Practice IMC

Seed: 270493

Problem 1 *Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.)*

Problem 2 *Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.*

Problem 3 *(András Gyárfás) Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.*

Problem 4 *Determine the least real number M such that the inequality*

$$\left|ab\left(a^2-b^2\right)+bc\left(b^2-c^2\right)+ca\left(c^2-a^2\right)\right| \leq M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b and c

Problem 5 *Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that*

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1}\frac{n}{n+1}.$$