Practice IMC

Seed: 507527

Problem 1 *Find the number of ordered pairs of integers (a, b) such that the sequence*

$$3, 4, 5, a, b, 30, 40, 50$$

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.

Problem 2 *A permutation of the set of positive integers $[n] = \{1, 2, \dots, n\}$ is a sequence $(a_1, a_2, \dots, a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. For example, $(3, 5, 1, 2, 4)$ is a permutation of $[5]$. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \leq k \leq n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.*

Problem 3 *For each positive integer n, let*

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ T_n &= S_1 + S_2 + S_3 + \cdots + S_n \\ U_n &= \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}. \end{aligned}$$

Find, with proof, integers $0 < a,\ b,\ c,\ d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

Problem 4 *There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car that starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).*

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 5 *Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that*

$F(n, r) = \frac{n+1}{r+1}$.