3. Exercise sheet

Due by: Friday, 3 May 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
You are expected to solve this week's exercises by hand

Problem 1. Consider the two-dimensional Gaussian PDF $\mathrm{n}(z; \bar{z}, P)$, $z = (x_1, x_2)$, with mean $\bar{z} = (\bar{x}_1, \bar{x}_2)$ and symmetric, positive definite covariance matrix

$$P = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{pmatrix},$$

where $\sigma_{12} = \sigma_{21}$.

(a) Obtain constants $\bar{x}_c$ and σ_c^2 such that

$$\begin{aligned} \mathrm{n}(z; \bar{z}, P) &= \frac{1}{2\pi |\det P|^{1/2}} \exp\left(-\frac{1}{2}(z - \bar{z})^T P^{-1} (z - \bar{z})\right) \\ &= \frac{1}{\sqrt{2\pi\sigma_c^2}} \exp\left(-\frac{1}{2\sigma_c^2}(x_1 - \bar{x}_c)^2\right) \frac{1}{\sqrt{2\pi\sigma_{22}^2}} \exp\left(-\frac{1}{2\sigma_{22}^2}(x_2 - \bar{x}_2)^2\right). \end{aligned}$$

(b) What are the corresponding formulas for the conditional PDF $\pi_{X_2}(x_2|x_1)$ and the marginal $\pi_{X_1}(x_1)$?

Problem 2. Let X_1 and X_2 be two random variables with joint PDF

$$\pi_{X_1 X_2}(x_1, x_2) = \frac{1}{Z} \exp\left(-x_1^2 - x_2^2 - x_1^2 x_2^2\right), \quad x_1, x_2 \in \mathbb{R},$$

where Z is a normalisation constant. Evaluate $\mathbb{E}[X_1 X_2^2 | X_1 = a]$, where $a \in \mathbb{R}$ is a constant.

Hint: Compute the marginal probability density $\pi_{X_1}(x_1)$ to obtain the conditional probability density $\pi_{X_2|X_1}(x_2|a)$. *(Hint corrected on May 1.)*

Problem 3. Let $p, q \colon \mathbb{R}^N \to \mathbb{R}$ be probability densities. Show that the Hellinger distance

$$d_{\mathrm{Hell}}(p, q) = \left(\frac{1}{2} \int_{\mathbb{R}^N} \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 \mathrm{d}x\right)^{1/2}$$

and the Kullback-Leibler divergence

$$D_{\mathrm{KL}}(p||q) = \int_{\mathbb{R}^N} \log\left(\frac{p(x)}{q(x)}\right) p(x) \, \mathrm{d}x$$

satisfy the inequality

$$d_{\mathrm{Hell}}(p, q)^2 \leq \frac{1}{2} D_{\mathrm{KL}}(p||q).$$

You may assume that the PDFs p and q are positive almost everywhere.

Hint: Begin by showing that $d_{\mathrm{Hell}}(p, q)^2 = \int_{\mathbb{R}^N} \left(1 - \sqrt{\frac{q(x)}{p(x)}}\right) p(x) \, \mathrm{d}x$ and then use the inequality $1 - \sqrt{x} \leq -\frac{1}{2} \log x$ for $x > 0$.