6. Exercise sheet

Due by: Friday, 31 May 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
You are expected to solve tasks 1–2 and part (a) of task 3 by hand

Problem 1. A very precise model of the weather models it as either sunny or rainy, i.e., $S = \{\text{sunny}, \text{rainy}\}$. The probability of having a sunny day after another sunny day is $\frac{2}{3}$. With a probability of $\frac{1}{3}$ it will rain on the next day. The probability that a rainy day is followed by another rainy day is $\frac{4}{5}$. We can express this with the following matrix

$$\mathrm{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{4}{5} \end{pmatrix}.$$

(a) Assume the weather today is sunny. What is the probability that it will be rainy on the day after tomorrow?

(b) Assume the weather today is sunny. Like a true meteorologist you wait for an infinitely long time and write down the weather every day. What is the relative frequency of sunny days?

(c) What is the invariant measure for the given Markov chain? Does the chain converge to its invariant measure?

Problem 2. Consider the Markov chain that takes values on $S = \{1, 2, 3, 4\}$. The transition probabilities p_{ij} from state j to state i are given as a matrix $P = [p_{ij}]_{i,j=1}^{4}$ such that

$$P = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

(a) Let $X_0 = 1$. Compute the probability mass function of X_2.

(b) Does the probability distribution of X_n converge as n goes to infinity? If so, find the limit. If not, explain why.

Problem 3. In this exercise, your task is to implement the Euler-Maruyama method. It is very similar to the Euler scheme from the first exercise sheet, but this time we are interested in stochastic differential equations (SDEs) instead of ODEs.

Consider a continuous time process $\{X_t \in \mathbb{R} : t \geq 0\}$ governed by the SDE:

$$\mathrm{d}X_t = -X_t \,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t, \quad X_0 = x_0,$$

or, in its integral form,

$$X_t = x_0 - \int_0^t X_s \, \mathrm{d}s + \sqrt{2}\, B_t,$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion. In order to apply the Euler scheme, consider a time-discretization Δt and approximate

$$X_{t+\Delta t} - X_t \approx -X_t \Delta t + \sqrt{2}\big(B_{t+\Delta t} - B_t\big).$$

The idea is to exploit the fact that the distribution of $B_{t+\Delta t} - B_t$ is Gaussian with mean 0 and variance Δt. Hence we discretize the SDE by

$$x_{n+1} = x_n - x_n \Delta t + \sqrt{2\Delta t}\, \xi_n,$$

where $\xi_n \sim \mathrm{N}(0, 1)$ for $n = 0, 1, \ldots$.

(a) What are the distributions of x_1 and x_2?

(b) Set $\Delta t = 0.01$ and $x_0 = 2$. Run 10 000 Monte Carlo simulations of the above. Save and plot histogram of the empirical distributions at times $t = 0$, $t = 0.5$, $t = 1$ and $t = 10$.

(c) How do the distributions look at the different times? Can you tell if it converges to something? You may run the simulation further in time if necessary.