Sheet 03

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Problem 1

ⓐ Given,

the two dimensional Gaussian PDF $n(z; \tilde{z}, P)$

$$= \frac{1}{2\pi\sqrt{|\det P|}} \exp\left(-\tfrac{1}{2}(z-\bar{z})^T P^{-1}(z-\bar{z})\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_c^2}} \exp\left(-\frac{1}{2\sigma_c^2}(x_1-\bar{x}_c)^2\right) \frac{1}{\sqrt{2\pi\sigma_{22}^2}} \exp\left(-\frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right)$$

This is equivalent to assuming that the two variables x_1 and x_2 are independent, with x_1 having a variance of σ_c^2 and x_2 having a variance of σ_{22}^2.
We need to equate the coefficients of the exponentials in the two expressions. This gives us:

$$\frac{1}{2}(z-\bar{z})^T P^{-1}(z-\bar{z}) = \frac{1}{2\sigma_c^2}(x_1-\bar{x}_c^2) + \frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2$$

By comparing the terms, we can see

$$\sigma_c^2 = \text{variance of } x_1$$

$$\therefore \sigma_c^2 = \sigma_{11}^2 - \frac{\sigma_{12}^2}{\sigma_{22}^2}\sigma_{21}^2$$

| using the formula for the variance of conditional Gaussian distribution

In the given equation, x_1 and x_2 are independent variables since the PDF of x_1 and x_2 is expressed as a ~~joint~~ product of two seperate Gaussian PDFs.
In a multivariate Gaussian distribution, if the joint PDF can be expressed as a product of the marginal PDFs, then the variables are statistically independent.

So, $\bar{x}_c$ = mean of the variable $x_1 = \bar{x}_1$

ⓑ The corresponding formulas for conditional PDF:

$$\pi_{x_2}(x_2|x_1) = \frac{\pi_{x_1x_2}(x_1, x_2)}{\pi_{x_1}(x_1)}$$

$$\pi_{x_1x_2}(x_1, x_2) = \pi_{x_1}(x_1|x_2)\,\pi_{x_2}(x_2) = \pi_{x_2}(x_2|x_1)\,\pi_{x_1}(x_1)$$

and $\pi_{x_1}(x_1) = E[\pi_{x_1}(x_1|x_2)]$

marginal:

$$\pi_{x_1}(x_1) = \int_{-\infty}^{\infty} \pi_{x_1x_2}(x_1, x_2)\,dx_2$$

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Exercise 3

Problem 2.

(2) First we find ~~$\pi_{x_2 t}$~~ $\pi_{X_2}(x_2)$:

$$\pi_{X_2}(x_2) = \int_R \pi_{(X_1, X_2)}(x_1, x_2)\, dx_1$$

$$= \frac{1}{Z} e^{-x_2^2} \int_R e^{-x_1^2(1+x_2^2)}\, dx_1$$

$$= \frac{1}{Z} e^{-x_2^2} \frac{\sqrt{\pi}}{\sqrt{(1+x_2^2)}}$$

where the integral is solved using polar co-ordinates, Now we can compute the conditional expectation.

$$E[X_1^2 X_2 | X_2 = a] = a E[X_1^2 | X_2 = a]$$

$$= a \int_R x_1^2 \pi_{X_1|X_2=a}(x_1|a)\, dx_1$$

$$= a \int_R \frac{x_1^2 \pi_{(X_1, X_2)}(x_1, a)}{\pi_{X_2}(a)}\, dx_1$$

$$= a \int_R x_1^2 \frac{1}{Z} e^{-x_1^2 - a^2 - x_1^2 a^2} \frac{Z\sqrt{1+a^2}\, e^{a^2}}{\sqrt{\pi}}\, dx_1$$

$$= a \int_R x_1^2 e^{-x_1^2(1+a^2)} \frac{\sqrt{1+a^2}}{\sqrt{\pi}}\, dx_1$$

$$= \frac{a}{2(1+a^2)}$$

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Sheet (3) MD Borhan Uddin

Problem (3):

Given two probability densities (p) and (q), defined on $\mathbb{R}^N \to \mathbb{R}$. Now, we have to prove inequality of Hellinger distance.

$$d_{Hell}(p,q) = \left(1/2 \int_{\mathbb{R}^N} \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx\right)^{1/2}$$

And, Kullback-Lieber divergence,

$$D_{KL}(p \| q) = \int_{\mathbb{R}^N} \log\left(\frac{p(x)}{q(x)}\right) p(x)\, dx$$

First of all,
we have a different form of hellinger distance expression,

$$\Rightarrow d_{Hell}(p,q)^2 = \int_{\mathbb{R}^n} \left(1 - \sqrt{\frac{q(x)}{p(x)}}\right) p(x)\, dx$$

Now, Applying inequality $\left(1 - \sqrt{x} \leq -\frac{1}{2}\log x \text{ for } x > 0\right)$ to above expression we get,

$$\Rightarrow d_{Hell}(p,q)^2 \leq \int_{\mathbb{R}^N} \left(-\frac{1}{2}\log\left(\frac{q(x)}{p(x)}\right)\right)^2 p(x)\, dx$$

$$\Rightarrow d_{Hell}(p,q)^2 \leq 1/4 \int_{\mathbb{R}^N} \left(\log\left(\frac{p(x)}{q(x)}\right)\right)^2 p(x)\, dx \quad \text{(i)}$$

[Simplify Right hand side]

Now, Recalling back kullback-leiber divergence.

$$D_{KL}(P\|q) = \int_{\mathbb{R}^N} \log\left(\frac{p(x)}{q(x)}\right) p(x)\,dx$$

And, Expression (i). $\left[d_{Hell}(p,q)^2 \leq \frac{1}{4}\int_{\mathbb{R}^N} \left(\log\left(\frac{p(x)}{q(x)}\right)\right)^2 p(x)\,dx\right]$,

proved that, Right hand of the inequality involving the Hellinger distance is half of kullback-leiber divergence:

$$d_{Hell}(p,q)^2 \leq \frac{1}{2} D_{KL}(P\|q).$$

Hence, Hellinger distance squared is bounded by half of the kullback-leiber divergence.