Lecture 1: Functions and Their Properties

Summary:
This lecture introduces the fundamental concept of functions and their key properties. We explore three essential types of functions:

- Injective (one-to-one) functions, where each element in the codomain has at most one pre-image
- Surjective (onto) functions, where each element in the codomain has at least one pre-image
- Bijective functions, which are both injective and surjective

Topics Covered: function, set, Identity function, inverse function, injectivity, surjectivity, bijectivity, cardinality, image, rationals, function composition

We also examine important properties of functions, including:

- Function composition and its properties
- The identity function and its role
- Inverse functions and their relationship to bijective functions
- The conditions under which a function has an inverse

These concepts form the foundation for understanding more complex mathematical structures and transformations that will be explored in later lectures.

Cardinality

- The cardinality of a (finite) set A, denoted $|A|$ or $\#A$ is the numbenr of elements in the set.

Note that

$$|A \times B| = |A| \cdot |B|$$

Functions

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2$$

The graph of the function is

$$\{(x, y) \in \mathbb{R}^2 : \ y = f(x)\}$$

Cartesian product

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Note that

$$|A \times B| = |A| \cdot |B|$$

$|A|$ is the number of elements in a set

Definition - function: Let S and T be two sets. A function from S to T is a subset $F \subseteq S \times T$ such that $\forall x \in S. \ \exists y \in T. \ (x, y) \in F$.

S is called the domain of F

T is the codomain, or range of F

$$(x, y) \in F \iff f(x) = y$$

Example:

$$f : \mathbb{Q} \to \mathbb{Z}$$

$$f\left(\frac{m}{n}\right) = m$$

Note that this is not a function because a rational number can have multiple representations

$$f\left(\frac{1}{3}\right) = 1,\ f\left(\frac{2}{6}\right) = 2$$

And $\frac{1}{3} = \frac{2}{6}$, so, it is not the case that for any $r \in \mathbb{Q}.\ \exists! y \in \mathbb{Z}.\ f(r) = y$.

Definition: Let $A \subseteq S$. The **inclusion function** $\iota : A \to S$ is such that $\forall a \in A.\ \iota(a) = a$.

Definition: Let $f : S \to T$ and $g : T \to U$ be two functions. We define the composition

$$(g \circ f)(x) = g(f(x))$$

> Try writing the composition of functions in cartesian product language.

Proposition: Let $f : S \to T, g : T \to U, R : U \to V$. Then,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

In other words, function composition is associative.

This is easy to prove using the definition of composition.

Definition: The function $f : S \to T$ is **onto** means $\operatorname{im}(f) = T$. In other words, $\forall t \in T.\ \exists s \in S.\ f(s) = t$. This is also called a **surjection**.

Definition: The function $f : S \to T$ is **one-to-one** means $f(x_1) = f(x_2) \implies x_1 = x_2$. This is also called an injection.

Definition: The function $f : S \to T$ is a **one-to-one correspondence** means f is injective and surjective. This is also called a bijection.

Example:

$$f : \{1, 2\} \to \{3, 4, 5\}$$

Note that f cannot be a surjection.

f can be an injection, for example:

$$1 \to 3$$

$$3 \to 5$$

Consider the bijection

$$g : \{1, 2, 3\} \to \{3, 4, 5\}$$

$$1 \to 4$$

$$3 \to 3$$

$$2 \to 5$$

Note that on the left, each elements appears exactly once, and same on the right side.

Identity function:

$$1_S : S \to S$$

$$\forall x \in S.\ 1_S(x) = x$$

Note that for any function f, we have

$$1_S \circ f = f \circ 1_S = f$$

Inverse function:

Let $f : S \to T$ and let $g : T \to S$ be functions. The functions f and g are inverses of each other means

$$f \circ g = 1_T$$
$$g \circ f = 1_S$$

Proposition: If f has an inverse, then the inverse is unique.

Proof:

Suppose that $g, h : T \to S$ are inverses of f.

We want to show that $g = h$.

Note that

$$\underbrace{(h \circ f)}_{1_S} \circ g = h \circ \underbrace{(f \circ g)}_{1_T}$$
$$\implies 1_S \circ g = h \circ 1_T$$
$$\implies g = h$$
$$QED$$

Proposition: $f : S \to T, g : T \to U$. Then,

1. f and g are onto $\implies g \circ f$ is onto
2. f and g are one-to-one $\implies g \circ f$ one-to-one

Proof of 1:

$\forall z \in U.\ \exists t \in T.\ g(t) = u$, since g is onto.

$\forall t \in T.\ \exists s \in S.\ f(s) = t$, since f is onto

Therefore,

$$(g \circ f)s(s) = g(f(s)) = u \implies g \circ f \text{ is onto}$$
$$QED$$

Proof of 2:

$$g(f(x_1)) = g(f(x_2),\ x_1, x_2 \in S$$

Since g is one-to-one $\implies f(x_1) = f(x_2)$

$$x_1 = x_2$$
$$QED$$

Corollary: If f, g are bijections $\implies g \circ f$ is a bijection.

Proposition: Let $f : S \to T$ be a function.

f has an inverse $\iff f$ is bijective.

$\implies$ **proof:**

Suppose that f has an inverse $g : T \to S$ such that

$$g \circ f = 1_S,\ f \circ g = 1_T$$

Let $y \in T$. Note that

$$y = 1_T(y) = (f \circ g)(y) = f(g(y))$$
$$\forall y \in T.\ f : g(y) \to y$$

So, f is onto

Let $x_1, x_2 \in S$ and suppose

$$f(x_1) = f(x_2)$$
$$\implies g(f(x_1)) = g(f(x_2))$$
$$\implies (g \circ f)(x_1) = (g \circ f)(x_2)$$
$$\implies x_1 = x_2$$

Since $(g \circ f)$ is the identity function

$$QED$$

$\impliedby$ **proof:**

Suppose that f is a bijection. We want to show that g has an inverse.

Define $g : T \to S$ as follows

$$\forall y \in T.\ \exists x \in S.\ f(x) = y$$

Define $g(y) = x$.

We propose that g is the inverse of f.

We must check that

$$f \circ g = 1_T$$
$$g \circ f = 1_S$$