10. Exercise sheet

Due by: Friday, 28 June 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
Next week's exercise sheet will be a mock exam. You can return your solutions to the mock exam to receive bonus points in case you have not received cumulative 50% *points from the 10 exercise sheets.*

In addition, please note that all participants are required to present at least one solution in the exercise sessions. The exercise sessions on July 8–9 will be the final opportunity to fulfill this criterion for passing the course.

Problem 1. Let us revisit task 3 from the 9th exercise sheet, however, this time we will use the ensemble Kalman filter to solve the problem.

We wish to track the state $x_k := \begin{bmatrix} p_k \\ v_k \end{bmatrix} \in \mathbb{R}^2$ of a moving particle. The first component p_k corresponds to the position of the particle while the second component v_k is its velocity at time $k = 0, 1, 2, \ldots$. You may assume that you know the initial state of the particle perfectly: $x_0 = \mathbb{E}[x_0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ and $C_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. The evolution model for the particle is given by $M = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, with time step $\Delta t = 0.01$, and the innovation term is given by $\Sigma = \begin{bmatrix} \frac{1}{4}(\Delta t)^4 + 10^{-10} & \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 & (\Delta t)^2 + 10^{-10} \end{bmatrix}$. Note also that the prior distribution $x_0 \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \in \mathbb{R}^{2\times 2}$. Meanwhile, we only measure the location of the particle so the observation model is given by $H = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2}$ and the observational noise variance is assumed to be $\Gamma = [1] \in \mathbb{R}^{1\times 1}$.

Implement the ensemble Kalman filter with ensemble size $N = 100$ for this model problem and plot the (approximate) filtered positions $(t_k, \mathbb{E}[p_k | y_1, \ldots, y_k])_{k=1}^{2000}$ and velocities $(t_k, \mathbb{E}[v_k | y_1, \ldots, y_k])_{k=1}^{2000}$ as a function of time $t_k = k\Delta t$, $k = 1, \ldots, 2000$. To simulate the noisy measurements, assume that the true trajectory of the particle is given by the function $x(t) = 0.1(t^2 - t)$ for $t \in [0, 20]$, and the measurements are given by $y_k = x(t_k) + \eta_k$, where $\eta_k \sim \mathrm{N}(0, \Gamma)$ is additive i.i.d. noise for $k = 1, \ldots, 2000$.

Remark. The covariance Σ in last week's exercise sheet is positive semidefinite, so in order to avoid problems sampling from the distribution $\mathrm{N}(0, \Sigma)$, the definition has been altered slightly. Note also that the prior distribution $x_0 \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$ is degenerate, so you should use the initial ensemble $x_0^{(j)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $j = 1, \ldots, N$.

Problem 2. The model in this exercise is

$$Z^{n+1} = Z^n + \delta t\, dZ^n + \delta t\, b + \sqrt{2\delta t}\, \Xi^n$$

with $d = -2$, $b = 1$, $\delta t = 0.01$ and $\Xi^n \sim \mathrm{N}(0, 1)$. Generate a reference trajectory

$\{Z^k\}_{k=1}^N$ starting at $Z_0 = 10$ with $N = 1000$. Generate the observations $\{Y^k\}_{k=1}^N$ at the time instances $t_k = k\,\delta t$ by

$$Y^k = Z^k + \Sigma^k$$

with $\Sigma^k \sim \mathrm{N}(0, 1)$. All noise processes are assumed to be independent.

(a) Run the Kalman filter with initial distribution $Z_0 \sim \mathrm{N}(1, 1)$. Plot the analysis mean and the observations into one plot.

(b) Run the Kalman filter but with a misspecified model where you set $d = -0.4$. The observations $\{Y^k\}_{k=1}^N$ are still the same ones as before, i.e., they are generated with $d = -2$. Plot the analysis mean and the observations into one plot. How does the analysis mean differ from the plot you obtained in part (a)?

(c) Compute the root-mean-square error for the analysis means you obtained in parts (a) and (b). How do you interpret the results?