200/2132 Discrete Mathematics Project

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Problem Statement and Solution

In this problem, we will be using binary predicates F(x, y), G(x, y), etc. to represent functions f, g : U $\rightarrow$ U, etc., where U is the universe. Thus,F(x, y) holds iff y = f(x), G(x, y)holds iff y = g(x), etc..

1 .Write predicate statements that expresses the following facts:

1.1 F represents a function:

This means every entry in the domain has at most one image from the codomain.

$$\forall x \exists ! y F(x, y)$$

1.2 F represents a one-to-one function:

This means that two alike images **must** imply that their entries from the domain is the same

$$\forall x_1 \forall x_2 \forall y (F(x_1, y) \wedge F(x_2, y)) \rightarrow (x = x_2)$$

1.3 F represents an onto function:

This means that all images in the codomain are mapped.

$$\forall x \exists ! y F(x, y) \wedge \forall y \exists x F(x, y)$$

1.4 F and G represent inverse functions of one another:

G inverse of F means:

$$((F \circ G)(x) = x) \tag{1}$$

F inverse of G means:

$$((G \circ F)(x) = x) \tag{2}$$

Therefore, from 1 and 2 we conclude that:

$$\forall x \forall y (F(x, y) \Leftrightarrow G(y, x))$$

1.5 H represents the composition function f $\circ g$:

With two functions f and g, where we take x from the domain of g and thus the composition leads to: H(x) = (f o g)(x) = f(g(x))
Therefore, H will look like this:

$$\forall x_1 \forall x_2 \forall y [H(x_1, y) \Leftrightarrow F(x_2, y) \wedge G(x_1, x_2)]$$

2 Use binary predicates representing functions to give formal proofs (in the style of Sec 1.6) of the following statements:

2.1 If f and g are one-to-one functions, then so is f o g.

$$\forall x_1 \forall x_2 \forall y (F(x_1, y) \wedge F(x_2, y)) \rightarrow (x_1 = x_2) \tag{1}$$

$$\forall x_1 \forall x_2 \forall y (G(x_1, y) \wedge G(x_2, y)) \rightarrow x_1 = x_2 \tag{2}$$

$$\forall x_1 \forall x_2 \forall y [H(x_1, y) \Leftrightarrow F(x_2, y) \wedge G(x_1, x_2)] \tag{3}$$

from (3) we get :

$$\forall x_1 \forall x_2 \forall y [H(x_2, y) \Leftrightarrow F(x_1, y) \wedge G(x_2, x_1)] \tag{4}$$

Using **Addition Rule** between (3) and (4)(from Predicates Chapter) we get:

$$\forall x_1 \forall x_2 \forall y [H(x_2, y) \wedge H(x_1, y) \Leftrightarrow F(x_2, y) \wedge G(x_1, x_2) \wedge F(x_1, y) \wedge G(x_2, x_1))] \tag{5}$$

Using **Simplification Rule** on (5) we get:

$$\forall x_1 \forall x_2 \forall y [H(x_2, y) \wedge H(x_1, y) \Leftrightarrow F(x_2, y) \wedge F(x_1, y)] \tag{6}$$

Using **Hypothetical Syllogism** between (1) and (6) we get:

$$\forall x_1 \forall x_2 \forall y [H(x_2, y) \wedge H(x_1, y) \Leftrightarrow (x_1 = x_2)] \tag{7}$$

Therefore:

$$[\forall x_1 \forall x_2 \forall y [H(x_1, y) \Leftrightarrow F(z, y) \wedge G(x, z)]] \rightarrow [\forall x_1 \forall x_2 \forall y (H(x_1, y) \wedge H(x_2, y)) \rightarrow x_1 = x_2] \tag{8}$$

and **Hence** f o g is one-to-one

2.2 If f and g are one-to-one functions, then so is f o g.

$$\forall x_! \exists ! y F(x_1, y) \wedge \forall y \exists x_1 F(x_1, y) \tag{1}$$

From (1):

$$\forall x_2 \exists ! y F(x_2, y) \wedge \forall y \exists x_2 F(x_2, y) \tag{2}$$

Using **Simplification Rule** on 2:

$$\forall x_2 \exists ! y F(x_2, y) \tag{3}$$

Using **Simplification Rule** again on 2:

$$\forall y \exists x_2 F(x_2, y) \tag{4}$$

$$\forall x_1 \exists ! y G(x_1, y) \wedge \forall y \exists x_1 G(x_1, y) \tag{5}$$

From (5) we get:

$$\forall x_1 \exists ! x_2 G(x_1, x_2) \wedge \forall x_2 \exists x_1 G(x_1, x_2) \tag{6}$$

Using **Simplification Rule** on (6):

$$\forall x_1 \exists ! x_2 G(x_1, x_2) \tag{7}$$

Using **Simplification Rule** again on (6):

$$\forall x_2 \exists x_1 G(x_1, x_2) \tag{8}$$

Using **Addition Rule** between (3) and (7) :

$$\forall x_1 \exists ! y [F(x_2, y) \wedge G(x_1, x_2)] \tag{9}$$

Using **Addition Rule** from (4) and (8):

$$\forall y \exists x [F(x_2, y) \wedge G(x_1, x_2)] \tag{10}$$

Referring to **Question 1** , **part e** we get:

$$\forall x_1 \forall x_2 \forall y [H(x_1, y) \Leftrightarrow F(x_2, y) \wedge G(x_1, x_2)] \tag{11}$$

From (9), (10) and (11) we get:

$$\forall x_1 \exists ! y H(x_1, y) \wedge \forall y \exists x_1 H(x_1, y) \tag{12}$$

Therefore:

$\forall x_1 \forall x_2 \forall y [H(x_1, y) \Leftrightarrow F(x_2, y) \wedge G(x_1, x_2)] \rightarrow [\forall x \exists ! y H(x_1, y) \wedge \forall y \exists ! x_1 H(x_1, y)]$

and **Hence**: f o g is an onto function.