## Problem 1

a) Let's consider the case where x1 and xe are jointly Gaussian with

$$\mathbb{E}[(X_1, X_2)] = (\overline{\chi}_1, \overline{\chi}_2)$$
 and  $Var[(X_1, X_2)] = \begin{pmatrix} \sigma_1^2 & c \\ c & \sigma_2^2 \end{pmatrix}$  where  $c = covariance$ 

.. C = COV [x1, x2] = E[X1 x2]-E[X1] E[X2]

Now, considering the square of the L2-wassenstein distance for the sake of simplicity:

$$(W_2(\pi x_1, \pi x_2))^2 = \inf_{c} \mathbb{E}[(x_1 - x_2)^2]$$

We know,  $Var(x) = E[x^2] - (E[x])^2$ 

$$E[X^{2}] = (E[X])^{2} + Var(X) = X_{1}$$

$$E[X_{1}^{2}] = \overline{\chi}_{1}^{2} + \sigma_{1}^{2} \mid C = E[X_{1} \times 2] - E[X_{1}][X_{2}]$$
and 
$$E[X_{2}^{2}] = \overline{\chi}_{2}^{2} + \sigma_{2}^{2} \mid E[X_{1} \times 2] = C + \overline{\chi}_{1} \overline{\chi}_{2}$$

From 
$$(1) \Rightarrow (W_2(\pi x_1, \pi x_2))^2 = \inf_{C} (\bar{\chi}_1^2 + \sigma_1^2) - \underbrace{\pm e}_{C} + (\bar{\chi}_2^2 + \sigma_2^2) - \underbrace{2(c + \bar{\chi}_1 \bar{\chi}_2)}_{C})$$

= int 
$$(\bar{\chi}_1 - \bar{\chi}_2)^2 + \sigma_1^2 + \sigma_2^2 - 2c$$

The maximum eligible value of c is given by the cauchy-schwarz inequality:

(Cov[x1,x2])2 ≤ var [x,] var[x2] > c ≤ 0102

Therefore,  $(W_2(\pi x_1, \pi x_2))^2 = (\bar{\chi}_1 - \bar{\chi}_2)^2 + \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2$  $= (\bar{\chi}_1 - \bar{\chi}_2)^2 + (\sigma_1 - \sigma_2)^2$ 

(b) The Kullback-Leibler (KL) divergence of 
$$\pi x_1$$
 from  $\pi x_2$  is given by:
$$D_{KL}(\pi x_1 || \pi x_2) := \int_{R} log\left(\frac{\pi x_1(x)}{\pi x_2(x)}\right) \pi x_1(x) dx$$

$$P_{KL}(\pi_{X_1} \parallel \pi_{X_2}) := \int_{R} \log \left( \frac{\pi \chi_1(x)}{\pi \chi_2(x)} \right) \pi \chi_1(x) dx$$

$$= \int_{R} \left[ \log \pi \chi_1(x) - \log \pi \chi_2(x) \right] \pi \chi_1(x) dx - 0$$

Fon Gaussian distribution, PDF  $\Re X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\chi - \bar{\chi})^2}{2\sigma^2}\right)$ 

Substituting the formula for the Kt divengence gives: DKL (+x, 11 +x2) = f Taking Logarithm on both sides, we get,

 $\log \pi \times (x) = \log \left( \frac{1}{\sqrt{2\pi} \delta^2} \exp \left( -\frac{(x-\overline{x})^2}{2\delta^2} \right) \right)$ 

 $\pi x_1(x) dx$ 

$$= \log\left(\frac{1}{\sqrt{2\pi\sigma^{2}}}\right) + \log\left(\exp\left(-\frac{(x-\bar{x})^{2}}{2\sigma^{2}}\right)\right)$$

$$= -\frac{1}{2}\log\left(2\pi\sigma^{2}\right) - \frac{(x-\bar{x})^{2}}{2\sigma^{2}}$$

$$\leq 0, 0 \Rightarrow \int_{\mathbb{R}} \left[-\frac{1}{2}\log\left(2\pi\sigma_{1}^{2}\right) - \frac{(x-\bar{x})^{2}}{2\sigma_{1}^{2}} + \frac{1}{2}\log\left(2\pi\sigma_{2}^{2}\right) + \frac{(x-\bar{x})^{2}}{2\sigma_{1}^{2}}\right]$$

 $= \mathbb{E} \left[ \frac{1}{2} \log \frac{\sigma_{2}^{2}}{\sigma_{1}^{2}} - \frac{1}{2} \sigma_{1}^{2} (x_{1} - \bar{x}_{1})^{2} + \frac{1}{2} \sigma_{2}^{2} (x_{1} - \bar{x}_{2})^{2} \right]$ Here, the first term is constant, and the second term

Here, the first term is constant, is the variance of 
$$X_1$$
, that is,
$$\frac{1}{2} = \Gamma(X_1 - \overline{X}_2)^2 = 1$$

For the third term,

 $\frac{1}{2\sigma_{i}^{2}} \mathbb{E}\left[\left(X_{1} - \bar{X}_{1}\right)^{2}\right] = \frac{1}{2\sigma_{i}^{2}} \cdot \sigma_{1}^{2} = \frac{1}{2}$ 

$$E[(x_1 - \bar{x}_2)^2] = E[x_1^2 - 2\bar{x}_2 x_1 + \bar{x}_1^2]$$

$$= (\sigma_1^2 + \bar{x}_1^2) - 2\bar{x}_1\bar{x}_2 + \bar{x}_2^2$$

$$= \sigma_1^2 + (\bar{x}_1^2 - \bar{x}_2^2)^2$$

Collecting all three terms, we conclude that,

 $D_{KL}(\pi x_1 || \pi x_2) = \frac{1}{2} \left[ \log \frac{\sigma_1^2}{\sigma_1^2} + \frac{\sigma_1^2 + (\bar{x}_1 - \bar{x}_2)^2}{\sigma_2^2} - 1 \right]$ 

$$= \frac{1}{2} \left[ \log \frac{\sigma_1^2}{\sigma_1^2} + \frac{\sigma_1^2}{\sigma_1^2} + \frac{(\bar{\chi}_1 - \bar{\chi}_2)^2}{\sigma_1^2} - 1 \right]$$

Note that this value becomes zero, if  $\bar{\chi}_1 = \bar{\chi}_2$  and  $\sigma_1^2 = \sigma_2^2$ 

## Problem 2(a):

Given the sels of,  $X_1 = \frac{1}{3}a_1 = 1$ ,  $a_1 = 2$ ,  $a_2 = 3$   $x_2 = \frac{1}{3}b_1 = 1.5$ ,  $b_2 = 2$ ,  $b_3 = -1$ and,  $p(a_i) = p(b_i) = \frac{1}{3}$ , where  $a_1 = 1.2.13$ .  $T = [t_i]_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$   $(t_i)_{\geq 0}$ .

To save this problem, we assume that we have three water bottle that filled with 1/3 unit of water. From the baseline the bottle heights are as, az, az, az, now, we target to re distribute the water into another sets of water bottle at height bi, bz, bz. The required energy required to move a unit from one bottle to Another is proportion of square of height difference. To find a strategy that minimizes energy Consumption, if Dij energy consumption per unit from ay to by, Dij can be expressed by following marrix

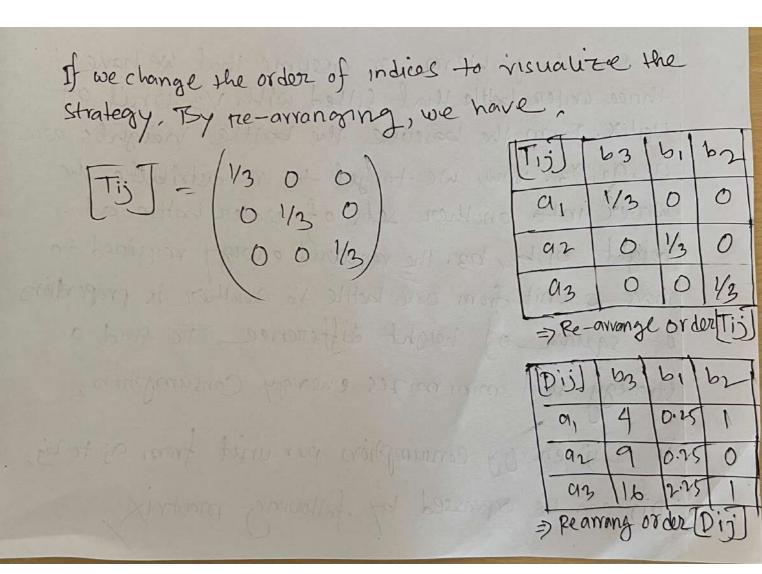
In the third column, the cost from anywhere to be is higher in every seenarios. So, from here we can start reavenge trick. We want to fill be as much as from a, and if it not pissible, the we may use water from a 2 and 20 on.

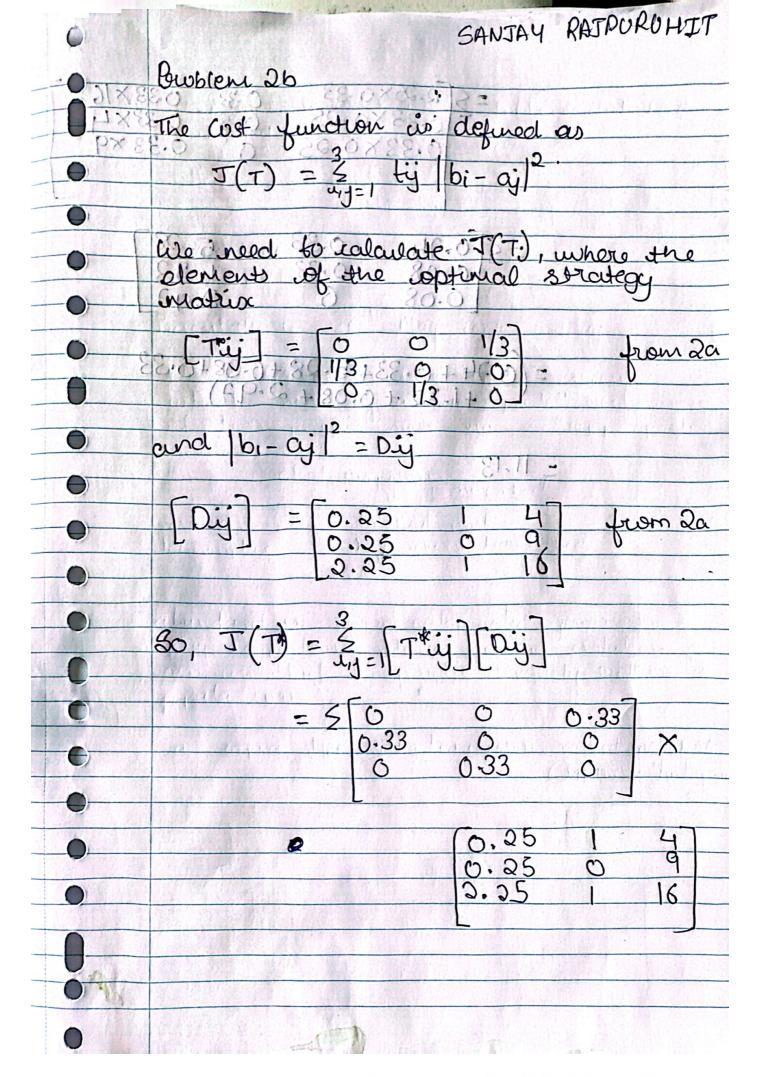
In the Question, we have exactly 1/3 unid in a and exactly 1/3 will be transferred into 63, 50,

NOW, we repeat for the rest of bothles. Now cost to fill by dominates one required to fill by we optimize by first by Assign 1/2 from any to by, namely \$12=1/2.

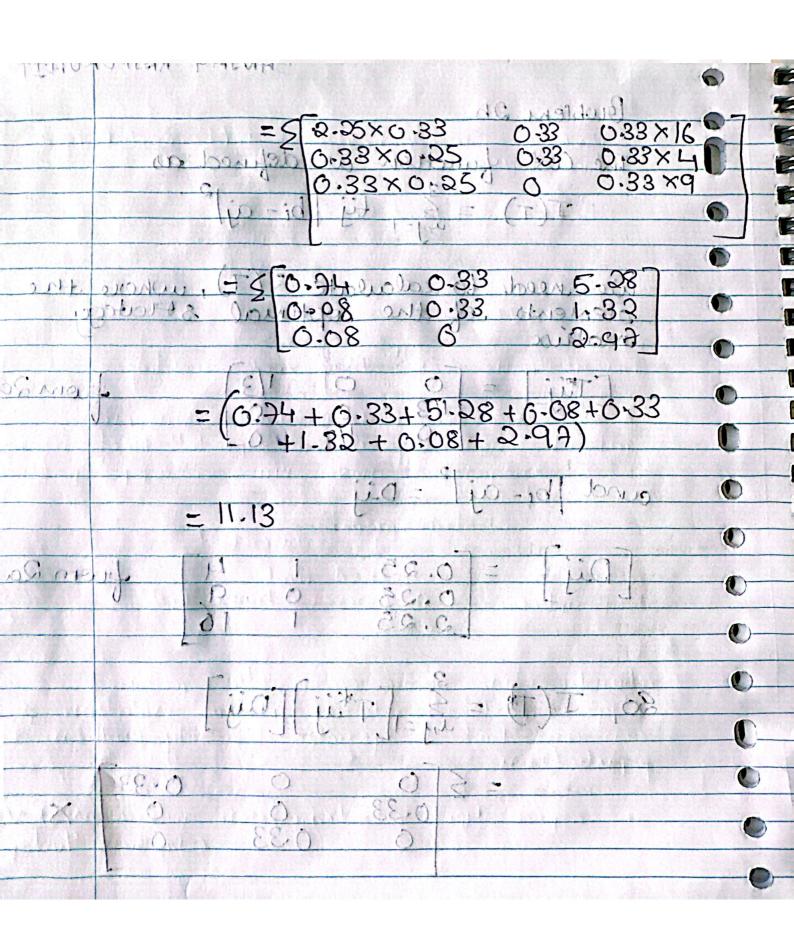
So, the resulting optimal strategy is,

Tist = (1/3 0 0).





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