Lecture 10.5: Counting

Summary:
We study fundamental counting principles, permutations, and subsets. We establish that n elements have n! permutations and 2^n subsets, and introduce binomial coefficients and the binomial theorem.

Topics Covered: addition principle, binomial coefficient, binomial theorem, counting subset, multiplication principle, permutation

Two basic counting principles

- Multiplication principle
  - If there are a ways of performing task A and b ways of performing task B, then, there are ab ways of performing A then B.
- Addition principle
  - If there are a ways of performing task A and b ways of performing task B, then, there are $a+b$ ways of performing A or B

Permutations

$$S_n = \{\text{permutations of } [n]\}$$

$$|S_n| = n!$$

Subsets

$$2^S = \{\text{Subsets of } S\}$$

Note that

$$|2^S| = 2^{|S|}$$

Note that there is a bijection

$$2^S \to \{0,1\}^{|S|}$$

Binomial coefficient

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

$\binom{n}{k}$ is the number of ways we can pick a k-subset from a set of size n.

Generating functions

The multivariate generating function for subsets of $[n]$ is

$$\sum_{A\subseteq[n]} \prod_{i\in A} x_i = (1+x_1)(1+x_2)\dots.(1+x_n)$$

Plug in $x_1 = x_2 = \dots. = x_n = x$ to get

$$\sum_{A\subseteq[n]} x^{|A|} = (1+x)^n$$

Binomial theorem:

$$\sum_{i=0}^{n} \binom{n}{k} x^k = (1+x)^n$$