Practice IMC

Seed: 710357

Problem 1 *Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers*

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Problem 2 *A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.*

For which n does it follow that the numbers on the cards are all equal?

Problem 3 *For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.*

Problem 4 *Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?*

Problem 5 *There are $4n$ pebbles of weights $1, 2, 3, ..., 4n$. Each pebble is colored in one of n colors and there are four pebbles of each color. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:*

The total weights of both piles are the same. Each pile contains two pebbles of each color.