Practice IMC

Seed: 514641

Problem 1 *For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:*

- *no two consecutive digits are equal, and*
- *the last digit is a prime.*

Problem 2 *A 2×3 rectangle has vertices as $(0,0), (2,0), (0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).*

Problem 3 *Find all functions $f : (0,\infty) \mapsto (0,\infty)$ (so f is a function from the positive real numbers) such that*

$\frac{(f(w))^2+(f(x))^2}{f(y^2)+f(z^2)} = \frac{w^2+x^2}{y^2+z^2}$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 4 *Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.*

Problem 5 *For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \le k \le n$. Prove that for some constant C, independent of n,*

$$n^{n^2/2-Cn}e^{-n^2/4} \le f(n) \le n^{n^2/2+Cn}e^{-n^2/4}.$$