ΜΑΘ-ΙΙ　　2η Σειρά Ασκήσεων

1. Να υπολογιστούν τα διπλά ολοκληρώματα πάνω στο ορθογώνιο R

(α′)

$$\iint_R xy\sqrt{1+x^2+y^2}dxdy,\ R: 0 \leq x \leq 1,\ 0 \leq y \leq 1$$

Λύση:

$$\begin{aligned}
I &= \iint_R xy\sqrt{1+x^2+y^2}dxdy \\
&= \int_0^1 dy \int_0^1 dx\ xy\sqrt{1+x^2+y^2} \\
&= \int_0^1 dy\ y \left[\frac{1}{3}(1+x^2+y^2)^{\frac{3}{2}}\right]_{x=0}^{x=1} \\
&= \frac{1}{3}\left(\int_0^1 y(2+y^2)^{\frac{3}{2}}dy - \int_0^1 y(1+y^2)^{\frac{3}{2}}dy\right) \\
&= \frac{1}{3}\left[\frac{1}{5}(2+y^2)^{\frac{5}{2}}\right]_0^1 - \left[\frac{1}{5}(1+y^2)^{\frac{5}{2}}\right]_0^1 \\
&= \frac{9\sqrt{3}-8\sqrt{2}+1}{15}
\end{aligned}$$

(β′)

$$\iint_R \frac{1}{(x+y+1)^3}dxdy,\ R: 0 \leq x \leq 2,\ 0 \leq y \leq 1$$

Λύση:

$$\begin{aligned}
I &= \iint_R \frac{1}{(x+y+1)^3}dxdy \\
&= \int_0^1 dy \int_0^2 dx \frac{1}{(x+y+1)^3} \\
&= \int_0^1 dy \left[-\frac{1}{2(x+y+1)^2}\right]_0^2 \\
&= \int_0^1 -\frac{1}{2(y+3)^2} + \frac{1}{2(y+1)^2}dy \\
&= \left[\frac{1}{2(y+3)} - \frac{1}{2(y+1)}\right]_0^1 \\
&= \frac{5}{24}
\end{aligned}$$

(γ′)

$$\iint_R xsin(xy)dxdy,\ R: 0 \leq x \leq 1,\ \pi \leq y \leq 2\pi$$

Λύση:

$$\begin{aligned}
I &= \iint_R xsin(xy)dxdy \\
&= \int_{\pi}^{2\pi} dy \int_0^1 dxxysin(xy) \\
&= \int_{\pi}^{2\pi} dy \left[-\frac{1}{y}xcos(xy) + \frac{1}{y^2}sin(xy)\right]_0^1 \\
&= \int_{\pi}^{2\pi} -\frac{1}{y}0cos(xy) + \frac{1}{y^2}sin(0y) + \frac{1}{y}cos(y) - \frac{1}{y^2}sin(y)dy \\
&= \int_{\pi}^{2\pi} \frac{1}{y}cos(y) - \frac{1}{y^2}sin(y)dy \\
&= \left[\frac{1}{y}sin(y)\right]_{\pi}^{2\pi} \\
&= 0
\end{aligned}$$

(δ′)

$$\iint_R (2x - 3y^2)dxdy,\ R : -1 \leq x \leq 1,\ 0 \leq y \leq 2$$

Λύση:

$$\begin{aligned}
I &= \iint_R (2x - 3y^2)dxdy \\
&= \int_0^2 dy \int_{-1}^1 dx(2x - 3y^2) \\
&= \int_0^2 \left[x^2 - 3y^2x\right]_{-1}^1 dy \\
&= \int_0^2 1 - 3y^2 - 1 - 3y^2dy \\
&= \int_0^2 -6y^2dy \\
&= \left[-2y^3\right]_0^2 \\
&= -16
\end{aligned}$$

(ε′)

$$\iint_R xcos(x^2 + y)dxdy,\ R : -\sqrt{\pi} \leq x \leq 0,\ 0 \leq y \leq \pi$$

Λύση:

$$\begin{aligned} I &= \iint_R xcos(x^2+y)dxdy \\ &= \int_0^\pi dy \int_{-\sqrt{\pi}}^0 dxxcos(x^2+y) \\ &= \int_0^\pi \left[\frac{1}{2}sin(x^2+y)\right]_{-\sqrt{\pi}}^0 dy \\ &= \int_0^\pi \frac{1}{2}sin(y) - \frac{1}{2}sin(\pi+y)dy \\ &= \int_0^\pi \frac{1}{2}sin(y) + \frac{1}{2}sin(y)dy \\ &= [-cos(y)]_0^\pi \\ &= 2 \end{aligned}$$

(ϛ′)

$$\iint_R x^2ye^{xy}dxdy, \ R: 0 \le x \le 1, \ 0 \le y \le 2$$

Λύση:

$$\begin{aligned} I &= \iint_R x^2ye^{xy}dxdy \\ &= \int_0^2 dy \int_0^1 dxx^2ye^{xy} \\ Use\ factorial\ integration\ two\ times. &= \int_0^2 dy\ y \int_0^1 dxx^2e^{xy} \\ Evaluate\ the\ integral\ of\ each\ term. &= \int_0^2 e^y - \frac{2e^y}{y} + \frac{2e^y-2}{y^2}dy \\ &= 2 \end{aligned}$$

2. Γράψτε τα ακόλουθα ολοκληρώματα σαν γινόμενο απλών ολοκληρωμάτων και να υπολογιστούν.

(α′)

$$\iint_R \frac{x^2}{1+y^2}dxdy, \ R: 0 \le x \le 1, \ 0 \le y \le 1$$

Λύση:

$$\begin{aligned} I &= \iint_R \frac{x^2}{1+y^2}dxdy \\ &= \int_0^1 dy\frac{1}{1+y^2}\int_0^1 dxx^2 \\ &= \left[tan^{-1}(y)\right]_0^1 * \left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{\pi}{12} \end{aligned}$$

(β′)

$$\iint_R \frac{x}{y}dxdy, \ R: 0 \le x \le 2, \ 1 \le y \le e$$

Λύση:

$$\begin{aligned} I &= \iint_R \frac{x}{y} dxdy \\ &= \int_1^e dy \frac{1}{y} \int_0^2 x dx \\ &= [ln|y|]_1^e * \left[\frac{x^2}{2}\right]_0^2 \\ &= 2 \end{aligned}$$

(γ′)

$$\iint_R e^{x-y} dxdy,\ R : -1 \leq x \leq 1,\ -1 \leq y \leq 1$$

Λύση:

$$\begin{aligned} I &= \iint_R e^{x-y} dxdy \\ &= \int_{-1}^1 dy e^{-y} \int_{-1}^1 e^x dx \\ &= \left[-e^{-y}\right]_{-1}^1 * [e^x]_{-1}^1 \\ &= -2 + e^2 + e^{-2} \end{aligned}$$

(δ′)

$$\iint_R xy(x^2+y^2) dxdy,\ R : 0 \leq x \leq 1,\ 0 \leq y \leq 1$$

Λύση:

$$\begin{aligned} I &= \iint_R xy(x^2+y^2) dxdy \\ &= \int_0^1 \int_0^1 x^3y + xy^3 dxdy \\ &= \int_0^1 xy^3 dy \int_0^1 x^3 y dx \\ &= \int_0^1 x dy \int_0^1 y^3 dy + \int_0^1 x^3 dx \int_0^1 y dy \\ &= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^4}{4}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \\ &= \tfrac{1}{4} \end{aligned}$$

(ε′)

$$\iint_R cos(x+y) dxdy,\ R : -\frac{\pi}{4} \leq x \leq \frac{\pi}{4},\ 0 \leq y \leq \frac{\pi}{4}$$

Λύση:

$$\begin{aligned}
I &= \iint_R cos(x+y)dxdy \\
&= \int_0^{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cos(x+y)dxdy \\
&= \int_0^{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cosxcosy - sinxsinydxdy \\
&= \int_0^{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cosxcosydxdy - \int_0^{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sinxsinydxdy \\
&= \int_0^{\frac{\pi}{4}} cosxdx \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cosydy - \int_0^{\frac{\pi}{4}} sinxdx \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sinydy \\
&= [sinx]_0^{\frac{\pi}{4}} [siny]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - [-cosx]_0^{\frac{\pi}{4}} [-cosy]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= 1
\end{aligned}$$

(ϛ′)

$$\iint_R xyln\frac{x}{y}dxdy,\ R: 1 \leq x \leq e,\ 1 \leq y \leq\ 2$$

Λύση:

$$\begin{aligned}
I &= \iint_R xyln\frac{x}{y}dxdy \\
&= \int_1^2 \int_1^e xylnx - xylnydxdy \\
&= \int_1^2 ydy \int_1^e xlnxdx - \int_1^2 ylnydy \int_1^e xdx \\
&= \left[\frac{y^2}{2}\right]_1^2 * \left[\frac{x^2}{2}lnx - \frac{x^2}{4}\right]_1^e - \left[\frac{y^2}{2}lny - \frac{y^2}{4}\right]_1^2 * \left[\frac{x^2}{2}\right]_1^e \\
&= e^2(\frac{3}{4} - ln2) + ln2
\end{aligned}$$

3. Γράψτε αναλυτικά την έκφραση I και να σχεδιάσεται τα χωρία ολοκλήρωσης, I $= \iint f(x,y)dxdy$.

(α′)

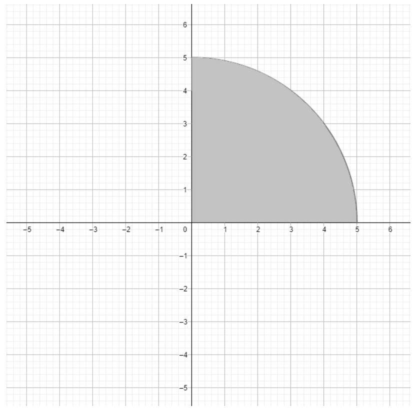
$$x = 0,\ y = 0,\ x^2 + y^2 = 25,\ 0 \leq x,\ 0 \leq y$$

Λύση:

$$Solve\ x^2 + y^2 = 25\ for\ y.$$

$$\begin{aligned}
x &> 0 \\
y &> 0 \\
y &= \sqrt{25 - x^2} \\
x &> 5
\end{aligned}$$

$$I = \int_0^5 (\int_0^{\sqrt{25-x^2}} f(x,y)\ dy)\ dx$$



(β′)

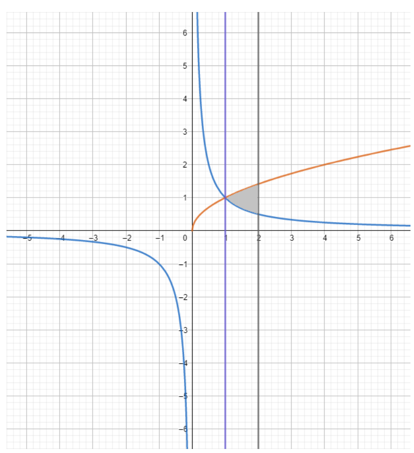
$$y = \frac{1}{x},\ y = \sqrt{x},\ x = 2$$

Λύση:

$$x = 2$$

$$\begin{aligned} y &= y \\ 1/x &= \sqrt{x} \\ x &= 1 \end{aligned}$$

$$I = \int_1^2 \left(\int_{\frac{1}{x}}^{\sqrt{x}} f(x,y)\ dy \right) dx$$



(γ′)

$$x^2 + y = 2,\ y^3 = x^2,\ x = 0$$

Λύση:

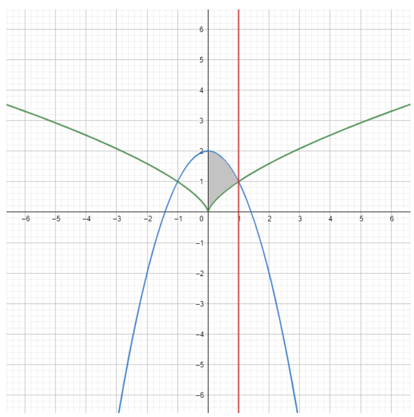
$$x = 0$$

$$y^3 = x^2$$
$$y = x^{\frac{2}{3}}$$

$$x^2 + y = 2$$
$$y = 2 - x^2$$

$$y = y$$
$$x^{\frac{2}{3}} = 2 - x^2,\ x > 0$$
$$x = 1$$

$$I = \int_0^1 (\int_{x^{\frac{2}{3}}}^{2-x^2} f(x,y)\ dy)\ dx$$



4. Αντιστρέψτε τη σειρά ολοκλήρωσης στα ακόλουθα ολοκληρώματα και να σχεδιάσεται τα χωρία ολοκλήρωσης.

(α′)

$$\int_0^1 dx \int_0^{lnx} 1\ dy$$

Λύση:

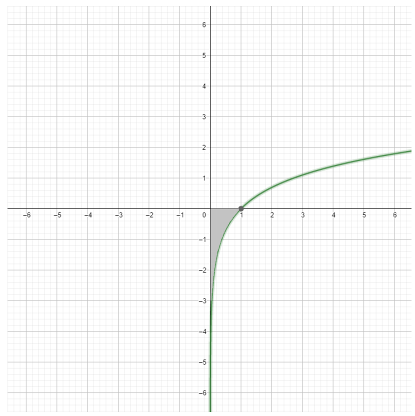
$$R : 0 \leq x \leq 1,\ 0 \leq y \leq\ lnx$$
$$R : e \leq x \leq e^y,\ 0 \leq y \leq\ 1$$

$$\begin{aligned} I &= \int_0^{-\infty} dy \int_0^{e^y} dx \\ &= \int_0^{-\infty} e^y dy \\ &= -1 \end{aligned}$$

$$
\begin{aligned}
I &= \int_0^1 lnx\ dx \int_0^{e^y} dx \\
&= [(x * lnx) - x]_0^1 \\
&= -1 - [(x * lnx) - x)]_{x=0}
\end{aligned}
$$

= -1

$(\lim_{x\to 0}((x * lnx) - x) = 0)$

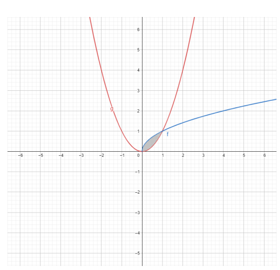


(β′)

$$
\int_0^1 dy \int_{y^2}^{\sqrt{y}} 1\ dx
$$

Λύση:

$$
\begin{aligned}
I &= \int_0^1 dx \int_{x^2}^{\sqrt{x}} 1\ dy \\
&= \int_0^1 [y]_{x^2}^{\sqrt{x}}\ dx \\
&= \int_0^1 \sqrt{x} - x^2\ dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

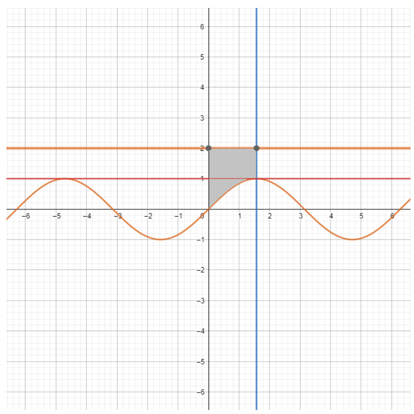


(γ′)

$$\int_0^{\pi/2} dx \int_{sinx}^{2} 1\ dy$$

Λύση:

$$\begin{aligned} I &= \int_0^{\pi/2} dx \int_{sinx}^{1} 1\ dy + \int_0^{\pi/2} dx \int_1^2 1\ dy \\ &= \int_0^1 dy \int_0^{arcsinx} 1\ dx + \int_1^2 dy \int_0^{\pi/2} 1\ dx \\ &= \int_0^1 [x]_0^{arcsinx}\, dy + \int_1^2 [x]_0^{\pi/2}\, dy \\ &= \int_0^1 arcsiny\ dy + \int_1^2 \pi/2\ dy \\ &= \left[y * arcsiny + (1-y^2)^{\frac{1}{2}}\right]_0^1 + \left[\frac{\pi}{2} * x\right]_1^2 \\ &= \pi - 1 \end{aligned}$$



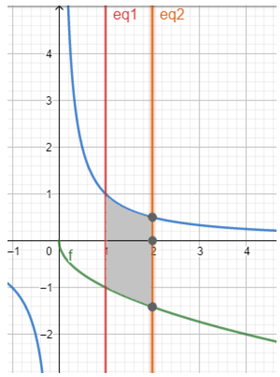
5. Να υπολογιστούν τα ακόλουθα ολοκληρώματα και να σχεδιάσετε τα χωρία ολοκλήρωσης.

(α′)

$$D : y = -\sqrt{x}, y = \frac{1}{x}, x = 1, x = 2$$

Λύση :

$$\begin{aligned} I &= \int_1^2 dx \int_{-\sqrt{x}}^{\frac{1}{x}} x^2 y dy \\ &= \int_1^2 \left[x^2 \frac{y^2}{2}\right]_{-\sqrt{x}}^{\frac{1}{x}} dx \\ &= \int_1^2 (x^2 \frac{1}{2x^2} - x^2 \frac{(-\sqrt{x})^2}{2}) dx \\ &= \int_1^2 (\frac{1}{2} - \frac{x^3}{2}) dx \\ &= \left[\frac{1}{2}x - \frac{x^4}{8}\right]_1^2 \\ &= -\frac{11}{8} \end{aligned}$$

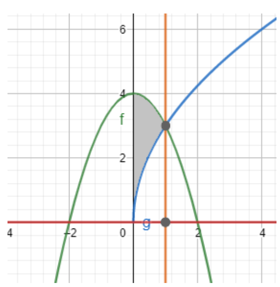


(β′)

$$D : y = -x^2 + 4, y = 3\sqrt{x}, y = 0$$

Λύση :

$$\begin{aligned} I &= \int_0^1 \int_{3\sqrt{x}}^{-x^2+4} xydydx \\ &= \int_0^1 \left[x\frac{y^2}{2} \right]_{3\sqrt{x}}^{-x^2+4} dx \\ &= \int_0^1 (-\frac{9x^2}{2} + \frac{x^5 - 8x^3 + 16x}{2})dx \\ &= \frac{1}{2} \left[\frac{x^6}{6} - \frac{8x^4}{4} - \frac{9x^3}{3} + \frac{16x^2}{2} \right]_0^1 \\ &= \frac{19}{12} \end{aligned}$$

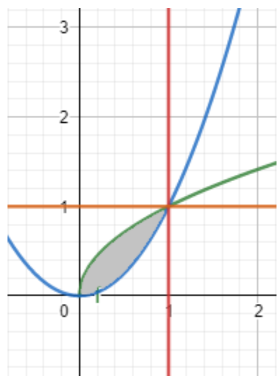


(γ′)

$$D : y = x^2, y^2 = x, x = 1, x = 2$$

Λύση :

$$\begin{aligned}
I &= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+y)dy \\
&= \int_0^1 \left[x^2 y + \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 (x^2\sqrt{x} + \frac{x}{2} - x^4 - \frac{x^4}{2})dx \\
&= \int_0^1 (x^{\frac{5}{2}} + \frac{x}{2} - 3\frac{x^4}{2})dx \\
&= \left[\frac{2}{7}x^{\frac{7}{2}} + \frac{x^2}{4} - 3\frac{x^5}{10}\right]_0^1 \\
&= \frac{33}{140}
\end{aligned}$$

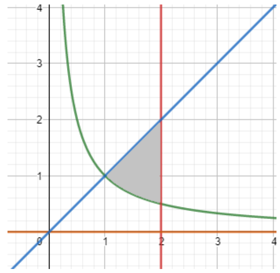


(δ′)

$$D : y = \frac{1}{x}, y = x, x = 2$$

Λύση :

$$\begin{aligned}
I &= \int_1^2 dx \int_x^{\frac{1}{x}} \frac{x^2}{y^2} dy \\
&= \int_1^2 \left[-\frac{x^2}{y}\right]_x^{\frac{1}{x}} dx \\
&= \int_1^2 (-\frac{x^2}{\frac{1}{x}} + \frac{x^2}{x})dx \\
&= \int_1^2 (-x^3 + x)dx \\
&= \left[-\frac{x^4}{4} + \frac{x^2}{2}\right]_1^2 \\
&= \frac{9}{4}
\end{aligned}$$

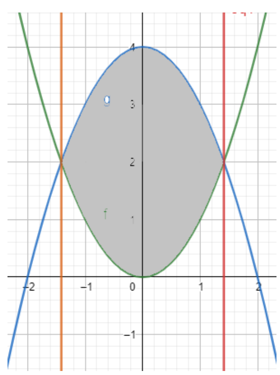


(ε′)

$$D : y = x^2, y = 4 - x^2$$

Λύση :

$$\begin{aligned} I &= \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{4-x^2} 1 dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} [y]_{4-x}^{x^2} \, dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} (2x^2 - 4) dx \\ &= \left[2\frac{x^3}{3} - 4x \right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{16\sqrt{2}}{3} \end{aligned}$$

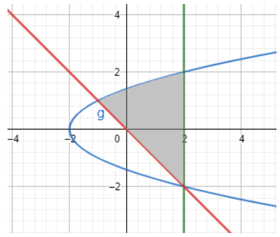


(ϛ′)

$$D : y^2 = x + 2, y = -x, x = 2$$

Λύση :

$$\begin{aligned} I &= \int_{-1}^{2} \int_{-x}^{\sqrt{x+2}} 2x dy dx \\ &= \int_{-1}^{2} [2xy]_{-x}^{\sqrt{x+2}} \, dx \\ &= \left[\frac{10x^3 + 20x(x+2)^{\frac{3}{2}} - 8(x+2)^{\frac{5}{2}}}{15} \right]_{-1}^{2} \\ &= \frac{182}{15} \end{aligned}$$

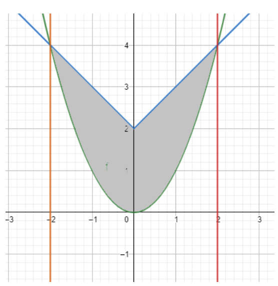


(ζ′)

$$D : y = x^2, y = 2 + |x|$$

Λύση :

$$\begin{aligned} I &= \int_{-2}^{2} dx \int_{x^2}^{|x|+2} 2xydy \\ &= \int_{-2}^{2} \left[xy^2\right]_{x^2}^{|x|+2} dx \\ &= \int_{-2}^{2} (x(|x|+2)^2 - x^5)dx \\ &= \int_{-2}^{0} x(x^2-2x+4) - x^5 dx + \int_{0}^{2} x(x^2+2x+4) - x^5 dx \\ &= \int_{-2}^{0} x^3 - 2x^2 + 4x - x^5 dx + \int_{0}^{2} x^3 + 2x^2 + 4x - x^5 dx \\ &= \left[\frac{x^4}{4} - \frac{2}{3}x^3 + 2x^2 - \frac{x^6}{6}\right]_{-2}^{0} + \left[\frac{x^4}{4} + \frac{2}{3}x^3 + 2x^2 - \frac{x^6}{6}\right]_{0}^{2} \\ &= 0 \end{aligned}$$

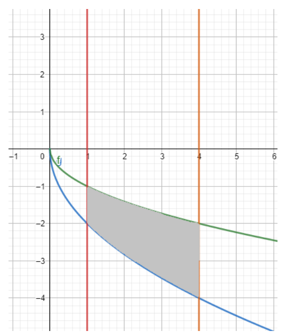


(η′)

$$D : y = -\sqrt{x}, y = -2\sqrt{x}, 1 \le x \le 4$$

Λύση :

$$
\begin{aligned}
I &= \int_1^4 dx \int_{-2\sqrt{x}}^{-\sqrt{x}} (x+2y)dy \\
&= \int_1^4 \left[xy+y^2\right]_{-\sqrt{x}}^{-2\sqrt{x}} dx \\
&= \int_1^4 (-x\sqrt{x}+x-(-2x\sqrt{x}+(-2\sqrt{x})^2)dx \\
&= \int_1^4 x\sqrt{x}-3xdx \\
&= \int_1^4 x^{\frac{3}{2}}-3xdx \\
&= \left[\frac{2}{3}x^{\frac{5}{2}}\right]_4^1 - \left[\frac{3}{2}x^2\right]_4^1 \\
&= -\frac{101}{10}
\end{aligned}
$$



6. Να υπολογιστούν τα ακόλουθα ολοκληρώματα με χρήση πολικών συντεταγμένων και να σχεδιαστούν τα χωρία ολοκλήρωσης.

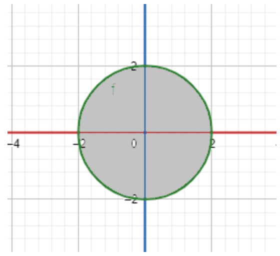
(α′)

$$\int\int (x^2+y^2)dxdy \; D: x^2+y^2 \leq 4$$

Λύση :

$$x = pcost, x^2+y^2=p^2, y=psint, 0 \leq p \leq 2, 0 \leq t \leq 2\pi$$

$$
\begin{aligned}
I &= \int\int \frac{j(x,y)}{j(p,t)} p^2 dpdt \\
&= \int_0^{2\pi} dt \int_0^2 p^3 dp \\
&= \int_0^{2\pi} dt \left[\frac{1}{4}p^4\right]_2^0 dp \\
&= \int_0^{2\pi} 4dt \\
&= [4t]_0^{2\pi} \\
&= 8\pi
\end{aligned}
$$



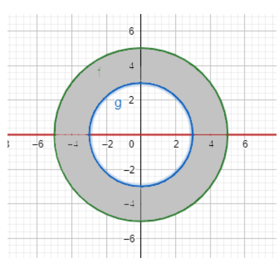
(β′)

$$\int\int \sqrt{x^2+y^2-9}dxdy \;\; D: x^2+y^2 \leq 25, 9 \leq x^2+y^2$$

Λύση :

$$x = pcost, x^2+y^2 = p^2, y = psint$$
$$3 \leq p \leq 5, 0 \leq t \leq 2\pi, \frac{j(x,y)}{j(p,t)} = p$$

$$\begin{aligned} I &= \int_0^{2\pi} dt \int_3^5 \frac{j(x,y)}{j(p,f)}\sqrt{p^2-9}dp \\ &= \int_0^{2\pi} dt \int_3^5 p\sqrt{p^2-9}dp \\ &= \frac{1}{2}\frac{2}{3}\int_0^{2\pi}\left[(p^2-9)^{\frac{3}{2}}\right]_3^5 dt \\ &= \frac{1}{3}\int_0^{2\pi} 16^{\frac{3}{2}}dt \\ &= \frac{1}{3}\int_0^{2\pi} 64dt \\ &= \frac{1}{3}\left[64t\right]_0^{2\pi} \\ &= \frac{128}{3}\pi \end{aligned}$$



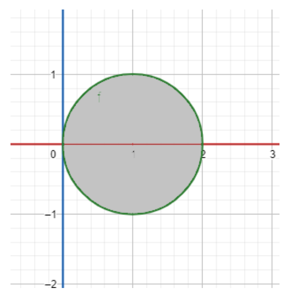
(γ′)

$$\int\int (x+y)dxdy \;\; D: x^2+y^2 \leq 2x$$

Λύση :

$$x = 1 + pcost, y = psint, (x-1)^2 + y^2 \leq 1$$
$$0 \leq t \leq 2\pi, \frac{j(x,y)}{j(p,t)} = p$$

$$\begin{aligned}
I &= \int\int (pcost + 1 + psint)\frac{j(x,y)}{j(p,t)} dpdt \\
&= \int_0^{\pi} dt \int_0^1 p + p^2(sint + cost)dt \\
&= \int_0^{2\pi} dt \left[\frac{p^2}{2} + \frac{p^3}{3}(sint + cost)\right]_0^1 \\
&= \int_0^{2\pi} dt(\frac{1}{2} + \frac{1}{3}(sint + cost)) \\
&= \pi + 0 \\
&= \pi
\end{aligned}$$



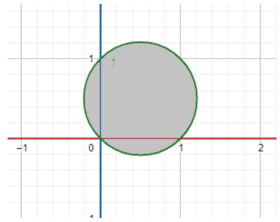
(δ′)

$$\int\int (x+y)dxdy \; D : x^2 + y^2 \leq x + y$$

Λύση :

$$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{2}, p = \sqrt{\frac{1}{2}}$$
$$x = \frac{1}{2} + psint, \; y = \frac{1}{2} + pcost$$

$$\begin{aligned}
I &= \int\int (\frac{1}{2} + pcost + \frac{1}{2} + psint)\frac{j(x,y)}{j(p,t)} dpdt \\
&= \int_0^{2\pi} dt \int_0^{\frac{1}{\sqrt{2}}} (\frac{1}{2} + pcost + \frac{1}{2} + psint)dp \\
&= \int_0^{2\pi} dt \int_0^{\sqrt{2}} p + p^2(cost + sint)dp \\
&= \int_0^{2\pi} \left[\frac{p^2}{2} + \frac{p^3}{3}(cost + sint)\right]_0^{\frac{1}{\sqrt{2}}} dt \\
&= \int_0^{2\pi} \left[\frac{1}{4} + \frac{1}{6\sqrt{2}}(cost + sint)\right] dt \\
&= \frac{\pi}{2}
\end{aligned}$$



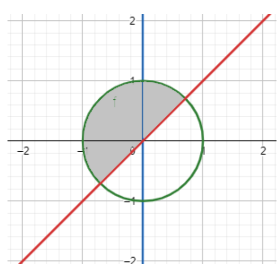
(εʹ)

$$\int\int x^2\sqrt{x^2+y^2}dxdy \;\; D: x^2+y^2 \le 1, x \le y$$

Λύση :

$$x = pcost, x^2+y^2=p^2, y = psint$$
$$0 \le p \le 1, 0 \le t \le \pi, \frac{j(x,y)}{j(p,t)} = p$$

$$\begin{aligned}
I &= \int_0^\pi dt \int_0^1 \frac{j(x,y)}{j(p,t)}\sqrt{p^2}(pcost)^2 dp \\
&= \int_0^\pi dt \int_0^1 p^4(cost)^2 dp \\
&= \int_0^\pi \frac{(cost)^2}{5} dt \\
&= \frac{1}{5}\int_0^\pi \frac{1+cos2t}{2} dt \\
&= \frac{1}{10}\int_0^\pi (t+\frac{sin4t}{2})dt \\
&= \frac{\pi}{10} + \left[\frac{sin2t}{4}\right]_0^\pi \\
&= \frac{\pi}{10}
\end{aligned}$$



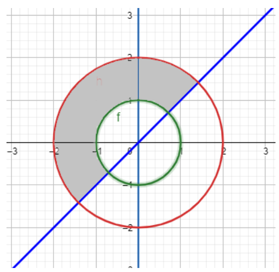
(ϛʹ)

$$\int\int \frac{1}{\sqrt{x^2+y^2}}dxdy \;\; D: 1 \le x^2+y^2, x^2+y^2 \le 4, x \le y$$

Λύση :

$$x = pcost, x^2 + y^2 = p^2, y = psint$$
$$1 \leq p \leq 2, 0 \leq t \leq \pi, \frac{j(x,y)}{j(p,t)} = p$$

$$\begin{aligned} I &= \int\int \sqrt{t^2} \frac{j(x,y)}{j(p,t)} dtdp \\ &= \int_0^\pi dt \int_1^2 \frac{1}{\sqrt{p^2}} pdp \\ &= \int_0^\pi [p]_1^2 \, dt \\ &= \int_0^\pi 1dt \\ &= [t]_0^\pi \\ &= \pi \end{aligned}$$



7. Να υπολογιστούν οι όγκοι των ακόλουθων στερεών.

(α′)

$$y = x^2, y = 1, z = 0, z = 2y$$

Λύση:

$$\begin{aligned} I &= \int_{-1}^1 \int_{x^2}^1 dxdy2y \\ &= \int_{-1}^1 dx \, [2y]_{x^2}^1 \\ &= \int_{-1}^1 (1 - x^4) dx \\ &= \left[x - \frac{x^5}{5} \right]_{-1}^1 \\ &= \frac{8}{5} \end{aligned}$$

(β′)

$$x = 0, y = 0, x + y = 1, z = 0, z = 2xy$$

Λύση:

$$\begin{aligned}
I &= \int_0^1 dx \int_0^{1-x} 2xydy \\
&= \int_0^1 dx \left[xy^2\right]_0^{1-x} \\
&= \int_0^1 dx(1-x^2)^2 \\
&= \int_0^1 dx(x+x^3-2x^2) \\
&= \left[\frac{x^2}{2}+\frac{x^4}{4}-\frac{2x}{3}\right]_0^1 \\
&= \frac{1}{12}
\end{aligned}$$

(γ′)

$$x=0, y=1, 2x+y=5, z=0, z=xy$$

Λύση:

$$\begin{aligned}
I &= \int_0^2 dx \int_1^{5-2x} xydy \\
&= \int_0^2 dx \left[x\frac{y^2}{2}\right]_1^{5-2x} \\
&= \int_0^2 x(\frac{(5-2x)^2}{2}-\frac{1}{2})dx \\
&= \int_0^2 x(\frac{25+4x^2-20x}{2}-\frac{1}{2})dx \\
&= \int_0^2 x(12x+2x^3-10x^2)dx \\
&= \left[6x^2+\frac{x^4}{2}-\frac{10x^3}{3}\right]_0^2 \\
&= \frac{16}{3}
\end{aligned}$$