①

Sheet (5)

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problem (1):

We want to find the decomposition of the form, $f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2)$

where,

$$f_0 = \iint_{[0,1]^2} f(x_1, x_2)\, dx_1 dx_2$$

$$f_1(x_1) = \int_{[0,1]} f(x_1, x_2)\, dx_2 - f_0$$

$$f_2(x_2) = \int_{[0,1]} f(x_1, x_2)\, dx_1 - f_0$$

$$f_{12}(x_1, x_2) = f(x_1, x_2) - f_1(x_1) - f_2(x_2) - f_0$$

Given that,

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

To compute,

$$\int_{[0,1]} f(x_1, x_2)\, dx_2 = \int_{[0,1]} 12x_1 + 6x_2 - 6x_1x_2\, dx_2$$

$$= 12x_1x_2 + 3x_2^2 - 3x_1x_2^2 \Big|_0^1$$

$$= 9x_1 + 3$$

So,

$$f_0 = \int_0^1 9x_1 + 3\, dx_1 = \frac{15}{2}$$

$$f_1(x_1) = 9x_1 + 3 - \frac{15}{2} = 9x_1 - \frac{9}{2}$$

Therefore,

$$f_2(x_2) = \int_{[0,1]} 12x_1 + 6x_2 - 6x_1x_2 dx_1 - f_0$$

$$= 6x^2 + 6x_2x - 3x_2x^2 \Big|_0^1 - \frac{15}{2}$$

$$= 3x_2 - 3/2$$

Hence, $f_{12}(x_1, x_2) = f(x_1, x_2) - f_1(x_1) - f_2(x_2) - f_0$

$$= 12x_1 + 6x_2 - 6x_1x_2 - 9x_1 \frac{9}{2} - 3x_2 + 3/2 - \frac{15}{2}$$

$$= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}$$

Now,

Variance computation, To do that, we know,

$$E[X_i] = 1/2, \ Var[X_i] = E[X_i^2] - E[X_i]^2$$

$$= 1/3 - \frac{1}{4} = 1/12$$

For $i = 1, 2$, where X_1 and X_2 are independent.

So, $Cov[X_1, X_2] = 0$

Now, $\sigma_1^2 = Var[f_1(X_1)] = Var[9X_1 \frac{9}{2}]$

$$= 81 Var[X_1] = \frac{27}{4}$$

$$\sigma_2^2 = Var[f_2(X_2)] = Var[3X_2 - 3/2]$$
$$= 9 var[X_2]$$
$$= 3/4$$

$$\therefore \sigma_{12}^2 = Var[f_{12}(X_1, X_2)]$$
$$= Var\left[3X_1 + 3X_2 - 6X_1X_2 - \frac{3}{2}\right]$$
$$= Var[3X_1] + var[3X_2] + var[6X_1X_2]$$
$$+ 2 cov[3X_1, 3X_2] - 2 cov[3X_1, 6X_1X_2]$$
$$- 2 cov[3X_2, 6X_1X_2]$$
$$= 9 var[X_1] + 9 var[X_2] + 36 Var[X_1X_2] - 36 cov[X_1, X_1X_2]$$
$$- 36 cov[X_2, X_1X_2]$$

Let's compute,
$$Var[X_1X_2] = E[(X_1X_2)^2] - E[X_1X_2]^2$$
$$= E[X_1^2]E[X_2^2] - (E[X_1]E[X_2])^2$$
$$= (1/12 + 1/4)(1/12 + 1/4) - (1/4)^2$$
$$= \frac{7}{144}$$

$$cov[X_1, X_1X_2] = E[X_1X_1X_2] - E[X_1]E[X_1X_2]$$
$$= E[X_1^2]E[X_2] - E[X_1]^2 E[X_2]$$
$$= (1/12 + 1/4)\,1/2 - (1/2)^2\,1/2$$
$$= 1/24$$

So, Now, $\sigma_{12}^2 = 9\,{}^1\!/_{12} + 9\,{}^1\!/_{12} + 36\,{}^7\!/_{144} - 72\,{}^1\!/_{24}$

$= {}^1\!/_4$

Hence, we can see that,

$\sigma_1^2 = {}^{27}\!/_4 > \sigma_2^2 = \frac{3}{4} > \sigma_{12}^2 = \frac{1}{4}$

f_{12} contributes least on the variance of $f(x_1 x_2)$.

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Excercise 5

Q3.)

~~Set X~~

Set be $X \sim N(1,3)$ and $f(x) = 1+2X+X^2$

Expectation:

$E[f(x)] = E[1+2X+X^2] = 1+2E[X]+E[X^2]$

$= 1+2+4$

$= 7$

where we use linearity of expectation

$+ E[X^2] = V[X] + E[X^2]$

$= 3+1$

$= 4$

Variance:

We first observe that $\left(\frac{X-1}{3}\right)^2 \sim \chi^2(1)$ $(V[\chi^2(1)] = 2$

and rewrite $f(x) = (X-1)^2 + 4X$, then

$V[f(x)] = V[(X-1)^2 \frac{3}{3}] + 16V[X] + 2\,\mathrm{Cov}[(X-1)^2, 4X]$

$= 9V[\chi^2(1)] + 16\times 3 + 8(\mathrm{Cov}[X, X^2] + \mathrm{Cov}[X,1] - 2V[X])$

$= 9\times 2 + 16\times 3 + 8(E[X^3] - E[X]E[X^2] - 2\times 3)$

$= 18 + 16\times 3 - 8(10-4-6)$

$= 66$

where the third moment of X, given that $X-1 \sim N(0,3)$ (then every odd moment is 0 due to density's symmetry) was computed this way.

$E[(X-1)^3] = 0 \Longleftrightarrow E[X^3] = 3E[X^2] - 3E[X] + 1 = 10$

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