MAT1002 Midterm Examination

Saturday, March 26, 2022

Time: 1:00 - 3:00 PM

Notes and Instructions

1. This exam is open-book. Please refer to the detailed regulation posted on Blackboard.

*2. The total score of this examination is **100**.*

*3. There are **ten** questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

5. State your answers in exact form, e.g., write $\sqrt{2}$ instead of 1.414.

*6. Show your intermediate steps **except Questions 1 and 2** — answers without intermediate steps will receive no marks.*

MAT1002 Midterm Examination Questions

1. [**10**] True (T) or False (F)? No explanation is required.

   (i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both divergent series, then $\sum_{n=1}^{\infty}(a_n + b_n)$ also diverges.

   (ii) For any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{p}$ in the 3D space, the following three vectors are coplanar (lying on the same plane): $\mathbf{a} \times \mathbf{p}$, $\mathbf{b} \times \mathbf{p}$, and $\mathbf{c} \times \mathbf{p}$.

   (iii) Fix $x > 0$, and define the sequence $\{a_n\}$ by $a_1 = \sqrt[3]{x}$ and $a_k = \sqrt[3]{x a_{k-1}}$ all integers $k > 1$. Then $\{a_n\}$ must converge.

   (iv) Suppose that C is a smooth curve in the xy-plane given by a parametrization $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$. Then $\mathbf{r}'(t)$ must be continuous and never the zero vector on the interval $[a, b]$.

   (v) Let $f(x)$ be infinitely differentiable on the real line, and let $\sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series of f. Then $\sum_{n=0}^{\infty} c_n x^n = f(x)$ must hold for at least one real number x.

2. [**12**] Short questions. No explanation or intermediate steps are required.

   (i) Let L be a real number. Which of the following statements is equivalent to saying $\lim_{n\to\infty} a_n = L$? Choose all that apply.

   A) For any $\epsilon > 0$, there exists a positive integer N such that for all integers $n > N$, we have $|a_n - L| \le \epsilon$.
   B) For any ϵ with $0 < \epsilon < 1$, there is a positive integer N such that for all integers $n > N$, we have $|a_n - L| < \epsilon$.
   C) For any positive integer m, there exists a positive integer N such that for all integers $n > N$, we have $|a_n - L| < \frac{1}{m}$.
   D) For any $\epsilon > 0$, there are infinitely many positive integers n such that $|a_n - L| < \epsilon$.

   (ii) Given three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that are perpendicular to each other, assume that their lengths are 1, 2, and 2, respectively. What is the angle between $\mathbf{a} + \mathbf{b} + \mathbf{c}$ and $\mathbf{c}$?

   (iii) Consider the curve C given by the parametric equations

   $$x = 3t^2,\ y = 2t^3,\ t \ge 0.$$

   Express the curve as a polar curve $r = f(\theta)$.

(iv) Consider the curve C in (iii) above. Suppose the portion of the curve C for $t \in [0,1]$ is revolved about the vertical axis $x = 3$. Find the area of the surface generated by this revolution.

3. [**4+2+4**] Consider a particle travelling along the curve C with position vector $\mathbf{r}(t)$ given by

$$\mathbf{r}(t) = (\tan^{-1} t)\mathbf{i} + (2e^{2t})\mathbf{j} + (8te^{t})\mathbf{k},$$

where t denotes time and $\tan^{-1}$ denotes the arctan (inverse tangent) function.

(i) Find the time t_0 when its instantaneous direction of motion is orthogonal to the plane M given by $x + 4y + 8z = 16$.

(ii) Find the point $P(a, b, c)$ on the curve C where the particle is at time t_0.

(iii) Suppose that at time t_0 above, the particle leaves the curve C and continues travelling in a straight line in the direction of the velocity vector $\mathbf{v}(t_0)$. Show that the particle will hit the plane M at some time $t \geq t_0$, and find the time.

4. [**6+4**] Consider a travelling particle in the space, with position vector $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ at time t. Suppose that its initial position is $(0, -2, 0)$, and that

$$\mathbf{a}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}, \quad \mathbf{v}(0) = -2\mathbf{i} + v_0\mathbf{k},$$

where v_0 is a positive constant.

(i) Find the time T_0 that it takes for the particle to travel a distance of L_0 along its trajectory.

(ii) What is the position of the particle at time T_0?

5. [**6+2**] Let a and b satisfy $0 < a < 1$ and $0 < b < 1$.

(i) Give a geometric justification of the following inequality:

$$\sqrt{a^2+b^2}+\sqrt{(1-a)^2+b^2}+\sqrt{a^2+(1-b)^2}+\sqrt{(1-a)^2+(1-b)^2} \geq \sqrt{8}.$$

(ii) Find the values of a and b for which the above holds with equality. (You do not need to explain it.)

6. [**3+3+4**] Suppose the straight line l is the intersection of the following two planes where k is a constant:

$$\begin{cases} x - 1 = y \\ x + z = k \end{cases}.$$

Assume that the line passes through the point $P(1, 0, 1)$, and let M denote the plane given by $x - y + 2z = 0$.

(i) Write down the parametric equations for the line l.

(ii) Find the intersection of this line l with the plane M.

(iii) Find the line obtained by projecting l onto the plane M.

7. [**5+5+5**] For each of the following series, determine whether it converges or diverges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{a}{b + c^n}$, where a, b, and c are all positive constants.

(ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{48e^n + n^\pi}{n! + (\ln n)^2}$

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^{1+\alpha}}$, where $\alpha > 0$.

8. [**4+2+6**] Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)3^{n+1}}$.

(i) Determine all values of x for which the series converges.

(ii) For each value in part (i), state whether the convergence is absolute or conditional.

(iii) Find the value of the series (the value of the infinite sum) whenever it converges.

9. [**6**] Find the following limit:

$$\lim_{x \to 0} \frac{e^{x^2} + (x/2) - \sqrt{1 + x}}{2x \cos x - \tan^{-1} x - \ln(1 + x)}.$$

Here $\tan^{-1}$ denotes the arctan function.

10. [**4+3**] Consider the function $f(x) = \sqrt[5]{1+x}$.

(i) Write down the first four terms of the Maclaurin series of $f(x)$.

(ii) Consider approximating the value of $\sqrt[5]{1.8}$ using the Maclaurin series of $f(x)$. How many terms would you need to take at least to ensure that the error is less than 0.01? Justify.