

Density Ratio Estimation in Variational Bayesian Machine Learning

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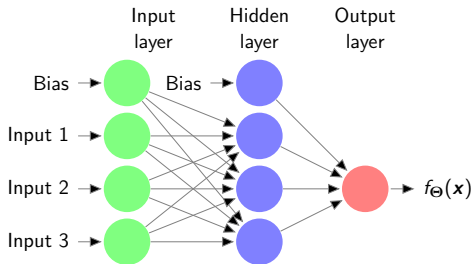
Statistics Honours, 2018

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 - (Amortized) Variational Inference
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Neural Networks

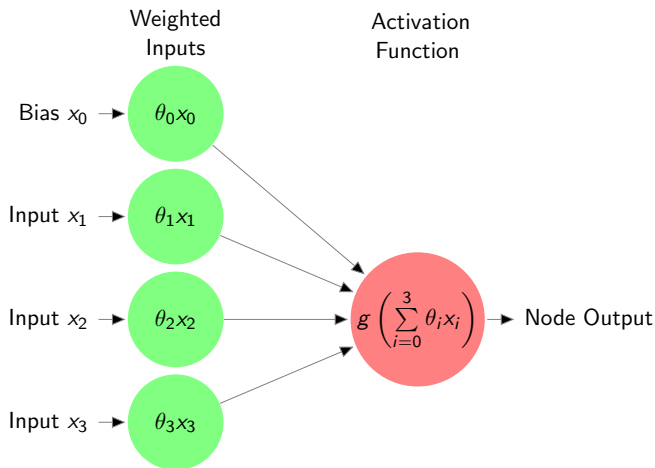
Overall Structure

- Objective is to approximate a function f^* using mapping with parameters Θ : $f_{\Theta}(x)$.
- Universal Approximation Theorem states a neural network can **approximate (almost) any function** if it is complex enough.
- Each node output is a transformed, weighted sum of previous node outputs.



Neural Networks

Individual Node



- Weights trained such that (ideally convex) loss function is minimized
e.g. Mean Squared Error: $\min_{\Theta} \frac{1}{2} \|\mathbf{y} - \mathbf{f}_{\Theta}(\mathbf{x})\|_2^2$.
- Back-propagation finds partial derivatives of loss function with respect to weights.
- **Gradient descent** uses these partial derivatives to optimize network.

(Amortized) Variational Inference

Bayesian Inference

- Fundamental problem in Bayesian computation is to **estimate posterior densities** $p(z|x)$:

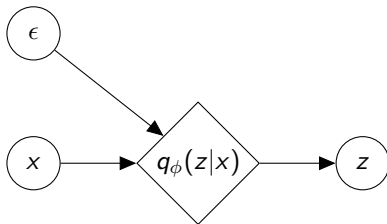
$$p(z|x) \propto \underbrace{p(z)}_{\text{Prior}} \underbrace{p(x|z)}_{\text{Likelihood}} .$$

- Typical MCMC methods are slow with high dimensional data or large datasets.
- (Amortized) Variational Inference is a solution.

(Amortized) Variational Inference

Introduction

- Amortized variational inference approximates $p(z|x)$ with a different density $q_\phi(z|x)$.
- $q_\phi(z|x)$ is a **neural network** with parameters ϕ .
- Random noise ϵ makes the network probabilistic.



(Amortized) Variational Inference

Network Training

- Train network by minimizing the reverse KL divergence:

$$KL(q_\phi(z|x)||p(z|x)) := \mathbb{E}_{q_\phi(z|x)} \left[\log \frac{q_\phi(z|x)}{p(z|x)} \right].$$

- This is the same as solving:

$$\min_{\phi} \mathbb{E}_{q^*(x)} \left[\underbrace{-\mathbb{E}_{q_\phi(z|x)} [\log p(x|z)]}_{\text{Likelihood}} + \underbrace{KL(q_\phi(z|x)||p(z))}_{\text{Log Density Ratio}} \right].$$

- We call this $NELBO(q)$ as it is the **n**egative of **e**vidence **l**ower **b**ound.

$$NELBO(q) \geq -\log p(x)$$

- $q^*(x)$ is the density of the dataset.

(Amortized) Variational Inference

Problems with Implicit Distributions

Consider our log density ratio term

$$KL(q_\phi(z|x)||p(z)) = \mathbb{E}_{q_\phi(z|x)} \left[\log \frac{q_\phi(z|x)}{p(z)} \right].$$

- $q_\phi(z|x)$ is **implicit**.
- Use density ratio estimation to evaluate $\frac{q_\phi(z|x)}{p(z)}$ in $KL(q_\phi(z|x)||p(z))$.
- Density ratio estimation only requires **samples**.

Density Ratio Estimation

Introduction

- There exist different methods of estimating a density ratio $\frac{q(u)}{p(u)}$.
- Many of them use a neural network $r_\alpha(u) \simeq \frac{q(u)}{p(u)}$.
- Various different loss functions used to train network.

Example

The following loss function

$$\min_{\alpha} -\mathbb{E}_{q(u)}[\log D_{\alpha}(u)] - \mathbb{E}_{p(u)}[\log(1 - D_{\alpha}(u))]$$

trains a network $D_{\alpha}(u)$ estimating $\frac{q(u)}{q(u)+p(u)}$.

Density Ratio Estimation

Theorem

Theorem (Nguyen, 2010, Adapted)

If f is a convex function with derivative f' and convex conjugate f^ , and $r_\alpha(u)$ is a neural network, then we have the lower bound for the f -divergence between densities $p(u)$ and $q(u)$:*

$$D_f[p(u)||q(u)] \geq \sup_{\alpha} \{ \mathbb{E}_{q(u)}[f'(r_\alpha(u))] - \mathbb{E}_{p(u)}[f^*(f'(r_\alpha(u)))] \},$$

with equality when $r_\alpha(u) = q(u)/p(u)$.

Example

For the reverse KL divergence, we have:

$$KL[q(u)||p(u)] \geq \sup_{\alpha} \{ \mathbb{E}_{q(u)}[1 + \log r_\alpha(u)] - \mathbb{E}_{p(u)}[r_\alpha(u)] \}.$$

Density Ratio Estimation

Algorithm Generalisation

- Apply theorem to generalise density ratio estimator loss functions.
- Choose **f -divergence bound**:
 - Reverse KL Divergence
 - GAN Divergence

and **estimator parametrisation**:

- Direct Ratio Estimator: $r_\alpha(u) \simeq \frac{q(u)}{p(u)}$
- Class Probability Estimator: $D_\alpha(u) = \frac{r_\alpha(u)}{r_\alpha(u)+1} \iff \frac{q(u)}{p(u)} \simeq \frac{D_\alpha(u)}{1-D_\alpha(u)}$
- Direct Log Ratio Estimator: $T_\alpha(u) = \log r_\alpha(u) \iff \frac{q(u)}{p(u)} \simeq e^{T_\alpha(u)}$

To train a variational posterior network:

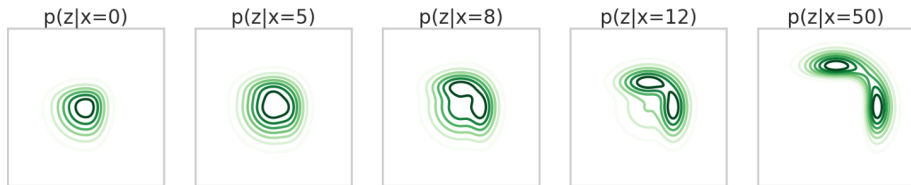
- 1 Train estimator network until convergence.
- 2 Use estimator network to calculate intractable term in $NELBO$.
- 3 Take one optimisation step of posterior network.
- 4 Repeat until posterior convergence.

Undertrained Estimator Experiment

Experiment Outline

$$(z_1, z_2) \sim \mathcal{N}(0, \sigma^2 I_{2 \times 2})$$

$$x|z \sim \text{Exp}(3 + \max(0, z_1)^3 + \max(0, z_2)^3)$$



- Posterior is unusually-shaped and bimodal.
- Use Gaussian KDE to find 'true' KL divergence for $q_\phi(z|x = 0, 5, 8, 12, 50)$.

Undertrained Estimator Experiment

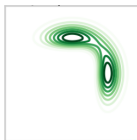
Experiment Outline

- In a previous experiment we found that all three estimator parametrisations lead to similar results when optimised effectively.
- What if they are poorly optimised?
- Training parameters:
 - High posterior training rate.
 - Low estimator training rate.
 - Low estimator to posterior iteration ratio (11:1).

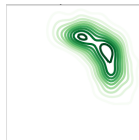
Undertrained Estimator Experiment

Results

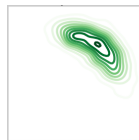
Algorithm		Mean KL Divergence	Standard Deviation
Reverse KL	$D_\alpha(z, x)$	1.3786	0.0286
	$r_\alpha(z, x)$	1.3934	0.0410
	$T_\alpha(z, x)$	1.4133	0.0597
GAN	$D_\alpha(z, x)$	1.4017	0.0286
	$r_\alpha(z, x)$	1.4086	0.0555
	$T_\alpha(z, x)$	1.4214	0.0518



True



Reverse KL

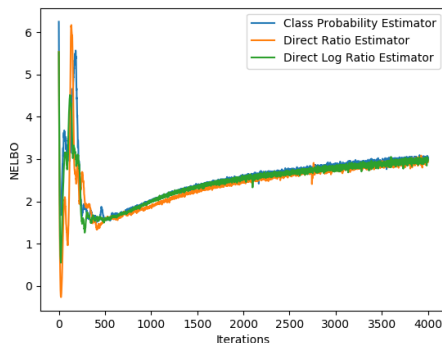


GAN

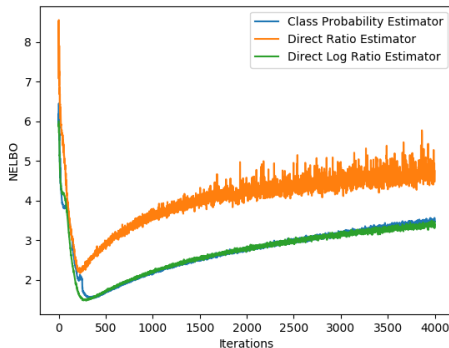
- Reverse KL divergence better than GAN divergence.
- $D_\alpha(z, x) < r_\alpha(z, x) < T_\alpha(z, x)$

Undertrained Estimator Experiment

NELBO Plots



(a) Reverse KL Divergence



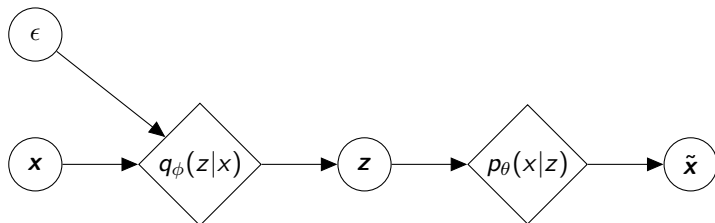
(b) GAN Divergence

- Reverse KL Divergence has initial instability.

Autoencoder Experiment

Autoencoders

- Posterior $q_\phi(z|x)$ 'compresses' data x into z .
- Likelihood $p_\theta(x|z)$ 'reconstructs' data \tilde{x} from z .

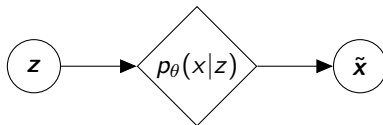


$$\min_{\theta, \phi} \mathbb{E}_{q^*(x)} \left[\underbrace{-\mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x|z)]}_{\text{Likelihood}} + \underbrace{KL(q_\phi(z|x) || p(z))}_{\text{Log Density Ratio}} \right]$$

Autoencoder Experiment

Autoencoders

- Generate z from $p(z)$.
- Typically $p(z)$ is $\mathcal{N}(0, I)$.



Autoencoder Experiment

Experiment Outline



- MNIST dataset - 28×28 grey-scale images of handwritten digits
- Again use undertrained estimator.
- Use reconstruction error $\|x - \tilde{x}\|^2$ as metric.

Generation Experiment

Results - 20-dimensional latent space

Algorithm	Mean Reconstruction Error	Standard Deviation
Reverse KL - $D_\alpha(z, x)$	0.0647	0.0191
GAN - $D_\alpha(z, x)$	0.0444	0.0017



Reverse KL

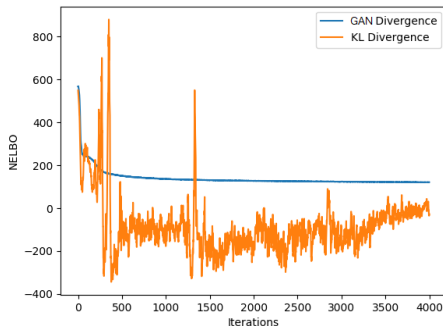


GAN

- Density ratios too big for direct ratio and log ratio estimators.
- Exponential of $T_\alpha(z, x)$ taken in loss function.
- $D_\alpha(z, x)$ ranges in $(0, 1)$.

Autoencoder Experiment


NELBO plot





- Reverse KL divergence fails to stabilise by the end of runtime.

- The class probability estimator $D_\alpha(u) \simeq \frac{q(u)}{q(u)+p(u)}$ is the ‘best’ parametrisation as it can store the **highest density ratios**.
- Reverse KL divergence upper bound may be **unstable** but leads to **faster convergence** when stable.
- Future Research
 - Still unclear exactly why reverse KL divergence is more unstable but more accurate when stable.
 - Several more f -divergences exist which have unknown stability when undertrained.
 - Alternate density ratio estimation algorithms e.g. denoisers, k-nearest neighbours

For Further Reading I

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