Sheet(6)

MD Borhan Uddin

Problem: 1

a) Following scenario,

Given the matrix p:

$$p = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix}$$

Since, we assume today is sunny (Day 0), the initial vector is:

$$V_0 = [1 \quad 0]$$

we can get Day 1 by multiplying V_0 by P,

$$V_1 = V_0 \cdot P = [1, 0] \cdot \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} = \begin{bmatrix} 2/3 & 1/5 \end{bmatrix}$$

Likewise, for Day 2,

$$V_2 = V_1 \cdot P = \begin{bmatrix} 2/3 & 1/5 \end{bmatrix} \cdot \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix}$$

Entries of V_2,

$$V_2 = \left[2/3 \cdot \frac{2}{3} + 1/5 \cdot \frac{1}{3} \quad 2/3 \cdot 1/5 + 1/5 \cdot 4/5 \right]$$

$$= \left[4/9 + 1/15 \quad 2/15 + 4/25 \right]$$

By converting a common Deno~~mi~~minator,

$$V_2 = \left[\frac{20}{45} + \frac{3}{45} \quad \frac{10}{75} + \frac{12}{75}\right]$$

(Denominator 45, 75)

$$= \left[\frac{23}{45} \quad \frac{22}{75}\right]$$

So, here, the probability that it will be rainy the day after tomorrow is the second element of V_2.

Hence, the probability of rainy day on the day after tomorrow is $\frac{22}{75}$.

b) To find the relative frequency, we need to find stationary distribution of markov chain.

Given that,
$$P = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix}$$

We know,
$$\pi = [\pi_S \quad \pi_R]$$

and it satisfy, $\pi P = \pi$, Normalized condition $[\pi_S + \pi_R = 1]$

So, from $\pi P = \pi$,
$$[\pi_S \quad \pi_R] \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} = [\pi_S \quad \pi_R]$$

From this we found following system of linear equation,

$$\pi_S \cdot 2/3 + \pi_R \cdot 1/3 = \pi_S \quad \text{— (i)}$$

$$\pi_S \cdot 1/5 + \pi_R \cdot 4/5 = \pi_R$$

$$\pi_S + \pi_R = 1$$

Now, From (i),

$$\pi_S \cdot 2/3 + \pi_R \, 1/3 = \pi_S$$

$$\Rightarrow \pi_R \cdot 1/3 = \pi_S \left(1 - 2/3\right) \quad \boxed{\text{Rearrange to istolate } \pi_R}$$

$$\pi_R \cdot \frac{1}{3} = \pi_S \cdot 1/3$$

$$\pi_R = \pi_S$$

Normalize condition is

$$\pi_S + \pi_R = 1$$

$$\pi_S + \pi_S = 1 \quad [\pi_R = \pi_S]$$

$$2\pi_S = 1$$

$$\pi_S = 1/2$$

Since, $\pi_R = \pi_S$;

so, the stationary distribution $\pi = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$

This means that over an infinitely long period, the relative frequency of sunny day is $\pi_S = 1/2$

c) To find the invariant measure for the given Marcov chain, we need to solve,

$$\pi = (x, y), \quad \pi P = \pi$$

Where P is,

$$P \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix}$$

This gives the following equation,

$$\frac{2}{3}x + \frac{1}{3}y = x \quad \text{——} \ (1)$$

$$\frac{1}{5}x + \frac{4}{5}y = y \quad \text{——} \ (11)$$

From (1) equation,

$$\frac{2}{3}x + \frac{1}{3}y = x$$

By simplify the it,

$$\Rightarrow \frac{1}{3}x = \frac{1}{3}y$$

$$\Rightarrow x = y$$

Now, Substituting $(x = y)$ into equation (11),

$$\frac{1}{5}x + \frac{4}{5}y = x$$

$$\Rightarrow \frac{5}{5}x = x$$

$$\Rightarrow \cancel{x = y} \ x = 1$$

So, the invariant measure for the given marcov chain is $\pi = (1, 1)$

Since, $\pi = (1,1)$, NOW, to determine if the chain converges to its invariant measure, we need to verify the conditions of irreducibility, aperiodicity and finiteness.

1) irreducibility; $\pi(1,1)$ is possible to transition between sunny and rainy, so, it meets the condition.

2) Aperiodicity: Along with, $\pi(1,1)$, its possible to transition from sunny to Rainy, its also meets a vice versa transition. so its Aperodicity

3) Finiteness: The chain is finite since there are only two possible states. (Sunny, Rainy)

~~Since~~, Hence, it converge to its invariant measure $\pi = (1,1)$.

=o=

Exercise 6

Lamia Islam

Problem 2

Let p_n be the probability distribution of X_n (in column vector). Then we have

$$p_{n+1} = P p_n$$

The Markov chain has a unique invariant measure

$$[1/4, 1/4, 1/4, 1/4]^T.$$

① If $X_0 = 1$,

$$p_0 = [1, 0, 0, 0]^T$$

$$\therefore p_1 = P p_0 = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

and $$p_2 = P p_1 = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$$

$\therefore$ Probability Mass function of $X_2 = [1/2, 0, 1/2, 0]^T$

② It does not converge to any probability distribution.
It is straight forward that for all $n \geq 1$,

$$p_n = \begin{cases} [1/2, 0, 1/2, 0]^T & \text{if } n \text{ is even} \\ [0, 1/2, 0, 1/2]^T & \text{if } n \text{ is odd.} \end{cases}$$

This shows an alternating pattern between the states, which does not settle down to a fixed distribution.

Since the probability distribution alternates between two vectors depending on whether n is even or odd, the Markov Chain does not converge to the invariant measure. The sum of probabilities for state 1 and 3 remains equal to the sum for states 2 and 4 at each step. Therefore, since $p_0(1)+p_0(3) \neq p_0(2)+p_0(4)$. p_n will never converge.

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Exercise 6 SANJAY RAJPUROHIT

Problem 3

a.) Given $X_0 = x_0$ we have,

$X_1 = x_0(1-\Delta t) + \sqrt{2\Delta t}N(0,1) \Rightarrow X_1 \sim N(x_0(1-\Delta t), 2\Delta t)$

Since a linear transformation of a normal r.v is still a normal r.v

b.) $X_2 = X_1(1-\Delta t) + \sqrt{2\Delta t}N(0,1) \Rightarrow X_2 \sim N(x_0(1-\Delta t)^2, ((1-\Delta t)^2+1)\Delta t)$

Since the sum of two independent normal r.v is still a normal r.v

~~Problem~~

b.) Code added to pdf.

c.) We see that the distribution, except for the time=0, where $X_0 = x_0$ a.s at time $t \in \{0.5, 1, 10\}$, given $X_0 = x_0$ are all normal and they seem to converge to a standard normal distribution

→ This makes sense since X_t has mean $x_0(1-\Delta t)^{\frac{t}{\Delta t}}$ that if $t \to \infty$ goes to and its normally distributed since the sum of independent normal ~~r.v~~ (X_{t-1}, ε_t) is still a normal variable

Therefore the distribution over time converge to a normal distribution with mean as 0 and variance 1, since most support is seen inside the values of

-2.75 and 2.75.