Guaranteeing Accuracy of a Square Root Approximation

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Suppose we have an algorithm to approximate the square root of a number S to a specified precision ϵ. Let $\alpha = \sqrt{S}$ and let x be our approximation of α. How do we check how close x is to α when we do not know the true value of α? The first idea may be to square the approximation and see how closely it aligns to the original number. However, $|S - x^2| = |\alpha^2 - x^2|$ is the error between the squares, not the error of the estimate itself, which is $|\alpha - x|$. Perhaps there is some relationship between the two?

Theorem 1. *Let $\alpha^2 = S \in \mathbb{R}^+$, and $x \approx \alpha$. If $|S - x^2| < \epsilon^2$ then $|\alpha - x| < \epsilon$*

Proof. Suppose $|S - x^2| < \epsilon^2$

$$\begin{aligned} |S - x^2| < \epsilon^2 &\iff -\epsilon^2 < S - x^2 < \epsilon^2 \\ &\iff -\epsilon^2 < S - x^2 \text{ and } S - x^2 < \epsilon^2 \end{aligned}$$

Taking the left inequality,

$$\begin{aligned} -\epsilon^2 < S - x^2 &\iff x^2 < \alpha^2 + \epsilon^2 \\ &\iff x < \sqrt{\alpha^2 + \epsilon^2} \\ &\implies x < \sqrt{\alpha^2 + 2\alpha\epsilon + \epsilon^2} \text{ (monotonicity of square root)} \\ &\iff x < \sqrt{(\alpha + \epsilon)^2} \\ &\implies x < \alpha + \epsilon \\ &\iff -\epsilon < \alpha - x \end{aligned} \tag{1}$$

A similar manipulation of the right inequality yields

$$\begin{aligned} S - x^2 < \epsilon &\iff \alpha^2 < x^2 + \epsilon^2 \\ &\iff \alpha < \sqrt{x^2 + \epsilon^2} \\ &\implies \alpha < \sqrt{x^2 + 2x\epsilon + \epsilon^2} \\ &\iff \alpha < \sqrt{(x + \epsilon)^2} \\ &\implies \alpha < x + \epsilon \\ &\iff \alpha - x < \epsilon \end{aligned} \tag{2}$$

Combining (1) $\epsilon < \alpha - x$ and (2) $\alpha - x < \epsilon$ yields $|\alpha - x| < \epsilon$

$\therefore |S - x^2| < \epsilon^2 \implies |\alpha - x| < \epsilon$ □

Thus, if we make sure $|S - x^2| < \epsilon^2$, we can conclude $|\alpha - x| < \epsilon$, which is our requirement. There may be tighter bounds for $|\alpha^2 - x^2|$, but using ϵ^2 was an easy, clean place to start.