Practice IMC

Seed: 995214

Problem 1 *A sequence is defined over non-negative integral indexes in the following way:* $a_0 = a_1 = 3$, $a_{n+1}a_{n-1} = a_n^2 + 2007$.

Find the greatest integer that does not exceed $\frac{a_{2006}^2 + a_{2007}^2}{a_{2006}a_{2007}}$.

Problem 2 *Find all positive integers* x, y, z *and* t *such that* $2^x 3^y + 5^z = 7^t$.

Problem 3 *Let* n *be a positive integer, and let* $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. *Prove that* f_n *has no roots in the closed unit disk* $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

Problem 4 *The 30 edges of a regular icosahedron are distinguished by labeling them* $1, 2, \ldots, 30$. *How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color? [Note: the top matter on each exam paper included the logo of the Mathematical Association of America, which is itself an icosahedron.]*

Problem 5 *In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.*