2. Exercise sheet

Due by: Monday, 29 April 2024, 12:00 noon (CEST) **(deadline extended)**
Please refer to **Assignment Submission Guideline** on Moodle
Modified on April 24, 2024

Problem 1. Implement the numerical model for the Lorenz-63 example (Example 1.1 in the textbook)

$$z^{n+1} = z^n + \delta t(f(z^n) + g(t_n)), \quad z^n = (\mathrm{x}^n, \mathrm{y}^n, \mathrm{z}^n)^{\mathrm{T}},$$

with step-size $\delta t = 0.01$, initial condition $z^0 = (\mathrm{x}^0, \mathrm{y}^0, \mathrm{z}^0) = (-0.587, -0.563, 16.870)^{\mathrm{T}}$, $f(z) = (10(\mathrm{y} - \mathrm{x}), \mathrm{x}(28 - \mathrm{z}) - \mathrm{y}, \mathrm{xy} - \frac{8}{3}\mathrm{z})^{\mathrm{T}}$, and $g(t_n) = g^n = (g_1^n, g_2^n, g_3^n)^{\mathrm{T}}$ defined by

$$g_i^{n+1} = \begin{cases} 1.99999 g_i^n + a/2 & \text{if } g_i^n \in [-a/2, 0), \\ -1.99999 g_i^n + a/2 & \text{otherwise,} \end{cases} \qquad i \in \{1, 2, 3\}, \; n \geq 0,$$

where $(g_1^0, g_2^0, g_3^0) = (a(2^{-1/2} - 1/2), a(3^{-1/2} - 1/2), a(5^{-1/2} - 1/2))$ and $a = 1/\sqrt{\delta t}$.

(a) Plot the x-component of your solution over the time interval $[0, 10]$ similarly to Figure 1.3 in the textbook.

(b) Store the resulting reference trajectory in time intervals of $\Delta t_{\text{out}} = 0.05$ over 4000 cycles (i.e., between $t = 0$ and $t = 200$) in a file for later use in other examples. Print the mean and standard deviation of the three matrix rows corresponding to x, y, and z. (Do not store the system state from every single timestep as this becomes very inefficient, even for low dimensional problems; it is much better to overwrite the state vector on each timestep, and take a copy of the vector when you need to store it. The resulting data set should be stored in a matrix of size 3×4001.)

Problem 2. Implement the numerical observation process as defined for the Lorenz-63 example using the reference trajectory generated in Exercise 1. That is, contaminate the observations with simulated noise via

$$x_{\text{obs}}(\mathrm{t}_k) = x(\mathrm{t}_k) + \frac{1}{20} \sum_{i=1+(k-1)20}^{20k} \xi_{10i}, \qquad k = 1, \ldots, 4000,$$

where $\mathrm{t}_k = k\Delta t_{\text{out}}$ for $k = 1, \ldots, 4000$ and the process $(\xi_k)_{k \geq 0}$ is generated using the tent map iteration

$$\xi_{k+1} = \begin{cases} 1.99999 \xi_k + a/2 & \text{if } \xi_k \in [-a/2, 0), \\ -1.99999 \xi_k + a/2 & \text{otherwise,} \end{cases} \qquad k \geq 0,$$

where $a = 4$ and $\xi_0 = a(2^{-1/2} - 1/2)$.

(a) Plot the observed x-components and the corresponding measurement errors over the time interval $[0, 10]$ similarly to Figure 1.3 in the textbook.

(b) Store the numerically generated observation values $y_{\text{obs}}(\mathrm{t}_k) = \mathrm{x}_{\text{obs}}(\mathrm{t}_k)$ for $k = 1, \ldots, N_{\text{obs}} = 4000$, in a file for later use. Print the mean and standard deviation of the vector $(y_{\text{obs}}(\mathrm{t}_k))_{k=1}^{4000}$.

Hint: You might obtain a trajectory different from the one displayed in the lecture notes. Differences can arise even for mathematically identical implementations due to round-off errors.

Problem 3. Follow the example on pages 11–12 of the lecture slides from week 2 (see also Example 1.4 in the textbook) and use linear extrapolation in order to produce forecasts for forecast intervals $\Delta t_{\text{out}} = 0.05$ and $3\Delta t_{\text{out}} = 0.15$, respectively, from the observations produced in Exercise 2. Plot your results over the time interval $[100, 105]$ as in the lecture slides. Compute the time averaged RMSE and discuss your findings.