7. Exercise sheet

Due by: Friday, 7 June 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
You are expected to solve part (a) of task 2 and parts (a) and (b) of task 3 by hand

Problem 1. Assume you are measuring the heights of members of a certain animal species. The probability distribution of the heights, x, is given by

$$\pi(x) = \frac{1}{C} \exp\big(-V(x)\big), \tag{1}$$

where

$$V(x) = \big((x-4)^2 - 2\big)^2$$

and C is the normalization constant such that $\int_{\mathbb{R}} \pi(x)\,\mathrm{d}x = 1$.

(a) Plot the unnormalized density $\pi(x)$ for $x \in [1,7]$ (set $C = 1$ in formula (1)).

(b) Consider the following SDE:

$$\mathrm{d}X_t = -V'(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t, \quad X_0 = x_0 = 1,$$

where $\{B_t : t \geq 0\}$ is standard Brownian motion. Use the Euler-Maruyama method and run 10 000 Monte-Carlo simulations with $\Delta t = 0.01$ up to time $T = 100$.

(c) Produce a plot which compares the histogram of the simulated samples with the graph of $\pi(x)$ by choosing a suitable constant factor C in (1) so that the graph of $\pi(x)$ is scaled similarly to the histogram. Print also the value you used for C.

(d) Using the samples, estimate the proportion of animals which have height greater than 6.

Problem 2. We solve a toy Bayesian inference problem. Consider the hidden state variable X with prior $X \sim \mathrm{N}(1,1)$. The observable is given by

$$Y = X^2 + W$$

where $W \sim \mathrm{N}(0,1)$. The random variables X and W are assumed to be independent.

(a) Using Bayes' formula, find the expression for $\pi_{X|Y=y}(x)$, the conditional density of X at x given $Y = y$. You may omit the explicit formula for the normalization constant.

(b) Plot the conditional PDF given $y = 2$ and find the *maximum a posteriori* (MAP) estimator of X given $Y = y$. (A numerical solution is OK.)

The exercises continue on the next page.

Problem 3. Let $y \in \mathbb{R}^2$, $x \in \mathbb{R}$, $\gamma > 0$, and

$$y = \begin{pmatrix} 2 \\ 1 \end{pmatrix} x + \eta, \quad \eta \sim \mathrm{N}(0, \gamma^2 I_2),$$

where $I_2 \in \mathbb{R}^{2\times 2}$ is the identity matrix. Suppose that the prior distribution is given by $x \sim \mathrm{N}(0, 2)$, with x and η assumed independent.

(a) Solve the posterior distribution when we observe $y = (1, 2)^{\mathrm{T}}$. What is the posterior variance?

(b) What happens to the posterior distribution and variance under decreasing noise ($\gamma \to 0$)? How do you interpret the limiting distribution?