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M9: Méthodes Numériques

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1 erreur9 avertissements

Table des matières

Liste des figures

Liste des tables

Introduction

FE

Chapitre 1

The Exact Boundary controllability

1.1 Formulation of the control problem

Let Ω be a bounded domain (that is, non-empty open connected set) in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ "Sufficiently smooth", Γ_1 be an open nonempty subset of Γ and $\Gamma_2 = \Gamma \backslash \Gamma_1$. With T a given positive number, we consider the following non-homogeneous wave equation :

$$\left\{ \begin{array}{rclcll} \dfrac{\partial^2 y}{\partial t^2} - \Delta y & = & 0 & \text{for} & (x,t) \in \Omega \times (0,T), & (1.1.1) \\ y(x,t) & = & 0 & \text{for} & (x,t) \in \Gamma_2 \times (0,T), & (1.1.2) \\ y(x,t) & = & v(x,t) & \text{for} & (x,t) \in \Gamma_1 \times (0,T), & (1.1.2) \\ y(x,0) & = & y^0, & \dfrac{\partial y}{\partial t}(x,0) = y^1 & \text{for} \quad x \in \Omega & (1.1.3) \end{array} \right. \tag{1.1}$$

$(1.1.2)$: The boundary conditions.
$(1.1.3)$ The initial conditions.
In (**1.1**), $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, such that $H^{-1}(\Omega)$ is the topological dual space of $H_0^{-1}(\Omega)$ and Δ is the Laplacian operator.
We shall now define the exact boundary controllability for the system (**1.1**).

Definition 1.1.1. *System* (**1.1**) *is controllable in time* $T > 0$ *if for every initial data* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *we can find a control function* $v \in L^2(\Gamma_1 \times (0,T))$ *such that the corresponding solution* ($y, y^{'}$) *of* (**1.1**) *verifies*

$$y(.,T) \quad = \quad y^{'}(.,T) = 0. \tag{1.2}$$

Remark 1.1.1. *For the existence of solution see section* $[1.2]$

Remark 1.1.2. *If the solution of* (**1.1**) *verifies* (**1.2**) *is also said to be null controllable in time* $T > 0$, *for more details* (*see* $[1]$, *page* 100).

The problem that we consider is the following one :
is it possible to find $T > 0$ sufficiently large or optimal and $v \in L^2(\Gamma_1 \times (0,T))$ a boundary

control function such that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we have (1.2). But before that, we have to prove that the system (1.1) admits a solution.

1.2 Existence and uniqueness of solutions

In this section we consider the general case of (1.1) whith v not necessarily null and a bounded domain Ω in $\mathbb{R}^n$.

Definition 1.2.1. *For (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(\Gamma_1 \times [0, +\infty))$ a function $y \in C([0, +\infty), L^2(\Omega)) \bigcap C^1([0, +\infty), H^{-1}(\Omega))$ is called a weak solution of (1.1) if the relation*

$$\begin{aligned} \int_\Omega y(x,t)\varphi(x)dx - \int_\Omega y^0(x)\varphi(x)dx - t < y^1, \varphi >_{-1,1} &= \int_0^t \int_0^s \int_\Omega y(x,\xi)\Delta\varphi(x)dxd\xi ds \\ &- \int_0^t \int_0^s \int_{\Gamma_1} v(x,\xi)\frac{\partial \varphi}{\partial n}(x)d\sigma d\xi ds, \end{aligned} \tag{1.3}$$

holds for every $t \geq 0$ and every $\varphi \in H^2(\Omega) \bigcap H_0^1(\Omega)$.

Remark 1.2.1. *For x in Γ and φ in $H^2(\Omega)$ we have*

$$\frac{\partial \varphi}{\partial n}(x) = \triangledown\varphi(x).\overrightarrow{n}(x),$$

is called the normal derivative, with $\overrightarrow{n}(x)$ is the normal vector at x

Remark 1.2.2. *The normal vector exists because the boundary Γ is sufficiently smooth.*

The main result for the existence of solutions of (1.1) is the following :

Theorem 1.2.1. *For every v in $L^2(\Gamma_1 \times (0,T))$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}$ system (1.1) has a unique weak solution*

$$(y, y^{'}) \in C([0,T], L^2(\Omega) \times H^{-1}),$$

moreover, there exists a constant $C = C(T) > 0$ such that

$$||(y, y^{'})||_{L^\infty([0,T],L^2(\Omega)\times H^{-1})} \leq C \left[||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}} + ||v||_{L^2(\Gamma_1\times(0,T))}\right]$$

Proof :
The theorem is a consequence of the theory of nonhomogeneous evolution equations.

$*$ Proof with variational method, see [2].
$*$ Proof with Semigroups operator, see [3].

1.3 Controllability by minimization

In this section we will give a necessary and sufficient condition for the exact controllability of (**1.1**) and also we will transform the controllability problem to a minimization problem, but before that we need to proof the following lemma :

Lemma 1.3.1. *For every* (φ^0, φ^1) *in* $H_0^1(\Omega) \times L^2(\Omega)$ *the following wave equation*

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi &= \quad 0 \qquad\qquad\qquad \text{in} \quad \Omega \times (0,T), \\ \qquad\quad \varphi &= \quad 0 \qquad\qquad\qquad \text{in} \quad \Gamma \times (0,T), \\ \quad \varphi(.,0) &= \quad \varphi^0, \quad \dfrac{\partial \varphi}{\partial t}(.,0) = \varphi^1 \quad \text{in} \quad \Omega \quad , \end{cases} \tag{1.4}$$

has a unique solution, moreover (**1.4**) *generates a group of isometries in* $H_0^1(\Omega) \times L^2(\Omega)$.

Proof :
this lemma is a consequence of the following classic theorem :

Theorem 1.3.1. ***(Stone, 1930)***
Let H *be a Hilbert space and* A *be a linear operator on* H *with dense domain, then* A *generates a* C_0*-group of unitary operators if and only if* A *is skewadjoint* ($A^{'} = -A$)

Let $w(t) = (\varphi(t), \varphi^{'}(t))$ and the state space is $H_0^1(\Omega) \times L^2(\Omega)$,
with the scalar product

$$< (\varphi_1, \varphi_2), (\psi_1, \psi_2) >= \int_\Omega \nabla\varphi_1 \nabla \psi_1 + \int_\Omega \varphi_2 \psi_2,$$

then the corresponding norm on H given by :

$$||(\varphi_1, \varphi_2)||^2 = \int_\Omega || \nabla \varphi_1||^2 + \int_\Omega ||\varphi_2||^2.$$

By the Poincaré inequality, this norm is eqivalent to the usual norm on H.
We define the linear operator A by :

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{and}$$

$$A = \begin{pmatrix} \bigcirc & I \\ \Delta & \bigcirc \end{pmatrix}.$$

Firstly the domain $D(A)$ is dense in H.
Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D(A)$, then

$$
\begin{aligned}
< A(\varphi_1, \varphi_2), (\psi_1, \psi_2) > &=< (\varphi_2, \Delta\varphi_1), (\psi_1, \psi_2) > \\
&= \int_\Omega \nabla\varphi_2 \nabla \psi_1 + \int_\Omega \Delta\varphi_1\psi_2 \\
&= \int_\Omega \nabla\varphi_2 \nabla \psi_1 - \int_\Omega \nabla\varphi_1 \nabla \psi_2 \\
&+ \underbrace{\int_\Gamma \frac{\partial\varphi_1}{\partial n}\psi_2}_{0} \\
&= - \int_\Omega \nabla\varphi_1 \nabla \psi_2 - \int_\Omega \varphi_2\Delta\psi_1 \\
&= - < (\varphi_1, \varphi_2), (\psi_2, \Delta\psi_1) > \\
&= - < (\varphi_1, \varphi_2), A(\psi_1, \psi_2) > .
\end{aligned}
$$

Which implies that the operator A is skewadjoint on H and thus generates a unitary C_0-group on H by Stone's theorem.
Then if (φ^0, φ^1) in $D(A)$ the system (**1.4**) has a unique strict solution by Semi-group theory, but in the case (φ^0, φ^1) in $H\backslash D(A)$, (**1.4**) has a unique weak solution by follows the proof of lemma.

The key result in this section is the variational characterization of the controllability of (**1.1**) given by the following lemma :

Lemma 1.3.2. *The initial data $(y^0, y^1) \in Ł^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there existes $v \in L^2(\Gamma_1 \times (0, T))$ such that :*

$$
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} v d\sigma dt =< y^1, \varphi^0 >_{-1,1} - \int_\Omega y^0\varphi^1 dx, \tag{1.5}
$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of the adjoint problem (**1.4**)

Proof :
For $(y^0, y^1) \in D(\Omega) \times D(\Omega)$, $(\varphi^0, \varphi^1) \in D(\Omega) \times D(\Omega)$ and $v \in D(\Gamma_1 \times (0, T))$, then y and φ are regular solutions. Multiplying the equation (1.1.1) in system (**1.1**) by φ and integrating, we have

$$\begin{aligned}
0 &= \int_0^T \int_\Omega (y^{''} - \Delta y)\varphi dx\, dt \\
&= \int_\Omega \int_0^T y^{''} \varphi dt\, dx - \int_0^T \int_\Omega \Delta y \varphi dx\, dt \\
&= \int_\Omega \left[\left[y^{'}\varphi \right]_0^T - \int_0^T y^{'}\varphi^{'} dt \right] dx + \int_0^T \int_\Omega \nabla y \nabla \varphi dx\, dt \\
&= \int_\Omega (y^{'}(T)\varphi(T) - y^{'}(0)\varphi(0))dx - \int_\Omega \left[y\varphi^{'} \right]_0^T dx + \int_\Omega \int_0^T y\varphi^{''} dt\, dx \\
&- \int_0^T \int_\Omega \Delta\varphi y dx\, dt + \int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt \\
&= \int_\Omega (y^{'}(T)\varphi(T) - y^{'}(0)\varphi(0))dx - \int_\Omega (y(T)\varphi^{'}(T) - y(0)\varphi^{'}(0))dx + \int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt
\end{aligned}$$

.

Hence,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &- \int_\Omega (y^{'}(T)\varphi(T) - y^{'}(0)\varphi(0))dx \\
&+ \int_\Omega (y(T)\varphi^{'}(T) - y(0)\varphi^{'}(0))dx,
\end{aligned}$$

finally we have,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &\int_\Omega y^1\varphi^0 dx - \int_\Omega y^0\varphi^1 dx \\
&+ \int_\Omega y(T)\varphi^{'}(T)dx - \int_\Omega y^{'}(T)\varphi(T)dx.
\end{aligned}$$

We Know that $D(\Omega) \times D(\Omega)$ dense in $L^2(\Omega) \times H^{-1}(\Omega)$ as well as in $H_0^1(\Omega) \times L^2(\Omega)$ and $D(\Gamma_1 \times (0,T))$ dense in $L^2(\Gamma_1 \times (0,T))$, and also we can show that there exists a constant $\hat{C} > 0$ such that :

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial\varphi}{\partial n} \right|^2 d\sigma\, dt \leq \hat{C} ||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.6}$$

then, $\frac{\partial\varphi}{\partial n}|_{\Gamma_1} \in L^2(\Gamma_1) \times (0,T)$ is a hidden regularity result, that may not be obtained by standard trace results $(see, [.])$. From a density argument we deduce that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ we have,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &< y^1\varphi^0 >_{-1,1} - \int_\Omega y^0, \varphi^1 dx \\
&+ < (\varphi(T), \varphi^{'}(T)), (y(T), y^{'}(T)) > .
\end{aligned}$$

With $< .,. >_{-1,1}$ be the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For all $(\varphi^0, \varphi^1) \in$

$H_0^1(\Omega) \times L^2(\Omega)$, $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we introduce the duality product

$$< (\varphi^0, \varphi^1), (y^0, y^1) >= \int_\Omega y^0 \varphi^1 - < y^1, \varphi^0 >_{-1,1} .$$

Such that the wave equation generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$, then (**1.5**) holds if and anly if (y^0, y^1) is controllable to zero in time $T > 0$. This completes the proof.
From lemme $1.3.2$, the equality (**1.5**) can be seen as an optimality condition for the minimants of the functional $\mathcal{J} \ : \ H_0^1(\Omega) \times L^2(\Omega) \ \rightarrow \ \mathbb{R}$, defined by

$$\mathcal{J} \ = \ \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma \ dt + < (\varphi^0, \varphi^1), (y^0, y^1) >, \tag{1.7}$$

where $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the corresponding solution of (**1.4**).
Before introducing the main theorem in this section we need to proof that the functional $\mathcal{J}$ has a minimant .

Definition 1.3.1. *System* (**1.4**) *is said to be observable in time $T > 0$ if there exists a positive a positive constant $C > 0$ such that*

$$C||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega) \times L^2(\Omega)} \ \leq \ \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma \ dt, \tag{1.8}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (**1.4**) *with initial data (φ^0, φ^1).*

In the following we assume that there is a positive time T^* such that for any $T > T^*$ the system (**1.4**) is observable.
On the other hand, the functional $\mathcal{J}$ is continuous, strictly convex and coercive.
It is easy to see that functional $\mathcal{J}$ is continuous, now let $(\varphi^0, \varphi^1), (\psi^0, \psi^1)$ in $H_0^1(\Omega) \times L^2(\Omega)$ and $\lambda \in]0, 1[$, we have,

$$\begin{aligned}
\mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1)) \ &= \ \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \lambda \frac{\partial \varphi}{\partial n} + (1-\lambda) \frac{\partial \psi}{\partial n} \right|^2 d\sigma \ dt \\
&+ \ < \lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1), (y^0, y^1) > \\
&= \ \lambda \mathcal{J}((\varphi^0, \varphi^1)) + (1-\lambda) \mathcal{J}((\psi^0, \psi^1)) \\
&- \ \frac{\lambda(1-\lambda)}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} - \frac{\partial \psi}{\partial n} \right|^2 d\sigma \ dt.
\end{aligned}$$

Using the observability inequation, we obtain,

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial}{\partial n} (\varphi - \psi) \right|^2 d\sigma \ dt \geq C||(\varphi^0 - \psi^0, \varphi^1 - \psi^1)||^2_{H_0^1(\Omega) \times L^2(\Omega)},$$

then if $(\varphi_0, \varphi_1) \neq (\psi_0, \psi_1)$, then,

$$\mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1)) \quad < \quad \lambda \mathcal{J}((\varphi^0, \varphi^1)) + (1-\lambda)\mathcal{J}((\psi^0, \psi^1)).$$

Hence $\mathcal{J}$ is strictly convex.
For the coercivity of the the functional $\mathcal{J}$, we have,

$$\begin{aligned}
\mathcal{J}((\varphi^0, \varphi^1)) \quad &\geq \quad \frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma\, dt - ||(\varphi^0, \varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} \\
&\geq \quad \frac{C}{2}||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} - ||(\varphi^0, \varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)},
\end{aligned}$$

then $\lim\limits_{||(\varphi^0,\varphi^1)||\to+\infty} \mathcal{J}((\varphi^0, \varphi^1)) = \infty.$
We conclude that the functional $\mathcal{J}$ has a unique minimizer $(\hat{\varphi^0}, \hat{\varphi^1})$ in $H_0^1(\Omega) \times L^2(\Omega)$, we have,

Theorem 1.3.2. *Let (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $(\hat{\varphi^0}, \hat{\varphi^1})$ in $H_0^1(\Omega) \times L^2(\Omega)$ be the unique minimizer of the functional $\mathcal{J}$, then the function $\hat{v}$ defined on $\Gamma_1 \times (0, T)$ by :*

$$\hat{v}(x,t) = \frac{\partial \hat{\varphi}}{\partial n}(x,t), \quad (x,t) \in \Gamma_1 \times (0,T),$$

is a control which leads (y^0, y^1) to zero in time $T > 0$.

Proof :
The functional $\mathcal{J}$ achieves its minimum at $(\hat{\varphi^0}, \hat{\varphi^1})$, then

$$\lim_{h\to 0} \frac{1}{h}\left[\mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1}) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1}))\right] = 0, \tag{1.9}$$

for all (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0, φ^1).
On the other hand, we have,

$$\begin{aligned}
\mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1}) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1})) = &\frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \hat{\varphi}}{\partial n} + h\frac{\partial \varphi}{\partial n}\right|^2 d\sigma\, dt + < (\hat{\varphi^0}, \hat{\varphi^1}) + h(\varphi^0, \varphi^1), (y^0, y^1) > \\
&- \frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \hat{\varphi}}{\partial n}\right|^2 d\sigma\, dt - < (\hat{\varphi^0}, \hat{\varphi^1}), (y^0, y^1) >,
\end{aligned}$$

hence,

$$\begin{aligned}
\frac{1}{h}\left[\mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1}) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi^0}, \hat{\varphi^1}))\right] = &\frac{h}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma\, dt + \int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n}\frac{\partial \varphi}{\partial n} d\sigma\, dt \\
&+ < (\varphi^0, \varphi^1), (y^0, y^1) >,
\end{aligned}$$

and from (**1.8**) we deduce that

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma\, dt = - < (\varphi^0, \varphi^1), (y^0, y^1) >$$
$$= - \int_\Omega y^0 \varphi^1 dx + < y^1, \varphi^0 >_{-1,1},$$

for every (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$.
From lemma 1.3.2, it follows that $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ is the control for (**1.1**). This complete the proof.
Now we can find the control of the wave equation by minimization of the functional $\mathcal{J}$, moreover, this control is the control of minimal L^2-norm :

Proposition 1.3.1. *Let $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ be the control given by minimizing the functional $\mathcal{J}$. If $w \in L^2(\Gamma_1 \times (0,T))$ is any other control for* (**1.1**), *then*

$$||\hat{v}||_{L^2(\Gamma_1 \times (0,T))} \leq ||w||_{L^2(\Gamma_1 \times (0,T))}. \tag{1.10}$$

Proof :
Let $(\hat{\varphi^0}, \hat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega)$ the minimizer of the functional $\mathcal{J}$ and w is a control function of (**1.1**). By taking $(\hat{\varphi^0}, \hat{\varphi^1})$ as initial data for (**1.4**), lemma 1.3.2 gives,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \hat{\varphi}}{\partial n} d\sigma\ dt = - \int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

hence,

$$||\hat{v}||^2_{L^2(\Gamma_1 \times (0,T))} = - \int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1} .$$

On the other hand,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} w\ d\sigma\ dt = - \int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

then,

$$||\hat{v}||^2_{L^2(\Gamma_1 \times (0,T))} = \int_0^T \int_{\Gamma_1} \hat{v}\ w\ d\sigma\ dt$$
$$\leq ||\hat{v}||_{L^2(\Gamma_1 \times (0,T))} ||w||_{L^2(\Gamma_1 \times (0,T))}.$$

Consequently, (**1.10**) is verified and the proof finishes.

Chapitre 2

Conjugate Gradient Algorithm

2.1 Variational Problem

In the previous chapter, we transferred the problem of controllability of (1.1) to a problem of minimization :

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \mathcal{J}((\varphi^0,\varphi^1)), \tag{2.1}$$

where the functional $\mathcal{J}$ is defined by (1.7).
Problem (2.1) can be written as follows :

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \left(\frac{1}{2}a((\varphi^0,\varphi^1),(\varphi^0,\varphi^1)) - L((\varphi^0,\varphi^1))\right), \tag{2.2}$$

where a is defined on $(H_0^1(\Omega)\times L^2(\Omega))\times(H_0^1(\Omega)\times L^2(\Omega))$ by

$$a((\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})) = \int_0^T\int_{\Gamma_1}\frac{\partial\varphi}{\partial n}\frac{\partial\widetilde{\varphi}}{\partial n}d\sigma\, dt, \quad \forall\, (\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega),$$

such that φ,$\widetilde{\varphi}$ respectively the solutions of (1.4) with initial data (φ^0,φ^1) and $(\widetilde{\varphi^0},\widetilde{\varphi^1})$, and L is defined on $H_0^1(\Omega)\times L^2(\Omega)$ by

$$\begin{aligned} L((\varphi^0,\varphi^1)) &= -<(\varphi^0,\varphi^1),(y^0,y^1)> \\ &= -\int_\Omega y^0\varphi^1 + <y^1,\varphi^0>_{-1,1}, \end{aligned}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ and (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$. we have,

Lemma 2.1.1. ♦ *The operator a is a bilinear form, continuous and $H_0^1(\Omega)\times L^2(\Omega)$-elliptic.*
♦ *For all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, we have $L\in L^2(\Omega)\times H^{-1}(\Omega)$.*

Proof :
• it easy to proof that L is linear continuous and a is bilinear
• The continuity of a follows from the inequality (1.6) and the coercivity follows from the inequality of observability (1.8).
Moreover the bilinear form a is symmetric, by follows and with the theorem of Lax-Milgram

the problem (2.2) reads as follows :

$$\begin{cases} \text{Find } (\widehat{\varphi^0}, \widehat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a((\widehat{\varphi^0}, \widehat{\varphi^1}), (\varphi^0, \varphi^1)) = L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega). \end{cases} \tag{2.3}$$

2.2 General conjugate gradient algorithm

Let H a Hilbert space, a a continuous, symmetric and coercive bilinear form on $H \times H$, and L a continuous linear form on H, the variational problem problem (2.3) is a particular case of the following general variational problem :

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u, v) = L(v), \quad \forall v \in H. \end{cases} \tag{2.4}$$

With the above hypotheses, problem (2.4) has a unique solution.
For the numerical solution of (2.4) we need the following algorithm, say the conjugate gradient algorithm :
(1) $u^{(0)}$: any arbitrarily vector in H ;

(2) solve

$$\begin{cases} \widetilde{u}^{(0)} \in H \\ < \widetilde{u}^{(0)}, v >_H = a(u^{(0)}, v) - L(v), \quad \forall v \in H. \end{cases} \tag{2.5}$$

(3) • If $\widetilde{u}^{(0)}$ is small ($\dfrac{||\widetilde{u}^{(0)}||}{||u^{(0)}||} < \epsilon$), take $u = u^{(0)}$;
• If not,set $\breve{u}^{(0)} = \widetilde{u}^{(0)}$;

Assuming that $u^{(n)}$, $\widetilde{u}^{(n)}$, $\breve{u}^{(n)}$ are known, compute $u^{(n+1)}$, $\widetilde{u}^{(n+1)}$, $\breve{u}^{(n+1)}$:

(4) $\rho_n = \dfrac{||\widetilde{u}^{(n)}||^2}{a(\breve{u}^{(n)}, \breve{u}^{(n)})}$;

(5) $u^{(n+1)} = u^{(n)} - \rho_n \breve{u}^{(n)}$;

(6) solve

$$\begin{cases} \widetilde{u}^{(n+1)} \in H \\ < \widetilde{u}^{(n+1)}, v >_H = < \widetilde{u}^{(n)}, v >_H - \rho_n a(\breve{u}^{(n)}, v), \quad \forall v \in H. \end{cases} \tag{2.6}$$

(7) • If $\widetilde{u}^{(n+1)}$ is small ($\dfrac{||\widetilde{u}^{(n+1)}||}{||\widetilde{u}^{(0)}||} < \epsilon$), take $u = u^{(n+1)}$;
• If not,
$\star$ $\gamma_n = \dfrac{||\widetilde{u}^{(n+1)}||^2}{||\widetilde{u}^{(n)}||^2}$;

$\star$ $\breve{u}^{(n+1)} = \widetilde{u}^{(n+1)} + \gamma_n \breve{u}^{(n)}$;
(8) $n = n + 1$ and go to (4);

2.3 Application of conjugate gradient algorithm

Let us now apply the general conjugate gradient algorithm to the solution of (2.3).
(1) $(\varphi_0^0, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$: Initialization ;

(2) solve

$$\begin{cases} (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}) \in H \\ < (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}), (\varphi^0, \varphi^1) >_H = a((\varphi_0^0, \varphi_0^1), (\varphi^0, \varphi^1)) - L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H, \end{cases} \tag{2.7}$$

consider the following non-homogeneous backward wave equation :

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 &= & 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_0 &= & \dfrac{\partial \varphi_0}{\partial n} & \text{in} \quad \Gamma_1 \times (0,T), \\ \psi_0 &= & 0 & \text{in} \quad \Gamma_2 \times (0,T), \\ \psi_0(.,T) &= & 0, & \dfrac{\partial \psi_0}{\partial t}(.,T) = 0 \quad \text{in} \quad \Omega \quad , \end{cases}$$

From the lemma $1.3.2$, we have,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \varphi_0}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma \; dt = < \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx,$$

for every $(\varphi^0, \varphi^1) \in H$, then we obtain,

$$\begin{aligned} \int_\Omega \nabla \widetilde{\varphi_0^0} \nabla \varphi^0 dx + \int_\Omega \widetilde{\varphi_0^1} \varphi^1 dx = &< \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx \\ &+ \int_\Omega y^0 \varphi^1 dx - < y^1, \varphi^0 >_{-1,1}, \end{aligned}$$

hence,

$$< -\Delta \widetilde{\varphi_0^0}, \varphi^0 >_{-1,1} - < \psi_0'(0) - y^1, \varphi^0 >_{-1,1} = \int_\Omega (y^0 - \psi_0(0) - \widetilde{\varphi_0^1}) \varphi^1 dx,$$

finally,

$$< (\varphi^0, \varphi^1), (y^0 - \psi_0(0) - \widetilde{\varphi_0^1} \; , -\Delta \widetilde{\varphi_0^0} - (\psi_0'(0) - y^1)) >= 0.$$

Its follows : (2)

$$\begin{cases} -\Delta \widetilde{\varphi_0^0} = \psi_0'(0) - y^1 \\ \widetilde{\varphi_0^1} = y^0 - \psi_0(0) \end{cases} \tag{2.8}$$

(3) • If $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$ is small , take $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_0^0, \varphi_0^1)$;

• If not, set $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$,

assuming that $(\varphi_n^0, \varphi_n^1)$, $(\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1})$, $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ and φ_n, ψ_n are known, compute $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$, $(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1})$, φ_{n+1}, ψ_{n+1}.

we knew that the form linear $(\varphi^0,\varphi^1)\in H\longmapsto a((\check{\varphi}^0_n,\check{\varphi}^1_n),(\varphi^0,\varphi^1))$ is continuous, then by Riesz's theorem there exists unique $(\underline{\varphi^0_n},\underline{\varphi^1_n})$ in H, such that

$$a((\check{\varphi}^0_n,\check{\varphi}^1_n),(\varphi^0,\varphi^1)) =< (\underline{\varphi^0_n},\underline{\varphi^1_n}),(\varphi^0,\varphi^1)>,\quad \forall(\varphi^0,\varphi^1)\in H.$$

Like the previous case, we can find $(\underline{\varphi^0_n},\underline{\varphi^1_n})$ by :

(4) solve

$$\begin{cases} \dfrac{\partial^2\check{\varphi}_n}{\partial t^2}-\Delta\check{\varphi}_n = 0 & \text{in } \Omega\times(0,T),\\ \check{\varphi}_n = 0 & \text{in } \Gamma\times(0,T),\\ \check{\varphi}_n(.,0) = \check{\varphi}^0_n, \quad \dfrac{\partial\check{\varphi}_n}{\partial t}(.,0)=\check{\varphi}^1_n & \text{in } \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2\check{\psi}_n}{\partial t^2}-\Delta\check{\psi}_n = 0 & \text{in } \Omega\times(0,T),\\ \check{\psi}_n = \dfrac{\partial\check{\varphi}_n}{\partial n} & \text{in } \Gamma_1\times(0,T),\\ \check{\psi}_n = 0 & \text{in } \Gamma_2\times(0,T),\\ \check{\psi}_n(.,T) = 0, \quad \dfrac{\partial\check{\psi}_n}{\partial t}(.,T)=0 & \text{in } \Omega \end{cases}$$

Compute now $(\underline{\varphi^0_n},\underline{\varphi^1_n})\in H$ by :

$$\begin{cases} -\Delta\underline{\varphi^0_n} = \check{\psi}_n{}'(0) \quad \text{in } \Omega\\ \\ \underline{\varphi^1_n} = -\check{\psi}_n(0) \end{cases} \tag{2.9}$$

(5)

$$\rho_n=\frac{1}{2}$$

Once ρ_n is know, compute :

(6)

$*\ (\varphi^0_{n+1},\varphi^1_{n+1}) =$

$*\ \varphi_{n+1} =$

$*\ \psi_{n+1} =$

$*\ (\widetilde{\varphi^0}_{n+1},\widetilde{\varphi^1}_{n+1}) =$

(7) If $(\widetilde{\varphi^0}_{n+1},\widetilde{\varphi^1}_{n+1})$ small, take $(\widehat{\varphi^0},\widehat{\varphi^1})=(\varphi^0_{n+1},\varphi^1_{n+1})$;

Conclusion

OK

Bibliographie

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