4. Exercise sheet

Due by: Friday, 10 May 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle

Problem 1. Let $X_1 \sim \mathrm{N}(\bar{x}_1, \sigma_1^2)$ and $X_2 \sim \mathrm{N}(\bar{x}_2, \sigma_2^2)$ be two Gaussian random variables, and $\pi_{X_1}(x_1)$ and $\pi_{X_2}(x_2)$ are PDFs of X_1 and X_2, respectively.

(a) Derive an explicit formula for the L^2-Wasserstein distance between π_{X_1} and π_{X_2}. (See Eq. (2.16) and Example 2.31 in the course textbook.)

(b) Derive an explicit formula for Kullback-Leibler divergence of π_{X_1} from π_{X_2} given by

$$D_{KL}(\pi_{X_1} \| \pi_{X_2}) := \int_{\mathbb{R}} \log\left(\frac{\pi_{X_1}(x)}{\pi_{X_2}(x)}\right) \pi_{X_1}(x)\, \mathrm{d}x.$$

You are expected to solve Problem 1 by hand.

Problem 2. Consider the two sets

$$\mathcal{X}_1 := \{a_1 = 1, a_2 = 2, a_3 = 3\} \quad \text{and} \quad \mathcal{X}_2 := \{b_1 = 1.5, b_2 = 2, b_3 = -1\}$$

with uniform probability mass $\mathbb{P}(a_i) = \mathbb{P}(b_i) = 1/3$ for $i = 1, 2, 3$. A coupling is defined by a matrix $T = [t_{ij}]_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$ with $t_{ij} \geq 0$ and

$$\sum_{i=1}^{3} t_{ij} = \sum_{j=1}^{3} t_{ij} = 1/3.$$

(a) Find the coupling that minimizes

$$\mathsf{J}(T) = \sum_{i,j=1}^{3} t_{ij} |b_i - a_j|^2.$$

What do you notice about the sparsity structure of the optimal coupling matrix T^*?

(b) Compute the cost $\mathsf{J}(T^*)$ corresponding to the minimizer.

You can solve Problem 2 either by hand or using Python (without using any special libraries). If you solve this problem using Python, your program should print the optimal coupling matrix T^* (part a) and the cost $\mathsf{J}(T^*)$ (part b).

Hint: If you wish to use the method discussed during the lecture (which is also used in the course textbook and Explain-Video on the course Moodle page) in part (a), note that the elements in the target state space $\mathcal{X}_2$ need to be *ordered.* However, you can permute the elements in $\mathcal{X}_2$ from smallest to largest, apply the algorithm discussed during the lecture to the permuted states, and then use the inverse permutation to recover the coupling corresponding to the original labeling of the target states.