

# MAT2040 Linear Algebra

Midterm Exam  
SSE, CUHK(SZ)

31 Oct 2020

Seat No.: \_\_\_\_\_ Student ID: \_\_\_\_\_

- Answer each question in the space after the question.
- For questions in Part I, you can give answers without explanations.
- For questions in Part II, be sure to give **full explanations** for your answers. The solution only is not supposed to be a complete answer.

Question	Points	Score
1	5	
2	6	
3	6	
4	10	
5	7	
6	12	

Question	Points	Score
7	10	
8	10	
9	10	
10	12	
11	12	
Total:	100	

This page has no questions, but a table of notations.
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### Table of Notations

$\mathbb{R}$	set of real numbers
	Without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	set of all (column) vectors of $n$ entries from $\mathbb{R}$ $n$ -dimensional Euclidean vector spaces
$0, 1, \dots$	scalar values
$\mathbf{0}$	zero vector in $\mathbb{R}^n$ , where $n$ is implied in the context
$I, I_n$	the $n \times n$ identity matrix
$A^T$	transpose of matrix $A$
$\det(A), \det A$	determinant of matrix $A$
$\text{Col}(A)$	column space of matrix $A$
$\text{Null}(A)$	null space of matrix $A$
$\text{Span}(\mathcal{A})$	linear span of the set of vectors $\mathcal{A}$
$\text{rank}(A)$	rank of a matrix $A$
$\dim(\mathcal{V})$	dimension of a vector space $\mathcal{V}$
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$

# I Multiple Choices and True/False

Answer the following questions following the instructions.

**Question 1** ..... 5 points

Consider two symmetric matrices  $A$  and  $B$  of the same size. Which of the following matrices are symmetric?

- (a)  $A^2 - B^2$
- (b)  $(A + B)(A - B)$
- (c)  $ABA$
- (d)  $ABAB$

**Question 2** ..... 6 points

Consider the following matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B = \begin{bmatrix} a_{14} & a_{13} & a_{12} & a_{11} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{34} & a_{33} & a_{32} & a_{31} \\ a_{44} & a_{43} & a_{42} & a_{41} \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where  $A$  is invertible, and then  $B^{-1} = (\text{_____})$

- (a)  $A^{-1}P_1P_2$
- (b)  $P_1A^{-1}P_2$
- (c)  $P_1P_2A^{-1}$
- (d)  $P_2A^{-1}P_1$

**Question 3** ..... 6 points

Consider a matrix  $A \in \mathbb{R}^{n \times n}$  and a vector  $\mathbf{a} \in \mathbb{R}^{n \times 1}$ , with

$$\text{rank} \begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \mathbf{0} \end{bmatrix} = \text{rank}(A).$$

Then, (\_\_\_\_\_) is always true.

- (a)  $A\mathbf{x} = \mathbf{a}$  has infinite number of solutions.
- (b)  $A\mathbf{x} = \mathbf{a}$  has a unique solution.
- (c)  $\begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \mathbf{0}$  only has the trivial solution.

(d)  $\begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \mathbf{0}$  has a non-trivial solution.

**Question 4** ..... 10 points

Determine whether each of the following statement is equivalent to that an  $n \times n$  matrix  $A$  is nonsingular (invertible) by marking True or False. For the statement marked False, please correct the statement so that it is equivalent to  $A$  is nonsingular.

- (1) \_\_\_\_\_  $A\mathbf{x} = \mathbf{0}$  is consistent.
- (2) \_\_\_\_\_  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any  $\mathbf{b} \in \mathbb{R}^n$ .
- (3) \_\_\_\_\_  $A$  is row equivalent to  $I$ .
- (4) \_\_\_\_\_ There exists a matrix  $E$  such that  $EA = I$ .
- (5) \_\_\_\_\_  $A$  is a product of a sequence of finite elementary matrices.
- (6) \_\_\_\_\_ The columns of  $A$  span  $\mathbb{R}^n$ .
- (7) \_\_\_\_\_ The rows of  $A$  form a basis of  $\mathbb{R}^n$ .
- (8) \_\_\_\_\_  $A$  is full rank, i.e.,  $\text{rank}(A) = n$ .
- (9) \_\_\_\_\_  $\dim(\text{Null}(A)) = 1$ .
- (10) \_\_\_\_\_  $\det(A) > 0$ .

## II Calculations and Proofs

(Please give the details of your proofs and solutions. Your answer will be only judged based on your writing on the paper.)

**Question 5** ..... 7 points

Let  $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ . Calculate  $A^2$ ,  $A^3$  and  $A^n$ . Suppose you know that

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

**Question 6** ..... 12 points

Let  $A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$ . Consider the following sequence of row operations:

$$A \xrightarrow{\text{op1}} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{\text{op2}} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{\text{op3}} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

- (a) (5 points) Specify the operation performed in each step of the row operations, and give the corresponding elementary matrices.
- (b) (3 points) Give an LU decomposition of  $A$ .
- (c) (4 points) Suppose that  $a = 1$ ,  $b = 2$ ,  $c = 3$  and  $d = 4$ . Solve the system of

linear equations:  $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ .

**Question 7** ..... 10 points

Denote by  $P_4$  the set of polynomial of degree less than 4. Let  $\mathcal{A}$  be a subset of  $P_4$ :

$$\mathcal{A} = \{1 + 2x^3, 2 + x - 3x^2, -x + 2x^2 + x^3, 1 + x - 3x^2 - 2x^3\},$$

where  $x$  is the indeterminate (variable) of the polynomials.

- (a) (5 points) Find the dimension of the subspace spanned by  $\mathcal{A}$ , together with a basis.
- (b) (5 points) Verify whether the polynomial  $p_1 = 1 + 2x + 2x^2$  is in  $\text{Span}(\mathcal{A})$  or not. If it is, give the coordinates with respect to the basis you find in (a).

**Question 8** ..... 10 points

Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

- (a) (5 points) Find the change-of-basis transition matrix from  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  to  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ .
- (b) (5 points) For  $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$ , determine the coordinates of  $\mathbf{x}$  with respect to  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ .

**Question 9** ..... 10 points

For the following matrix, find a basis for the row space, a basis for the column space and a basis for the null space.

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

**Question 10** ..... 12 points

Consider the following linear system

$$\begin{cases} (a_1 + b)x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = 0 \\ a_1x_1 + (a_2 + b)x_2 + a_3x_3 + \cdots + a_nx_n = 0 \\ a_1x_1 + a_2x_2 + (a_3 + b)x_3 + \cdots + a_nx_n = 0 \\ \cdots \\ a_1x_1 + a_2x_2 + a_3x_3 + \cdots + (a_n + b)x_n = 0 \end{cases},$$

where  $\sum_{i=1}^n a_i \neq 0$ , and  $a_i$  and  $b$  are some scalars. Questions:

- (a) (6 points) When this linear system only has the trivial solution? (For which  $a_i$  and  $b$ ?)
- (b) (6 points) When this linear system has non-trivial solutions? Then, obtain the solutions.

**Question 11** ..... 12 points

- (a) (4 points) Show that for  $\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{R}^m$  (where  $m \geq n+1$ ), if  $\mathbf{y}_1, \dots, \mathbf{y}_n$  are linearly independent and  $\mathbf{x} \notin \text{Span}\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ , then  $\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n$  are linearly independent.
- (b) (8 points) Suppose  $n \geq 3$  and  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}_3, \dots, \mathbf{d}_n \in \mathbb{R}^n$ . Show that if  $\mathbf{a} \notin \text{Span}\{\mathbf{b}, \mathbf{c}, \mathbf{d}_3, \dots, \mathbf{d}_n\}$ ,  $\mathbf{b} \notin \text{Span}\{\mathbf{a}, \mathbf{c}, \mathbf{d}_3, \dots, \mathbf{d}_n\}$ , and  $\mathbf{c} \notin \text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{d}_3, \dots, \mathbf{d}_n\}$ , then  $\mathbf{d}_3, \dots, \mathbf{d}_n$  are linearly dependent.