Practice IMC

Seed: 958512

Problem 1 *Let $a, b, x,$ and y be real numbers with $a > 4$ and $b > 1$ such that*

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 16} = \frac{(x-20)^2}{b^2 - 1} + \frac{(y-11)^2}{b^2} = 1.$$

Find the least possible value of $a + b$.

Problem 2 *Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that*

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Problem 3 *Show that, for any fixed integer $n \geq 1$, the sequence*

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \dots \pmod n$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2,\ a_{i+1} = 2^{a_i}$. Also $a_i \pmod n$ means the remainder which results from dividing a_i by n.]

Problem 4 *Let P be a regular 2006-gon. A diagonal of P is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.*

Problem 5 *Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f: \mathbb{R} \to \mathbb{R}$ satisfying the equation*

$f(x + f(x+y)) + f(xy) = x + f(x+y) + yf(x)$

for all real numbers x and y.

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