Practice IMC

Seed: 961016

Problem 1 *A gardener plants three maple trees, four oaks, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let* $\frac{m}{n}$ *in lowest terms be the probability that no two birch trees are next to one another. Find* $m+n$*.*

Problem 2 *The equation* $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ *has exactly two real roots, one of which is* $\frac{m+\sqrt{n}}{r}$*, where* m*,* n *and* r *are integers,* m *and* r *are relatively prime, and* $r > 0$*. Find* $m+n+r$*.*

Problem 3 *Prove that in the set* $\{1, 2, \ldots, 1989\}$ *can be expressed as the disjoint union of subsets* $A_i, \{i = 1, 2, \ldots, 117\}$ *such that*

i.) each A_i *contains 17 elements*

ii.) the sum of all the elements in each A_i *is the same.*

Problem 4 *Let* $n > 6$ *be an integer and* $a_1, a_2, \cdots, a_k$ *be all the natural numbers less than* n *and relatively prime to* n*. If*

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n *must be either a prime number or a power of* 2*.*

Problem 5 *Fix an integer* $b \geq 2$*. Let* $f(1) = 1$*,* $f(2) = 2$*, and for each* $n \geq 3$*, define* $f(n) = nf(d)$*, where* d *is the number of base-*b *digits of* n*. For which values of* b *does*

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?