Practice IMC

Seed: 929701

Problem 1 *For a nonempty set S of integers, let $\sigma(S)$ be the sum of the elements of S. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \leq 1500$, there is a subset S of A for which $\sigma(S) = n$. What is the smallest possible value of a_{10}?*

Problem 2 *Chords AA', BB', and CC' of a sphere meet at an interior point P but are not contained in the same plane. The sphere through A, B, C, and P is tangent to the sphere through A', B', C', and P. Prove that $AA' = BB' = CC'$.*

Problem 3 *Evaluate*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Problem 4 *Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.*

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Problem 5 *Determine the least real number M such that the inequality*

$$\left|ab\left(a^2 - b^2\right) + bc\left(b^2 - c^2\right) + ca\left(c^2 - a^2\right)\right| \leq M\left(a^2 + b^2 + c^2\right)^2$$

holds for all real numbers a, b and c