9. Exercise sheet

Due by: Friday, 21 June 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
Implementational details about the Kalman filter algorithm can be found in Chapter 6 of the textbook as well as the note "A brief note on Bayesian and Kalman filtering" on the course Moodle page.

You are expected to solve task 1 and part (c) of task 2 by hand. The other tasks can be solved using Python.

Problem 1. In this exercise, we will do a very simple filtering and smoothing step by hand. The forward map is given by

$$X_{n+1} = \frac{1}{2}X_n + 1 + \Xi_n,$$

where $\Xi_n \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$. The observation operator is given by

$$Y_n = X_n + \sqrt{2}\Sigma_n$$

with $\Sigma_n \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$. Assume $X_0 \sim \mathrm{N}(-1,2)$. All noise processes are independent.

(a) Prediction step: compute the distribution of X_1.

(b) Filtering step: compute the distribution of X_1 conditioned on $Y_1 = 2$.

(c) Smoothing step: compute the distribution of X_0 conditioned on $Y_1 = 2$.

(d) Now we want to write this down in pseudocode. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim \mathrm{N}(m,1)$$

with the same observation and noise model as above. You also observe that $Y_1 = y$. Write a pseudocode that, given the inputs α, β, m and y, will return as output the distribution of

i) X_1

ii) X_1 conditioned on $Y_1 = y$, and

iii) X_0 conditioned on $Y_1 = y$.

Problem 2. In this exercise, we will implement a particle filter for the previous model.

(a) Prediction step: Draw $N = 1000$ independent samples of X_0^i and Ξ_0^i for $i = 1, \ldots, N$. Compute X_1^i for each sample path.

(b) Plot the histogram of particles and compare with the theoretical PDF you obtained in Problem 1.

The exercises continue on the next page.

(c) Obtain (by hand) the formula for the likelihood

$$\tilde{w}_i := \pi_{Y_1|X_1}(y \mid X_1^i).$$

Now implement it for the observation $Y_1 = 2$. You do not have to compute the common constant factor as it will be normalized in the next step.

(d) Filtering step: compute the normalized weight w_i for $i = 1, \dots, N$, where

$$w_i := \frac{\tilde{w}_i}{\sum_{i=1}^N \tilde{w}_i}.$$

(e) Compute the weighted mean and the variance

$$\bar{m}_1 := \sum_{i=1}^N w_i X_1^i, \quad \bar{V}_1 := \sum_{i=1}^N w_i (X_1^i - \bar{m}_1)^2.$$

(f) Compute the "effective sample size" defined by

$$ESS := \frac{1}{\sum_{i=1}^N w_i^2}.$$

Problem 3. Consider the *evolution-observation model*

$$x_{j+1} = M x_j + \xi_{j+1}, \quad \xi_{j+1} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \Sigma), \tag{1}$$

$$y_{j+1} = H x_{j+1} + \eta_{j+1}, \quad \eta_{j+1} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \Gamma). \tag{2}$$

We are interested in finding the probability distribution of $x_{j+1}|y_1, \dots, y_{j+1}$ (i.e., we wish to estimate the state at some future time step $j+1$ given measurements at all previous time steps $1, 2, \dots, j+1$).

Your task is to implement the Kalman filter for the following model problem:

We wish to track the state $x_k := \begin{bmatrix} p_k \\ v_k \end{bmatrix} \in \mathbb{R}^2$ of a moving particle. The first component p_k corresponds to the position of the particle while the second component v_k is its velocity at time $k = 0, 1, 2, \dots$. You may assume that you know the initial state of the particle perfectly: $x_0 = \mathbb{E}[x_0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ and $C_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. The evolution model for the particle is given by $M = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, with time step $\Delta t = 0.01$, and the innovation term is given by $\Sigma = \begin{bmatrix} \frac{1}{4}(\Delta t)^4 & \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 & (\Delta t)^2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Meanwhile, we only measure the location of the particle so the observation model is given by $H = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2}$ and the observational noise variance is assumed to be $\Gamma = [1] \in \mathbb{R}^{1\times 1}$.

The exercises continue on the next page.

Implement the Kalman filter for this model problem and plot the filtered positions $(t_k, \mathbb{E}[p_k|y_1, \ldots, y_k])_{k=1}^{2000}$ and velocities $(t_k, \mathbb{E}[v_k|y_1, \ldots, y_k])_{k=1}^{2000}$ as a function of time $t_k = k\Delta t$, $k = 1, \ldots, 2000$. To simulate the noisy measurements, assume that the true trajectory of the particle is given by the function $x(t) = 0.1(t^2 - t)$ for $t \in [0, 20]$, and the measurements are given by $y_k = x(t_k) + \eta_k$, where $\eta_k \sim \mathrm{N}(0, \Gamma)$ is additive i.i.d. noise for $k = 1, \ldots, 2000$.