Response Surface Methodology for Optimizing A Paper Helicopter

MASTERS PROJECT

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Outline

Outline

Response Surface Methodology (RSM)

RSM is a collection of mathematical and statistical techniques useful for developing, improving, and optimizing processes [1]. It consists of:

- Experimental strategy for exploring the space of the process or independent variables
- Empirical statistical modeling to develop an appropriate approximating relationship between the response and the variables
- Optimization methods for finding the values of the process variables that produce desirable values of the responses

A three-dimensional response surface showing the expected yield (%) as a function of temperature (x_1) and pressure (x_2)

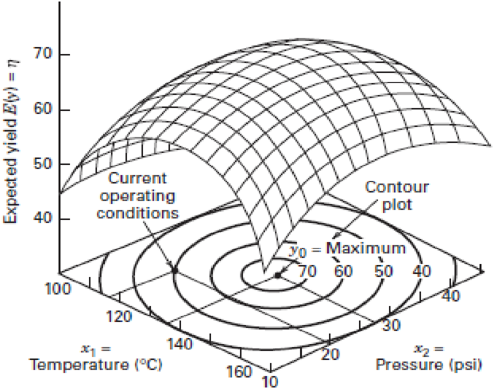

Figure: Contour plot of a response surface [2]

Outline

Advantages of RSM

- Explains the relationship between predictor variables and the response variables using a relatively small number of experiments
- Easy to implement for various experimental settings and response models
- Part of Multidisciplinary Optimization useful in engineering and industrial and manufacturing settings where process optimization is a key concern

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Disadvantages of RSM

- Linearity and normality assumptions of the model can be violated and the reliability of the model can be compromised
- Small variations in the experimental conditions can lead to large differences in the estimated response surface
- May not be suitable for experimental settings with a large number of predictor variables or highly nonlinear response surfaces

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RSM Implementation

- Step 1: Design of experiment i.e Central Composite Design (CCD) and Box Behnken
- Step 2: Statistical and regression analysis to develop model equations that represent the response surface modeling
- Step 3: Parameter estimation using least squares method and response optimization over the region of exploration

RSM Implementation

- First order model for the response

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \epsilon \tag{1}$$

- Second order model for the response

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_i i x_i^2 + \sum \sum_{i<j} \beta_{ij} x_i x_j + \epsilon \tag{2}$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ iid, and k is the number of factors

- Coded variables are found using actual variables such that

$$x_i = \frac{\text{actual level - } \frac{\text{high level} + \text{low level}}{2}}{\frac{\text{high level} - \text{low level}}{2}} \tag{3}$$

Central Composite Design CCD

Consists of a 2^k factorial treatment design:

- To reduce the number of factors to a manageable number
- To fit first-order models and determine the path of steepest ascent

Runs at a center point where all factors are at level 0

- To provide an inexpensive way of checking curvature
- To obtain an independent estimate of the (pure) error variance

Uses Axial points

- $(\pm\alpha, 0, ..., 0), ..., (0, ..., \pm\alpha)$ where α can be 1 or $\sqrt{k}$ or other values
- To estimate the parameters for the second order terms

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Problem Description

The objective of this study is to find an improved helicopter design giving longer flight times using RSM.

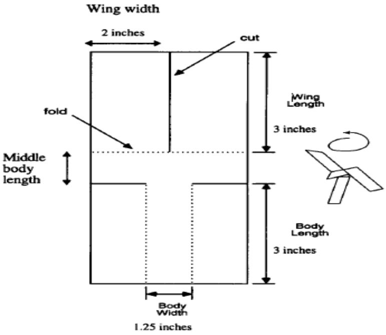

Figure: Paper Helicopter Design [3].

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Experimental Design

A 2^4 Factorial Arrangement of Treatment is used where a single helicopter was made for each combination of the following factors:

Symbol	Factor	-1	1
A(x_1)	Wing area (in^2)	11.80	13.00
B(x_2)	Wing length to width ratio	2.25	2.78
C(x_2)	Base width (in)	1.00	1.50
D(x_2)	Base length (in)	1.50	2.50

Each helicopter was flown 4 times and the average flight time and the standard deviation of the flight time were recorded.

Problem Description

CCD is used with 6 runs at the center point and 8 axial points with $\alpha = \sqrt{4} = \pm 2$.

Symbol	Factor	-2	0	2
A(x_1)	Wing area (in^2)	11.20	12.40	13.60
B(x_2)	Wing length to width ratio	1.98	2.52	3.04
C(x_2)	Base width (in)	0.75	1.25	1.75
D(x_2)	Base length (in)	1.00	2.00	3.00

A total of 30 observations of 2 responses:

y_1: Average flight time

y_2: Standard deviation flight time

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Second Order Model of Average Flight Time

Fitting a full quadratic model using PROC RSREG in SAS

Regression	DF	SS	R-Square	F Value	$Pr > F$
Linear	4	0.151	0.494	29.16	$< .0001$
Quadratic	4	0.024	0.0789	4.66	0.0121
Crossproduct	6	0.111	0.3641	14.34	$< .0001$
Total Model	14	0.286	0.9365	15.81	$< .0001$

Residual	DF	SS	MS	F Value	$Pr > F$
Lack of Fit	10	0.010	0.001	0.57	0.7907
Pure Error	5	0.009	0.002		
Total Error	15	0.019	0.001		

$R^2 = 0.9365$

Second Order Model of Average Flight Time

Parameter	DF	Estimate	Std Err	t Value	$Pr > \|t\|$
Intercept	1	3.71	0.015	252.47	$<.0001$
A	1	-8.33e^{-4}	0.007	-0.11	0.9112
B	1	0.051	0.007	6.92	$<.0001$
C	1	2.5e^{-3}	0.007	0.34	0.7383
D	1	-0.061	0.007	-8.28	$<.0001$
A^2	1	-0.018	0.007	-2.61	0.0198
$A * B$	1	-0.029	0.009	-3.20	0.0060
B^2	1	-0.014	0.007	-2.06	0.0570
$A * C$	1	-0.037	0.009	-4.17	0.0008
$B * C$	1	0.046	0.009	5.14	0.0001
C^2	1	-0.022	0.007	-3.34	0.0045
$A * D$	1	0.044	0.009	4.86	0.0002
$D * B$	1	-0.015	0.009	-1.67	0.1161
$C * D$	1	-0.021	0.009	-2.36	0.0321
D^2	1	8.33e^{-4}	0.007	0.12	0.9051

Second Order Model of Average Flight Time

Using the principle of hierarchy, the results of the partial t-tests with $\alpha = 0.1$

$H_0 : \beta_{11} = 0$	$\hat{\alpha} = 0.0198$	A^2 Significant
$H_0 : \beta_{22} = 0$	$\hat{\alpha} = 0.0570$	B^2 Significant
$H_0 : \beta_{33} = 0$	$\hat{\alpha} = 0.0045$	C^2 Significant
$H_0 : \beta_{44} = 0$	$\hat{\alpha} = 0.9051$	D^2 Not Significant
$H_0 : \beta_{12} = 0$	$\hat{\alpha} = 0.0060$	$A * B$ Significant
$H_0 : \beta_{13} = 0$	$\hat{\alpha} = 0.0008$	$A * C$ Significant
$H_0 : \beta_{14} = 0$	$\hat{\alpha} = 0.0002$	$A * D$ Significant
$H_0 : \beta_{23} = 0$	$\hat{\alpha} = 0.0001$	$B * C$ Significant
$H_0 : \beta_{24} = 0$	$\hat{\alpha} = 0.1161$	$B * D$ Not Significant
$H_0 : \beta_{34} = 0$	$\hat{\alpha} = 0.0321$	$C * D$ Significant

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2$$
$$+ \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{14} x_1 x_4 + \beta_{34} x_3 x_4 + \epsilon$$

Residual Analysis of Average Flight Time Model

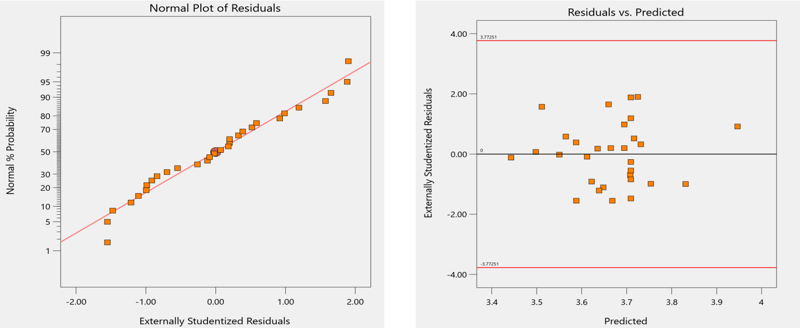
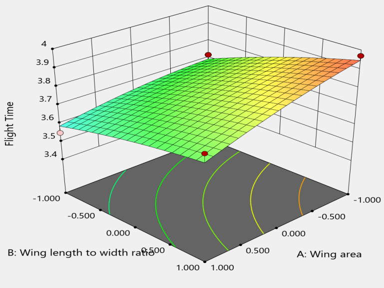


Figure: Residual plots do not suggest strong violations of the model assumptions.

Response Surface For Average Flight Time

From PROC GLM $R^2 = 0.924708$ so $\Delta R^2 = 0.011$

$$\hat{y} = 3.71 - 8.33e^{-4}x_1 + 0.051x_2 + 2.5e^{-3}x_3 - 0.061x_4$$
$$- 0.018x_1^2 - 0.014 * x_2^2 - 0.022x_3^2$$
$$- 0.029x_1x_2 - 0.037x_1x_3 + 0.046x_2x_3 + 0.044x_1x_4 - 0.021x_3x_4$$



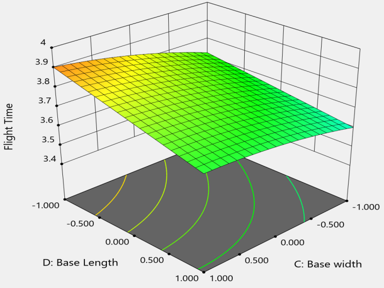


Figure: Response surface of average flight time.

Maximum Average Flight Time

A maximum value of 3.97 is achieved for

	Variable	Coded	Actual
A(x_1)	wing area (in^2)	-1	11.80
B(x_2)	wing-length to width ratio	1	2.78
C(x_3)	base width (in)	1	1.50
D(x_4)	base length (in)	-1	1.50

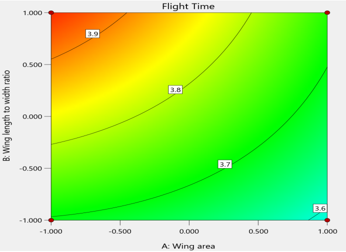


Figure: Contour plots of average flight time.

Outline

Second Order Model of Standard Deviation of Flight Time

Fitting a full quadratic model using PROC RSREG in SAS

Regression	DF	SS	R-Square	F Value	$Pr > F$
Linear	4	0.000865	0.0543	0.38	0.8168
Quadratic	4	0.003961	0.2484	1.76	0.1901
Crossproduct	6	0.002661	0.1669	0.79	0.5937
Total Model	14	0.007487	0.4696	0.95	0.5367

Residual	DF	SS	MS	F Value	$Pr > F$
Lack of Fit	10	0.007166	0.000717	2.78	0.1354
Pure Error	5	0.001289	0.000258		
Total Error	15	0.008455	0.000564		

$R^2 = 0.4696$

Second Order Model of Standard Deviation of Flight Time

Using the principle of hierarchy, the results of the partial t-tests with $\alpha = 0.1$

$$
\begin{array}{lll}
H_0 : \beta_{11} = 0 & \hat{\alpha} = 0.0353 & A^2 \text{ Significant} \\
H_0 : \beta_{22} = 0 & \hat{\alpha} = 0.3784 & B^2 \text{ Not significant} \\
H_0 : \beta_{33} = 0 & \hat{\alpha} = 0.6670 & C^2 \text{ Not significant} \\
H_0 : \beta_{44} = 0 & \hat{\alpha} = 0.4831 & D^2 \text{ Not significant} \\
H_0 : \beta_{12} = 0 & \hat{\alpha} = 0.4420 & A * B \text{ Not significant} \\
H_0 : \beta_{13} = 0 & \hat{\alpha} = 0.7643 & A * C \text{ Not significant} \\
H_0 : \beta_{14} = 0 & \hat{\alpha} = 0.5171 & A * D \text{ Not significant} \\
H_0 : \beta_{23} = 0 & \hat{\alpha} = 0.8441 & B * C \text{ Not significant} \\
H_0 : \beta_{24} = 0 & \hat{\alpha} = 0.6572 & B * D \text{ Not significant} \\
H_0 : \beta_{34} = 0 & \hat{\alpha} = 0.0885 & C * D \text{ Significant}
\end{array}
$$

$$y_2 = \beta_0 + \beta_1 x_1 + \beta_3 x_3 + \beta_4 x_4 + \beta_{11} x_1^2 + \beta_{34} x_3 x_4 + \epsilon$$

Residual Analysis of Standard Deviation of Flight Time Model

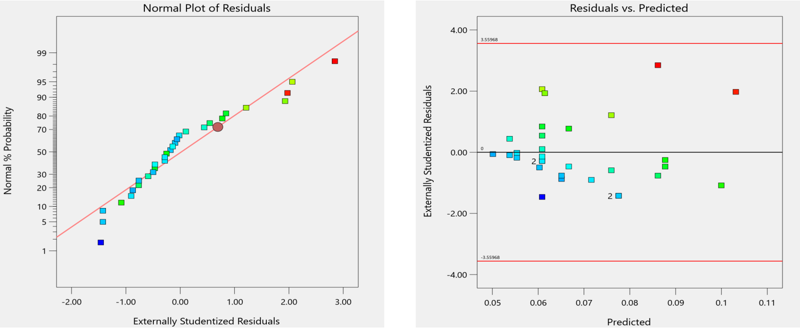


Figure: Residual plots of standard deviation of flight time.

Second Order Model of Standard Deviation of Flight Time

From PROC GLM $R^2 = 0.358608$ so $\Delta R^2 = 0.11$

$$\hat{y} = 0.061 - 7.917e^{-4}x_1 + 2.917e^{-3}x_3 + 5.375e^{-3}x_4 + 0.01x_1^2 - 0.011x_3x_4$$

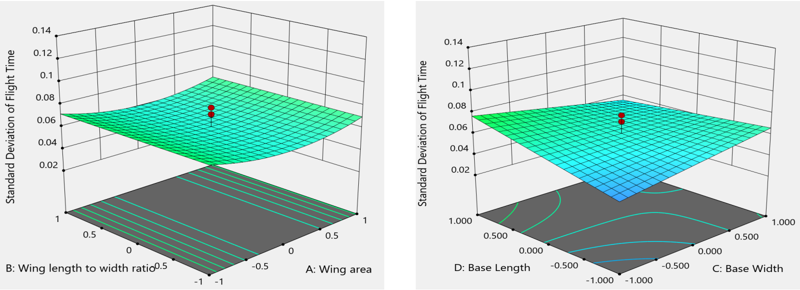


Figure: Response surface of standard deviation of flight time.

Minimum Standard Deviation of Flight Time

A minimum value of **0.045** is achieved for any value of wing length to width ratio B(x_2) and:

Factor	Coded Value	Actual Value
A(x_1): wing area (in^2)	0	12.4
C(x_3): base width (in)	-1	1
D(x_4): base length (in)	-1	1.5

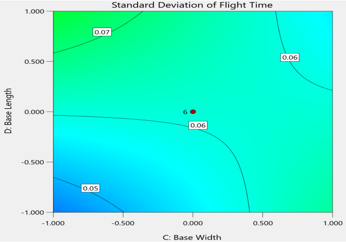


Figure: Contour plots of standard deviation of flight time.

Outline

Maximizing Flight Time and Minimizing its Standard Deviation

- Design Expert's Optimization tool is based on the optimization technique by Derringer and Suich (1980) [2].
- First each response y_i is converted into an individual desirability function d_i such that $0 \leq d_i \leq 1$
- $d_i = 1$ represents the ideal case i.e. y_i is at its goal (max, min,...).
- $d_i = 0$ indicates that one or more responses fall outside desirable limits.
- Then the design variables are chosen to maximize the overall desirability for m responses $D = (d_1 \cdot \cdots \cdot d_m)^{1/m}$

Desirability Functions

- Individual desirability function for **maximizing** the average flight time:

$$d_1 = \begin{cases} 0 & y_1 < L \\ \left(\frac{y_1 - L}{T - L}\right)^r & L \leq y_1 \leq T \\ 1 & y_1 > T \end{cases} \tag{4}$$

- Individual desirability function for **minimizing** the standard deviation of flight time:

$$d_2 = \begin{cases} 1 & y_2 < T \\ \left(\frac{U - y_2}{U - T}\right)^r & T \leq y_2 \leq U \\ 0 & y_2 > U \end{cases} \tag{5}$$

- The design variables are chosen to maximize the overall desirability $D = (d_1 \cdot d_2)^{1/2}$

Numerical Optimization: Results

Name	Goal	Lower Limit	Upper Limit
Flight Time	Maximize	3.44	3.97
Std Dev of flight time	Minimize	0.032	0.132

The highest desirability achieved was 0.873 for the following values:

Name	Coded Value	Actual Value
Average flight time		**3.97**
Standard deviation of flight time		**0.056**
A(x_1): wing area (in^2)	-0.738	11.957
B(x_2): wing length to width ratio	2.00	3.04
C(x_3): base width (in)	-0.02	1.245
D(x_4): base length (in)	-2	1.00

Numerical Optimization: Overall Desirability

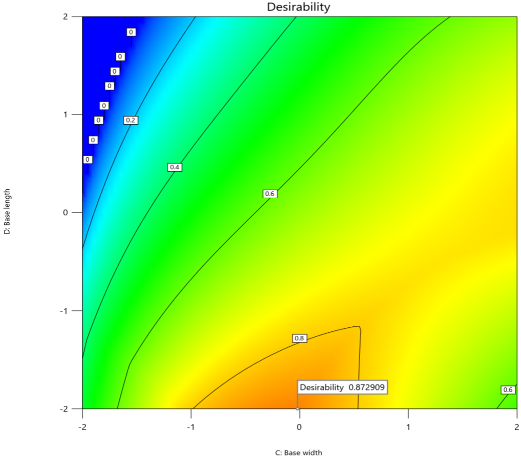


Figure: Contour plots of the overall desirability function

Numerical Optimization: Average Flight Time

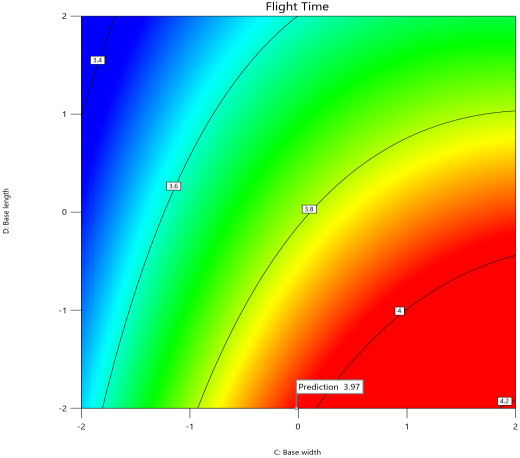


Figure: Contour plots of the average flight time

Numerical Optimization: Standard Deviation of Flight Time

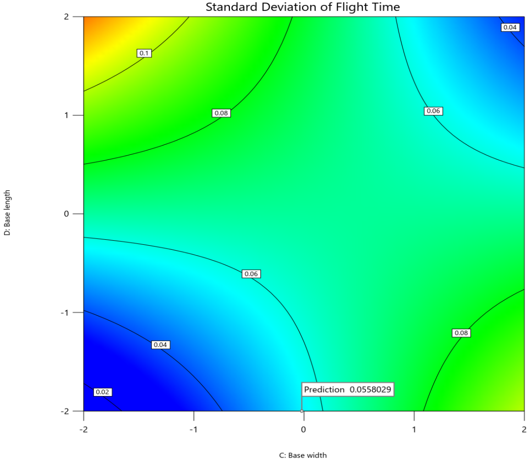
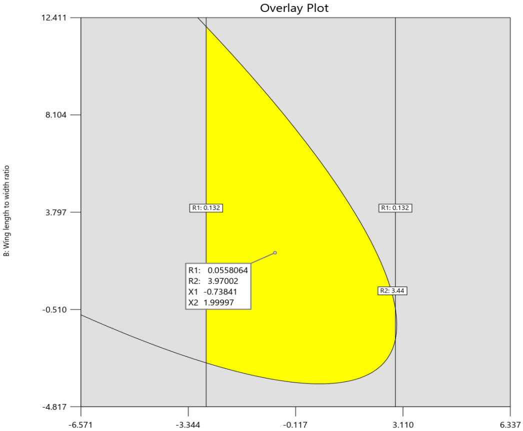


Figure: Contour plots of the standard deviation of flight time

Graphical Optimization

The yellow region shows the values of wing area and base length that satisfy the requirements of both responses for C[x_3] = -0.02 and D[x_4] = -2 .



Conclusion and Future Work

- An overview of RSM is presented and the method used on the paper helicopter example.
- Second order model of the average flight time response and its standard deviation are performed.
- Maximum flight time of the paper helicopter and the minimum standard deviation are found
- A region of optimal variable values that maximize the flight time while simultaneously minimizing its standard deviation is obtained.
- For future work, ridge analysis can be used to solve the minimax problem and find new optimum conditions using the method of Steepest Ascent.

Thank you for your attention
Any questions ?

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