香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

PHY1001 Mechanics

2022-2023 Term 2

Midterm Examination

March 19th, 2023; Time Allowed: 3 Hours

NAME (print) ______________________________

CUHKSZ ID ______________________________

ZOOM session No. ______________________________

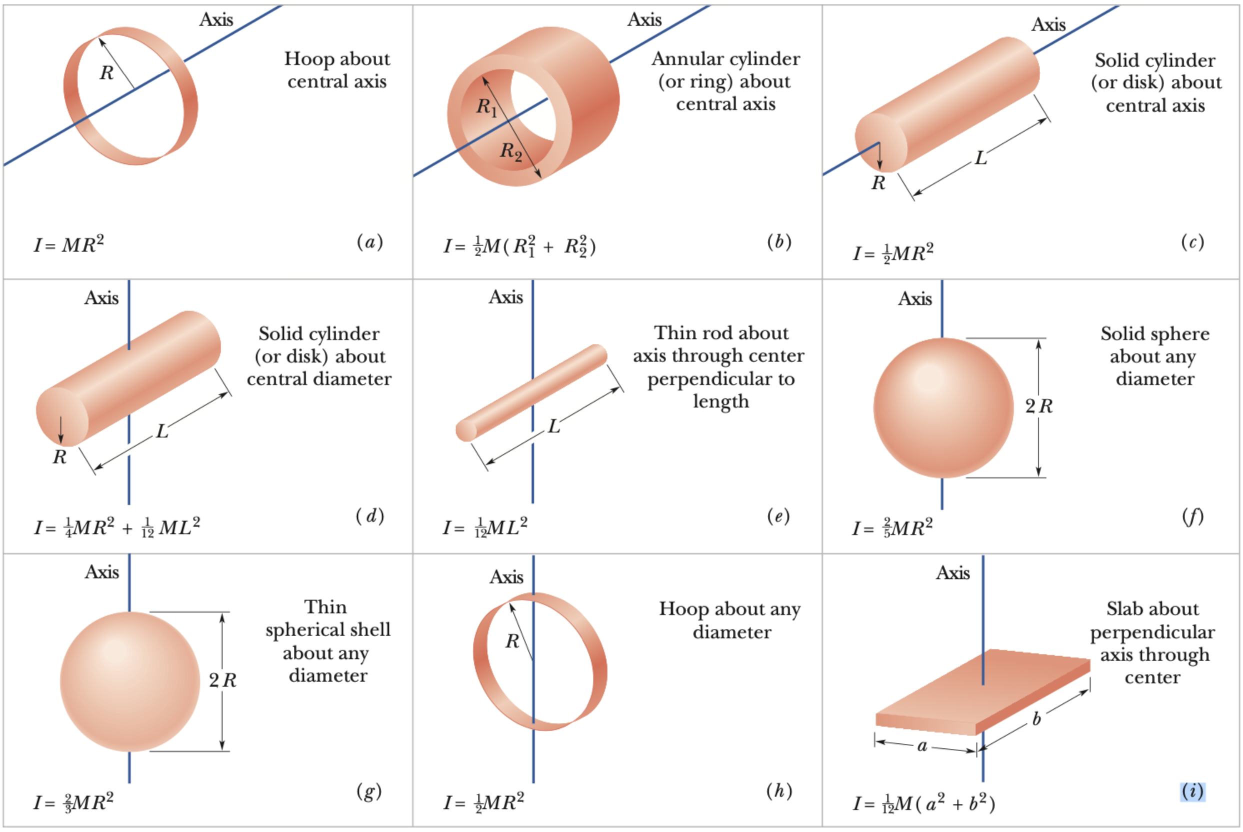
- **Show all your work**. Correct answers with little supporting work will not be given credit.
- Closed Book Exam: One piece of double-sided A4 reference paper, a scientific calculator, and a paper-based dictionary are allowed.
- Unless approved by the instructors, students who arrive more than 30 minutes late will NOT be admitted.
- The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	TOTAL
10	10	10	10	10	10	10	10	10	10	10	10	120

Summary of Basic Calculus:

$$\sin 2\theta = 2\sin\theta\cos\theta\,,$$
$$\frac{d}{dx}x^n = nx^{n-1},$$
$$\frac{d}{dx}e^{ax} = ae^{ax},$$
$$\frac{d}{dx}\ln ax = \frac{1}{x},$$
$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v,$$
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ \ (n \neq -1),$$
$$\int \frac{dx}{x} = \ln x + C,$$
$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C, \quad C \text{ is a constant.}$$

Table of Moments of Inertia:



1. Given the two vectors

$$\vec{a} = 4\hat{\imath} + 3\hat{\jmath} + 5\hat{k}, \quad \text{and} \quad \vec{b} = -6\hat{\imath} + 3\hat{\jmath} - 4\hat{k},$$

answer the following questions.(All the numbers are exact in this problem. No need to worry about the significant figures.) **(10 pts)**

(a) Find $\vec{a} + \vec{b}$. **(2 pts)**

(b) Find the vector product $\vec{a} \times (3\vec{b})$. **(4 pts)**

(c) Find the scalar product of the above two vectors $(\vec{a} + \vec{b}) \cdot (\vec{a} \times 3\vec{b})$. **(4 pts)**

Solution:

(a)

2 points for $\vec{a} + \vec{b} = -2\hat{\imath} + 6\hat{\jmath} + \hat{k}$

(b)

correct answer gets 4 pts.

4 points for $\vec{a} \times 3\vec{b} = 3(-27\hat{\imath} - 14\hat{\jmath} + 30\hat{k}) = -81\hat{\imath} - 42\hat{\jmath} + 90\hat{k}$

steps: $\vec{a} \times 3\vec{b} = (3a_y b_z - 3a_z b_y)\hat{i} + \cdots \rightsquigarrow$ 2pts.

4 points for $(\vec{a} + \vec{b}) \cdot (\vec{a} \times 3\vec{b}) = 162 - 252 + 90 = 0$

correct answer gets 4 pts

step: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \rightarrow$ 2pts.

or $\vec{a} \perp \vec{a} \times \vec{b}$ and $\vec{b} \perp \vec{a} \times \vec{b}$ → final dot product must be zero. → 4pts

2. An object of 2.00 kg moves from point O with initial velocity of 5.00 m/s along positive y direction. After 2.00 seconds, it reaches point A, which is 20.0 m away from point O as shown below. Suppose that there is only one non-zero constant force acting on this object. **(10 pts)**

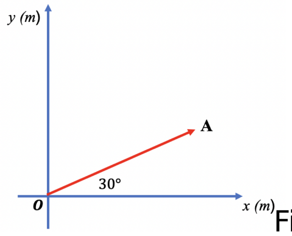


Fig. 1

(a) What kind of motion does this object do? (Hint: constant velocity, constant acceleration, rotation, free fall or any other motions.) **(2 pts)**

(b) Find the force acting on the object. Please indicate the magnitude and direction of the force. **(4 pts)**

(c) Draw (sketch) the trajectory of the object in the above figure, and find the trajectory function relation between x and y. **(4 pts)**

Solution:

a. (2 points) constant net force → constant acceleration motion

b. (1 points) Along x-axis

$$r_x\,(2s) = 20\cos 30°\, m = v_{0x}t + \frac{1}{2}a_x t^2 = \left(0 + \frac{1}{2}a_x 2^2\right) m$$

$$a_x = 8.66\ m/s^2$$

(1 points) Along y-axis

$$r_y\,(2s) = 20\sin 30°\ \ m = v_{0y}t + \frac{1}{2}a_y t^2 = \left(5\times 2 + \frac{1}{2}a_y 2^2\right) m$$

$$a_y = 0$$

(2 points) The force acting on it

$$\vec{F} = m\vec{a} = 2\times 8.66\hat{\imath} + 0\hat{\jmath} = 17.32\hat{\imath}\ N$$

The magnitude is 17.3 N, the direction is along positive x-axis.

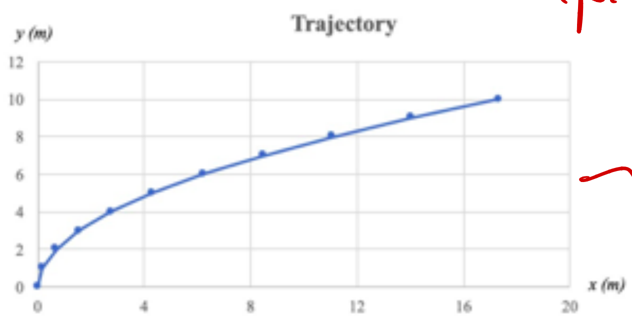
c. (4 points)

$$r_x = v_{0x}t + \frac{1}{2}a_x t^2 = \frac{1}{2}\times 8.66t^2 = 4.33(m/s^2)t^2$$ → 1pt

$$r_y = v_{0y}t + \frac{1}{2}a_y t^2 = 5(m/s)t$$ → 1pt

The trajectory is a parabola with $r_x = 0.1732r_y^2$ → 1pt

Drawing on the figure above is fine!



→ 1pt

3. As shown in Fig. 2, an inventor designs a pendulum clock using a bob with mass 10.0 kg at the end of a thin wire of length $L = 1.00$ m. Instead of swing back and forth, the bob is to move in a horizontal circle with constant speed v with the wire making a fixed angle $\beta = 30.0°$ with the vertical direction. This is called a conical pendulum because the suspending wire traces out a cone. **(10 pts)**

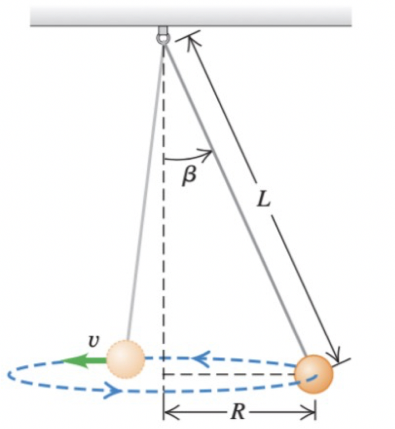


Fig. 2

(a) Find the speed v of the bob. **(4 pts)**

(b) Find the period T, which is the time needed for the bob to complete a circle. **(2 pts)**

(c) If a new bob with mass 20.0 kg is used to replace the original one, will the period be affected? **(4 pts)**

Solution:

a. (2 points) The centripetal force

$$F = mg\tan\beta = mv^2/R$$

(1 point) The radius

$$R = L\sin\beta$$

(1 points) The speed is

$$v = \sqrt{gR\tan\beta} = \sqrt{gL\sin\beta\tan\beta} = \sqrt{9.8\times 1.00\times \sin 30°\tan 30°}$$
$$= 1.67\ m/s$$

b. (2 Points) The bob is doing constant speed circular motion → uniform circular motion

$$T = \frac{2\pi R}{v}$$ → 1pt

$$T = \frac{2\pi R}{v} = 2\pi\frac{L\sin\beta}{v} = \frac{2\times 3.14\times 1.00\times \frac{1}{2}}{1.67}s = 1.87\ s$$ → 1pt

c. (2 Points) The period is

1pt ←
$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R}{g\tan\beta}} = 2\pi\sqrt{\frac{L\sin\beta}{g\tan\beta}} = 2\pi\sqrt{\frac{L\cos\beta}{g}}$$ = T 1pt

2pts
(2 Points) It won't be affected by the mass.
With a new bob of mass 20 kg, the T remains unchanged (T=1.87 s).

4. Only one force is acting on a 2.0 kg particle-like object whose velocity is given by **(10 pts)**

$$v = 5.0\,m/s + (4.0\,m/s^2)t - (3.0\,m/s^3)t^2,$$

with v in m/s and t in seconds.

(a) What is the displacement from $t = 0$ s to $t = 1.0$ s? **(2 pts)**

$$S = \int_{0s}^{1.0s} v\,dt \qquad (1\,pt)$$

$$= \left[(5.0 m/s)t + (2.0 m/s^2)t^2 - (1.0 m/s^3)t^3\right]\Big|_{0s}^{1.0s} \qquad (0.5pt)$$

$$= 6.0\,\cancel{m/s}\,m \qquad (0.5pt).$$

(b) What is the kinetic energy of this object at $t = 1.0$ s? **(2 pts)**

At $t = 1.0s$, $v = 6.0\,m/s$ (0.5pt)

$K = \frac{1}{2} m v^2$ (0.5pt)

$= 36.0\,J$ (1 pt)

36 is OK

(c) What is the work done by the force from $t = 0$ s to $t = 1.0$ s? **(2 pts)**

At $t = 0s$. $K = \frac{1}{2} m v_0^2 = \frac{1}{2} \times (2.0kg) \times (5.0 m/s)^2 = 25.0 J$ (0.5pt)

Work - Energy Theorem : $W = \Delta K$ (0.5pt)

$= 36.0J - 25.0J$

$= 11.0J$ 11 is OK (1 pt).

(d) What is the instantaneous acceleration of this object at $t = 1.0$ s? **(2 pts)**

$$a(t) = \frac{dv}{dt} = 4.0\,m/s^2 - 6.0\,m/s^3 \cdot t \qquad (1pt)$$

At $t = 1.0s$, $a = -2.0\,m/s^2$ (1pt)

(e) What is the magnitude of the force at $t = 1.0$ s? **(2 pts)**

At $t = 1.0\,s$, $F = ma$ (1 pt)

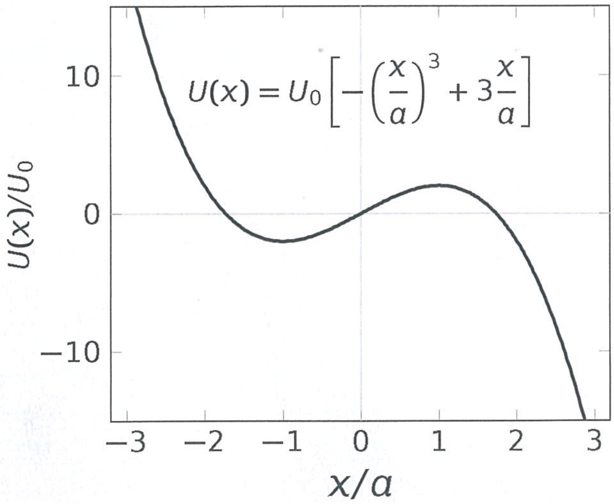
$= 2.0kg \times (-2.0\,m/s^2)$

$= -4.0\,N$ (1pt)

5. A force $F(x)$ acted on a particle is represented by the potential-energy function **(10 pts)**

$$U(x) = U_0\left[-\left(\frac{x}{a}\right)^3 + 3\frac{x}{a}\right],$$

where a is constant with the dimension of length. $U(x)$ is plotted in the figure below.



(a) Find the force $F(x) = -\dfrac{dU(x)}{dx}$ from the potential given above. **(2 pts)**

$$F(x) = -\frac{dU(x)}{dx}$$

$$= \frac{3U_0}{a}\left(\frac{x^2}{a^2} - 1\right) \quad (2pts)$$

(b) At what value of x is the force $F(x)$ zero? **(3 pts)**

$F(x) = 0 \Rightarrow \quad x = a \quad (1.5pt)$

or $x = -a \quad (1.5pt).$

Here assume $U_0 > 0$ 8.

(c) The location where the force equals zero is called the equilibrium of this potential. Are these equilibrium positions stable or unstable? Explain why. **(5 pts)**

At $x = a$, equilibrium is unstable, (1 pt)

because $\left.\dfrac{d^2U}{dx^2}\right|_{x=a} < 0$ (or $U(x)$ is at maximum.) (1.5pt)

At $x = -a$, equilibrium is stable, (1pt)

because $\left.\dfrac{d^2U}{dx^2}\right|_{x=-a} > 0$ (or $U(x)$ is at minimum) (1.5pt)

6. A wedge of mass M is initially at rest on a frictionless and horizontal surface. A block of mass m is moving on the horizontal surface toward the wedge with an initial velocity v_0. The block can move upward along the wedge, which also has a frictionless surface, until the block's center of mass reaches the highest position with a height h relative to its initial position (see figure below). **(10 pts)**

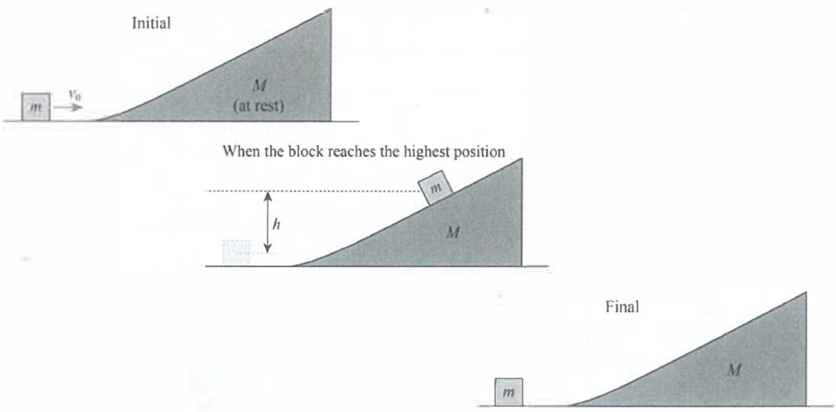


Fig. 4

(a) What are the speeds of the block and the wedge when the block reaches the highest position? **(2 pts)**

The block ad the wedge share the same speed v_s (0.5pt)

Conservation of momentum: $m v_0 = (m+M) v_s$ (0.5pt)

$\Rightarrow v_s = \frac{m}{m+M} v_0$ (1 pt)

(b) What is the height h for the highest position of the block? **(3 pts)**

Conservation of energy (0.5pt)

$\frac{1}{2} m v_0^2 = \frac{1}{2}(m+M) v_s^2 + mgh$ (0.5pt)

$\Rightarrow h = \frac{M}{m+M} \frac{v_0^2}{2g}$ (1 pt)

(c) Later on, the block moves downward along the wedge and separates with the wedge at the end. What are the final speeds of the block (v_B) and of the wedge (v_W)? **(4 pts)**

Conservation of momentum: $m v_0 = m v_B + M v_W$ (1 pt)

or $m v_0 = -m v_B + M v_W$

Conservation of energy: $\frac{1}{2} m v_0^2 = \frac{1}{2} m v_B^2 + \frac{1}{2} M v_W^2$ (1 pt)

$\Rightarrow v_B = \frac{m-M}{m+M} v_0$ or $\frac{M-m}{m+M} v_0$ (1 pt)

$\Rightarrow v_W = \frac{2m}{m+M} v_0$ (1 pt).

(d) Check your calculation plausibility by considering the limiting case when $M \gg m$. **(1 pts)**

When $M \gg 0$, $v_B = -v_0$ (or $v_B = v_0$ in opposite direction) (0.5pt)

$v_W = 0$ (0.5pt).

7. A water storage truck is moving <u>forward</u> at v = 3.6 km/h along a straight and frictionless street when its driver decides to spray water. Water is sprayed at a constant rate of 2.0 kg/s at a speed (relative to the truck) of v_s = 2.5 m/s. The initial mass of the truck with water is $M = 2.0 \times 10^3$ kg. (Hint: This is a mass-changing problem.) **(10 pts)**

(a) If the truck <u>accelerates forward</u> when the drive starts spraying water, does he spray water forward or backward? Why? **(2 pts)**

(1 point) Backward.

(1 point) Conservation of momentum

(b) If the truck sprays water forward, what is the force applied to the truck? Specify the magnitude and direction. **(3 pts)**

(1 point) Assuming the speed of water just sprayed out of the truck is v_w,

$v_s = v_W - v \rightarrow v_W = v + v_s$

Since the truck sprays water forward, the truck is decelerating. Similar to the rocket propulsion, from the conservation of momentum,

(1 point) $(M - dm)(v - dv) + (v + v_s)dm = Mv \rightarrow M\frac{dv}{dt} = v_s\frac{dm}{dt}$

(1 point) $\rightarrow F = M\frac{dv}{dt} = v_s\frac{dm}{dt} = 2.5\ \mathrm{m/s} \times 2.0\ \mathrm{kg/s} = 5.0\ \mathrm{N}$

(c) If the truck sprays water backward, what is the instantaneous acceleration of the truck at the moment when it just starts spraying water? **(2 pts)**

[redacted] Assuming the speed of water just sprayed out of the truck is v_w,

$-v_s = v_W - v \rightarrow v_W = v - v_s$

If the truck sprays water backward, the truck is accelerating. Similar to the rocket propulsion, from the conservation of momentum,

(1 point) $(M - dm)(v + dv) + (v - v_s)dm = Mv \rightarrow M\frac{dv}{dt} = v_s\frac{dm}{dt}$

(1 point) $\rightarrow a = \frac{dv}{dt} = \frac{v_s}{M}\frac{dm}{dt} = \frac{2.5\mathrm{m/s}}{2.0\times10^3 kg} \times 2.0\ \mathrm{kg/s} = 2.5 \times 10^{-3}\mathrm{m/s^2}$

(d) If the truck keeps spraying water backward for a while, <u>find</u> the <u>time-dependent</u> acceleration of the truck. (Hint: find the time dependence of the truck's mass $M(t)$ first.) **(3 pts)**

~~(1 point)~~ The time-dependent mass of the truck is

1.5 pt

$$M_t = M - \frac{dm}{dt}t = 2.0 \times 10^3 kg - 2.0\ \mathrm{kg/s}\ \ t$$

1.5 pt

~~(1 point)~~ The time dependent acceleration is

$$a = \frac{dv}{dt} = \frac{v_s}{M_t}\frac{dm}{dt} = \frac{2.5\mathrm{m/s}}{2.0\times10^3 kg - 2.0\ \mathrm{kg/s}\ t} \times 2.0\ \mathrm{kg/s} = \frac{5.0\ \mathrm{N}}{2.0\times10^3 kg - 2.0\ \mathrm{kg/s}\ t}$$

8. Rotating Wheel: A wheel with radius r is lying in the plane of your paper and rotating counter-clockwise with angular velocity $\vec{\omega}$. Choose a point P on the rim and draw a vector $\vec{r}$ from the center of the wheel to that point. **(10 pts)**

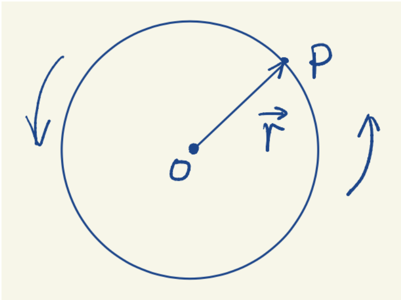


Fig. 5

(a) What is the direction of angular velocity $\vec{\omega}$? (Upward, perpendicular to the paper or Downward, Perpendicular to the paper) **(2 pts)**

(b) Find both the direction (indicate or describe) and magnitude of the velocity $\vec{v}$ at point P. **(2 pts)**

(c) Suppose ω is a constant, find the magnitude and direction of the linear acceleration for the point P point on the wheel. **(2 pts)**

(d) Suppose the angular acceleration α is in the same direction of ω, which means the angular velocity increases, find the magnitude and direction of the total acceleration for the point P on the wheel. (Draw a diagram to show the direction.) **(4 pts)**

Solution:

a. (2 points) upward, perpendicular to the paper.

b. (1 point) Magnitude of speed:

$$v = \omega R$$

(1 point) Direction: Perpendicular to $\vec{r}$, upward. Or draw the direction in the graph.

c. (1 Point) constant ω, no tangential acceleration.

$$\vec{a} = \vec{a_r}$$

$$a_r = \omega^2 r$$

(1 point) direction: pointing towards O

d. (1 Point) The radial acceleration

$$a_r = \omega^2 r$$

(1 Point) The tangential acceleration

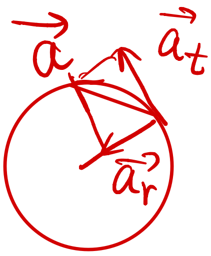
$$a_t = \alpha r$$

(1 Point) The magnitude of linear acceleration

$$\vec{a} = \vec{a_r} + \vec{a_t}$$

$$a = \sqrt{a_r^2 + a_t^2} = r\sqrt{\omega^4 + \alpha^2}$$

(1 Point) for the drawing



9. Fig. 6 shows an early method of measuring the speed of light that makes use of a rotating slotted wheel. A beam of light passes through one of the slots at the outside edge of the wheel, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such slotted wheel has a radius of 5.00 cm and 500 slots around its edge. Measurements taken when the mirror is $L = 500$ m from the wheel indicate a speed of light of 3.00×10^8 m/s. **(10 pts)**

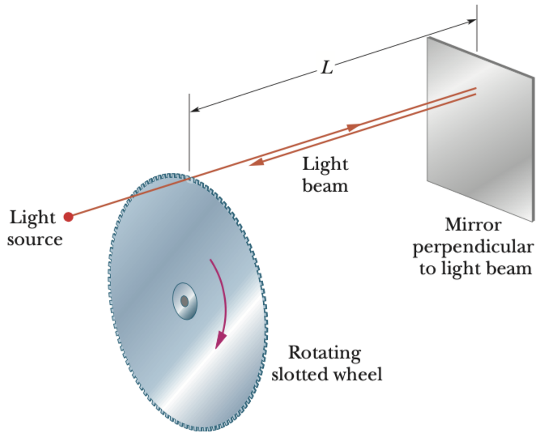


Fig. 6

(a) How much time does the light take to go from the wheel to the mirror and back again? **(2 pts)**

$\Delta t = \frac{2L}{c}$ → 1pt

$= \underline{\underline{3.33}} \times 10^{-6}$ (s) 1pt. 3.3 or 3, 3.333..., etc. OK.

(b) What is the angle between two slots next to each other? **(2 pts)**

$\Delta\theta = \frac{2\pi}{500}$ → 1pt

$= \underline{\underline{1.25}} \times 10^{-2}$ (rad) 1pt

$\boxed{\frac{\pi}{250} \text{ is ok. } 1.2, 1.3 \text{ OK}}$

(c) What is the angular speed (constant) of the wheel? **(3 pts)**

$\omega = \frac{\Delta\theta}{\Delta t}$ → 2pt

$= 3.75 \times 10^3$ rad/s → 1pt. $\underline{3.8}$, $\underline{4.}$ OK

(d) What is the linear speed of a point on the edge of the wheel? **(3 pts)**

$v = \omega r$ → 2 pt

$= 1.88 \times 10^2$ m/s

188 OK
190 OK.

10. Moments of Inertia (Rotational Inertia) of thin rods. **(10 pts)**

(a) Demonstrate that the moment of inertia of a thin uniform rod is $I_{cm} = \frac{1}{12}ML^2$ when it is rotating about the axis through the center, perpendicular to the rod. **(2 pts)**

$$I_a = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{1}{12} ML^2$$

1pt 1pt

Different methods with correct answer get full credit!

(b) Find the moment of inertia of a thin uniform rod of length L and mass M when it is rotating about its left end, perpendicular to rod, as shown below. **(2 pts)**

Fig. 7

Method 1: $I_b = \int_0^L dx\, x^2 \frac{M}{L} = \frac{1}{3} ML^2$

1pt 1pt

Method 2: $I_b = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$

1pt 1pt

(c) Now consider a non-uniform rod with the linear density σx where x is the distance from the origin shown in part (b). Given the length L and mass M for this rod, find σ in terms L and M. **(3 pts)**

$M = \int_0^L \sigma x\, dx$ → 1pt

$= \frac{1}{2} \sigma L^2$ → 1pt

$\sigma = \frac{2M}{L^2}$ → 1pt

(d) Find the moment of inertia of the non-uniform rod described in part (c), when it is rotating about its left end, perpendicular to rod. **(3 pts)**

$I_d = \int_0^L \sigma x^3 dx$ → 1pt

$= \frac{1}{4} \sigma L^4$ → 1pt

$= \frac{1}{2} ML^2$ → 1pt.

11. Figure below shows a disk of mass M and radius R is held up by a massless string. **(10 pts)**

Different methods with correct answer get full credit!

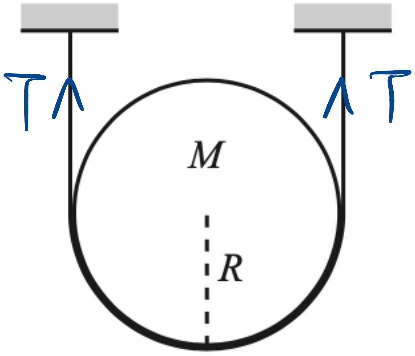


Fig. 8

(a) Suppose the surface of the disk is frictionless. What is the tension T in the string supporting the disk? **(2 pts)**

$2T = Mg \rightsquigarrow 1\text{pt}.$

$T = \frac{1}{2}Mg \rightsquigarrow 1\text{pt}.$

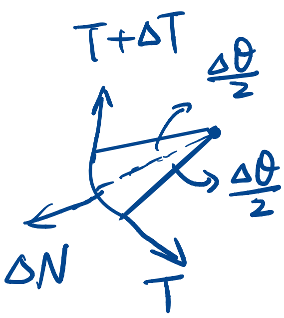
(b) Following part (a), is the magnitude of the tension along the string the same everywhere? YES/NO. **(1 pts)**

Yes

(c) Let there now be friction between the disk and the string, with coefficient μ. Is the magnitude of the tension along the string the same everywhere? YES/NO. **(1 pts)**

No

(d) Consider the friction between the disk and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point? **(5 pts)**



$\Delta N = (T+\Delta T)\sin\frac{\Delta\theta}{2} + T\sin\frac{\Delta\theta}{2}$

$= T\Delta\theta \rightsquigarrow 1\text{ pt}$

The $=$ equal sign is OK.

$(T+\Delta T)\cos\frac{\Delta\theta}{2} - T\cos\frac{\Delta\theta}{2} \le \mu\Delta N \rightsquigarrow 1\text{pt}.$

$\Delta T \le \mu T\Delta\theta \rightsquigarrow 1\text{ pt}.$

$\Rightarrow \int_{T_{min}}^{T_0}\frac{dT}{T} \le \int_0^{\pi/2}\mu\, d\theta \Rightarrow \ln\frac{T_0}{T_{min}} = \mu\frac{\pi}{2}$ 1pt

$\Rightarrow T_{min} = T_0 e^{-\mu\frac{\pi}{2}} \rightsquigarrow 1\text{pt}$

(e) Consistency check: What is the minimum tension T_{min} when $\mu \to \infty$? **(1 pts)**

when $\mu\to\infty$, $T_{min} = 0$ as expected.

12. Moment of inertia (rotational inertia) of the regular polygon:

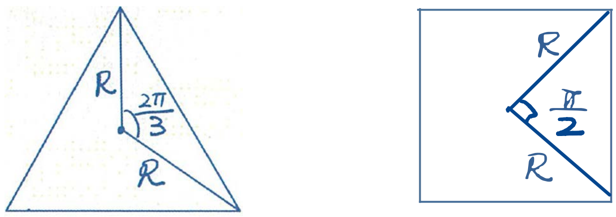
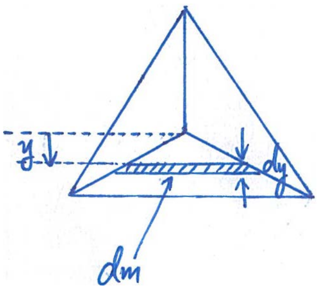


Fig. 9

(a) Find the moment of inertia of a regular triangle (as shown in above left figure) plate (with radius R and mass M) about an axis through its center perpendicular to the plate. (Hint: You may find the parallel axis theorem useful in solving this problem.) **(2 pts)**



$$\sigma = \frac{M}{\frac{3\sqrt{3}}{4}R^2}$$ (0.5pt)

$$dm = \sigma \cdot 2\sqrt{3}\, y\, dy$$ (0.5pt)

$$dI = \frac{1}{12}\cdot(2\sqrt{3}y)^2 dm + y^2 dm = \sigma 4\sqrt{3}\, y^3\, dy$$ (parallel axis theorem) (0.5pt)

$$I = 3\int_0^{\frac{R}{2}} 4\sqrt{3}\,\sigma\, y^3\, dy$$

$$= 12\sqrt{3}\,\frac{M}{\frac{3\sqrt{3}}{4}R^2}\left(\frac{y^4}{4}\Big|_0^{\frac{R}{2}}\right)$$

$$= \frac{1}{4}MR^2$$ (0.5pt)

Different methods with correct answer get full credit!

(b) Find the rotational inertia of a square plate with radius R and mass M as shown in above right figure) about an axis through its center perpendicular to the plate. (Hint: You may find the table on Page 1 useful and you can use the result without derivations.) **(2 pts)**

$$I = \frac{1}{12}M\left[(\sqrt{2}R)^2 + (\sqrt{2}R)^2\right]$$

$$= \frac{1}{3}MR^2$$ (2pts).

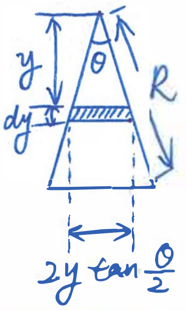
Direct calculation can also get full credit as long as students get

$$I = \frac{1}{3}MR^2$$

partial credit can be given accordingly.

(c) Find the rotational inertia of a regular polygon[1] with radius R and mass M about an axis through its center perpendicular to the plate. **(5 pts)**

$\theta = \frac{2\pi}{N}$



Polygon with N sides can be devided into N triangles.

(1pt)

$$\sigma = \frac{M}{N \cdot \frac{1}{2} R^2 \sin\theta} \qquad (1pt)$$

$$dm = \sigma \cdot 2y \tan\frac{\theta}{2}\, dy \qquad (1pt)$$

$$dI = \frac{1}{12}\left(2y \tan\frac{\theta}{2}\right)^2 dm + y^2 dm \quad \text{(parallel axis theorem)}$$

$$= \frac{1}{3} y^2 \tan^2\frac{\theta}{2}\, dm + y^2 dm. \qquad (1pt)$$

$$I_{total} = N \cdot I_\Delta = N \int_\Delta dI$$

$$= N \int_0^{R\cos\frac{\theta}{2}} \left(1 + \frac{1}{3}\tan^2\frac{\theta}{2}\right) y^2 \cdot \sigma \cdot 2y \tan\frac{\theta}{2}\, dy$$

$$= 2N\sigma \left(1 + \frac{1}{3}\tan^2\frac{\theta}{2}\right) \tan\frac{\theta}{2} \int_0^{R\cos\frac{\theta}{2}} y^3\, dy$$

$$= 2N \frac{M}{NR^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}} \left(1 + \frac{1}{3}\tan^2\frac{\theta}{2}\right) \tan\frac{\theta}{2} \cdot \frac{1}{4} R^4 \cos^4\frac{\theta}{2}$$

$$= \frac{1}{2} M R^2 \left(\cos^2\frac{\pi}{N} + \frac{1}{3}\sin^2\frac{\pi}{N}\right) \qquad (1pt)$$

Check: $N=3$, $I = \frac{1}{2}MR^2\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}MR^2$

$N=4$, $I = \frac{1}{2}MR^2\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}MR^2$

(d) What is the corresponding moment of inertia when one takes $N \to \infty$? (Hint: what does a regular polygon look like when $N \to \infty$?) **(1 pts)**

$N \to \infty \Rightarrow \cos\frac{\pi}{N} \to 1$, $\sin\frac{\pi}{N} \to 0$ (1pt)

$\Rightarrow I_{N\to\infty} = \frac{1}{2}MR^2$

(thin disk).

[1] A regular polygon is a polygon that is direct equiangular (all angles are equal) and equilateral (all sides have the same length).