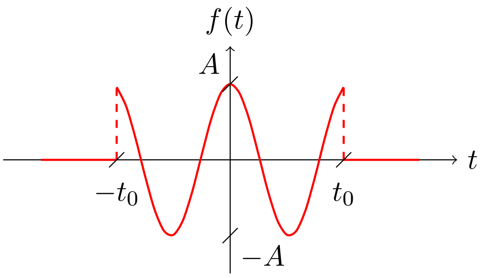
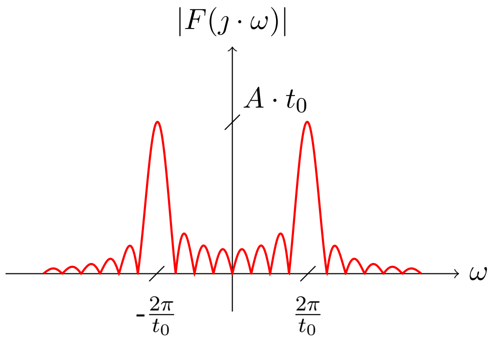
Teoria Sygnałów w zadaniach





$$f(t) = A \cdot \Pi\left(\frac{t}{2 \cdot t_0}\right) \cdot cos\left(\frac{2\pi}{t_0} \cdot t\right)$$

$$F(\jmath\omega) = A \cdot t_0 \cdot [\ Sa\left(\omega \cdot t_0 + 2\pi\right) \\ - Sa\left(\omega \cdot t_0 - 2\pi\right)]$$

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5 kwietnia 2020

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ISBN 978-83-939620-1-3
Wydrukowano w Polsce

Rozdział 1

Podstawowe własności sygnałów

1.1 Podstawowe własności sygnałów

1.1.1 Wartość średnia

1.1.2 Energia sygnału

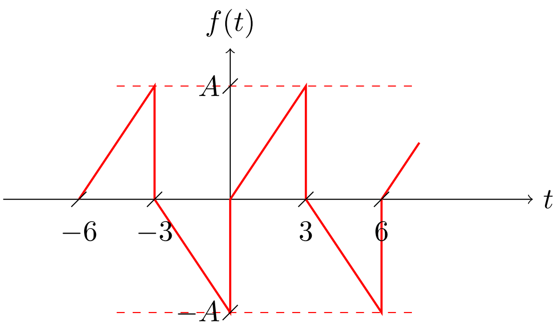
1.1.3 Moc sygnału

Rozdział 2

Analiza sygnałów okresowych za pomocą szeregów ortogonalnych

2.1 Trygonometryczny szerego Fouriera

Zadanie 1. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = a \cdot t + b \tag{2.1}$$

In the first interval of the first period (e.g. $t \in (0;3)$), linear function crosses two points: $(0,0)$ and $(3,A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 0 + b \\ A = a \cdot 3 + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot 3 + b \end{cases}$$

$$\begin{cases} 0 = b \\ A = a \cdot 3 + 0 \end{cases}$$

$$\begin{cases} 0 = b \\ \frac{A}{3} = a \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{3} \cdot t$$

In the second interval of the first period (e.g. $t \in (3;6)$), linear function crosses other two points:: $(3,0)$ and $(6,-A)$. So, in order to derive a and b, the following system of the equations has to be solved.

$$\begin{cases} 0 = a \cdot 3 + b \\ -A = a \cdot 6 + b \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 6 \cdot a - 3 \cdot a \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -A = 3 \cdot a \end{cases}$$
$$\begin{cases} -3 \cdot a = b \\ -\frac{A}{3} = a \end{cases}$$
$$\begin{cases} -3 \cdot (-\frac{A}{3}) = b \\ -\frac{A}{3} = a \end{cases}$$
$$\begin{cases} A = b \\ -\frac{A}{3} = a \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{A}{3} \cdot t + A$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{3} \cdot t & for \quad t \in (0;3) \\ -\frac{A}{3} \cdot t + A & for \quad t \in (3;6) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{3} \cdot (t - k \cdot 6) & for \quad t \in (0 + k \cdot 6; 3 + k \cdot 6) \\ -\frac{A}{3} \cdot (t - k \cdot 6) + A & for \quad t \in (3 + k \cdot 6; 6 + k \cdot 6) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \qquad (2.2)$$

For the period $t \in (0; 6)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T f(t)\cdot dt\\
&= \frac{1}{6}\cdot\left[\int_0^3 \frac{A}{3}\cdot t\cdot dt + \int_3^6\left(-\frac{A}{3}\cdot t + A\right)\cdot dt\right] =\\
&= \frac{A}{18}\cdot\int_0^3 t\cdot dt - \frac{A}{18}\cdot\int_3^6 t\cdot dt + \frac{A}{6}\cdot\int_3^6 dt =\\
&= \frac{A}{18}\cdot\left.\frac{t^2}{2}\right|_0^3 - \frac{A}{18}\cdot\left.\frac{t^2}{2}\right|_3^6 + \frac{A}{6}\cdot t|_3^6 =\\
&= \frac{A}{36}\cdot\left(3^2-0^2\right) - \frac{A}{36}\cdot\left(6^2-3^2\right) + \frac{A}{6}\cdot(6-3) =\\
&= \frac{A}{36}\cdot 9 - \frac{A}{36}\cdot 27 + \frac{A}{6}\cdot 3 =\\
&= \frac{9\cdot A}{36} - \frac{27\cdot A}{36} + \frac{18\cdot A}{36} =\\
&= 0
\end{aligned}$$

The a_0 coefficient equals 0.

The a_k coefficients are defined as:

$$a_k = \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.3}$$

For the period $t \in (0; 6)$, i.e. $k = 0$, we get:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\\
&= \frac{2}{6}\cdot\left[\int_0^3\frac{A}{3}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt + \int_3^6\left(-\frac{A}{3}\cdot t + A\right)\cdot cos\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt\right] =\\
&= \frac{A}{9}\cdot\int_0^3 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt - \frac{A}{9}\cdot\int_3^6 t\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt + \frac{A}{3}\cdot\int_3^6 cos\left(k\cdot\frac{\pi}{3}\cdot t\right)dt =\\
&= \begin{Bmatrix} u &= t & dv &= cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\\ du &= dt & v &= \frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\end{Bmatrix} =\\
&= \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3 - \int_0^3\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] -\\
&- \frac{A}{9}\cdot\left[t\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 - \int_3^6\frac{3}{k\cdot 3\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right] +\\
&+ \frac{A}{3}\cdot\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6 =\\
&= \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) - 0\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 0\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3\right] -\\
&- \frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - 3\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right) + \frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6\right] +\\
&+ \frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot\frac{\pi}{3}\cdot 6\right) - sin\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right] =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot sin\left(k\cdot\pi\right)-0+\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)-\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 0\right)\right]-\\
&-\frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot sin\left(k\cdot 2\pi\right)-3\cdot sin\left(k\pi\right)+\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 6\right)-\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right]+\\
&+\frac{A}{k\cdot\pi}\cdot\left[sin\left(k\cdot 2\pi\right)-sin\left(k\cdot\pi\right)\right]=\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[3\cdot 0+\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot cos\left(0\right)\right]-\\
&-\frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot 0-3\cdot 0+\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\right)-\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right]+\\
&+\frac{A}{k\cdot\pi}\cdot\left[0-0\right]=\\
&= \frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot 1\right]-\\
&-\frac{A}{3\cdot k\cdot\pi}\cdot\left[\frac{3}{k\cdot\pi}\cdot 1-\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)\right]=\\
&= \frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right)-\frac{A}{k^2\cdot\pi^2}-\frac{A}{k^2\cdot\pi^2}+\frac{A}{k^2\cdot\pi^2}\cdot cos\left(k\cdot\pi\right)=\\
&= \frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left(cos\left(k\cdot\pi\right)-1\right)=\\
&= \frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left(\left(-1\right)^k-1\right)
\end{aligned}
$$

The a_k coefficients equal to $\frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left(\left(-1\right)^k-1\right)$.
The b_k coefficients are defined as:

$$
b_k=\frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.4}
$$

For the period $t\in(0;6)$, i.e. $k=0$, we get:

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_T f(t)\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\\
&= \frac{2}{6}\cdot\left[\int_0^3\frac{A}{3}\cdot t\cdot sin\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt+\int_3^6\left(-\frac{A}{3}\cdot t+A\right)\cdot sin\left(k\cdot\frac{2\pi}{6}\cdot t\right)\cdot dt\right]=\\
&= \frac{A}{9}\cdot\int_0^3 t\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt-\frac{A}{9}\cdot\int_3^6 t\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt+\frac{A}{3}\cdot\int_3^6 sin\left(k\cdot\frac{\pi}{3}\cdot t\right)dt=\\
&= \left\{\begin{matrix} u & =t & dv & =sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\\ du & =dt & v & =-\frac{3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\end{matrix}\right\}=\\
&= \frac{A}{9}\cdot\left[t\cdot\frac{-3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3-\int_0^3\frac{-3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right]-\\
&-\frac{A}{9}\cdot\left[t\cdot\frac{-3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6-\int_3^6\frac{-3}{k\cdot 3\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\cdot dt\right]+\\
&+\frac{A}{3}\cdot\frac{-3}{k\cdot\pi}\cdot cos\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6=\\
&= \frac{-3\cdot A}{9\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)-0\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 0\right)-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_0^3\right]+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3\cdot A}{9\cdot k\cdot\pi}\cdot\left[6\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 6\right)-3\cdot cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot t\right)\Big|_3^6\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[cos\left(k\cdot\frac{\pi}{3}\cdot 6\right)-cos\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)-0-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right)+\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 0\right)\right]+\\
&+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot cos\left(k\cdot 2\pi\right)-3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 6\right)+\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\frac{\pi}{3}\cdot 3\right)\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[cos\left(k\cdot 2\pi\right)-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\pi\right)+\frac{3}{k\cdot\pi}\cdot sin\left(0\right)\right]+\\
&+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6\cdot 1-3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\right)+\frac{3}{k\cdot\pi}\cdot sin\left(k\cdot\pi\right)\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot 0+\frac{3}{k\cdot\pi}\cdot 0\right]+\\
&+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)-\frac{3}{k\cdot\pi}\cdot 0+\frac{3}{k\cdot\pi}\cdot 0\right]-\\
&-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{3\cdot k\cdot\pi}\cdot\left[3\cdot cos\left(k\cdot\pi\right)\right]+\frac{A}{3\cdot k\cdot\pi}\cdot\left[6-3\cdot cos\left(k\cdot\pi\right)\right]-\frac{A}{k\cdot\pi}\cdot\left[1-cos\left(k\cdot\pi\right)\right]=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{2\cdot A}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)-\frac{A}{k\cdot\pi}+\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)=\\
&=\frac{-A}{k\cdot\pi}\cdot cos\left(k\cdot\pi\right)+\frac{A}{k\cdot\pi}=\\
&=\frac{-A}{k\cdot\pi}\cdot(-1)^k+\frac{A}{k\cdot\pi}=\\
&=\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)
\end{aligned}
$$

The b_k coefficients equal to $\frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$
\begin{aligned}
a_0 &= 0\\
a_k &= \frac{2\cdot A}{k^2\cdot\pi^2}\cdot\left((-1)^k-1\right)\\
b_k &= \frac{A}{k\cdot\pi}\cdot\left(1-(-1)^k\right)
\end{aligned}
$$

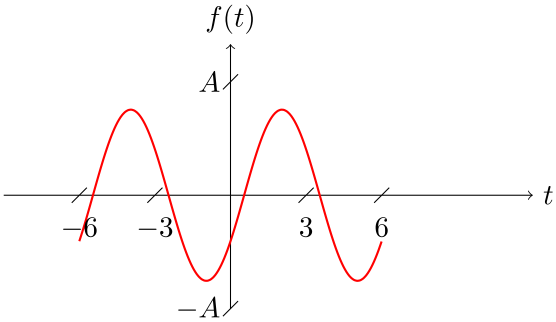
The first six coefficients are equal to:

k	1	2	3	4	5	6
a_k	$\frac{-4\cdot A}{\pi^2}$	0	$\frac{-4\cdot A}{9\cdot\pi^2}$	0	$\frac{-4\cdot A}{25\cdot\pi^2}$	0
b_k	$\frac{2\cdot A}{\pi}$	0	$\frac{2\cdot A}{3\cdot\pi}$	0	$\frac{2\cdot A}{5\cdot\pi}$	0

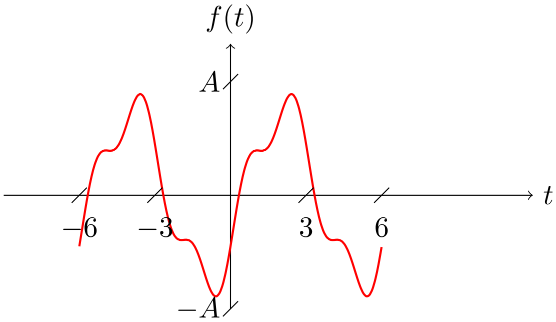
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right]$$
$$f(t) = \sum_{k=1}^{\infty} \left[\frac{2 \cdot A}{k^2 \cdot \pi^2} \cdot \left((-1)^k - 1\right) \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + \left(\frac{A}{k \cdot \pi} \cdot \left(1 - (-1)^k\right)\right) \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right] \tag{2.5}$$

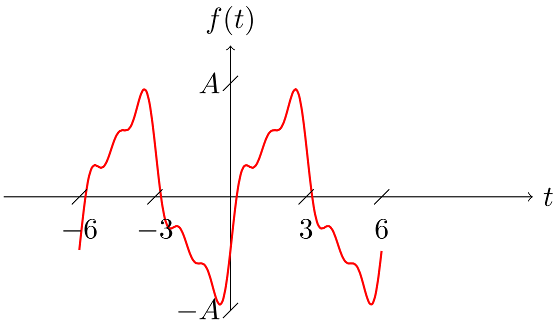
A partial approximation of the $f(t)$ signal for $k_{max} = 1$ results in:



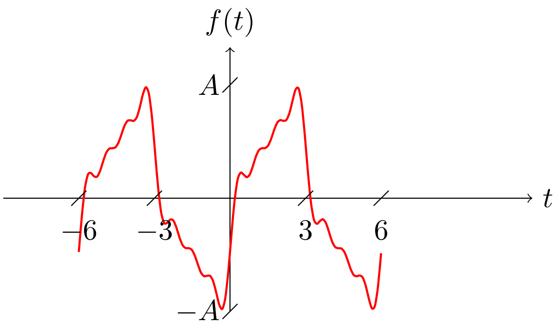
A partial approximation of the $f(t)$ signal for $k_{max} = 3$ results in:



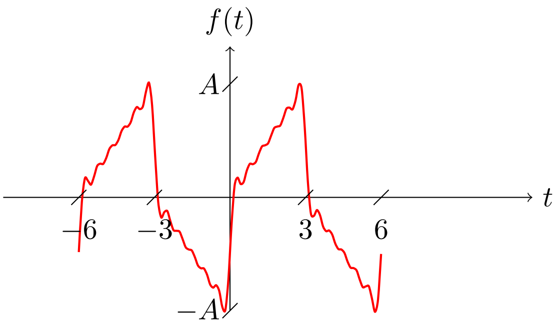
A partial approximation of the $f(t)$ signal for $k_{max} = 5$ results in:



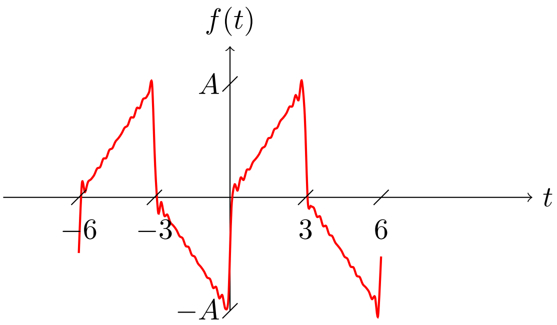
A partial approximation of the $f(t)$ signal for $k_{max} = 7$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 11$ results in:

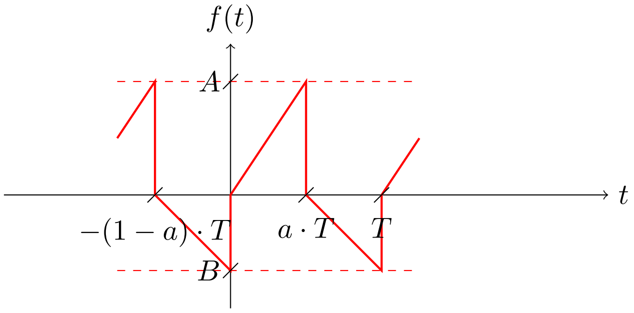


A partial approximation of the $f(t)$ signal for $k_{max} = 21$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

Zadanie 2. Calculate coefficients of the periodic signal $f(t)$ shown below for the expansion into a trigonometric Fourier series.



First of all, the definition of $f(t)$ signal has to be derived. This is periodic piecewise function, piecewise linear function to be precise. The simplest form of linear function is:

$$f(t) = m \cdot t + b \tag{2.6}$$

In the first interval of the first period (e.g. $t \in (0; a \cdot T)$), linear function crosses two points: $(0, 0)$ and $(a \cdot T, A)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot 0 + b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + b \end{cases}$$
$$\begin{cases} 0 = b \\ A = m \cdot a \cdot T + 0 \end{cases}$$
$$\begin{cases} 0 = b \\ \frac{A}{a \cdot T} = m \end{cases}$$

As a result we get:

$$f(t) = \frac{A}{a \cdot T} \cdot t$$

In the second interval of the first period (e.g. $t \in (a \cdot T; T)$), linear function crosses other two points: $(a \cdot T, 0)$ and $(T, -B)$. So, in order to derive m and b, the following system of the equations has to be solved:

$$\begin{cases} 0 = m \cdot a \cdot T + b \\ -B = m \cdot T + b \end{cases}$$
$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot T - m \cdot a \cdot T \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -B = m \cdot (T - a \cdot T) \end{cases}$$

$$\begin{cases} -m \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{T-a\cdot T} \cdot a \cdot T = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T-a\cdot T} = m \end{cases}$$

$$\begin{cases} \frac{B}{1-a} \cdot a = b \\ -\frac{B}{T\cdot(1-a)} = m \end{cases}$$

As a result, second interval of the first period is described by:

$$f(t) = -\frac{B}{T \cdot (1-a)} \cdot t + \frac{B}{1-a} \cdot a$$

As a result the piecewise linear function in the first period is given by:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot t & dla \quad t \in (0; a \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot t + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T; T) \end{cases}$$

For the whole periodic signal $f(t)$ we get:

$$f(t) = \begin{cases} \frac{A}{a\cdot T} \cdot (t - k \cdot T) & dla \quad t \in (0 + k \cdot T; a \cdot T + k \cdot T) \\ -\frac{B}{(1-a)\cdot T} \cdot (t - k \cdot T) + \frac{B}{1-a} \cdot a & dla \quad t \in (a \cdot T + k \cdot T; T + k \cdot T) \end{cases} \wedge k \in Z$$

The a_0 coefficient is defined as:

$$a_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.7}$$

For the period $t \in (0; T)$ we get:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\ &= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} \left(-\frac{B}{(1-a) \cdot T} \cdot t + \frac{B}{1-a} \cdot a \right) \cdot dt \right) = \\ &= \frac{1}{T} \left(\int_0^{a\cdot T} \frac{A}{a \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} -\frac{B}{(1-a) \cdot T} \cdot t \cdot dt + \int_{a\cdot T}^{T} \frac{B}{1-a} \cdot a \cdot dt \right) = \\ &= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \int_0^{a\cdot T} t \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a\cdot T}^{T} t \cdot dt + \frac{B}{1-a} \cdot a \cdot \int_{a\cdot T}^{T} dt \right) = \\ &= \frac{1}{T} \left(\frac{A}{a \cdot T} \cdot \left. \frac{t^2}{2} \right|_0^{a\cdot T} - \frac{B}{(1-a) \cdot T} \cdot \left. \frac{t^2}{2} \right|_{a\cdot T}^{T} + \frac{B}{1-a} \cdot a \cdot \left. t \right|_{a\cdot T}^{T} \right) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{(a\cdot T)^2}{2}-\frac{0^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{(a\cdot T)^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot(T-a\cdot T)\right)=\\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}-\frac{0}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot\left(\frac{T^2}{2}-\frac{a^2\cdot T^2}{2}\right)+\frac{B}{1-a}\cdot a\cdot T\cdot(1-a)\right)=\\
&= \frac{1}{T}\left(\frac{A}{a\cdot T}\cdot\left(\frac{a^2\cdot T^2}{2}\right)-\frac{B}{(1-a)\cdot T}\cdot T^2\cdot\left(\frac{1}{2}-\frac{a^2}{2}\right)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot\left(\frac{a\cdot T}{2}\right)-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot\left(1-a^2\right)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot\frac{a\cdot T}{2}-\frac{B}{1-a}\cdot T\cdot\frac{1}{2}\cdot(1-a)\cdot(1+a)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot(1+a)+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}\cdot a+B\cdot a\cdot T\right)=\\
&= \frac{1}{T}\left(A\cdot a\cdot T\cdot\frac{1}{2}-B\cdot T\cdot\frac{1}{2}+B\cdot T\cdot\frac{1}{2}\cdot a\right)=\\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}+B\cdot\frac{1}{2}\cdot a=\\
&= A\cdot a\cdot\frac{1}{2}-B\cdot\frac{1}{2}(1-a)=\\
&= \frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)
\end{aligned}
$$

The a_0 coefficient equals $\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$.

The a_k coefficients are defined as:

$$
a_k=\frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt \tag{2.8}
$$

For the period $t\in(0;T)$ we get:

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_T f(t)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt=\\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}\left(-\frac{B}{(1-a)\cdot T}\cdot t+\frac{B}{1-a}\cdot a\right)\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\left(\int_0^{a\cdot T}\frac{A}{a\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\int_{a\cdot T}^{T}-\frac{B}{(1-a)\cdot T}\cdot t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\int_{a\cdot T}^{T}\frac{B}{1-a}\cdot a\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \frac{2}{T}\cdot\left(\frac{A}{a\cdot T}\cdot\int_0^{a\cdot T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt-\frac{B}{(1-a)\cdot T}\cdot\int_{a\cdot T}^{T}t\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt+\right.\\
&\left.+\frac{B}{1-a}\cdot a\cdot\int_{a\cdot T}^{T}cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot dt\right)=\\
&= \left\{\begin{array}{c} z=k\cdot\frac{2\pi}{T}\cdot t\\ dz=k\cdot\frac{2\pi}{T}\cdot dt\\ \frac{dz}{k\cdot\frac{2\pi}{T}}=dt\end{array}\right\}=
\end{aligned}
$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \int_{a \cdot T}^{T} cos\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \int_{a \cdot T}^{T} cos\,(z) \cdot dz \right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\, sin\,(z)|_{a \cdot T}^{T} \right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi}\, sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} \right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{ll} u = t & dv = cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \int_0^{a \cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} - \int_{a \cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a \cdot T}^{T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a \cdot T}^{T} sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$

$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a \cdot T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a \cdot T} sin\,(z) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) + \right.$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Bigg|_{a\cdot T}^{T} - \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a\cdot T}^{T} sin\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) +$$

$$+ \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(sin\left(k \cdot 2\pi\right) - sin\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) - 0 \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot -\right)\right) + \right.\right.$$

$$\left. - \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_{0}^{a\cdot T} sin\left(z\right) \cdot dz \right) +$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(\left(T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot T\right) - a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right) + \right.$$

$$\left. - \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_{a\cdot T}^{T} sin\left(z\right) \cdot dz \right) +$$

$$+ \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(0 - sin\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - 0\right) + \right.\right.$$

$$\left. - \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(-cos\left(z\right)\right)\big|_{0}^{a\cdot T} \right) +$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(\left(T \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi\right) - a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) + \right.$$

$$\left. - \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(-cos\left(z\right)\right)\big|_{a\cdot T}^{T} \right) +$$

$$+ \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(-sin\left(k \cdot 2\pi \cdot a\right)\right)\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \right.\right.$$

$$\left. - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Bigg|_{0}^{a\cdot T} \right) +$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot 0 - a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \right.$$

$$\left. - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right)\Bigg|_{a\cdot T}^{T} \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \right.\right.$$

$$\left. - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) + cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right)\right) +$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(0 - a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \right.$$

$$
\begin{aligned}
&- \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) + cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\Bigg) + \\
&- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\Bigg) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi \cdot a\right) + cos\left(0\right)\right)\right) + \right. \\
&- \frac{B}{(1-a) \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi\right) + cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot T^2 \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(-cos\left(k \cdot 2\pi \cdot a\right) + 1\right)\right) + \right. \\
&- \frac{B}{(1-a) \cdot T} \cdot T^2 \left(-a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(-1 + cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2}{T} \cdot \left(\frac{A}{a} \cdot T \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \right. \\
&+ \frac{B}{1-a} \cdot T \left(a \cdot \frac{1}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&- \left.\frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right)\right) = \\
&= \frac{2 \cdot A}{a} \left(\frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&+ \frac{2 \cdot B}{1-a} \left(\frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) + \\
&- \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) = \\
&= \frac{2 \cdot A}{a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{2 \cdot A}{a} \cdot \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&+ \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) - \frac{2 \cdot B}{1-a} \cdot \frac{1}{k^2 \cdot 4\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{2 \cdot B}{1-a} \cdot \frac{a}{k \cdot 2\pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) = \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) + \\
&- \frac{A}{a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot \frac{1}{2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) + \\
&- \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot \frac{1}{2} \cdot \left(1 - cos\left(k \cdot 2\pi \cdot a\right)\right) \\
&= \left\{ \; \tfrac{1}{2} \cdot \left(1 - cos\left(x\right)\right) = sin^2\left(\tfrac{x}{2}\right) \; \right\} = \\
&= \frac{A}{k \cdot \pi} \cdot sin\left(k \cdot 2\pi \cdot a\right) +
\end{aligned}
$$

$$- \frac{A}{a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot sin^2 \left(k \cdot \pi \cdot a\right) +$$
$$- \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot \pi^2} \cdot sin^2 \left(k \cdot \pi \cdot a\right)$$
$$= \frac{A}{k \cdot \pi} \cdot sin \left(k \cdot 2\pi \cdot a\right) - \left(\frac{A}{a} + \frac{B}{1-a}\right) \cdot \frac{1}{k^2 \cdot \pi^2} \cdot sin^2 \left(k \cdot \pi \cdot a\right)$$

The a_k coefficients equal to $\frac{A}{k \cdot \pi} \cdot sin \left(k \cdot 2\pi \cdot a\right) - \left(\frac{A}{a} + \frac{B}{1-a}\right) \cdot \frac{1}{k^2 \cdot \pi^2} \cdot sin^2 \left(k \cdot \pi \cdot a\right)$.
The b_k coefficients are defined as:

$$b_k = \frac{2}{T} \int_T f(t) \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \tag{2.9}$$

For the period $t \in (0; T)$ we get:

$$b_k = \frac{2}{T} \int_T f(t) \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt =$$
$$= \frac{2}{T} \cdot \left(\int_0^{a \cdot T} \frac{A}{a \cdot T} \cdot t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{a \cdot T}^{T} \left(-\frac{B}{(1-a) \cdot T} \cdot t + \frac{B}{1-a} \cdot a\right) \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \left(\int_0^{a \cdot T} \frac{A}{a \cdot T} \cdot t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{a \cdot T}^{T} -\frac{B}{(1-a) \cdot T} \cdot t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \int_{a \cdot T}^{T} \frac{B}{1-a} \cdot a \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \int_{a \cdot T}^{T} sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) =$$
$$= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \int_{a \cdot T}^{T} sin \left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \int_{a \cdot T}^{T} sin \left(z\right) \cdot dz \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$
$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \, cos \left(z\right) \Big|_{a \cdot T}^{T} \right) =$$
$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a \cdot T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a) \cdot T} \cdot \int_{a \cdot T}^{T} t \cdot sin \left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} \Bigg) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \int_0^{a\cdot T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt - \frac{B}{(1-a)\cdot T} \cdot \int_{a\cdot T}^{T} t \cdot sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt + \right.$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{ll} u = t & dv = sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \\ du = dt & v = -\frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \int_0^{a\cdot T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \int_{a\cdot T}^{T} \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a\cdot T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) + \right.$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a\cdot T}^{T} cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot dt \right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) - cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right)\right) =$$

$$= \left\{ \begin{array}{l} z = k \cdot \frac{2\pi}{T} \cdot t \\ dz = k \cdot \frac{2\pi}{T} \cdot dt \\ \frac{dz}{k \cdot \frac{2\pi}{T}} = dt \end{array} \right\} =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_0^{a\cdot T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_0^{a\cdot T} cos\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) + \right.$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(-t \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\Big|_{a\cdot T}^{T} + \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \int_{a\cdot T}^{T} cos\left(z\right) \cdot \frac{dz}{k \cdot \frac{2\pi}{T}} \right) +$$

$$\left. - \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \left(cos\left(k \cdot 2\pi\right) - cos\left(k \cdot 2\pi \cdot a\right)\right)\right) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right) + 0 \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot 0\right)\right) + \right.\right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_0^{a\cdot T} cos\left(z\right) \cdot dz \right) +$$

$$- \frac{B}{(1-a)\cdot T} \cdot \left(\left(-T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot T\right) + a \cdot T \cdot \frac{1}{k \cdot \frac{2\pi}{T}} \cdot cos\left(k \cdot \frac{2\pi}{T} \cdot a \cdot T\right)\right) + \right.$$

$$\left. + \frac{1}{\left(k \cdot \frac{2\pi}{T}\right)^2} \cdot \int_{a\cdot T}^{T} cos\left(z\right) \cdot dz \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(\left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + 0 \right) + \right.\right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot (sin\,(z))|_0^{a \cdot T} \right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi) + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right) + \right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot (sin\,(z))|_{a \cdot T}^{T} \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \right) \Bigg|_0^{a \cdot T} \right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(\left(-T \cdot \frac{T}{k \cdot 2\pi} \cdot 1 + a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) \right) + \right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin \left(k \cdot \frac{2\pi}{T} \cdot t \right) \right) \Bigg|_{a \cdot T}^{T} \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot T \cdot \frac{T}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin \left(k \cdot \frac{2\pi}{T} \cdot a \cdot T \right) - sin \left(k \cdot \frac{2\pi}{T} \cdot 0 \right) \right) \right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(T \cdot \frac{T}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \right.$$

$$\left. + \frac{1}{k^2 \cdot \frac{4\pi^2}{T^2}} \cdot \left(sin \left(k \cdot \frac{2\pi}{T} \cdot T \right) - sin \left(k \cdot \frac{2\pi}{T} \cdot a \cdot T \right) \right) \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right.$$

$$\left. + \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi \cdot a) - sin\,(0)) \right) +$$

$$- \frac{B}{(1-a) \cdot T} \cdot \left(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \right.$$

$$\left. + \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi) - sin\,(k \cdot 2\pi \cdot a)) \right) +$$

$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Big) =$$

$$= \frac{2}{T} \cdot \left(\frac{A}{a \cdot T} \cdot \left(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \right.\right.$$

$$+ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (sin\,(k \cdot 2\pi \cdot a) - 0)\Bigg) +$$
$$- \frac{B}{(1-a) \cdot T} \cdot \Bigg(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) +$$
$$+ \frac{T^2}{k^2 \cdot 4\pi^2} \cdot (0 - sin\,(k \cdot 2\pi \cdot a))\Bigg) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Bigg) =$$
$$= \frac{2}{T} \cdot \Bigg(\frac{A}{a \cdot T} \cdot \Bigg(-a \cdot \frac{T^2}{k \cdot 2\pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{(1-a) \cdot T} \cdot \Bigg(\frac{T^2}{k \cdot 2\pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{T^2}{k^2 \cdot 4\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Bigg) =$$
$$= \frac{2}{T} \cdot \Bigg(\frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a))\Bigg) =$$
$$= \frac{2}{T} \cdot \frac{A}{a \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{2}{T} \cdot \frac{B}{(1-a) \cdot T} \cdot \frac{T^2}{2} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{2}{T} \cdot \frac{B}{1-a} \cdot a \cdot \frac{T}{k \cdot 2\pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) =$$
$$= \frac{A}{a} \cdot \Bigg(-a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{1-a} \cdot \Bigg(\frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) - \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a)\Bigg) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) =$$
$$= -\frac{A}{a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$
$$- \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot (-1 + a \cdot cos\,(k \cdot 2\pi \cdot a)) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot (1 - cos\,(k \cdot 2\pi \cdot a)) =$$
$$= -\frac{A}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$
$$+ \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} - \frac{B}{1-a} \cdot \frac{1}{k \cdot \pi} \cdot a \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$
$$- \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} + \frac{B}{1-a} \cdot a \cdot \frac{1}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) =$$
$$= -\frac{A}{k \cdot \pi} \cdot cos\,(k \cdot 2\pi \cdot a) + \frac{A}{a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$
$$+ \frac{B}{1-a} \cdot \frac{1}{k^2 \cdot 2\pi^2} \cdot sin\,(k \cdot 2\pi \cdot a) +$$

$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}-\frac{B}{1-a}\cdot a\cdot\frac{1}{k\cdot\pi}=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+$$
$$+\frac{B}{1-a}\cdot\frac{1}{k\cdot\pi}\cdot(1-a)=$$
$$=-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)+\frac{B}{k\cdot\pi}$$

The b_k coefficients equal to $\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$.
To sum up, coefficients for the expansion into trigonometric Fourier series are given by:

$$a_0=\frac{1}{2}\cdot A\cdot a-\frac{1}{2}\cdot B\cdot(1-a)$$
$$a_k=\frac{A}{k\cdot\pi}\cdot sin\left(k\cdot 2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot\pi^2}\cdot sin^2\left(k\cdot\pi\cdot a\right)$$
$$b_k=\frac{B}{k\cdot\pi}-\frac{A}{k\cdot\pi}\cdot cos\left(k\cdot 2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{k^2\cdot 2\pi^2}\cdot sin\left(k\cdot 2\pi\cdot a\right)$$

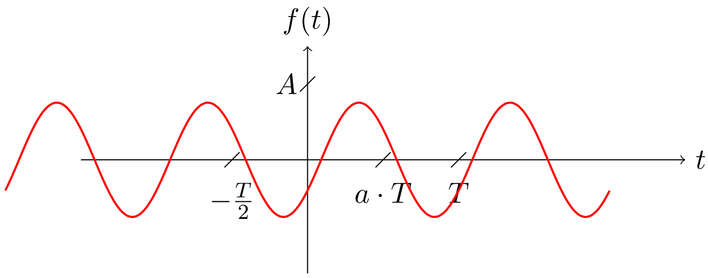
The first six coefficients are equal to:

k	a_k	b_k
1	$\frac{A}{\pi}\cdot sin\left(2\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{\pi^2}\cdot sin^2\left(\pi\cdot a\right)$	$\frac{B}{\pi}-\frac{A}{\pi}\cdot cos\left(2\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{2\pi^2}\cdot sin\left(2\pi\cdot a\right)$
2	$\frac{A}{2\pi}\cdot sin\left(4\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{4\pi^2}\cdot sin^2\left(2\pi\cdot a\right)$	$\frac{B}{2\pi}-\frac{A}{2\pi}\cdot cos\left(4\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{8\pi^2}\cdot sin\left(4\pi\cdot a\right)$
3	$\frac{A}{3\pi}\cdot sin\left(6\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{9\pi^2}\cdot sin^2\left(3\pi\cdot a\right)$	$\frac{B}{3\pi}-\frac{A}{3\pi}\cdot cos\left(6\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{18\pi^2}\cdot sin\left(6\pi\cdot a\right)$
4	$\frac{A}{4\pi}\cdot sin\left(8\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{16\pi^2}\cdot sin^2\left(4\pi\cdot a\right)$	$\frac{B}{4\pi}-\frac{A}{4\pi}\cdot cos\left(8\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{32\pi^2}\cdot sin\left(8\pi\cdot a\right)$
5	$\frac{A}{5\pi}\cdot sin\left(10\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{25\pi^2}\cdot sin^2\left(5\pi\cdot a\right)$	$\frac{B}{5\pi}-\frac{A}{5\pi}\cdot cos\left(10\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{50\pi^2}\cdot sin\left(10\pi\cdot a\right)$
6	$\frac{A}{6\pi}\cdot sin\left(12\pi\cdot a\right)-\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{36\pi^2}\cdot sin^2\left(6\pi\cdot a\right)$	$\frac{B}{6\pi}-\frac{A}{6\pi}\cdot cos\left(12\pi\cdot a\right)+\left(\frac{A}{a}+\frac{B}{1-a}\right)\cdot\frac{1}{72\pi^2}\cdot sin\left(12\pi\cdot a\right)$

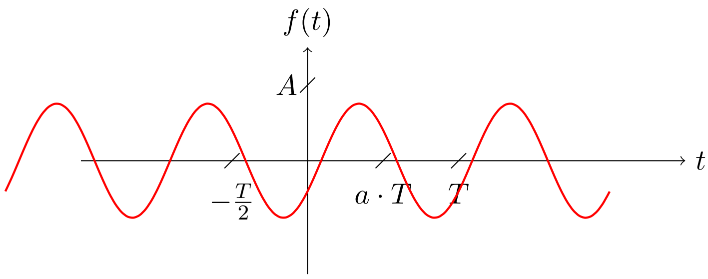
Hence, the signal $f(t)$ may be expressed as the sum of the harmonic series

$$f(t)=a_0+\sum_{k=1}^{\infty}\left[a_k\cdot cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)+b_k\cdot sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\right] \tag{2.10}$$

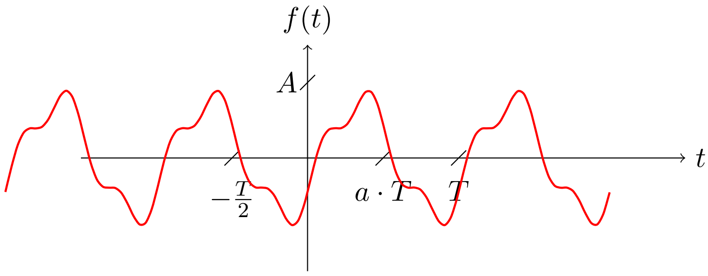
A partial approximation of the $f(t)$ signal for $k_{max}=1$ results in:



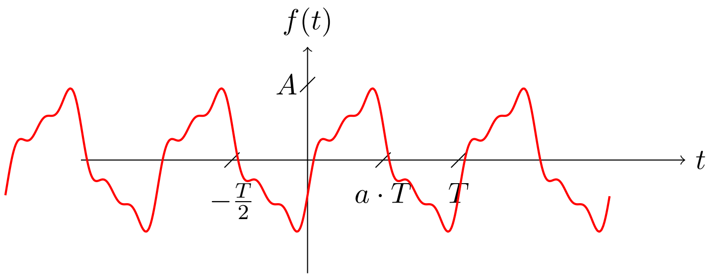
A partial approximation of the $f(t)$ signal for $k_{max}=2$ results in:



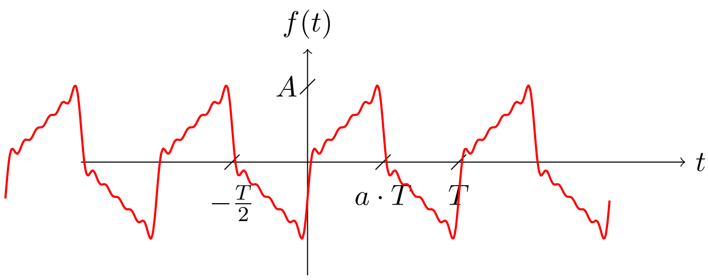
A partial approximation of the $f(t)$ signal for $k_{max} = 4$ results in:



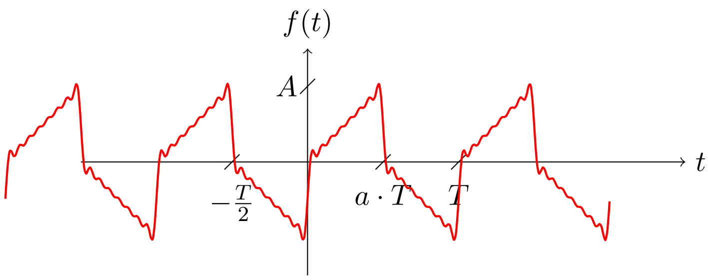
A partial approximation of the $f(t)$ signal for $k_{max} = 6$ results in:



A partial approximation of the $f(t)$ signal for $k_{max} = 12$ results in:



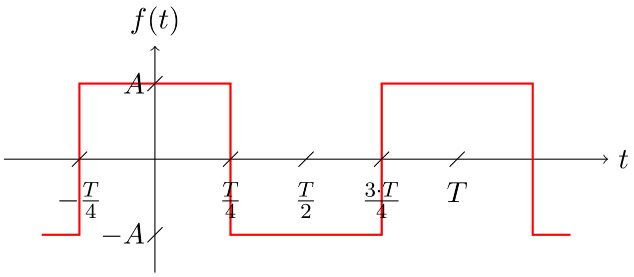
A partial approximation of the $f(t)$ signal for $k_{max} = 16$ results in:



Approximation of the $f(t)$ signal for $k_{max} = \infty$ results in oryginal signal.

2.2 Zespolony szerego Fouriera

Zadanie 1. Wyznacz współczynniki zespolonego szeregu fouriera dla okresowego sygnału $f(t)$ przedstawionego na rysunku



W pierwszej kolejności należy opisać sygnał za pomocą wzoru.

$$f(x) = \begin{cases} A & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in C \tag{2.11}$$

Współczynnik F_0 wyznaczamy ze wzoru

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.12}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} dt + 0 \right) = \\
&= \frac{1}{T} \left(A \cdot t|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{T} \cdot t|_0^{\frac{T}{2}} = \\
&= \frac{A}{T} \cdot \left(\frac{T}{2} - 0 \right) = \\
&= \frac{A}{T} \cdot \left(\frac{T}{2} \right) = \\
&= \frac{A}{\cancel{T}} \cdot \left(\frac{\cancel{T}}{2} \right) = \\
&= \frac{A}{2}
\end{aligned} \tag{2.13}$$

Wartość współczynnika F_0 wynosi $\frac{A}{2}$

Współczynnik F_k wyznaczamy ze wzoru

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.14}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$
\begin{aligned}
F_k &= \frac{1}{T}\int_T f(t)\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{1}{T}\int_0^{\frac{T}{2}} A\cdot e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot dt = \\
&= \left\{\begin{matrix} z & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t \\ dz & = -\jmath\cdot k\cdot\frac{2\pi}{T}\cdot dt \\ dt & = \frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} \end{matrix}\right\} = \\
&= \frac{A}{T}\int_0^{\frac{T}{2}} e^z\cdot\frac{dz}{-\jmath\cdot k\cdot\frac{2\pi}{T}} = \\
&= -\frac{A}{T\cdot\jmath\cdot k\cdot\frac{2\pi}{T}}\int_0^{\frac{T}{2}} e^z\cdot dz = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^z\big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\, e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot\frac{T}{2}} - e^{-\jmath\cdot k\cdot\frac{2\pi}{T}\cdot 0}\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - e^0\right) = \\
&= -\frac{A}{\jmath\cdot k\cdot 2\pi}\left(e^{-\jmath\cdot k\cdot\pi} - 1\right) = \\
&= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\pi} - 1\right)
\end{aligned}
$$

Wartość współczynnika F_k wynosi $\jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\pi} - 1\right)$

Współczynniki zespolonego szeregu fouriera dla funkcji przedstawionej na rysunku przyjmują wartości

$$
\begin{aligned}
F_0 &= \frac{A}{2} \\
F_k &= \jmath\cdot\frac{A}{k\cdot 2\pi}\cdot\left(e^{-\jmath\cdot k\cdot\pi} - 1\right)
\end{aligned}
$$

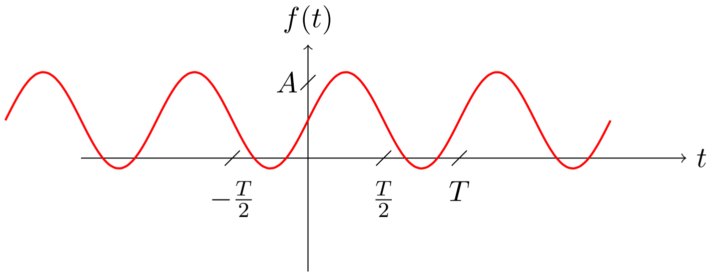
Możemy wyznaczyć kilka wartości współczynników F_k

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	$\jmath\cdot\frac{A}{5\pi}$	0	$\jmath\cdot\frac{A}{3\pi}$	0	$\jmath\cdot\frac{A}{\pi}$	0	$-\jmath\cdot\frac{A}{\pi}$	0	$-\jmath\cdot\frac{A}{3\pi}$	0	$-\jmath\cdot\frac{A}{5\pi}$
$\|F_k\|$	$\frac{A}{5\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{\pi}$	0	$\frac{A}{\pi}$	0	$\frac{A}{3\pi}$	0	$\frac{A}{5\pi}$
$Arg\{F_k\}$	π	0	π	0	π	0	$-\pi$	0	$-\pi$	0	$-\pi$

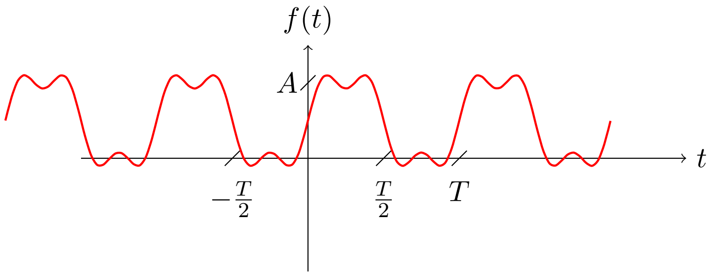
Podstawiając to wzoru aproksymacyjnego funkcje $f(t)$ możemy wyrazić jako

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.15}$$

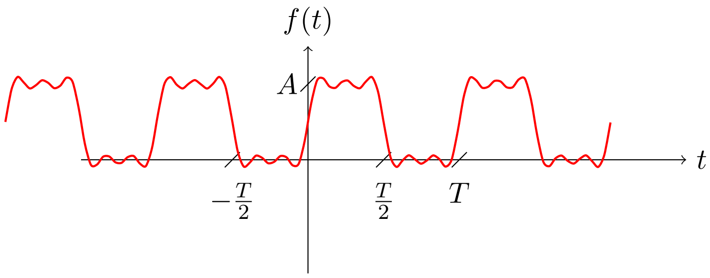
W przypadku sumowania od $k_{min} = -1$ do $k_{max} = 1$ otrzymujemy



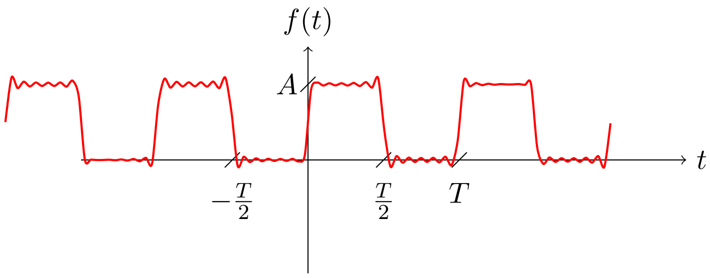
W przypadku sumowania od $k_{min} = -3$ do $k_{max} = 3$ otrzymujemy



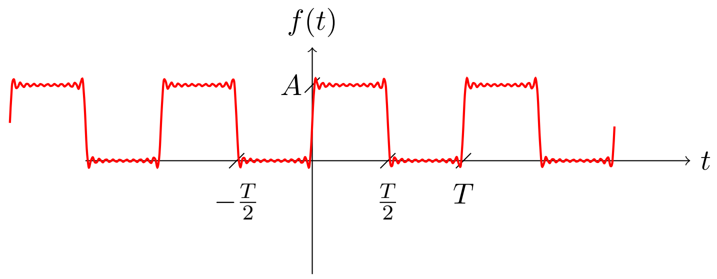
W przypadku sumowania od $k_{min} = -5$ do $k_{max} = 5$ otrzymujemy



W przypadku sumowania od $k_{min} = -11$ do $k_{max} = 11$ otrzymujemy

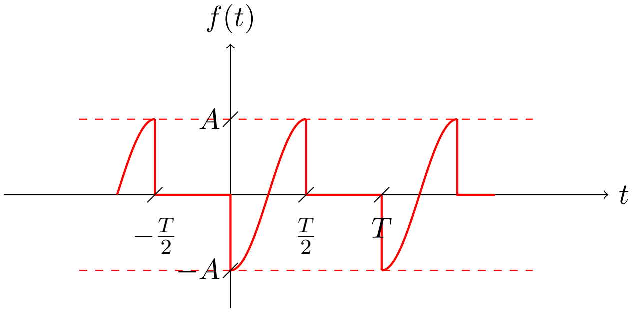


W przypadku sumowania od $k_{min} = -21$ do $k_{max} = 21$ otrzymujemy



W granicy sumowania od $k_{min} = -\infty$ do $k_{max} = \infty$ otrzymujemy oryginalny sygnał.

Zadanie 2. Wyznacz wszystkie współczynniki zespolonego szeregu fouriera dla okresowego sygnału $f(t)$ będącego przekształceniem sygnału sinusoidalnego przedstawionego na rysunku.



W pierwszej kolejności należy opisać sygnał za pomocą wzoru:

$$f(x) = \begin{cases} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) & t \in \left(0 + k \cdot T; \frac{T}{2} + k \cdot T\right) \\ 0 & t \in \left(\frac{T}{2} + k \cdot T; T + k \cdot T\right) \end{cases} \wedge k \in C \tag{2.16}$$

Współczynnik F_0 wyznaczamy ze wzoru

$$F_0 = \frac{1}{T} \int_T f(t) \cdot dt \tag{2.17}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$\begin{aligned}
F_0 &= \frac{1}{T} \int_T f(t) \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt + 0 \right) = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot dt = \\
&= \left\{ \begin{matrix} z & = \frac{2\pi}{T} \cdot t \\ dz & = \frac{2\pi}{T} \cdot dt \\ dt & = \frac{1}{\frac{2\pi}{T}} \cdot dz \\ dt & = \frac{T}{2\pi} \cdot dz \end{matrix} \right\} = \\
&= \frac{A}{T} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot \frac{T}{2\pi} \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot \int_0^{\frac{T}{2}} cos\left(z\right) \cdot dz = \\
&= \frac{A}{T} \cdot \frac{T}{2\pi} \cdot sin\left(z\right)\Big|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot sin\left(\frac{2\pi}{T} \cdot t\right)\Bigg|_0^{\frac{T}{2}} = \\
&= \frac{A}{2\pi} \cdot \left(sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) - sin\left(\frac{2\pi}{T} \cdot 0\right) \right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{2\pi} \cdot (sin\,(pi) - sin\,(0)) = \\
&= \frac{A}{2\pi} \cdot (0 - 0) = \\
&= \frac{A}{2\pi} \cdot 0 = \\
&= 0
\end{aligned}
$$

Wartość współczynnika F_0 wynosi 0

Współczynnik F_k wyznaczamy ze wzoru

$$F_k = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \tag{2.18}$$

Podstawiamy do wzoru wzór naszej funkcji w pierwszym okresie $k = 0$

$$
\begin{aligned}
F_k &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t\right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \right) \cdot dt = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{cc} z_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t & z_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t \\ dz_1 = \jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot dt & dz_2 = -\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot dt \\ dt = \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 & dt = \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \end{array} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z_1} \cdot \frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot dz_1 + \int_0^{\frac{T}{2}} e^{z_2} \cdot \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{1}{\jmath \cdot \frac{2\pi}{T} \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 + \frac{1}{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{\jmath \cdot 2\pi \cdot (1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{T}{\jmath \cdot 2\pi \cdot (1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{\jmath \cdot 2\pi} \cdot \left(\frac{1}{(1-k)} \cdot \int_0^{\frac{T}{2}} e^{z_1} \cdot dz_1 - \frac{1}{(1+k)} \cdot \int_0^{\frac{T}{2}} e^{z_2} \cdot dz_2 \right) =
\end{aligned}
$$

$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{z_1} \Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{z_2} \Big|_0^{\frac{T}{2}} \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot t} \Big|_0^{\frac{T}{2}} - \frac{1}{(1+k)} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot t} \Big|_0^{\frac{T}{2}} \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot \frac{T}{2}} - e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-k) \cdot 0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+k) \cdot 0} \right) \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi \cdot (1-k)} - e^{0} \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi \cdot (1+k)} - 1 \right) \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(e^{\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(e^{-\jmath \cdot \pi} \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(-1 \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{1}{(1-k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) - \frac{1}{(1+k)} \cdot \left(- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 \right) \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1+k)}{(1-k) \cdot (1+k)} - \frac{\left(-e^{-\jmath \cdot k \cdot \pi} - 1 \right) \cdot (1-k)}{(1-k) \cdot (1+k)} \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{(1-k) \cdot (1+k)} - \frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 + k \cdot e^{-\jmath \cdot k \cdot \pi} + k}{(1-k) \cdot (1+k)} \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{- \cdot e^{-\jmath \cdot k \cdot \pi} - 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k + e^{-\jmath \cdot k \cdot \pi} + 1 - k \cdot e^{-\jmath \cdot k \cdot \pi} - k}{1-k^2} \right) =$$
$$= \frac{A}{\jmath \cdot 4\pi} \cdot \left(\frac{-2 \cdot k \cdot e^{-\jmath \cdot k \cdot \pi} - 2 \cdot k}{1-k^2} \right) =$$
$$= -\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1-k^2} \right)$$

Wartość współczynnika F_k wynosi $-\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi}+1}{1-k^2} \right)$.
Dla $k = 1$ i $k = -1$ trzeba wyzanczyć wartość współczynnika raz jeszcze wprost ze wzoru

$$F_1 = \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt =$$
$$= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot 1 \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) =$$
$$= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos \left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) =$$
$$= \left\{ cos\,(x) = \tfrac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} =$$
$$= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t}}{2} \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + 0 \right) =$$
$$= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot e^{-\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt =$$
$$= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot t - \jmath \cdot \frac{2\pi}{T} \cdot t} \right) \cdot dt =$$
$$= \frac{A}{2 \cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath \cdot \frac{2\pi}{T} \cdot (1-1) \cdot t} + e^{-\jmath \cdot \frac{2\pi}{T} \cdot (1+1) \cdot t} \right) \cdot dt =$$

$$
\begin{aligned}
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath \cdot \frac{2\pi}{T} \cdot 0 \cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{2\pi}{T} \cdot 2 \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{0} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} 1 \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{-\jmath \cdot \frac{4\pi}{T} \cdot t} \cdot dt \right) = \\
&= \left\{ \begin{array}{c} z = -\jmath \cdot \frac{4\pi}{T} \cdot t \\ dz = -\jmath \cdot \frac{4\pi}{T} \cdot dt \\ dt = \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \end{array} \right\} = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \int_0^{\frac{T}{2}} e^{z} \cdot \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\int_0^{\frac{T}{2}} dt + \frac{1}{-\jmath \cdot \frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(t\Big|_0^{\frac{T}{2}} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\left(\frac{T}{2} - 0 \right) - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot e^{-\jmath \cdot \frac{4\pi}{T} \cdot t}\Big|_0^{\frac{T}{2}} \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot \frac{4\pi}{T} \cdot \frac{T}{2}} - e^{-\jmath \cdot \frac{4\pi}{T} \cdot 0} \right) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot \left(e^{-\jmath \cdot 2\pi} - e^{0} \right) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot (1 - 1) \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - \frac{1}{\jmath \cdot \frac{4\pi}{T}} \cdot 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \left(\frac{T}{2} - 0 \right) = \\
&= \frac{A}{2 \cdot T} \cdot \frac{T}{2} = \\
&= \frac{A}{4}
\end{aligned}
$$

Wartość współczynnika F_1 wynosi $\frac{A}{4}$.

$$
\begin{aligned}
F_{-1} &= \frac{1}{T} \int_T f(t) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt = \\
&= \frac{1}{T} \left(\int_0^{\frac{T}{2}} A \cdot cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot e^{-\jmath \cdot (-1) \cdot \frac{2\pi}{T} \cdot t} \cdot dt \right) = \\
&= \frac{1}{T} \left(A \cdot \int_0^{\frac{T}{2}} cos\left(\frac{2\pi}{T} \cdot t \right) \cdot e^{\jmath \cdot \frac{2\pi}{T} \cdot t} \cdot dt + \int_{\frac{T}{2}}^{T} 0 \cdot dt \right) = \\
&= \left\{ cos\left(x \right) = \frac{e^{\jmath \cdot x} + e^{-\jmath \cdot x}}{2} \right\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{T}\left(A \cdot \int_0^{\frac{T}{2}} \frac{e^{\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}}{2} \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot dt + 0\right) = \\
&= \frac{A}{2\cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t}\right) \cdot e^{\jmath\cdot\frac{2\pi}{T}\cdot t} \cdot dt = \\
&= \frac{A}{2\cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot t+\jmath\cdot\frac{2\pi}{T}\cdot t}\right) \cdot dt = \\
&= \frac{A}{2\cdot T} \cdot \int_0^{\frac{T}{2}} \left(e^{\jmath\cdot\frac{2\pi}{T}\cdot(1+1)\cdot t} + e^{-\jmath\cdot\frac{2\pi}{T}\cdot(-1+1)\cdot t}\right) \cdot dt = \\
&= \frac{A}{2\cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{2\pi}{T}\cdot 2\cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{-\jmath\cdot\frac{2\pi}{T}\cdot 0\cdot t} \cdot dt\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t} \cdot dt + \int_0^{\frac{T}{2}} e^{0} \cdot dt\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t} \cdot dt + \int_0^{\frac{T}{2}} 1 \cdot dt\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{\jmath\cdot\frac{4\pi}{T}\cdot t} \cdot dt + \int_0^{\frac{T}{2}} dt\right) = \\
&= \left\{\begin{matrix} z = \jmath\cdot\frac{4\pi}{T}\cdot t \\ dz = \jmath\cdot\frac{4\pi}{T}\cdot dt \\ dt = \frac{1}{\jmath\cdot\frac{4\pi}{T}}\cdot dz \end{matrix}\right\} = \\
&= \frac{A}{2\cdot T} \cdot \left(\int_0^{\frac{T}{2}} e^{z} \cdot \frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot \int_0^{\frac{T}{2}} e^{z} \cdot dz + \int_0^{\frac{T}{2}} dt\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot e^{z}\Big|_0^{\frac{T}{2}} + t\Big|_0^{\frac{T}{2}}\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot e^{\jmath\cdot\frac{4\pi}{T}\cdot t}\Big|_0^{\frac{T}{2}} + \left(\frac{T}{2} - 0\right)\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot \left(e^{\jmath\cdot\frac{4\pi}{T}\cdot\frac{T}{2}} - e^{\jmath\cdot\frac{4\pi}{T}\cdot 0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot \left(e^{\jmath\cdot 2\pi} - e^{0}\right) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot (1-1) + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(\frac{1}{\jmath\cdot\frac{4\pi}{T}} \cdot 0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T} \cdot \left(0 + \frac{T}{2}\right) = \\
&= \frac{A}{2\cdot T} \cdot \frac{T}{2} = \\
&= \frac{A}{4}
\end{aligned}
$$

Wartość współczynnika F_{-1} wynosi $\frac{A}{4}$.

Tak wiec ostatecznie współczynniki zespolonego szeregu fouriera

$$F_0 = 0$$
$$F_1 = \frac{A}{4}$$
$$F_{-1} = \frac{A}{4}$$
$$F_k = -\frac{A \cdot k}{\jmath \cdot 2\pi} \cdot \left(\frac{e^{-\jmath \cdot k \cdot \pi} + 1}{1 - k^2} \right)$$

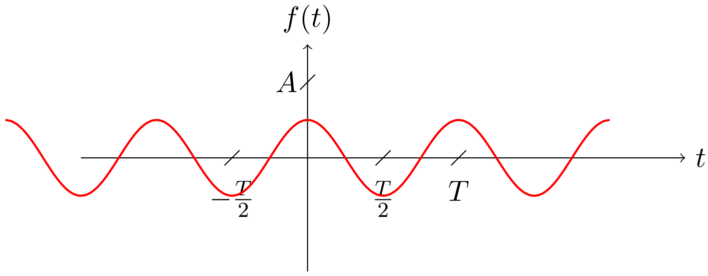
Możemy wyznaczyć kilka wartości współczynników F_k

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_k	0	$\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0	$\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$-\jmath \cdot \frac{2 \cdot A}{3 \cdot \pi}$	0	$-\jmath \cdot \frac{4 \cdot A}{15 \cdot \pi}$	0
$\lvert F_k \rvert$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0	$\frac{2 \cdot A}{3 \cdot \pi}$	$\frac{A}{4}$	0	$\frac{A}{4}$	$\frac{2 \cdot A}{3 \cdot \pi}$	0	$\frac{4 \cdot A}{15 \cdot \pi}$	0
$Arg\{F_k\}$	0	π	0	π	0	0	0	$-\pi$	0	$-\pi$	0

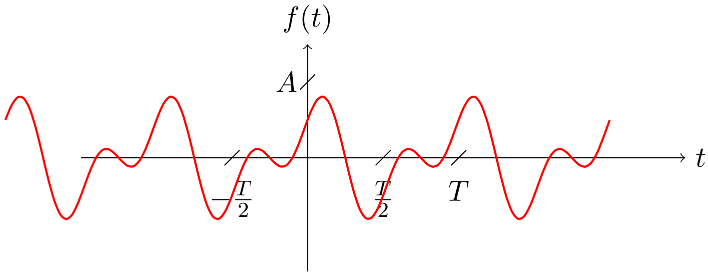
Podstawiając to wzoru aproksymacyjnego funkcje $f(t)$ możemy wyrazić jako

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot e^{\jmath \cdot k \cdot \frac{2\pi}{T} \cdot t} \tag{2.19}$$

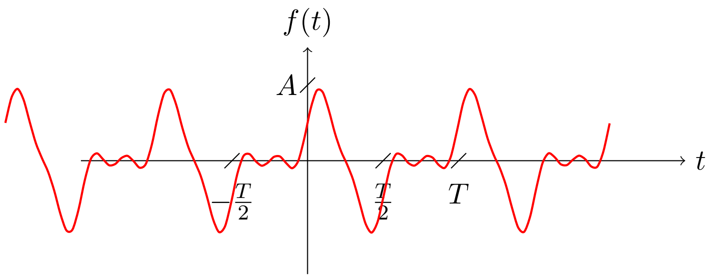
W przypadku sumowania od $k_{min} = -1$ do $k_{max} = 1$ otrzymujemy



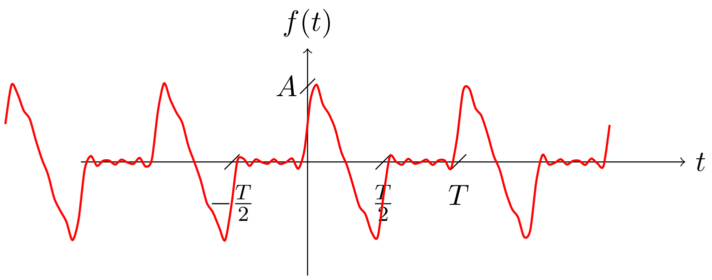
W przypadku sumowania od $k_{min} = -2$ do $k_{max} = 2$ otrzymujemy



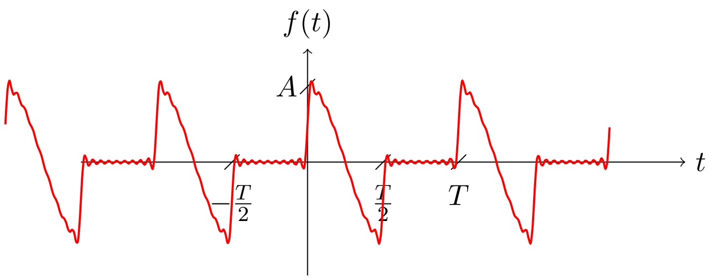
W przypadku sumowania od $k_{min} = -4$ do $k_{max} = 4$ otrzymujemy



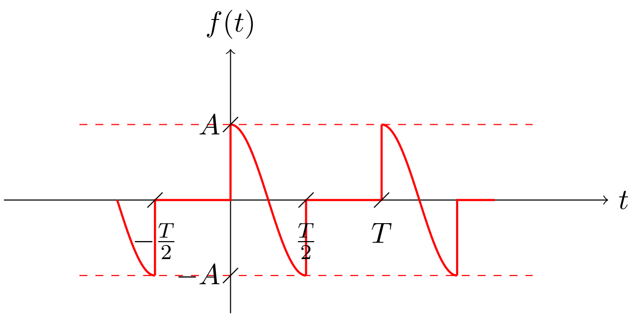
W przypadku sumowania od $k_{min} = -10$ do $k_{max} = 10$ otrzymujemy



W przypadku sumowania od $k_{min} = -20$ do $k_{max} = 20$ otrzymujemy



W granicy sumowania od $k_{min} = -\infty$ do $k_{max} = \infty$ otrzymujemy oryginalny sygnał.



2.3 Obliczenia mocy sygnałów - twierdzenie Parsevala

Rozdział 3

Analiza sygnałów nieokresowych. Transformata Fouriera

3.1 Wyznaczanie transformaty Fouriera z definicji

3.2 Wykorzystanie twierdzeń do obliczeń transformaty Fouriera

3.3 Obliczenia energii sygnału za pomocą transformaty Fouriera. Twierdzenie Parsevala

Rozdział 4

Przetwarzanie sygnałów za pomocą układów LTI

4.1 Obliczanie splotu ze wzoru

4.2 Filtry

ISBN 978-83-939620-1-3