5. Exercise sheet

Due by: Friday, 24 May 2024, 11:59 pm (CEST)
Please refer to **Assignment Submission Guideline** on Moodle
The next exercise sessions will be held 27.–28.5.2024

Problem 1. Determine the ANOVA decomposition for

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

and compute the associated variances σ_1^2, σ_2^2 and σ_{12}^2. The underlying measure is the uniform probability measure on $[0,1]^2$. (See also pp. 71–72 in the course textbook for an explanation about the ANOVA decomposition.)

Problem 2. Let $\pi_X(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$. In this task, we will consider the problem of approximating integrals

$$\int_{-\infty}^{\infty} f(x)\pi_X(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

by generating an appropriate quadrature rule with positive weights $(w_i)_{i=1}^n$ and nodes $(x_i)_{i=1}^n$.

In this case, the sequence of polynomials H_k orthogonal with respect to the inner product $\langle p, q\rangle_{\pi_X} = \int_{-\infty}^{\infty} p(x)q(x)\pi_X(x)\,\mathrm{d}x$ can be characterized by the three-term recurrence

$$\begin{aligned}
&H_0(x) = 1,\\
&H_1(x) = (x-\alpha_1)H_0(x),\\
&H_{k+1}(x) = (x-\alpha_{k+1})H_k(x) - \beta_{k+1}H_{k-1}(x), \quad k \geq 1,
\end{aligned}$$

where $\alpha_k = 0$ for all $k \geq 1$ and $\beta_k = k-1$ for all $k \geq 2$.

(a) Let $n = 10$, form the tridiagonal matrix

$$A = \begin{bmatrix} \alpha_1 & \sqrt{\beta_2} & & & \\ \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & & \\ & \sqrt{\beta_3} & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_n} \\ & & & \sqrt{\beta_n} & \alpha_n \end{bmatrix},$$

and solve the eigenvalues x_j and eigenvectors $\boldsymbol{q}_j = [q_{1,j}, \ldots, q_{n,j}]^{\mathrm{T}}$, $j = 1, \ldots, n$ (cf., e.g., `numpy.linalg.eig`). The Golub–Welsch algorithm states that the nodes of the quadrature rule (1) are precisely the eigenvalues of matrix A and the corresponding quadrature weights are $w_j = q_{1,j}^2$. Print the nodes and weights that you obtain using this method.

(b) The even moments of a Gaussian random variable $X \sim \mathcal{N}(0,1)$ satisfy

$$I_{2k} = \mathbb{E}[X^{2k}] = \frac{2^k}{\sqrt{\pi}}\Gamma\big(k + \tfrac{1}{2}\big), \quad k \geq 0, \tag{2}$$

where Γ denotes the *gamma function* (cf., e.g., `scipy.special.gamma`).

Using the quadrature rule (1) with the nodes and weights you obtained in part (a) with $n = 10$, compute

$$Q_{2k} = \sum_{i=1}^{n} w_i x_i^{2k},$$

and compare these values with the analytical solution (2) for $k = 0, 1, \ldots, 9$. That is, print the absolute differences $|I_{2k} - Q_{2k}|$ for $k = 0, 1, \ldots, 9$. What happens when $k = 10$? Why?

Problem 3. Let $X \sim \mathrm{N}(1,3)$ and $f(x) = 1 + 2x + x^2$.

(a) Calculate $\mathbb{E}[f(X)]$ and $\mathrm{Var}[f(X)]$ by hand.

(b) Implement the Monte Carlo method to approximate the expected value of f, i.e.

$$\mathbb{E}[f(X)] \approx f_M := \frac{1}{M}\sum_{i=1}^{M} f(x_i), \qquad x_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(1,3).$$

Let $M = 1, 2, 4, 8, \ldots, 256$. For each M, compute $N = 10000$ simulations (realizations) of f_M. For each M, calculate the mean and the variance of f_M over the N rounds. Visualize your results by plotting the mean and variance of f_M over the N rounds as functions of M in two separate plots.