Practice IMC

Seed: 629228

Problem 1 *A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.*

Problem 2 *A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.*

Problem 3 *(a) Prove that if the six dihedral (i.e. angles between pairs of faces) of a given tetrahedron are congruent, then the tetrahedron is regular.*

(b) Is a tetrahedron necessarily regular if five dihedral angles are congruent?

Problem 4 *Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths*

$a, f(b)$ and $f(b + f(a) - 1)$.

(A triangle is non-degenerate if its vertices are not collinear.)

Author: Bruno Le Floch, France

Problem 5 *For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$