MAT1002 Midterm Reference Solution (2022)

1. (i) F (ii) T (iii) T (iv) F (v) T

2. (i) ABC (ii) $\arccos\left(\frac{2}{3}\right)$ (or $\cos^{-1}\left(\frac{2}{3}\right)$).

(iii) $r = \frac{27}{4}\frac{\sin^2\theta}{\cos^3\theta}$ $\left(= \frac{27}{4}\tan^2\theta\sec\theta\right)$, $0 \le \theta \le \frac{\pi}{2}$.

(iv) $\frac{12}{5}\pi\left(8\sqrt{2}-7\right)$

$$
\begin{aligned}
A &= \int_0^1 2\pi(3-x)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi\int_0^1 (3-3t^2)\sqrt{(6t)^2+(6t^2)^2}\,dt \\
&= 2\pi\int_0^1 3(1-t^2)\,6t\sqrt{1+t^2}\,dt \\
&= 36\pi\int_0^1 (1-u)\sqrt{1+u}\,\frac{1}{2}\,du
\end{aligned}
$$

$u = t^2$
$du = 2t\,dt$

$$
= 18\pi\int_1^2 (2-v)\sqrt{v}\,dv
$$

$v = 1+u$
$dv = du$

$$
\begin{aligned}
&= 18\pi\left(\frac{4}{3}v^{\frac{3}{2}}-\frac{2}{5}v^{\frac{5}{2}}\right)\Bigg|_{v=1}^{2} \\
&= \frac{12}{5}\pi\left(8\sqrt{2}-7\right)
\end{aligned}
$$

No need to show.

3. (i) $\vec{r}'(t) = \left\langle \frac{1}{1+t^2}, 4e^{2t}, 8te^t + 8e^t \right\rangle$.

Want $\vec{r}'(t) = \lambda\langle 1,4,8\rangle$ for some λ.

Solve $\begin{cases} \frac{1}{1+t^2} = \lambda & ① \\ 4e^{2t} = 4\lambda & ② \\ 8e^t(t+1) = 8\lambda & ③ \end{cases}$.

②, ③ $\Rightarrow t+1 = e^t$. It's easy to check that $f(t) = e^t - t - 1$ has a unique minimum at $t=0$ and $f(0)=0$. Hence $t+1 = e^t \iff t=0$.

Easy to check that $\vec{r}'(0) = \langle 1,4,8\rangle$ works. So $t_0 = 0$.

(ii) $\vec{r}(0) = \langle 0,2,0\rangle$, so point is $P(0,2,0)$.

(iii) $\vec{v}(0) = \vec{r}'(0) = \langle 1,4,8\rangle$, so for $t \geq 0$, line of movement is

$$x = t, \quad y = 2+4t, \quad z = 8t, \quad t \geq 0$$

(Here t is indeed time)

Sub into $x+4y+8z=16$ yields

$$t + 8 + 16t + 64t = 16 \Rightarrow t = \frac{8}{81}.$$

Hence it will hit the plane at time $t = \frac{8}{81}$.

4. (i) $\vec{v}(t)-\vec{v}(0)=\int_0^t \vec{a}(u)\,du=\int_0^t \langle 2\sin u,\ 2\cos u,\ 0\rangle du$

$= \langle 2\cos u\big|_t^0,\ 2\sin u\big|_0^t,\ 0\rangle$

$= \langle 2-2\cos t,\ 2\sin t,\ 0\rangle$

$\because\ \vec{v}(0)=\langle -2,0,v_0\rangle$

$\therefore\ \vec{v}(t)=\langle -2\cos t,\ 2\sin t,\ v_0\rangle.$

Then $L_0=\int_0^{T_0}|\vec{v}(t)|\,dt=\int_0^{T_0}\sqrt{4\cos^2 t+4\sin^2 t+v_0^2}\,dt$

$=\sqrt{4+v_0^2}\;T_0.$

$\Rightarrow\ T_0=\dfrac{L_0}{\sqrt{4+v_0^2}}$

(ii) $\vec{r}(t)-\vec{r}(0)=\int_0^t \vec{v}(u)\,du=\int_0^t\langle -2\cos u,\ 2\sin u,\ v_0\rangle du$

$=\langle 2\sin u\big|_t^0,\ 2\cos u\big|_t^0,\ v_0 t\rangle$

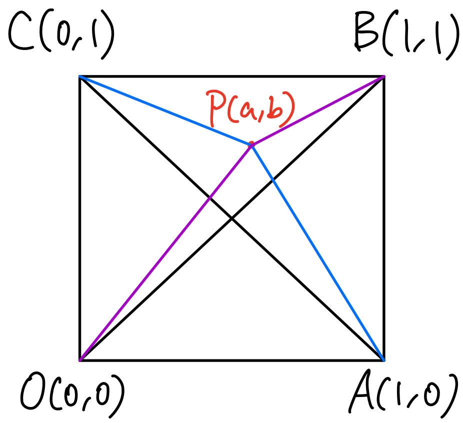
$=\langle -2\sin t,\ 2-2\cos t,\ v_0 t\rangle$

$\because\ \vec{r}(0)=\langle 0,-2,0\rangle$

$\therefore\ \vec{r}(t)=\langle -2\sin t,\ -2\cos t,\ v_0 t\rangle$

$\therefore\ \vec{r}(T_0)=\langle -2\sin T_0,\ -2\cos T_0,\ v_0T_0\rangle.$

5. (i) Consider the following diagram:



By triangle inequality

$\because |CP|+|PA| \geq |AC| \quad \therefore \sqrt{a^2+(1-b)^2}+\sqrt{(1-a)^2+b^2} \geq \sqrt{2}$. ①

$\because |OP|+|PB| \geq |OB| \quad \therefore \sqrt{a^2+b^2}+\sqrt{(1-a)^2+(1-b)^2} \geq \sqrt{2}$. ②

① + ② $\Rightarrow$ desired inequality.

(ii) $a = b = \frac{1}{2}$.

6. (i) Plug in $(1,0,1)$ to $x+z=k$ gives $k=2$.

Pick another point on ℓ, say $P_1(0,-1,2)$ (set $x=0$).

One possible set of parametric equations (with $P=(1,0,1)$ and direction $\overrightarrow{PP_1}=\langle -1,-1,1\rangle$) is

$$x=1-t,\quad y=-t,\quad z=1+t,\qquad t\in\mathbb{R}. \qquad (*)$$

(ii) Sub $(*)$ into $x-y+2z=0$:

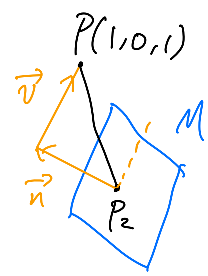
$$(1-t)+t+2(1+t)=0 \;\Rightarrow\; 3+2t=0 \;\Rightarrow\; t=-\frac{3}{2}.$$

Point is $(x,y,z)=(2.5,1.5,-0.5)$.

(iii) Let $P_2 := (2.5,1.5,-0.5)$ be on $M\cap\ell$.

Consider $P := (1,0,1)$ on ℓ. $\langle 2,1,-0.5\rangle$

Project $\overrightarrow{P_2P}$ onto plane normal $\vec{n} := \langle 1,-1,2\rangle$:



$$\operatorname{proj}_{\vec{n}}\overrightarrow{P_2P}=\frac{\overrightarrow{P_2P}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}=\frac{(-1.5)+1.5+3}{6}\vec{n}=\frac{1}{2}\vec{n}.$$

Then $\vec{v} := \overrightarrow{P_2P}-\frac{1}{2}\vec{n}$ is a direction of the projected line.

$$\vec{v}=\langle -1.5,-1.5,1.5\rangle-\langle 0.5,-0.5,1\rangle=\langle -2,-1,0.5\rangle.$$

Line of projection is $x=2.5-2t,\ y=1.5-t,\ z=-0.5+0.5t,$

$t\in\mathbb{R}$.

7. (a) · If $c<1$, then $\lim_{n\to\infty} \frac{a}{b+c^n} = \frac{a}{b} \neq 0$, so series diverges.

· If $c=1$, then $\lim_{n\to\infty} \frac{a}{b+c^n} = \frac{a}{b+1} \neq 0$, so series diverges.

· If $c>1$, then $0 < \frac{a}{b+c^n} \leq \frac{a}{c^n}$.

Since $\sum_{n=1}^{\infty} \frac{a}{c^n} = a \sum_{n=1}^{\infty} \left(\frac{1}{c}\right)^n$ converges as a geometric series ($\left|\frac{1}{c}\right| < 1$), original series converges.

(b) Consider $\sum_{n=1}^{\infty} \frac{48e^n + n^\pi}{n! + (\ln n)^2} =: \sum_{n=1}^{\infty} a_n$.

Then $0 < a_n \leq \underbrace{\frac{48e^n}{n!}}_{b_n} + \underbrace{\frac{n^\pi}{n!}}_{c_n}$.

Since $\frac{b_{n+1}}{b_n} = \frac{48e^{n+1}}{(n+1)!} \cdot \frac{n!}{48e^n} = \frac{e}{n+1} \to 0$ as $n\to\infty$

and $\frac{c_{n+1}}{c_n} = \frac{(n+1)^\pi}{(n+1)!} \cdot \frac{n!}{n^\pi} = \frac{1}{n+1}\left(\frac{n+1}{n}\right)^\pi \to 0 \cdot 1 = 0$ as $n\to\infty$,

$\sum(b_n + c_n) = \sum b_n + \sum c_n$ converges by ratio test.

By comparison test, $\sum a_n$ converges. Hence

$\sum (-1)^n a_n$ converges (absolutely).

(c) Consider $f(x) := \dfrac{1}{x \ln x (\ln(\ln x))^{1+\alpha}}$, positive, continuous, and decreasing on $[3, \infty)$.

$u = \ln x$
$du = \dfrac{dx}{x}$

$$\int_3^b \frac{1}{x \ln x (\ln(\ln x))^{1+\alpha}}\, dx = \int_{\ln 3}^{\ln b} \frac{du}{u (\ln u)^{1+\alpha}}$$

$w = \ln u$
$dw = \dfrac{du}{u}$

$$= \int_{\ln\ln 3}^{\ln\ln b} \frac{dw}{w^{1+\alpha}}$$

$$= -\frac{1}{\alpha} \frac{1}{w^\alpha} \Big|_{w=\ln\ln 3}^{\ln\ln b}$$

$$= \frac{1}{\alpha}\left[\frac{1}{(\ln\ln 3)^\alpha} - \frac{1}{(\ln\ln b)^\alpha}\right] \xrightarrow{b\to\infty} \frac{1}{\alpha}\frac{1}{(\ln\ln 3)^\alpha}.$$

By the integral test, series converges since $\int_3^\infty f(x)dx$ converges.

8. Let $S(x) := \sum_{n=0}^{\infty} \frac{x^n}{\underbrace{(n+1)3^{n+1}}_{a_n}}$

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+2)3^{n+2}} \cdot \frac{(n+1)3^{n+1}}{|x|^n} = |x|\left(\frac{n+1}{n+2}\right)\frac{1}{3}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\frac{1}{3} \begin{cases} <1, & \text{if } |x|<3 \\ >1, & \text{if } |x|>3 \end{cases}.$$

By ratio test, Series converges on $(-3,3)$.

For $x=3$, Series $= \sum_{n=0}^{\infty} \frac{1}{3}\frac{1}{n+1}$ diverges (harmonic).

For $x=-3$, Series $= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}\frac{1}{3}$ Converges (Alt. harmonic).

Hence, Series Converges only for $-3 \le x < 3$.

(b) For $x \in (-3,3)$, Convergence is absolute;

for $x=-3$, Convergence is Conditional.

(c) Let $x \in (-3,3)$. Then

$$\sum_{n=0}^{\infty} \frac{x^n}{\underbrace{(n+1)3^{n+1}}_{a_n}}$$

$$xS(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)3^{n+1}}$$

$$\Rightarrow (xS(x))' = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \frac{1}{3} = \frac{1}{3}\frac{1}{1-\frac{x}{3}} = \frac{1}{3-x}.$$

$$\Rightarrow \quad xS(x) = \int_0^x (tS(t))' dt = \int_0^x \frac{1}{3-t} dt = -\ln(3-t)\Big|_0^x$$

$$= \ln 3 - \ln(3-x) = \ln\frac{3}{3-x}.$$

$$\left(\text{or } = -(\ln(3-x) - \ln 3) = -\ln\frac{3-x}{3} = -\ln\left(1-\frac{x}{3}\right). \right)$$

$$\Rightarrow \quad S(x) = \frac{\ln\left(\frac{3}{3-x}\right)}{x} \quad \text{if} \quad x \neq 0.$$

If $x=0$, then $S(0) = \frac{1}{3}$ is clear.

If $x=-3$, then

$$S(-3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} \frac{1}{3} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \frac{\ln 2}{3}.$$

Hence

$$S(x) = \begin{cases} \frac{\ln\left(\frac{3}{3-x}\right)}{x}, & \text{if } x \in [-3,3) \setminus \{0\}; \\ \frac{1}{3}, & \text{if } x = 0. \end{cases}$$

9. Let $f(x) := \dfrac{e^{x^2} + \frac{x}{2} - \sqrt{1+x}}{2x\cos x - \arctan x - \ln(1+x)}$.

Then $f(x) = \dfrac{1+x^2+O(x^4)+\frac{x}{2}-1-\frac{1}{2}x+\frac{1}{8}x^2+O(x^3)}{2x(1+O(x^2))-x+O(x^3)-x+\frac{1}{2}x^2+O(x^3)}$

$$= \frac{\frac{9}{8}x^2+O(x^3)+O(x^4)}{2xO(x^2)+O(x^3)+\frac{1}{2}x^2+O(x^3)},$$

where big-Oh is used as $x \to 0$.

Then $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\frac{9}{8}x^2+O(x^3)+O(x^4)}{2xO(x^2)+O(x^3)+\frac{1}{2}x^2+O(x^3)}$

$$= \lim_{x\to 0} \frac{\frac{9}{8}+O(x)+O(x^2)}{2O(x)+O(x^3)+\frac{1}{2}+O(x^3)}$$

$$= \frac{\frac{9}{8}+0+0}{2\cdot 0+0+\frac{1}{2}+0} = \frac{9}{4}.$$

10. (a) $\binom{\frac{1}{5}}{0} = 1$, $\binom{\frac{1}{5}}{1} = \frac{1}{5}$, $\binom{\frac{1}{5}}{2} = \dfrac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2} = \dfrac{-2}{25}$,

$$\binom{\frac{1}{5}}{3} = \frac{\frac{1}{5}\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)}{3!} = \frac{6}{125}$$

First four terms are $1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3$.

(b) $$\sqrt[5]{1.8} = (1+0.8)^{\frac{1}{5}} = \sum_{n=0}^{\infty} \binom{\frac{1}{5}}{n} (0.8)^n = 1 + \underbrace{\sum_{n=1}^{\infty} \binom{\frac{1}{5}}{n} (0.8)^n}_{\sum a_n}.$$

Note that $\sum a_n$ is alternating.

Since $$a_4 = \binom{\frac{1}{5}}{4}(0.8)^4 = \binom{\frac{1}{5}}{3}\frac{\left(-\frac{14}{5}\right)}{4}(0.8)^4 = \frac{6}{125} \cdot \frac{-14}{20} \cdot (0.8)^4$$

$$|a_n| > 0.01$$

$$a_5 = \frac{6}{125} \cdot \frac{-14}{20} \cdot \frac{\left(-\frac{19}{5}\right)}{5} (0.8)^5 < 0.009 < 0.01,$$

by alternating series approximation, we need to take five terms at least,

i.e., take $$(1.8)^{\frac{1}{5}} \approx \sum_{n=0}^{4} \binom{\frac{1}{5}}{n} (0.8)^n.$$