Practice IMC

Seed: 283502

Problem 1 *Each vertex of a regular dodecagon (12-gon) is to be colored either red or blue, and thus there are 2^{12} possible colorings. Find the number of these colorings with the property that no four vertices colored the same color are the four vertices of a rectangle.*

Problem 2 *Let S be the set of ordered pairs (x, y) such that $0 < x \leq 1, 0 < y \leq 1$, and $\left[\log_2\left(\frac{1}{x}\right)\right]$ and $\left[\log_5\left(\frac{1}{y}\right)\right]$ are both even. Given that the area of the graph of S is m/n, where m and n are relatively prime positive integers, find $m + n$. The notation $[z]$ denotes the greatest integer that is less than or equal to z.*

Problem 3 *Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.*

Problem 4 *Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer*

$$p^{(j)}(k) = \left. \frac{d^j}{dx^j} p(x) \right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

Problem 5 *let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.*