香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

PHY1001 Mechanics

2023-2024 Term 2

Midterm Examination

March 30th, 2024;   Time Allowed: 3 Hours

NAME (print) ______________________________

CUHKSZ ID ______________________________

Room No. ______________________________

Seat No. ______________________________

- **Show all your work**. Correct answers with little supporting work will not be given credit.
- Closed Book Exam: One piece of double-sided A4 reference paper, a scientific calculator, and a paper-based dictionary are allowed.
- Unless approved by the instructors, students who arrive more than 30 minutes late will NOT be admitted.
- The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	TOTAL
10	10	10	10	10	10	10	10	10	10	10	10	120

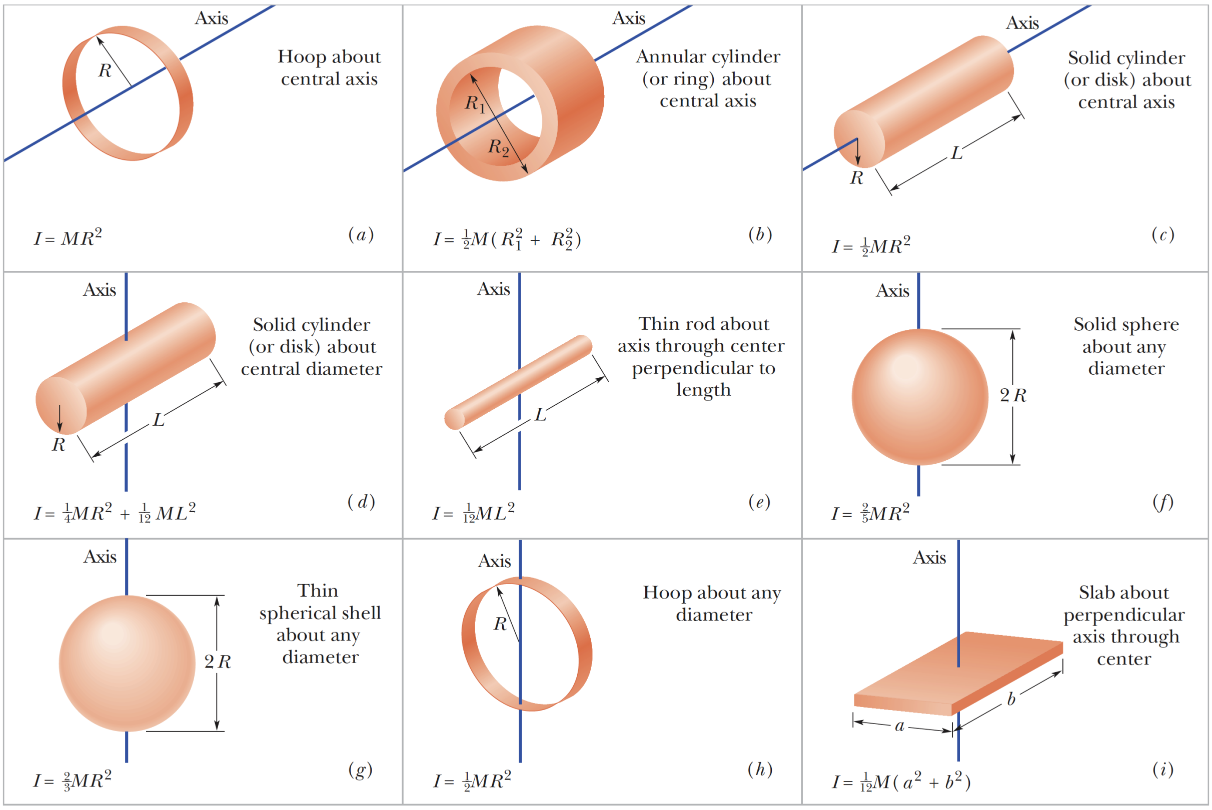
$g = 9.8\,\mathrm{m/s^2}$ for all questions that are involved with gravitational acceleration g.

Summary of Basic Calculus:

$$
\begin{aligned}
& e^{ix} = \cos x + i \sin x, \\
& \frac{d}{dx} x^n = n x^{n-1}, \\
& \frac{d}{dx} e^{ax} = a e^{ax}, \\
& \frac{d}{dx} \ln ax = \frac{1}{x}, \\
& \frac{d}{dx}(uv) = v \frac{d}{dx} u + u \frac{d}{dx} v, \\
& \int x^n dx = \frac{x^{n+1}}{n+1} + C, \ \ (n \neq -1), \\
& \int \frac{dx}{x} = \ln x + C, \\
& \int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad C \text{ is a constant.}
\end{aligned}
$$

Taylor series:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots (\text{all}\, x) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots (\text{all}\, x) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots (\text{all}\, x).
\end{aligned}
$$



1. A rock is thrown vertically upward from the top of a tree at time $t = 0$. At $t_1 = 1.0\text{s}$ it reaches its maximum height, and 1.5s later at $t_2 = 2.5\text{s}$ it reaches the ground. **(10 pts)**

(a) What is initial velocity of the rock? **(5 pts)**

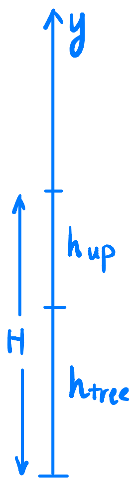
$V_h = V_0 + at = V_0 - gt_1$ 2'

$0 = V_0 - 9.8\,m/s^2 \times 1.0\,s$

$V_0 = +9.8\,m/s$ 2'

The magnitude of the initial velocity is 9.8 m/s, with the direction pointing vertically up. 1'

(b) What is the height of the tree? **(5 pts)**



$h_{up} = V_0 t_1 - \frac{1}{2} g t_1^2 = 9.8\,m/s \times 1.0\,s - \frac{1}{2} \times 9.8\,m/s^2 \times 1.0\,s^2 = 4.9\,m$ 2'

$H = h_{up} + h_{tree} = \frac{1}{2} g (t_2 - t_1)^2 = \frac{1}{2} \times 9.8\,m/s^2 \times (1.5\,s)^2 = 11.03\,m$ 2'

$h_{tree} = H - h_{up} = 11.03\,m - 4.9\,m = 6.13\,m \approx 6.1\,m$ 1'

2. The initial velocity of a car is 0 at $t = 0$, and the acceleration is $a = \beta t$ with $\beta = 1.0\mathrm{m/s^3}$. **(10 pts)**

(a) What's the velocity v of the car at $t = 2.0\mathrm{s}$? **(5 pts)**

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt \Rightarrow v = v_0 + \int_0^t a\,dt \quad 2'$$

$$v(2.0s) = 0 + \int_0^{2.0} \beta t\,dt = \frac{1}{2}\beta t^2 = \frac{1}{2} \times 1.0\,m/s^3 \times (2.0)^2 s^2 \quad 2'$$

$$v(2.0s) = 2.0\,m/s \quad 1'$$

(b) What's the displacement Δx of the car at $t = 3.0\mathrm{s}$? **(5 pts)**

$$v = \frac{dx}{dt} \Rightarrow \underbrace{\Delta x = \int_0^{3.0} v\,dt}_{} \overset{2'}{=} \int_0^{3.0} \frac{1}{2}\beta t^2\,dt$$

$$\Delta x(3.0s) = \frac{1}{6}\beta t^3 \Big|_0^{3.0s} = \frac{1}{6} \times 1.0\,m/s^3 \times (27-0)\,s^3 \quad 2'$$

$$\Delta x(3.0s) = 4.5\,m \quad 1'$$

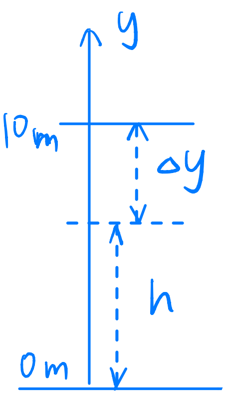
3. A platform is 10 m above water. A diver jumps from the edge of the platform with a horizontal speed of 2.0 m/s. **(10 pts)**

(a) When $t = 0.80$ s, what is the horizontal distance between the diver and the edge of the platform? **(3 pts)**

$$\underline{L = V_0 t} = 2.0\,m/s \times 0.8\,s = \underline{1.6\,m}$$

2' 1'

(b) When $t = 0.80$ s, what is the vertical distance between the diver and water? **(3 pts)**



$$\Delta y = \frac{1}{2}at^2 = -\frac{1}{2}gt^2 = -\frac{1}{2}\times 9.8\,m/s^2 \times (0.80\,s)^2 = -3.1\,m \quad 2'$$

$$h = H + \Delta y = 10\,m - 3.1\,m = 6.9\,m \quad 1'$$

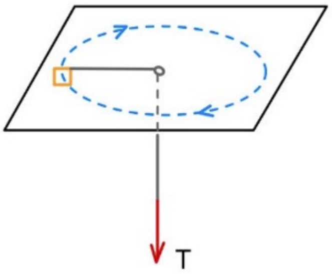
(c) When the diver strikes water, what is the horizontal distance between the diver and the edge of the platform? **(4 pts)**

$$\Delta y' = -H = -\frac{1}{2}gt'^2 \Rightarrow H = \frac{1}{2}gt'^2 \Rightarrow t' = \sqrt{\frac{2H}{g}}$$

$$t' = \sqrt{\frac{2\times 10}{9.8}}\,s = 1.43\,s \quad 2'$$

$$L' = V_0 t' = 2.0\,m/s \times 1.43\,s = 2.86\,m \ (\text{or } 2.9\,m) \quad 2'$$

4. A small block with a mass of 0.190 kg is attached to a cord passing through a hole in a frictionless, horizontal surface. The block is originally revolving at a distance of 0.40 m from the hole in uniform circular motion with a speed of 0.90 m/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.10 m. At this new distance, block is in uniform circular motion with a speed of 3.60 m/s. **(10 pts)**



(a) What is the tension in the cord in the original situation, when the block has speed of 0.90 m/s? **(3 pts)**

$$T_1 = \underline{\frac{m v_1^2}{r_1}} = 0.190\,kg \times \frac{(0.90\,m/s)^2}{0.40\,m} = \underline{0.38\,N}$$

2' 1'

(b) What is the tension in the cord in the final situation, when the block has speed of 3.60 m/s? **(3 pts)**

$$\underline{T_2 = \frac{m v_2^2}{r_2}} = 0.190\,kg \times \frac{(3.60\,m/s)^2}{0.10\,m} = \underline{24.6\,N}$$

2' 1'

(c) How much work is done by the person who pulled on the cord? **(4 pts)**

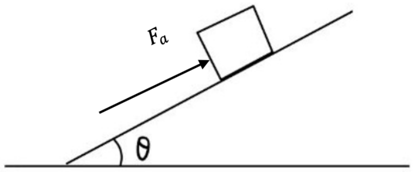
$$\underline{W = \Delta K} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \underline{\frac{1}{2} m (v_2^2 - v_1^2)}$$ 1'

2'

$$= \frac{1}{2} \times 0.190\,kg \times (3.60^2 - 0.90^2)\,m^2/s^2$$

$$= \underline{1.15\,J}$$ 1'

5. Suppose a force is applied to a crate with mass of m along the slope surface pointing upwards at an angle θ above the horizontal. **(10 pts)**



(a) Given the coefficient of static friction μ_s. Please find the range of the applied force to keep the crate stay at rest. ($\mu_s \leq \tan\theta$) **(3 pts)**

① If f_s points upward

$$\begin{cases} f_s + F_a = mg\sin\theta \\ f_s \leq f_{s,max} = \mu_s mg\cos\theta \end{cases}$$

$F_a \geq mg(\sin\theta - \mu_s\cos\theta)$ 1.5'

② If f_s points downward

$$\begin{cases} F_a = mg\sin\theta + f_s \\ f_s \leq f_{s,max} = \mu_s mg\cos\theta \end{cases}$$

$F_a \leq mg(\sin\theta + \mu_s\cos\theta)$ 1.5'

The range for F_a is $mg(\sin\theta - \mu_s\cos\theta) \leq F_a \leq mg(\sin\theta + \mu_s\cos\theta)$

(b) Given the coefficient of kinetic friction μ_k. How hard must you pull to keep it moving upwards at a constant velocity? **(3 pts)**

$F_a = mg\sin\theta + f_k \Rightarrow F_a = mg(\sin\theta + \mu_k\cos\theta)$

3'

(c) The applied force keeps the block moving upwards at a constant velocity. Suppose the angle θ of the slope is adjustable, find the angle θ with which the magnitude of the applied force has the maximum value?. **(4 pts)**

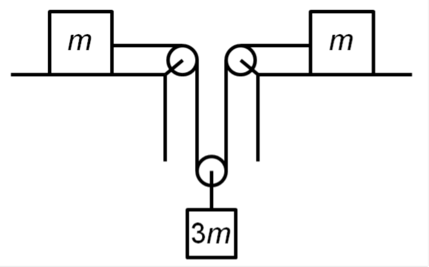
$F_a = mg(\sin\theta + \mu_k\cos\theta)$

The F_a reaches its maximum value, if $dF_a/d\theta$ is zero.

$\frac{dF_a}{d\theta} = mg(\cos\theta - \mu_k\sin\theta) = 0$ 2'

$\cos\theta = \mu_k\sin\theta \Rightarrow \tan\theta = \frac{1}{\mu_k}$ 2'

6. Two particles of masses m are placed on two horizontal tables. A string, which connects these two masses, hangs over three pulleys, which suspends a particle of mass $3m$, as shown below. The pulley has negligible mass. The two parts of the string on the table are parallel and perpendicular to the edge of the table. The hanging parts of the string are vertical. **(10 pts)**



(a) Assuming that the surface is <u>frictionless</u>, find the acceleration of the particle of mass 3m after releasing it from the equilibrium position. **(5 pts)**

$$\begin{cases} 3mg - 2T = 3m \cdot a & 1.5' \\ T = ma & 1.5' \end{cases}$$

$$a = \frac{3mg}{5m} = \frac{3}{5}g = 5.9\ m/s^2 \quad 1'$$

The acceleration has a magnitude of 5.88 m/s^2 with direction pointing <u>vertically downward</u>. 1'

(b) Assuming that the surface is rough and all particles would move after releasing them from the equilibrium position, find the acceleration of the three particles. The coefficient of kinetic friction μ_k between the block and the surface is 0.50. **(5 pts)**

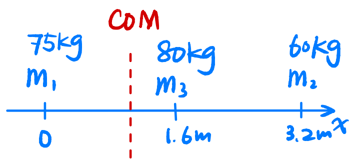
$$\begin{cases} 3mg - 2T = 3m \cdot a & 1' \\ T - \mu mg = ma & 1' \end{cases}$$

$$a = \frac{(3-2\mu) \cdot mg}{5m} = \frac{3-2\mu}{5}g = \frac{2}{5}g = 3.9\ m/s^2 \quad 1'$$

All 3 blocks have same magnitudes of accelerations.
Directions of accelerations for each block:
Left block (m) : Right 1'
Right block (m) : Left 1'
Large block (3m) : Down

7. Two people, one of mass 75 kg and the other of mass 60 kg, sit in a boat of mass 80 kg. The boat has a uniform density. With the boat initially at rest, the two people, who have been sitting at opposite ends of the boat, 3.2 m apart from each other, exchange seats. Ignore the resistance between boat and water. **(10 pts)**

(a) Find the center of mass of the system. **(3 pts)**



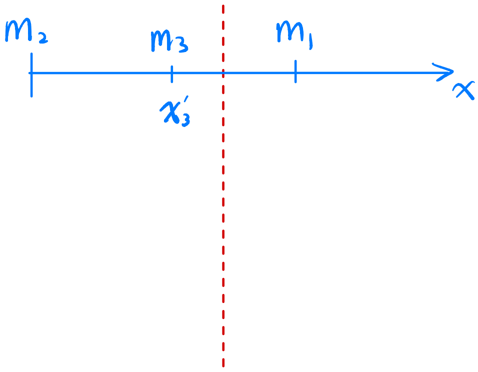
Build a coordinate system with m_1 (75kg) at $x=0$

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \quad 2'$$

$$= \frac{75kg \times 0 + 60kg \times 3.2m + 80kg \times 1.6m}{70 kg + 60kg + 80 kg}$$

$$= 1.5 \text{ m} \quad 1'$$

(b) In what direction will the boat move during this process? **(2 pts)**



There is no net force acting on this system.

⇒ Linear Momentum is conserved.

$\vec{P} = M\vec{V}_{com} = 0$ ⇒ X_{com} doesn't change. 1'

The Boat itself would move to left to maintain X_{com} unchanged. 1'

(c) How far will the boat move? **(5 pts)**

Method 1: x_3' and x_3 is symmetric about X_{com} 3'

$x_3' = 1.49 - (1.60 - 1.49)$ m $= 1.38$ m 1'

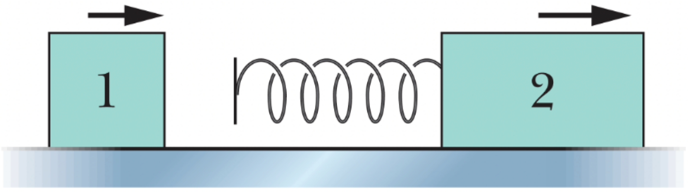
$\Delta x_3 = x_3' - x_3 = 1.38$ m $- 1.60$ m $= -0.22$m 1'

Method 2:

$$\begin{cases} x_{com} = \dfrac{m_2 \cdot x_2' + m_3 \cdot x_3' + m_1 x_1'}{m_1 + m_2 + m_3} = 1.49m \ (\text{or } 1.50m) & 2' \\ x_1' = x_2' + 3.2\,m & 1' \\ x_3' = x_2' + 1.6m & 1' \end{cases} \Rightarrow \begin{cases} x_1' = +2.98\,m \\ x_2' = -0.22\,m \\ x_3' = +1.38\,m \end{cases}$$

$\Delta x_3 = x_3' - x_3 = -0.22m$ 1'

8. In the figure below, block 1 (mass 2.0 kg) is moving rightward at 10 m/s and block 2 (mass 5.0 kg) is moving rightward at 3.0 m/s. The surface is frictionless, and an ideal spring with a spring constant k of 1.1×10^3 N/m is fixed to block 2. The mass of spring is negligible. **(10 pts)**



(a) Since block 1 is faster, it will collide with and compress the spring later. What are the velocities of these two blocks when the compression of spring is maximum? (Hint: The compression of the spring is maximum when the two blocks have the same velocity.) **(3 pts)**

$$m_1 V_1 + m_2 V_2 = (m_1 + m_2) V_f \quad 2'$$

$$V_f = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2} = \frac{2.0kg \times 10 m/s + 5.0 kg \times 3.0 m/s}{2.0 kg + 5.0 kg}$$

$$= 5.0 m/s \quad 1'$$

(b) What is the elastic potential energy of the spring when the compression is the maximum? **(3 pts)**

$$\Delta E_{mec} = 0 \Rightarrow \Delta K + \Delta U = 0 \Rightarrow \Delta U = -\Delta K \quad 1'$$

$$\Delta U = -\Delta K = -[\tfrac{1}{2}(m_1+m_2)V_f^2 - \tfrac{1}{2}m_1V_1^2 - \tfrac{1}{2}m_2V_2^2] \quad 1'$$

$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 - \tfrac{1}{2}(m_1+m_2)V_f^2$$

$$= \tfrac{1}{2}\times 2.0 \times 10^2 + \tfrac{1}{2}\times 5.0 \times 3.0^2 - \tfrac{1}{2}\times 7.0 \times 5.0^2 \text{ J}$$

$$= 35 \text{J} \quad 1'$$

(c) Find the maximum compression. **(4 pts)**

$$\Delta U = +\tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2$$

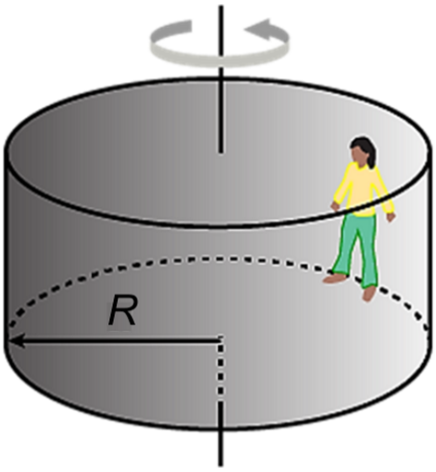
$$x_i = 0 \Rightarrow \Delta U = \tfrac{1}{2}kx_f^2 \quad 2'$$

or 0.25 m

$$x_f = \pm\sqrt{\frac{2\Delta U}{k}} = \pm\sqrt{\frac{2\times 35}{1.1\times 10^3}} \text{ m} = \pm 0.25 \text{ m} \quad 2'$$

The maximum compression is 0.25m.

9. A hollow thin cylinder has a radius $R = 5.0$ m. A rider stands against the wall as shown in the figure below. At $t = 0$ the cylinder starts to rotate and the rider rotates together with the cylinder. The angular position $\theta(t)$ of the rider is $\theta = ct^3$, with $c = 0.10\,\text{rad/s}^3$. When $t = 2.0\,\text{s}$, let's determine the magnitudes of the following of the rider. **(10 pts)**



(a) angular speed ω **(2 pts)**

$$\omega = \frac{d\theta}{dt} = 3ct^2 = 3 \times 0.1 \times 4 \text{ rad/s} = 1.2 \text{ rad/s}$$

1' 1'

(b) linear speed v **(1 pts)**

$$v = \omega R = 1.2 \text{ rad/s} \times 5.0 \text{ m} = 6.0 \text{ m/s}$$

1'

(c) angular acceleration α **(2 pts)**

$$\alpha = \frac{d\omega}{dt} = 6ct = 6 \times 0.10 \text{ rad/s}^3 \times 2.0 \text{ s} = 1.2 \text{ rad/s}^2$$

1' 1'

(d) tangential acceleration a_t **(1 pts)**

$$a_t = \alpha R = 1.2\ rad/s^2 \times 5m = 6.0\ m/s^2$$ 1'

(e) radial acceleration a_r **(2 pts)**

$$a_r = \underline{\omega^2 R} = (1.2\ rad/s)^2 \times 5\ m = \underline{7.2\ m/s^2}$$

1' 1'

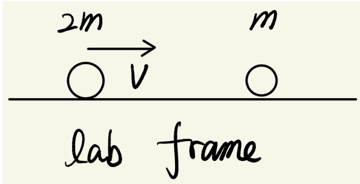
(f) total linear acceleration a **(2 pts)**

$$\underline{a = \sqrt{a_t^2 + a_r^2}} = \sqrt{6^2 + 7.2^2}\ m/s^2 = \underline{9.4\ m/s^2}$$

1' 1'

10. Consider the following one-dimensional collision. On a physics lab table with no friction, a mass $2m$ moving with velocity $+v$ to the right collides elastically with a mass m at rest. **(10 pts)**

(a) Find their final lab-frame velocities by working in the lab frame. **(3 pts)**



Elastic collision

$\vec{P}_i = \vec{P}_f$ 1'

$K_i = K_f$ 1'

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} = \frac{2m - m}{2m + m} \cdot V = \frac{1}{3} V \quad 0.5'$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} = \frac{2 \times 2m}{2m + m} \cdot V = \frac{4}{3} V \quad 0.5'$$

(b) Find the Center of Mass (CM) velocity of this two body system. **(3 pts)**

$$\underbrace{V_{com} = \frac{P_i}{2m + m}}_{2'} = \frac{2mV}{3m} = \underbrace{\frac{2}{3} V}_{1'}$$

(c) Find the initial velocities of these two objects in the CM frame. Does it agree with the physics results that you find in part (a)? **(4 pts)**

$V_{1i}' = V_{1i} - V_{com} = V - \frac{2}{3}V = \frac{1}{3}V$ 1'

$V_{2i}' = V_{2i} - V_{com} = 0 - \frac{2}{3}V = -\frac{2}{3}V$ 1'

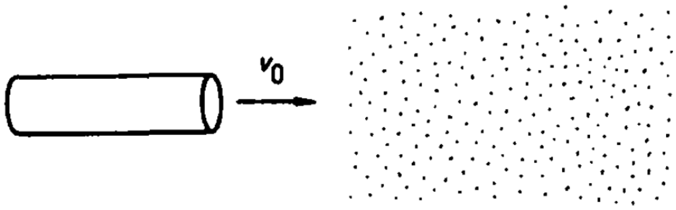
1'

$$\boxed{\begin{aligned} &\left\{ \begin{aligned} K_i &= K_f \\ P_i &= P_f \end{aligned} \right. \Rightarrow \quad v_{1f}' = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}' + \frac{2m_2}{m_1 + m_2} v_{2i}' = -\frac{1}{3} V \\ &\qquad\qquad\qquad v_{2f}' = \frac{2m_1}{m_1 + m_2} v_{1i}' + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}' = \frac{2}{3} V \\ &V_{1f}' = V_{1f} - V_{com} \Rightarrow V_{1f} = V_{1f}' + V_{com} = -\frac{1}{3}V + \frac{2}{3}V = \frac{1}{3}V \\ &V_{2f}' = V_{2f} - V_{com} \Rightarrow V_{2f} = V_{2f}' + V_{com} = \frac{2}{3}V + \frac{2}{3}V = \frac{4}{3}V \end{aligned}}$$

The results $\underbrace{\text{agree}}_{1'}$ with (a).

11. In the vast expanse of space, the legendary Starship Enterprise, with a mass of m_0 and a cross-section of A, is cruising through the cosmos at a velocity v_0 when it collides with a stationary dust cloud of density ρ at $t = 0$. **(10 pts)**

(a) As the dust particles stick to the spacecraft and no other resistance is present, determine the acceleration of the Enterprise. Express it in terms of m_0, v_0, ρ, A, and v. **(4 pts)**



Linear Momentum Conservation

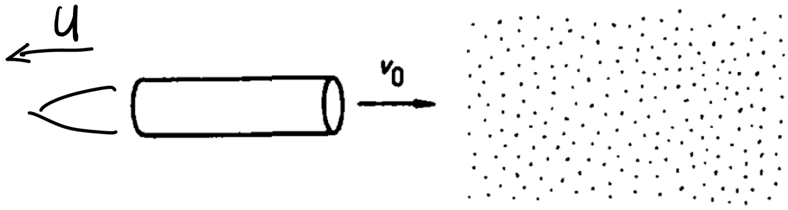
$m_0 v_0 = mv = (m+dm)(v+dv) \Rightarrow m\,dv = -dm\,v$ 1'

$$\left.\begin{aligned} & mv = m_0 v_0 \Rightarrow m = \frac{m_0 v_0}{v} \\ & dm = \rho \cdot A \cdot v \cdot dt \end{aligned}\right\} \Rightarrow \frac{m_0 v_0}{v}\,dv = -\rho A v \cdot dt\, v$$

(1' beside each of the two lines)

$$a = \frac{dv}{dt} = -\frac{\rho A v^2}{m_0 v_0 / v} = -\frac{\rho A v^3}{m_0 v_0}$$ 1'

(b) Captain Kirk, renowned for his decisive actions, commands the ignition of the engine to maintain a constant velocity v_0 for the ship. The engine expels exhaust at a consistent relative velocity u in the opposite direction to provide propulsion. Calculate the rate at which fuel is being burned to sustain this boost in speed. **(3 pts)**



$$M \cdot V = (M + dM + dm)(V + dV) - dM \cdot V' \qquad 1'$$

$$U = V + dV - V' \Rightarrow V' = V + dV - U$$

$$\cancel{MV} = \cancel{MV} + MdV + \cancel{dM \cdot V} + \cancel{dM\, dV} + dm \cdot V + \cancel{dm\, dV}$$

$$- \cancel{dM \cdot V} - \cancel{dM \cdot dV} + dM \cdot U$$

$$M dV + \rho \cdot A \cdot V \cdot dt \cdot V + dM \cdot U = 0$$

$$M \cdot \frac{dV}{dt} + \rho A V^2 + U \cdot \frac{dM}{dt} = 0 \qquad 1'$$

Since $V = V_0 =$ constant, $dV/dt = 0$

$$\rho A V_0^2 + U \frac{dM}{dt} = 0 \Rightarrow \frac{dM}{dt} = -\frac{\rho A V_0^2}{U} \qquad 1'$$

if dM is a positive value

$$M V = (M - dM + dm)(V + dV) + dM V'$$

$$\frac{dM}{dt} = \frac{\rho A V_0^2}{U}$$

Also correct.

(c) The Enterprise moves at the constant speed v_0. Compute the mass of the Enterprise after it traverses a distance of L into the dust cloud. (Hint: need to consider two contributions, one of which is due to burnt fuel and the other is due to dust cloud.) **(3 pts)**

Total Mass change Rate

$$dM_{tot} = dM + dm \quad 1'$$

$$\frac{dM_{tot}}{dt} = \frac{dM+dm}{dt} = \frac{dM}{dt} + \frac{dm}{dt} = -\frac{\rho A v_0^2}{u} + \rho A v$$

Due to constant Velocity v_0

$$t = L/v_0 \qquad v = v_0$$

$$\frac{dM_{tot}}{dt} = -\frac{\rho A v_0^2}{u} + \rho A v_0 \quad 1'$$

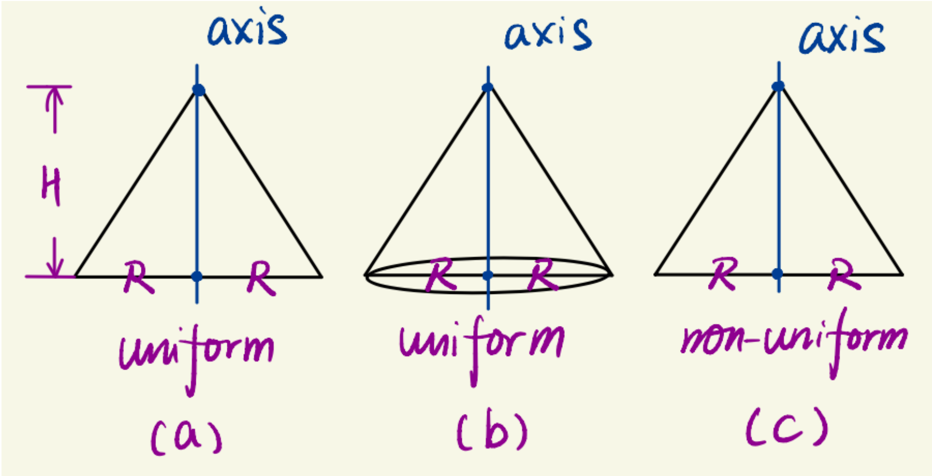
$$\Delta M_{tot}(t) = \int_0^{L/v_0} \left(-\frac{\rho A v_0^2}{u} + \rho A v_0\right) dt$$

$$= \rho A \cancel{v_0} \left(1 - \frac{v_0}{u}\right) \frac{L}{\cancel{v_0}}$$

$$M_{tot}(t) = m_0 + \rho A L \left(1 - \frac{v_0}{u}\right) \quad 1'$$

if the student misses $\frac{dM}{dt}$ or $\frac{dm}{dt}$, give $1'$ in total.

12. Moment of inertia (rotational inertia). **(10 pts)**



(a) Find the moment of inertia of an isosceles triangle (a triangle with two equal sides) uniform plate (constant density σ) (with bottom side of length $2R$, height H and mass M) about an in-plane axis through center as shown in the figure (a). **(3 pts)**

$M = \sigma \cdot \frac{1}{2} \cdot 2R \cdot H = \sigma R H$

For thin rod located at position y.

$\frac{r}{R} = \frac{y}{H} \Rightarrow r = \frac{R}{H} \cdot y$

$\underline{dm} = 2r \cdot dy \cdot \sigma = 2\sigma r dy = 2\sigma \cdot \frac{R}{H} y \cdot dy$ 1'

$\underline{dI} = \frac{1}{12} dm \cdot (2r)^2 = \frac{1}{\cancel{12}_{3}} \cdot 2\sigma \frac{R}{H} y \cdot dy \cdot \cancel{4} \cdot \frac{R^2}{H^2} \cdot y^2$ 1'

$= \frac{2}{3} \frac{R^3}{H^3} \cdot \sigma \, y^3 \cdot dy$

$I = \int_0^H \frac{2}{3} \frac{R^3}{H^3} \cdot \sigma \cdot y^3 dy = \frac{1}{4} H^4 \times \frac{2}{3} \frac{R^3}{H^3} \cdot \sigma = \frac{1}{6} \sigma R H \cdot R^2$

$= \frac{1}{6} M R^2$ 1'

(b) Calculate the moment of inertia of a uniform solid cone about an axis through its center (see Figure (b) above). The cone has mass M and height H. The radius of its circular base is R. (Hint: Volume of a cone with a radius of R and height of H: $V = \frac{1}{3}\pi R^2 H$) **(3 pts)**

$M = \frac{1}{3}\pi R^2 \cdot H \cdot \rho$

Thin disk

$\frac{r}{R} = \frac{y}{H} \Rightarrow r = \frac{R}{H} y$

$dm = \rho \pi r^2 \cdot dy = \rho \pi \cdot \frac{R^2}{H^2} y^2 \cdot dy$ 1'

$dI = \frac{1}{2} dm \cdot r^2$

$= \frac{1}{2} \rho \cdot \pi \frac{R^2}{H^2} \cdot y^2 \cdot \frac{R^2}{H^2} \cdot y^2 \cdot dy$

$dI = \frac{1}{2} \rho \pi \frac{R^4}{H^4} y^4 dy$ 1'

Solid cone

$I = \int_0^H \frac{1}{2} \rho \pi \frac{R^4}{H^4} \cdot y^4 dy$

$= \frac{1}{5} \times \frac{1}{2} \rho \pi \frac{R^4}{H^4} \cdot H^5$

$= \frac{1}{10} \rho \pi R^2 \cdot H \cdot R^2$

$= \frac{3}{10} M R^2$ 1'

(c) Find the moment of inertia of an isosceles triangle (a triangle with two equal sides) non-uniform plate about an in-plane axis through the center as shown in figure (c). The density of the plate is given by $\sigma(r) = \alpha r$, with α being a constant and r representing the distance to the axis. This means that the density grows linearly with the distance to the axis. Again, this triangular plate has a bottom side of length $2R$, height H, and mass M. **(3 pts)**

For a thin Rod. $r = \frac{R}{H} \cdot y$

$$dm_{rod} = 2\int_0^r \sigma(r) \cdot dr \cdot dy = 2\int_0^r \alpha r dr dy = 2 \times \frac{1}{2} \alpha r^2 \cdot dy = \alpha r^2 dy \quad 1'$$

$$M = \int dm_{rod} = \int_0^H \alpha r^2 dy = \int_0^H \alpha \cdot \frac{R^2}{H^2} \cdot y^2 dy = \frac{1}{3} \alpha \frac{R^2}{H^2} \cdot H^3 = \frac{1}{3} \alpha R^2 \cdot H$$

$$dI_{rod} = 2\int_0^r r^2 \cdot \sigma(r) \cdot dr \cdot dy = 2\int_0^r r^2 \times \alpha r \times dr dy = \frac{1}{2} \alpha r^4 dy \quad 1'$$

$$I = \int dI_{rod} = \int_0^H \frac{1}{2} \alpha r^4 dy = \int_0^H \frac{1}{2} \alpha \cdot \frac{R^4}{H^4} \cdot y^4 dy = \frac{1}{10} \alpha \frac{R^4}{H^4} \times H^5$$

$$= \frac{1}{10} \alpha H R^2 \cdot R^2 = \frac{3}{10} M R^2 \quad 1'$$

(d) Compare the moments of inertia that you found in part (b) and (c). You should find that they are the same, explain why? Is it possible to find another mass density distribution $\sigma(r)$ that gives the same moment of inertia? (Yes/No) **(1 pts)**

Explanation: Same mass distribution with respect to rotational axis. 0.5'

Yes/No: [✓] Yes, [_] No 0.5'