MIDTERM EXAM SOLUTION

MAT 3007

Nov 6, 2019

INSTRUCTIONS

a) Write ALL your answers in this exam paper.

b) One piece of note is allowed. No computer or calculator is allowed.

c) The exam time is 10:30am - 12:00pm.

d) There are 6 questions and 100 points in total. Except the true or false questions, write down the reasonings for your answers.

In taking this examination, I acknowledge and accept the instructions.

NAME (signed) ________________________________

NAME (printed) ________________________________

For grading use. Don't write in this part

	1 (18pts)	2 (20pts)	3 (16pts)	4 (22pts)	5 (14pts)	6 (10pts)	Total
Points							

Problem 1: True/False (18pts)

State whether each of the following statements is True or False. For each part, only your answer, which should be one of True or False, will be graded. Explanations will not be read.

(a) For a linear optimization problem in the standard form, suppose it has a finite optimal solution, then if one increases an objective coefficient (e.g., from 1 to 2), the optimal value will not decrease. **True**

(b) In a linear optimization problem, two different basis must correspond to two different vertices. **False**

(c) For linear optimization, if a primal problem is feasible, then the dual problem must be feasible too. **False**

(d) Let $\boldsymbol{x}$ be an optimal solution to a linear optimization problem. Now we add another constraint. If $\boldsymbol{x}$ satisfies the additional constraint, then it must still be optimal. **True**

(e) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to a linear optimization problem and its dual, respectively. Then if they satisfy all the complementarity conditions, they must both be optimal solutions. **True**

(f) In the simplex method for linear optimization problem, degeneracy happens when there are multiple optimal solutions. **False**

Problem 2: Solve Linear Optimization Problem (20pts)

Use two-phase simplex method to solve the following linear optimization problem:

$$\begin{array}{rrrrcl} \text{maximize} & x_1 & +2x_2 & +x_3 & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & = & 7 \\ & 2x_1 & -2x_2 & +x_3 & \geq & 10 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Answer. We first convert it to the standard form (2pts):

$$\begin{array}{rrrrrcl} \text{minimize} & -x_1 & -2x_2 & -x_3 & & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & & = & 7 \\ & 2x_1 & -2x_2 & +x_3 & -x_4 & = & 10 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{array}$$

Then we construct the phase-one problem (2pts):

$$\begin{array}{rrrrrrrcl} \text{minimize} & & & & & x_5 & +x_6 & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & & +x_5 & & = & 7 \\ & 2x_1 & -2x_2 & +x_3 & -x_4 & & +x_6 & = & 10 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq & 0. \end{array}$$

We write down the initial tableau for the phase-one problem (the top row worths 2 points)

B	-3	1	-2	1	0	0	-17
5	1	1	1	0	1	0	7
6	2	-2	1	-1	0	1	10

Then in the first iteration, we choose the number 2 to be the pivot element, the next table becomes (1pt):

B	0	-2	-1/2	-1/2	0	3/2	-2
5	0	2	1/2	1/2	1	-1/2	2
1	1	-1	1/2	-1/2	0	1/2	5

We choose the second column to be the incoming column, and the number 2 to be the pivot element, the next table becomes (1pt)

B	0	0	0	0	1	1	0
2	0	1	1/4	1/4	1/2	-1/4	1
1	1	0	3/4	-1/4	1/2	1/4	6

The optimal solution to the phase one problem is $x_1 = 6$, $x_2 = 1$, $x_3 = x_4 = x_5 = x_6 = 0$ and the optimal value is 0 (4pts).

Now we construct the initial tableau for the original problem (phase two problem). We need to calculate the reduced cost for x_3 and x_4. We have (note that the order of B is (2,1))

$$\bar{c}_3 = c_3 - [c_2, c_1]^T A_B^{-1} A_3 = -1 - [-2, -1]^T [1/4, 3/4] = 1/4 \quad \text{(2pts)}$$
$$\bar{c}_4 = c_4 - [c_2, c_1]^T A_B^{-1} A_4 = 0 - [-2, -1]^T [1/4, -1/4] = 1/4 \quad \text{(2pts)}$$

Therefore, the initial tableau for the phase two problem is

B	0	0	1/4	1/4	8
2	0	1	1/4	1/4	1
1	1	0	3/4	-1/4	6

The reduced costs are all non-negative, thus the solution is already optimal. Thus the optimal solution to the original problem is $x_1 = 6$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$. And the optimal value is 8 (4pts).

Problem 3: Duality and Complementarity Conditions (16pts)

Continue to consider the linear program in the previous question:

$$\begin{array}{rrrrcl} \text{maximize} & x_1 & +2x_2 & +x_3 & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & = & 7 \\ & 2x_1 & -2x_2 & +x_3 & \geq & 10 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

(a) Write down its dual problem. (5pts)

Answer. The dual problem is (each error costs 1 point)

$$\begin{array}{rrrcl} \text{minimize} & 7y_1 & +10y_2 & & \\ \text{s.t.} & y_1 & +2y_2 & \geq & 1 \\ & y_1 & -2y_2 & \geq & 2 \\ & y_1 & +y_2 & \geq & 1 \\ & y_1 \text{ free}, & y_2 \leq 0. & & \end{array}$$

(b) Write down the complementarity conditions. (6pts)

Answer. The complementarity conditions include (each wrong equation costs 2 points)

$$\begin{aligned} x_1 \cdot (y_1 + 2y_2 - 1) &= 0 \\ x_2 \cdot (y_1 - 2y_2 - 2) &= 0 \\ x_3 \cdot (y_1 + y_2 - 1) &= 0 \\ y_2 \cdot (2x_1 - 2x_2 + x_3 - 10) &= 0 \end{aligned}$$

(c) Use the complementarity conditions and the answer to the previous question to find out the dual optimal solution. (5pts)

Answer. From the previous question, we know that the optimal solution is $x_1 = 6$, $x_2 = 1$, $x_3 = 0$. Thus by the complementarity condition, we must have at optimal solution $y_1 + 2y_2 - 1 = 0$ and $y_1 - 2y_2 - 2 = 0$. Solve these equations, we have $y_1 = 3/2$, $y_2 = -1/4$. (5pts)

If one solves problem 2 incorrectly but uses the right idea here, then one can get as much as 3 points.

Problem 4: Sensitivity Analysis (22pts)

Consider the following linear optimization problem:

$$\begin{array}{llllll} \text{minimize} & 2x_1 & +x_2 & +2x_3 & -3x_4 & \\ \text{subject to} & 8x_1 & -4x_2 & -x_3 & +3x_4 & \le 10 \\ & 2x_1 & +3x_2 & +x_3 & -x_4 & \le 7 \\ & & -2x_2 & -x_3 & +4x_4 & \le 12 \\ & x_1, & x_2, & x_3, & x_4 & \ge 0 \end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B	12/5	0	7/5	0	0	1/5	4/5	11
5	10	0	1/2	0	1	1	-1/2	11
2	4/5	1	3/10	0	0	2/5	1/10	4
4	2/5	0	-1/10	1	0	1/5	3/10	5

(a) What is the optimal solution and the optimal value to the original problem? (4pts)

Answer. The optimal solution is $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, $x_4 = 5$ (2pts), and the optimal value is -11 (2pts).

(b) What is the optimal solution to the dual problem? (4pts)

Answer. The dual optimal solution is $[0, -1/5, -4/5]$ (4pts).

(c) In what range can we change the second right hand side number 7 so that the current optimal basis is still the optimal basis? (7pts)

Answer. Consider we change the current $\boldsymbol{b}$ to $\boldsymbol{b} + \Delta\boldsymbol{b}$ with $\Delta\boldsymbol{b} = [0; \lambda; 0]$. We need to ensure that $A_B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) \ge 0$. (2pts)

Thus we have

$$A_B^{-1}(\boldsymbol{b}+\Delta\boldsymbol{b}) = \begin{bmatrix} 11 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 2/5 & 1/10 \\ 0 & 1/5 & 3/10 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2/5 \\ 1/5 \end{bmatrix} \ge 0 \quad \text{(2pts)},$$

which implies

$$\lambda \ge -11, \lambda \ge -10, \lambda \ge -25.$$

Thus overall $\lambda \ge -10$ (1pt), i.e., the range is $[-3, \infty)$ (2pts).

(d) In what range can we change the second objective coefficient 1 so that the current optimal basis is still the optimal basis? (7pts)

Answer. Consider we change $\boldsymbol{c}$ to $\boldsymbol{c} + \Delta\boldsymbol{c}$ where $\Delta\boldsymbol{c} = [0, \lambda, 0, 0]$ and study the range of λ such that the reduced costs are still nonnegative. We look at the reduced costs under the new objective coefficients (2pts). We have (the basis is $\{5, 2, 4\}$)

$$
\begin{aligned}
\bar{\boldsymbol{c}}_N^{\text{new}} &= \bar{\boldsymbol{c}}_N - \Delta\boldsymbol{c}_B^T A_B^{-1} A_N \\
&= [12/5, 7/5, 1/5, 4/5] - [0, \lambda, 0]^T \cdot \begin{bmatrix} 10 & 1/2 & 1 & -1/2 \\ 4/5 & 3/10 & 2/5 & 1/10 \\ 2/5 & -1/10 & 1/5 & 3/10 \end{bmatrix} \\
&= [12/5, 7/5, 1/5, 4/5] - \lambda \cdot [4/5, 3/10, 2/5, 1/10] \geq 0 \quad \text{(3pts)}
\end{aligned}
$$

Solving from the above, we have $\lambda \leq 3$, $\lambda \leq 14/3$, $\lambda \leq 1/2$, $\lambda \leq 8$. Thus overall $\lambda \leq 1/2$ and the range for the coefficient is $(-\infty, 3/2]$. (2pts)

Problem 5: Cutting Stock Problem (14pts)

Consider a factory that produces certain lengths of tubes. The raw materials are tubes of W meters and there are I types of products each with length w_i $(i = 1, ..., I)$. The demand of product type i is d_i. The objective is to use the minimal number of raw materials to meet the demand.

To solve this question, we consider a set of P cutting patterns (we consider the set of cutting patterns as given). A cutting pattern p can cut a raw material into a_{pi} number of product i. For example, if $W = 20$, $w_1 = 5$, $w_2 = 7$, then there are 3 patterns which are $\{a_{11} = 4, a_{12} = 0\}$, $\{a_{21} = 2, a_{22} = 1\}$ and $\{a_{31} = 1, a_{32} = 2\}$ (other patterns are not efficient in using materials). In the following, please write an optimization formulation to determine the best way of producing. That is, to write an optimization problem to determine how many each patterns to use in order to use the minimal number of materials to meet the demand.

Answer. Let $x_p \in Z$ be the decision variable indicating the number of pattern p used in the production. (3pts)

The objective is then to minimize $x_1 + \cdots + x_P$ (3pts).

The constraint is that for each type of product, we satisfy the demand, that is $\sum_{p=1}^{P} x_p a_{pi} \geq d_i$ (4pts).

Therefore, the optimization problem is (4pts):

$$\begin{array}{ll} \text{minimize} & \sum_{p=1}^{P} x_p \\ \text{subject to} & \sum_{p=1}^{P} x_p a_{pi} \geq d_i, \quad \forall i = 1, ..., I \\ & x_p \geq 0, x_p \in \mathcal{Z} \end{array}$$

Remark. If one didn't write the integer constraint, then 2 points will be deducted.

Problem 6: Robust Linear Program (10pts)

Consider the following so-called robust linear program:

$$\begin{array}{rc} \text{maximize}_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}^T\boldsymbol{x} \le b + \boldsymbol{u}^T\boldsymbol{x}; \quad \forall \boldsymbol{u} \in \{\boldsymbol{v} : A\boldsymbol{v} \le \boldsymbol{w}\} \\ & \boldsymbol{x} \ge 0 \end{array}$$

where $\boldsymbol{c} \in R^n$, $\boldsymbol{a} \in R^n$, $\boldsymbol{w} \in R^k$ are given column vectors; b is a given scalar; and $A \in R^{k\times n}$ is a given matrix. In this problem, $\boldsymbol{u} \in R^n$ is an uncertain state vector and varies in the hypercube $\{\boldsymbol{v} : A\boldsymbol{v} \le \boldsymbol{w}\}$. The problem is to find an $\boldsymbol{x} \ge 0$ such that $\boldsymbol{a}^T\boldsymbol{x} \le b + \boldsymbol{u}^T\boldsymbol{x}$ for all possible $\boldsymbol{u}$ in the hypercube and the objective function $\boldsymbol{c}^T\boldsymbol{x}$ is maximized. Reformulate the problem as a single linear optimization problem with a finite number of constraints (currently it has infinite many constraints indexed by $\boldsymbol{u}$) and variables and without the state vector $\boldsymbol{u}$ appearing in the problem. (Hint: Apply duality theory on $\boldsymbol{u}^T\boldsymbol{x}$.)

Answer. For the constraint, basically it means that $\boldsymbol{a}^T\boldsymbol{x} \le b + \min_{\boldsymbol{v}:A\boldsymbol{v}\le\boldsymbol{w}} \boldsymbol{v}^T\boldsymbol{x}$. (2pts)

Now we apply the dual to the inside optimization problem

$$\begin{array}{rc} \text{minimize} & \boldsymbol{x}^T\boldsymbol{v} \\ \text{subject to} & A\boldsymbol{v} \le \boldsymbol{w} \end{array}$$

Let the dual variable be η. The dual problem is (3pts)

$$\begin{array}{rc} \text{maximize} & \boldsymbol{w}^T\eta \\ \text{subject to} & A^T\eta = \boldsymbol{x} \\ & \eta \le 0 \end{array}$$

Then the original problem can be written as (5pts)

$$\begin{array}{rc} \text{maximize}_{\boldsymbol{x},\eta} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}^T\boldsymbol{x} \le b + \boldsymbol{w}^T\eta \\ & A^T\eta = \boldsymbol{x} \\ & \boldsymbol{x} \ge 0, \eta \le 0 \end{array}$$

Solving the above problem will give the optimal solution to the original problem.