

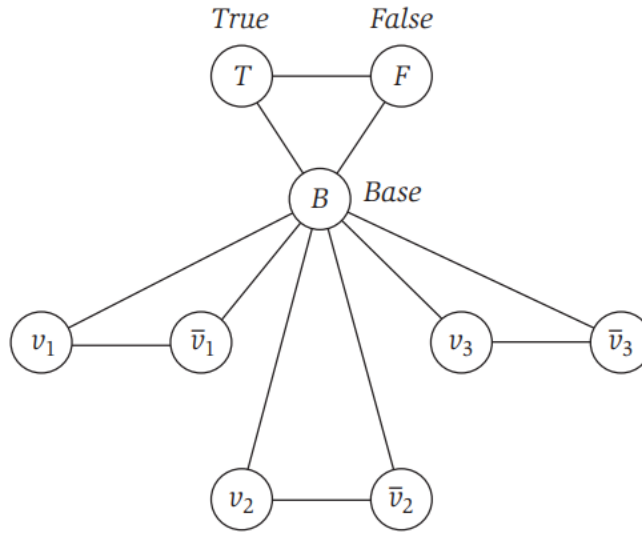
Assignment 05: Reduction Report

$3 - SAT \leq_p 3 - Color$

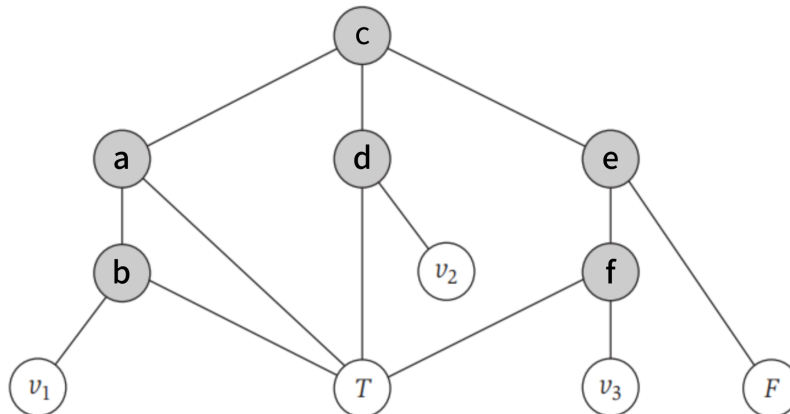
To prove the above problem, we just have to prove that for any arbitrary instance of $3 - SAT$, we can construct a $3 - Color$ instance, whose true value is the same the instance of $3 - SAT$.

To achieve this, we need to make use of the ability of $3 - Color$ for encoding Boolean expressions. In a $3 - Color$ graph, if we want the colors of two nodes, we can add an edge between them, so we just have to take the logic variables as nodes and connect the nodes with different true values together.

First, we define three special nodes *True*, *False*, *Base*. For each logic variable v_i in $3 - SAT$, we connect $v_i, \bar{v}_i, Base$ together as an triangle. Also connect *True*, *False*, *Base* together as an triangle. Now we have a graph ensuring that no pairs of v_i, \bar{v}_i have the same color, so they will not be both true or both false at the same time. And the node *Base* has a color different from all the other nodes. Also we can see that *True* and *False* get the different color. We simply name the color of *True* as true color, the other one as false color. For this reason, each pair of v_i, \bar{v}_i can have the true color for one node, and the false color for another node.



Now we need to deal with the graph according to the clauses in the $3 - SAT$ instance. For each clause, we need to guarantee that at least one of the three nodes can get the true color. For such a clause with $v_1 \vee v_2 \vee v_3$, we the following sub-graph with additional six nodes can ensure the property. This is easy to prove. We have already know that in the main graph, any v_i, \bar{v}_i cannot have the base color, so all the v_i here can only be true or false. Let us assume that all the three v_i are false, then *b*, *d* and *f* can only be base color. So *a* is the false color, *e* is the true color. Then *c* comes to a situation that it can picks no colors. So the case crushes. The three logic variables cannot be false at the same time. We can easily prove that other cases with at least one variable being true can color this sub-graph. So for every clauses in the $3 - SAT$ instance, we can add extra six nodes to the main graph and connect edges in the way this sub-graph does. When we apply $3 - Color$ to the graph, we can guarantee that all the clauses are satisfied.



The above process completes the construction. For such a graph, if a $3 - Color$ solution exists, then all the clauses in the $3 - SAT$ instance have at least one logic variables picking the true color. If no $3 - Color$ solutions exist, the the $3 - SAT$ instance is not satisfiable. Also the size of the $3 - Color$ instances are polynomial in the size of the corresponding $3 - SAT$ instances, so we can get $3 - SAT \leq_p 3 - Color$.