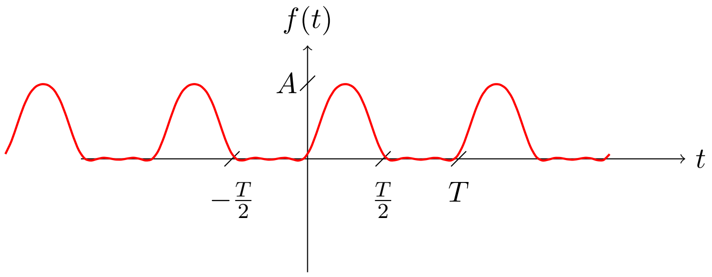
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



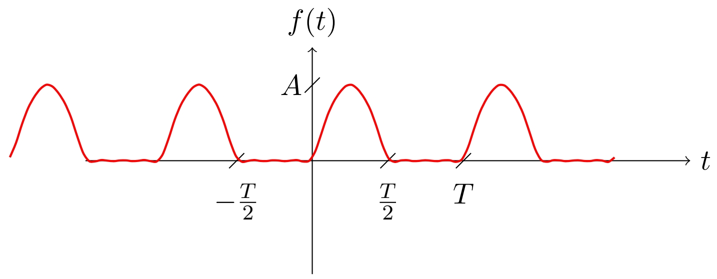
A partial approximation of the $f(t)$ signal for $k_{max} = 2$ results in:



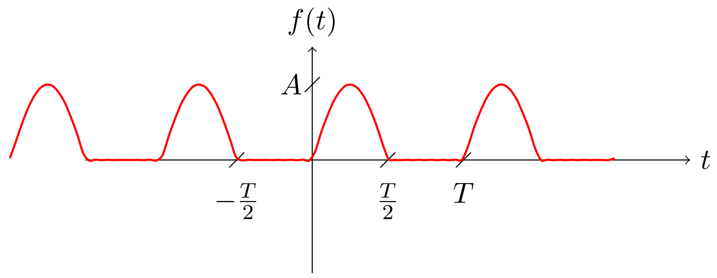
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:

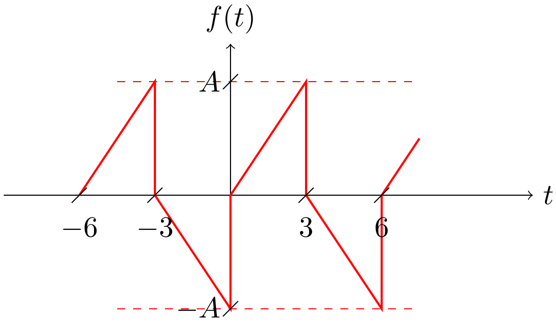


A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 5. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of a linear function is:

$$f(t) = a \cdot t + b \tag{2.22}$$

In the first interval of the first period (e.g. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (e.g. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$

$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.23}$$

For the period $t \in (0;6)$, i.e. $k = 0$, we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt \\ &= \frac{1}{6} \cdot \left[\int_0^3 \frac{A}{3} \cdot t \cdot dt + \int_3^6 \left(-\frac{A}{3} \cdot t + A \right) \cdot dt \right] = \\ &= \frac{A}{18} \cdot \int_0^3 t \cdot dt - \frac{A}{18} \cdot \int_3^6 t \cdot dt + \frac{A}{6} \cdot \int_3^6 dt = \\ &= \frac{A}{18} \cdot \left. \frac{t^2}{2} \right|_0^3 - \frac{A}{18} \cdot \left. \frac{t^2}{2} \right|_3^6 + \frac{A}{6} \cdot t|_3^6 = \\ &= \frac{A}{36} \cdot \left(3^2 - 0^2 \right) - \frac{A}{36} \cdot \left(6^2 - 3^2 \right) + \frac{A}{6} \cdot (6 - 3) = \\ &= \frac{A}{36} \cdot 9 - \frac{A}{36} \cdot 27 + \frac{A}{6} \cdot 3 = \\ &= \frac{9 \cdot A}{36} - \frac{27 \cdot A}{36} + \frac{18 \cdot A}{36} = \end{aligned}$$

$$= 0$$

The a_0 coefficient equals 0.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.24}$$

For the period $t \in (0;6)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\\
&= \frac{2}{6}\cdot\left[\int_0^3 \frac{A}{3}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt + \int_3^6\left(-\frac{A}{3}\cdot t + A\right)\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt\right] =\\
&= \frac{A}{9}\cdot\int_0^3 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt - \frac{A}{9}\cdot\int_3^6 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt + \frac{A}{3}\cdot\int_3^6 cos\left(k\cdot\frac{\pi}{3}\cdot t\right)dt =\\
&= \left\{\begin{matrix} u & = t & dv & = cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\\ du & = dt & v & = \frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\end{matrix}\right\} =\\
&= \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3 - \int_0^3\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] -\\
&- \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 - \int_3^6\frac{3}{k\cdot 3\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] +\\
&+ \frac{A}{3}\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 =\\
&= \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) - 0\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 0\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3\right] -\\
&- \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - 3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - sin\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\pi\right) - 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 0\right)\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot 2\pi\right) - 3\cdot sin\left(k\pi\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 6\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot 2\pi\right) - sin\left(k\cdot\pi\right)\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right) - \frac{3}{k\cdot\pi}\cdot cos\left(0\right)\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot 0 - 3\cdot 0 + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\right) - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[0 - 0\right] =\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right) - \frac{3}{k\cdot\pi}\cdot 1\right] -\\
&- \frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot 1 - \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right] =\\
&= \frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right) - \frac{A}{k^2\cdot\pi^2} - \frac{A}{k^2\cdot\pi^2} + \frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \left(cos\left(k \cdot \pi\right) - 1\right) = \\
&= \frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \Big((-1)^k - 1\Big)
\end{aligned}
$$

The a_k coefficients equal to $\frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \Big((-1)^k - 1\Big)$.
The b_k coefficients are defined as:

$$
b_k = \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.25}
$$

For the period $t \in (0; 6)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
b_k &= \frac{2}{T} \int_T f(t) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\
&= \frac{2}{6} \cdot \left[\int_0^3 \frac{A}{3} \cdot t \cdot sin\left(k \cdot \frac{2\pi}{6} \cdot t\right) \cdot dt + \int_3^6 \left(-\frac{A}{3} \cdot t + A\right) \cdot sin\left(k \cdot \frac{2\pi}{6} \cdot t\right) \cdot dt\right] = \\
&= \frac{A}{9} \cdot \int_0^3 t \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt - \frac{A}{9} \cdot \int_3^6 t \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt + \frac{A}{3} \cdot \int_3^6 sin\left(k \cdot \frac{\pi}{3} \cdot t\right) dt = \\
&= \left\{\begin{matrix} u & = t & dv & = sin\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt \\ du & = dt & v & = -\frac{3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right) \end{matrix}\right\} = \\
&= \frac{A}{9} \cdot \left[t \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_0^3 - \int_0^3 \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt\right] - \\
&- \frac{A}{9} \cdot \left[t \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6 - \int_3^6 \frac{-3}{k \cdot 3\pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right) \cdot dt\right] + \\
&+ \frac{A}{3} \cdot \frac{-3}{k \cdot \pi} \cdot cos\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6 = \\
&= \frac{-3 \cdot A}{9 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 3\right) - 0 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 0\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_0^3\right] + \\
&+ \frac{3 \cdot A}{9 \cdot k \cdot \pi} \cdot \left[6 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 6\right) - 3 \cdot cos\left(k \cdot \frac{\pi}{3} \cdot 3\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot t\right)\Bigg|_3^6\right] - \\
&- \frac{A}{k \cdot \pi} \cdot \left[cos\left(k \cdot \frac{\pi}{3} \cdot 6\right) - cos\left(k \cdot \frac{\pi}{3} \cdot 3\right)\right] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\left(k \cdot \pi\right) - 0 - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 3\right) + \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 0\right)\right] + \\
&+ \frac{A}{3 \cdot k \cdot \pi} \cdot \left[6 \cdot cos\left(k \cdot 2\pi\right) - 3 \cdot cos\left(k \cdot \pi\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 6\right) + \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \frac{\pi}{3} \cdot 3\right)\right] - \\
&- \frac{A}{k \cdot \pi} \cdot \left[cos\left(k \cdot 2\pi\right) - cos\left(k \cdot \pi\right)\right] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\left(k \cdot \pi\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \pi\right) + \frac{3}{k \cdot \pi} \cdot sin\left(0\right)\right] + \\
&+ \frac{A}{3 \cdot k \cdot \pi} \cdot \left[6 \cdot 1 - 3 \cdot cos\left(k \cdot \pi\right) - \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi\right) + \frac{3}{k \cdot \pi} \cdot sin\left(k \cdot \pi\right)\right] - \\
&- \frac{A}{k \cdot \pi} \cdot \left[1 - cos\left(k \cdot \pi\right)\right] = \\
&= \frac{-A}{3 \cdot k \cdot \pi} \cdot \left[3 \cdot cos\left(k \cdot \pi\right) - \frac{3}{k \cdot \pi} \cdot 0 + \frac{3}{k \cdot \pi} \cdot 0\right] +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot 0+\frac{3}{k\cdot\pi}\cdot 0\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)\right]+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)\right]-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{2\cdot A}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)-\frac{A}{k\cdot\pi}+\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{A}{k\cdot\pi}=\\
&=\frac{-A}{k\cdot\pi}\cdot\left(-1\right)^k+\frac{A}{k\cdot\pi}=\\
&=\frac{A}{k\cdot\pi}\cdot\left(1-\left(-1\right)^k\right)
\end{aligned}
$$

The b_k coefficients equal to $\frac{A}{k\cdot\pi}\cdot\left(1-\left(-1\right)^k\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$
\begin{aligned}
a_0&=0\\
a_k&=\frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left(\left(-1\right)^k-1\right)\\
b_k&=\frac{A}{k\cdot\pi}\cdot\left(1-\left(-1\right)^k\right)
\end{aligned}
$$

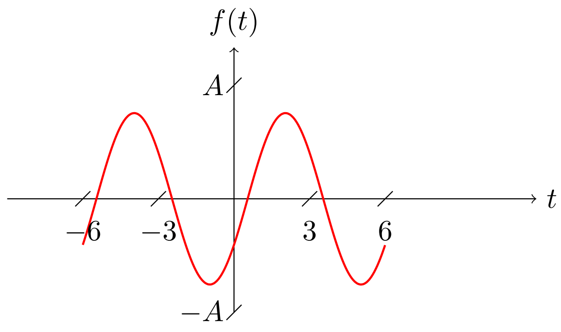
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	$\frac{-4\cdot A}{\pi^2}$	0	$\frac{-4\cdot A}{9\cdot\pi^2}$	0	$\frac{-4\cdot A}{25\cdot\pi^2}$	0
b_k	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$	0

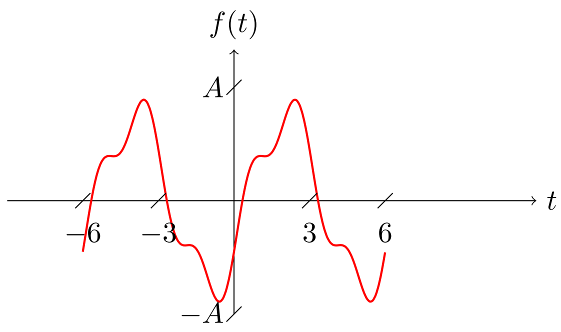
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t)&=a_0+\sum_{k=1}^{\infty}\left[a_k\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+b_k\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right]\\
f(t)&=\sum_{k=1}^{\infty}\left[\frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left(\left(-1\right)^k-1\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+\left(\frac{A}{k\cdot\pi}\cdot\left(1-\left(-1\right)^k\right)\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right]
\end{aligned}
\tag{2.26}
$$

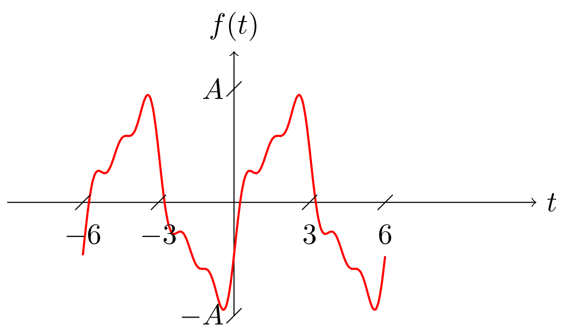
A partial approximation of the $f(t)$ signal for $k_{max}=1$ results in:



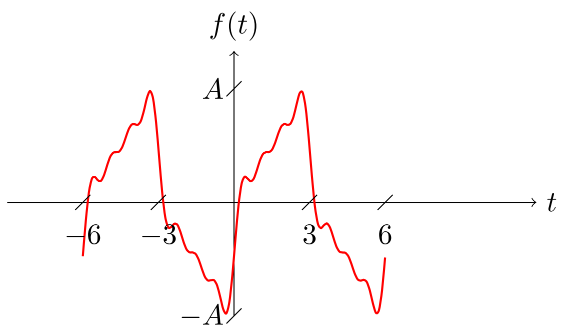
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



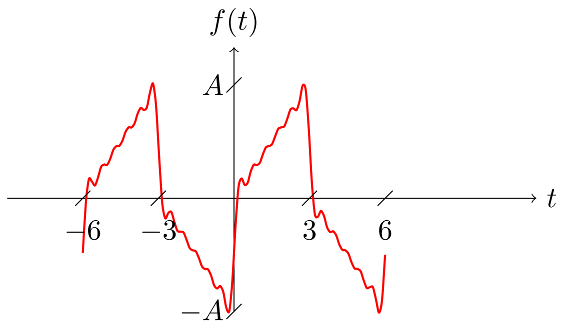
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



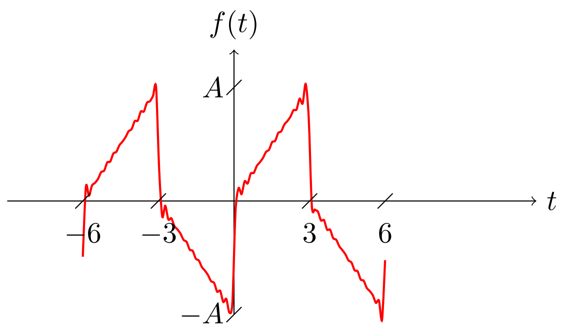
A partial approximation of the $f(t)$ signal for $k_{max} = 7$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

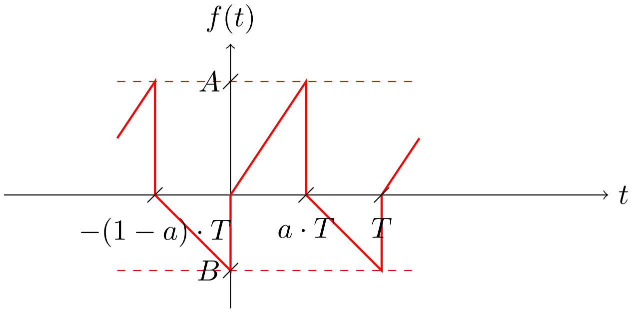


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Task 6. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{2.27}$$

In the first interval of the first period (e.g. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a\cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T\cdot(1-a)} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T \cdot (1-a)} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.28}$$

For the period $t \in (0; T)$ we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} \left(-\frac{B}{(1-a) \cdot T} \cdot t + \frac{B}{1-a} \cdot a \right) \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} -\frac{B}{(1-a) \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} \frac{B}{1-a} \cdot a \cdot dt \right) = \\ &= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \int_0^{a\cdot T} t \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a\cdot T}^{T} t \cdot dt + \frac{B}{1-a} \cdot a \cdot \int_{a\cdot T}^{T} dt \right) = \\ &= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \left. \frac{t^2}{2} \right|_0^{a\cdot T} - \frac{B}{(1-a) \cdot T} \cdot \left. \frac{t^2}{2} \right|_{a\cdot T}^{T} + \frac{B}{1-a} \cdot a \cdot \left. t \right|_{a\cdot T}^{T} \right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{(a\cdot T)^2}{2}-\frac{0^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{(a\cdot T)^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot(T-a\cdot T)\right)=\\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}-\frac{0}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{a^2\cdot T^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot T\cdot(1-a)\right)=\\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot T^2\cdot\left(\frac{1}{2}-\frac{a^2}{2}\right)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot\left(\frac{a\cdot T}{2}\right)-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot\left(1-a^2\right)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot\frac{a\cdot T}{2}-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot(1-a)\cdot(1+a)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot(1+a)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot a+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}+B\cdot T\cdot\frac{1}{2}\cdot a\right)=\\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}+B\cdot\frac{1}{2}\cdot a=\\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}(1-a)=\\
&= \frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)
\end{aligned}
$$

The a_0 coefficient equals $\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$.

The a_k coefficients are defined as:

$$
a_k=\frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.29}
$$

For the period $t\in(0;T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt=\\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}\left(-\frac{B}{(1-a)\cdot T}\cdot t+\frac{B}{1-a}\cdot a\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}-\frac{B}{(1-a)\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\int_{a\cdot T}^{T}\frac{B}{1-a}\cdot a\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \left\{\begin{array}{l} z=k\cdot\frac{2\pi}{T}\cdot t\\ dz=k\cdot\frac{2\pi}{T}\cdot dt\\ \frac{dz}{k\cdot\frac{2\pi}{T}}=dt\end{array}\right\}=
\end{aligned}
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \right.
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \int_{a \cdot T}^{T} cos\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \right.
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \int_{a \cdot T}^{T} cos\left(z\right) \cdot dz \right) =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \right.
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \, sin\left(z\right)\Big|_{a \cdot T}^{T} \right) =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \right.
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \, sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_{a \cdot T}^{T} \right) =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt + \right.
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin \left(k \cdot \frac{2\pi}{T} \cdot T \right) - sin \left(k \cdot \frac{2\pi}{T} \cdot a \cdot T \right) \right) \right) =
$$

$$
= \left\{ \begin{array}{ll} u = t & dv = cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array} \right\} =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_0^{a \cdot T} - \int_0^{a \cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) + \right.
$$

$$
- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_{a \cdot T}^{T} - \int_{a \cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) +
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin \left(k \cdot \frac{2\pi}{T} \cdot T \right) - sin \left(k \cdot \frac{2\pi}{T} \cdot a \cdot T \right) \right) \right) =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) + \right.
$$

$$
- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_{a \cdot T}^{T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a \cdot T}^{T} sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \cdot dt \right) +
$$

$$
\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin \left(k \cdot \frac{2\pi}{T} \cdot T \right) - sin \left(k \cdot \frac{2\pi}{T} \cdot a \cdot T \right) \right) \right) =
$$

$$
= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =
$$

$$
= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \Bigg|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) + \right.
$$

$$
\begin{aligned}
&- \frac{B}{(1-a)\cdot T}\cdot\left(\left. t\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right|_{a\cdot T}^{T} - \frac{1}{k\cdot\frac{2\pi}{T}}\cdot\int_{a\cdot T}^{T} sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}}\right) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(sin\left(k\cdot 2\pi\right) - sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\Bigg(\frac{A}{a\cdot T}\cdot\Bigg(\left(a\cdot T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right) - 0\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot -\right)\right) + \\
&- \frac{1}{\left(k\cdot\frac{2\pi}{T}\right)^2}\cdot\int_{0}^{a\cdot T} sin\left(z\right)\cdot dz\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\Bigg(\left(T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot T\right) - a\cdot T\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right)\right) + \\
&- \frac{1}{\left(k\cdot\frac{2\pi}{T}\right)^2}\cdot\int_{a\cdot T}^{T} sin\left(z\right)\cdot dz\Bigg) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(0 - sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\Bigg(\frac{A}{a\cdot T}\cdot\Bigg(\left(a\cdot T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) - 0\right) + \\
&- \frac{1}{k^2\cdot\frac{4\pi^2}{T^2}}\cdot\left.\left(-cos\left(z\right)\right)\right|_{0}^{a\cdot T}\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\Bigg(\left(T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\right) - a\cdot T\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\right) + \\
&- \frac{1}{k^2\cdot\frac{4\pi^2}{T^2}}\cdot\left.\left(-cos\left(z\right)\right)\right|_{a\cdot T}^{T}\Bigg) + \\
&+ \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\left(-sin\left(k\cdot 2\pi\cdot a\right)\right)\Bigg) = \\
&= \frac{2}{T}\cdot\Bigg(\frac{A}{a\cdot T}\cdot\Bigg(a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left.\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right)\right|_{0}^{a\cdot T}\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\Bigg(\frac{T^2}{k\cdot 2\pi}\cdot 0 - a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left.\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right)\right|_{a\cdot T}^{T}\Bigg) + \\
&- \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\Bigg) = \\
&= \frac{2}{T}\cdot\Bigg(\frac{A}{a\cdot T}\cdot\Bigg(a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \\
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right) + cos\left(k\cdot\frac{2\pi}{T}\cdot 0\right)\right)\Bigg) + \\
&- \frac{B}{(1-a)\cdot T}\cdot\Bigg(0 - a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) +
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot\frac{2\pi}{T}\cdot T\right)+cos\left(k\cdot\frac{2\pi}{T}\cdot a\cdot T\right)\right)\Bigg) + \\
&- \frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\Bigg) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\left(a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot 2\pi\cdot a\right)+cos\left(0\right)\right)\right)\right. + \\
&- \frac{B}{(1-a)\cdot T}\cdot\left(-a\cdot\frac{T^2}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{T^2}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot 2\pi\right)+cos\left(k\cdot 2\pi\cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\right) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot T^2\left(a\cdot\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(-cos\left(k\cdot 2\pi\cdot a\right)+1\right)\right)\right. + \\
&- \frac{B}{(1-a)\cdot T}\cdot T^2\left(-a\cdot\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(-1+cos\left(k\cdot 2\pi\cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\right) = \\
&= \frac{2}{T}\cdot\left(\frac{A}{a}\cdot T\left(a\cdot\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right)\right)\right. + \\
&+ \frac{B}{1-a}\cdot T\left(a\cdot\frac{1}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)\right) = \\
&= \frac{2\cdot A}{a}\left(\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right)\right) + \\
&+ \frac{2\cdot B}{1-a}\left(\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right)\right) + \\
&- \frac{2\cdot B}{1-a}\cdot\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) = \\
&= \frac{2\cdot A}{a}\cdot\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{2\cdot A}{a}\cdot\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) + \\
&+ \frac{2\cdot B}{1-a}\cdot\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\frac{2\cdot B}{1-a}\cdot\frac{1}{k^2\cdot 4\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) + \\
&- \frac{2\cdot B}{1-a}\cdot\frac{a}{k\cdot 2\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) = \\
&= \frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \\
&- \frac{A}{a}\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) + \\
&- \frac{B}{1-a}\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) \\
&= \frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) + \\
&- \frac{A}{a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot\frac{1}{2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) + \\
&- \frac{B}{1-a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot\frac{1}{2}\cdot\left(1-cos\left(k\cdot 2\pi\cdot a\right)\right) \\
&= \left\{\ \tfrac{1}{2}\cdot\left(1-cos\left(x\right)\right)=sin^2\left(\tfrac{x}{2}\right)\ \right\} = \\
&= \frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right) +
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{A}{a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)+\\
&-\frac{B}{1-a}\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)\\
&=\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)
\end{aligned}
$$

The a_k coefficients equal to $\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)$.
The b_k coefficients are defined as:

$$
b_k=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.30}
$$

For the period $t\in(0;T)$ we get:

$$
\begin{aligned}
b_k&=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt=\\
&=\frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}\left(-\frac{B}{(1-a)\cdot T}\cdot t+\frac{B}{1-a}\cdot a\right)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}-\frac{B}{(1-a)\cdot T}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\int_{a\cdot T}^{T}\frac{B}{1-a}\cdot a\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&=\left\{\begin{array}{l} z=k\cdot\frac{2\pi}{T}\cdot t\\ dz=k\cdot\frac{2\pi}{T}\cdot dt\\ \frac{dz}{k\cdot\frac{2\pi}{T}}=dt\end{array}\right\}=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}sin\left(z\right)\cdot\frac{dz}{k\cdot\frac{2\pi}{T}}\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\frac{1}{k\cdot\frac{2\pi}{T}}\int_{a\cdot T}^{T}sin\left(z\right)\cdot dz\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.-\frac{B}{1-a}\cdot a\cdot\frac{T}{k\cdot 2\pi}\,cos\left(z\right)|_{a\cdot T}^{T}\right)=\\
&=\frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.
\end{aligned}
$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T}\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a\cdot T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a\cdot T}^{T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{\begin{array}{ll} u = t & dv = sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = -\frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array}\right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \int_0^{a\cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \int_{a\cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a\cdot T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a\cdot T}^{T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{\begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array}\right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a\cdot T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}}\right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a\cdot T}^{T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}}\right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\,(k \cdot 2\pi) - cos\,(k \cdot 2\pi \cdot a)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) + 0 \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right) + \right.\right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_0^{a\cdot T} cos\,(z) \cdot dz\right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) + a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right) + \right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_{a\cdot T}^{T} cos\,(z) \cdot dz\right) +$$

$$
\begin{aligned}
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + 0 \right) + \right.\right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\,(z)\right)\Big|_0^{a \cdot T} \right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi) + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right) + \right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\,(z)\right)\Big|_{a \cdot T}^{T} \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Bigg|_0^{a \cdot T} \right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot 1 + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right) + \right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Bigg|_{a \cdot T}^{T} \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(T \cdot \frac{T}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \right. \\
&\left. + \ \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right. \\
&\left. + \ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi \cdot a) - sin\,(0)) \right) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \right. \\
&\left. + \ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi) - sin\,(k \cdot 2\pi \cdot a)) \right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi \cdot a) - 0)\Bigg) + \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \right. \\
&+ \left. \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (0 - sin\,(k \cdot 2\pi \cdot a))\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Bigg) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \right. \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \left. \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \left(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \right. \\
&- \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \left(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \left. \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\right) = \\
&= \frac{2}{T} \cdot \frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \left(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \frac{2}{T} \cdot \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \left(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \frac{2}{T} \cdot \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) = \\
&= \frac{A}{a} \cdot \left(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \frac{B}{1-a} \cdot \left(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\right) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) = \\
&= -\frac{A}{a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) = \\
&= -\frac{A}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) + \\
&+ \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} - \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot a \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) = \\
&= -\frac{A}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) + \\
&+ \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +
\end{aligned}
$$

$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}-\frac{B}{1-a}\cdot a\cdot\frac{1}{k\cdot\pi}=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+$$
$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}\cdot(1-a)=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+\frac{B}{k\cdot\pi}$$

The b_k coefficients equal to $\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0=\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$$
$$a_k=\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)$$
$$b_k=\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$$

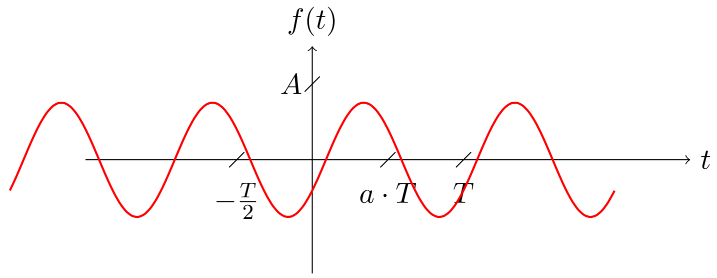
The first six coefficients are equal to:

k	a_k	b_k
1	$\frac{A}{\pi}\cdot sin\left(2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{\pi^2}\cdot sin^2\left(\pi\cdot a\right)$	$\frac{B}{\pi}-\frac{A}{\pi}\cdot cos\left(2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{2\pi^2}\cdot sin\left(2\pi\cdot a\right)$
2	$\frac{A}{2\pi}\cdot sin\left(4\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{4\pi^2}\cdot sin^2\left(2\pi\cdot a\right)$	$\frac{B}{2\pi}-\frac{A}{2\pi}\cdot cos\left(4\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{8\pi^2}\cdot sin\left(4\pi\cdot a\right)$
3	$\frac{A}{3\pi}\cdot sin\left(6\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{9\pi^2}\cdot sin^2\left(3\pi\cdot a\right)$	$\frac{B}{3\pi}-\frac{A}{3\pi}\cdot cos\left(6\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{18\pi^2}\cdot sin\left(6\pi\cdot a\right)$
4	$\frac{A}{4\pi}\cdot sin\left(8\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{16\pi^2}\cdot sin^2\left(4\pi\cdot a\right)$	$\frac{B}{4\pi}-\frac{A}{4\pi}\cdot cos\left(8\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{32\pi^2}\cdot sin\left(8\pi\cdot a\right)$
5	$\frac{A}{5\pi}\cdot sin\left(10\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{25\pi^2}\cdot sin^2\left(5\pi\cdot a\right)$	$\frac{B}{5\pi}-\frac{A}{5\pi}\cdot cos\left(10\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{50\pi^2}\cdot sin\left(10\pi\cdot a\right)$
6	$\frac{A}{6\pi}\cdot sin\left(12\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{36\pi^2}\cdot sin^2\left(6\pi\cdot a\right)$	$\frac{B}{6\pi}-\frac{A}{6\pi}\cdot cos\left(12\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{72\pi^2}\cdot sin\left(12\pi\cdot a\right)$

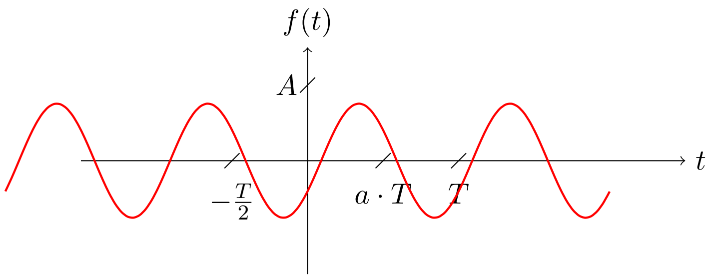
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t)=a_0+\sum_{k=1}^{\infty}\left[a_k\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+b_k\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \tag{2.31}$$

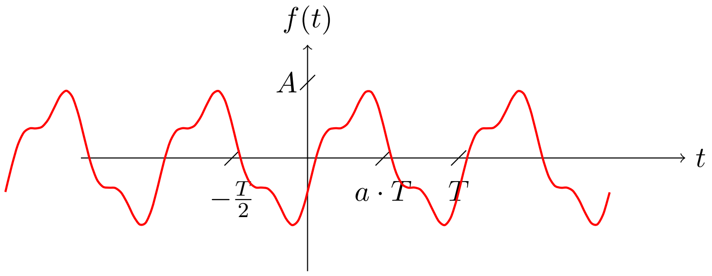
A partial approximation of the $f(t)$ signal for $k_{max}=1$ results in:



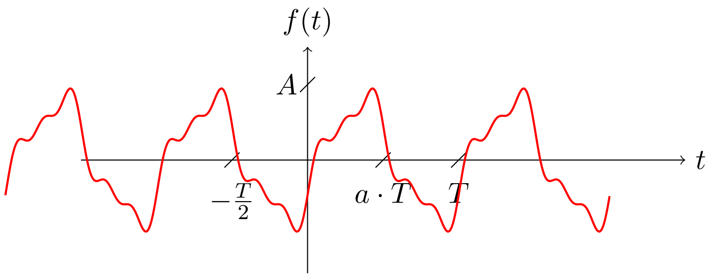
A partial approximation of the $f(t)$ signal for $k_{max}=2$ results in:



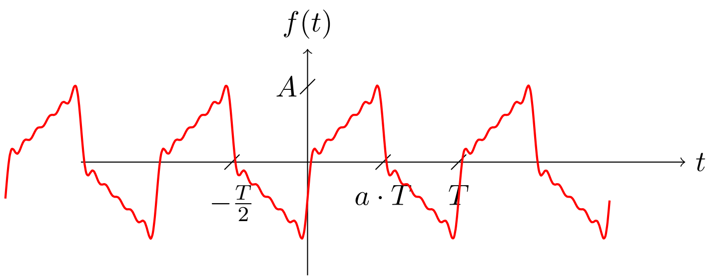
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



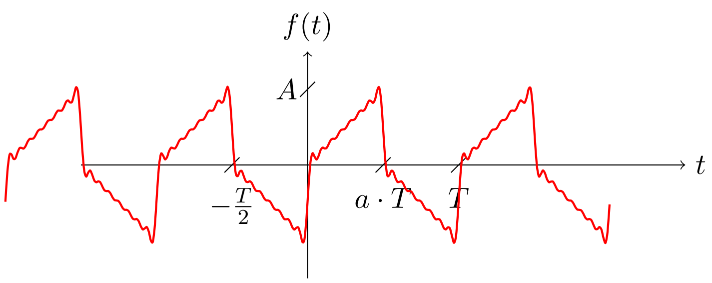
A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



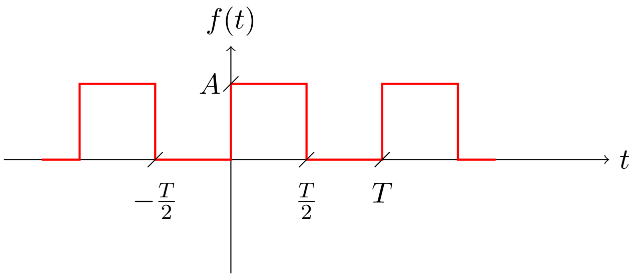
A partial approximation of the $f(t)$ signal for $k_{max} = 16$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

2.2 Complex exponential Fourier series

Task 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.32}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.33}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{1}{T} \left(A \cdot t|_0^{\frac{T}{2}} \right) = \\ &= \frac{A}{T} \cdot t|_0^{\frac{T}{2}} = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A}{2} \end{aligned} \tag{2.34}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.35}$$

For the period $t \in (0;T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \begin{Bmatrix} z &= -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz &= -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt &= \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{Bmatrix} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^{z}\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left.e^{z}\right|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left.e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^{0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - 1\right) = \\
&= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\pi} - 1\right) \\
&= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)
\end{aligned}
$$

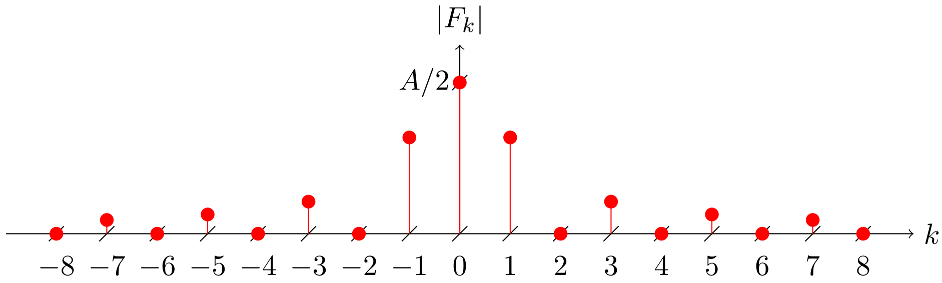
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t} \\
f(t) &= \frac{A}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[\jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left((-1)^k - 1\right)\right]\cdot e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}
\end{aligned}
\tag{2.36}
$$

The first several coefficients are equal to:

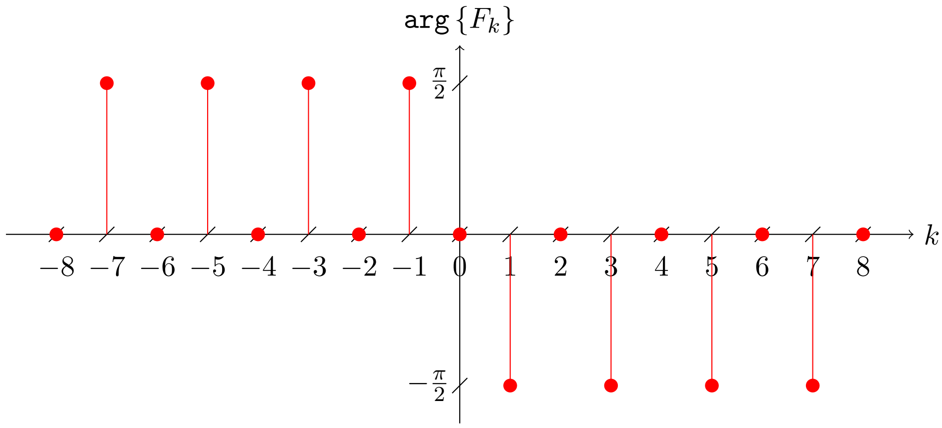
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	0	$\jmath \cdot \frac{A}{\pi}$	$\frac{A}{2}$	$-\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$
$\lvert F_k \rvert$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{2}$	$\frac{A}{\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$
$Arg\{F_k\}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



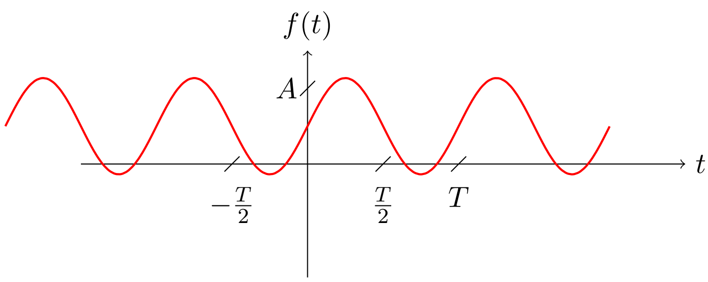
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

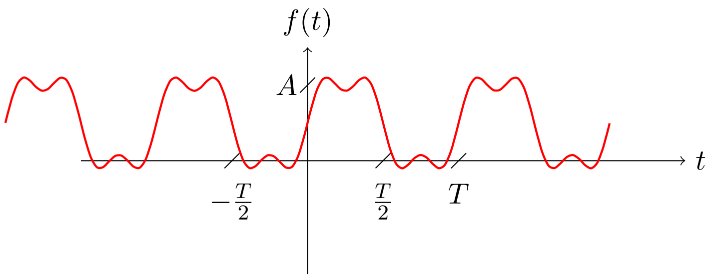


The phase spectrum of a real signal is an odd-symmetric function of k.

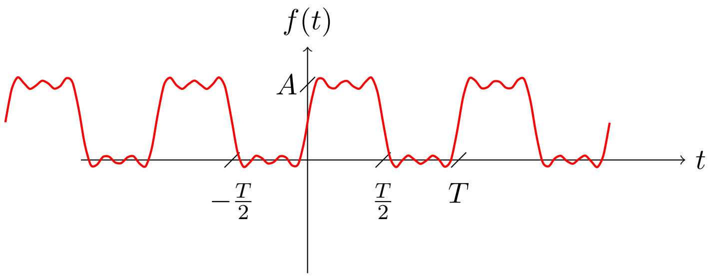
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



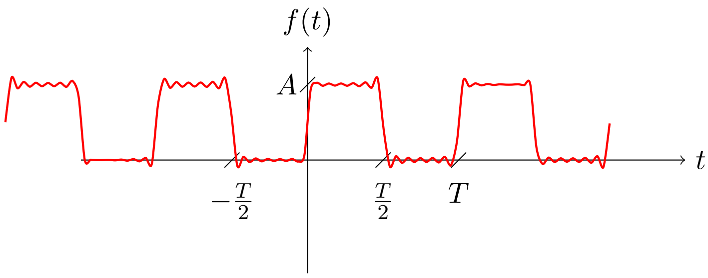
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



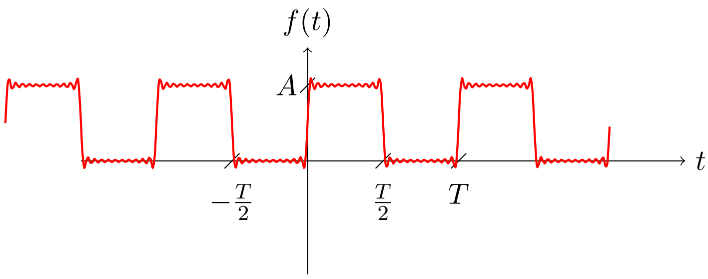
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

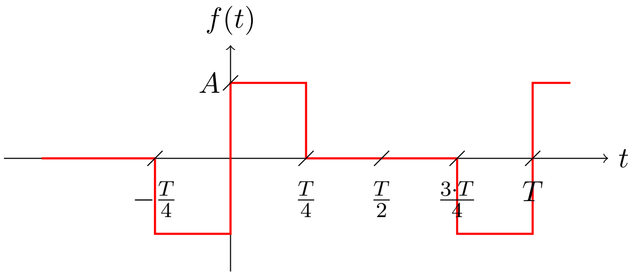


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} -A & t \in \left(-\frac{T}{4} + k \cdot T; 0 + k \cdot T\right) \\ A & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; \frac{3\cdot T}{4} + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.37}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.38}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0 \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{0} -A \cdot dt + \int_0^{\frac{T}{4}} A \cdot dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot \int_{-\frac{T}{4}}^{0} dt + A \cdot \int_0^{\frac{T}{4}} dt + 0 \right) = \\ &= \frac{1}{T} \left(-A \cdot t|_{-\frac{T}{4}}^{0} + A \cdot t|_0^{\frac{T}{4}} \right) = \\ &= \frac{1}{T} \left(-A \cdot \left(0 - \left(-\frac{T}{4}\right)\right) + A \cdot \left(\frac{T}{4} - 0\right) \right) = \\ &= \frac{1}{T} \left(-A \cdot \frac{T}{4} + A \cdot \frac{T}{4} \right) = \\ &= \frac{1}{T} (0) = \\ &= 0 \end{aligned} \tag{2.39}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.40}$$

For the period $t \in (-\frac{T}{4}; \frac{3\cdot T}{4})$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{0} -A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{0}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + A\cdot\int_{0}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} 0\cdot dt\right) = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T}\left(-A\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + A\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} + 0\right) = \\
&= \frac{1}{T}\left(-\frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{-\frac{T}{4}}^{0} e^{z}\cdot dz + \frac{A}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\int_{0}^{\frac{T}{4}} e^{z}\cdot dz\right) = \\
&= \frac{1}{T}\cdot\frac{A}{\jmath\cdot k\cdot\frac{2\pi}{T}}\cdot\left(e^{z}\big|_{-\frac{T}{4}}^{0} - e^{z}\big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{0} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{0}^{\frac{T}{4}}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0} - e^{\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(e^{0} - e^{\jmath\cdot k\cdot\frac{2\pi}{4}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{2\pi}{4}} - e^{0}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) - \left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 1\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(1 - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}} + 1\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot\left(2 - \left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right)\right) = \\
&= \frac{A}{\jmath\cdot k\cdot 2\pi}\cdot 2\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - \frac{e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}}{2}\right) = \\
&= \left\{cos(x) \quad = \frac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{A}{\jmath\cdot k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= -\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(1 - cos\left(k\cdot\frac{\pi}{2}\right)\right) = \\
&= \jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath\cdot\frac{A}{k\cdot\pi}\cdot\left(cos\left(k\cdot\frac{\pi}{2}\right) - 1\right)$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
&F_0 = 0 \\
&F_k = \jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1 \right)
\end{aligned}
\tag{2.41}
$$

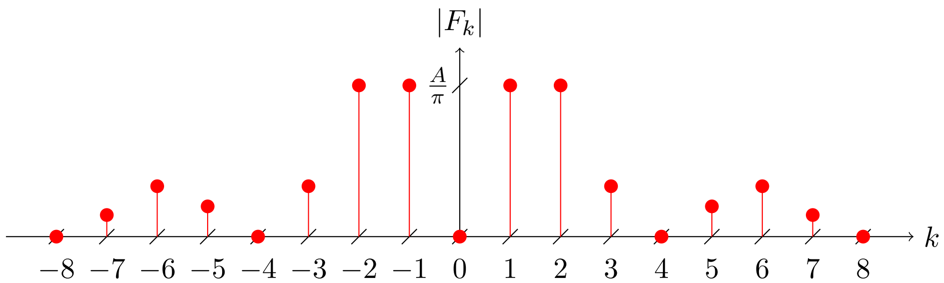
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
&f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{k \cdot \frac{2\pi}{T} \cdot t} \\
&f(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot \pi} \cdot \left(cos\left(k \cdot \frac{\pi}{2}\right) - 1 \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.42}
$$

The first several coefficients are equal to:

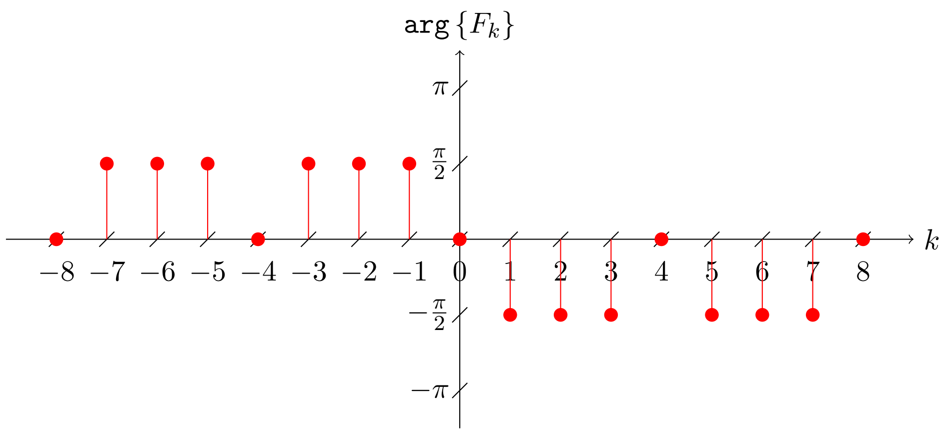
k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{5\pi}$	0	$\jmath \cdot \frac{A}{3\pi}$	$\jmath \cdot \frac{A}{\pi}$	$\jmath \cdot \frac{A}{\pi}$	0	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{\pi}$	$-\jmath \cdot \frac{A}{3\pi}$	0	$-\jmath \cdot \frac{A}{5\pi}$	$-\jmath \cdot \frac{A}{3\pi}$
$\|F_k\|$	$\frac{A}{3\pi}$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{\pi}$	$\frac{A}{\pi}$	0	$\frac{A}{\pi}$	$\frac{A}{\pi}$	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$	$\frac{A}{3\pi}$
$Arg\left\{F_k\right\}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



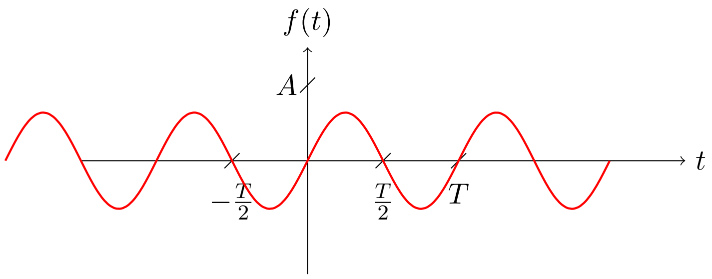
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

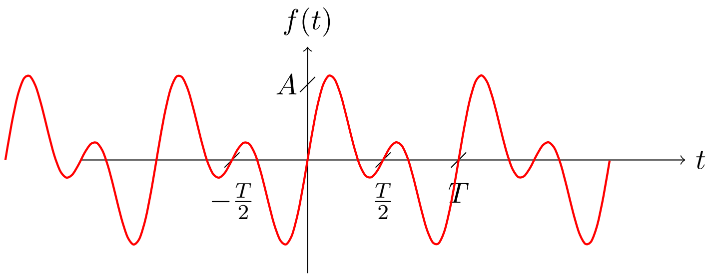


The phase spectrum of a real signal is an odd-symmetric function of k.

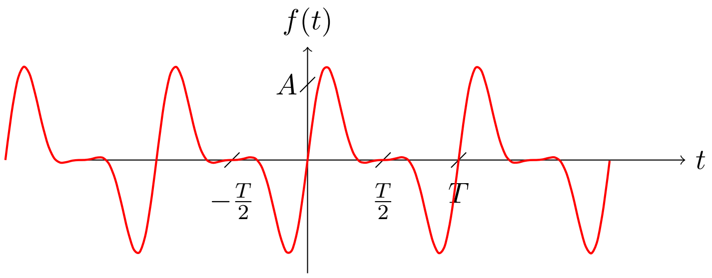
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



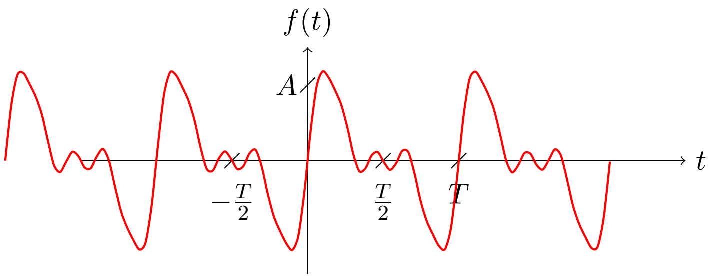
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



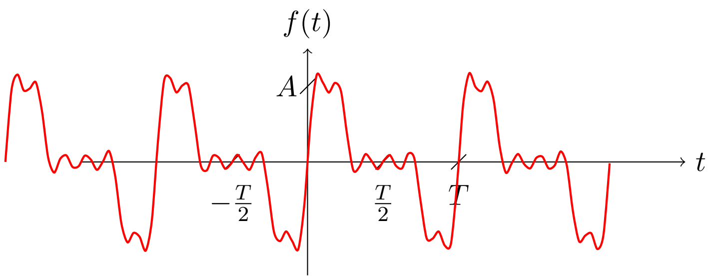
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



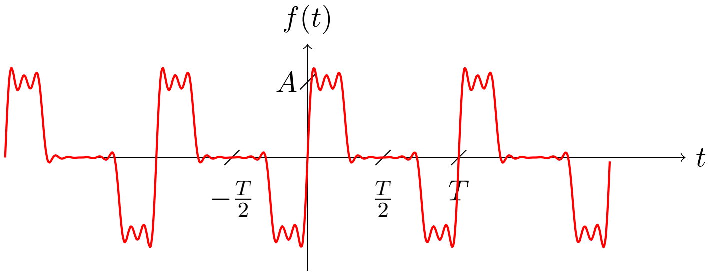
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



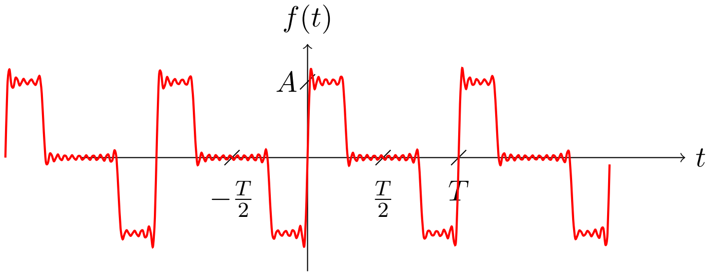
A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

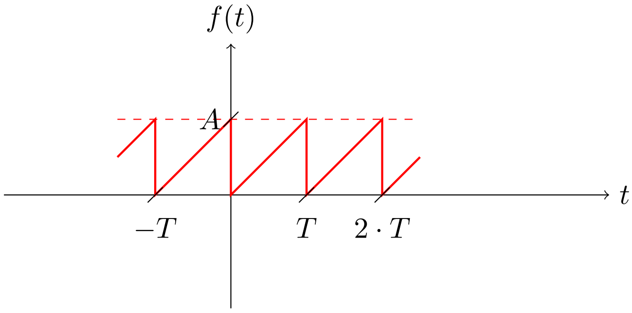


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 3. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.43}$$

In the first period (i.e. $t \in (0; T)$), linear function crosses two points: $(0,0)$ and (T, A). So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{T} = a \end{cases}$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \frac{A}{T} \cdot t$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \tfrac{A}{T} \cdot (t - k \cdot T) \quad t \in (0 + k \cdot T; T + k \cdot T) \wedge k \in Z$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.44}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T}\int_T f(t)\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot dt = \\
&= \frac{A}{T^2}\int_0^T t\cdot dt = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot t^2\Big|_0^T = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot\left(T^2-0^2\right) = \\
&= \frac{A}{T^2}\cdot\frac{1}{2}\cdot T^2 = \\
&= \frac{A}{2}
\end{aligned}$$

The F_0 coefficient equals $\frac{A}{2}$.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \tag{2.45}$$

For the period $t\in(0;T)$, i.e. $k=0$, we get:

$$\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^T \frac{A}{T}\cdot t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1\cdot A}{T^2}\int_0^T t\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \begin{Bmatrix} u & = t & dv & = e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt \\ du & = dt & v & = \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\end{Bmatrix} = \\
&= \frac{A}{T^2}\cdot\left(t\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T - \int_0^T \frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{A}{T^2}\cdot\left(\left(T\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - 0\cdot\frac{T}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) + \frac{T^2}{\left(-\jmath\cdot k\cdot 2\pi\right)^2}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^T\right) = \\
&= \frac{A}{T^2}\cdot\left(\frac{T^2}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} + \frac{T^2}{-\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot T} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi}\cdot e^{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot\left(e^{-\jmath\cdot k\cdot 2\pi} - e^0\right)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot(1-1)\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - \frac{1}{\left(k\cdot 2\pi\right)^2}\cdot 0\right) = \\
&= A\cdot\left(\frac{1}{-\jmath\cdot k\cdot 2\pi} - 0\right) =
\end{aligned}$$

$$= \frac{A}{-\jmath \cdot k \cdot 2\pi} =$$
$$= \jmath \cdot \frac{A}{k \cdot 2\pi}$$

The F_k coefficients equal to $\jmath \cdot \frac{A}{k \cdot 2\pi}$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi}$$

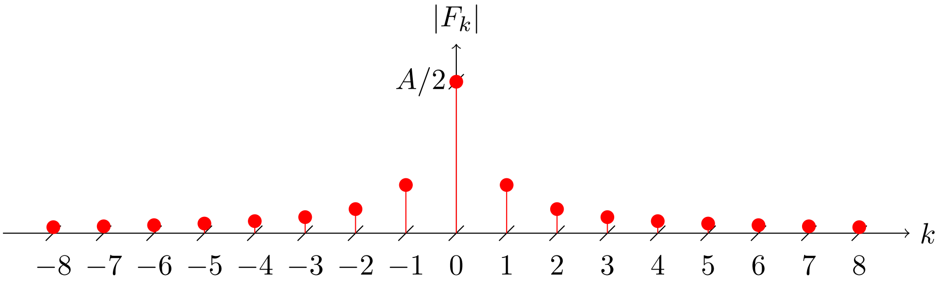
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}$$
$$f(t) = \frac{A}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\jmath \cdot \frac{A}{k \cdot 2\pi}\right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.46}$$

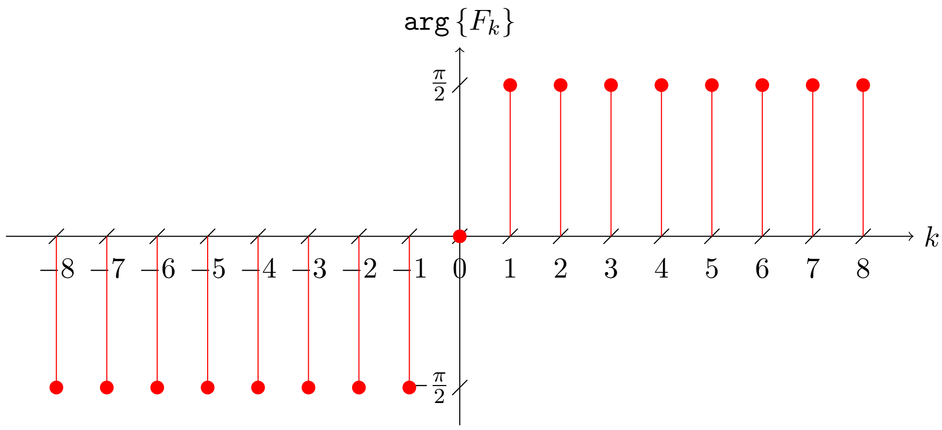
The first several coefficients are equal to:

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$-\jmath \cdot \frac{A}{10 \cdot \pi}$	$-\jmath \cdot \frac{A}{8 \cdot \pi}$	$-\jmath \cdot \frac{A}{6 \cdot \pi}$	$-\jmath \cdot \frac{A}{4 \cdot \pi}$	$-\jmath \cdot \frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\jmath \cdot \frac{A}{2 \cdot \pi}$	$\jmath \cdot \frac{A}{4 \cdot \pi}$	$\jmath \cdot \frac{A}{6 \cdot \pi}$	$\jmath \cdot \frac{A}{8 \cdot \pi}$	$\jmath \cdot \frac{A}{10 \cdot \pi}$
$\lvert F_k \rvert$	$\frac{A}{10 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{2}$	$\frac{A}{2 \cdot \pi}$	$\frac{A}{4 \cdot \pi}$	$\frac{A}{6 \cdot \pi}$	$\frac{A}{8 \cdot \pi}$	$\frac{A}{10 \cdot \pi}$
$Arg(F_k)$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

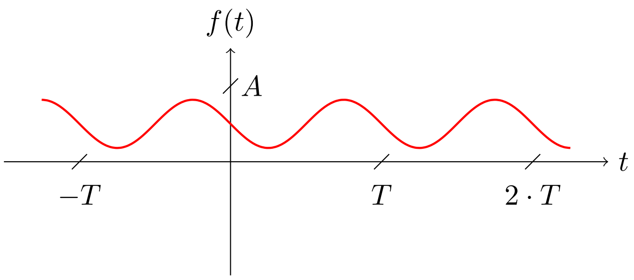


The magnitude spectrum of a real signal is an even-symmetric function of k.
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

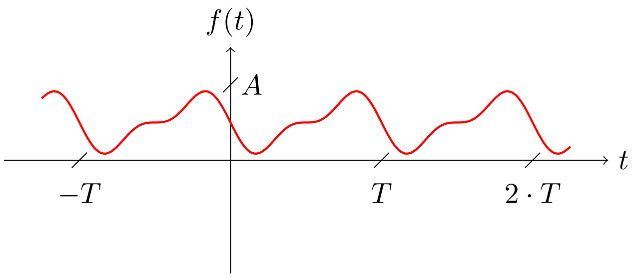


The phase spectrum of a real signal is an odd-symmetric function of k.

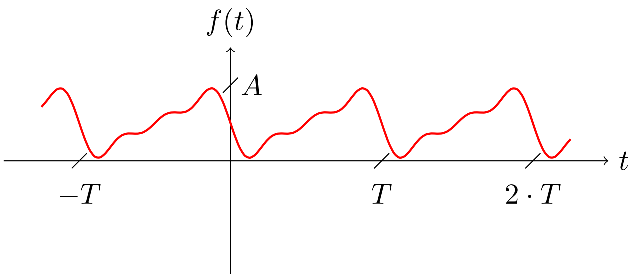
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



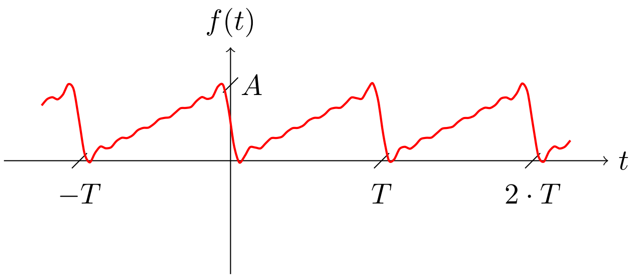
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



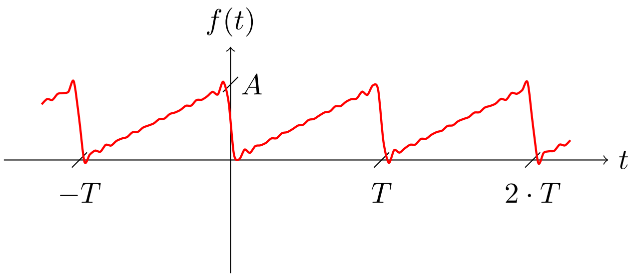
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -7$ to $k_{max} = 7$ results in:

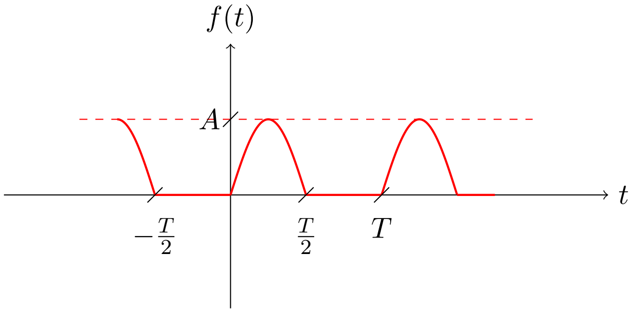


A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 4. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.47}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.48}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T} \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{A}{T} \left(\int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{2\pi}{T}} \end{array} \right\} = \\
&= \frac{A}{T} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot \frac{dz}{\frac{2\pi}{T}} = \\
&= \frac{A}{T \cdot \frac{2\pi}{T}} \int_0^{\frac{T}{2}} sin\left(z\right) \cdot dz = \\
&= \frac{A}{2\pi} \cdot \left(-cos\left(z\right)|_0^{\frac{T}{2}} \right) = \\
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} \right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{A}{2\pi} \cdot \left(cos\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - cos\left(\frac{2\pi}{T} \cdot 0\right)\right) = \\
&= -\frac{A}{2\pi} \cdot (cos\,(\pi) - cos\,(0)) = \\
&= -\frac{A}{2\pi} \cdot (-1 - 1) = \\
&= -\frac{A}{2\pi} \cdot (-2) = \\
&= \frac{A}{\pi}
\end{aligned}
$$

The F_0 coefficient equals $\frac{A}{\pi}$.

The F_k coefficients are defined as:

$$
F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.49}
$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-k)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+k)} \cdot dt \right) = \\
&= \left\{ \begin{array}{llll} z_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 & = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 & = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt & = \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} & dt & = \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \end{array} \right\} = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{dz_1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} - \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{dz_2}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\frac{1}{1-k} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{1+k} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} + \frac{1}{1+k} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1}{1-k} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^0 \right) + \frac{1}{1+k} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - e^0 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{1+k}{(1-k) \cdot (1+k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + \frac{1-k}{(1-k) \cdot (1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right)}{(1-k) \cdot (1+k)} + \frac{(1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{(1+k) \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - 1 \right) + (1-k) \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 1 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} - k + e^{-\jmath \cdot \pi \cdot (1+k)} - 1 - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)} + k}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi \cdot (1-k)} - 2 + k \cdot e^{\jmath \cdot \pi \cdot (1-k)} + e^{-\jmath \cdot \pi \cdot (1+k)} - k \cdot e^{-\jmath \cdot \pi \cdot (1+k)}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - 2 + k \cdot e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} + e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k} - k \cdot e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-1 \cdot e^{-\jmath \cdot \pi \cdot k} - 2 + k \cdot (-1) \cdot e^{-\jmath \cdot \pi \cdot k} - 1 \cdot e^{-\jmath \cdot \pi \cdot k} - k \cdot (-1) \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-e^{-\jmath \cdot \pi \cdot k} - 2 - k \cdot e^{-\jmath \cdot \pi \cdot k} - e^{-\jmath \cdot \pi \cdot k} + k \cdot e^{-\jmath \cdot \pi \cdot k}}{1-k^2} \right) = \\
&= \frac{A}{-4 \cdot \pi} \cdot \left(\frac{-2 \cdot e^{-\jmath \cdot \pi \cdot k} - 2}{1-k^2} \right) = \\
&= \frac{A}{4 \cdot \pi} \cdot \left(\frac{2 \cdot e^{-\jmath \cdot \pi \cdot k} + 2}{1-k^2} \right) = \\
&= \frac{A}{4 \cdot \pi} \cdot 2 \cdot \left(\frac{e^{-\jmath \cdot \pi \cdot k} + 1}{1-k^2} \right) = \\
&= \frac{A}{2 \cdot \pi} \cdot \left(\frac{e^{-\jmath \cdot \pi \cdot k} + 1}{1-k^2} \right) \\
&= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$ for $k \neq 1 \wedge k \neq -1$.
We have to calculate F_k for $k = 1$ directly by definition:

$$F_1 = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt =$$

$$= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$

$$= \left\{ sin\,(x) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} =$$

$$= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) =$$

$$= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) =$$

$$= \begin{Bmatrix} z & = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \end{Bmatrix} =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \int_0^{\frac{T}{2}} e^{z} \cdot \frac{dz}{-\jmath \cdot \frac{4\pi}{T}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} dt - \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(t\Big|_0^{\frac{T}{2}} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{2} - 0 \right) + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) =$$

$$= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) =$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{T}{2} + 0 \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= \frac{A}{4\jmath} = \\
&= -\jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $-\jmath \cdot \frac{A}{4}$.

We have to calculate F_k for $k = -1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot sin\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} sin\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ sin\,(x) \quad = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T} \cdot \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2\jmath} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{1}{T} \cdot \left(\frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \cdot \frac{A}{2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t + \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \right) \cdot dt = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1+1)} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot (1-1)} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 2} \cdot dt - \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t \cdot 0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} e^{0} \cdot dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt - \int_0^{\frac{T}{2}} 1 \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z & = \jmath \cdot \frac{4\pi}{T} \cdot t \\ dz & = \jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\jmath \cdot \frac{4\pi}{T}} \end{array} \right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{\jmath \cdot \frac{4\pi}{T}} - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^z \cdot dz - \int_0^{\frac{T}{2}} dt \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^z\Big|_0^{\frac{T}{2}} - t\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) - \left(\frac{T}{2} - 0 \right) \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^0 \right) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 - \frac{T}{2} \right) = \\
&= \frac{A}{T \cdot 2\jmath} \cdot \left(0 - \frac{T}{2} \right) = \\
&= -\frac{A}{T \cdot 2\jmath} \cdot \frac{T}{2} = \\
&= -\frac{A}{4\jmath} = \\
&= \jmath \cdot \frac{A}{4}
\end{aligned}
$$

The F_{-1} coefficients equal to $\jmath \cdot \frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= \frac{A}{\pi} \\
F_{-1} &= \jmath \cdot \frac{A}{4} \\
F_1 &= -\jmath \cdot \frac{A}{4} \\
F_k &= \frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}
$$

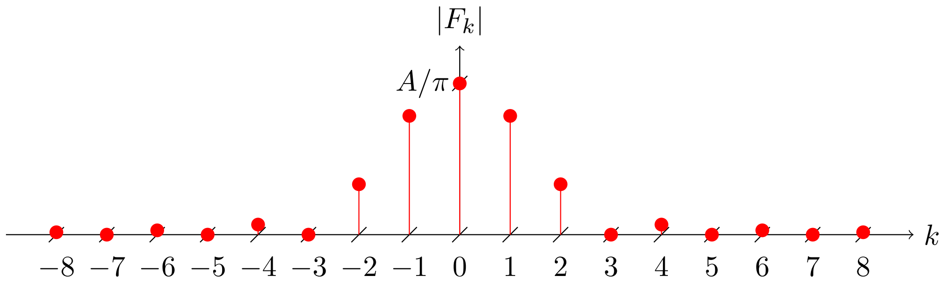
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{\pi} + \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \jmath \cdot \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\frac{A}{2 \cdot \pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \qquad (2.50)
\end{aligned}
$$

The first several coefficients are equal to:

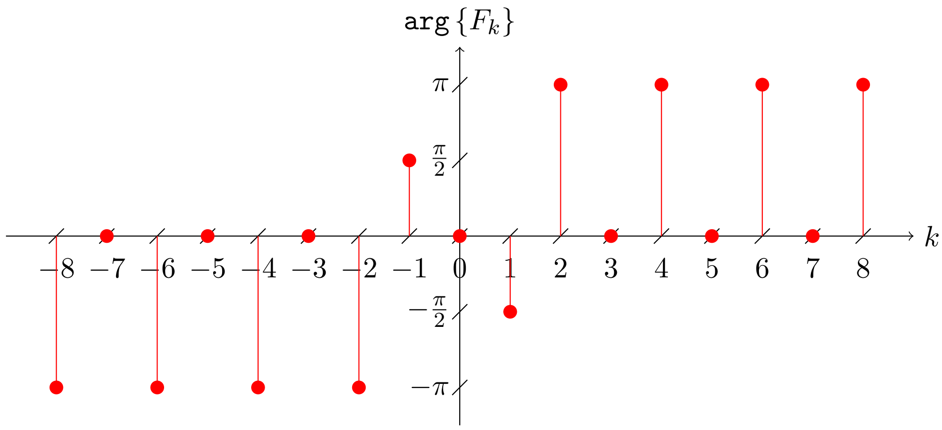
F_k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_k	$-\frac{A}{35\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{3\pi}$	$\jmath\cdot\frac{A}{4}$	$\frac{A}{\pi}$	$-\jmath\cdot\frac{A}{4}$	$-\frac{A}{3\pi}$	0	$-\frac{A}{15\pi}$	0	$-\frac{A}{35\pi}$
$\|F_k\|$	$\frac{A}{35\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{3\pi}$	$\frac{A}{4}$	$\frac{A}{\pi}$	$\frac{A}{4}$	$\frac{A}{3\pi}$	0	$\frac{A}{15\pi}$	0	$\frac{A}{35\pi}$
$Arg\left\{F_k\right\}$	$-\pi$	0	$-\pi$	0	$-\pi$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	π	0	π	0	π

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



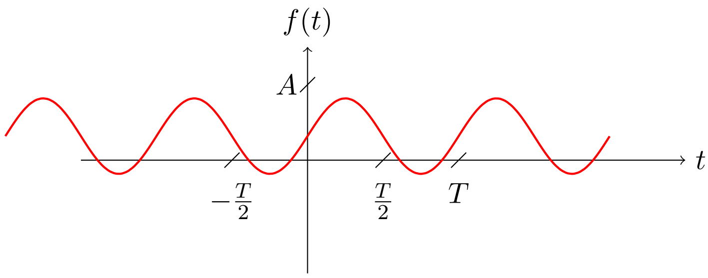
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\left\{F_k\right\}$ of the $f(t)$ signal.

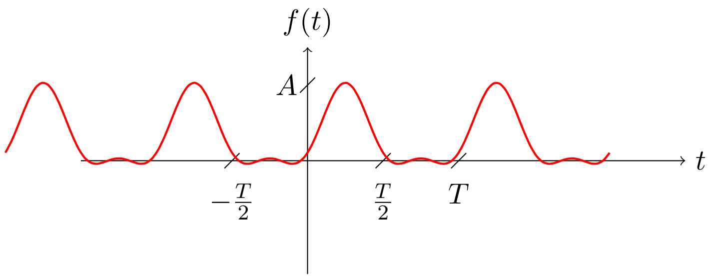


The phase spectrum of a real signal is an odd-symmetric function of k.

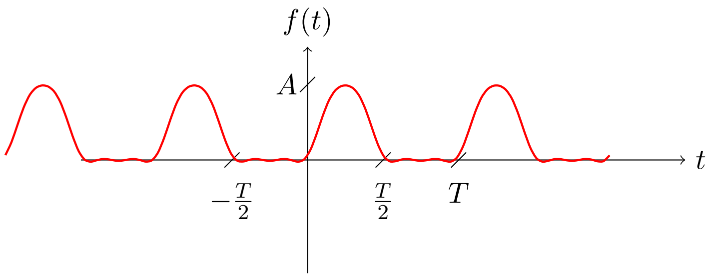
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



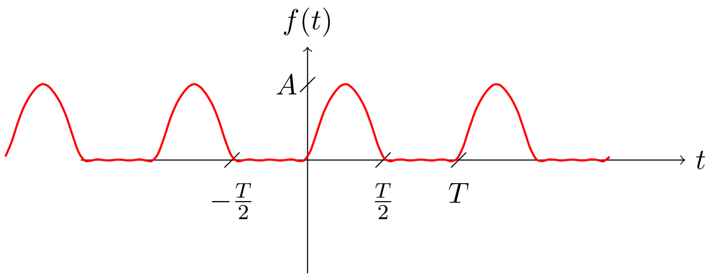
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



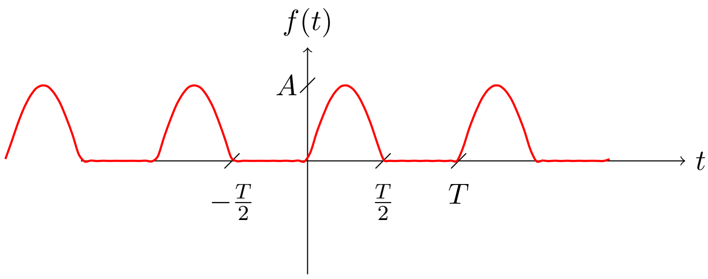
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -6$ to $k_{max} = 6$ results in:

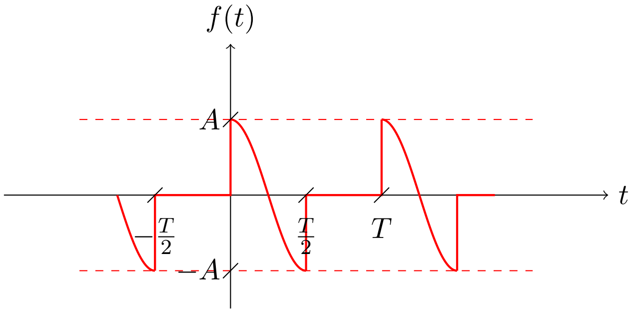


A partial approximation of the $f(t)$ signal from $k_{min} = -12$ to $k_{max} = 12$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 5. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.51}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.52}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{aligned} z &= \tfrac{2\pi}{T} \cdot t \\ dz &= \tfrac{2\pi}{T} \cdot dt \\ dt &= \tfrac{1}{\frac{2\pi}{T}} \cdot dz \\ dt &= \tfrac{T}{2\pi} \cdot dz \end{aligned} \right\} = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{A}{2\pi} \cdot (0 - 0) = \\
&= \frac{A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.53}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{cc} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{array} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) = \\
&= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) = \\
&= -\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right) \\
&= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1-k^2} \right)
\end{aligned}
$$

The F_k coefficients equal to $\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k+1}{1-k^2} \right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt+\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{0}\cdot dt+\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} 1\cdot dt+\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} dt+\int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{matrix} z=-\jmath\cdot\frac{4\pi}{T}\cdot t\\ dz=-\jmath\cdot\frac{4\pi}{T}\cdot dt\\ dt=\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\end{matrix}\right\}=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} dt+\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} dt+\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(t\Big|_0^{\frac{T}{2}}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\left(\frac{T}{2}-0\right)-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)=\\
&= \frac{A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)=\\
&= \frac{A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{A}{4}$.

We have to calculate F_k for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{1}{T}\left(A\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) = \\
&= \left\{\begin{matrix} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\} = \\
&= \frac{A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T}\cdot\frac{T}{2} =
\end{aligned}
$$

$$= \frac{A}{4}$$

The F_{-1} coefficients equal to $\frac{A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= \frac{A}{4} \\
F_{-1} &= \frac{A}{4} \\
F_k &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}$$

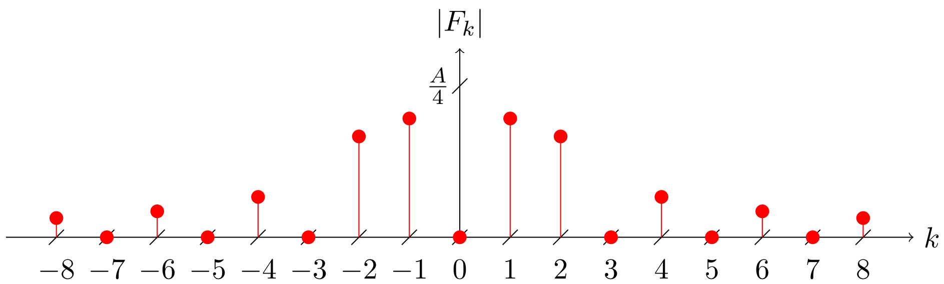
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} + \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \frac{A}{2} \cdot cos\left(\frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.54}$$

The first several coefficients are equal to:

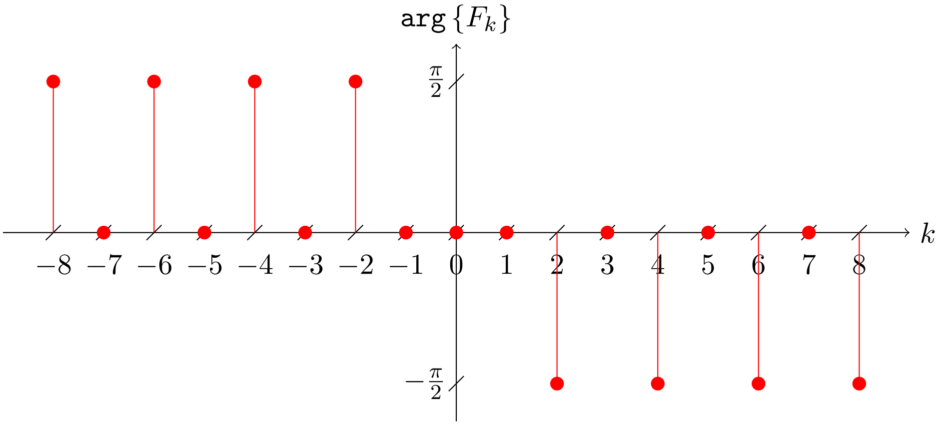
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	0	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

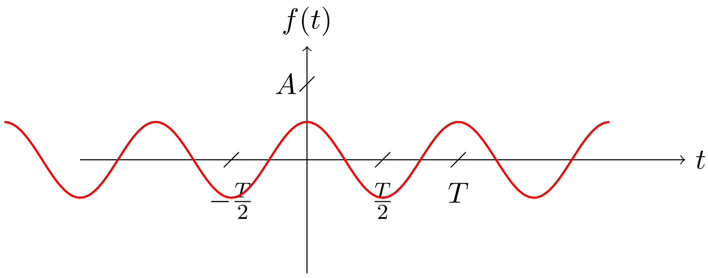


The magnitude spectrum of a real signal is an even-symmetric function of k.

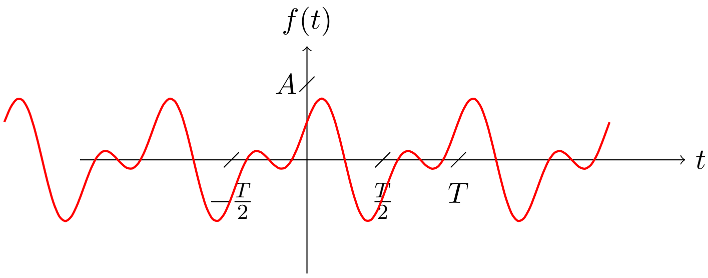
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



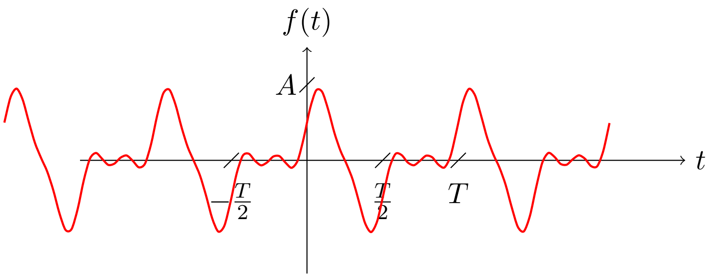
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



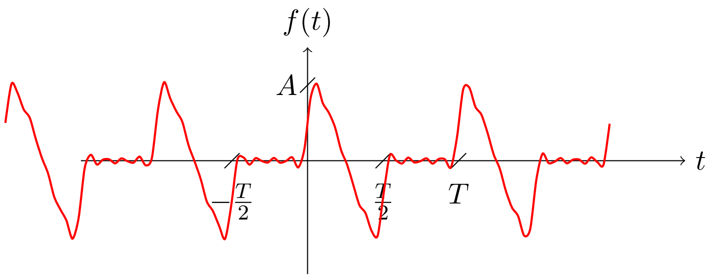
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



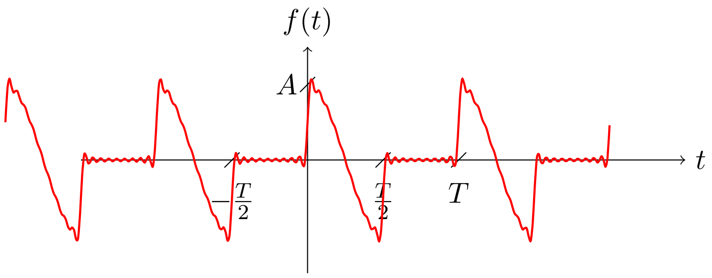
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

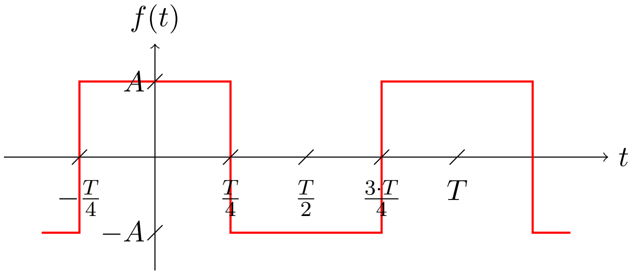


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 6. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function assuming period $t \in \left(\frac{-T}{4}; \frac{3 \cdot T}{4}\right)$ is given by:

$$f(x) = \begin{cases} A & t \in \left(-\frac{T}{4} + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ -A & t \in \left(\frac{T}{4} + k \cdot T; \frac{3 \cdot T}{4} + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.55}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.56}$$

For the period $t \in (-\frac{T}{4}; \frac{3 \cdot T}{4})$, i.e. $k = 0$, we get:

$$\begin{aligned} F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A \cdot dt + \int_{\frac{T}{4}}^{\frac{3 \cdot T}{4}} (-A) \cdot dt \right) = \\ &= \frac{1}{T} \left(A \cdot \int_{-\frac{T}{4}}^{\frac{T}{4}} dt - A \cdot \int_{\frac{T}{4}}^{\frac{3 \cdot T}{4}} dt \right) = \\ &= \frac{A}{T} \left(t|_{-\frac{T}{4}}^{\frac{T}{4}} - t|_{\frac{T}{4}}^{\frac{3 \cdot T}{4}} \right) = \\ &= \frac{A}{T} \cdot \left[\left(\frac{T}{4} - \left(-\frac{T}{4} \right) \right) - \left(\frac{3 \cdot T}{4} - \frac{T}{4} \right) \right] = \\ &= \frac{A}{T} \cdot \left[\frac{T}{4} + \frac{T}{4} - \frac{3 \cdot T}{4} + \frac{T}{4} \right] = \\ &= \frac{A}{T} \cdot [0] = \\ &= 0 \end{aligned} \tag{2.57}$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.58}$$

For the period $t \in (-\frac{T}{4}; \frac{3 \cdot T}{4})$, i.e. $k = 0$, we get:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt =$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}}(-A)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \frac{1}{T}\left(A\cdot\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - A\cdot\int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \frac{A}{T}\left(\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) =\\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t\\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt\\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\end{matrix}\right\} =\\
&= \frac{A}{T}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}}\right] =\\
&= \frac{-A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\left[\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{z}\cdot dz - \int_{\frac{T}{4}}^{\frac{3\cdot T}{4}} e^{z}\cdot dz\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{z}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{z}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left[e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_{\frac{T}{4}}^{\frac{3\cdot T}{4}}\right] =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\left(-\frac{T}{4}\right)} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{3\cdot T}{4}} + e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{4}}\right) =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \left\{e^{-\jmath\cdot k\cdot\frac{3\pi}{2}} = e^{-\jmath\cdot k\cdot\left(2\pi-\frac{\pi}{2}\right)} = e^{-\jmath\cdot k\cdot 2\pi}\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = 1\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}} = e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right\} =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{\jmath\cdot k\cdot\frac{\pi}{2}} + e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{-A}{\jmath\cdot k\cdot 2\pi}\left(2\cdot e^{-\jmath\cdot k\cdot\frac{\pi}{2}} - 2\cdot e^{\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{2\cdot A}{\jmath\cdot k\cdot 2\pi}\left(e^{\jmath\cdot k\cdot\frac{\pi}{2}} - e^{-\jmath\cdot k\cdot\frac{\pi}{2}}\right) =\\
&= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

The F_k coefficients equal to $\frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)$.
To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
F_0 &= 0\\
F_k &= \frac{2\cdot A}{k\cdot\pi}\left(sin\left(k\cdot\frac{\pi}{2}\right)\right)
\end{aligned}
$$

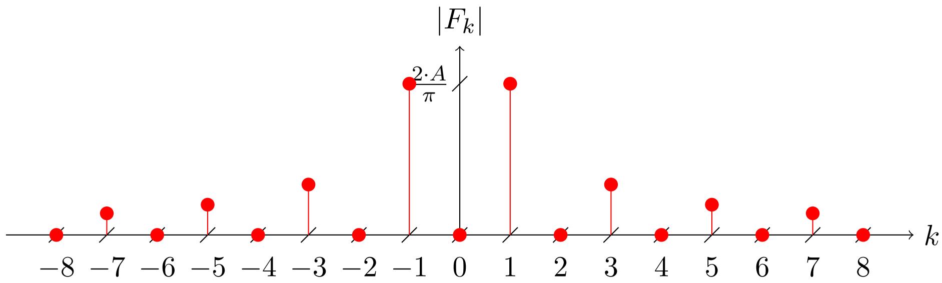
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left[\frac{2 \cdot A}{k \cdot \pi} \left(sin \left(k \cdot \frac{\pi}{2} \right) \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned}
\tag{2.59}
$$

The first several coefficients are equal to:

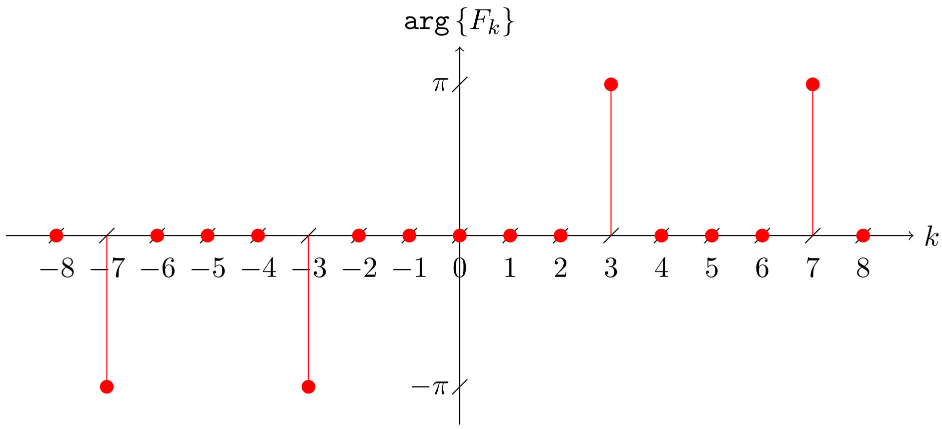
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\frac{2 \cdot A}{5 \cdot \pi}$	0	$-\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$-\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$
$\|F_k\|$	$\frac{2 \cdot A}{5 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{\pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{2 \cdot A}{5 \cdot \pi}$
$Arg\{F_k\}$	0	0	$-\pi$	0	0	0	0	0	π	0	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.



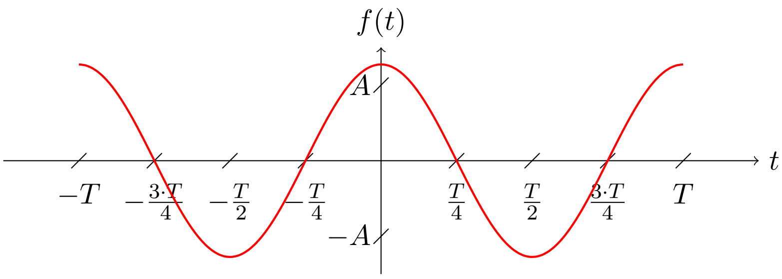
The magnitude spectrum of a real signal is an even-symmetric function of k.

Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.

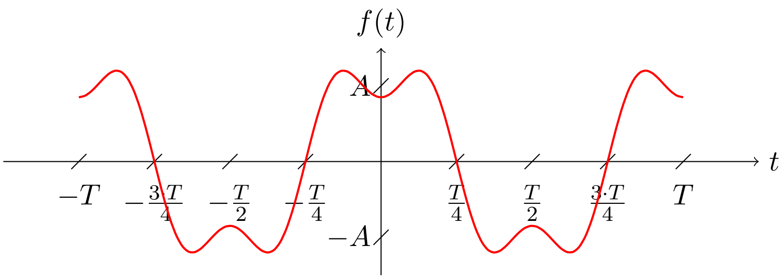


The phase spectrum of a real signal is an odd-symmetric function of k.

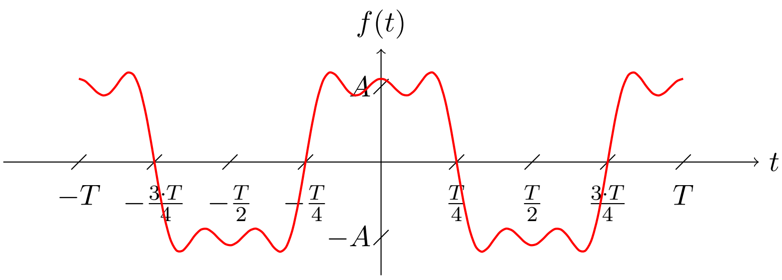
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



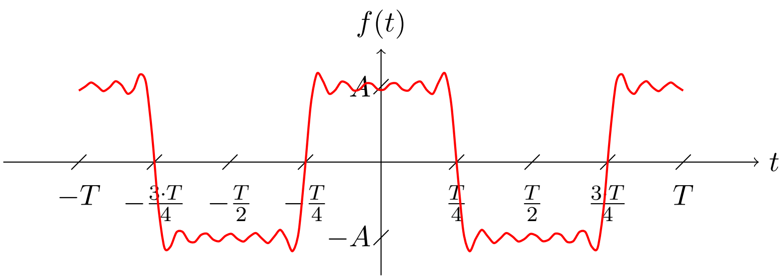
A partial approximation of the $f(t)$ signal from $k_{min} = -3$ to $k_{max} = 3$ results in:



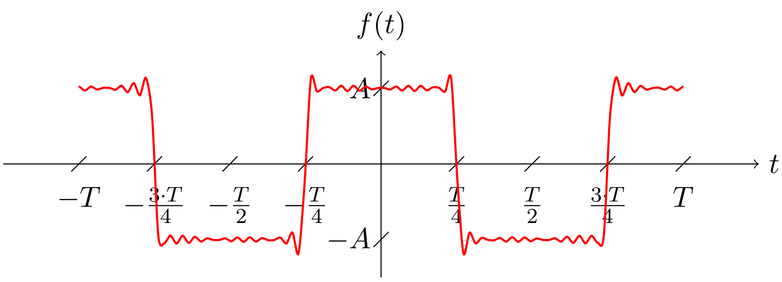
A partial approximation of the $f(t)$ signal from $k_{min} = -5$ to $k_{max} = 5$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -11$ to $k_{max} = 11$ results in:

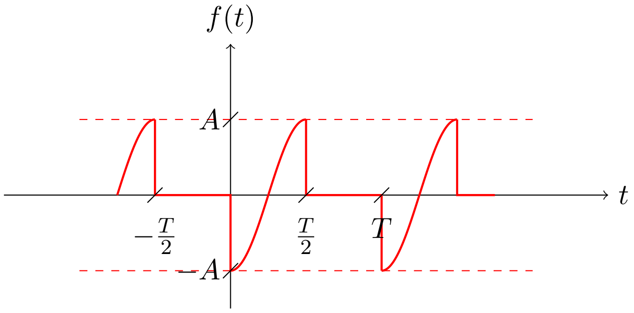


A partial approximation of the $f(t)$ signal from $k_{min} = -21$ to $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 7. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(x) = \begin{cases} -A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.60}$$

The F_0 coefficient is defined as:

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.61}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{array}{lcl} z & = & \frac{2\pi}{T} \cdot t \\ dz & = & \frac{2\pi}{T} \cdot dt \\ dt & = & \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = & \frac{T}{2\pi} \cdot dz \end{array} \right\} = \\
&= \frac{-A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{-A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{-A}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{-A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{-A}{2\pi} \cdot (0 - 0) = \\
&= \frac{-A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

The F_0 coefficient equals 0.

The F_k coefficients are defined as:

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.62}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} (-A) \cdot cos \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} cos \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left((-A) \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{matrix} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{matrix} \right\} = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{-A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot e^{z_1}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)}\cdot e^{z_2}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot t}\Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot t}\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-k)\cdot 0}\right) - \frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot\frac{T}{2}} - e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+k)\cdot 0}\right)\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\pi\cdot(1-k)} - e^{0}\right) - \frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\pi\cdot(1+k)} - 1\right)\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot k\cdot\pi} - 1\right) - \frac{1}{(1+k)}\cdot\left(e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot k\cdot\pi} - 1\right)\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(-1\cdot e^{-\jmath\cdot k\cdot\pi} - 1\right) - \frac{1}{(1+k)}\cdot\left(-1\cdot e^{-\jmath\cdot k\cdot\pi} - 1\right)\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{1}{(1-k)}\cdot\left(-e^{-\jmath\cdot k\cdot\pi} - 1\right) - \frac{1}{(1+k)}\cdot\left(-e^{-\jmath\cdot k\cdot\pi} - 1\right)\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{\left(-e^{-\jmath\cdot k\cdot\pi} - 1\right)\cdot(1+k)}{(1-k)\cdot(1+k)} - \frac{\left(-e^{-\jmath\cdot k\cdot\pi} - 1\right)\cdot(1-k)}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{-e^{-\jmath\cdot k\cdot\pi} - 1 - k\cdot e^{-\jmath\cdot k\cdot\pi} - k}{(1-k)\cdot(1+k)} - \frac{-e^{-\jmath\cdot k\cdot\pi} - 1 + k\cdot e^{-\jmath\cdot k\cdot\pi} + k}{(1-k)\cdot(1+k)}\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{-e^{-\jmath\cdot k\cdot\pi} - 1 - k\cdot e^{-\jmath\cdot k\cdot\pi} - k + e^{-\jmath\cdot k\cdot\pi} + 1 - k\cdot e^{-\jmath\cdot k\cdot\pi} - k}{1-k^2}\right) = \\
&= \frac{-A}{\jmath\cdot 4\pi}\cdot\left(\frac{-2\cdot k\cdot e^{-\jmath\cdot k\cdot\pi} - 2\cdot k}{1-k^2}\right) = \\
&= \frac{A\cdot k}{\jmath\cdot 2\pi}\cdot\left(\frac{e^{-\jmath\cdot k\cdot\pi} + 1}{1-k^2}\right) \\
&= -\jmath\cdot\frac{A\cdot k}{2\pi}\cdot\left(\frac{(-1)^k + 1}{1-k^2}\right)
\end{aligned}
$$

The F_k coefficients equal to $-\jmath\cdot\frac{A\cdot k}{2\pi}\cdot\left(\frac{(-1)^k+1}{1-k^2}\right)$.
We have to calculate F_k for $k = 1$ directly by definition:

$$
\begin{aligned}
F_1 &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}}(-A)\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot e^{-\jmath\cdot 1\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) = \\
&= \left\{cos\,(x) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}}\frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1-1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}\right)\cdot dt=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}e^{0}\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}1\cdot dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=\\
&= \left\{\begin{matrix} z=-\jmath\cdot\frac{4\pi}{T}\cdot t\\ dz=-\jmath\cdot\frac{4\pi}{T}\cdot dt\\ dt=\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\end{matrix}\right\}=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\int_0^{\frac{T}{2}}e^{z}\cdot\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}}dt+\frac{1}{-\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}}e^{z}\cdot dz\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(t|_0^{\frac{T}{2}}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\left(\frac{T}{2}-0\right)-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}}\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot 0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1)\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\left(\frac{T}{2}-0\right)=\\
&= \frac{-A}{2\cdot T}\cdot\frac{T}{2}=\\
&= \frac{-A}{4}
\end{aligned}
$$

The F_1 coefficients equal to $\frac{-A}{4}$.
We have to calculate F_k for $k=-1$ directly by definition:

$$
\begin{aligned}
F_{-1} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt=\\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}}(-A)\cdot cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{2}}^{T}0\cdot e^{-\jmath\cdot(-1)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=
\end{aligned}
$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{2}}^{T} 0\cdot dt\right) =$$

$$= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} =$$

$$= \frac{1}{T}\left((-A)\cdot\int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\int_0^{\frac{T}{2}}\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t}+e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right)\cdot dt =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} e^{0}\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} 1\cdot dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \left\{\begin{array}{c} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{array}\right\} =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\int_0^{\frac{T}{2}} e^{z}\cdot\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\int_0^{\frac{T}{2}} e^{z}\cdot dz + \int_0^{\frac{T}{2}} dt\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2}-0\right)\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot\left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot(1-1) + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(\frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot 0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\left(0 + \frac{T}{2}\right) =$$

$$= \frac{-A}{2\cdot T}\cdot\frac{T}{2} =$$

$$= \frac{-A}{4}$$

The F_{-1} coefficients equal to $\frac{-A}{4}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= \frac{-A}{4} \\
F_{-1} &= \frac{-A}{4} \\
F_k &= -\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right)
\end{aligned}$$

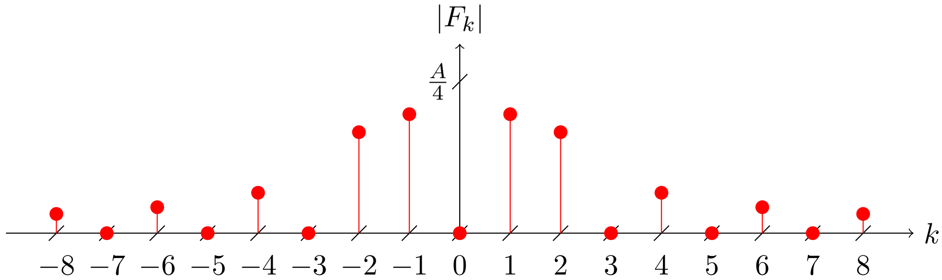
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{4} \cdot e^{\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} - \frac{A}{4} \cdot e^{\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \\
f(t) &= -\frac{A}{2} \cdot cos\left(\frac{2\pi}{T} \cdot t \right) + \sum_{\substack{k=-\infty \\ k \neq 0 \\ k \neq -1 \wedge k \neq 1}}^{\infty} \left[-\jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \right] \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t}
\end{aligned} \tag{2.63}$$

The first several coefficients are equal to:

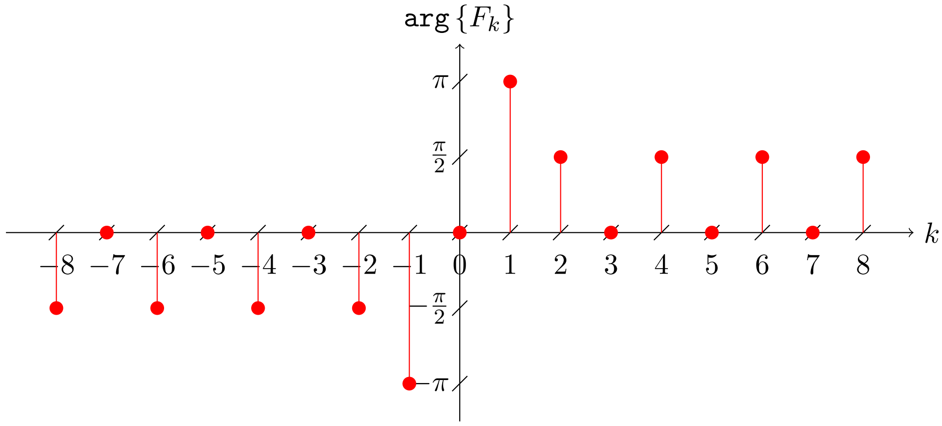
k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{-A}{4}$	0	$\frac{-A}{4}$	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$-\pi$	0	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0

Based on coefficients F_k we can plot magnitude spectrum $|F_k|$ of the $f(t)$ signal.

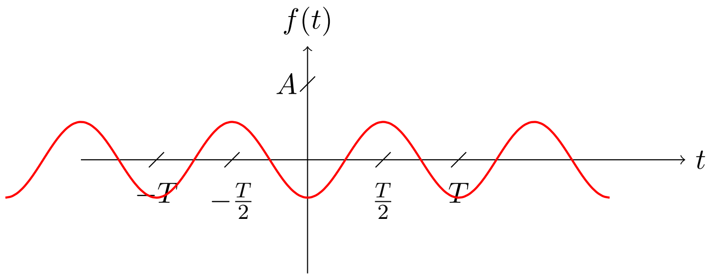


The magnitude spectrum of a real signal is an even-symmetric function of k.

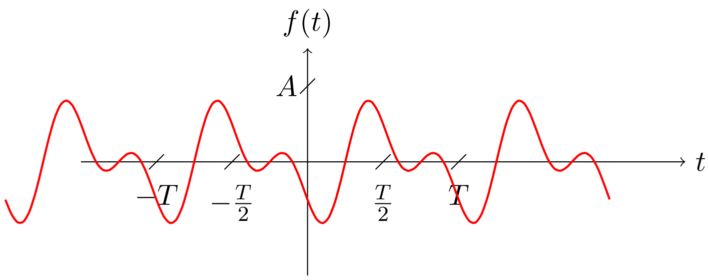
Based on coefficients F_k we can plot phase spectrum $\texttt{arg}\{F_k\}$ of the $f(t)$ signal.



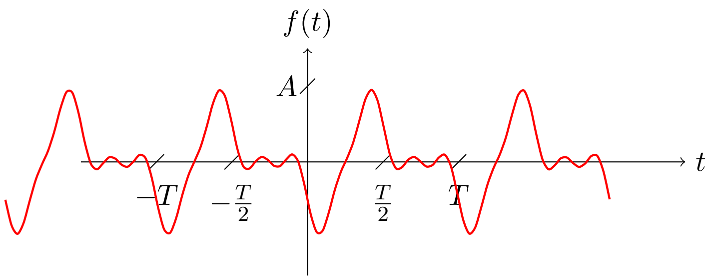
The phase spectrum of a real signal is an odd-symmetric function of k.
A partial approximation of the $f(t)$ signal from $k_{min} = -1$ to $k_{max} = 1$ results in:



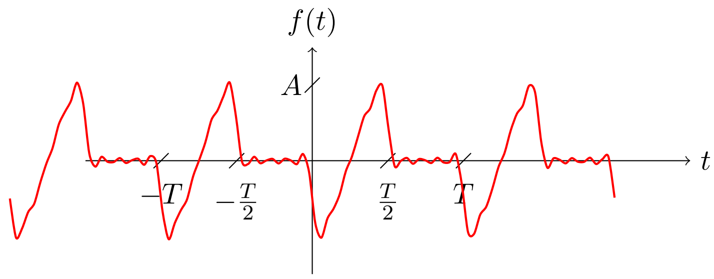
A partial approximation of the $f(t)$ signal from $k_{min} = -2$ to $k_{max} = 2$ results in:



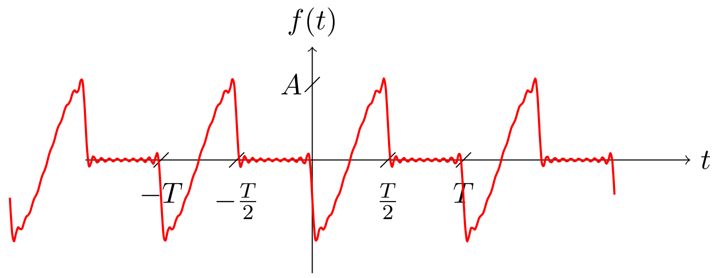
A partial approximation of the $f(t)$ signal from $k_{min} = -4$ to $k_{max} = 4$ results in:



A partial approximation of the $f(t)$ signal from $k_{min} = -10$ to $k_{max} = 10$ results in:

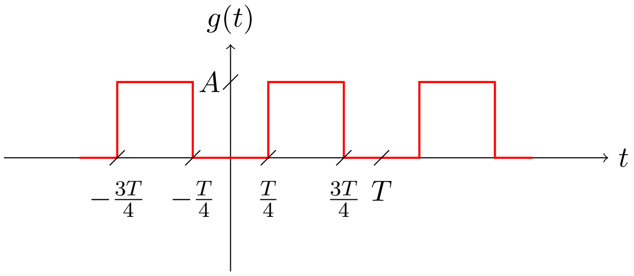


A partial approximation of the $f(t)$ signal from $k_{min} = -20$ to $k_{max} = 20$ results in:



Approximation of the $f(t)$ signal for from $k_{min} = -\infty$ to $k_{max} = \infty$ results in original signal.

Task 8. Calculate coefficients of the periodic signal $g(t)$ shown below for the expansion into a complex exponential Fourier series. Use properties of the complex series and coefficients calculated in task 1.



Periodic signal $g(t)$, as a piecewise linear function, is given by:

$$g(x) = \begin{cases} 0 & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ A & t \in \left(\frac{T}{4} + k \cdot T; \frac{3T}{4} + k \cdot T\right) \wedge k \in Z \\ 0 & t \in \left(\frac{3T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \tag{2.64}$$

You may notice, that the $g(t)$ is the $f(t)$ from task 1 shifted in time by $\frac{T}{4}$:

$$g(t) = f\left(t - \frac{T}{4}\right)$$

The coefficients for the expansion into a complex exponential Fourier series of $f(t)$ signal from task 1 are equal to:

$$F_0 = \frac{A}{2}$$
$$F_k = \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right)$$

Based on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$g(t) = f(t - t_0)$$
$$G_k = F_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0}$$

Right now the G_k coefficients are equal to:

$$\begin{aligned} G_k &= F_k \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot t_0} = \\ &= \left\{ \; t_0 = \frac{T}{4} \; \right\} = \\ &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot k \cdot \frac{T}{4}} \\ &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \cdot e^{-\jmath \cdot \pi \cdot k \cdot \frac{1}{2}} \end{aligned}$$

$$= \jmath \cdot \frac{A \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot k}}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big)$$

Similarly, we have to calculate G_0 coefficient:

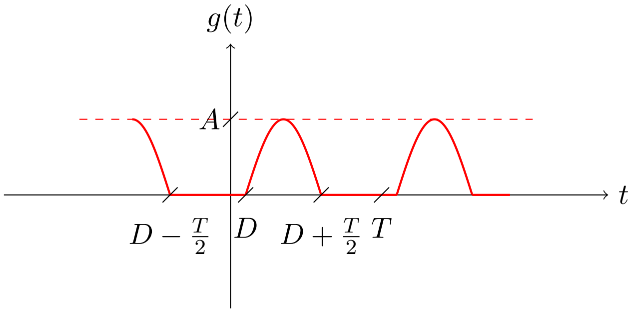
$$\begin{aligned} G_0 &= F_0 \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0 \cdot t_0} = \\ &= F_0 \cdot e^0 = \\ &= F_0 \cdot 1 = \\ &= F_0 = \\ &= \frac{A}{2} \end{aligned}$$

Please note that $G_0 = F_0$. It is obvious, if you recall, that complex exponential Fourier series coefficient for $k = 0$ is equal to mean value of the signal. The mean value does not change with shifting of the signal it time.

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$\begin{aligned} G_0 &= \frac{A}{2} \\ G_k &= \jmath \cdot \frac{A \cdot e^{-\jmath \cdot \frac{\pi}{2} \cdot k}}{k \cdot 2\pi} \cdot \Big((-1)^k - 1 \Big) \end{aligned}$$

Task 9. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{12\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.65}$$

$$\begin{aligned} g(t) &= f(t) \cdot sin\left(\frac{12\pi}{T} \cdot t\right) \\ &= \left\{ \ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \ \right\} = \\ &= f(t) \cdot \frac{e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}}{2 \cdot \jmath} \\ &= \frac{1}{2 \cdot \jmath} \left(f(t) \cdot e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - f(t) \cdot e^{-\jmath \cdot \frac{12\pi}{T} \cdot t} \right) \end{aligned}$$

$$\begin{aligned} F_0 &= \frac{A}{2} \\ F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \end{aligned}$$

$$\begin{aligned} g^1(t) &= f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0 \cdot t} \\ G_k^1 &= F_{k-k_0} \end{aligned}$$

$$\begin{aligned} g(t) &= \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^1 \cdot t} - \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^2 \cdot t} \\ g(t) &= g^1(t) - g^2(t) \\ G_k &= G_k^1 - G_k^2 \\ G_k &= \frac{1}{2 \cdot \jmath} \left(F_{k-k_0^1} - F_{k-k_0^2} \right) \end{aligned}$$

$$
\begin{aligned}
e^{\jmath\cdot\frac{12\pi}{T}\cdot t} &= e^{\jmath\cdot\frac{2\ cdot6\pi}{T}\cdot t}\\
&= e^{\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t} \Rightarrow k_0^1 = 6
\end{aligned}
$$

$$
\begin{aligned}
e^{-\jmath\cdot\frac{12\pi}{T}\cdot t} &= e^{-\jmath\cdot\frac{2\ cdot6\pi}{T}\cdot t}\\
&= e^{-\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t}\\
&= e^{\jmath\cdot\frac{2\pi}{T}\cdot(-6)\cdot t} \Rightarrow k_0^2 = -6
\end{aligned}
$$

$$
\begin{aligned}
G_k &= \frac{1}{2\cdot\jmath}\left(F_{k-k_0^1} - F_{k-k_0^2}\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-k_0^1)\cdot 2\pi}\cdot\left((-1)^{k-k_0^1}-1\right) - \jmath\cdot\frac{A}{(k-k_0^2)\cdot 2\pi}\cdot\left((-1)^{k-k_0^2}-1\right)\right) =\\
&= \left\{\ k_0^1 = 6 \quad k_0^2 = -6\ \right\} =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k-(-6))\cdot 2\pi}\cdot\left((-1)^{k-(-6)}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k+6}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{6}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right)\right) =\\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi} - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{1}{2\cdot\jmath}\cdot\jmath\cdot\frac{A}{2\pi}\left(\frac{1}{k-6}-\frac{1}{k+6}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{k+6}{(k-6)\cdot(k+6)}-\frac{k-6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{k+6-k+6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{4\pi}\left(\frac{12}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{A}{\pi}\left(\frac{3}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) =\\
&= \frac{3\cdot A}{\pi\cdot(k^2-36)}\cdot\left((-1)^{k}-1\right)
\end{aligned}
$$

$$
G_6 = \frac{1}{2\cdot\jmath}\left(F_{6-k_0^1} - F_{6-k_0^2}\right) =
$$

$$
\begin{aligned}
&= \left\{ \; k_0^1 = 6 \quad k_0^2 = -6 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \Big(F_{6-6} - F_{6-(-6)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{12} \right)
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \Big(F_0 - F_{12)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(\frac{A}{2} - \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \Big((-1)^{12} - 1 \Big) \right) = \\
&= \frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} - \frac{1}{2 \cdot \jmath} \cdot \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \Big((-1)^{12} - 1 \Big) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (1 - 1) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (0) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot 0 = \\
&= \frac{A}{4 \cdot \jmath} - 0 = \\
&= \frac{A}{4 \cdot \jmath}
\end{aligned}
$$

$$
\begin{aligned}
G_{-6} &= \frac{1}{2 \cdot \jmath} \Big(F_{-6-k_0^1} - F_{-6-k_0^2} \Big) = \\
&= \left\{ \; k_0^1 = 6 \quad k_0^2 = -6 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \Big(F_{-6-6} - F_{-6-(-6)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_{-6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_0 \right)
\end{aligned}
$$

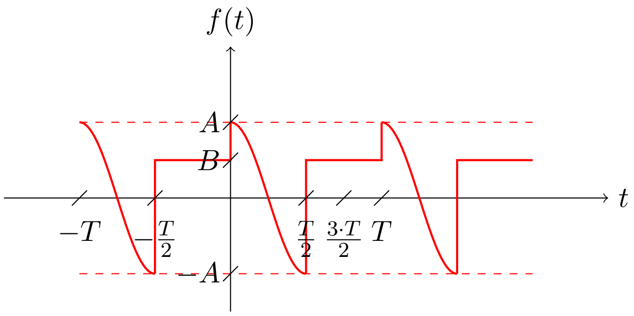
$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \Big(F_{-12} - F_{0)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{-12 \cdot 2\pi} \cdot \Big((-1)^{-12} - 1 \Big) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (1 - 1) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (0) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(0 - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(-\frac{A}{2} \right) =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\cdot \jmath}\cdot\frac{A}{2} = \\
&= -\frac{A}{4\cdot \jmath}
\end{aligned}
$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$
\begin{aligned}
G_{-6} &= -\frac{A}{4\cdot \jmath} \\
G_{6} &= \frac{A}{4\cdot \jmath} \\
G_{k} &= \frac{3\cdot A}{\pi\cdot (k^2-36)}\cdot\Big((-1)^k-1\Big)
\end{aligned}
$$

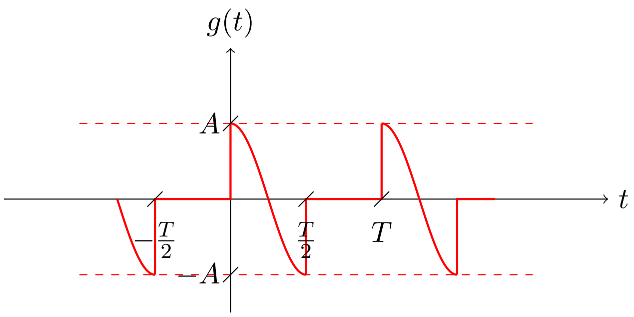
Task 10. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.

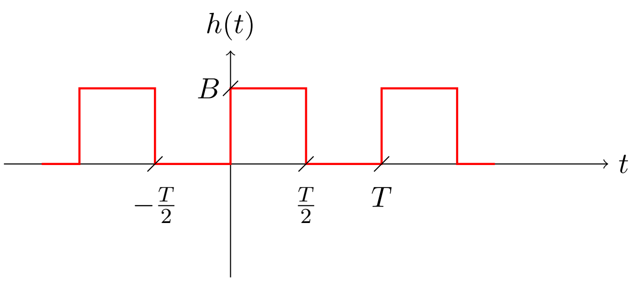


Periodic signal $f(t)$, as a piecewise function, is given by:

$$f(t) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ B & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.66}$$

If we look carefully, signal $f(t)$ may be decomposed into two signals $g(t)$ and $h(t)$ for which we have already calculated Fourier series coefficients. The signals are given below:





To be precise, the $f(t)$ signal will be the sum of $g(t)$ and $h(t)$ shifted in time by $\frac{T}{2}$:

$$f(t) = g(t) + h\left(t - \frac{T}{2}\right) \tag{2.67}$$

Based on linearity of complex exponential Fourier series and on the effect of signal shift in time on the complex exponential Fourier series, we can write:

$$F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot \frac{T}{2}}$$

$$F_k = G_k + H_k \cdot e^{-\jmath \cdot k \cdot \pi}$$
$$F_k = G_k + H_k \cdot (-1)^k$$

From previous tasks we know, that coefficients for the expansion into a complex exponential Fourier series of $g(t)$ and $h(t)$ signals are equal to:

$$\begin{aligned} G_0 &= 0 \\ G_1 &= \frac{A}{4} \\ G_{-1} &= \frac{A}{4} \\ G_k &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) \end{aligned}$$

$$\begin{aligned} H_0 &= \frac{B}{2} \\ H_k &= \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \left((-1)^k - 1 \right) \end{aligned}$$

Right now we know everything to calculate F_k coefficients:

$$\begin{aligned} F_k &= G_k + H_k \cdot (-1)^k = \\ &= \jmath \cdot \frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \jmath \cdot \frac{B}{k \cdot 2\pi} \cdot \left((-1)^k - 1 \right) \cdot (-1)^k = \\ &= \jmath \cdot \left[\frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \frac{B}{k \cdot 2\pi} \cdot \left(1 - (-1)^k \right) \right] \end{aligned}$$

Similarly, we have to calculate F_0:

$$\begin{aligned} F_0 &= G_0 + H_0 \cdot (-1)^0 = \\ &= 0 + \frac{B}{2} \cdot 1 = \\ &= \frac{B}{2} \end{aligned}$$

Also F_1 and F_{-1} have to be calculated separately:

$$F_1 = G_1 + H_1 \cdot (-1)^1 =$$

$$
\begin{aligned}
&= \frac{A}{4} + \jmath \cdot \frac{B}{1 \cdot 2\pi} \cdot \Big((-1)^1 - 1\Big) \cdot (-1) = \\
&= \frac{A}{4} + \jmath \cdot \frac{B}{\pi}
\end{aligned}
$$

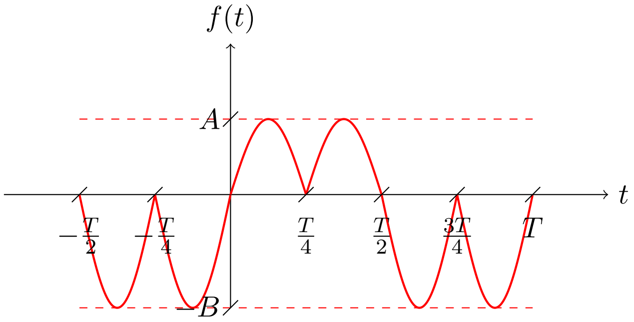
$$
\begin{aligned}
F_{-1} &= G_{-1} + H_{-1} \cdot (-1)^{-1} = \\
&= \frac{A}{4} + \jmath \cdot \frac{B}{(-1) \cdot 2\pi} \cdot \Big((-1)^{-1} - 1\Big) \cdot (-1) = \\
&= \frac{A}{4} - \jmath \cdot \frac{B}{\pi}
\end{aligned}
$$

To sum up, complex exponential Fourier coefficients for $f(t)$ are equal to:

$$
\begin{aligned}
F_0 &= \frac{B}{2} \\
F_1 &= \frac{A}{4} + \jmath \cdot \frac{B}{\pi} \\
F_{-1} &= \frac{A}{4} - \jmath \cdot \frac{B}{\pi} \\
F_k &= \jmath \cdot \left[\frac{A \cdot k}{2\pi} \cdot \left(\frac{(-1)^k + 1}{1 - k^2} \right) + \frac{B}{k \cdot 2\pi} \cdot \Big(1 - (-1)^k\Big) \right]
\end{aligned}
$$

Task 11.

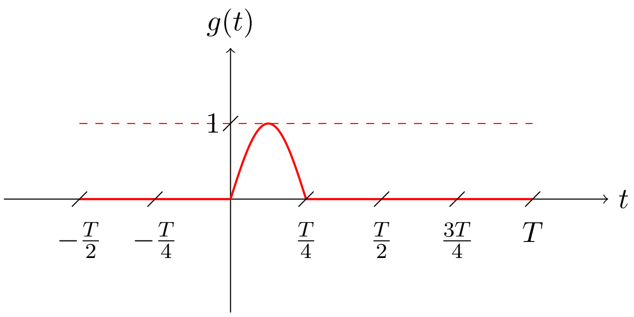
Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Use knowledge about linearity of complex exponential Fourier series and about the effect of signal shift in time on the complex exponential Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, which may be describe as:

$$f(t) = \begin{cases} A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ -A \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{4} + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{T}{2} + k \cdot T; \frac{3T}{4} + k \cdot T\right) \\ B \cdot sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(\frac{3T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.68}$$

The F_k coefficients may be calculated directly by definition. However, four integrals have to be solved, each for single interval of one period of the $f(t)$ signal. If we look carefully, signal $f(t)$ may be decomposed into linear combination of shifted in time $g(t)$ signals, for $g(t)$ signal given below:



This is periodic piecewise function, which may be describe as:

$$g(t) = \begin{cases} sin\left(\frac{4\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{4} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{4} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.69}$$

For such a definition of $g(t)$ signal, our $f(t)$ may be described as:

$$g(t) = A \cdot g\left(t\right) + A \cdot g\left(t - \frac{T}{4}\right) - B \cdot g\left(t - \frac{T}{2}\right) - B \cdot g\left(t - \frac{3T}{4}\right) \tag{2.70}$$

Right now, it is enough to calculate G_k - complex exponential Fourier coefficients of $g(t)$ signal. Then, based on linearity and on the effect of signal shift in time on the complex exponential Fourier series, we will be able to derive F_k of $f(t)$ signal.

The G_0 coefficient is defined as:

$$G_0 = \frac{1}{T}\int_T g(t) \cdot dt \tag{2.71}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
G_0 &= \frac{1}{T}\int_T g(t) \cdot dt = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt + \frac{1}{T}\int_{\frac{T}{4}}^{T} 0 \cdot dt\right) = \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt + 0\right) = \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{\begin{matrix} z & = \frac{4\pi}{T} \cdot t \\ dz & = \frac{4\pi}{T} \cdot dt \\ dt & = \frac{dz}{\frac{4\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} sin\left(z\right) \cdot \frac{dz}{\frac{4\pi}{T}} = \\
&= \frac{1}{T \cdot \frac{4\pi}{T}}\int_0^{\frac{T}{4}} sin\left(z\right) \cdot dz = \\
&= \frac{1}{4\pi} \cdot \left(-cos\left(z\right)\Big|_0^{\frac{T}{4}}\right) = \\
&= -\frac{1}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{4}}\right) = \\
&= -\frac{1}{4\pi} \cdot \left(cos\left(\frac{4\pi}{T} \cdot \frac{T}{4}\right) - cos\left(\frac{4\pi}{T} \cdot 0\right)\right) = \\
&= -\frac{1}{4\pi} \cdot \left(cos\left(\pi\right) - cos\left(0\right)\right) = \\
&= -\frac{1}{4\pi} \cdot \left(-1 - 1\right) = \\
&= -\frac{1}{4\pi} \cdot \left(-2\right) = \\
&= \frac{1}{2\pi}
\end{aligned}$$

The G_0 coefficient equals $\frac{1}{2\pi}$.

The G_k coefficients are defined as:

$$G_k = \frac{1}{T}\int_T g(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.72}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$
\begin{aligned}
G_k &= \frac{1}{T}\int_T g(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \left\{ sin\left(x\right) \quad = \frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath} \right\} = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2-k)}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2+k)}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2-k)}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot t\cdot(2+k)}\cdot dt\right) = \\
&= \left\{\begin{matrix} z_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot t & z_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot t \\ dz_1 & = \jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot dt & dz_2 & = -\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot dt \\ dt & = \frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)} & dt & = \frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)} \end{matrix}\right\} = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{z_1}\cdot\frac{dz_1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)} - \int_0^{\frac{T}{4}} e^{z_2}\cdot\frac{dz_2}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)}\cdot\int_0^{\frac{T}{4}} e^{z_1}\cdot dz_1 - \frac{1}{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)}\cdot\int_0^{\frac{T}{4}} e^{z_2}\cdot dz_2\right) = \\
&= \frac{1}{T\cdot 2\jmath\cdot\jmath\cdot\frac{2\pi}{T}}\cdot\left(\frac{1}{2-k}\cdot\int_0^{\frac{T}{4}} e^{z_1}\cdot dz_1 + \frac{1}{2+k}\cdot\int_0^{\frac{T}{4}} e^{z_2}\cdot dz_2\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot e^{z_1}\Big|_0^{\frac{T}{4}} + \frac{1}{2+k}\cdot e^{z_2}\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot t}\Big|_0^{\frac{T}{4}} + \frac{1}{2+k}\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot t}\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot\left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot\frac{T}{4}}-e^{\jmath\cdot\frac{2\pi}{T}\cdot(2-k)\cdot 0}\right) + \frac{1}{2+k}\cdot\left(e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot\frac{T}{4}}-e^{-\jmath\cdot\frac{2\pi}{T}\cdot(2+k)\cdot 0}\right)\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{1}{2-k}\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-e^{0}\right) + \frac{1}{2+k}\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-e^{0}\right)\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2+k}{(2-k)\cdot(2+k)}\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-1\right) + \frac{2-k}{(2-k)\cdot(2+k)}\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-1\right)\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{(2+k)\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-1\right)}{(2-k)\cdot(2+k)} + \frac{(2-k)\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-1\right)}{(2-k)\cdot(2+k)}\right) = \\
&= \frac{1}{-4\cdot\pi}\cdot\left(\frac{(2+k)\cdot\left(e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-1\right)+(2-k)\cdot\left(e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-1\right)}{(2-k)\cdot(2+k)}\right) =
\end{aligned}
$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-2+k\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-k+2\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-2-k\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}+k}{4-k^2}\right)=$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}-4+k\cdot e^{\jmath\cdot\frac{\pi}{2}\cdot(2-k)}+2\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}-k\cdot e^{-\jmath\cdot\frac{\pi}{2}\cdot(2+k)}}{4-k^2}\right)=$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4+k\cdot e^{\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+2\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-k\cdot e^{-\jmath\cdot\pi}\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$

$$= \begin{Bmatrix} e^{\jmath\cdot\pi} & = cos\left(\pi\right)+\jmath\cdot sin\left(\pi\right)=-1 \\ e^{-\jmath\cdot\pi} & = cos\left(\pi\right)-\jmath\cdot sin\left(\pi\right)=-1 \end{Bmatrix}=$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{2\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4+k\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+2\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-k\cdot(-1)\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{-2\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4-k\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-2\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+k\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$

$$= \frac{1}{-4\cdot\pi}\cdot\left(\frac{-4\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}-4}{4-k^2}\right)=$$

$$= \frac{1}{4\cdot\pi}\cdot\left(\frac{4\cdot e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+4}{4-k^2}\right)=$$

$$= \frac{1}{4\cdot\pi}\cdot 4\cdot\left(\frac{e^{-\jmath\cdot\frac{k\cdot\pi}{2}}+1}{4-k^2}\right)=$$

$$= \frac{1}{\pi}\cdot\left(\frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{4-k^2}\right)=$$

$$= \frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{\pi\left(4-k^2\right)}$$

The G_k coefficients are equal to $\frac{1+e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{\pi(4-k^2)}$ for $k\neq 2\wedge k\neq -2$.
We have to calculate G_k for $k=2$ directly by definition:

$$G_2 = \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt =$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot 2\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+\int_{\frac{T}{4}}^{T} 0\cdot dt\right)=$$
$$= \left\{sin\left(x\right) \quad = \frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\}=$$
$$= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt+0\right)=$$
$$= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right)=$$
$$= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=$$
$$= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}}\left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt=$$

$$
\begin{aligned}
&= \frac{1}{T \cdot 2\jmath} \cdot \int_0^{\frac{T}{4}} \left(e^{\jmath \cdot \frac{4\pi}{T} \cdot t \cdot (1-1)} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot t \cdot (1+1)} \right) \cdot dt = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t \cdot (1-1)} \cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t \cdot (1+1)} \cdot dt \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^{\jmath \cdot \frac{4\pi}{T} \cdot t \cdot 0} \cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t \cdot 2} \cdot dt \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} e^{0} \cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath \cdot \frac{8\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} 1 \cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath \cdot \frac{8\pi}{T} \cdot t} \cdot dt \right) = \\
&= \begin{Bmatrix} z & = -\jmath \cdot \frac{8\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{8\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{8\pi}{T}} \end{Bmatrix} = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} dt - \int_0^{\frac{T}{4}} e^{z} \cdot \frac{dz}{-\jmath \cdot \frac{8\pi}{T}} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\int_0^{\frac{T}{4}} dt - \frac{1}{-\jmath \cdot \frac{8\pi}{T}} \cdot \int_0^{\frac{T}{4}} e^{z} \cdot dz \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(t\Big|_0^{\frac{T}{4}} + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{4}} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\left(\frac{T}{4} - 0 \right) + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot e^{-\jmath \cdot \frac{8\pi}{T} \cdot t}\Big|_0^{\frac{T}{4}} \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{T}{4} + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{8\pi}{T} \cdot \frac{T}{4}} - e^{-\jmath \cdot \frac{8\pi}{T} \cdot 0} \right) \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{T}{4} + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) = \\
&= \left\{ e^{-\jmath \cdot 2\pi} \quad = cos\left(2\pi\right) - \jmath \cdot sin\left(2\pi\right) = 1 \right\} = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{T}{4} + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot (1-1) \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{T}{4} + \frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot 0 \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \left(\frac{T}{4} + 0 \right) = \\
&= \frac{1}{T \cdot 2\jmath} \cdot \frac{T}{4} = \\
&= \frac{1}{8\jmath} = \\
&= \frac{-\jmath}{8}
\end{aligned}
$$

The G_2 coefficients equal to $\frac{-\jmath}{8}$.

We have to calculate G_k for $k = -2$ directly by definition:

$$
\begin{aligned}
G_{-2} &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot e^{-\jmath\cdot(-2)\cdot\frac{2\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} sin\left(\frac{4\pi}{T}\cdot t\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + \int_{\frac{T}{4}}^{T} 0\cdot dt\right) = \\
&= \left\{sin\left(x\right) \quad = \tfrac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\jmath}\right\} = \\
&= \frac{1}{T}\cdot\left(\int_0^{\frac{T}{4}} \frac{e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}}{2\jmath}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt + 0\right) = \\
&= \frac{1}{T}\cdot\left(\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot dt\right) = \\
&= \frac{1}{T}\cdot\frac{1}{2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t}\cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t+\jmath\cdot\frac{4\pi}{T}\cdot t}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\int_0^{\frac{T}{4}} \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}-e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\right)\cdot dt = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1+1)}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot(1-1)}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 2}\cdot dt - \int_0^{\frac{T}{4}} e^{-\jmath\cdot\frac{4\pi}{T}\cdot t\cdot 0}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt - \int_0^{\frac{T}{4}} e^{0}\cdot dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{\jmath\cdot\frac{8\pi}{T}\cdot t}\cdot dt - \int_0^{\frac{T}{4}} 1\cdot dt\right) = \\
&= \left\{\begin{matrix} z & = \jmath\cdot\frac{8\pi}{T}\cdot t \\ dz & = \jmath\cdot\frac{8\pi}{T}\cdot dt \\ dt & = \frac{dz}{\jmath\cdot\frac{8\pi}{T}} \end{matrix}\right\} = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\int_0^{\frac{T}{4}} e^{z}\cdot\frac{dz}{\jmath\cdot\frac{8\pi}{T}} - \int_0^{\frac{T}{4}} dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\int_0^{\frac{T}{4}} e^{z}\cdot dz - \int_0^{\frac{T}{4}} dt\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{z}\Big|_0^{\frac{T}{4}} - t\Big|_0^{\frac{T}{4}}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot e^{-\jmath\cdot\frac{8\pi}{T}\cdot t}\Big|_0^{\frac{T}{4}} - \left(\frac{T}{4}-0\right)\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot\frac{8\pi}{T}\cdot\frac{T}{4}}-e^{-\jmath\cdot\frac{8\pi}{T}\cdot 0}\right) - \frac{T}{4}\right) = \\
&= \frac{1}{T\cdot 2\jmath}\cdot\left(\frac{1}{\jmath\cdot\frac{8\pi}{T}}\cdot\left(e^{-\jmath\cdot 2\pi}-e^{0}\right) - \frac{T}{4}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ e^{-\jmath\cdot 2\pi} \quad = cos\left(2\pi\right) - \jmath \cdot sin\left(2\pi\right) = 1 \right\} = \\
&= \frac{A}{T\cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot (1-1) - \frac{T}{4} \right) = \\
&= \frac{1}{T\cdot 2\jmath} \cdot \left(\frac{1}{\jmath \cdot \frac{8\pi}{T}} \cdot 0 - \frac{T}{4} \right) = \\
&= \frac{1}{T\cdot 2\jmath} \cdot \left(0 - \frac{T}{4} \right) = \\
&= -\frac{1}{T\cdot 2\jmath} \cdot \frac{T}{4} = \\
&= -\frac{1}{8\jmath} = \\
&= \frac{\jmath}{8}
\end{aligned}
$$

The G_{-2} coefficients equal to $\frac{\jmath}{8}$.

To sum up, coefficients for the expansion into a complex exponential Fourier series for $g(t)$ signal are given by:

$$
\begin{aligned}
G_0 &= \frac{1}{2\pi} \\
G_2 &= \frac{-\jmath}{8} \\
G_{-2} &= \frac{\jmath}{8} \\
G_k &= \frac{1 + e^{-\jmath\cdot\frac{k\cdot\pi}{2}}}{\pi\left(4-k^2\right)}
\end{aligned}
$$

Right now, we may go back to the description of the $f(t)$ signal with shifted in time $g(t)$ signals:

$$
f(t) = A\cdot g\left(t\right) + A\cdot g\left(t - \frac{T}{4}\right) - B\cdot g\left(t-\frac{T}{2}\right) - B\cdot g\left(t - \frac{3T}{4}\right) \tag{2.73}
$$

Recall the linearity and the effect of signal shift in time on the complex exponential Fourier series coefficients:

$$
\begin{aligned}
n(t) &\to N_k \\
m(t) &= A\cdot n(t-t_0) \\
M_k &= A\cdot N_k \cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot t_0}
\end{aligned}
$$

Applying mentioned theorems for $f(t)$ signal, we may write:

$$
\begin{aligned}
F_k &= A\cdot G_k + A\cdot G_k\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot\frac{T}{4}} - B\cdot G_k\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot\frac{T}{2}} - B\cdot G_k\cdot e^{-\jmath\cdot\frac{2\pi}{T}\cdot k\cdot\frac{3T}{4}} = \\
&= A\cdot G_k + A\cdot G_k\cdot e^{-\jmath\cdot\frac{k\pi}{2}} - B\cdot G_k\cdot e^{-\jmath\cdot k\cdot\pi} - B\cdot G_k\cdot e^{-\jmath\cdot\frac{3\cdot k\cdot\pi}{2}} =
\end{aligned}
$$

$$
\begin{aligned}
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left(e^{-\jmath \cdot k\pi} + e^{-\jmath \cdot \frac{3k\pi}{2}}\right) = \\
&= \left\{\begin{matrix} e^{-\jmath \cdot k\pi} & = cos\left(k\pi\right) + \jmath \cdot sin\left(k\pi\right) = (-1)^k \\ e^{-\jmath \cdot \frac{3 \cdot k\pi}{2}} & = e^{-\jmath \cdot \left(\frac{2 \cdot k\pi}{2} + \frac{k\pi}{2}\right)} = e^{-\jmath \cdot \frac{2 \cdot k\pi}{2}} \cdot e^{-\jmath \cdot \frac{k\pi}{2}} = (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}} \end{matrix}\right\} = \\
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot \left((-1)^k + (-1)^k \cdot e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\
&= A \cdot G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) - B \cdot G_k \cdot (-1)^k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) = \\
&= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}
$$

Now, we may insert G_k coefficients into F_k equation:

$$
\begin{aligned}
F_k &= G_k \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\
&= \frac{1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}}{\pi\left(4 - k^2\right)} \cdot \left(1 + e^{-\jmath \cdot \frac{k\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^k\right) = \\
&= \frac{\left(1 + e^{-\jmath \cdot \frac{k \cdot \pi}{2}}\right)^2}{\pi\left(4 - k^2\right)} \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}
$$

Similarly, we may calculate F_0 coefficient:

$$
\begin{aligned}
F_0 &= G_0 \cdot \left(1 + e^{-\jmath \cdot \frac{0 \cdot \pi}{2}}\right) \cdot \left(A - B \cdot (-1)^0\right) = \\
&= \frac{1}{2\pi} \cdot \left(1 + e^0\right) \cdot (A - B \cdot 1) = \\
&= \frac{1}{2\pi} \cdot (1 + 1) \cdot (A - B) = \\
&= \frac{1}{2\pi} \cdot (2) \cdot (A - B) = \\
&= \frac{A - B}{\pi}
\end{aligned}
$$

Similarly, we may calculate F_2 coefficient:

$$
\begin{aligned}
F_2 &= G_2 \cdot \left(1 + e^{-\jmath \cdot \frac{2 \cdot \pi}{2}}\right) \cdot \left(A - B \cdot (-1)^2\right) = \\
&= \frac{-\jmath}{8} \cdot \left(1 + e^{-\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{e^{-\jmath \cdot \pi} \quad = cos\left(\pi\right) - \jmath \cdot sin\left(\pi\right) = -1\right\} = \\
&= \frac{-\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{-\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}
$$

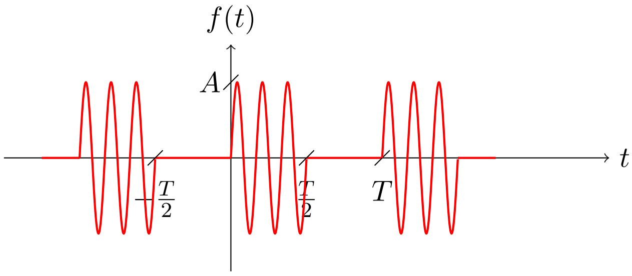
Similarly, we may calculate F_{-2} coefficient:

$$
\begin{aligned}
F_{-2} &= G_{-2} \cdot \left(1 + e^{-\jmath \cdot \frac{(-2)\cdot\pi}{2}}\right) \cdot \left(A - B \cdot (-1)^{-2}\right) = \\
&= \frac{\jmath}{8} \cdot \left(1 + e^{\jmath \cdot \pi}\right) \cdot (A - B \cdot 1) = \\
&= \left\{e^{\jmath \cdot \pi} \quad = cos\,(\pi) + \jmath \cdot sin\,(\pi) = -1\right\} = \\
&= \frac{\jmath}{8} \cdot (1 - 1) \cdot (A - B) = \\
&= \frac{\jmath}{8} \cdot (0) \cdot (A - B) = \\
&= 0
\end{aligned}
$$

To sum up, coefficients for the expansion into a complex exponential Fourier series for $f(t)$ signal are given by:

$$
\begin{aligned}
F_0 &= \frac{A - B}{\pi} \\
F_2 &= 0 \\
F_{-2} &= 0 \\
F_k &= \frac{\left(1 + e^{-\jmath \cdot \frac{k\cdot\pi}{2}}\right)^2}{\pi\,(4 - k^2)} \cdot \left(A - B \cdot (-1)^k\right)
\end{aligned}
$$

Task 12. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{12\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.74}$$

$$\begin{aligned} g(t) &= f\left(t\right) \cdot sin\left(\frac{12\pi}{T} \cdot t\right) \\ &= \left\{ \ sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \ \right\} = \\ &= f\left(t\right) \cdot \frac{e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}}{2 \cdot \jmath} \\ &= \frac{1}{2 \cdot \jmath}\left(f\left(t\right) \cdot e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - f\left(t\right) \cdot e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}\right) \end{aligned}$$

$$\begin{aligned} F_0 &= \frac{A}{2} \\ F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \end{aligned}$$

$$\begin{aligned} g^1(t) &= f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0 \cdot t} \\ G_k^1 &= F_{k-k_0} \end{aligned}$$

$$\begin{aligned} g(t) &= \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^1 \cdot t} - \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0^2 \cdot t} \\ g(t) &= g^1(t) - g^2(t) \\ G_k &= G_k^1 - G_k^2 \\ G_k &= \frac{1}{2 \cdot \jmath}\left(F_{k-k_0^1} - F_{k-k_0^2}\right) \end{aligned}$$

$$e^{\jmath \cdot \frac{12\pi}{T} \cdot t} = e^{\jmath \cdot \frac{2\ cdot6\pi}{T} \cdot t}$$

$$= e^{\jmath \cdot \frac{2\pi}{T} \cdot 6 \cdot t} \Rightarrow k_0^1 = 6$$

$$\begin{aligned}
e^{-\jmath \cdot \frac{12\pi}{T} \cdot t} &= e^{-\jmath \cdot \frac{2\ cdot 6\pi}{T} \cdot t} \\
&= e^{-\jmath \cdot \frac{2\pi}{T} \cdot 6 \cdot t} \\
&= e^{\jmath \cdot \frac{2\pi}{T} \cdot (-6) \cdot t} \Rightarrow k_0^2 = -6
\end{aligned}$$

$$\begin{aligned}
G_k &= \frac{1}{2 \cdot \jmath}\left(F_{k-k_0^1} - F_{k-k_0^2}\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-k_0^1) \cdot 2\pi} \cdot \left((-1)^{k-k_0^1} - 1\right) - \jmath \cdot \frac{A}{(k-k_0^2) \cdot 2\pi} \cdot \left((-1)^{k-k_0^2} - 1\right)\right) = \\
&= \left\{ \ k_0^1 = 6 \quad k_0^2 = -6 \ \right\} = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} \cdot \left((-1)^{k-6} - 1\right) - \jmath \cdot \frac{A}{(k-(-6)) \cdot 2\pi} \cdot \left((-1)^{k-(-6)} - 1\right)\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} \cdot \left((-1)^{k-6} - 1\right) - \jmath \cdot \frac{A}{(k+6) \cdot 2\pi} \cdot \left((-1)^{k+6} - 1\right)\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} \cdot \left((-1)^{k} \cdot (-1)^{-6} - 1\right) - \jmath \cdot \frac{A}{(k+6) \cdot 2\pi} \cdot \left((-1)^{k} \cdot (-1)^{6} - 1\right)\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} \cdot \left((-1)^{k} \cdot 1 - 1\right) - \jmath \cdot \frac{A}{(k+6) \cdot 2\pi} \cdot \left((-1)^{k} \cdot 1 - 1\right)\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} \cdot \left((-1)^{k} - 1\right) - \jmath \cdot \frac{A}{(k+6) \cdot 2\pi} \cdot \left((-1)^{k} - 1\right)\right) = \\
&= \frac{1}{2 \cdot \jmath}\left(\jmath \cdot \frac{A}{(k-6) \cdot 2\pi} - \jmath \cdot \frac{A}{(k+6) \cdot 2\pi}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{1}{2 \cdot \jmath} \cdot \jmath \cdot \frac{A}{2\pi}\left(\frac{1}{k-6} - \frac{1}{k+6}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{A}{4\pi}\left(\frac{k+6}{(k-6) \cdot (k+6)} - \frac{k-6}{(k-6) \cdot (k+6)}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{A}{4\pi}\left(\frac{k+6-k+6}{(k-6) \cdot (k+6)}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{A}{4\pi}\left(\frac{12}{k^2-36}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{A}{\pi}\left(\frac{3}{k^2-36}\right) \cdot \left((-1)^{k} - 1\right) = \\
&= \frac{3 \cdot A}{\pi \cdot (k^2-36)} \cdot \left((-1)^{k} - 1\right)
\end{aligned}$$

$$\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath}\left(F_{6-k_0^1} - F_{6-k_0^2}\right) = \\
&= \left\{ \ k_0^1 = 6 \quad k_0^2 = -6 \ \right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\cdot\jmath}\left(F_{6-6} - F_{6-(-6)}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(F_{0} - F_{6+6}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(F_{0} - F_{12}\right)
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= \frac{1}{2\cdot\jmath}\left(F_0 - F_{12)}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\frac{A}{2} - \jmath\cdot\frac{A}{12\cdot 2\pi}\cdot\left((-1)^{12}-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\cdot\frac{A}{2} - \frac{1}{2\cdot\jmath}\cdot\jmath\cdot\frac{A}{12\cdot 2\pi}\cdot\left((-1)^{12}-1\right) = \\
&= \frac{A}{4\cdot\jmath} - \frac{A}{12\cdot 4\pi}\cdot(1-1) = \\
&= \frac{A}{4\cdot\jmath} - \frac{A}{12\cdot 4\pi}\cdot(0) = \\
&= \frac{A}{4\cdot\jmath} - \frac{A}{12\cdot 4\pi}\cdot 0 = \\
&= \frac{A}{4\cdot\jmath} - 0 = \\
&= \frac{A}{4\cdot\jmath}
\end{aligned}
$$

$$
\begin{aligned}
G_{-6} &= \frac{1}{2\cdot\jmath}\left(F_{-6-k_0^1} - F_{-6-k_0^2}\right) = \\
&= \left\{\ k_0^1 = 6 \quad k_0^2 = -6\ \right\} = \\
&= \frac{1}{2\cdot\jmath}\left(F_{-6-6} - F_{-6-(-6)}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(F_{-12} - F_{-6+6}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(F_{-12} - F_{0}\right)
\end{aligned}
$$

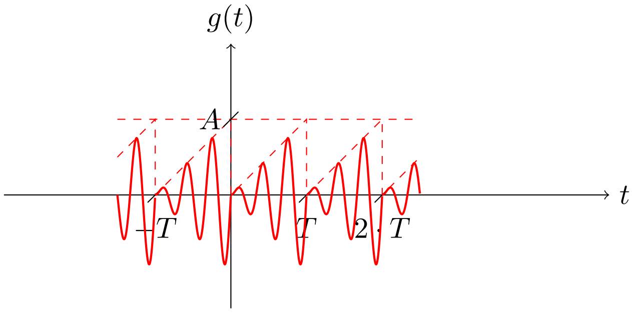
$$
\begin{aligned}
G_6 &= \frac{1}{2\cdot\jmath}\left(F_{-12} - F_{0)}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{-12\cdot 2\pi}\cdot\left((-1)^{-12}-1\right) - \frac{A}{2}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{12\cdot 2\pi}\cdot(1-1) - \frac{A}{2}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{12\cdot 2\pi}\cdot(0) - \frac{A}{2}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(0 - \frac{A}{2}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(-\frac{A}{2}\right) =
\end{aligned}
$$

$$= -\frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} =$$
$$= -\frac{A}{4 \cdot \jmath}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$G_{-6} = -\frac{A}{4 \cdot \jmath}$$
$$G_6 = \frac{A}{4 \cdot \jmath}$$
$$G_k = \frac{3 \cdot A}{\pi \cdot (k^2 - 36)} \cdot \Big((-1)^k - 1\Big)$$

Task 13. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a complex exponential Fourier series. Draw magnitude and phase spectra.



Periodic signal $f(t)$, as a piecewise linear function, is given by:

$$f(x) = \begin{cases} A \cdot sin\left(\frac{12\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.75}$$

$$\begin{aligned} g(t) &= f\left(t\right) \cdot sin\left(\frac{12\pi}{T} \cdot t\right) \\ &= \left\{ \; sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \; \right\} = \\ &= f\left(t\right) \cdot \frac{e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}}{2 \cdot \jmath} \\ &= \frac{1}{2 \cdot \jmath}\left(f\left(t\right) \cdot e^{\jmath \cdot \frac{12\pi}{T} \cdot t} - f\left(t\right) \cdot e^{-\jmath \cdot \frac{12\pi}{T} \cdot t}\right) \end{aligned}$$

$$\begin{aligned} F_0 &= \frac{A}{2} \\ F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \end{aligned}$$

$$\begin{aligned} g^1(t) &= f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k_0 \cdot t} \\ G^1_k &= F_{k-k_0} \end{aligned}$$

$$\begin{aligned} g(t) &= \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k^1_0 \cdot t} - \frac{1}{2 \cdot \jmath} f(t) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot k^2_0 \cdot t} \\ g(t) &= g^1(t) - g^2(t) \\ G_k &= G^1_k - G^2_k \\ G_k &= \frac{1}{2 \cdot \jmath}\left(F_{k-k^1_0} - F_{k-k^2_0}\right) \end{aligned}$$

$$\begin{aligned} e^{\jmath\cdot\frac{12\pi}{T}\cdot t} &= e^{\jmath\cdot\frac{2\ cdot6\pi}{T}\cdot t} \\ &= e^{\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t} \Rightarrow k_0^1 = 6 \end{aligned}$$

$$\begin{aligned} e^{-\jmath\cdot\frac{12\pi}{T}\cdot t} &= e^{-\jmath\cdot\frac{2\ cdot6\pi}{T}\cdot t} \\ &= e^{-\jmath\cdot\frac{2\pi}{T}\cdot 6\cdot t} \\ &= e^{\jmath\cdot\frac{2\pi}{T}\cdot(-6)\cdot t} \Rightarrow k_0^2 = -6 \end{aligned}$$

$$\begin{aligned}
G_k &= \frac{1}{2\cdot\jmath}\left(F_{k-k_0^1} - F_{k-k_0^2}\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-k_0^1)\cdot 2\pi}\cdot\left((-1)^{k-k_0^1}-1\right) - \jmath\cdot\frac{A}{(k-k_0^2)\cdot 2\pi}\cdot\left((-1)^{k-k_0^2}-1\right)\right) = \\
&= \left\{\ k_0^1 = 6 \quad k_0^2 = -6\ \right\} = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k-(-6))\cdot 2\pi}\cdot\left((-1)^{k-(-6)}-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k+6}-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{-6}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot(-1)^{6}-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}\cdot 1-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right) - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\cdot\left((-1)^{k}-1\right)\right) = \\
&= \frac{1}{2\cdot\jmath}\left(\jmath\cdot\frac{A}{(k-6)\cdot 2\pi} - \jmath\cdot\frac{A}{(k+6)\cdot 2\pi}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{1}{2\cdot\jmath}\cdot\jmath\cdot\frac{A}{2\pi}\left(\frac{1}{k-6} - \frac{1}{k+6}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{A}{4\pi}\left(\frac{k+6}{(k-6)\cdot(k+6)} - \frac{k-6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{A}{4\pi}\left(\frac{k+6-k+6}{(k-6)\cdot(k+6)}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{A}{4\pi}\left(\frac{12}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{A}{\pi}\left(\frac{3}{k^2-36}\right)\cdot\left((-1)^{k}-1\right) = \\
&= \frac{3\cdot A}{\pi\cdot(k^2-36)}\cdot\left((-1)^{k}-1\right)
\end{aligned}$$

$$G_6 = \frac{1}{2\cdot\jmath}\left(F_{6-k_0^1} - F_{6-k_0^2}\right) =$$

$$
\begin{aligned}
&= \left\{ \; k_0^1 = 6 \quad k_0^2 = -6 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \Big(F_{6-6} - F_{6-(-6)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_0 - F_{12} \right)
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \Big(F_0 - F_{12)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(\frac{A}{2} - \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \Big((-1)^{12} - 1 \Big) \right) = \\
&= \frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} - \frac{1}{2 \cdot \jmath} \cdot \jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot \Big((-1)^{12} - 1 \Big) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (1 - 1) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot (0) = \\
&= \frac{A}{4 \cdot \jmath} - \frac{A}{12 \cdot 4\pi} \cdot 0 = \\
&= \frac{A}{4 \cdot \jmath} - 0 = \\
&= \frac{A}{4 \cdot \jmath}
\end{aligned}
$$

$$
\begin{aligned}
G_{-6} &= \frac{1}{2 \cdot \jmath} \Big(F_{-6-k_0^1} - F_{-6-k_0^2} \Big) = \\
&= \left\{ \; k_0^1 = 6 \quad k_0^2 = -6 \; \right\} = \\
&= \frac{1}{2 \cdot \jmath} \Big(F_{-6-6} - F_{-6-(-6)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_{-6+6} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(F_{-12} - F_0 \right)
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= \frac{1}{2 \cdot \jmath} \Big(F_{-12} - F_{0)} \Big) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{-12 \cdot 2\pi} \cdot \Big((-1)^{-12} - 1 \Big) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (1 - 1) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(\jmath \cdot \frac{A}{12 \cdot 2\pi} \cdot (0) - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(0 - \frac{A}{2} \right) = \\
&= \frac{1}{2 \cdot \jmath} \left(-\frac{A}{2} \right) =
\end{aligned}
$$

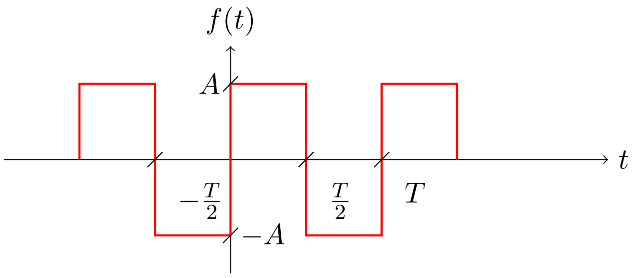
$$= -\frac{1}{2 \cdot \jmath} \cdot \frac{A}{2} =$$
$$= -\frac{A}{4 \cdot \jmath}$$

To sum up, coefficients for the expansion into a complex exponential Fourier series are given by:

$$G_{-6} = -\frac{A}{4 \cdot \jmath}$$
$$G_{6} = \frac{A}{4 \cdot \jmath}$$
$$G_{k} = \frac{3 \cdot A}{\pi \cdot (k^2 - 36)} \cdot \Big((-1)^k - 1\Big)$$

2.3 Computing the power of a signal – the Parseval's theorem

Task 1. Compute the percentage contribution of the fundamental (first) harmonic in the total power of the periodic square signal shown in the figure below:



$$\frac{P_1}{P} = ? \tag{2.76}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(t) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ -A & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.77}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.78}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |-A|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + A^2 \cdot \int_{\frac{T}{2}}^{T} dt \right) = \\ &= \frac{A^2}{T} \cdot \left(t|_0^{\frac{T}{2}} + t|_{\frac{T}{2}}^{T} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 + T - \frac{T}{2} \right) = \\ &= \frac{A^2}{T} \cdot (T) = \\ &= A^2 \end{aligned}$$

The total power of the $f(t)$ signal equals A^2.

Based on Parseval theorem, power of the fundamental harmonic is defined as:

$$P_1 = |F_1|^2 + |F_{-1}|^2 \tag{2.79}$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.80}$$

In order to calculate the P_1, the F_1 coefficient has to be calculated:

$$F_1 = \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.81}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned}
F_1 &= \frac{1}{T} \cdot \int_T f(t) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} -A \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt - \int_{\frac{T}{2}}^{T} e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{ll} z & = -\jmath \cdot \frac{2\pi}{T} \cdot t \\ dz & = -\jmath \cdot \frac{2\pi}{T} \cdot dt \\ dt & = \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \end{array} \right\} = \\
&= \frac{A}{T} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} - \int_{\frac{T}{2}}^{T} e^z \cdot \frac{dz}{-\jmath \cdot \frac{2\pi}{T}} \right) = \\
&= -\frac{A}{T \cdot \jmath \cdot \frac{2\pi}{T}} \cdot \left(\int_0^{\frac{T}{2}} e^z \cdot dz - \int_{\frac{T}{2}}^{T} e^z \cdot dz \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^z \Big|_0^{\frac{T}{2}} - e^z \Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \Big|_0^{\frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \Big|_{\frac{T}{2}}^{T} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot 0} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot T} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot \frac{T}{2}} \right) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot \left(e^{-\jmath \cdot \pi} - e^0 - e^{-\jmath \cdot 2 \cdot \pi} + e^{-\jmath \cdot \pi} \right) = \\
&= \left\{ \begin{array}{ll} e^{-\jmath \cdot 2 \cdot \pi} & = cos(2\pi) - \jmath \cdot sin(2\pi) = 1 \\ e^{-\jmath \cdot \pi} & = cos(\pi) - \jmath \cdot sin(\pi) = -1 \end{array} \right\} = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-1 - 1 - 1 - 1) = \\
&= -\frac{A}{\jmath \cdot 2\pi} \cdot (-4) = \\
&= \frac{2 \cdot A}{\jmath \cdot \pi} = \\
&= -\jmath \cdot \frac{2 \cdot A}{\pi}
\end{aligned}$$

The F_1 coefficient equals $-\jmath \cdot \frac{2 \cdot A}{\pi}$.

Thus, the P_1 may be calculated:

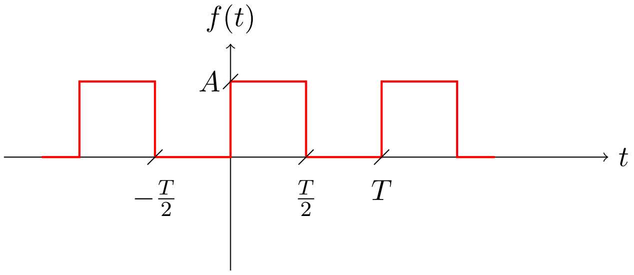
$$\begin{aligned} P_1 &= 2 \cdot |F_1|^2 = \\ &= 2 \cdot \left| -\jmath \cdot \frac{2 \cdot A}{\pi} \right|^2 = \\ &= 2 \cdot \left(\frac{2 \cdot A}{\pi} \right)^2 = \\ &= 2 \cdot \frac{4 \cdot A^2}{\pi^2} = \\ &= \frac{8 \cdot A^2}{\pi^2} \end{aligned}$$

The power of the fundamental harmonic equals $P_1 = \frac{8 \cdot A^2}{\pi^2}$.

Finally, the percentage contribution of the fundamental harmonic in the total power of the $f(t)$ signal is equal to:

$$\frac{P_1}{P} = \frac{\frac{8 \cdot A^2}{\pi^2}}{A^2} = \frac{8}{\pi^2} \approx 81\% \tag{2.82}$$

Task 2. Calculate the percentage contribution of the power of the higher harmonics ($k > 1$) to the total average power of the periodic signal shown below.



$$\frac{P_{>1}}{P} = ? \tag{2.83}$$

First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise linear function, which may be describe as:

$$f(t) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in Z \tag{2.84}$$

The total power of the signal is defined as:

$$P = \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt \tag{2.85}$$

For the period $t \in (0; T)$, i.e. $k = 0$, we get:

$$\begin{aligned} P &= \frac{1}{T} \cdot \int_T |f(t)|^2 \cdot dt = \\ &= \frac{1}{T} \cdot \left(\int_0^{\frac{T}{2}} |A|^2 \cdot dt + \int_{\frac{T}{2}}^{T} |0|^2 \cdot dt \right) = \\ &= \frac{1}{T} \cdot \left(A^2 \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(t\big|_0^{\frac{T}{2}} \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\ &= \frac{A^2}{T} \cdot \left(\frac{T}{2} \right) = \\ &= \frac{A^2}{2} \end{aligned}$$

The total power of the $f(t)$ signal equals $\frac{A^2}{2}$.

Based on Parseval theorem, the power of the higher harmonics is defined as:

$$P_{>1} = P - P_0 - P_1 \tag{2.86}$$

where:

$$\begin{aligned} P_0 &= |F_0|^2 \\ P_1 &= |F_1|^2 + |F_{-1}|^2 \end{aligned}$$

Because the $f(t) \in R$, thus $|F_1| = |F_{-1}|$ and the power of the fundamental harmonic may be calculated as

$$P_1 = 2 \cdot |F_1|^2 \tag{2.87}$$

In order to calculate P_0 and P_1, the F_0 and F_1 coefficients have to be calculated. The F_k coefficients have been calculated in task 1 and are equal to:

$$\begin{aligned} F_0 &= \frac{A}{2} \\ F_k &= \jmath \cdot \frac{A}{k \cdot 2\pi} \cdot \left((-1)^k - 1\right) \end{aligned}$$

Now, we may calculate the P_0 and P_1:

$$\begin{aligned} P_0 &= |F_0|^2 \\ &= \left|\frac{A}{2}\right|^2 \\ &= \frac{A^2}{4} \end{aligned}$$

$$\begin{aligned} P_1 &= 2 \cdot |F_1|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{1 \cdot 2\pi} \cdot \left((-1)^1 - 1\right)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{2\pi} \cdot (-1 - 1)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{A}{2\pi} \cdot (-2)\right|^2 \\ &= 2 \cdot \left|\jmath \cdot \frac{-A}{\pi}\right|^2 \\ &= 2 \cdot \left(\frac{A}{\pi}\right)^2 \\ &= 2 \cdot \frac{A^2}{\pi^2} \end{aligned}$$

Finally, the power of the higher harmonics is defined as:

$$
\begin{aligned}
P_{>1} &= P - P_0 - P_1 \\
&= \frac{A^2}{2} - \frac{A^2}{4} - 2 \cdot \frac{A^2}{\pi^2} \\
&= \frac{2 \cdot A^2 \cdot \pi^2}{4\pi^2} - \frac{A^2 \cdot \pi^2}{4\pi^2} - \frac{8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \pi^2 - 8 \cdot A^2}{4\pi^2} \\
&= \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}
\end{aligned}
$$

The power of the fundamental harmonic equals $P_{>1} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}$.

Finally, the percentage contribution of the higher harmonics in the total power of the $f(t)$ signal is equal to:

$$
\frac{P_{>1}}{P} = \frac{\frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2}}{\frac{A^2}{2}} = \frac{A^2 \cdot \left(\pi^2 - 8\right)}{4\pi^2} \cdot \frac{2}{A^2} = \frac{\pi^2 - 8}{2\pi^2} = \approx 9\% \tag{2.88}
$$

Task 3. For a certain real-valued periodic signal, its coefficients of expansion to a complex exponential Fourier series are:

$$F_k = \frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2} \wedge k > 0 \tag{2.89}$$

Compute the mean value ($\bar{f}$), knowing that the effective (RMS) value is $U = \frac{A\sqrt{6}}{60}$. During calculation use:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \tag{2.90}$$

From the theoretical considerations we know that:

$$F_0 = \bar{f}$$
$$U = \sqrt{P}$$

In order to calculate $\bar{f}$ we have to calculate F_0. But we know values of the F_k for $k > 0$ only. However, based on Parseval theorem, the power of the signal is defined as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2 \tag{2.91}$$

This equation may be rewritten as:

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2$$
$$P = \sum_{k=-\infty}^{-1} |F_k|^2 + |F_0|^2 + \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=-\infty}^{-1} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$

Because the $f(t) \in R$, thus $|F_k| = |F_{-k}|$ and we may write:

$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_{-k}|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - \sum_{k=1}^{\infty} |F_k|^2 - \sum_{k=1}^{\infty} |F_k|^2$$
$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

Now, we can calculate the F_0:

$$|F_0|^2 = P - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2$$

$$
\begin{aligned}
|F_0|^2 &= U^2 - 2 \cdot \sum_{k=1}^{\infty} |F_k|^2 \\
|F_0|^2 &= \left(\frac{A\sqrt{6}}{60}\right)^2 - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot pi^2}\right|^2 \\
|F_0|^2 &= \frac{A^2 \cdot 6}{3600} - 2 \cdot \sum_{k=1}^{\infty} \left|\frac{A}{\jmath \cdot k^2 \cdot 4 \cdot \pi^2}\right|^2 \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \left(\frac{A}{k^2 \cdot 4 \cdot \pi^2}\right)^2 \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \sum_{k=1}^{\infty} \frac{A^2}{k^4 \cdot 16 \cdot \pi^4} \\
|F_0|^2 &= \frac{A^2}{600} - 2 \cdot \frac{A^2}{16 \cdot \pi^4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^4} \\
|F_0|^2 &= \frac{A^2}{600} - \frac{A^2}{8 \cdot \pi^4} \cdot \frac{\pi^4}{90} \\
|F_0|^2 &= \frac{A^2}{600} - \frac{A^2}{720} \\
|F_0|^2 &= \frac{720 \cdot A^2}{600 \cdot 720} - \frac{600 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{720 \cdot A^2 - 600 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{120 \cdot A^2}{600 \cdot 720} \\
|F_0|^2 &= \frac{A^2}{5 \cdot 720} \\
|F_0|^2 &= \frac{A^2}{3600} \\
|F_0| &= \sqrt{\frac{A^2}{3600}} \\
|F_0| &= \frac{A}{60} \\
F_0 &= \pm\frac{A}{60}
\end{aligned}
$$

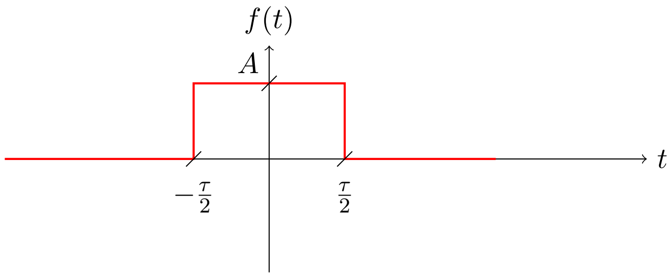
The mean value is equal to $\bar{f} = \pm\frac{A}{60}$.

Chapter 3

Analysis of non-periodic signals. Fourier Transformation and Transform

3.1 Calculation of Fourier Transform by definition

Task 1. Compute the Fourier transform of a rectangular impulse shown below. Compute and draw magnitude and phase spectra.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \Pi(\frac{t}{\tau}) \tag{3.1}$$

Which can be expresed as:

$$f(t) = \begin{cases} 0 & \text{if } t \in (-\infty; -frac\tau 2) \\ A & \text{if } t \in (-frac\tau 2; frac\tau 2) \\ 0 & \text{if } t \in (frac\tau 2; \infty) \end{cases} \tag{3.2}$$

The Fourier transform is defined as:

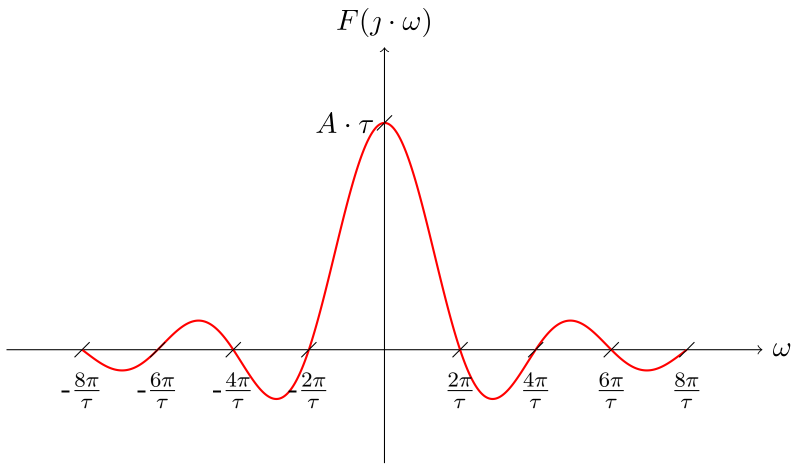
$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.3}$$

For the given $f(t)$ signal we get:

$$
\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t)\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt = \\
&= \int_{-\infty}^{\infty} A\cdot \Pi(\frac{t}{\tau})\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt = \\
&= \int_{-\infty}^{-\frac{\tau}{2}} 0\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt + \int_{\frac{\tau}{2}}^{\infty} 0\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt = \\
&= \int_{-\infty}^{-\frac{\tau}{2}} 0\cdot dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt + \int_{\frac{\tau}{2}}^{\infty} 0\cdot dt = \\
&= 0 + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt + 0 = \\
&= A\cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot\omega\cdot t \\ dz & = -\jmath\cdot\omega\cdot dt \\ dt & = \frac{1}{-\jmath\cdot\omega}\cdot dz \end{matrix}\right\} = \\
&= A\cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{z}\cdot \frac{1}{-\jmath\cdot\omega}\cdot dz = \\
&= A\cdot \frac{1}{-\jmath\cdot\omega}\cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cdot e^{z}\cdot dz = \\
&= A\cdot \frac{1}{-\jmath\cdot\omega}\cdot e^{z}|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \\
&= A\cdot \frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \\
&= \frac{A}{-\jmath\cdot\omega}\cdot \left(e^{-\jmath\cdot\omega\cdot\frac{\tau}{2}} - e^{-\jmath\cdot\omega\cdot(-\frac{\tau}{2})}\right) = \\
&= \frac{A}{\jmath\cdot\omega}\cdot \left(e^{\jmath\cdot\omega\cdot\frac{\tau}{2}} - e^{-\jmath\cdot\omega\cdot\frac{\tau}{2}}\right) = \\
&= \left\{sin(x) = \frac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath}\right\} = \\
&= \frac{2\cdot A}{\omega}\cdot sin\left(\omega\cdot\frac{\tau}{2}\right) = \\
&= \left\{\frac{sin(x)}{x} = Sa(x)\right\} = \\
&= A\cdot\tau\cdot Sa\left(\omega\cdot\frac{\tau}{2}\right)
\end{aligned}
$$

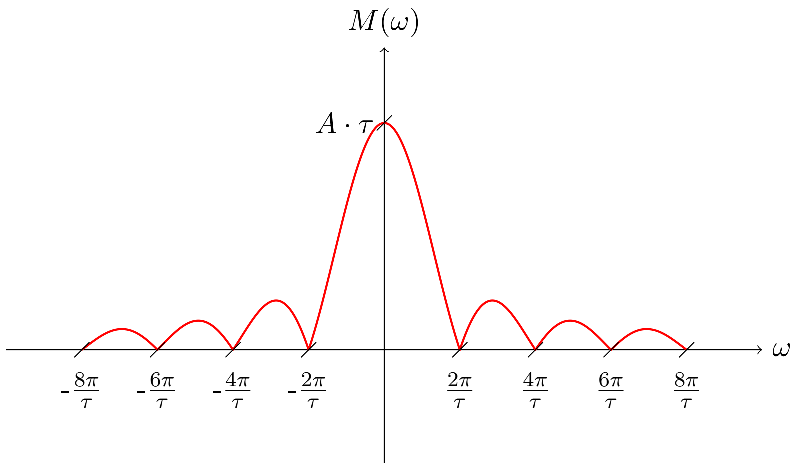
The Fourier transform of the $f(t) = A\cdot\Pi(\frac{t}{\tau})$ is equal to $F(\jmath\omega) = A\cdot\tau\cdot Sa\left(\omega\cdot\frac{\tau}{2}\right)$.
Draw complex spectrum of the $f(t) = A\cdot\Pi(\frac{t}{\tau})$:

$$
F(\jmath\omega) = A\cdot\tau\cdot Sa\left(\omega\cdot\frac{\tau}{2}\right) \tag{3.4}
$$



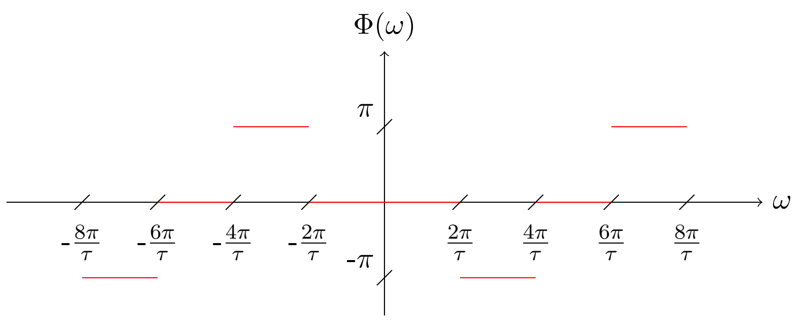
The magnitude spectrum is defined as:

$$M(\omega) = |F(j \cdot \omega)| \tag{3.5}$$



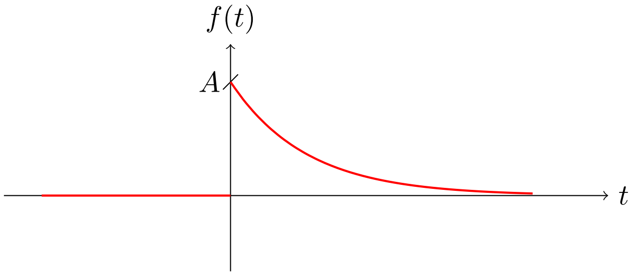
The magnitude spectrum of a real signal is an even-symmetric function of k.
The phase spectrum is defined as:

$$\Phi(\omega) = arctan2(\frac{Im\{F(j \cdot \omega)\}}{Re\{F(j \cdot \omega)\}}) \tag{3.6}$$



The phase spectrum of a real signal is an odd-symmetric function of k.

Task 2. Compute the Fourier transform of a impulse shown below. Compute and draw magnitude and phase spectra.



The signal $f(t)$, as a piecewise function, is given by:

$$f(t) = \begin{cases} 0 & \text{if } t \in (-\infty; 0) \\ A \cdot e^{-a \cdot t} & \text{if } t \in (0; \infty) \end{cases} \tag{3.7}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.8}$$

For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 0 + \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{0}^{\infty} A \cdot e^{-a \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{0}^{\infty} e^{-(a + \jmath \cdot \omega) \cdot t} \cdot dt = \\
&= \lim_{\tau \to \infty} A \cdot \int_{0}^{\tau} e^{-(a + \jmath \cdot \omega) \cdot t} \cdot dt = \\
&= \left\{ \begin{array}{ll} z & = -(a + \jmath \cdot \omega) \cdot t \\ dz & = -(a + \jmath \cdot \omega) \cdot dt \\ dt & = \frac{1}{-(a + \jmath \cdot \omega)} \cdot dz \end{array} \right\} = \\
&= \lim_{\tau \to \infty} A \cdot \int_{0}^{\tau} e^{z} \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot dz = \\
&= A \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} \int_{0}^{\tau} e^{z} \cdot dz = \\
&= A \cdot \frac{1}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} e^{z} \big|_{0}^{\tau} = \\
&= \frac{A}{-(a + \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} e^{-(a + \jmath \cdot \omega) \cdot t} \Big|_{0}^{\tau} =
\end{aligned}$$

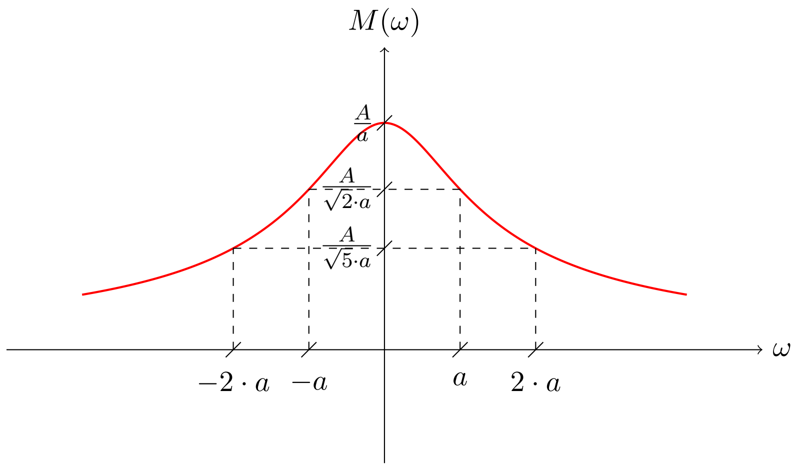
$$
\begin{aligned}
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau} - e^{-(a+\jmath\cdot\omega)\cdot 0}\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau} - e^{0}\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\lim_{\tau\to\infty}\left(e^{-(a+\jmath\cdot\omega)\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty} e^{-(a+\jmath\cdot\omega)\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty} e^{-a\cdot\tau+\jmath\cdot\omega\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty} e^{-a\cdot\tau}\cdot e^{\jmath\cdot\omega\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(\lim_{\tau\to\infty} e^{-a\cdot\tau}\cdot\lim_{\tau\to\infty} e^{\jmath\cdot\omega\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot\left(0\cdot\lim_{\tau\to\infty} e^{\jmath\cdot\omega\cdot\tau} - 1\right) = \\
&= \frac{A}{-(a+\jmath\cdot\omega)}\cdot(0-1) = \\
&= \frac{A}{a+\jmath\cdot\omega}
\end{aligned}
$$

The Fourier transform of the $f(t) = A\cdot\Pi(\frac{t}{\tau})$ is equal to $F(\jmath\omega) = \frac{A}{a+\jmath\cdot\omega}$.
Let's explicitly determine the real and imaginary part:

$$
\begin{aligned}
F(\jmath\omega) &= \frac{A}{(a+\jmath\cdot\omega)} = \\
&= \frac{A}{(a+\jmath\cdot\omega)}\cdot\frac{(a-\jmath\cdot\omega)}{(a-\jmath\cdot\omega)} = \\
&= \frac{A\cdot(a-\jmath\cdot\omega)}{(a^2+\omega^2)} = \\
&= \frac{A\cdot a}{(a^2+\omega^2)} - \jmath\cdot\frac{A\cdot\omega}{(a^2+\omega^2)}
\end{aligned}
$$

The magnitude spectrum is defined as:

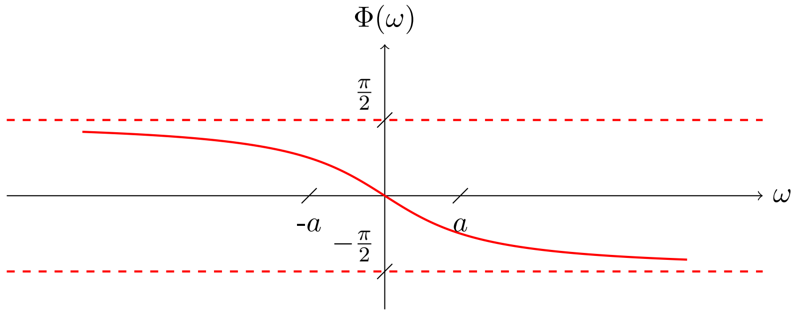
$$
\begin{aligned}
M(\omega) = |F(j\omega)| &= \\
&= \sqrt{\left(\frac{A\cdot a}{(a^2+\omega^2)}\right)^2 + \left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\right)^2} = \\
&= \sqrt{\frac{A^2\cdot(a^2+\omega^2)}{(a^2+\omega^2)^2}} = \\
&= \sqrt{\frac{A^2}{(a^2+\omega^2)}} = \\
&= \frac{A}{\sqrt{a^2+\omega^2}}
\end{aligned}
$$



The magnitude spectrum of a real signal is an even-symmetric function of k.

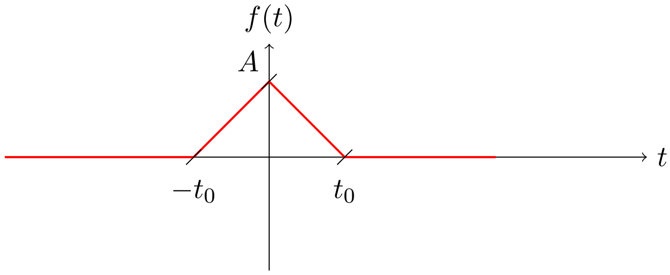
The phase spectrum is defined as:

$$\begin{aligned}
\Phi(\omega) &= arctan2(\frac{Im\{F(\jmath\omega)\}}{Re\{F(\jmath\omega)\}}) = \\
&= arctan2\left(\frac{\left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\right)}{\left(\frac{A\cdot a}{(a^2+\omega^2)}\right)}\right) = \\
&= arctan2\left(\frac{-A\cdot\omega}{(a^2+\omega^2)}\cdot\frac{(a^2+\omega^2)}{A\cdot a}\right) = \\
&= arctan2\left(-\frac{\omega}{a}\right)
\end{aligned}$$



The phase spectrum of a real signal is an odd-symmetric function of k.

Task 3. Compute the Fourier transform of a triangle impulse shown below.



First of all, describe the $f(t)$ signal using elementary signals:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.9}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.10}$$

In order to integrate the $f(t)$ signal, we need to describe it as a piecewise signal.

The simplest form of a linear function is:

$$f(t) = m \cdot t + b \tag{3.11}$$

In the first interval (e.g. $t \in (-t_0; 0)$), linear function crosses two points: $(-t_0, 0)$ and $(0, A)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot (-t_0) + b \\ A = m \cdot 0 + b \end{cases}$$
$$\begin{cases} -b = m \cdot (-t_0) \\ A = b \end{cases}$$
$$\begin{cases} \frac{b}{t_0} = m \\ A = b \end{cases}$$
$$\begin{cases} A = b \\ \frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{t_0} \cdot t + A$$

In the second interval (e.g. $t \in (0; t_0)$), linear function crosses two points: $(0; A)$ and $(t_0, 0)$. So, in order to derive m and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = m \cdot t_0 + b \\ A = m \cdot 0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot t_0 \\ A = b \end{cases}$$

$$\begin{cases} -\frac{b}{t_0} = m \\ A = b \end{cases}$$

$$\begin{cases} A = b \\ -\frac{A}{t_0} = m \end{cases}$$

As a result we get:

$$f(t) = -\frac{A}{t_0} \cdot t + A$$

As a result the piecewise linear function is given by:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) = \begin{cases} 0 & for \quad t \in (-\infty; -t_0) \\ \frac{A}{t_0} \cdot t + A & for \quad t \in (-t_0; 0) \\ -\frac{A}{t_0} \cdot t + A & for \quad t \in (0; t_0) \\ 0 & for \quad t \in (t_0; \infty) \end{cases} \tag{3.12}$$

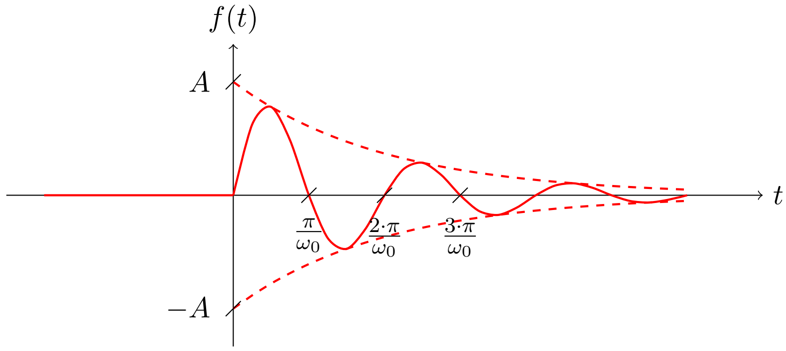
For the given $f(t)$ signal we get:

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} \left(\frac{A}{t_0} \cdot t + A\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \left(-\frac{A}{t_0} \cdot t + A\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt + \int_{-t_0}^{0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-t_0}^{0} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} -\frac{A}{t_0} \cdot t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{t_0} A \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 + \frac{A}{t_0} \cdot \int_{-t_0}^{0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + A \cdot \int_{-t_0}^{0} e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&- \frac{A}{t_0} \cdot \int_{0}^{t_0} t \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + A \cdot \int_{0}^{t_0} e^{-\jmath\cdot\omega\cdot t} \cdot dt + 0 = \\
&= \left\{ \begin{matrix} u & = t & dv & = e^{-\jmath\cdot\omega\cdot t} \cdot dt \\ du & = dt & v & = \frac{1}{-\jmath\cdot\omega} \cdot e^{-\jmath\cdot\omega\cdot t} \end{matrix} \right\} = \\
&= \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t} \Big|_{-t_0}^{0} - \int_{-t_0}^{0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) = \\
&+ A \cdot \left(\frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath\cdot\omega\cdot t} \Big|_{-t_0}^{0} \right) =
\end{aligned}$$

$$
\begin{aligned}
&-\frac{A}{t_0}\cdot\left(t\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_0^{t_0}-\int_0^{t_0}\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\cdot dt\right)=\\
&+A\cdot\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_0^{t_0}\right)=\\
&=\frac{A}{t_0}\cdot\left(0\cdot e^{-\jmath\cdot\omega\cdot 0}-(-t_0)\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot(-t_0)}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_{-t_0}^{0}\right)\right)=\\
&+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)=\\
&-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0\cdot e^{-\jmath\cdot\omega\cdot 0}+\frac{1}{\jmath\cdot\omega}\left(\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t}\Big|_0^{t_0}\right)\right)=\\
&+\frac{A}{-\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)=\\
&=\frac{A}{t_0}\cdot\left(0-t_0\cdot\frac{1}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot 0}-e^{-\jmath\cdot\omega\cdot(-t_0)}\right)\right)=\\
&-\frac{A}{\jmath\cdot\omega}\cdot\left(1-e^{\jmath\cdot\omega\cdot t_0}\right)=\\
&-\frac{A}{t_0}\cdot\left(t_0\cdot\frac{1}{-\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}-0-\frac{1}{\jmath^2\cdot\omega^2}\left(e^{-\jmath\cdot\omega\cdot t_0}-e^{-\jmath\cdot\omega\cdot 0}\right)\right)=\\
&-\frac{A}{\jmath\cdot\omega}\cdot\left(e^{-\jmath\cdot\omega\cdot t_0}-1\right)=\\
&=-\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{\jmath\cdot\omega\cdot t_0}-\frac{A}{\jmath\cdot\omega}+\frac{A}{\jmath\cdot\omega}\cdot e^{\jmath\cdot\omega\cdot t_0}=\\
&+\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot e^{-\jmath\cdot\omega\cdot t_0}-\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}-\frac{A}{\jmath\cdot\omega}\cdot e^{-\jmath\cdot\omega\cdot t_0}+\frac{A}{\jmath\cdot\omega}=\\
&=-\frac{2\cdot A}{t_0\cdot\jmath^2\cdot\omega^2}+\frac{A}{t_0\cdot\jmath^2\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}-\frac{A}{t_0\cdot\omega^2}\cdot\left(e^{\jmath\cdot\omega\cdot t_0}+e^{-\jmath\cdot\omega\cdot t_0}\right)=\\
&=\left\{cos(x)=\tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\}=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}-\frac{2\cdot A}{t_0\cdot\omega^2}\cdot cos(\omega\cdot t_0)=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}\cdot(1-cos(\omega\cdot t_0))=\\
&=\left\{\begin{matrix} sin^2(x)=\frac{1}{2}-\frac{1}{2}\cdot cos(2\cdot x)\\ cos(2\cdot x)=1-2\cdot sin^2(x)\end{matrix}\right\}=\\
&=\frac{2\cdot A}{t_0\cdot\omega^2}\cdot(1-1+2\cdot sin^2(\frac{\omega\cdot t_0}{2}))=\\
&=\frac{4\cdot A}{t_0\cdot\omega^2}\cdot sin^2(\frac{\omega\cdot t_0}{2})=\\
&=\frac{A\cdot t_0}{\frac{t_0^2\cdot\omega^2}{4}}\cdot sin^2(\frac{\omega\cdot t_0}{2})=\\
&=\left\{\tfrac{sin(x)}{x}=Sa(x)\right\}=\\
&=A\cdot t_0\cdot Sa^2(\frac{\omega\cdot t_0}{2})
\end{aligned}
$$

The Fourier transform of the $f(t)=A\cdot\Lambda(\frac{t}{t_0})$ is equal to $F(\jmath\omega)=A\cdot t_0\cdot Sa^2(\frac{\omega\cdot t_0}{2})$.

Task 4. Compute the Fourier transform of a signal shown below. Compute and draw magnitude and phase spectra.



The $f(t)$ signal, as a piecewise function is given by:

$$f(t) = \begin{cases} 0 & \text{if } t \in (-\infty; 0) \\ e^{-a\cdot t} \cdot sin(\omega_0 \cdot t) & \text{if } t \in (0; \infty) \end{cases} \tag{3.13}$$

The Fourier transform is defined as:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.14}$$

For the given $f(t)$ signal we get:

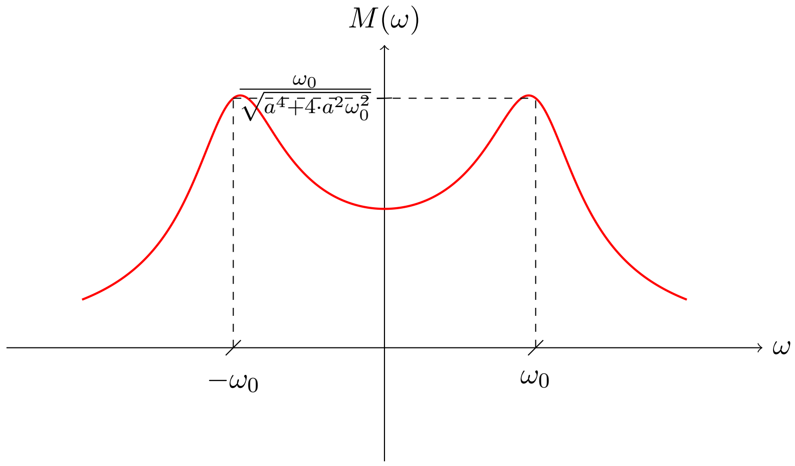
$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} 0 \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{\infty} e^{-a\cdot t} \cdot sin(\omega_0 \cdot t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ sin(x) = \tfrac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath} \right\} = \\
&= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} e^{-a\cdot t} \cdot \left(\frac{e^{\jmath\cdot\omega_0\cdot t} - e^{-\jmath\cdot\omega_0\cdot t}}{2\cdot\jmath} \right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= 0 + \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \left(\int_{0}^{\tau} e^{-a\cdot t} \cdot e^{\jmath\cdot\omega_0\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt - \int_{0}^{\tau} e^{-a\cdot t} \cdot e^{-\jmath\cdot\omega_0\cdot t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \right) = \\
&= \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \left(\int_{0}^{\tau} e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t} \cdot dt - \int_{0}^{\tau} e^{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{lclllcl} z &=& (-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t & & w &=& (-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t \\ dz &=& (-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot dt & & dw &=& (-a-\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot dt \\ dt &=& \frac{1}{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot dz & & dt &=& \frac{1}{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)}\cdot dw \end{array} \right\} = \\
&= \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \int_{0}^{\tau} e^{z} \cdot \frac{dz}{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} - \lim_{\tau\to\infty} \frac{1}{2\cdot\jmath} \int_{0}^{\tau} e^{w} \cdot \frac{dw}{(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \int_{0}^{\tau} e^{z} \cdot dz - \frac{1}{2\cdot\jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \int_{0}^{\tau} e^{w} \cdot dw = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{z}\Big|_{0}^{\tau} - \frac{1}{2\cdot\jmath\cdot(-a-\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{w}\Big|_{0}^{\tau} = \\
&= \frac{1}{2\cdot\jmath\cdot(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{(-a+\jmath\cdot\omega_0-\jmath\cdot\omega)\cdot t}\Big|_{0}^{\tau} +
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} e^{(-a - \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot t} \Big|_0^\tau \\
&= \frac{1}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} \left(e^{(-a + \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - e^{(-a + \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot 0} \right) + \\
&- \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \lim_{\tau \to \infty} \left(e^{(-a - \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - e^{(-a - \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot 0} \right) = \\
&= \frac{1}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(\lim_{\tau \to \infty} \left(e^{-a \cdot \tau} \cdot e^{(\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} \right) - \lim_{\tau \to \infty} 1 \right) + \\
&- \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(\lim_{\tau \to \infty} \left(e^{-a \cdot \tau} \cdot e^{(-\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} \right) - \lim_{\tau \to \infty} 1 \right) = \\
&= \frac{1}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(\lim_{\tau \to \infty} (e^{-a \cdot \tau}) \cdot \lim_{\tau \to \infty} e^{(\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - 1 \right) + \\
&- \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(\lim_{\tau \to \infty} (e^{-a \cdot \tau}) \cdot \lim_{\tau \to \infty} e^{(-\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - 1 \right) = \\
&= \frac{1}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(0 \cdot \lim_{\tau \to \infty} e^{(\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - 1 \right) + \\
&- \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} \cdot \left(0 \cdot \lim_{\tau \to \infty} e^{(-\jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot \tau} - 1 \right) = \\
&= \frac{-1}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)} + \frac{1}{2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} = \\
&= \frac{-(2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)) + 2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega)}{2 \cdot \jmath \cdot (-a + \jmath \cdot \omega_0 - \jmath \cdot \omega) \cdot 2 \cdot \jmath \cdot (-a - \jmath \cdot \omega_0 - \jmath \cdot \omega)} = \\
&= \frac{2 \cdot \jmath \cdot a + 2 \cdot \jmath^2 \cdot \omega_0 + 2 \cdot \jmath^2 \cdot \omega - 2 \cdot \jmath \cdot a + 2 \cdot \jmath^2 \cdot \omega_0 - 2 \cdot \jmath^2 \cdot \omega}{4 \cdot \jmath^2 \cdot (a^2 + a \cdot \jmath \cdot \omega_0 + a \cdot \jmath \cdot \omega - a \cdot \jmath \cdot \omega_0 - \jmath^2 \cdot \omega_0^2 - \jmath^2 \cdot \omega_0 \cdot \omega + a \cdot \jmath \cdot \omega + \jmath^2 \cdot \omega_0 \cdot \omega + \jmath^2 \cdot \omega^2)} = \\
&= \frac{4 \cdot \jmath^2 \cdot \omega_0}{4 \cdot \jmath^2 \cdot (a^2 + 2 \cdot a \cdot \jmath \cdot \omega - \jmath^2 \cdot \omega_0^2 + \jmath^2 \cdot \omega^2)} = \\
&= \frac{\omega_0}{a^2 + 2 \cdot a \cdot \jmath \cdot \omega + \omega_0^2 - \omega^2} = \\
&= \frac{\omega_0}{\omega_0^2 + (a^2 + 2 \cdot a \cdot \jmath \cdot \omega - \omega^2)} = \\
&= \frac{\omega_0}{\omega_0^2 + (a + \jmath \cdot \omega)^2}
\end{aligned}
$$

The Fourier transform of the $f(t)$ is equal to $F(\jmath\omega) = \frac{\omega_0}{\omega_0^2 + (a + \jmath \cdot \omega)^2}$.
The magnitude spectrum is defined as:

$$
\begin{aligned}
M(\omega) = |F(j\omega)| &= \\
&= \left| \frac{\omega_0}{\omega_0^2 + (a + \jmath \cdot \omega)^2} \right| = \\
&= \left| \frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega + (\jmath \cdot \omega)^2} \right| = \\
&= \left| \frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega - \omega^2} \right| = \\
&= \left| \frac{\omega_0}{\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega} \right| = \\
&= \left\{ \left| \tfrac{z_1}{z_2} \right| = \tfrac{|z_1|}{|z_2|} \right\} = \\
&= \frac{|\omega_0|}{|\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega|} =
\end{aligned}
$$

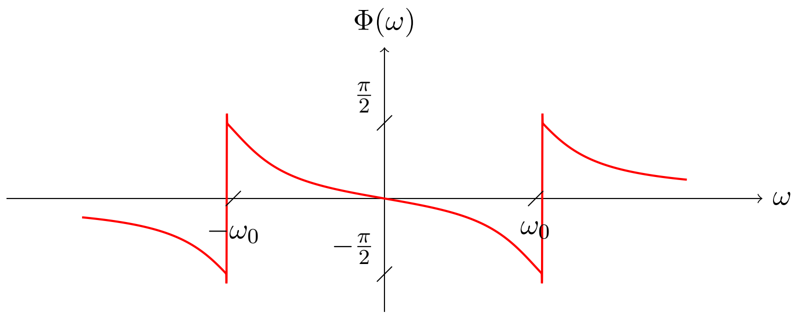
$$= \left\{ |a + \jmath \cdot b| = \sqrt{a^2 + b^2} \right\} =$$
$$= \frac{\omega_0}{\sqrt{\left(\omega_0^2 - \omega^2 + a^2\right)^2 + \left(2 \cdot a \cdot \omega\right)^2}}$$



The magnitude spectrum of a real signal is an even-symmetric function of k.

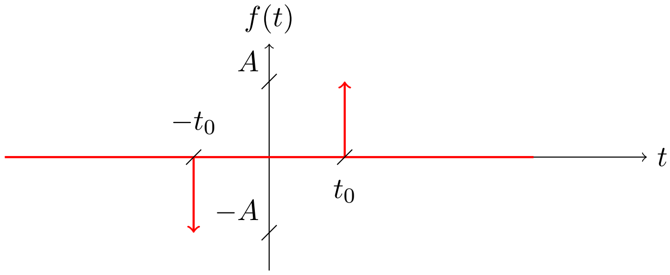
The phase spectrum is defined as:

$$\Phi(\omega) = arg\left(\frac{\omega_0}{\omega_0^2 + (a + \jmath \cdot \omega)^2}\right) =$$
$$= arg\left(\frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega + (\jmath \cdot \omega)^2}\right) =$$
$$= arg\left(\frac{\omega_0}{\omega_0^2 + a^2 + 2 \cdot a \cdot \jmath \cdot \omega - \omega^2}\right) =$$
$$= arg\left(\frac{\omega_0}{\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega}\right) =$$
$$= \left\{ arg\left(\tfrac{z_1}{z_2}\right) = arg\left(z_1\right) - arg\left(z_2\right) \right\} =$$
$$= arg\left(\omega_0\right) - arg\left(\omega_0^2 - \omega^2 + a^2 + \jmath \cdot 2 \cdot a \cdot \omega\right) =$$
$$= \left\{ arg\left(a + \jmath \cdot b\right) = arctan(\tfrac{b}{a}) \right\} =$$
$$= arctan\left(\frac{0}{\omega_0}\right) - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) =$$
$$= arctan\left(0\right) - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) =$$
$$= 0 - arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right) =$$
$$= -arctan\left(\frac{2 \cdot a \cdot \omega}{\omega_0^2 - \omega^2 + a^2}\right)$$



The phase spectrum of a real signal is an odd-symmetric function of k.

Task 5. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku oraz narysuj jego widmo amplitudowe i fazowe



W pierwszej kolejności opiszmy sygnał za pomocą sygnałów elementarnych:

$$f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0) \tag{3.15}$$

Transformatę Fouriera obliczamy ze wzoru:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.16}$$

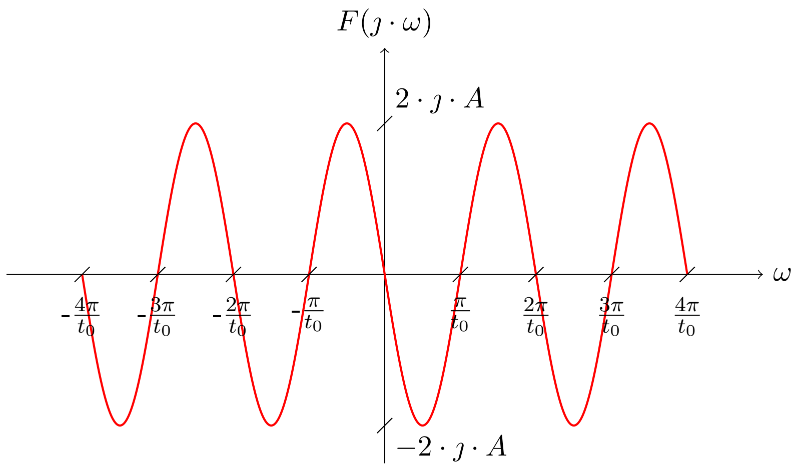
Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} A \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \int_{-\infty}^{\infty} A \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - A \cdot \int_{-\infty}^{\infty} \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot t_0} - A \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot t_0} - A \cdot e^{\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(e^{-\jmath \cdot \omega \cdot t_0} - e^{\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(-e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= -A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= -A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= -2 \cdot \jmath \cdot A \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0)
\end{aligned}$$

Transformata sygnału $f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)$ to $F(\jmath\omega) = -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0)$

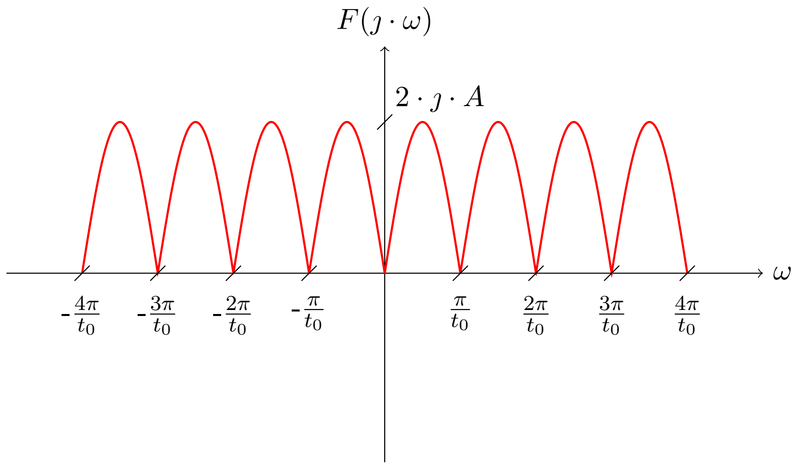
Narysujmy widmo sygnału $f(t) = A \cdot \delta(t - t_0) - A \cdot \delta(t + t_0)$ czyli:

$$F(\jmath\omega) = -2 \cdot \jmath \cdot A \cdot sin(\omega \cdot t_0) \tag{3.17}$$



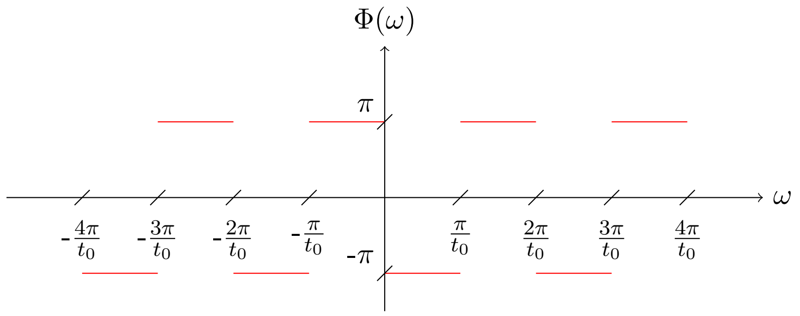
Widmo amplitudowe obliczamy ze wzoru:

$$M(\omega) = |F(j \cdot \omega)| \tag{3.18}$$

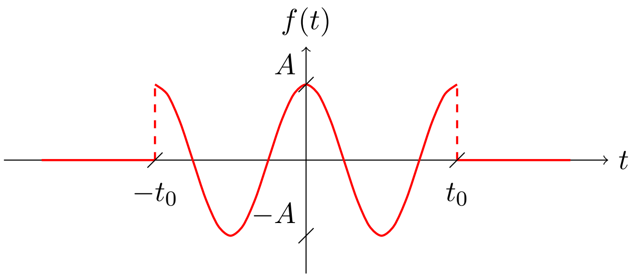


Widmo fazowe obliczamy ze wzoru:

$$\Phi(\omega) = arctg(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}) \tag{3.19}$$



Task 6. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku oraz narysuj jego widmo amplitudowe i fazowe



$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.20}$$

Transformatę Fouriera obliczamy ze wzoru:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.21}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-t_0}^{t_0} A \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ cos(x) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt + \int_{-t_0}^{t_0} A \cdot \frac{e^{\jmath \cdot \frac{2\pi}{t_0} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{t_0} \cdot t}}{2} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 + \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath \cdot \frac{2\pi}{t_0} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{t_0} \cdot t} \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + 0 = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath \cdot \frac{2\pi}{t_0} \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} + e^{-\jmath \cdot \frac{2\pi}{t_0} \cdot t} \cdot e^{-\jmath \cdot \omega \cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath \cdot \frac{2\pi}{t_0} \cdot t - \jmath \cdot \omega \cdot t} + e^{-\jmath \cdot \frac{2\pi}{t_0} \cdot t - \jmath \cdot \omega \cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \int_{-t_0}^{t_0} \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t} + e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t} \cdot dt + \int_{-t_0}^{t_0} e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{rr} z_1 = \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t & z_2 = -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t \\ dz_1 = \jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot dt & dz_2 = -\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot dt \\ dt = \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \end{array} \right\} =
\end{aligned}$$

$$= \frac{A}{2} \cdot \left(\int_{-t_0}^{t_0} e^{z_1} \cdot \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot dz_1 + \int_{-t_0}^{t_0} e^{z_2} \cdot \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot dz_2 \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \int_{-t_0}^{t_0} e^{z_2} \cdot dz_2 \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{z_1}\Big|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{z_2}\Big|_{-t_0}^{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t}\Bigg|_{-t_0}^{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot (-t_0)} \right) + \right.$$

$$\left. + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot (-t_0)} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \right.$$

$$\left. + \frac{1}{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) + \right.$$

$$\left. + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \right) \cdot \frac{2}{2} + \right.$$

$$\left. + \frac{1}{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right)} \cdot \left(e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) \cdot \frac{2}{2} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0}}{2 \cdot \jmath} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot \frac{e^{\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} - e^{-\jmath \cdot \left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0}}{2 \cdot \jmath} \right) =$$

$$= \left\{ sin(x) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} =$$

$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \right) =$$

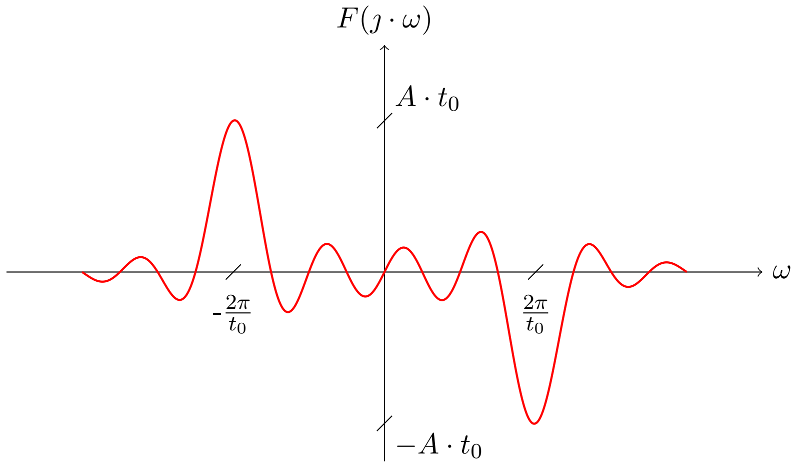
$$= \frac{A}{2} \cdot \left(\frac{2}{\left(\frac{2\pi}{t_0} - \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) \cdot \frac{t_0}{t_0} + \frac{2}{\left(\frac{2\pi}{t_0} + \omega\right)} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \cdot \frac{t_0}{t_0} \right) =$$

$$= \frac{A}{2} \cdot \left(\frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} - \omega \right) \cdot t_0 \right) + \frac{2 \cdot t_0}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \cdot sin\left(\left(\frac{2\pi}{t_0} + \omega \right) \cdot t_0 \right) \right) =$$

$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0} + 2 \cdot t_0 \cdot \frac{sin\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)}{\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0} \right) =$$
$$= \left\{ Sa(x) = \frac{sin(x)}{x} \right\} =$$
$$= \frac{A}{2} \cdot \left(2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + 2 \cdot t_0 \cdot Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)\right) =$$
$$= A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)\right)$$

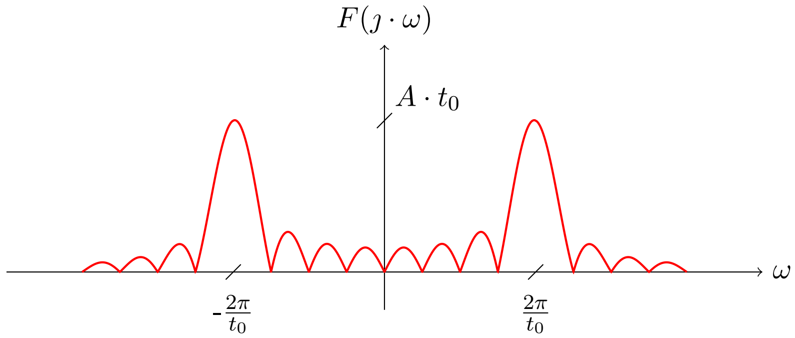
Transformata sygnału $f(t)$ to $F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)\right)$
Narysujmy widmo sygnału $f(t)$ czyli:

$$F(\jmath\omega) = A \cdot t_0 \cdot \left(Sa\left(\left(\frac{2\pi}{t_0} - \omega\right) \cdot t_0\right) + Sa\left(\left(\frac{2\pi}{t_0} + \omega\right) \cdot t_0\right)\right) \tag{3.22}$$



Widmo amplitudowe obliczamy ze wzoru:

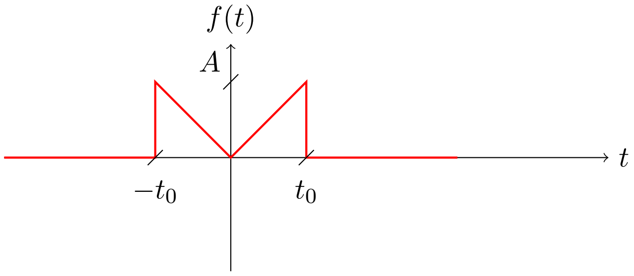
$$M(\omega) = |F(j \cdot \omega)| \tag{3.23}$$



Widmo fazowe obliczamy ze wzoru:

$$\Phi(\omega) = arctg(\frac{Im\{F(\jmath \cdot \omega)\}}{Re\{F(\jmath \cdot \omega)\}}) \tag{3.24}$$

Task 7. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku



Transformatę Fouriera obliczamy ze wzoru:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.25}$$

Do obliczenia całki potrzebujemy jawnej postaci funkcji $f(t)$. Funkcja ta jest określona za pomocą równań opisujących proste na odcinkach $(-t_0, 0)$ oraz $(0, t_0)$

Ogólne równanie prostej to:

$$f(t) = m \cdot t + b \tag{3.26}$$

Dla pierwszego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(-t_0, A)$ oraz $(0, 0)$. Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} A = m \cdot (-t_0) + b \\ 0 = m \cdot 0 + b \end{cases}$$

$$\begin{cases} A = m \cdot (-t_0) + b \\ 0 = b \end{cases}$$

$$\begin{cases} A = m \cdot (-t_0) + 0 \\ 0 = b \end{cases}$$

$$\begin{cases} -\frac{A}{t_0} = m \\ 0 = b \end{cases}$$

Równianie prostej dla t z zakresu $(-t_0, 0)$ to:

$$f(t) = -\frac{A}{t_0} \cdot t$$

Dla drugiego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(0; 0)$ oraz $(t_0; A)$. Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} A = m \cdot t_0 + b \\ 0 = m \cdot 0 + b \end{cases}$$

$$\begin{cases} A = m \cdot t_0 + b \\ 0 = b \end{cases}$$

$$\begin{cases} A = m \cdot t_0 + 0 \\ 0 = b \end{cases}$$

$$\begin{cases} \frac{A}{t_0} = m \\ 0 = b \end{cases}$$

Równianie prostej dla t z zakresu $(0, t_0)$ to:

$$f(t) = \frac{A}{t_0} \cdot t$$

Podsumowując, sygnał $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & dla \quad t \in (-\infty; -t_0) \\ -\frac{A}{t_0} \cdot t & dla \quad t \in (-t_0; 0) \\ \frac{A}{t_0} \cdot t & dla \quad t \in (0; t_0) \\ 0 & dla \quad t \in (t_0; \infty) \end{cases} \tag{3.27}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
F(\jmath\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-t_0}^{0} \left(-\frac{A}{t_0} \cdot t \right) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{-t_0} 0 \cdot dt - \int_{-t_0}^{0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&+ \int_{0}^{t_0} \frac{A}{t_0} \cdot t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{t_0}^{\infty} 0 \cdot dt = \\
&= 0 - \frac{A}{t_0} \cdot \int_{-t_0}^{0} t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \frac{A}{t_0} \cdot \int_{0}^{t_0} t \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + 0 = \\
&= \left\{ \begin{matrix} u & = t & dv & = e^{-\jmath \cdot \omega \cdot t} \cdot dt \\ du & = dt & v & = \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \end{matrix} \right\} = \\
&= -\frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{-t_0}^{0} - \int_{-t_0}^{0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&+ \frac{A}{t_0} \cdot \left(t \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{0}^{t_0} - \int_{0}^{t_0} \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= -\frac{A}{t_0} \cdot \left(0 \cdot e^{-\jmath \cdot \omega \cdot 0} - (-t_0) \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} + \frac{1}{\jmath \cdot \omega} \left(\frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{-t_0}^{0} \right) \right) + \\
&+ \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - 0 \cdot e^{-\jmath \cdot \omega \cdot 0} + \frac{1}{\jmath \cdot \omega} \left(\frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t} \Big|_{0}^{t_0} \right) \right) =
\end{aligned}$$

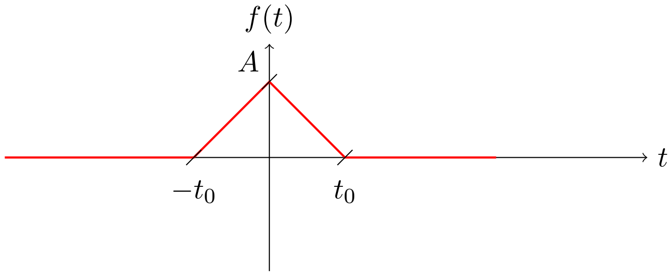
$$
\begin{aligned}
&= -\frac{A}{t_0} \cdot \left(0 - t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot 0} - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&+ \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{-\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - 0 - \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot 0}\right)\right) = \\
&= \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{0} - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&+ \frac{A}{t_0} \cdot \left(-t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot t_0} - e^{0}\right)\right) = \\
&= \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2}\left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right)\right) + \\
&- \frac{A}{t_0} \cdot \left(t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right)\right) = \\
&= \frac{A}{t_0} \cdot t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2}\left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right) + \\
&- \frac{A}{t_0} \cdot t_0 \cdot \frac{1}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{\jmath \cdot \omega} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2}\left(1 - e^{-\jmath \cdot \omega \cdot (-t_0)}\right) - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2}\left(e^{-\jmath \cdot \omega \cdot t_0} - 1\right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}\right) + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} = \\
&= \frac{2 \cdot A}{2 \cdot \jmath \cdot \omega} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}\right) + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{\jmath \cdot \omega \cdot t_0} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ 2 \cdot \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} - \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - \frac{2}{2} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}\right)\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} + \\
&+ \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - 2 \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}}{2}\right) = \\
&= \begin{Bmatrix} sin\left(x\right) = \frac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \\ cos\left(x\right) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \end{Bmatrix} = \\
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot \left(2 - 2 \cdot cos\left(\omega \cdot t_0\right)\right) = \\
&= \frac{2 \cdot A}{\omega} \cdot sin\left(\omega \cdot t_0\right) + \frac{A}{t_0} \cdot \frac{1}{\jmath^2 \cdot \omega^2} \cdot 4 \cdot \left(\frac{1}{2} - \frac{1}{2} \cdot cos\left(\omega \cdot t_0\right)\right) = \\
&= \left\{sin^2\left(x\right) = \tfrac{1}{2} - \tfrac{1}{2} \cdot cos\left(2 \cdot x\right)\right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\cdot A}{\omega}\cdot sin\left(\omega\cdot t_0\right) + \frac{A}{t_0}\cdot\frac{4}{\jmath^2\cdot\omega^2}\cdot sin^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right) = \\
&= \frac{2\cdot A}{\omega}\cdot\frac{t_0}{t_0} sin\left(\omega\cdot t_0\right) + \frac{A}{t_0}\cdot\frac{4}{\jmath^2\cdot\omega^2}\cdot\frac{t_0}{t_0}\cdot sin^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right) = \\
&= 2\cdot A\cdot t_0\cdot\frac{sin\left(\omega\cdot t_0\right)}{t_0\cdot\omega} + A\cdot t_0\cdot\frac{4}{-1\cdot\omega^2\cdot t_0^2}\cdot sin^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right) = \\
&= 2\cdot A\cdot t_0\cdot\frac{sin\left(\omega\cdot t_0\right)}{t_0\cdot\omega} + A\cdot t_0\cdot\frac{-1}{\frac{\omega^2\cdot t_0^2}{4}}\cdot sin^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right) = \\
&= 2\cdot A\cdot t_0\cdot\frac{sin\left(\omega\cdot t_0\right)}{t_0\cdot\omega} - A\cdot t_0\cdot\frac{sin^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right)}{\left(\frac{\omega\cdot t_0}{2}\right)^2} = \\
&= 2\cdot A\cdot t_0\cdot\frac{sin\left(\omega\cdot t_0\right)}{t_0\cdot\omega} - A\cdot t_0\cdot\left(\frac{sin\left(\frac{1}{2}\cdot\omega\cdot t_0\right)}{\frac{1}{2}\cdot\omega\cdot t_0}\right)^2 = \\
&= \left\{Sa\left(x\right) = \frac{sin(x)}{x}\right\} = \\
&= 2\cdot A\cdot t_0\cdot Sa\left(\omega\cdot t_0\right) - A\cdot t_0\cdot Sa^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right)
\end{aligned}
$$

Transformata sygnału $f(t)$ wynosi $F(\jmath\omega) = 2\cdot A\cdot t_0\cdot Sa\left(\omega\cdot t_0\right) - A\cdot t_0\cdot Sa^2\left(\frac{1}{2}\cdot\omega\cdot t_0\right)$

3.2 Exploiting properties of the Fourier transform

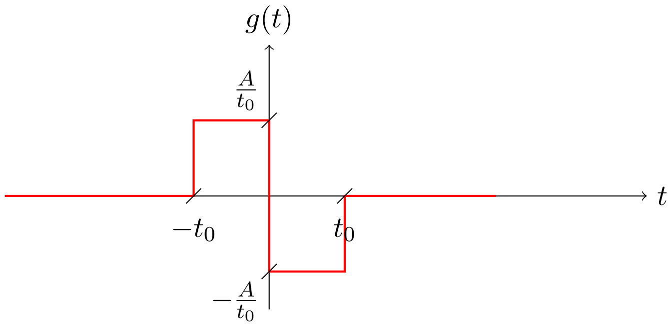
Task 1. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku wykorzystując twierdzenia opisujące własciwości transformacji Fouriera. Wykorzystaj informację o tym, że $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.



W pierwszej kolejności należy ustalić wzór funkcji przedstawionej na rysunku. Wykorzystując sygnały elementarne możemy napisać:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.28}$$

Wyznaczmy pochodną sygnału $f(t)$, czyli sygnał $g(t) = \frac{\partial}{\partial t} f(t)$.



Sygnał $g(t)$ można opisać, wykorzystując sygnały elementarne:

$$g(t) = \frac{A}{t_0} \cdot \Pi(\frac{t - (-\frac{t_0}{2})}{t_0}) - \frac{A}{t_0} \cdot \Pi(\frac{t - \frac{t_0}{2}}{t_0}) \tag{3.29}$$

Można sprawdzić, że całkując sygnał $g(t)$ otrzymamy sygnał $f(t)$, czyli:

$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.30}$$

Skoro tak jest, to transformatę sygnału $f(t)$ mozna wyznaczyć z twierdzenia o całkowaniu sygnału, w tym przypadku całkować będziemy sygnał $g(t)$:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.31}$$

Z powyzszego równania widać, że musimy znać $G(\jmath\omega)$, czyli transformatę sygnału $g(t)$:

$$g(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) - \frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0}) \tag{3.32}$$

Ponieważ transformacja Fouriera jest przekształceniem liniowym, dlatego można wyznaczyć osobno transformaty poszczególnych prostokątów, czyli:

$$g(t) = g_1(t) - g_2(t) \tag{3.33}$$

gdzie:

$$g_1(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0})$$
$$g_2(t) = \frac{A}{t_0} \cdot \Pi(\frac{t-\frac{t_0}{2}}{t_0})$$

Wyznaczmy transformtę sygnału $g_1(t)$, czyli $G_1(\jmath\omega)$.

Z tablic matematycznych wiemy, że: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot(-\frac{t_0}{2})} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} \frac{A}{t_0} \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\frac{A}{t_0} \cdot \Pi(\frac{t-(-\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} A \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$

Transformata sygnału $g_1(t)$ to:

$$G_1(\jmath\omega) = \mathcal{F}\{g_1(t)\} = A \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}} \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \tag{3.34}$$

Teraz wyznaczmy transformtę sygnału $g_2(t)$, czyli $G_2(\jmath\omega)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right)$$
$$\Pi(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi(\frac{t-(\frac{t_0}{2})}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot(\frac{t_0}{2})} \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$

$$\Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot t_0\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$
$$\frac{A}{t_0}\cdot\Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} \frac{A}{t_0}\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot t_0\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$
$$\frac{A}{t_0}\cdot\Pi(\frac{t-\frac{t_0}{2}}{t_0}) \xrightarrow{\mathcal{F}} A\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$

Transformata sygnału $g_2(t)$ to:

$$G_2(\jmath\omega)=\mathcal{F}\{g_2(t)\}=A\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right) \tag{3.35}$$

Czyli transformata sygnału $g(t)$ to:

$$G(\jmath\omega)=\mathcal{F}\{g(t)\}=A\cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)-A\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$
$$G(\jmath\omega)=A\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot\left(e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}-e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\right)$$
$$G(\jmath\omega)=A\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot\left(e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}-e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}\right)$$
$$\left\{sin(x)=\frac{e^{\jmath\cdot x}-e^{-\jmath\cdot x}}{2\cdot\jmath}\right\}$$
$$G(\jmath\omega)=A\cdot 2\cdot\jmath\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot sin\left(\frac{\omega\cdot t_0}{2}\right)$$

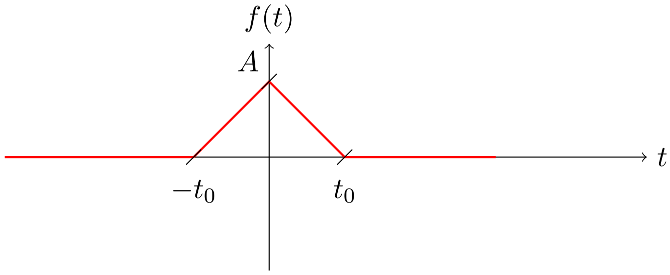
Mamy wyznaczoną transformatę $G(\jmath\omega)$. Teraz, z twierdzenia o całkowaniu sygnału, możemy wyznaczyc transformatę sygnału $f(t)$:

$$F(\jmath\omega)=\frac{1}{\jmath\cdot\omega}\cdot G(\jmath\omega)+\pi\cdot\delta(\omega)\cdot G(0) \tag{3.36}$$

$$\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath\cdot\omega}\cdot G(\jmath\omega)+\pi\cdot\delta(\omega)\cdot G(0)=\\
&= \frac{1}{\jmath\cdot\omega}\cdot A\cdot 2\cdot\jmath\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot sin\left(\frac{\omega\cdot t_0}{2}\right)+\pi\cdot\delta(\omega)\cdot G(0)=\\
&= \left\{\begin{array}{l} G(0)=A\cdot 2\cdot\jmath\cdot Sa\left(\frac{0\cdot t_0}{2}\right)\cdot sin\left(\frac{0\cdot t_0}{2}\right)\\ G(0)=A\cdot 2\cdot\jmath\cdot Sa(0)\cdot sin(0)\\ G(0)=A\cdot 2\cdot\jmath\cdot 1\cdot 0\\ G(0)=0\end{array}\right\}=\\
&= \frac{1}{\jmath\cdot\omega}\cdot A\cdot 2\cdot\jmath\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot sin\left(\frac{\omega\cdot t_0}{2}\right)=\\
&= \left\{\frac{sin(x)}{x}=Sa(x)\right\}=\\
&= \frac{A\cdot 2\cdot t_0}{\omega\cdot t_0}\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot sin\left(\frac{\omega\cdot t_0}{2}\right)=\\
&= A\cdot t_0\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)=\\
&= A\cdot t_0\cdot Sa^2(\frac{\omega\cdot t_0}{2})
\end{aligned}$$

Transformata sygnału $f(t)=A\cdot\Lambda(\frac{t}{t_0})$ to $F(\jmath\omega)=A\cdot t_0\cdot Sa^2(\frac{\omega\cdot t_0}{2})$

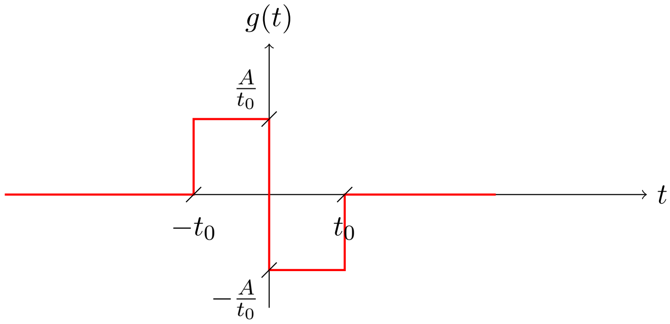
Task 2. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku wykorzystując twierdzenia opisujące własciwości transformacji Fouriera.



W pierwszej kolejności należy ustalić wzór funkcji przedstawionej na rysunku. Wykorzystując sygnały elementarne możemy napisać:

$$f(t) = A \cdot \Lambda(\frac{t}{t_0}) \tag{3.37}$$

Wyznaczmy pochodną sygnału $f(t)$, czyli sygnał $g(t) = \frac{\partial}{\partial t} f(t)$.



Można sprawdzić, że całkując sygnał $g(t)$ otrzymamy sygnał $f(t)$, czyli:

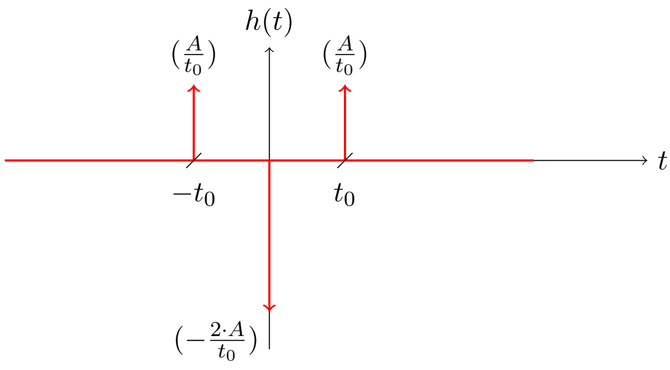
$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.38}$$

Skoro tak jest, to transformatę sygnału $f(t)$ mozna wyznaczyć z twierdzenia o całkowaniu sygnału, w tym przypadku całkować będziemy sygnał $g(t)$:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.39}$$

Pytanie, czy można dalej uprościc sygnał $g(t)$ dokonując jego rózniczkowania. Wyznaczmy pochodną sygnału $g(t)$, czyli drugą pochodną sygnału $f(t)$:

$$h(t) = \frac{\partial}{\partial t} g(t) = \frac{\partial^2}{\partial t^2} f(t) \tag{3.40}$$



Sygnał $h(t)$ można opisać, wykorzystując sygnały elementarne:

$$h(t) = \frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0)) \tag{3.41}$$

Można sprawdzić, że całkując sygnał $g(t)$ otrzymamy sygnał $f(t)$, czyli:

$$g(t) = \int_{-\infty}^{t} h(x) \cdot dx \tag{3.42}$$

Skoro tak jest, to transformatę sygnału $f(t)$ mozna wyznaczyć z twierdzenia o całkowaniu sygnału, w tym przypadku całkować będziemy sygnał $g(t)$:

$$G(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) \tag{3.43}$$

Z powyższego równania widać, że musimy znać $H(\jmath\omega)$, czyli transformatę sygnału $h(t)$:

$$h(t) = \frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0)) \tag{3.44}$$

Ponieważ transformacja Fouriera jest przekształceniem liniowym, dlatego można wyznaczyć osobno transformaty poszczególnych delt Diraca, czyli:

$$\begin{aligned}
H(\jmath\omega) &= \mathcal{F}\{h(t)\} = \\
&= \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (-t_0)) - \frac{2 \cdot A}{t_0} \cdot \delta(t) + \frac{A}{t_0} \cdot \delta(t - (t_0))\right\} = \\
&= \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (-t_0))\right\} - \mathcal{F}\left\{\frac{2 \cdot A}{t_0} \cdot \delta(t)\right\} + \mathcal{F}\left\{\frac{A}{t_0} \cdot \delta(t - (t_0))\right\} = \\
&= \frac{A}{t_0} \cdot \mathcal{F}\left\{\delta(t - (-t_0))\right\} - \frac{2 \cdot A}{t_0} \cdot \mathcal{F}\left\{\delta(t)\right\} + \frac{A}{t_0} \cdot \mathcal{F}\left\{\delta(t - (t_0))\right\} = \\
&= \left\{\begin{array}{l} \delta(t) \xrightarrow{\mathcal{F}} 1 \\ \delta(t - (-t_0)) \xrightarrow{\mathcal{F}} 1 \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} \\ \delta(t - (t_0)) \xrightarrow{\mathcal{F}} 1 \cdot e^{-\jmath \cdot \omega \cdot t_0} \end{array}\right\} = \\
&= \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - \frac{2 \cdot A}{t_0} \cdot 1 + \frac{A}{t_0} \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= \frac{A}{t_0} \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - 2 + e^{-\jmath \cdot \omega \cdot t_0}\right) = \\
&= \left\{cos(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2}\right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{t_0} \cdot (2 \cdot cos(\omega \cdot t_0) - 2) = \\
&= \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1)
\end{aligned}
$$

Czyli transformata sygnału $h(t)$ to:

$$H(\jmath\omega) = \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1) \tag{3.45}$$

Mamy wyznaczoną transformatę $H(\jmath\omega)$. Teraz, z twierdzenia o całkowaniu sygnału, możemy wyznaczyć transformatę $G(\jmath\omega)$:

$$
\begin{aligned}
G(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{t_0} \cdot (cos(\omega \cdot t_0) - 1) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \left\{ \begin{array}{l} H(0) = \frac{2 \cdot A}{t_0} \cdot (cos(0 \cdot t_0) - 1) \\ H(0) = \frac{2 \cdot A}{t_0} \cdot (cos(0) - 1) \\ H(0) = \frac{2 \cdot A}{t_0} \cdot (1 - 1) \\ H(0) = 0 \end{array} \right\} = \\
&= \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1)
\end{aligned}
$$

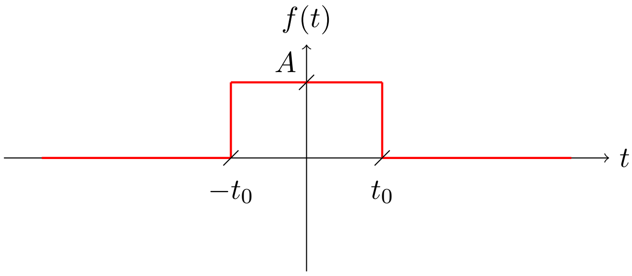
Mamy wyznaczoną transformatę $G(\jmath\omega)$. Teraz, kolejny raz z twierdzenia o całkowaniu sygnału, możemy wyznaczyć transformatę $F(\jmath\omega)$:

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{array}{l} G(0) = \frac{2 \cdot A}{\jmath \cdot 0 \cdot t_0} \cdot (cos(0 \cdot t_0) - 1) \\ G(0) = \frac{0}{0}!!! \\ G(0) = \int_{-\infty}^{\infty} g(t) \cdot dt = \int_{-t_0}^{0} \frac{A}{t_0} \cdot dt + \int_{0}^{t_0} (-\frac{A}{t_0}) \cdot dt \\ G(0) = \frac{A}{t_0} \cdot (0 - (-t_0)) - \frac{A}{t_0} \cdot (t_0 - 0) = A - A \\ G(0) = 0 \end{array} \right\} = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{2 \cdot A}{\jmath \cdot \omega \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) = \\
&= \frac{2 \cdot A}{\jmath^2 \cdot \omega^2 \cdot t_0} \cdot (cos(\omega \cdot t_0) - 1) = \\
&= \frac{2 \cdot A}{\omega^2 \cdot t_0} \cdot (1 - cos(\omega \cdot t_0)) = \\
&= \left\{ \begin{array}{l} sin^2(x) = \frac{1}{2} - \frac{1}{2} \cdot cos(2 \cdot x) \\ cos(2 \cdot x) = 1 - 2 \cdot sin^2(x) \end{array} \right\} = \\
&= \frac{2 \cdot A}{\omega^2 \cdot t_0} \cdot \left(1 - 1 + 2 \cdot sin^2\left(\frac{\omega \cdot t_0}{2}\right)\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4 \cdot A}{\omega^2 \cdot t_0} \cdot sin^2\left(\frac{\omega \cdot t_0}{2}\right) = \\
&= \left\{\frac{sin(x)}{x} = Sa(x)\right\} = \\
&= A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})
\end{aligned}
$$

Transformata sygnału $f(t) = A \cdot \Lambda(\frac{t}{t_0})$ to $F(\jmath\omega) = A \cdot t_0 \cdot Sa^2(\frac{\omega \cdot t_0}{2})$

Task 3. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



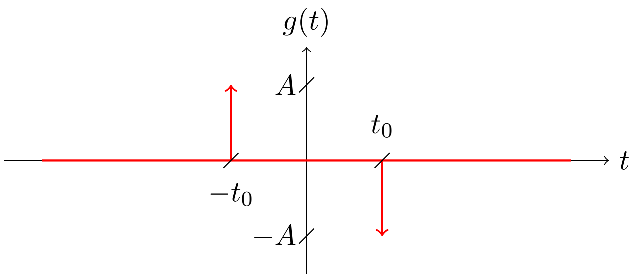
$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ A & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.46}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0) \tag{3.47}$$

czyli po prostu

$$g(t) = f'(t) = A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0) \tag{3.48}$$



Wyznaczanie transformaty sygnału $g(t)$ złożonego z delt diracka jest znacznie prostsze.

$$G(\jmath\omega) = \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.49}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned} G(\jmath\omega) &= \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\ &= \int_{-\infty}^{\infty} (A \cdot \delta(t + t_0) - A \cdot \delta(t - t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\ &= \int_{-\infty}^{\infty} \left(A \cdot \delta(t + t_0) \cdot e^{-\jmath \cdot \omega \cdot t} - A \cdot \delta(t - t_0) \cdot e^{-\jmath \cdot \omega \cdot t}\right) \cdot dt = \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} A \cdot \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - \int_{-\infty}^{\infty} A \cdot \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt - A \cdot \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot e^{-\jmath \cdot \omega \cdot (-t_0)} - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot e^{\jmath \cdot \omega \cdot t_0} - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= A \cdot 2 \cdot \jmath \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} - e^{-\jmath \cdot \omega \cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= A \cdot 2 \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0)
\end{aligned}
$$

Transformata sygnału $g(t)$ to $G(\jmath\omega) = \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $f(t)$ na podstawie transformaty sygnału $g(t) = f'(t)$

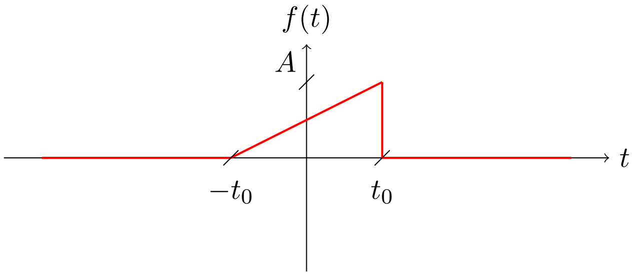
$$
\begin{gathered}
g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{gathered}
$$

Podstawiając obliczona wcześniej transformatę $G(\jmath\omega)$ sygnału $g(t)$ otrzymujemy transformatę $F(\jmath\omega)$ sygnału $f(t)$

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(0) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \jmath \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot sin\,(0 \cdot t_0) = \\
&= \frac{1}{\omega} \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot sin\,(0) = \\
&= \frac{1}{\omega} \cdot 2 \cdot A \cdot sin\,(\omega \cdot t_0) \cdot \frac{t_0}{t_0} + \pi \cdot \delta(0) \cdot \jmath \cdot 2 \cdot A \cdot 0 = \\
&= 2 \cdot A \cdot t_0 \cdot \frac{sin\,(\omega \cdot t_0)}{\omega \cdot t_0} + 0 = \\
&= \left\{ Sa\,(x) = \tfrac{sin(x)}{x} \right\} = \\
&= 2 \cdot A \cdot t_0 \cdot Sa\,(\omega \cdot t_0)
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = 2 \cdot A \cdot t_0 \cdot Sa\,(\omega \cdot t_0)$.

Task 4. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



W pierwszej kolejności trzeba wyznaczyć jawną postać równań opisujących funkcję $f(t)$.

W tym celu wyznaczamy równanie prostej na odcinku $(-t_0, t_0)$

Ogólne równanie prostej to:

$$f(t) = m \cdot t + b \tag{3.50}$$

Dla rozważanego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(-t_0, 0)$ oraz (t_0, A). Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} 0 = m \cdot (-t_0) + b \\ A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} -b = m \cdot (-t_0) \\ A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ A = \frac{b}{t_0} \cdot t_0 + b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ A = b + b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ A = 2 \cdot b \end{cases}$$

$$\begin{cases} \frac{b}{t_0} = m \\ \frac{A}{2} = b \end{cases}$$

$$\begin{cases} \frac{A}{2 \cdot t_0} = m \\ \frac{A}{2} = b \end{cases}$$

Równianie prostej dla t z zakresu $(-t_0, t_0)$ to:

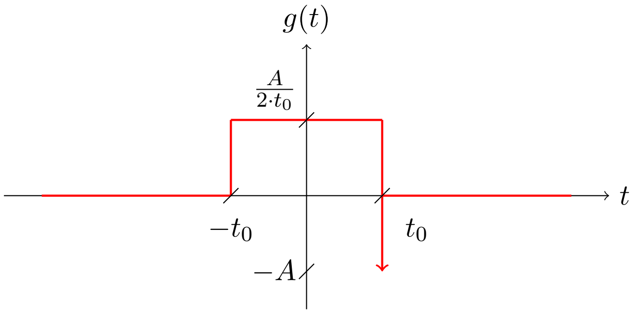
$$f(t) = \frac{A}{2 \cdot t_0} \cdot t + \frac{A}{2}$$

Podsumowując, sygnal $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} \cdot t + \frac{A}{2} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.51}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} - A \cdot \delta(t - t_0) \tag{3.52}$$

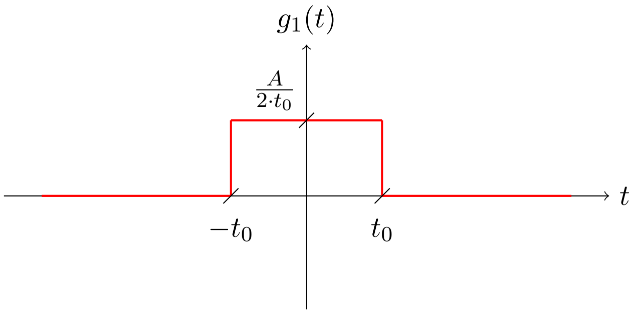


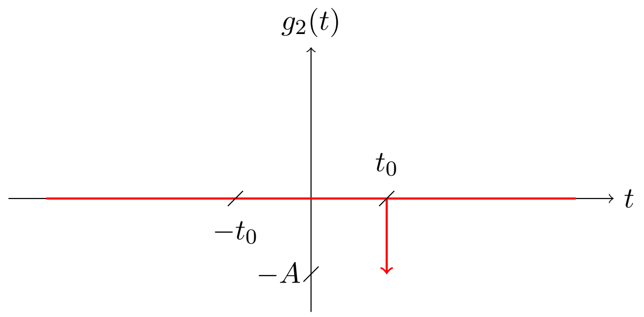
Funkcja $g(t)$ składa się z dwóch sygnałów $g_1(t)$ i $g_2(t)$

$$g(t) = g_1(t) + g_2(t) \tag{3.53}$$

$$g_1(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ \frac{A}{2 \cdot t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.54}$$

$$g_2(t) = -A \cdot \delta(t - t_0) \tag{3.55}$$





Wyznaczenie transformaty sygnału $g_2(t)$ złożonego z delty diracka jest znacznie prostsze.

$$G_2(\jmath\omega) = \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.56}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
G_2(\jmath\omega) &= \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (-A \cdot \delta(t - t_0)) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= -A \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= -A \cdot e^{-\jmath\cdot\omega\cdot t_0}
\end{aligned}$$

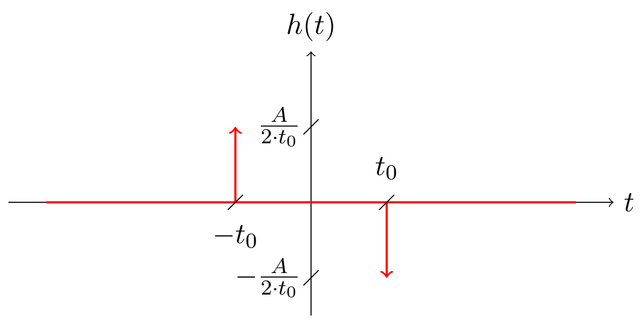
Transformata sygnału $g_2(t)$ to $G_2(\jmath\omega) = -A \cdot e^{-\jmath\cdot\omega\cdot t_0}$

Funkcja $g_1(t)$ jest jeszcze zbyt złożona tak wiec wyznaczamy pochodną raz jeszcze

$$h(t) = g_1'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + \frac{A}{2 \cdot t_0}\delta(t + t_0) - \frac{A}{2 \cdot t_0}\delta(t - t_0) \tag{3.57}$$

czyli po prostu

$$h(t) = g_1'(t) = \frac{A}{2 \cdot t_0}\delta(t + t_0) - \frac{A}{2 \cdot t_0}\delta(t - t_0) \tag{3.58}$$



Wyznaczanie transformaty sygnału $h(t)$ złożonego z delt diracka jest znacznie prostsze.

$$H(\jmath\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.59}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
H(\jmath\omega) &= \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(\frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) - \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(\frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) \cdot e^{-\jmath\cdot\omega\cdot t} - \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \right) \cdot dt = \\
&= \int_{-\infty}^{\infty} \frac{A}{2 \cdot t_0} \cdot \delta(t + t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt - \int_{-\infty}^{\infty} \frac{A}{2 \cdot t_0} \cdot \delta(t - t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot t_0} \cdot \int_{-\infty}^{\infty} \delta(t + t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt - \frac{A}{2 \cdot t_0} \cdot \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{ \textstyle\int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= \frac{A}{2 \cdot t_0} \cdot e^{-\jmath\cdot\omega\cdot(-t_0)} - \frac{A}{2 \cdot t_0} \cdot e^{-\jmath\cdot\omega\cdot t_0} = \\
&= \frac{A}{2 \cdot t_0} \cdot e^{\jmath\cdot\omega\cdot t_0} - \frac{A}{2 \cdot t_0} \cdot e^{-\jmath\cdot\omega\cdot t_0} = \\
&= \frac{A}{2 \cdot t_0} \cdot \left(e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0} \right) = \\
&= \frac{A}{2 \cdot t_0} \cdot \left(e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0} \right) \cdot \frac{\jmath}{\jmath} = \\
&= \frac{A}{t_0} \cdot \jmath \cdot \frac{e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \tfrac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath} \right\} = \\
&= \frac{A}{t_0} \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= \jmath \cdot \frac{A}{t_0} \cdot sin\,(\omega \cdot t_0)
\end{aligned}$$

Transformata sygnału $h(t)$ to $H(\jmath\omega) = \jmath \cdot \frac{A}{t_0} \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $g_1(t)$ na podstawie transformaty sygnału $h(t) = g_1'(t)$

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$
$$g_1(t) = \int_{-\infty}^{t} h(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0)$$

Podstawiając obliczona wcześniej transformatę $H(\jmath\omega)$ sygnału $h(t)$ otrzymujemy transformatę $G_1(\jmath\omega)$ sygnału $g_1(t)$

$$G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(0) \cdot H(0) =$$

$$
\begin{aligned}
&= \frac{1}{\jmath \cdot \omega} \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(\omega \cdot t_0\right) + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(0 \cdot t_0\right) = \\
&= \frac{1}{\omega} \cdot \frac{A}{t_0} \cdot sin\left(\omega \cdot t_0\right) + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot sin\left(0\right) = \\
&= A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} + \pi \cdot \delta(0) \cdot \jmath \cdot \frac{A}{t_0} \cdot 0 = \\
&= A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} + 0 = \\
&= \left\{ Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\
&= A \cdot Sa\left(\omega \cdot t_0\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $g_1(t)$ jest równa $G_1(\jmath\omega) = A \cdot Sa\left(\omega \cdot t_0\right)$.
Korzystając z jednorodności transformaty Fouriera

$$
\begin{aligned}
g_1(t) &\xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\
g_2(t) &\xrightarrow{\mathcal{F}} G_2(\jmath\omega) \\
g(t) = \alpha \cdot g_1(t) + \beta \cdot g_2(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) = \alpha \cdot G_1(\jmath\omega) + \beta \cdot G_2(\jmath\omega)
\end{aligned}
$$

można wyznaczyć transformatę Fouriera $G(\jmath\omega)$ funkcji $g(t)$

$$
\begin{aligned}
G(\jmath\omega) &= G_1(\jmath\omega) + G_2(\jmath\omega) = \\
&= A \cdot Sa\left(\omega \cdot t_0\right) - A \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= A \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)
\end{aligned}
$$

Znając transformatę $G(\jmath\omega)$ i korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $F(\jmath\omega)$ funkcji $f(t)$

$$
\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{aligned}
$$

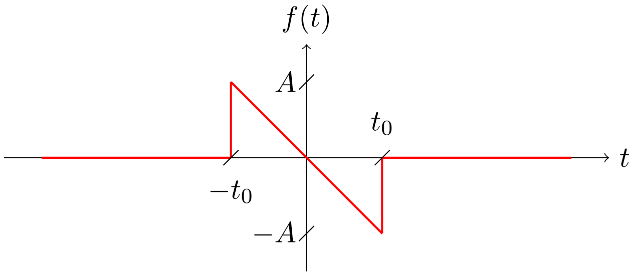
Podstawiając otrzymujemy

$$
\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(0) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot A \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot \left(Sa\left(0 \cdot t_0\right) - e^{-\jmath \cdot 0 \cdot t_0} \right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot \left(Sa\left(0\right) - e^{0} \right) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot (1 - 1) = \\
&= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + \pi \cdot \delta(0) \cdot A \cdot 0 =
\end{aligned}
$$

$$= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right) + 0 =$$
$$= \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = \frac{A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - e^{-\jmath \cdot \omega \cdot t_0} \right)$.

Task 5. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.



W pierwszej kolejności trzeba wyznaczyć jawną postać równań opisujących funkcję $f(t)$.

W tym celu wyznaczamy równanie prostej na odcinku $(-t_0, t_0)$

Ogólne równanie prostej to:

$$f(t) = m \cdot t + b \tag{3.60}$$

Dla rozważanego zakresu wartości t wykres funkcji jest prostą przechodzącą przez dwa punkty: $(-t_0, A)$ oraz $(t_0, -A)$. Możemy więc napisać układ równań, rozwiązać go i wyznaczyć parametry prostej m i b.

$$\begin{cases} A = m \cdot (-t_0) + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} A = -m \cdot t_0 + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} A - A = -m \cdot t_0 + b + m \cdot t_0 + b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = 2 \cdot b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 + b \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ -A = m \cdot t_0 \end{cases}$$

$$\begin{cases} 0 = b \\ -\frac{A}{t_0} = m \end{cases}$$

Równianie prostej dla t z zakresu $(-t_0, t_0)$ to:

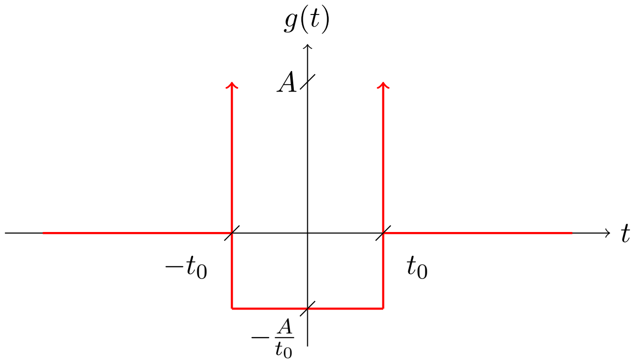
$$f(t) = -\frac{A}{t_0} \cdot t$$

Podsumowując, sygnal $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} \cdot t & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.61}$$

W pierwszej kolejności wyznaczamy pochodna sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} + A \cdot \delta(t + t_0) + A \cdot \delta(t - t_0) \tag{3.62}$$

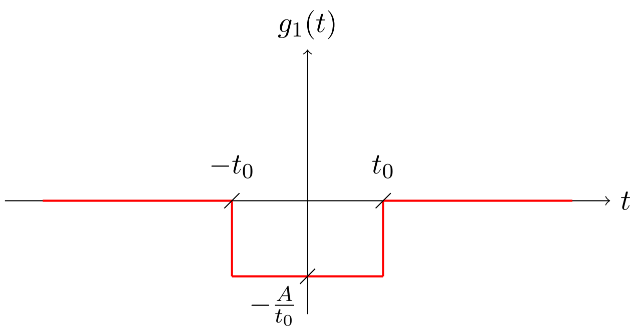


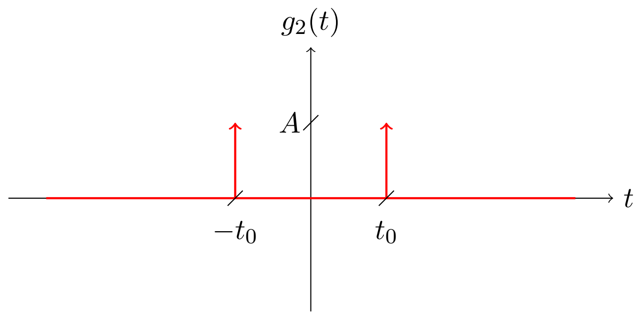
Funkcja $g(t)$ składa się z dwóch sygnałów $g_1(t)$ i $g_2(t)$

$$g(t) = g_1(t) + g_2(t) \tag{3.63}$$

$$g_1(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ -\frac{A}{t_0} & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} \tag{3.64}$$

$$g_2(t) = A \cdot \delta(t + t_0) + A \cdot \delta(t - t_0) \tag{3.65}$$





Wyznaczenie transformaty sygnału $g_2(t)$ złożonego z delt diracka jest znacznie prostsze.

$$G_2(\jmath\omega) = \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.66}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
G_2(\jmath\omega) &= \int_{-\infty}^{\infty} g_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (A \cdot \delta(t+t_0) + A \cdot \delta(t-t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \int_{-\infty}^{\infty} (\delta(t+t_0) + \delta(t-t_0)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= A \cdot \left(\int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= A \cdot \left(e^{-\jmath \cdot \omega \cdot (-t_0)} + e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0} \right) = \\
&= A \cdot \left(e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0} \right) \cdot \frac{2}{2} = \\
&= 2 \cdot A \cdot \frac{e^{\jmath \cdot \omega \cdot t_0} + e^{-\jmath \cdot \omega \cdot t_0}}{2} = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= 2 \cdot A \cdot cos\,(\omega \cdot t_0)
\end{aligned}$$

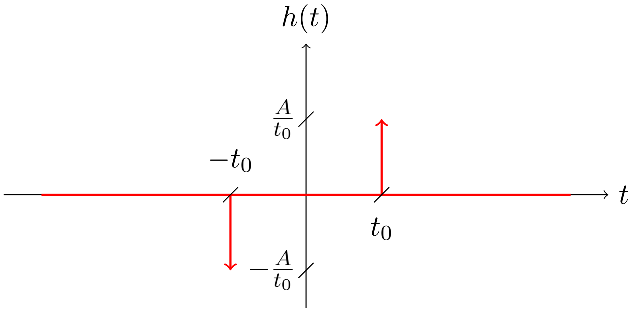
Transformata sygnału $g_2(t)$ to $G_2(\jmath\omega) = 2 \cdot A \cdot cos\,(\omega \cdot t_0)$

Funkcja $g_1(t)$ jest jeszcze zbyt złożona tak wiec wyznaczamy pochodną raz jeszcze

$$h(t) = g_1'(t) = \begin{cases} 0 & t \in (-\infty; -t_0) \\ 0 & t \in (-t_0; t_0) \\ 0 & t \in (t_0; \infty) \end{cases} - \frac{A}{t_0}\delta(t+t_0) + \frac{A}{t_0}\delta(t-t_0) \tag{3.67}$$

czyli po prostu

$$h(t) = g_1'(t) = -\frac{A}{t_0}\delta(t+t_0) + \frac{A}{t_0}\delta(t-t_0) \tag{3.68}$$



Wyznaczanie transformaty sygnału $h(t)$ złożonego z delt diracka jest znacznie prostsze.

$$H(\jmath\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt \tag{3.69}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
H(\jmath\omega) &= \int_{-\infty}^{\infty} h(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(-\frac{A}{t_0} \cdot \delta(t+t_0) + \frac{A}{t_0} \cdot \delta(t-t_0)\right) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} \left(-\frac{A}{t_0} \cdot \delta(t+t_0) \cdot e^{-\jmath\cdot\omega\cdot t} + \frac{A}{t_0} \cdot \delta(t-t_0) \cdot e^{-\jmath\cdot\omega\cdot t}\right) \cdot dt = \\
&= -\int_{-\infty}^{\infty} \frac{A}{t_0} \cdot \delta(t+t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{-\infty}^{\infty} \frac{A}{t_0} \cdot \delta(t-t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= -\frac{A}{t_0} \cdot \int_{-\infty}^{\infty} \delta(t+t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \frac{A}{t_0} \cdot \int_{-\infty}^{\infty} \delta(t-t_0) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \left\{\int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0)\right\} = \\
&= -\frac{A}{t_0} \cdot e^{-\jmath\cdot\omega\cdot(-t_0)} + \frac{A}{t_0} \cdot e^{-\jmath\cdot\omega\cdot t_0} = \\
&= -\frac{A}{t_0} \cdot e^{\jmath\cdot\omega\cdot t_0} + \frac{A}{t_0} \cdot e^{-\jmath\cdot\omega\cdot t_0} = \\
&= -\frac{A}{t_0} \cdot \left(e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0}\right) = \\
&= -\frac{A}{t_0} \cdot \left(e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0}\right) \cdot \frac{2\cdot\jmath}{2\cdot\jmath} = \\
&= -\frac{2\cdot A}{t_0} \cdot \jmath \cdot \frac{e^{\jmath\cdot\omega\cdot t_0} - e^{-\jmath\cdot\omega\cdot t_0}}{2\cdot\jmath} = \\
&= \left\{sin\,(x) = \frac{e^{\jmath\cdot x} - e^{-\jmath\cdot x}}{2\cdot\jmath}\right\} = \\
&= -\frac{2\cdot A}{t_0} \cdot \jmath \cdot sin\,(\omega \cdot t_0) = \\
&= -\jmath \cdot \frac{2\cdot A}{t_0} \cdot sin\,(\omega \cdot t_0)
\end{aligned}$$

Transformata sygnału $h(t)$ to $H(\jmath\omega) = -\jmath \cdot \frac{2\cdot A}{t_0} \cdot sin\,(\omega \cdot t_0)$

Następnie możemy wykorzystać twierdzenie o całkowaniu aby wyznaczyć transformatę sygnału $g_1(t)$ na podstawie transformaty sygnału $h(t) = g_1'(t)$

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$

$$g_1(t) = \int_{-\infty}^{t} h(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} G_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0)$$

Podstawiając obliczona wcześniej transformatę $H(\jmath\omega)$ sygnału $h(t)$ otrzymujemy transformatę $G_1(\jmath\omega)$ sygnału $g_1(t)$

$$\begin{aligned}
G_1(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot H(\jmath\omega) + \pi \cdot \delta(\omega) \cdot H(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \left(-\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(\omega \cdot t_0\right) \right) + \pi \cdot \delta(\omega) \cdot \left(-\jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(0 \cdot t_0\right) \right) = \\
&= -\frac{1}{\omega} \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(\omega \cdot t_0\right) - \pi \cdot \delta(\omega) \cdot \jmath \cdot \frac{2 \cdot A}{t_0} \cdot sin\left(0\right) = \\
&= -2 \cdot A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} - \pi \cdot \delta(\omega) \cdot \jmath \cdot \frac{2 \cdot A}{t_0} \cdot 0 = \\
&= -2 \cdot A \cdot \frac{sin\left(\omega \cdot t_0\right)}{\omega \cdot t_0} - 0 = \\
&= \left\{ Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\
&= -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right)
\end{aligned}$$

Ostatecznie transformata sygnału $g_1(t)$ jest równa $G_1(\jmath\omega) = -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right)$.

Korzystając z jednorodności transformaty Fouriera

$$\begin{gathered}
g_1(t) \xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\
g_2(t) \xrightarrow{\mathcal{F}} G_2(\jmath\omega) \\
g(t) = \alpha \cdot g_1(t) + \beta \cdot g_2(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \alpha \cdot G_1(\jmath\omega) + \beta \cdot G_2(\jmath\omega)
\end{gathered}$$

można wyznaczyć transformatę Fouriera $G(\jmath\omega)$ funkcji $g(t)$

$$\begin{aligned}
G(\jmath\omega) &= G_1(\jmath\omega) + G_2(\jmath\omega) = \\
&= -2 \cdot A \cdot Sa\left(\omega \cdot t_0\right) + 2 \cdot A \cdot cos\left(\omega \cdot t_0\right) = \\
&= -2 \cdot A \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right)
\end{aligned}$$

Znając transformatę $G(\jmath\omega)$ i korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $F(\jmath\omega)$ funkcji $f(t)$

$$\begin{gathered}
g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{gathered}$$

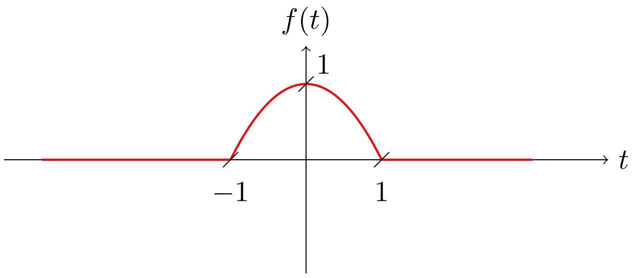
Podstawiając otrzymujemy

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) =$$

$$
\begin{aligned}
&= \frac{1}{\jmath \cdot \omega} \cdot \left(-2 \cdot A \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right)\right) + \pi \cdot \delta(\omega) \cdot \left(-2 \cdot A \cdot \left(Sa\left(0 \cdot t_0\right) - cos\left(0 \cdot t_0\right)\right)\right) = \\
&= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot \left(Sa\left(0\right) - cos\left(0\right)\right) = \\
&= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot (1-1) = \\
&= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right) - \pi \cdot \delta(\omega) \cdot 2 \cdot A \cdot 0 = \\
&= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right) - 0 = \\
&= -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = -\frac{2 \cdot A}{\jmath \cdot \omega} \cdot \left(Sa\left(\omega \cdot t_0\right) - cos\left(\omega \cdot t_0\right)\right)$.

Task 6. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń.

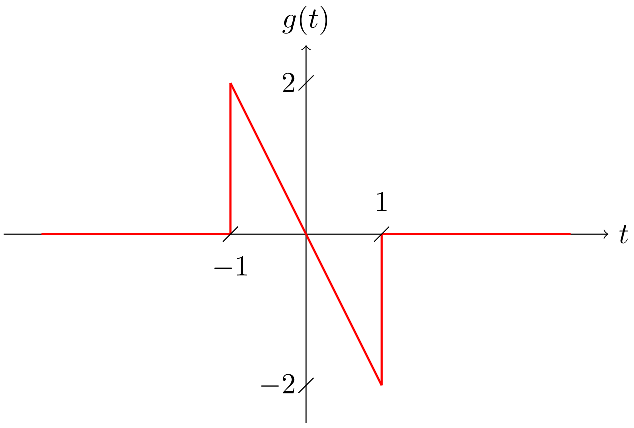


Sygnal $f(t)$ możemy opisać jako:

$$f(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ 1 - t^2 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} \tag{3.70}$$

W pierwszej kolejności wyznaczamy pochodną sygnału $f(t)$

$$g(t) = f'(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ -2 \cdot t & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} \tag{3.71}$$



Można sprawdzić, że całkując sygnał $g(t)$ otrzymamy sygnał $f(t)$, czyli:

$$f(t) = \int_{-\infty}^{t} g(x) \cdot dx \tag{3.72}$$

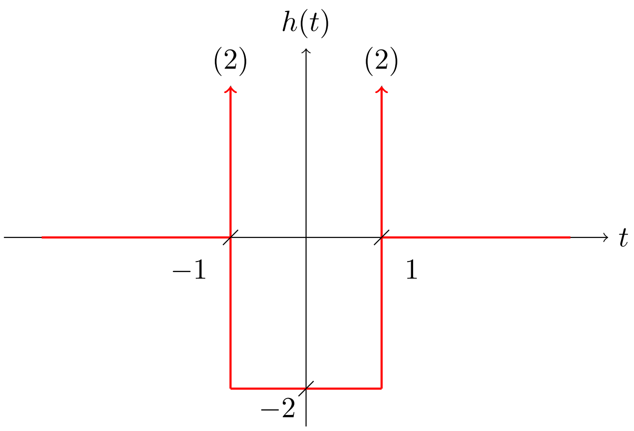
Skoro tak jest, to transformatę sygnału $f(t)$ mozna wyznaczyć z twierdzenia o całkowaniu sygnału, w tym przypadku całkować będziemy sygnał $g(t)$:

$$F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) \tag{3.73}$$

Pytanie, czy można dalej uprościc sygnał $g(t)$ dokonując jego rózniczkowania. Wyznaczmy pochodną sygnału $g(t)$, czyli drugą pochodną sygnału $f(t)$:

$$h(t) = \frac{\partial}{\partial t} g(t) = \frac{\partial^2}{\partial t^2} f(t) \tag{3.74}$$

$$h(t) = g'(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ -2 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} + 2 \cdot \delta(t+1) + 2 \cdot \delta(t-1) \tag{3.75}$$

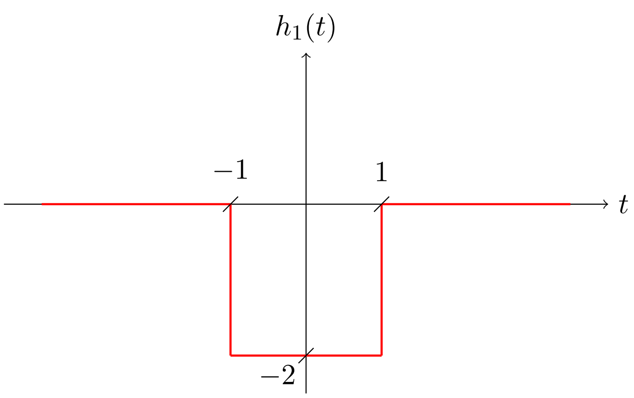


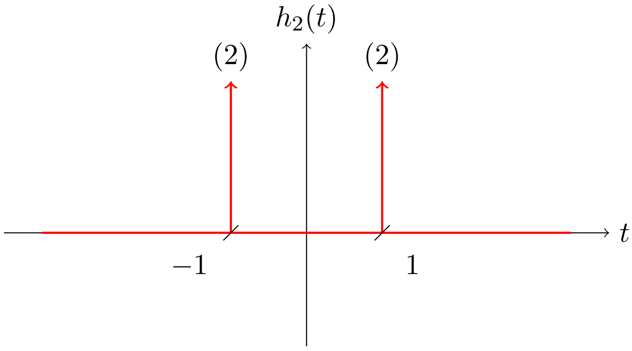
Funkcja $h(t)$ składa się z dwóch sygnałów $h_1(t)$ i $h_2(t)$

$$h(t) = h_1(t) + h_2(t) \tag{3.76}$$

$$h_1(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ -2 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} \tag{3.77}$$

$$h_2(t) = 2 \cdot \delta(t+1) + 2 \cdot \delta(t-1) \tag{3.78}$$





Wyznaczenie transformaty sygnału $h_2(t)$ złożonego z delt Diracka jest znacznie prostsze.

$$H_2(\jmath\omega) = \int_{-\infty}^{\infty} h_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.79}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
H_2(\jmath\omega) &= \int_{-\infty}^{\infty} h_2(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (2 \cdot \delta(t+1) + 2 \cdot \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \int_{-\infty}^{\infty} (\delta(t+1) + \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \left(\int_{-\infty}^{\infty} \delta(t+1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-\infty}^{\infty} \delta(t-1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= 2 \cdot \left(e^{-\jmath \cdot \omega \cdot (-1)} + e^{-\jmath \cdot \omega \cdot 1} \right) = \\
&= 2 \cdot \left(e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega} \right) = \\
&= 2 \cdot \left(e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega} \right) \cdot \frac{2}{2} = \\
&= 4 \cdot \frac{e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega}}{2} = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= 4 \cdot cos\,(\omega)
\end{aligned}$$

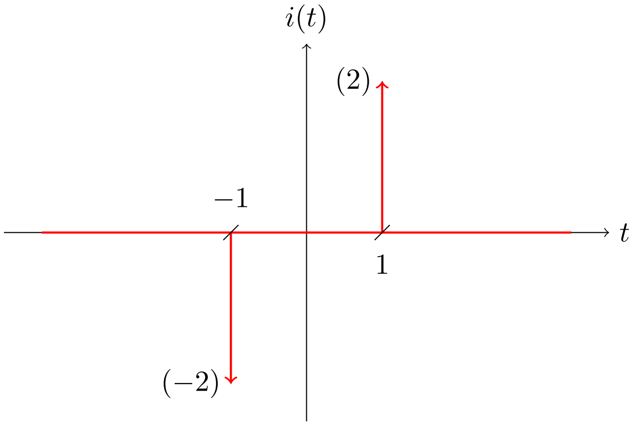
Transformata sygnału $h_2(t)$ to $G_2(\jmath\omega) = 4 \cdot cos\,(\omega)$

Funkcja $h_1(t)$ jest jeszcze zbyt złożona, więc wyznaczamy pochodną raz jeszcze

$$i(t) = h_1'(t) = \begin{cases} 0 & t \in (-\infty; -1) \\ 0 & t \in (-1; 1) \\ 0 & t \in (1; \infty) \end{cases} - 2\delta(t+1) + 2\delta(t-1) \tag{3.80}$$

,czyli po prostu:

$$i(t) = h_1'(t) = -2\delta(t+1) + 2\delta(t-1) \tag{3.81}$$



Wyznaczanie transformaty sygnału $i(t)$ złożonego z delt Diracka jest znacznie prostsze.

$$I(\jmath\omega) = \int_{-\infty}^{\infty} i(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \tag{3.82}$$

Podstawiamy do wzoru na transformatę wzór naszej funkcji

$$\begin{aligned}
I(\jmath\omega) &= \int_{-\infty}^{\infty} i(t) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} (-2 \cdot \delta(t+1) + 2 \cdot \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \int_{-\infty}^{\infty} (-\delta(t+1) + \delta(t-1)) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt = \\
&= 2 \cdot \left(\int_{-\infty}^{\infty} -\delta(t+1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt + \int_{-\infty}^{\infty} \delta(t-1) \cdot e^{-\jmath \cdot \omega \cdot t} \cdot dt \right) = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t-t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= 2 \cdot \left(-e^{-\jmath \cdot \omega \cdot (-1)} + e^{-\jmath \cdot \omega \cdot 1} \right) = \\
&= 2 \cdot \left(-e^{\jmath \cdot \omega} + e^{-\jmath \cdot \omega} \right) = \\
&= -2 \cdot \left(e^{\jmath \cdot \omega} - e^{-\jmath \cdot \omega} \right) \cdot \frac{2 \cdot \jmath}{2 \cdot \jmath} = \\
&= -4 \cdot \jmath \cdot \frac{e^{\jmath \cdot \omega} - e^{-\jmath \cdot \omega}}{2 \cdot \jmath} = \\
&= \left\{ sin\,(x) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= -4 \cdot \jmath \cdot sin\,(\omega)
\end{aligned}$$

Transformata sygnału $i(t)$ to $I(\jmath\omega) = -4 \cdot \jmath \cdot sin\,(\omega)$

Następnie możemy wykorzystać twierdzenie o całkowaniu, aby wyznaczyć transformatę sygnału $h_1(t)$ na podstawie transformaty sygnału $i(t) = h_1'(t)$

$$i(t) \xrightarrow{\mathcal{F}} I(\jmath\omega)$$

$$h_1(t) = \int_{-\infty}^{t} i(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} H_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot I(\jmath\omega) + \pi \cdot \delta(\omega) \cdot I(0)$$

Podstawiając obliczoną wcześniej transformatę $I(\jmath\omega)$ sygnału $i(t)$ otrzymujemy transformatę $H_1(\jmath\omega)$ sygnału $h_1(t)$

$$
\begin{aligned}
H_1(\jmath\omega) &= \frac{1}{\jmath\cdot\omega}\cdot I(\jmath\omega)+\pi\cdot\delta(\omega)\cdot I(0) = \\
&= \left\{\begin{matrix} I(0) = -4\cdot\jmath\cdot sin\left(0\right) \\ I(0) = -4\cdot\jmath\cdot 0 \\ I(0) = 0 \end{matrix}\right\} = \\
&= \frac{1}{\jmath\cdot\omega}\cdot\left(-4\cdot\jmath\cdot sin\left(\omega\right)\right)+0 = \\
&= -4\cdot\frac{sin\left(\omega\right)}{\omega} = \\
&= \left\{Sa\left(x\right)=\tfrac{sin(x)}{x}\right\} = \\
&= -4\cdot Sa\left(\omega\right)
\end{aligned}
$$

Ostatecznie transformata sygnału $h_1(t)$ jest równa $H_1(\jmath\omega) = -4\cdot Sa\left(\omega\right)$.

Korzystając z liniowości transformacji Fouriera

$$
\begin{aligned}
h_1(t) &\xrightarrow{\mathcal{F}} H_1(\jmath\omega) \\
h_2(t) &\xrightarrow{\mathcal{F}} H_2(\jmath\omega) \\
h(t)=\alpha\cdot h_1(t)+\beta\cdot h_2(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega)=\alpha\cdot H_1(\jmath\omega)+\beta\cdot H_2(\jmath\omega)
\end{aligned}
$$

można wyznaczyć transformatę Fouriera $H(\jmath\omega)$ funkcji $h(t)$

$$
\begin{aligned}
H(\jmath\omega) &= H_1(\jmath\omega)+H_2(\jmath\omega) = \\
&= -4\cdot Sa\left(\omega\right)+4\cdot cos\left(\omega\right) = \\
&= 4\cdot\left(cos\left(\omega\right)-Sa\left(\omega\right)\right)
\end{aligned}
$$

Znając transformatę $H(\jmath\omega)$ i korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $G(\jmath\omega)$ funkcji $g(t)$

$$
\begin{aligned}
h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) \\
g(t)=\int_{-\infty}^{t} h(\tau)\cdot d\tau &\xrightarrow{\mathcal{F}} G(\jmath\omega)=\frac{1}{\jmath\cdot\omega}\cdot H(\jmath\omega)+\pi\cdot\delta(\omega)\cdot H(0)
\end{aligned}
$$

Podstawiając odpowiednie dane otrzymujemy:

$$
\begin{aligned}
G(\jmath\omega) &= \frac{1}{\jmath\cdot\omega}\cdot H(\jmath\omega)+\pi\cdot\delta(\omega)\cdot H(0) = \\
&= \left\{\begin{matrix} H(0) = 4\cdot\left(cos\left(0\right)-Sa\left(0\right)\right) \\ H(0) = 4\cdot(1-1) \\ H(0) = 4\cdot 0 \\ H(0) = 0 \end{matrix}\right\} = \\
&= \frac{1}{\jmath\cdot\omega}\cdot\left(4\cdot\left(cos\left(\omega\right)-Sa\left(\omega\right)\right)\right)+0 =
\end{aligned}
$$

$$= \frac{4}{\jmath \cdot \omega} \cdot (cos\,(\omega) - Sa\,(\omega))$$

Ostatecznie transformata sygnału $g(t)$ jest równa $G(\jmath\omega) = \frac{4}{\jmath\cdot\omega} \cdot (cos\,(\omega) - Sa\,(\omega))$.

Znając transformatę $G(\jmath\omega)$ i kolejny raz korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę $F(\jmath\omega)$ funkcji $f(t)$

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)$$

Podstawiając odpowiednie dane otrzymujemy:

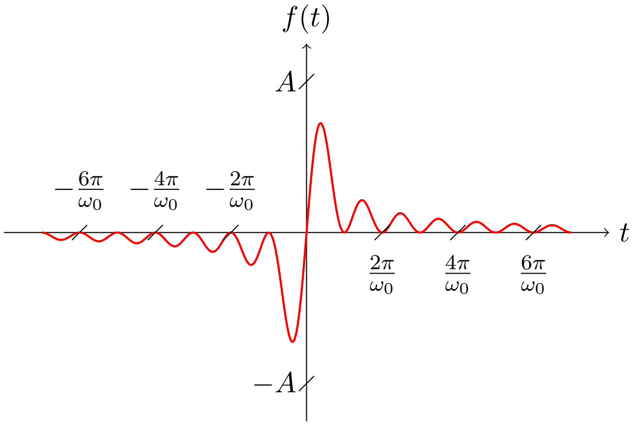
$$\begin{aligned}
F(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \left\{ \begin{array}{l} G(0) = \frac{4}{\jmath\cdot 0} \cdot (cos\,(0) - Sa\,(0)) \\ G(0) = \frac{0}{0}!!! \\ G(0) = \int_{-\infty}^{\infty} g(t) \cdot dt = \int_{-1}^{1} (-2) \cdot t \cdot dt = (-2) \cdot \frac{t^2}{2}\Big|_{-1}^{1} \\ G(0) = (-2) \cdot \left(\frac{1}{2} - \frac{1}{2}\right) = (-2) \cdot 0 \\ G(0) = 0 \end{array} \right\} = \\
&= \frac{1}{\jmath \cdot \omega} \cdot \frac{4}{\jmath \cdot \omega} \cdot (cos\,(\omega) - Sa\,(\omega)) + 0 = \\
&= \frac{4}{\jmath^2 \cdot \omega^2} \cdot (cos\,(\omega) - Sa\,(\omega)) = \\
&= \frac{4}{(-1) \cdot \omega^2} \cdot (cos\,(\omega) - Sa\,(\omega)) = \\
&= \frac{4}{\omega^2} \cdot (Sa\,(\omega) - cos\,(\omega))
\end{aligned}$$

Ostatecznie transformata sygnału $f(t)$ jest równa $F(\jmath\omega) = \frac{4}{\omega^2} \cdot (Sa\,(\omega) - cos\,(\omega))$.

Task 7. Oblicz transformatę Fouriera sygnału $f(t) = Sa\left(\omega_0 \cdot t\right) \cdot sin\left(\omega_0 \cdot t\right)$ za pomocą twierdzeń, wiedząc że transformata sygnału $\Pi(t)$ jest rowna $Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa\left(\omega_0 \cdot t\right) \cdot sin\left(\omega_0 \cdot t\right) \tag{3.83}$$

$$\Pi(t) \xrightarrow{F} Sa\left(\frac{\omega}{2}\right) \tag{3.84}$$



W pierwszej kolejności można funkcję $f(t)$ rozpisać następująco

$$\begin{aligned}
f(t) &= Sa\left(\omega_0 \cdot t\right) \cdot sin\left(\omega_0 \cdot t\right) = \\
&= \left\{ sin\left(x\right) = \tfrac{e^{\jmath \cdot x} - e^{-\jmath \cdot x}}{2 \cdot \jmath} \right\} = \\
&= Sa\left(\omega_0 \cdot t\right) \cdot \frac{e^{\jmath \cdot \omega_0 \cdot t} - e^{-\jmath \cdot \omega_0 \cdot t}}{2 \cdot \jmath} = \\
&= \frac{1}{2 \cdot \jmath} \cdot \left(Sa\left(\omega_0 \cdot t\right) \cdot e^{\jmath \cdot \omega_0 \cdot t} - Sa\left(\omega_0 \cdot t\right) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \right) = \\
&= \left\{ \begin{aligned} f_1(t) &= Sa\left(\omega_0 \cdot t\right) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\ f_2(t) &= Sa\left(\omega_0 \cdot t\right) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \end{aligned} \right\} = \\
&= \frac{1}{2 \cdot \jmath} \cdot \left(f_1(t) - f_2(t)\right)
\end{aligned}$$

Należy zauważyć iż funkcja $f_1(t)$ i $f_2(t)$ jest złożeniem funkcji Sa i funkcji wykładniczych.

$$\begin{aligned}
f_1(t) &= Sa\left(\omega_0 \cdot t\right) \cdot e^{\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\
f_2(t) &= Sa\left(\omega_0 \cdot t\right) \cdot e^{-\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{-\jmath \cdot \omega_0 \cdot t}
\end{aligned}$$

Znając transformatę sygnału $g(t) = Sa\left(\omega_0 \cdot t\right)$ możemy skorzystać z twierdzenia o przesunięciu w dziedzinie częstotliwości.

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$f(t) = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F(\jmath\omega) = G(\jmath\,(\omega - \omega_0))$$

Aby wyznaczyć transformatę sygnału $g(t)$ możemy skorzystać z twierdzenia o symetrii. Znając transformatę $H(\jmath\omega)$ sygnału $h(t)$ można wyznaczyć transformatę $G(\jmath\omega)$ sygnału $g(t)$

$$h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$$
$$g(t) = H(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = 2\pi \cdot h(-\omega)$$

Tak wiec zacznijmy od transformaty sygnału prostokątnego $h(t) = \Pi(t)$ i wyznaczymy transformatę funkcji Sa

$$h(t) = \Pi(t) \xrightarrow{F} H(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$$
$$g_1(t) = H(t) = Sa\left(\frac{t}{2}\right) \xrightarrow{F} G_1(\jmath\omega) = 2\pi \cdot h(-\jmath\omega) = \pi \cdot \Pi\,(-\omega) = 2\pi \cdot \Pi\,(\omega)$$

Wyznaczyliśmy transformatę funkcji $g_1(t)$. Jednak funkcja $g_1(t)$ nie ma takiej samej postaci jak funkcja $g(t)$

$$\begin{aligned}
g(t) = Sa\,(\omega_0 \cdot t) = \\
&= Sa\left(\omega_0 \cdot t \cdot \frac{2}{2}\right) = \\
&= Sa\left(2 \cdot \omega_0 \cdot \frac{t}{2}\right) = \\
&= Sa\left(\frac{2 \cdot \omega_0 \cdot t}{2}\right) = \\
&= \left\{a = 2 \cdot \omega_0\right\} = \\
&= Sa\left(\frac{a \cdot t}{2}\right) = \\
&= g_1(a \cdot t)
\end{aligned}$$

Znając transformatę funkcji $g_1(t)$ możemy wyznaczyć transformatę funkcji $g(t) = g_1(a \cdot t)$ za pomocą twierdzenia o zmianie skali.

$$g_1(t) \xrightarrow{\mathcal{F}} G_1(\jmath\omega)$$
$$g(t) = g_1(\alpha \cdot t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha})$$

Podstawiając wyznaczoną transformatę $G_1(\jmath\omega)$

$$\begin{aligned}
G(\jmath\omega) &= \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha}) = \\
&= \left\{\alpha = 2 \cdot \omega_0\right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|2\cdot\omega_0|}\cdot G_1(\frac{\omega}{2\cdot\omega_0}) = \\
&= \left\{G_1(\jmath\omega) = 2\pi\cdot\Pi\left(\omega\right)\right\} = \\
&= \frac{1}{2\cdot\omega_0}\cdot 2\pi\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right) = \\
&= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)
\end{aligned}
$$

Tak wiec transformata sygnału $g(t) = Sa\left(\omega_0\cdot t\right)$ jest równa $G(\jmath\omega) = \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)$
Kolejnym krokiem jest wyznaczenie transformaty dwóch sygnałów

$$
\begin{aligned}
f_1(t) &= Sa\left(\omega_0\cdot t\right)\cdot e^{\jmath\cdot\omega_0\cdot t} \\
f_2(t) &= Sa\left(\omega_0\cdot t\right)\cdot e^{-\jmath\cdot\omega_0\cdot t}
\end{aligned}
$$

Korzystając z twierdzenie o przesunięciu w dziedzinie częstotliwości

$$
\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f_1(t) = g(t)\cdot e^{\jmath\cdot\omega_0\cdot t} &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) = G(\jmath\left(\omega-\omega_0\right))
\end{aligned}
$$

otrzymujemy wprost

$$
\begin{aligned}
F_1(\jmath\omega) &= G\left(\jmath\left(\omega-\omega_0\right)\right) = \\
&= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_2(\jmath\omega) &= G\left(\jmath\left(\omega+\omega_0\right)\right) = \\
&= \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)
\end{aligned}
$$

Ostatecznie korzystając z liniowości transformaty Fouriera

$$
\begin{aligned}
f_1(t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) \\
f_2(t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega) \\
f(t) = \alpha\cdot f_1(t) + \beta\cdot f_2(t) &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha\cdot F_1(\jmath\omega) + \beta\cdot F_2(\jmath\omega)
\end{aligned}
$$

otrzymujemy

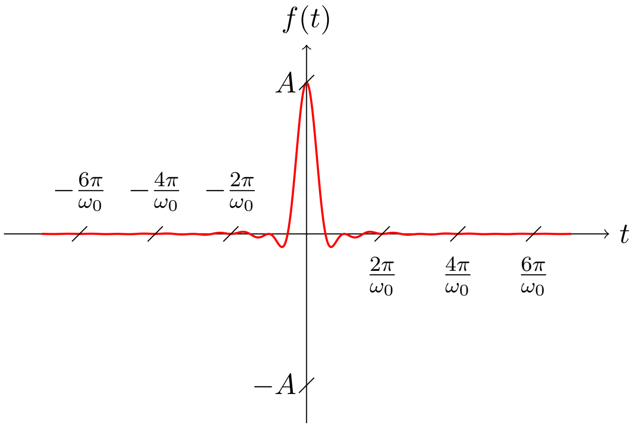
$$
\begin{aligned}
F(\jmath\omega) &= F_1(\jmath\omega) - F_2(\jmath\omega) = \\
&= \frac{1}{2\cdot\jmath}\cdot\left(\frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) - \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)
\end{aligned}
$$

Transformata Fouriera sygnału $f(t)$ jest równa $F(\jmath\omega) = \frac{1}{2\cdot\jmath}\cdot\left(\frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) - \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)$

Task 8. Oblicz transformatę Fouriera sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ za pomocą twierdzeń, wiedząc że transformata sygnału $\Lambda(t)$ jest rowna $Sa^2\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) \tag{3.85}$$

$$\Lambda(t) \xrightarrow{F} Sa^2\left(\frac{\omega}{2}\right) \tag{3.86}$$



W pierwszej kolejności można funkcję $f(t)$ rozpisać następująco

$$\begin{aligned}
f(t) &= Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) = \\
&= \left\{ cos(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= Sa^2(\omega_0 \cdot t) \cdot \frac{e^{\jmath \cdot \omega_0 \cdot t} + e^{-\jmath \cdot \omega_0 \cdot t}}{2} = \\
&= \frac{1}{2} \cdot \left(Sa^2(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} + Sa^2(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \right) = \\
&= \begin{Bmatrix} f_1(t) & = Sa^2(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\ f_2(t) & = Sa^2(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} \end{Bmatrix} = \\
&= \frac{1}{2} \cdot (f_1(t) + f_2(t))
\end{aligned}$$

Należy zauważyć iż funkcja $f_1(t)$ i $f_2(t)$ jest złożeniem funkcji Sa^2 i funkcji wykładniczych.

$$\begin{aligned}
f_1(t) &= Sa(\omega_0 \cdot t) \cdot e^{\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\
f_2(t) &= Sa(\omega_0 \cdot t) \cdot e^{-\jmath \cdot \omega_0 \cdot t} = g(t) \cdot e^{-\jmath \cdot \omega_0 \cdot t}
\end{aligned}$$

Znając transformatę sygnału $g(t) = Sa(\omega_0 \cdot t)$ możemy skorzystać z twierdzenia o przesunięciu w dziedzinie częstotliwości.

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$f(t) = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F(\jmath\omega) = G(\jmath\,(\omega - \omega_0))$$

Aby wyznaczyć transformatę sygnału $g(t)$ możemy skorzystać z twierdzenia o symetrii. Znając transformatę $H(\jmath\omega)$ sygnału $h(t)$ można wyznaczyć transformatę $G(\jmath\omega)$ sygnału $g(t)$

$$\begin{aligned} h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) \\ g(t) = H(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) = 2\pi \cdot h(-\omega) \end{aligned}$$

Tak wiec zacznijmy od transformaty sygnału prostokątnego $h(t) = \Pi(t)$ i wyznaczymy transformatę funkcji Sa

$$\begin{aligned} h(t) = \Lambda(t) &\xrightarrow{F} H(\jmath\omega) = Sa^2\left(\frac{\omega}{2}\right) \\ g_1(t) = H(t) = Sa^2\left(\frac{t}{2}\right) &\xrightarrow{F} G_1(\jmath\omega) = 2\pi \cdot h(-\jmath\omega) = \pi \cdot \Lambda\,(-\omega) = 2\pi \cdot \Lambda\,(\omega) \end{aligned}$$

Wyznaczyliśmy transformatę funkcji $g_1(t)$. Jednak funkcja $g_1(t)$ nie ma takiej samej postaci jak funkcja $g(t)$

$$\begin{aligned} g(t) &= Sa^2\,(\omega_0 \cdot t) = \\ &= Sa^2\left(\omega_0 \cdot t \cdot \frac{2}{2}\right) = \\ &= Sa^2\left(2 \cdot \omega_0 \cdot \frac{t}{2}\right) = \\ &= Sa^2\left(\frac{2 \cdot \omega_0 \cdot t}{2}\right) = \\ &= \Big\{a = 2 \cdot \omega_0\Big\} = \\ &= Sa^2\left(\frac{a \cdot t}{2}\right) = \\ &= g_1(a \cdot t) \end{aligned}$$

Znając transformatę funkcji $g_1(t)$ możemy wyznaczyć transformatę funkcji $g(t) = g_1(a \cdot t)$ za pomocą twierdzenia o zmianie skali.

$$\begin{aligned} g_1(t) &\xrightarrow{\mathcal{F}} G_1(\jmath\omega) \\ g(t) = g_1(\alpha \cdot t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha}) \end{aligned}$$

Podstawiając wyznaczoną transformatę $G_1(\jmath\omega)$

$$\begin{aligned} G(\jmath\omega) &= \frac{1}{|\alpha|} \cdot G_1(\jmath\frac{\omega}{\alpha}) = \\ &= \Big\{\alpha = 2 \cdot \omega_0\Big\} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|2 \cdot \omega_0|} \cdot G_1(\frac{\omega}{2 \cdot \omega_0}) = \\
&= \left\{ G_1(\jmath\omega) = 2\pi \cdot \Lambda\left(\omega\right) \right\} = \\
&= \frac{1}{2 \cdot \omega_0} \cdot 2\pi \cdot \Lambda\left(\frac{\omega}{2 \cdot \omega_0}\right) = \\
&= \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega}{2 \cdot \omega_0}\right)
\end{aligned}$$

Tak wiec transformata sygnału $g(t) = Sa\left(\omega_0 \cdot t\right)$ jest równa $G(\jmath\omega) = \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega}{2 \cdot \omega_0}\right)$
Kolejnym krokiem jest wyznaczenie transformaty dwóch sygnałów

$$\begin{aligned}
f_1(t) &= Sa^2\left(\omega_0 \cdot t\right) \cdot e^{\jmath \cdot \omega_0 \cdot t} \\
f_2(t) &= Sa^2\left(\omega_0 \cdot t\right) \cdot e^{-\jmath \cdot \omega_0 \cdot t}
\end{aligned}$$

Korzystając z twierdzenie o przesunięciu w dziedzinie częstotliwości

$$\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f_1(t) = g(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) = G(\jmath\left(\omega - \omega_0\right))
\end{aligned}$$

otrzymujemy wprost

$$\begin{aligned}
F_1(\jmath\omega) &= G\left(\jmath\left(\omega - \omega_0\right)\right) = \\
&= \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega - \omega_0}{2 \cdot \omega_0}\right)
\end{aligned}$$

$$\begin{aligned}
F_2(\jmath\omega) &= G\left(\jmath\left(\omega + \omega_0\right)\right) = \\
&= \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega + \omega_0}{2 \cdot \omega_0}\right)
\end{aligned}$$

Ostatecznie korzystając z liniowości transformaty Fouriera

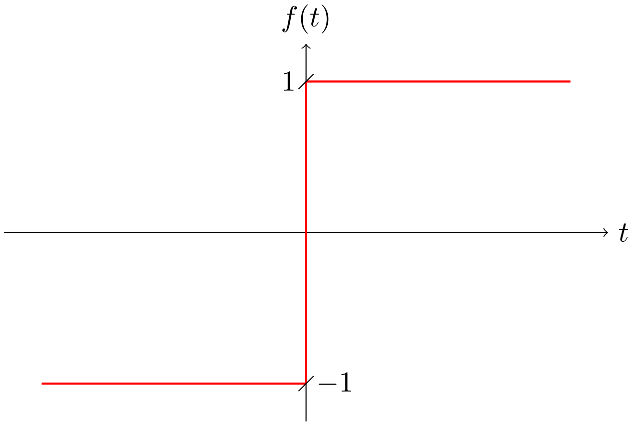
$$\begin{aligned}
f_1(t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) \\
f_2(t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega) \\
f(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t) &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot F_1(\jmath\omega) + \beta \cdot F_2(\jmath\omega)
\end{aligned}$$

otrzymujemy

$$\begin{aligned}
F(\jmath\omega) &= F_1(\jmath\omega) - F_2(\jmath\omega) = \\
&= \frac{1}{2} \cdot \left(\frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega - \omega_0}{2 \cdot \omega_0}\right) + \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega + \omega_0}{2 \cdot \omega_0}\right)\right)
\end{aligned}$$

Transformata Fouriera sygnału $f(t)$ jest równa $F(\jmath\omega) = \frac{1}{2} \cdot \left(\frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega - \omega_0}{2 \cdot \omega_0}\right) + \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega + \omega_0}{2 \cdot \omega_0}\right)\right)$

Task 9. Oblicz transformatę Fouriera sygnału $f(t) = sgn(t)$ za pomocą twierdzeń.

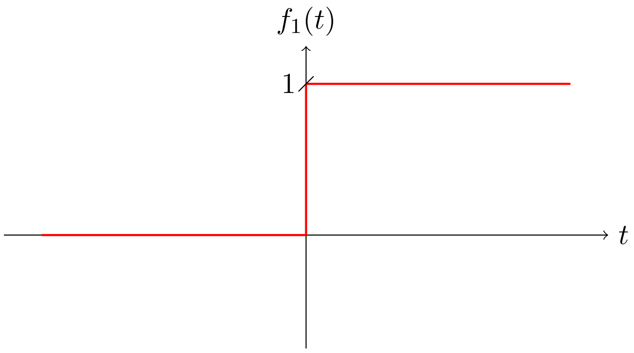


Sygnał $f(t)$ można zapisać jako

$$
\begin{aligned}
f(t) &= sgn(t) = \\
&= \mathbb{1}(t) - \mathbb{1}(-t) = \\
&= f_1(t) - f_2(t)
\end{aligned}
$$

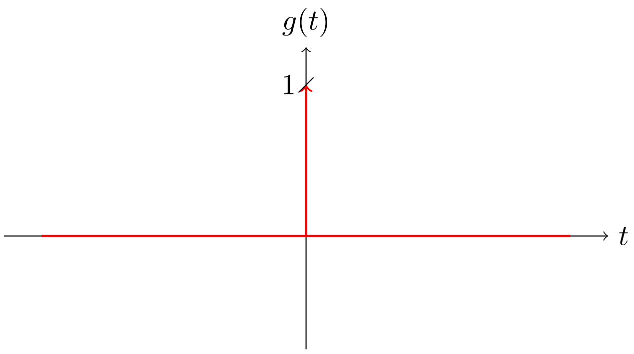
Wyraźnie widac iż funkcja jest złożeniem dwóch skoków jednostkowych

$$
\begin{aligned}
f_1(t) &= \mathbb{1}(t) \\
f_2(t) &= \mathbb{1}(-t)
\end{aligned}
$$



Transformaty sygnału $f_1(t) = \mathbb{1}(t)$ nie można wyznaczyć wprost ze wzoru. Ale łatwo można wyznaczyć pochodnią $f_1'(t)$

$$g(t) = f_1'(t) = \delta(t)$$



dla której w bardzo łatwy sposób można wyznaczyć transformatę Fouriera.

$$\begin{aligned}
G(\jmath\omega) &= \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath \cdot \omega \cdot t} = \\
&= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-\jmath \cdot \omega \cdot t} = \\
&= \left\{ \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) \cdot dt = f(t_0) \right\} = \\
&= e^{-\jmath \cdot \omega \cdot 0} = \\
&= e^{0} = \\
&= 1
\end{aligned}$$

Transformatą Fouriera sygnału $g(t) = \delta(t)$ jest $G(\jmath\omega) = 1$
Korzystając z twierdzenia o całkowaniu można wyznaczyć transformatę funkcji $f_1(t)$

$$\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega) \\
f_1(t) = \int_{-\infty}^{t} g(\tau) \cdot d\tau &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0)
\end{aligned}$$

Tak wiec mamy

$$\begin{aligned}
F_1(\jmath\omega) &= \frac{1}{\jmath \cdot \omega} \cdot G(\jmath\omega) + \pi \cdot \delta(\omega) \cdot G(0) = \\
&= \frac{1}{\jmath \cdot \omega} \cdot 1 + \pi \cdot \delta(\omega) \cdot 1 = \\
&= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)
\end{aligned}$$

A wiec transformata skoku jednostkowego jest $F_1(\jmath\omega) = \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)$
Funkcję $f_2(t)$ można zapisać jako

$$\begin{aligned}
f_2(t) &= \mathbb{1}(-t) = \\
&= \mathbb{1}(-1 \cdot t) = \\
&= f_1(-1 \cdot t)
\end{aligned}$$

A wiec transformatę funkcji $f_2(t)$ można wyznaczyć z twierdzenia o zmianie skali

$$
\begin{aligned}
f_1(t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) \\
f_2(t) = f_1(\alpha \cdot t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega) = \frac{1}{|\alpha|} \cdot F_1(\jmath\frac{\omega}{\alpha})
\end{aligned}
$$

$$
\begin{aligned}
F_2(\jmath\omega) &= \frac{1}{|a|} \cdot F_1(\jmath\frac{\omega}{a}) = \\
&= \Big\{a = -1\Big\} = \\
&= \frac{1}{|-1|} \cdot \frac{1}{\jmath \cdot \frac{\omega}{-1}} + \pi \cdot \delta(\frac{\omega}{-1}) = \\
&= \frac{1}{1} \cdot \frac{1}{-\jmath \cdot \omega} + \pi \cdot \delta(-\omega) = \\
&= -\frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)
\end{aligned}
$$

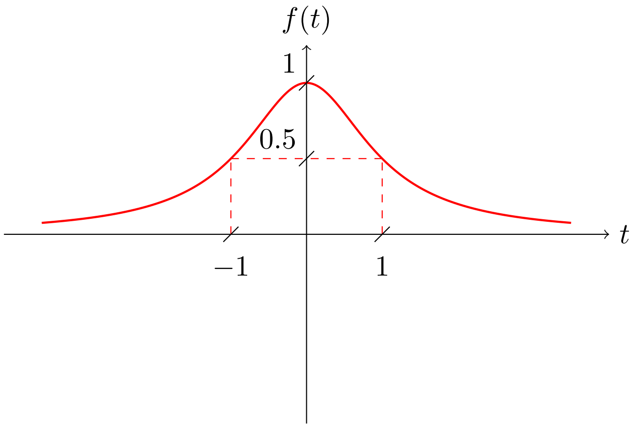
A więc transformata funkcji $f_2(t)$ jest równa $F_2(\jmath\omega) - \frac{1}{\jmath\cdot\omega} + \pi \cdot \delta(\omega)$
Transformatę funkcji $f(t)$ możemy wyznaczyć z twierdzenia o jednorodności

$$
\begin{aligned}
f_1(t) &\xrightarrow{\mathcal{F}} F_1(\jmath\omega) \\
f_2(t) &\xrightarrow{\mathcal{F}} F_2(\jmath\omega) \\
f(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t) &\xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot F_1(\jmath\omega) + \beta \cdot F_2(\jmath\omega)
\end{aligned}
$$

$$
\begin{aligned}
F(\jmath\omega) &= F_1(\jmath\omega) - F_2(\jmath\omega) = \\
&= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega) - \left(-\frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega)\right) = \\
&= \frac{1}{\jmath \cdot \omega} + \pi \cdot \delta(\omega) + \frac{1}{\jmath \cdot \omega} - \pi \cdot \delta(\omega) = \\
&= \frac{2}{\jmath \cdot \omega}
\end{aligned}
$$

Ostatecznie transformata funkcji $f(t)$ jest równa $F(\jmath\omega) = \frac{2}{\jmath\cdot\omega}$.

Task 10. Oblicz transformatę Fouriera sygnału $f(t) = \frac{1}{1+t^2}$ za pomocą twierdzeń.



Załóżmy sygnał $g(t) = e^{-|t|}$ i wyznaczmy jego transformatę.

$$
\begin{aligned}
G(\jmath\omega) &= \int_{-\infty}^{\infty} g(t) \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{\infty} e^{-|t|} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} e^{t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{\infty} e^{-t} \cdot e^{-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \int_{-\infty}^{0} e^{t-\jmath\cdot\omega\cdot t} \cdot dt + \int_{0}^{\infty} e^{-t-\jmath\cdot\omega\cdot t} \cdot dt = \\
&= \lim_{\tau\to\infty} \int_{-\tau}^{0} e^{(1-\jmath\cdot\omega)\cdot t} \cdot dt + \lim_{\tau\to\infty} \int_{0}^{\tau} e^{-(1+\jmath\cdot\omega)\cdot t} \cdot dt = \\
&= \left\{ \begin{matrix} z_1 = -(1+\jmath\cdot\omega)\cdot t & z_2 = (1-\jmath\cdot\omega)\cdot t \\ dz_1 = -(1+\jmath\cdot\omega)\cdot dt & dz_2 = (1-\jmath\cdot\omega)\cdot dt \\ dt = \frac{1}{-(1+\jmath\cdot\omega)} \cdot dz_1 & dt = \frac{1}{1-\jmath\cdot\omega} \cdot dz_2 \end{matrix} \right\} = \\
&= \lim_{\tau\to\infty} \int_{-\tau}^{0} e^{z_2} \cdot \frac{1}{1-\jmath\cdot\omega} \cdot dz_2 + \lim_{\tau\to\infty} \int_{0}^{\tau} e^{z_1} \cdot \frac{1}{-(1+\jmath\cdot\omega)} \cdot dz_1 = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} \int_{-\tau}^{0} e^{z_2} \cdot dz_2 + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \int_{0}^{\tau} e^{z_1} \cdot dz_1 = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} e^{z_2}\big|_{-\tau}^{0} + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{z_1}\big|_{0}^{\tau} = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} e^{(1-\jmath\cdot\omega)\cdot t}\Big|_{-\tau}^{0} + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} e^{-(1+\jmath\cdot\omega)\cdot t}\Big|_{0}^{\tau} = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} \left(e^{(1-\jmath\cdot\omega)\cdot 0} - e^{(1-\jmath\cdot\omega)\cdot(-\tau)}\right) + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \left(e^{-(1+\jmath\cdot\omega)\cdot\tau} - e^{-(1+\jmath\cdot\omega)\cdot 0}\right) = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} \left(e^{0} - e^{-(1-\jmath\cdot\omega)\cdot\tau}\right) + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \left(e^{-(1+\jmath\cdot\omega)\cdot\tau} - e^{0}\right) = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \lim_{\tau\to\infty} \left(1 - e^{-\tau+\jmath\cdot\omega\cdot\tau}\right) + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \lim_{\tau\to\infty} \left(e^{-\tau-\jmath\cdot\omega\cdot\tau} - 1\right) = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \left(\lim_{\tau\to\infty} 1 - \lim_{\tau\to\infty} e^{-\tau} \cdot e^{\jmath\cdot\omega\cdot\tau}\right) + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \left(\lim_{\tau\to\infty} e^{-\tau} \cdot e^{-\jmath\cdot\omega\cdot\tau} - \lim_{\tau\to\infty} 1\right) = \\
&= \frac{1}{1-\jmath\cdot\omega} \cdot \left(1 - \lim_{\tau\to\infty} e^{-\tau} \cdot \lim_{\tau\to\infty} e^{\jmath\cdot\omega\cdot\tau}\right) + \frac{1}{-(1+\jmath\cdot\omega)} \cdot \left(\lim_{\tau\to\infty} e^{-\tau} \cdot \lim_{\tau\to\infty} e^{-\jmath\cdot\omega\cdot\tau} - 1\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1-\jmath\cdot\omega}\cdot\left(1-0\cdot\lim_{\tau\to\infty} e^{\jmath\cdot\omega\cdot\tau}\right)+\frac{1}{-(1+\jmath\cdot\omega)}\cdot\left(0\cdot\lim_{\tau\to\infty} e^{-\jmath\cdot\omega\cdot\tau}-1\right)=\\
&= \frac{1}{1-\jmath\cdot\omega}\cdot(1-0)+\frac{1}{-(1+\jmath\cdot\omega)}\cdot(0-1)=\\
&= \frac{1}{1-\jmath\cdot\omega}+\frac{1}{-(1+\jmath\cdot\omega)}\cdot(-1)=\\
&= \frac{1}{1-\jmath\cdot\omega}+\frac{1}{1+\jmath\cdot\omega}=\\
&= \frac{(1+\jmath\cdot\omega)}{(1+\jmath\cdot\omega)\cdot(1-\jmath\cdot\omega)}+\frac{(1-\jmath\cdot\omega)}{(1+\jmath\cdot\omega)\cdot(1-\jmath\cdot\omega)}=\\
&= \frac{(1+\jmath\cdot\omega)+(1-\jmath\cdot\omega)}{(1+\jmath\cdot\omega)\cdot(1-\jmath\cdot\omega)}=\\
&= \frac{2}{1+\omega^2}
\end{aligned}
$$

Transformata sygnału $g(t)=e^{-|t|}$ jest równa $G(\jmath\omega)=\frac{2}{1+\omega^2}$. Postać funkcji $G(\jmath\omega)=\frac{2}{1+\omega^2}$ nie jest identyczna z postacią funkcji $f(t)$, funkcja różni się o współczynnik 2.

Z twierdzenia o liniowości transformaty

$$
\begin{aligned}
g(t) &\xrightarrow{\mathcal{F}} G(\jmath\omega)\\
h(t)=\alpha\cdot g(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega)=\alpha\cdot G(\jmath\omega)
\end{aligned}
$$

otrzymujemy

$$
\begin{aligned}
h(t) &= \frac{1}{2}\cdot e^{-|t|}\\
H(\jmath\omega) &= \frac{1}{2}\cdot\frac{2}{1+\omega^2}=\\
&= \frac{1}{1+\omega^2}
\end{aligned}
$$

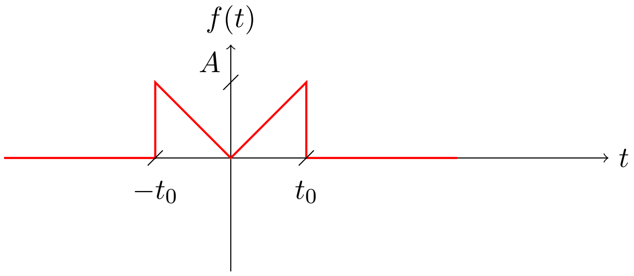
Na podstawie sygnału $h(t)$ i korzystając z twierdzenia o symetrii możemy wyznaczyć transformatę sygnału $f(t)$.

$$
\begin{aligned}
h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega)\\
f(t)=H(t) &\xrightarrow{\mathcal{F}} F(\jmath\omega)=2\pi\cdot h(-\omega)
\end{aligned}
$$

$$
\begin{aligned}
F(\jmath\omega) &= 2\pi\cdot h(-\omega)=\\
&= 2\pi\cdot\frac{1}{2}\cdot e^{-|-\omega|}=\\
&= \pi\cdot e^{-|\omega|}
\end{aligned}
$$

Transformata Fouriera sygnału $f(t)=\frac{1}{1+t^2}$ jest równa $F(\jmath\omega)=\pi\cdot e^{-|\omega|}$

Task 11. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku wykorzystując twierdzenia opisujące własciwości transformacji Fouriera. Wykorzystaj informację o tym, że $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$ oraz $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.



W pierwszej kolejności należy ustalić wzór funkcji przedstawionej na rysunku. Możemy zauważyć iż przedstawiony sygnał można otrzymać przez oodjęcie trójkąta od sygnału prostokątnego. Wykorzystując sygnały elementarne możemy to zapisać następująco:

$$f(t) = A \cdot \left(\Pi\left(\frac{t}{2 \cdot t_0}\right) - \Lambda\left(\frac{t}{t_0}\right)\right) \tag{3.87}$$

Ponieważ transformacja Fouriera jest przekształceniem liniowym, dlatego można wyznaczyć osobno transformaty poszczególnych sygnałów elementarnych, czyli:

$$f(t) = A \cdot (f_1(t) - f_2(t)) \tag{3.88}$$

gdzie:

$$f_1(t) = \Pi\left(\frac{t}{2 \cdot t_0}\right)$$
$$f_2(t) = \Lambda\left(\frac{t}{t_0}\right)$$

Wyznaczmy transformtę sygnału $f_1(t)$, czyli $F_1(\jmath\omega)$.

Z treści zadania wiemy, że: $\mathcal{F}\{\Pi(t)\} = Sa\left(\frac{\omega}{2}\right)$. Wykorzystując twierdzenie o zmianie skali mamy:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$f(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{2 \cdot t_0}\right|} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{2 \cdot t_0}}}{2}\right)$$
$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} 2 \cdot t_0 \cdot Sa\left(\frac{\omega \cdot 2 \cdot t_0}{2}\right)$$
$$\Pi(\frac{t}{2 \cdot t_0}) \xrightarrow{\mathcal{F}} 2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right)$$

Transformata sygnału $f_1(t)$ to:

$$F_1(\jmath\omega) = \mathcal{F}\{f_1(t)\} = 2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) \tag{3.89}$$

Teraz wyznaczmy transformtę sygnału $f_2(t)$, czyli $F_2(\jmath\omega)$.
Z treści zadania wiemy, że: $\mathcal{F}\{\Lambda(t)\} = Sa^2\left(\frac{\omega}{2}\right)$.
Wykorzystując twierdzenie o zmianie skali mamy:

$$\begin{gathered} g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\ f(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha}) \end{gathered}$$

$$\begin{gathered} \Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right) \\ \Lambda(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{t_0}\right|} \cdot Sa^2\left(\frac{\frac{\omega}{\frac{1}{t_0}}}{2}\right) \\ \Lambda(\frac{t}{t_0}) \xrightarrow{\mathcal{F}} t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right) \end{gathered}$$

Transformata sygnału $f_2(t)$ to:

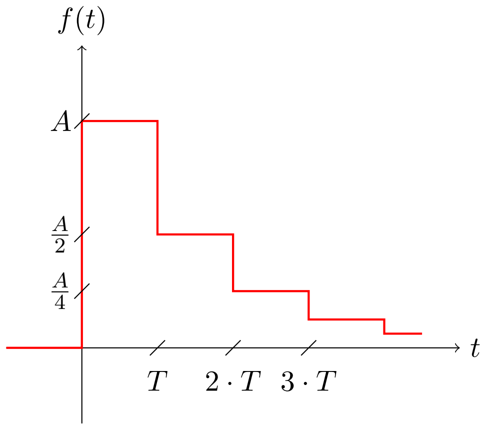
$$F_2(\jmath\omega) = \mathcal{F}\{f_2(t)\} = t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right) \tag{3.90}$$

Czyli transformata sygnału $f(t)$ to:

$$F(\jmath\omega) = \mathcal{F}\{f(t)\} = A \cdot \left(2 \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)\right)$$

Transformata sygnału $f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) - A \cdot \Lambda(\frac{t}{t_0})$ to $F(\jmath\omega) = 2 \cdot A \cdot t_0 \cdot Sa\left(\omega \cdot t_0\right) - A \cdot t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)$

Task 12. Oblicz transformatę Fouriera sygnału $f(t)$ przedstawionego na rysunku za pomocą twierdzeń, wiedząc że transformata sygnału prostokątnego $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa\left(\frac{\omega}{2}\right)$.



Sygnał zbudowany jest z ciągu poprzesuwanych sygnałów prostokątnych o wykładniczo malejącej amplitudzie.

$$f(t) = \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot \Pi\left(\frac{t - \frac{T}{2} - n \cdot T}{T}\right)$$

Nasz sygnał jest nieskończoną sumą funkcji prostokątnych. Korzystając z liniowość transformaty fouriera

$$f_1(t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega)$$
$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$
$$f(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = \alpha \cdot F_1(\jmath\omega) + \beta \cdot F_2(\jmath\omega)$$

możemy napisać że:

$$F(\jmath\omega) = \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot H_n(\jmath\omega)$$

gdzie $H_n(\jmath\omega)$ jest transformatą Fouriera odpowiednio przesuniętego sygnału prostokątnego $h_n(t) = \Pi\left(\frac{t-\frac{T}{2}-n \cdot T}{T}\right)$.

Transformata sygnału $g(t) = \Pi(t)$ jest równa $G(\jmath\omega) = Sa(\frac{\omega}{2})$. Postać funkcji $g(t)$ nie jest identyczna z postacią funkcji $h_n(t)$, funkcja różni się skalą i przesunięciem. Zacznijmy od skali.

Wyznaczanym transformaty funkcji przeskalowanej $h(t) = \Pi\left(\frac{t}{T}\right)$

Z twierdzenia o zmianie skali mamy

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$
$$h(t) = g(\alpha \cdot t) \xrightarrow{\mathcal{F}} H(\jmath\omega) = \frac{1}{|\alpha|} \cdot G(\jmath\frac{\omega}{\alpha})$$

a wiec otrzymujemy

$$
\begin{aligned}
h(t) = \Pi\left(\frac{t}{T}\right) &= \\
&= \Pi\left(\frac{1}{T} \cdot t\right) = \\
&= g\left(\frac{1}{T} \cdot t\right)
\end{aligned}
$$

$$\alpha = \frac{1}{T}$$

$$
\begin{aligned}
H(\jmath\omega) &= \frac{1}{\frac{1}{T}} \cdot G\left(\frac{\jmath\omega}{\frac{1}{T}}\right) = \\
&= \frac{1}{\frac{1}{T}} \cdot Sa\left(\frac{\frac{\omega}{\frac{1}{T}}}{2}\right) = \\
&= T \cdot Sa\left(\frac{\omega \cdot T}{2}\right)
\end{aligned}
$$

Dalej wyznaczanym transformaty funkcji przeskalowanej i przesuniętej $h_n(t) = \Pi\left(\frac{t - \frac{T}{2} - n \cdot T}{T}\right)$ Korzystając z twierdzenia o przesunięciu w dziedzinie czasu

$$
\begin{aligned}
h_n(t) &\xrightarrow{\mathcal{F}} H_n(\jmath\omega) \\
h(t) = h_n(t - t_0) &\xrightarrow{\mathcal{F}} H(\jmath\omega) = H_n(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0}
\end{aligned}
$$

możemy napisać że:

$$
\begin{aligned}
H_n(\jmath\omega) &= H(\jmath\omega) \cdot e^{-\jmath \cdot \omega \cdot t_0} = \\
&= T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \left(\frac{T}{2} + n \cdot T\right)}
\end{aligned}
$$

Ostatecznie wzór na transformatę sygnału $f(t)$ jest równy

$$
\begin{aligned}
F(\jmath\omega) &= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot H_n(\jmath\omega) = \\
&= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \left(\frac{T}{2} + n \cdot T\right)} = \\
&= \sum_{n=0}^{\infty} \frac{A}{2^n} \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot e^{-\jmath \cdot \omega \cdot n \cdot T} = \\
&= \sum_{n=0}^{\infty} T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{A}{2^n} \cdot e^{-\jmath \cdot \omega \cdot n \cdot T} =
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot A \cdot \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n
\end{aligned}$$

Można zauważyć że suma w rozwiązaniu to szereg geometryczny. Z wzoru na sumę szeregu geometrycznego mamy

$$\begin{aligned}
F(\jmath\omega) &= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}\right)^n = \\
&= \left\{\sum_{n=0}^{\infty} q^n = \tfrac{1}{1-q}\right\} = \\
&= A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}
\end{aligned}$$

Ostatecznie transformata sygnału $f(t)$ równa się:

$$F(\jmath\omega) = A \cdot T \cdot Sa\left(\frac{\omega \cdot T}{2}\right) \cdot e^{-\jmath \cdot \omega \cdot \frac{T}{2}} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-\jmath \cdot \omega \cdot T}}$$

3.3 Calculating energy of the signal from its Fourier transform. The Parseval's theorem

Task 1. Oblicz energię sygnału $f(t) = Sa\left(\omega_0 \cdot t\right)$, wiedząc że transformata sygnału $\Pi(t)$ jest równa $Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa\left(\omega_0 \cdot t\right) \tag{3.91}$$

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.92}$$

Energię sygnału można wyznaczyc ze wzoru:

$$E = \int_{-\infty}^{\infty} \left|f(t)\right|^2 \cdot dt \tag{3.93}$$

Podstawiając dany sygnał $f(t)$ do wzoru na energie otrzymujemy:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \left|f(t)\right|^2 \cdot dt = \\ &= \int_{-\infty}^{\infty} \left|Sa\left(\omega_0 \cdot t\right)\right|^2 \cdot dt = \\ &= \left\{ \quad Sa\left(x\right) = \tfrac{sin(x)}{x} \right\} = \\ &= \int_{-\infty}^{\infty} \left|\frac{sin\left(\omega_0 \cdot t\right)}{\left(\omega_0 \cdot t\right)}\right|^2 \cdot dt = \\ &= \int_{-\infty}^{\infty} \frac{sin^2\left(\omega_0 \cdot t\right)}{\left(\omega_0 \cdot t\right)^2} \cdot dt = \\ &= .... \end{aligned}$$

Próbując obliczyc energię tym sposobem musimy obliczyć całkę cykliczną. A może jest łatwiejszy sposób?

Spróbumy wykorzystać twierdzenie Parsevala:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \left|F(\jmath\omega)\right|^2 \cdot d\omega \tag{3.94}$$

W tym podejściu musimy obliczyć transformatę Fouriera sygnału $f(t)$, czyli $F(\jmath\omega)$. Skoro wiemy, że:

$$g(t) = \Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.95}$$

to, na podstawie twierdzenia o symetrii przekształcenia Fouriera:

$$\begin{gathered} g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega) \\ f(t) = G(t) \xrightarrow{\mathcal{F}} F(\jmath\omega) = 2\pi \cdot g(-\omega) \end{gathered}$$

otrzymujemy:

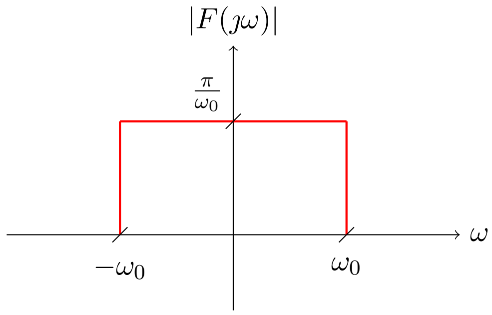
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega)$$
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$

Teraz musimy przeskalować $Sa\left(\frac{t}{2}\right)$ tak, aby otrzymać $Sa\left(\omega_0 \cdot t\right)$. W tym celu skorzystamy z twierdzenia o zmianie skali podstawiając $\alpha = 2 \cdot \omega_0$:

$$f(t) \xrightarrow{\mathcal{F}} F(\jmath\omega)$$
$$g(t) = f(\alpha \cdot t) \xrightarrow{\mathcal{F}} G(\jmath\omega) = \frac{1}{|\alpha|} \cdot F(\jmath\frac{\omega}{\alpha})$$

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$
$$Sa\left(2 \cdot \omega_0 \cdot \frac{t}{2}\right) \xrightarrow{\mathcal{F}} \frac{1}{2 \cdot \omega_0} \cdot 2\pi \cdot \Pi(\frac{\omega}{2 \cdot \omega_0})$$
$$Sa\left(\omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0} \cdot \Pi(\frac{\omega}{2 \cdot \omega_0})$$

Narysujmy widmo amplitudowe sygnału $f(t)$, czyli $|F(\jmath\omega)|$.



$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -\omega_0) \\ \frac{\pi}{\omega_0} & \omega \in (-\omega_0; \omega_0) \\ 0 & \omega \in (\omega_0; \infty) \end{cases}$$

Ponieważ energię wyznaczamy ze wzoru:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.96}$$

to wyznaczmy $|F(\jmath\omega)|^2$:

$$|F(\jmath\omega)|^2 = \begin{cases} 0 & \omega \in (-\infty; -\omega_0) \\ \left(\frac{\pi}{\omega_0}\right)^2 & \omega \in (-\omega_0; \omega_0) \\ 0 & \omega \in (\omega_0; \infty) \end{cases}$$

$$\begin{aligned}
E &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\int_{-\infty}^{-\omega_0} 0 \cdot d\omega + \int_{-\omega_0}^{\omega_0} \left(\frac{\pi}{\omega_0}\right)^2 \cdot d\omega + \int_{\omega_0}^{\infty} 0 \cdot d\omega \right) = \\
&= \frac{1}{2\pi} \cdot \left(0 + \left(\frac{\pi}{\omega_0}\right)^2 \cdot \int_{-\omega_0}^{\omega_0} d\omega + 0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{\pi}{\omega_0}\right)^2 \cdot \left(\omega|_{-\omega_0}^{\omega_0} \right) = \\
&= \frac{\pi}{2 \cdot \omega_0^2} \cdot (\omega_0 - (-\omega_0)) = \\
&= \frac{\pi}{2 \cdot \omega_0^2} \cdot (2 \cdot \omega_0) = \\
&= \frac{\pi}{\omega_0}
\end{aligned}$$

Energia sygnału $f(t) = Sa\,(\omega_0 \cdot t)$ równa się $E = \frac{\pi}{\omega_0}$.

Task 2. Oblicz, jaka część energi sygnału $f(t) = A{\cdot}Sa\left(2 \cdot \omega_0 \cdot t\right){\cdot}cos^2\left(2 \cdot \omega_0 \cdot t\right)$ przypada na wartości pulsacji $|\omega| < 2 \cdot \omega_0$. Wykorzystaj informację, że transformata sygnału $\Pi(t)$ jest równa $Sa\left(\frac{\omega}{2}\right)$.

$$f(t) = A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right) \tag{3.97}$$

$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.98}$$

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} = ? \tag{3.99}$$

Ponieważ musimy obliczyć energię tylko dla pewnego zakresu pulsacji, to wykorzystamy twierdzenie Parsevala:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \left|F(\jmath\omega)\right|^2 \cdot d\omega \tag{3.100}$$

W tym podejściu musimy obliczyć transformatę Fouriera sygnału $f(t)$, czyli $F(\jmath\omega)$.

Ponieważ możemy korzystać tylko ze znanych twierdzeń oraz wiedzy o transformacie sygnału $\Pi(t)$, to spróbujmy przekształcić sygnał $f(t)$ do postaci, w której wprost możemy zastosować twierdzenia. Zauważmy, że:

$$\begin{aligned}
f(t) &= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right) = \\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x}+e^{-\jmath\cdot x}}{2}\right\} = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{2\cdot\jmath\cdot\omega_0\cdot t} + e^{-2\cdot\jmath\cdot\omega_0\cdot t}}{2}\right)^2 = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{(e^{2\cdot\jmath\cdot\omega_0\cdot t})^2 + 2 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t} \cdot e^{-2\cdot\jmath\cdot\omega_0\cdot t} + (e^{-2\cdot\jmath\cdot\omega_0\cdot t})^2}{4}\right) = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + 2 \cdot e^{2\cdot\jmath\cdot\omega_0\cdot t - 2\cdot\jmath\cdot\omega_0\cdot t} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{4}\right) = \\
&= A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot \left(\frac{e^{4\cdot\jmath\cdot\omega_0\cdot t} + 2 \cdot e^{0} + e^{-4\cdot\jmath\cdot\omega_0\cdot t}}{4}\right) = \\
&= \frac{A}{4} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot e^{4\cdot\jmath\cdot\omega_0\cdot t} + \frac{A}{2} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) + \frac{A}{4} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot e^{-4\cdot\jmath\cdot\omega_0\cdot t} = \\
&= f_1(t) + f_2(t) + f_3(t)
\end{aligned}$$

Korzystając z liniowości przekształcenia Fouriera możemy niezależnie obliczyć transformaty dla sygnałów $f_1(t)$, $f_2(t)$ i $f_3(t)$, a następnie zsumować te transformaty. Zacznijmy od sygnału $f_2(t)$:

Skoro wiemy, że:

$$g(t) = \Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right) \tag{3.101}$$

to, na podstawie twierdzenia o symetrii przekształcenia Fouriera:

$$g(t) \xrightarrow{\mathcal{F}} G(\jmath\omega)$$

$$f_2(t) = G(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega) = 2\pi \cdot g(-\omega)$$

otrzymujemy:

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega)$$
$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$

Teraz musimy przeskalować $Sa\left(\frac{t}{2}\right)$ tak, aby otrzymać $Sa\left(2 \cdot \omega_0 \cdot t\right)$. W tym celu skorzystamy z twierdzenia o zmianie skali podstawiając $\alpha = 4 \cdot \omega_0$:

$$f(t) \xrightarrow{\mathcal{F}} F(\jmath\omega)$$
$$f_1(t) = f(\alpha \cdot t) \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = \frac{1}{|\alpha|} \cdot F(\jmath\frac{\omega}{\alpha})$$

$$Sa\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(\omega)$$
$$Sa\left(4 \cdot \omega_0 \cdot \frac{t}{2}\right) \xrightarrow{\mathcal{F}} \frac{1}{4 \cdot \omega_0} \cdot 2\pi \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$Sa\left(2 \cdot \omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{\pi}{2 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$f_2(t) = \frac{A}{2} \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) = F_2(\jmath\omega)$$

Podsumowując, transformata sygnału $f_2(t)$ to $F_2(\jmath\omega) = \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$.

Zauważmy, że $f_1(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t}$, czyli $f_1(t)$ to zmodulowany sygnał $f_2(t)$. Stosując twierdzenie o modulacji:

$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$
$$f_1(t) = f_2(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F_1(\jmath\omega) = F_2(\jmath\left(\omega - \omega_0\right))$$

otrzymujemy:

$$f_2(t) \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right)$$
$$f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$$
$$f_1(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{4 \cdot \jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right) = F_1(\jmath\omega)$$

Podsumowując, transformata sygnału $f_1(t)$ to $F_1(\jmath\omega) = \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$.

Podobnie, zauważmy, że $f_3(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t}$, czyli $f_3(t)$ to zmodulowany sygnał $f_2(t)$. Stosując twierdzenie o modulacji:

$$f_2(t) \xrightarrow{\mathcal{F}} F_2(\jmath\omega)$$

$$f_3(t) = f_2(t) \cdot e^{\jmath \cdot \omega_0 \cdot t} \xrightarrow{\mathcal{F}} F_3(\jmath\omega) = F_2(\jmath(\omega - \omega_0))$$

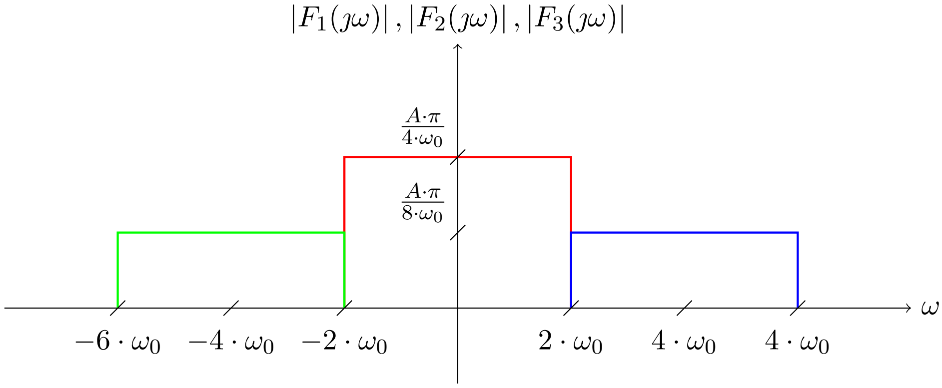
otrzymujemy:

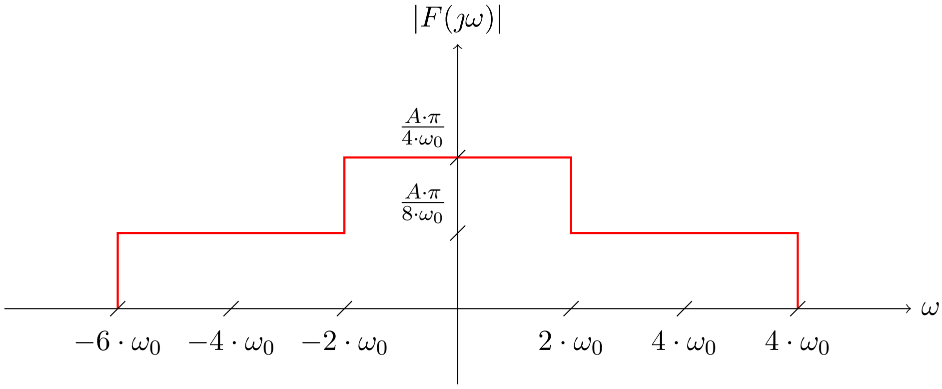
$$\begin{aligned} f_2(t) &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) \\ f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t} &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) \\ f_3(t) = \frac{1}{2} \cdot f_2(t) \cdot e^{-4 \cdot \jmath \cdot \omega_0 \cdot t} &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) = F_3(\jmath\omega) \end{aligned}$$

Podsumowując, transformata sygnału $f_3(t)$ to $F_3(\jmath\omega) = \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right)$.
Teraz możemy podać transformatę sygnału $f(t) = f_1(t) + f_2(t) + f_3(t)$,

$$\begin{aligned} F(\jmath\omega) &= F_1(\jmath\omega) + F_2(\jmath\omega) + F_3(\jmath\omega) = \\ &= \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega - 4 \cdot \omega_0}{4 \cdot \omega_0}\right) + \frac{A \cdot \pi}{4 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{4 \cdot \omega_0}\right) + \frac{A \cdot \pi}{8 \cdot \omega_0} \cdot \Pi\left(\frac{\omega + 4 \cdot \omega_0}{4 \cdot \omega_0}\right) \end{aligned}$$

Narysujmy widmo amplitudowe sygnału $f(t)$, czyli $|F(\jmath\omega)|$.





$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -6\cdot\omega_0) \\ \frac{A\cdot\pi}{8\cdot\omega_0} & \omega \in (-6\cdot\omega_0; -2\cdot\omega_0) \\ \frac{A\cdot\pi}{4\cdot\omega_0} & \omega \in (-2\cdot\omega_0; 2\cdot\omega_0) \\ \frac{A\cdot\pi}{8\cdot\omega_0} & \omega \in (2\cdot\omega_0; 6\cdot\omega_0) \\ 0 & \omega \in (6\cdot\omega_0; \infty) \end{cases}$$

Ponieważ energię wyznaczamy ze wzoru:

$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.102}$$

to wyznaczmy $|F(\jmath\omega)|^2$:

$$|F(\jmath\omega)|^2 = \begin{cases} 0 & \omega \in (-\infty; -6\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 & \omega \in (-6\cdot\omega_0; -2\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{4\cdot\omega_0}\right)^2 & \omega \in (-2\cdot\omega_0; 2\cdot\omega_0) \\ \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 & \omega \in (2\cdot\omega_0; 6\cdot\omega_0) \\ 0 & \omega \in (6\cdot\omega_0; \infty) \end{cases}$$

$$\begin{aligned}
E &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\int_{-\infty}^{-6\cdot\omega_0} 0 \cdot d\omega + \int_{-6\cdot\omega_0}^{-2\cdot\omega_0} \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 \cdot d\omega + \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left(\frac{A\cdot\pi}{4\cdot\omega_0}\right)^2 \cdot d\omega + \int_{2\cdot\omega_0}^{6\cdot\omega_0} \left(\frac{A\cdot\pi}{8\cdot\omega_0}\right)^2 \cdot d\omega + \int_{6\cdot\omega_0}^{\infty} 0 \cdot d\omega \right) = \\
&= \frac{1}{2\pi} \cdot \left(0 + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \int_{-6\cdot\omega_0}^{-2\cdot\omega_0} d\omega + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} d\omega + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \int_{2\cdot\omega_0}^{6\cdot\omega_0} d\omega + 0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \omega|_{-6\cdot\omega_0}^{-2\cdot\omega_0} + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot \omega|_{-2\cdot\omega_0}^{2\cdot\omega_0} + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot \omega|_{2\cdot\omega_0}^{6\cdot\omega_0} \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot (-2\cdot\omega_0 - (-6\cdot\omega_0)) + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot (2\cdot\omega_0 - (-2\cdot\omega_0)) + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot (6\cdot\omega_0 - 2\cdot\omega_0) \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot 4\cdot\omega_0 + \frac{A^2\cdot\pi^2}{16\cdot\omega_0^2} \cdot 4\cdot\omega_0 + \frac{A^2\cdot\pi^2}{64\cdot\omega_0^2} \cdot 4\cdot\omega_0 \right) = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2\cdot\pi^2}{16\cdot\omega_0} + \frac{A^2\cdot\pi^2}{4\cdot\omega_0} + \frac{A^2\cdot\pi^2}{16\cdot\omega_0} \right) = \\
&= \frac{1}{2\pi} \cdot \frac{A^2\cdot\pi^2}{4\cdot\omega_0} \cdot \left(\frac{1}{4} + 1 + \frac{1}{4} \right) = \\
&= \frac{A^2\cdot\pi}{8\cdot\omega_0} \cdot \left(\frac{2}{4} + 1 \right) =
\end{aligned}$$

$$= \frac{A^2 \cdot \pi}{8 \cdot \omega_0} \cdot \left(\frac{3}{2}\right) =$$
$$= \frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}$$

Energia sygnału $f(t) = A \cdot Sa\left(2 \cdot \omega_0 \cdot t\right) \cdot cos^2\left(2 \cdot \omega_0 \cdot t\right)$ równa się $E = \frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}$.

Energię sygnału dla pewnego zakresu pulsacji, także można wyznaczyć z twierdzenia Parsevala, ale zmieniając granice w całce zgodnie z oczekiwanym zakresem pulsacji, czyli dla pulsacji $|\omega| < 2 \cdot \omega_0$ otrzymamy wzór:

$$E_{|\omega|<2\cdot\omega_0} = \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.103}$$

Podstawiając dane dla naszego sygnału otrzymamy:

$$\begin{aligned}
E_{|\omega|<2\cdot\omega_0} &= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left|\frac{A \cdot \pi}{4 \cdot \omega_0}\right|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} \left(\frac{A \cdot \pi}{4 \cdot \omega_0}\right)^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A \cdot \pi}{4 \cdot \omega_0}\right)^2 \cdot \int_{-2\cdot\omega_0}^{2\cdot\omega_0} d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{A^2 \cdot \pi^2}{16 \cdot \omega_0^2}\right) \cdot \omega\Big|_{-2\cdot\omega_0}^{2\cdot\omega_0} = \\
&= \frac{A^2 \cdot \pi}{32 \cdot \omega_0^2} \cdot \left(2 \cdot \omega_0 - (-2 \cdot \omega_0)\right) = \\
&= \frac{A^2 \cdot \pi}{32 \cdot \omega_0^2} \cdot (4 \cdot \omega_0) = \\
&= \frac{A^2 \cdot \pi}{8 \cdot \omega_0}
\end{aligned}$$

Podsumowując $E_{|\omega|<\omega_0} = \frac{A^2 \cdot \pi}{8 \cdot \omega_0}$.

Teraz możemy obliczyć:

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} =? \tag{3.104}$$

Podstawiając nasze wcześniejsze wyniki otrzymujemy:

$$\frac{E_{|\omega|<2\cdot\omega_0}}{E} = \frac{\frac{A^2 \cdot \pi}{8 \cdot \omega_0}}{\frac{3 \cdot A^2 \cdot \pi}{16 \cdot \omega_0}} = \frac{A^2 \cdot \pi}{8 \cdot \omega_0} \cdot \frac{16 \cdot \omega_0}{3 \cdot A^2 \cdot \pi} = \frac{2}{3} \approx 66\%$$

Na pulsacje z zakresu $|\omega| < 2 \cdot \omega_0$ przypada około 66% energii sygnału.

Task 3.

Oblicz, jaka część energi sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ przypada na wartości pulsacji $|\omega| < \omega_0$. Wykorzystaj informację, że transformata sygnału $\Lambda(t)$ jest równa $Sa^2\left(\frac{\omega}{2}\right)$.

$$f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t) \tag{3.105}$$

$$\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right) \tag{3.106}$$

$$\frac{E_{|\omega|<\omega_0}}{E} = ? \tag{3.107}$$

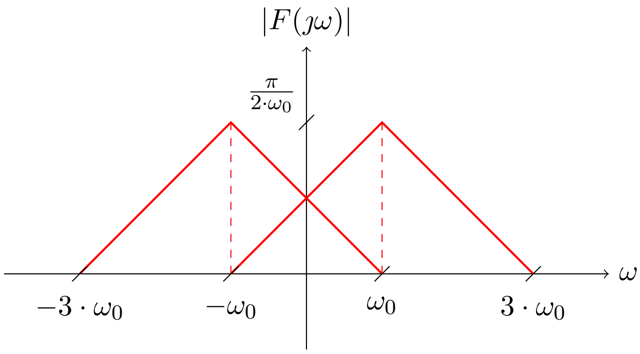
Całkowitą energię sygnału można wyznaczyc z twierdzenia Parsevala:

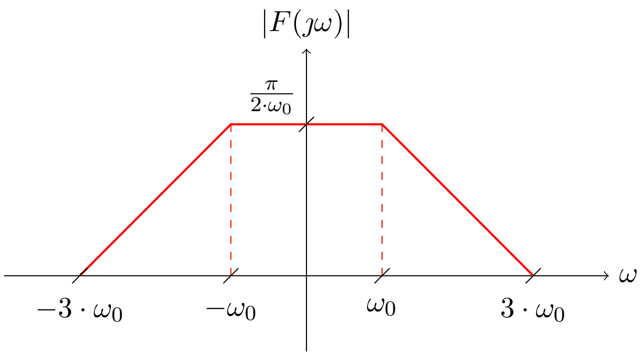
$$E = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.108}$$

W tym celu musimy wyznaczyc transformatę sygnału $f(t)$.

W jednym z wcześniejszych zadań obliczyliśmy, że transformata Fouriera sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ jest równa $F(\jmath\omega) = \frac{1}{2} \cdot \left(\frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega-\omega_0}{2\cdot\omega_0}\right) + \frac{\pi}{\omega_0} \cdot \Lambda\left(\frac{\omega+\omega_0}{2\cdot\omega_0}\right)\right)$.

Narysujmy widmo amplitudowe sygnału $f(t)$, czyli $|F(\jmath\omega)|$.





$$|F(\jmath\omega)| = \begin{cases} 0 & \omega \in (-\infty; -3\cdot\omega_0\rangle \\ \frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0} & \omega \in (-3\cdot\omega_0; -\omega_0\rangle \\ \frac{\pi}{2\cdot\omega_0} & \omega \in (-\omega_0; \omega_0\rangle \\ -\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0} & \omega \in (\omega_0; 3\cdot\omega_0\rangle \\ 0 & \omega \in (3\cdot\omega_0; \infty) \end{cases}$$

Podstawiając do wzoru na energię całkowitą, otrzymujemy:

$$E = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}|F(\jmath\omega)|^2\cdot d\omega =$$
$$= \frac{1}{2\pi}\cdot\left[\int_{-\infty}^{-3\cdot\omega_0}|0|^2\cdot d\omega + \int_{-3\cdot\omega_0}^{-\omega_0}\left|\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0}\right|^2\cdot d\omega + \int_{-\omega_0}^{\omega_0}\left|\frac{\pi}{2\cdot\omega_0}\right|^2\cdot d\omega + \right.$$
$$\left. + \int_{\omega_0}^{3\cdot\omega_0}\left|-\frac{\pi}{4\cdot\omega_0^2}\cdot\omega + \frac{3\cdot\pi}{4\cdot\omega_0}\right|^2\cdot d\omega + \int_{3\cdot\omega_0}^{\infty}|0|^2\cdot d\omega\right] =$$
$$= \frac{1}{2\pi}\cdot\left[0 + \int_{-3\cdot\omega_0}^{-\omega_0}\left(\left(\frac{\pi}{4\cdot\omega_0^2}\right)^2\cdot\omega^2 + 2\cdot\frac{\pi}{4\cdot\omega_0^2}\cdot\frac{3\cdot\pi}{4\cdot\omega_0}\cdot\omega + \left(\frac{3\cdot\pi}{4\cdot\omega_0}\right)^2\right)\cdot d\omega + \frac{\pi^2}{4\cdot\omega_0^2}\cdot\int_{-\omega_0}^{\omega_0}d\omega + \right.$$
$$\left. + \int_{\omega_0}^{3\cdot\omega_0}\left(\left(-\frac{\pi}{4\cdot\omega_0^2}\right)^2\cdot\omega^2 - 2\cdot\frac{\pi}{4\cdot\omega_0^2}\cdot\frac{3\cdot\pi}{4\cdot\omega_0}\cdot\omega + \left(\frac{3\cdot\pi}{4\cdot\omega_0}\right)^2\right)\cdot d\omega + 0\right] =$$
$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}\omega^2\cdot d\omega + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}\omega\cdot d\omega + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\int_{-3\cdot\omega_0}^{-\omega_0}d\omega + \frac{\pi^2}{4\cdot\omega_0^2}\cdot\omega|_{-\omega_0}^{\omega_0} + \right.$$
$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\int_{\omega_0}^{3\cdot\omega_0}\omega^2\cdot d\omega - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\int_{\omega_0}^{3\cdot\omega_0}\omega\cdot d\omega + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\int_{\omega_0}^{3\cdot\omega_0}d\omega\right] =$$
$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\left.\frac{\omega^3}{3}\right|_{-3\cdot\omega_0}^{-\omega_0} + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left.\frac{\omega^2}{2}\right|_{-3\cdot\omega_0}^{-\omega_0} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\omega|_{-3\cdot\omega_0}^{-\omega_0} + \frac{\pi^2}{4\cdot\omega_0^2}\cdot(\omega_0 - (-\omega_0)) + \right.$$
$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\left.\frac{\omega^3}{3}\right|_{\omega_0}^{3\cdot\omega_0} - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left.\frac{\omega^2}{2}\right|_{\omega_0}^{3\cdot\omega_0} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot\omega|_{\omega_0}^{3\cdot\omega_0}\right] =$$
$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\left(-\frac{\omega_0^3}{3} - \left(-\frac{27\cdot\omega_0^3}{3}\right)\right) + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(\frac{\omega_0^2}{2} - \frac{9\cdot\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot(-\omega_0 - (-3\cdot\omega_0)) + \right.$$
$$\left. + \frac{\pi^2}{4\cdot\omega_0^2}\cdot 2\cdot\omega_0 + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\left(\frac{27\cdot\omega_0^3}{3} - \frac{\omega_0^3}{3}\right) - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(\frac{9\cdot\omega_0^2}{2} - \frac{\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot(3\cdot\omega_0 - \omega_0)\right] =$$
$$= \frac{1}{2\pi}\cdot\left[\frac{\pi^2}{16\cdot\omega_0^4}\cdot\frac{26\cdot\omega_0^3}{3} + \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\left(-\frac{8\cdot\omega_0^2}{2}\right) + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot 2\cdot\omega_0 + \frac{\pi^2}{2\cdot\omega_0} + \right.$$
$$\left. + \frac{\pi^2}{16\cdot\omega_0^4}\cdot\frac{26\cdot\omega_0^3}{3} - \frac{6\cdot\pi^2}{16\cdot\omega_0^3}\cdot\frac{8\cdot\omega_0^2}{2} + \frac{9\cdot\pi^2}{16\cdot\omega_0^2}\cdot 2\cdot\omega_0\right] =$$
$$= \frac{1}{2\pi}\cdot\left[\frac{26\cdot\pi^2}{48\cdot\omega_0} - \frac{48\cdot\pi^2}{32\cdot\omega_0} + \frac{18\cdot\pi^2}{16\cdot\omega_0} + \frac{\pi^2}{2\cdot\omega_0} + \frac{26\cdot\pi^2}{48\cdot\omega_0} - \frac{48\cdot\pi^2}{32\cdot\omega_0} + \frac{18\cdot\pi^2}{16\cdot\omega_0}\right] =$$
$$= \frac{1}{2\pi}\cdot\frac{\pi^2}{2\cdot\omega_0}\cdot\left[\frac{26}{24} - \frac{48}{16} + \frac{18}{8} + 1 + \frac{26}{24} - \frac{48}{16} + \frac{18}{8}\right] =$$
$$= \frac{\pi}{4\cdot\omega_0}\cdot\left[\frac{52}{24} - \frac{96}{16} + \frac{36}{8} + 1\right] =$$

$$
\begin{aligned}
&= \frac{\pi}{4 \cdot \omega_0} \cdot \left[\frac{52}{24} - 6 + \frac{108}{24} + 1\right] = \\
&= \frac{\pi}{4 \cdot \omega_0} \cdot \left[\frac{160}{24} - 5\right] = \\
&= \frac{\pi}{4 \cdot \omega_0} \cdot \left[\frac{160}{24} - \frac{120}{24}\right] = \\
&= \frac{\pi}{4 \cdot \omega_0} \cdot \left[\frac{40}{24}\right] = \\
&= \frac{5 \cdot \pi}{12 \cdot \omega_0}
\end{aligned}
$$

Podsumowując, całkowita energia sygnału $f(t) = Sa^2(\omega_0 \cdot t) \cdot cos(\omega_0 \cdot t)$ to $E = \frac{5 \cdot \pi}{12 \cdot \omega_0}$.

Energię sygnału dla pewnego zakresu pulsacji, także można wyznaczyc z twierdzenia Parsevala, ale zmieniając granice w całce zgodnie z oczekiwanym zakresem pulsacji, czyli dla pulsacji $|\omega| < \omega_0$ otrzymamy wzór:

$$E_{|\omega|<\omega_0} = \frac{1}{2\pi} \cdot \int_{-\omega_0}^{\omega_0} |F(\jmath\omega)|^2 \cdot d\omega \tag{3.109}$$

Podstawiając dane dla naszego sygnału otrzymamy:

$$
\begin{aligned}
E_{|\omega|<\omega_0} &= \frac{1}{2\pi} \cdot \int_{-\omega_0}^{\omega_0} |F(\jmath\omega)|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-\omega_0}^{\omega_0} \left|\frac{\pi}{2 \cdot \omega_0}\right|^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \int_{-\omega_0}^{\omega_0} \left(\frac{\pi}{2 \cdot \omega_0}\right)^2 \cdot d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{\pi}{2 \cdot \omega_0}\right)^2 \cdot \int_{-\omega_0}^{\omega_0} d\omega = \\
&= \frac{1}{2\pi} \cdot \left(\frac{\pi^2}{4 \cdot \omega_0^2}\right) \cdot \omega|_{-\omega_0}^{\omega_0} = \\
&= \frac{\pi}{8 \cdot \omega_0^2} \cdot (\omega_0 - (-\omega_0)) = \\
&= \frac{\pi}{8 \cdot \omega_0^2} \cdot (2 \cdot \omega_0) = \\
&= \frac{\pi}{4 \cdot \omega_0}
\end{aligned}
$$

Podsumowując $E_{|\omega|<\omega_0} = \frac{\pi}{4 \cdot \omega_0}$.

Teraz możemy obliczyć:

$$\frac{E_{|\omega|<\omega_0}}{E} =? \tag{3.110}$$

Podstawiając nasze wcześniejsze wyniki otrzymujemy:

$$\frac{E_{|\omega|<\omega_0}}{E} = \frac{\frac{\pi}{4 \cdot \omega_0}}{\frac{5 \cdot \pi}{12 \cdot \omega_0}} = \frac{\pi}{4 \cdot \omega_0} \cdot \frac{12 \cdot \omega_0}{5 \cdot \pi} = \frac{12}{20} = \frac{6}{10} = 60\%$$

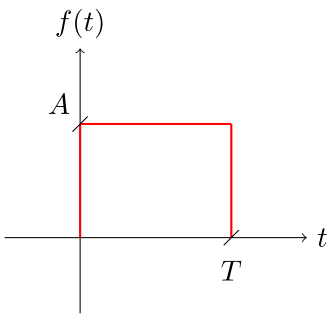
Na pulsacje z zakresu $|\omega| < \omega_0$ przypada 60% energi sygnału.

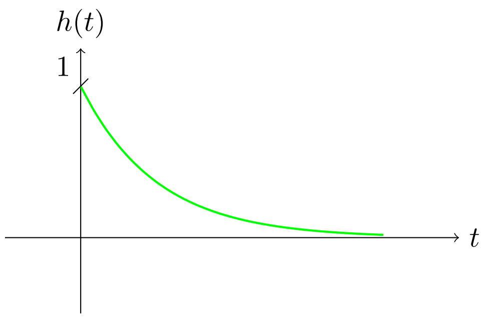
Chapter 4

Processing of signals by linear and time invariant (LTI) systems

4.1 Linear convolution

Task 1. Oblicz splot sygnałów $f(t) = A \cdot \Pi\left(\frac{t-T}{T}\right)$ i $h(t) = \mathbb{1}(t) \cdot e^{-a \cdot t}$





Wzór na slot sygnałów

$$y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) \cdot d\tau \tag{4.1}$$

Wzory sygnałów pod całką

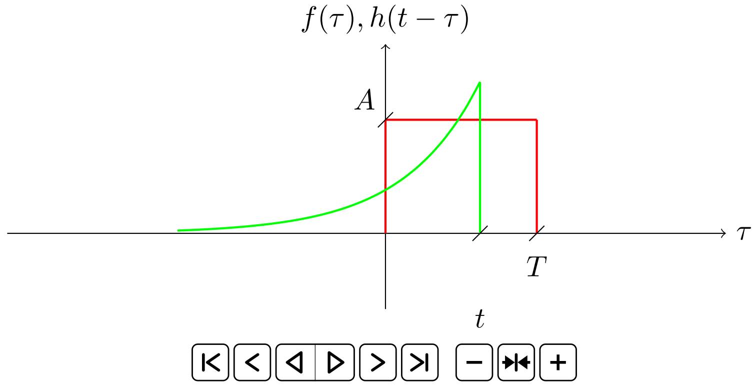
$$f(\tau) = A \cdot \Pi\left(\frac{\tau}{T}\right)$$

$$h(t-\tau) = \mathbb{1}(t) \cdot e^{-a \cdot (t-\tau)}$$

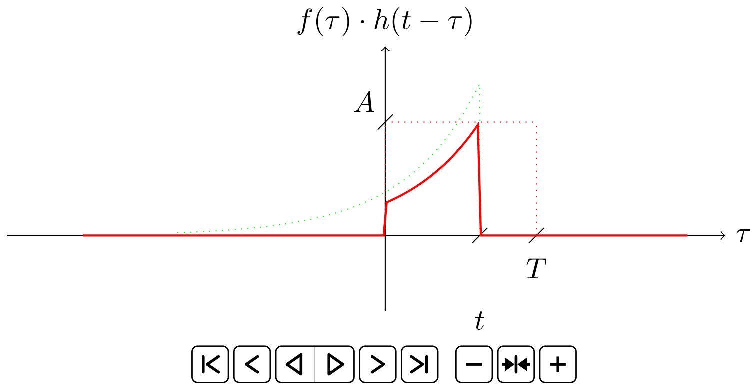
$$f(\tau) = \begin{cases} 0 & \tau \in (-\infty; 0) \\ A & \tau \in (0; T) \\ 0 & \tau \in (T; \infty) \end{cases}$$

$$h(t-\tau) = \begin{cases} e^{-a \cdot (t-\tau)} & \tau \in (-\infty; t) \\ 0 & \tau \in (t; \infty) \end{cases}$$

Wykresy obu funkcji w dziedzinie τ dla różnych wartości t:

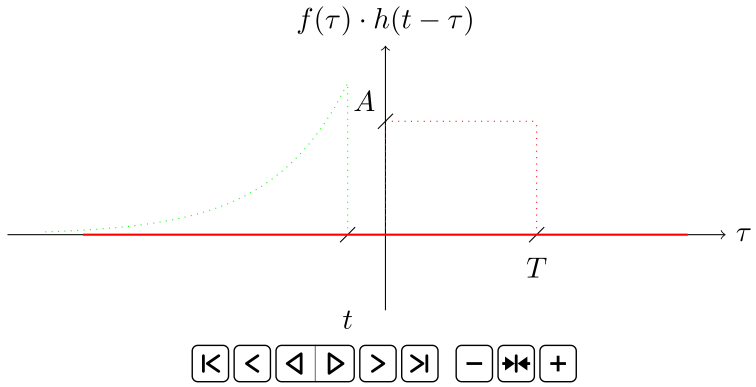


Po wymnożeniu obu funkcji, dla przykładowych wartości t, otrzymujemy (ciągła, czerwona linia):



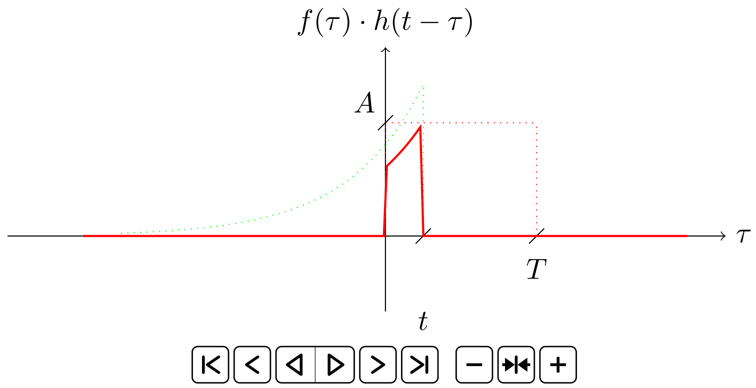
Z wykresu widać, że dla różnych wartości t otrzymujemy różny kształt funkcji podcałkowej $f(\tau) \cdot h(t-\tau)$. W związku z tym, wyznaczymy splot oddzielnie dla poszczególnych przedziałów wartości t

Przedział 1 Dla wartości t spełniających warunek $t < 0$ otrzymujemy:



$$y(t) = \int_{-\infty}^{\infty} 0 \cdot d\tau =$$
$$= 0$$

Przedział 2 Dla wartości t spełniających warunki $t \geq 0$ i $t < T$ otrzymujemy

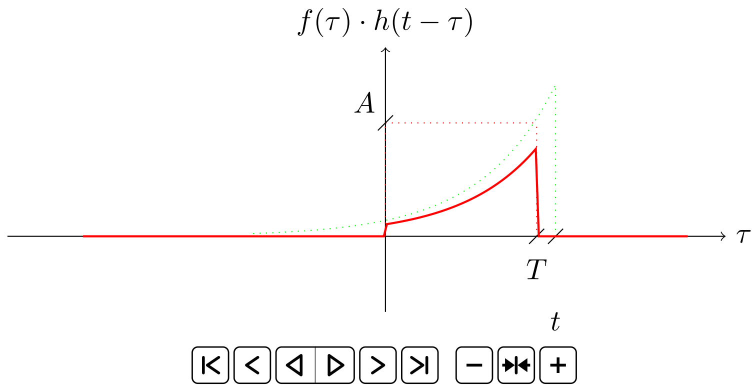


$$
f(\tau) \cdot h(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; 0) \\ A \cdot e^{-a \cdot (t-\tau)} & \tau \in (0; t) \\ 0 & \tau \in (t; \infty) \end{cases}
$$

Wartość splotu $y(t)$ wyznaczamy ze wzoru:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) \cdot d\tau = \\
&= \int_{-\infty}^{0} 0 \cdot d\tau + \int_{0}^{t} \left(A \cdot e^{-a \cdot (t-\tau)}\right) \cdot d\tau + \int_{t}^{\infty} 0 \cdot d\tau = \\
&= 0 + A \cdot \int_{0}^{t} \left(e^{-a \cdot t} \cdot e^{a \cdot \tau}\right) \cdot d\tau + 0 = \\
&= A \cdot e^{-a \cdot t} \cdot \int_{0}^{t} \left(e^{a \cdot \tau}\right) \cdot d\tau = \\
&= A \cdot e^{-a \cdot t} \cdot \frac{1}{a} \cdot e^{a \cdot \tau}\Big|_{0}^{t} = \\
&= \frac{A}{a} \cdot e^{-a \cdot t} \cdot \left(e^{a \cdot t} - e^{a \cdot 0}\right) = \\
&= \frac{A}{a} \cdot e^{-a \cdot t} \cdot \left(e^{a \cdot t} - 1\right) = \\
&= \frac{A}{a} \cdot \left(e^{a \cdot t} \cdot e^{-a \cdot t} - 1 \cdot e^{-a \cdot t}\right) = \\
&= \frac{A}{a} \cdot \left(e^{a \cdot t - a \cdot t} - e^{-a \cdot t}\right) = \\
&= \frac{A}{a} \cdot \left(e^{0} - e^{-a \cdot t}\right) = \\
&= \frac{A}{a} \cdot \left(1 - e^{-a \cdot t}\right)
\end{aligned}
$$

Przedział 3 Dla wartości t spełniających warunki $t \geq T$ otrzymujemy



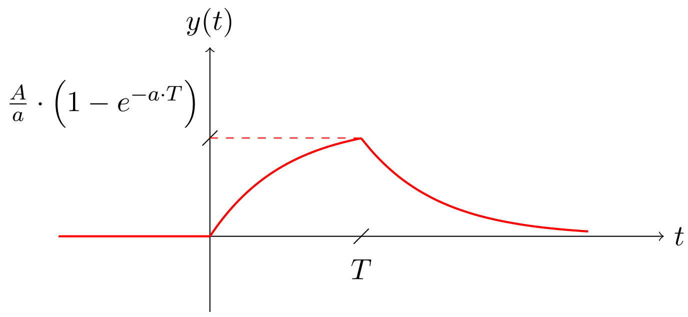
$$f(\tau)\cdot h(t-\tau)=\begin{cases}0 & \tau\in(-\infty;0)\\ A\cdot e^{-a\cdot(t-\tau)} & \tau\in(0;T)\\ 0 & \tau\in(T;\infty)\end{cases}$$

Wartość splotu $y(t)$ wyznaczamy ze wzoru:

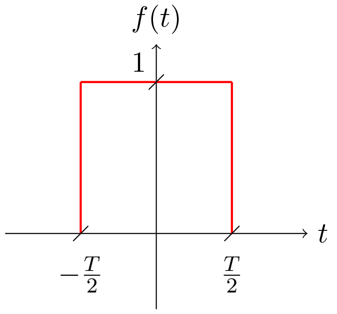
$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} f(\tau)\cdot h(t-\tau)\cdot d\tau =\\
&= \int_{-\infty}^{0} 0\cdot d\tau + \int_{0}^{T}\left(A\cdot e^{-a\cdot(t-\tau)}\right)\cdot d\tau + \int_{T}^{\infty} 0\cdot d\tau =\\
&= 0 + A\cdot\int_{0}^{T}\left(e^{-a\cdot t}\cdot e^{a\cdot\tau}\right)\cdot d\tau + 0 =\\
&= A\cdot e^{-a\cdot t}\cdot\int_{0}^{T}\left(e^{a\cdot\tau}\right)\cdot d\tau =\\
&= A\cdot e^{-a\cdot t}\cdot\frac{1}{a}\cdot e^{a\cdot\tau}\Big|_{0}^{T} =\\
&= \frac{A}{a}\cdot e^{-a\cdot t}\cdot\left(e^{a\cdot T}-e^{a\cdot 0}\right) =\\
&= \frac{A}{a}\cdot e^{-a\cdot t}\cdot\left(e^{a\cdot T}-1\right)
\end{aligned}$$

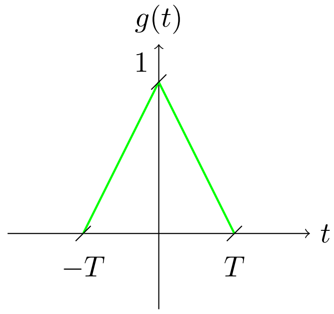
Podsumowując:

$$y(t)=\int_{-\infty}^{\infty} f(\tau)\cdot h(t-\tau)\cdot d\tau=\begin{cases}0 & t\in(-\infty;0)\\ \frac{A}{a}\cdot\left(1-e^{-a\cdot t}\right) & t\in(0;T)\\ \frac{A}{a}\cdot e^{-a\cdot t}\cdot\left(e^{a\cdot T}-1\right) & t\in(T;\infty)\end{cases}$$


$y(t)$
$\frac{A}{a} \cdot \left(1 - e^{-a \cdot T}\right)$
t
T

Task 2. Oblicz splot sygnałów $f(t) = \Pi\left(\frac{t}{T}\right)$ i $g(t) = \Lambda\left(\frac{t}{T}\right)$





Wzór na slot sygnałów

$$h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau \tag{4.2}$$

Wzory sygnałów pod całką

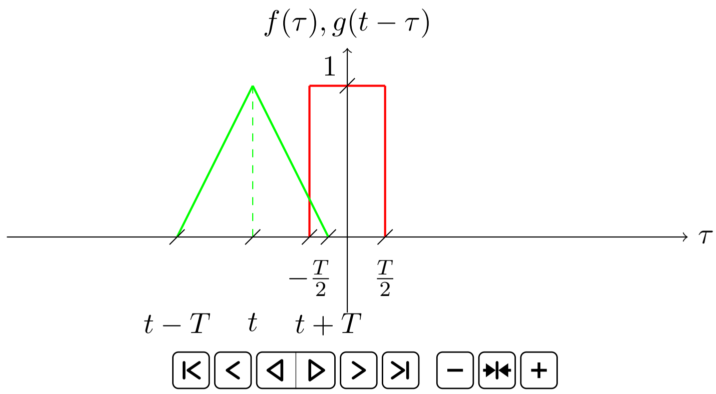
$$f(\tau) = \Pi\left(\frac{\tau}{T}\right)$$

$$g(t-\tau) = \Lambda\left(\frac{t-\tau}{T}\right)$$

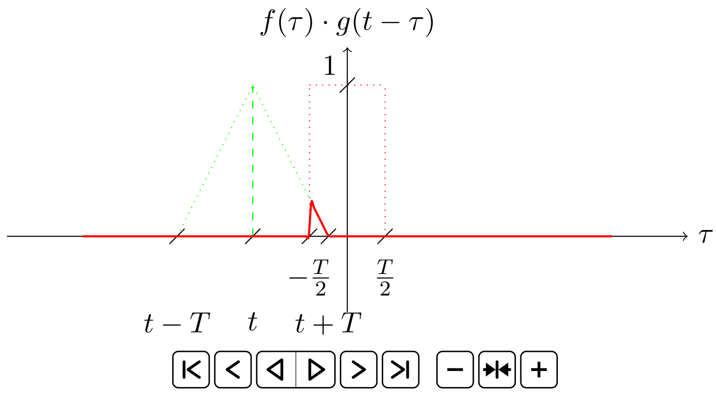
$$f(\tau) = \begin{cases} 0 & \tau \in \left(-\infty; -\frac{T}{2}\right) \\ A & \tau \in \left(-\frac{T}{2}; \frac{T}{2}\right) \\ 0 & \tau \in \left(\frac{T}{2}; \infty\right) \end{cases}$$

$$g(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; t-T); \\ \frac{1}{T} \cdot \tau - \frac{t-T}{T} & \tau \in (t-T; t) \\ -\frac{1}{T} \cdot \tau - \frac{-t-T}{T} & \tau \in (t; t+T) \\ 0 & \tau \in (t+T; \infty); \end{cases}$$

Wykresy obu funkcji dla różnych wartości t

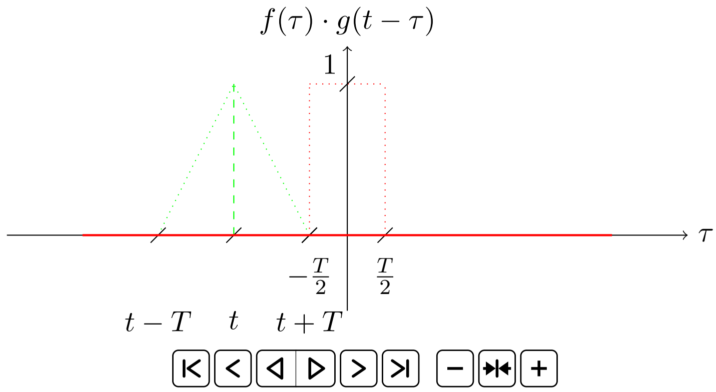


Po wymnożeniu obu funkcji dla przykładowych wartości t otrzymujemy



Jak widać dla różnych wartości t otrzymujemy różny kształt funkcji podcałkowej $f(\tau) \cdot g(t-\tau)$.

Przedział 1 .



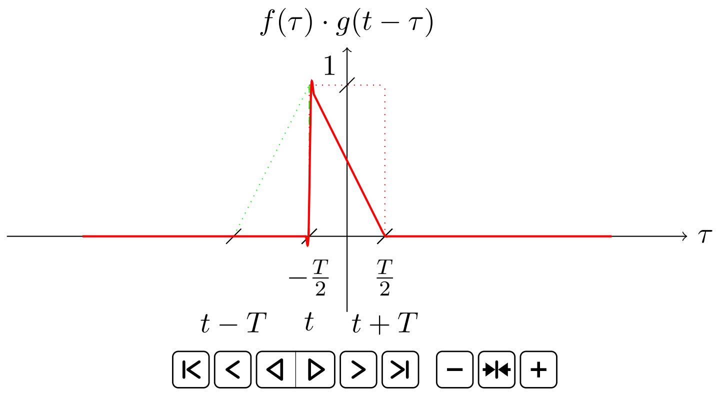
Dla wartości t spełniających warunek $t+T < -\frac{T}{2}$

$$
\begin{aligned}
t+T &< -\frac{T}{2} \\
t &< -\frac{T}{2} - T \\
t &< -\frac{3}{2} \cdot T
\end{aligned}
$$

w wyniku mnożenia otrzymyjemy 0 a więc wartość splotu jest także równa 0

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} 0 \cdot d\tau \\
&= 0
\end{aligned}
$$

Przedział 2 .



Dla wartości t spełniających warunki $t+T \geq -\frac{T}{2}$ i $t < -\frac{T}{2}$

$$\begin{aligned}
t+T &\geq -\frac{T}{2} & \wedge & & t &< -\frac{T}{2}\\
t &\geq -\frac{T}{2} - T & \wedge & & t &< -\frac{T}{2}\\
t &\geq -\frac{3}{2}\cdot T & \wedge & & t &< -\frac{T}{2}
\end{aligned}$$

a więc $t \in \left\langle -\frac{3}{2}\cdot T, -\frac{T}{2}\right)$
w wyniku mnożenia otrzymujemy prostą zdefiniowaną na odcinku $t \in \left(-\frac{T}{2}, t+T\right)$.

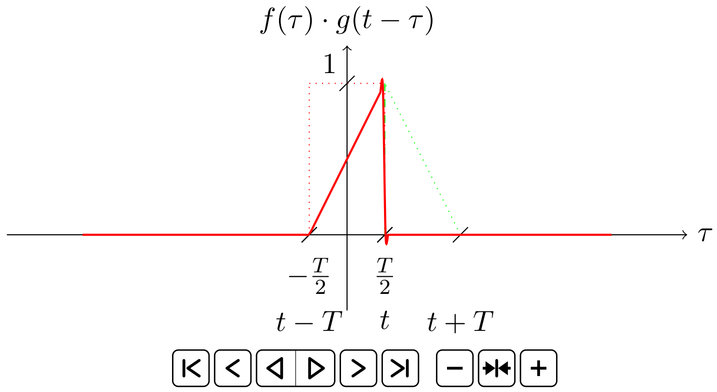
$$f(\tau)\cdot g(t-\tau) = \begin{cases} 0 & \tau \in \left(-\infty; -\frac{T}{2}\right) \\ -\frac{1}{T}\cdot\tau - \frac{-t-T}{T} & \tau \in \left(-\frac{T}{2}; t+T\right) \\ 0 & \tau \in (t+T;\infty) \end{cases}$$

wartość splotu wyznaczamy z ze wzoru

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\cdot d\tau = \\
&= \int_{-\infty}^{\frac{T}{2}} 0\cdot d\tau + \int_{-\frac{T}{2}}^{t+T}\left(-\frac{1}{T}\cdot\tau - \frac{-t-T}{T}\right)\cdot d\tau + \int_{t+T}^{\infty} 0\cdot d\tau = \\
&= 0 - \int_{-\frac{T}{2}}^{t+T}\frac{1}{T}\cdot\tau d\tau + \int_{-\frac{T}{2}}^{t+T}\frac{t+T}{T}\cdot d\tau + 0 = \\
&= -\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{t+T}\tau\cdot d\tau + \frac{t+T}{T}\cdot\int_{-\frac{T}{2}}^{t+T} d\tau = \\
&= -\frac{1}{T}\cdot\left(\frac{1}{2}\cdot\tau^2\right)_{-\frac{T}{2}}^{t+T} + \frac{t+T}{T}\cdot(\tau)_{-\frac{T}{2}}^{t+T} = \\
&= -\frac{1}{T}\cdot\frac{1}{2}\cdot\left((t+T)^2 - \left(-\frac{T}{2}\right)^2\right) + \frac{t+T}{T}\cdot\left(t+T-\left(-\frac{T}{2}\right)\right) = \\
&= -\frac{1}{2\cdot T}\cdot\left(t^2 + 2\cdot t\cdot T + T^2 - \frac{T^2}{4}\right) + \frac{t+T}{T}\cdot\left(t+T+\frac{T}{2}\right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{t+T}{T}\cdot\left(t+\frac{3}{2}\cdot T\right)=\\
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{1}{T}\cdot\left(t^2+\frac{3}{2}\cdot t\cdot T+t\cdot T+\frac{3}{2}\cdot T^2\right)=\\
&= -\frac{1}{2\cdot T}\cdot\left(t^2+2\cdot t\cdot T+\frac{3}{4}\cdot T^2\right)+\frac{2}{2\cdot T}\cdot\left(t^2+\frac{5}{2}\cdot t\cdot T+\frac{3}{2}\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(-t^2-2\cdot t\cdot T-\frac{3}{4}\cdot T^2\right)+\frac{1}{2\cdot T}\cdot\left(2\cdot t^2+5\cdot t\cdot T+3\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(-t^2-2\cdot t\cdot T-\frac{3}{4}\cdot T^2+2\cdot t^2+5\cdot t\cdot T+3\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot\left(t^2+3\cdot t\cdot T+2\frac{1}{4}\cdot T^2\right)=\\
&= \frac{1}{2\cdot T}\cdot t^2+\frac{1}{2\cdot T}\cdot 3\cdot t\cdot T+\frac{1}{2\cdot T}\cdot\frac{9}{4}\cdot T^2=\\
&= \frac{1}{2\cdot T}\cdot t^2+\frac{3}{2}\cdot t+\frac{9}{8}\cdot T
\end{aligned}
$$

Przedział 3 .



Dla wartości t spełniających warunki $t \geq -\frac{T}{2}$ i $t < \frac{T}{2}$

$$
t \geq -\frac{T}{2} \qquad \wedge \qquad t < \frac{T}{2}
$$

a więc $t \in \left\langle -\frac{1}{2}\cdot T, \frac{1}{2}\cdot T\right)$

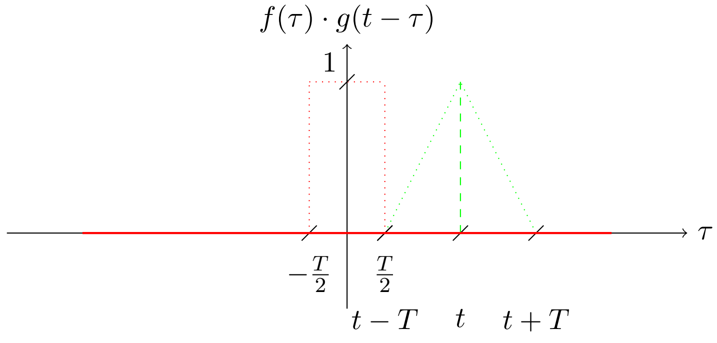
w wyniku mnożenia otrzymujemy dwie proste zdefiniowaną na odcinkach $t \in \left(-\frac{T}{2}, t\right)$ oraz $t \in \left(t, \frac{T}{2}\right)$.

$$
f(\tau)\cdot g(t-\tau)=\begin{cases}0 & \tau\in\left(-\infty;-\frac{T}{2}\right)\\ \frac{1}{T}\cdot\tau-\frac{t-T}{T} & \tau\in\left(-\frac{T}{2};t\right)\\ -\frac{1}{T}\cdot\tau-\frac{-t-T}{T} & \tau\in\left(t;\frac{T}{2}\right)\\ 0 & \tau\in\left(\frac{T}{2};\infty\right)\end{cases}
$$

wartość splotu wyznaczamy z ze wzoru

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\cdot d\tau = \\
&= \int_{-\infty}^{-\frac{T}{2}} 0\cdot d\tau + \int_{-\frac{T}{2}}^{t}\left(\frac{1}{T}\cdot\tau-\frac{t-T}{T}\right)\cdot d\tau + \int_{t}^{\frac{T}{2}}\left(-\frac{1}{T}\cdot\tau-\frac{-t-T}{T}\right)\cdot d\tau + \int_{\frac{T}{2}}^{\infty} 0\cdot d\tau = \\
&= 0 + \int_{-\frac{T}{2}}^{t}\frac{1}{T}\cdot\tau\cdot d\tau - \int_{-\frac{T}{2}}^{t}\frac{t-T}{T}\cdot d\tau + \int_{t}^{\frac{T}{2}}\left(-\frac{1}{T}\cdot\tau\right)\cdot d\tau - \int_{t}^{\frac{T}{2}}\frac{-t-T}{T}\cdot d\tau + 0 = \\
&= \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{t}\tau\cdot d\tau - \frac{t-T}{T}\cdot\int_{-\frac{T}{2}}^{t} d\tau - \frac{1}{T}\cdot\int_{t}^{\frac{T}{2}}\tau\cdot d\tau + \frac{t+T}{T}\cdot\int_{t}^{\frac{T}{2}} d\tau = \\
&= \frac{1}{T}\cdot\frac{1}{2}\cdot\tau^2\Big|_{-\frac{T}{2}}^{t} - \frac{t-T}{T}\cdot\tau|_{-\frac{T}{2}}^{t} - \frac{1}{T}\cdot\frac{1}{2}\cdot\tau^2\Big|_{t}^{\frac{T}{2}} + \frac{t+T}{T}\cdot\tau|_{t}^{\frac{T}{2}} = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2-\left(-\frac{T}{2}\right)^2\right) - \frac{t-T}{T}\cdot\left(t-\left(-\frac{T}{2}\right)\right) - \frac{1}{2\cdot T}\cdot\left(\left(\frac{T}{2}\right)^2-t^2\right) + \frac{t+T}{T}\cdot\left(\frac{T}{2}-t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2+\frac{1}{4}\cdot T^2\right) - \frac{t-T}{T}\cdot\left(t+\frac{T}{2}\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2-t^2\right) + \frac{t+T}{T}\cdot\left(\frac{T}{2}-t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2+\frac{1}{4}\cdot T^2\right) - \frac{2}{2\cdot T}\cdot(t-T)\cdot\left(t+\frac{T}{2}\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2-t^2\right) + \frac{2}{2\cdot T}\cdot(t+T)\cdot\left(\frac{T}{2}-t\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2+\frac{1}{4}\cdot T^2\right) - \frac{2}{2\cdot T}\cdot\left(t^2+\frac{1}{2}\cdot t\cdot T-t\cdot T-\frac{1}{2}\cdot T^2\right) - \frac{1}{2\cdot T}\cdot\left(\frac{1}{4}\cdot T^2-t^2\right) + \\
&+ \frac{2}{2\cdot T}\cdot\left(\frac{1}{2}\cdot t\cdot T-t^2+\frac{1}{2}\cdot T^2-t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2+\frac{1}{4}\cdot T^2\right) + \frac{1}{2\cdot T}\cdot\left(-2\cdot t^2-t\cdot T+2\cdot t\cdot T+T^2\right) + \frac{1}{2\cdot T}\cdot\left(-\frac{1}{4}\cdot T^2+t^2\right) + \\
&+ \frac{1}{2\cdot T}\cdot\left(t\cdot T-2\cdot t^2+T^2-2\cdot t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(t^2+\frac{1}{4}\cdot T^2-2\cdot t^2-t\cdot T+2\cdot t\cdot T+T^2-\frac{1}{4}\cdot T^2+t^2+t\cdot T-2\cdot t^2+T^2-2\cdot t\cdot T\right) = \\
&= \frac{1}{2\cdot T}\cdot\left(-2\cdot t^2+2\cdot T^2\right) = \\
&= \frac{1}{T}\cdot\left(-t^2+T^2\right) = \\
&= -\frac{1}{T}\cdot t^2+T
\end{aligned}
$$

Przedział 4 .



Dla wartości t spełniających warunki $t-T \geq -\frac{T}{2}$ i $t-T < \frac{T}{2}$

$$
\begin{aligned}
t - T &\geq -\frac{T}{2} && \wedge && t - T < \frac{T}{2} \\
t &\geq -\frac{T}{2} + T && \wedge && t < \frac{T}{2} + T \\
t &\geq \frac{1}{2} \cdot T && \wedge && t < \frac{3}{2} \cdot T
\end{aligned}
$$

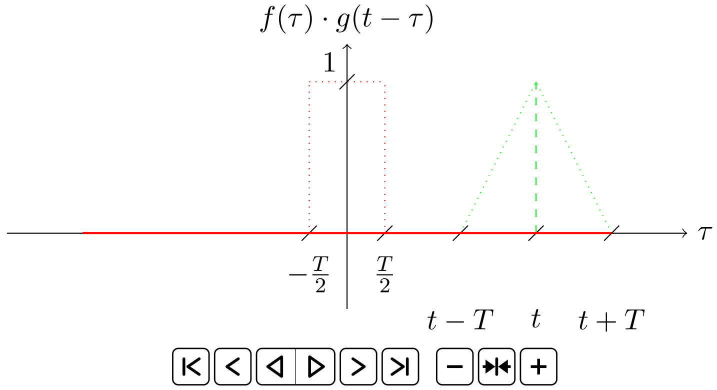
a więc $t \in \left\langle \frac{1}{2} \cdot T, \frac{3}{2} \cdot T \right)$
w wyniku mnożenia otrzymujemy prostą zdefiniowaną na odcinku $t \in \left(t - T, \frac{T}{2} \right)$.

$$
f(\tau) \cdot g(t-\tau) = \begin{cases} 0 & \tau \in (-\infty; t-T) \\ \frac{1}{T} \cdot \tau - \frac{t-T}{T} & \tau \in \left(t-T; \frac{T}{2} \right) \\ 0 & \tau \in \left(\frac{T}{2}; \infty \right) \end{cases}
$$

wartość splotu wyznaczamy z ze wzoru

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau = \\
&= \int_{-\infty}^{t-T} 0 \cdot d\tau + \int_{t-T}^{\frac{T}{2}} \left(\frac{1}{T} \cdot \tau - \frac{t-T}{T} \right) \cdot d\tau + \int_{\frac{T}{2}}^{\infty} 0 \cdot d\tau = \\
&= 0 + \int_{t-T}^{\frac{T}{2}} frac1T \cdot \tau \cdot d\tau - \int_{t-T}^{\frac{T}{2}} \frac{t-T}{T} \cdot d\tau + 0 = \\
&= \frac{1}{T} \cdot \int_{t-T}^{\frac{T}{2}} \tau \cdot d\tau - \frac{t-T}{T} \cdot \int_{t-T}^{\frac{T}{2}} d\tau = \\
&= \frac{1}{T} \cdot \frac{1}{2} \cdot \tau^2 \Big|_{t-T}^{\frac{T}{2}} - \frac{t-T}{T} \cdot \tau|_{t-T}^{\frac{T}{2}} = \\
&= \frac{1}{T} \cdot \frac{1}{2} \cdot \left(\left(\frac{T}{2} \right)^2 - (t-T)^2 \right) - \frac{t-T}{T} \cdot \left(\frac{T}{2} - (t-T) \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{1}{4} \cdot T^2 - \left(t^2 - 2 \cdot t \cdot T + T^2 \right) \right) - \frac{t-T}{T} \cdot \left(\frac{T}{2} - t + T \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{1}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T - T^2 \right) - \frac{1}{T} \cdot (t-T) \cdot \left(\frac{3}{2} \cdot T - t \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T \right) - \frac{2}{2 \cdot T} \cdot \left(\frac{3}{2} \cdot t \cdot T - t^2 - \frac{3}{2} \cdot T^2 + t \cdot T \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T \right) - \frac{1}{2 \cdot T} \cdot \left(\frac{6}{2} \cdot t \cdot T - 2 \cdot t^2 - \frac{6}{2} \cdot T^2 + 2 \cdot t \cdot T \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(-\frac{3}{4} \cdot T^2 - t^2 + 2 \cdot t \cdot T - \frac{6}{2} \cdot t \cdot T + 2 \cdot t^2 + \frac{6}{2} \cdot T^2 - 2 \cdot t \cdot T \right) = \\
&= \frac{1}{2 \cdot T} \cdot \left(\frac{9}{4} \cdot T^2 - 3 \cdot t \cdot T + t^2 \right) = \\
&= \frac{1}{2 \cdot T} \cdot \frac{9}{4} \cdot T^2 - \frac{1}{2 \cdot T} \cdot 3 \cdot t \cdot T + \frac{1}{2 \cdot T} \cdot t^2 = \\
&= \frac{9}{8} \cdot T - \frac{3}{2} \cdot t + \frac{1}{2 \cdot T} \cdot t^2
\end{aligned}
$$

Przedział 5 .



Dla wartości t spełniających warunek $t - T \geq \frac{T}{2}$.

$$\begin{aligned} t - T &\geq \frac{T}{2} \\ t &\geq \frac{T}{2} + T \\ t &\geq \frac{3}{2} \cdot T \end{aligned}$$

a więc $t \in \left\langle \frac{3}{2} \cdot T, \infty \right)$
w wyniku mnożenia otrzymujemy sygnał zerowy

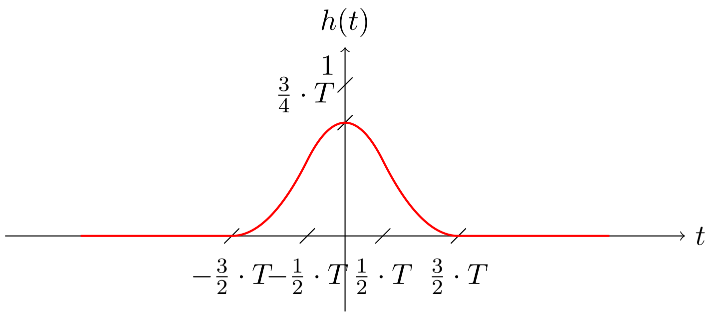
$$f(\tau) \cdot g(t-\tau) = 0$$

a więc wartość splotu wyznaczona ze wzoru

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau = \\ &= \int_{-\infty}^{\infty} 0 \cdot d\tau = \\ &= 0 \end{aligned}$$

Podsumowanie Zbierając wyniki, wynik splotu wyrażony jest jako funkcja o pięciu przedziałach

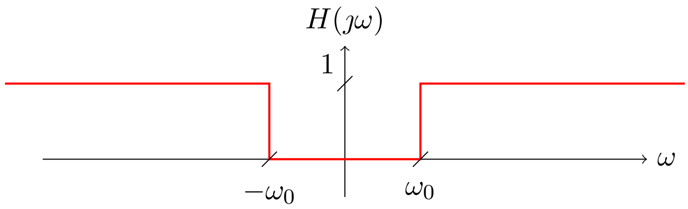
$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau = \\ &= \begin{cases} 0 & \tau \in \left(-\infty; -\frac{3}{2} \cdot T\right); \\ \frac{1}{2 \cdot T} \cdot t^2 + \frac{3}{2} \cdot t + \frac{9}{8} \cdot T & \tau \in \left(-\frac{3}{2} \cdot T; -\frac{1}{2} \cdot T\right); \\ -\frac{1}{T} \cdot t^2 + \frac{3}{4} \cdot T & \tau \in \left(-\frac{1}{2} \cdot T; \frac{1}{2} \cdot T\right); \\ \frac{9}{8} \cdot T - \frac{3}{2} \cdot t + \frac{1}{2 \cdot T} \cdot t^2 & \tau \in \left(\frac{1}{2} \cdot T; \frac{3}{2} \cdot T\right); \\ 0 & \tau \in \left(\frac{3}{2} \cdot T; \infty\right); \end{cases} \end{aligned}$$


$h(t)$
1
$\frac{3}{4} \cdot T$
t
$-\frac{3}{2} \cdot T$
$-\frac{1}{2} \cdot T$
$\frac{1}{2} \cdot T$
$\frac{3}{2} \cdot T$

4.2 Filters

Task 1.

Na układ LTI o transmitancji podanej poniżej, podano sygnał $u(t) = A \cdot Sa\left(3 \cdot \omega_0 \cdot t\right)$. Wyznacz odpowiedź układu $y(t)$ wiedząc, że $\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$.



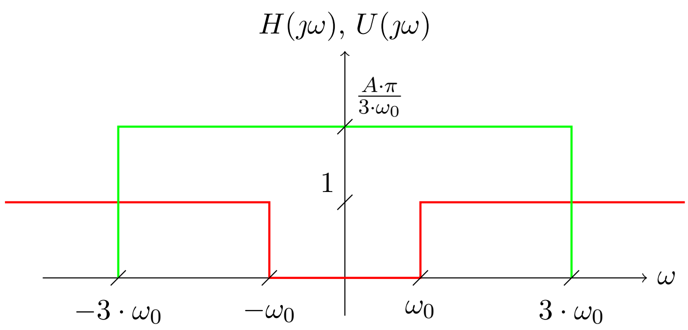
Wiemy, że odpowiedź układu LTI można obliczyć z zależności $y(t) = u(t) * h(t)$, gdzie $h(t)$ jest odpowiedzią impulsową układu. Wiemy także, że transformatę odpowiedzi układu można wyznaczyć ze wzoru $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$.

Ponieważ wyznaczenie splotu liniowego sygnałów jest bardziej skomplikowane niż operacja mnożenia, dlatego spróbujemy skorzystać z tej drugie zależności, czyli mnożenia transformat. W tym celu musimy wyznaczyć transformatę sygnału wejściowego $u(t)$, czyli $U(\jmath\omega)$.

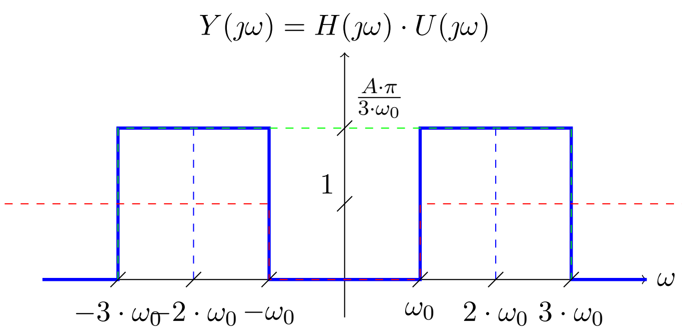
$$\begin{aligned}
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)\\
Sa\left(\frac{t}{2}\right) &\xrightarrow{\mathcal{F}} 2\pi \cdot \Pi(-\omega)\\
Sa\left(6 \cdot \omega_0 \cdot \frac{t}{2}\right) &\xrightarrow{\mathcal{F}} \frac{1}{|6 \cdot \omega_0|} \cdot 2\pi \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right)\\
A \cdot Sa\left(3 \cdot \omega_0 \cdot t\right) &\xrightarrow{\mathcal{F}} \frac{A \cdot \pi}{3 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right)
\end{aligned}$$

Transformata sygnału wejściowego $u(t)$ to $U(\jmath\omega) = \frac{A \cdot \pi}{3 \cdot \omega_0} \cdot \Pi\left(\frac{\omega}{6 \cdot \omega_0}\right)$.

Transformatę sygnału wyjściowego, czyli $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$ wyznaczymy graficznie, W tym celu na wykresie transmitancji $H(\jmath\omega)$ dodamy transformatę $U(\jmath\omega)$:



Teraz dokonujmy operacji mnożenia transformat $U(\jmath\omega)$ przez $H(\jmath\omega)$



$$Y(\jmath\omega) = \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega - 2\cdot\omega_0}{2\cdot\omega_0}\right) + \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega + 2\cdot\omega_0}{2\cdot\omega_0}\right) \tag{4.3}$$

Skoro zanmy transformatę sygnału wyjściowego, to spróbujmy wyznaczyć sygnał wyjściowy w dziedzinie czasu wykorzystując wcześniejsze obliczenia. Transformata $Y(\jmath\omega)$ to suma dwóch przeskalowanych prostokątów, przesuniętych na osi pulsacji. W takim razie można wnioskować, że sygnał w dziedzinie czasu to będzie suma dwóch zmodulowanych i przeskalowanych funkcji $Sa(t)$.

$$\begin{aligned}
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)\\
Sa\left(\frac{t}{2}\right) &\xrightarrow{\mathcal{F}} 2\pi\cdot\Pi(-\omega)\\
Sa\left(2\cdot\omega_0\cdot\frac{t}{2}\right) &\xrightarrow{\mathcal{F}} \frac{1}{|2\cdot\omega_0|}\cdot 2\pi\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)\\
Sa(\omega_0\cdot t) &\xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega}{2\cdot\omega_0}\right)\\
e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) &\xrightarrow{\mathcal{F}} \frac{\pi}{\omega_0}\cdot\Pi\left(\frac{\omega - 2\cdot\omega_0}{2\cdot\omega_0}\right)\\
\frac{A}{3}\cdot e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) &\xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega - 2\cdot\omega_0}{2\cdot\omega_0}\right)\\
\frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) &\xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega - (-2\cdot\omega_0)}{2\cdot\omega_0}\right)\\
\frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) &\xrightarrow{\mathcal{F}} \frac{A\cdot\pi}{3\cdot\omega_0}\cdot\Pi\left(\frac{\omega + 2\cdot\omega_0}{2\cdot\omega_0}\right)
\end{aligned}$$

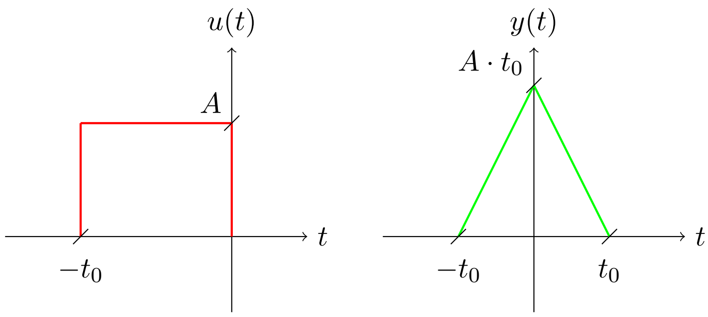
Podsumowując sygnał wyjściowy $y(t)$:

$$\begin{aligned}
y(t) &= \frac{A}{3}\cdot e^{(\jmath\cdot 2\cdot\omega_0\cdot t)}\cdot Sa(\omega_0\cdot t) + \frac{A}{3}\cdot e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\cdot Sa(\omega_0\cdot t) =\\
&= \frac{A}{3}\cdot Sa(\omega_0\cdot t)\cdot\left(e^{(\jmath\cdot 2\cdot\omega_0\cdot t)} + e^{(\jmath\cdot(-2\cdot\omega_0)\cdot t)}\right) =\\
&= \left\{cos\left(x\right) = \tfrac{e^{\jmath\cdot x} + e^{-\jmath\cdot x}}{2}\right\} =\\
&= \frac{2\cdot A}{3}\cdot Sa(\omega_0\cdot t)\cdot cos\left(2\cdot\omega_0\cdot t\right))
\end{aligned}$$

Odpowiedź układu to $y(t) = \frac{2\cdot A}{3}\cdot Sa(\omega_0\cdot t)\cdot cos\left(2\cdot\omega_0\cdot t\right))$.

Task 2.

Wyznacz odpowiedź implusową $h(t)$ układu LTI, wiedząc, że sygnały $u(t)$ oraz $y(t)$ wygladają jak na poniższych wykresach. Wykorzystaj informacje o transformatach sygnałów: $\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$ oraz $\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right)$.



Wiemy, że transformatę odpowiedzi układu można wyznaczyć ze wzoru $Y(\jmath\omega) = U(\jmath\omega) \cdot H(\jmath\omega)$ oraz że $h(t) \xrightarrow{\mathcal{F}} H(\jmath\omega)$. W związku z tym $H(\jmath\omega) = \frac{Y(\jmath\omega)}{U(\jmath\omega)}$ oraz $h(t) \xrightarrow{\mathcal{F}^{-1}} H(\jmath\omega)$.

W pierwszym kroku wyznaczmy transformaty sygnałów $u(t)$ oraz $y(t)$:

$$u(t) = A \cdot \Pi\left(\frac{t+\frac{t_0}{2}}{t_0}\right)$$
$$U(\jmath\omega) = \mathcal{F}\{u(t)\}$$
$$\Pi(t) \xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)$$
$$\Pi\left(\frac{1}{t_0} \cdot t\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right)$$
$$\Pi\left(\frac{t+\frac{t_0}{2}}{t_0}\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}$$
$$A \cdot \Pi\left(\frac{t+\frac{t_0}{2}}{t_0}\right) \xrightarrow{\mathcal{F}} A \cdot t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}$$

$$y(t) = A \cdot t_0 \cdot \Lambda\left(\frac{t}{t_0}\right)$$
$$Y(\jmath\omega) = \mathcal{F}\{y(t)\}$$
$$\Lambda(t) \xrightarrow{\mathcal{F}} Sa^2\left(\frac{\omega}{2}\right)$$
$$\Lambda\left(\frac{1}{t_0} \cdot t\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)$$
$$A \cdot t_0 \cdot \Lambda\left(\frac{t}{t_0}\right) \xrightarrow{\mathcal{F}} A \cdot t_0^2 \cdot Sa^2\left(\frac{\omega \cdot t_0}{2}\right)$$

Skoro zanmy transformaty sygnałów wejściowego i wyjściowego, to możemy wyznaczyc transmitancję układu, czyli $H(\jmath\omega)$.

$$\begin{aligned}
H(\jmath\omega) = \frac{Y(\jmath\omega)}{U(\jmath\omega)} &= \\
&= \frac{A \cdot t_0^2 \cdot Sa^2\left(\frac{\omega\cdot t_0}{2}\right)}{A \cdot t_0 \cdot Sa\left(\frac{\omega\cdot t_0}{2}\right) \cdot e^{\jmath\cdot\omega\cdot\frac{t_0}{2}}} = \\
&= t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}
\end{aligned}$$

Teraz możemy wyznaczć odpowiedź implusową układu $h(t)$:

$$\begin{aligned}
h(t) &\xrightarrow{\mathcal{F}} H(\jmath\omega) \\
? &\xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega \cdot t_0}{2}\right) \cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}} \\
\Pi(t) &\xrightarrow{\mathcal{F}} Sa\left(\frac{\omega}{2}\right)
\end{aligned}$$

$$\Pi\left(\frac{1}{t_0}\cdot t\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)$$
$$\Pi\left(\frac{t-\frac{t_0}{2}}{t_0}\right) \xrightarrow{\mathcal{F}} t_0 \cdot Sa\left(\frac{\omega\cdot t_0}{2}\right)\cdot e^{-\jmath\cdot\omega\cdot\frac{t_0}{2}}$$

Odpowiedź implusowa układu to $h(t) = \Pi\left(\frac{t-\frac{t_0}{2}}{t_0}\right)$.

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