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Chapitre 1

The Exact Boundary controllability

1.1 Formulation of the control problem

Let Ω be a bounded domain (that is, non-empty open connected set) in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ "Sufficiently smooth", Γ_1 be an open nonempty subset of Γ and $\Gamma_2 = \Gamma \backslash \Gamma_1$. With T a given positive number, we consider the following non-homogeneous wave equation :

$$\left\{\begin{array}{rcllr} \dfrac{\partial^2 y}{\partial t^2} - \Delta y &=& 0 & \text{for} \quad (x,t) \in \Omega \times (0,T), & (1.1.1)\\ y(x,t) &=& 0 & \text{for} \quad (x,t) \in \Gamma_2 \times (0,T), & (1.1.2)\\ y(x,t) &=& v(x,t) & \text{for} \quad (x,t) \in \Gamma_1 \times (0,T), & (1.1.2)\\ y(x,0) &=& y^0, & \dfrac{\partial y}{\partial t}(x,0) = y^1 \quad \text{for} \quad x \in \Omega. & (1.1.3) \end{array}\right. \tag{1.1}$$

$(1.1.2)$: The boundary conditions.
$(1.1.3)$ The initial conditions.
In (**1.1**), $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, such that $H^{-1}(\Omega)$ is the topological dual space of $H_0^{-1}(\Omega)$ and Δ is the Laplacian operator.
We shall now define the exact boundary controllability for the system (**1.1**).

Definition 1.1.1. *System* (**1.1**) *is controllable in time* $T > 0$ *if for every initial data* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *we can find a control function* $v \in L^2(\Gamma_1 \times (0,T))$ *such that the corresponding solution* ($y, y^{'}$) *of* (**1.1**) *verifies*

$$y(.,T) \;=\; y^{'}(.,T) = 0. \tag{1.2}$$

Remark 1.1.1. *For the existence of solution see section* $[1.2]$

Remark 1.1.2. *If the solution of* (**1.1**) *verifies* (**1.2**) *is also said to be null controllable in time* $T > 0$, *for more details* (*see* $[1]$, *page* 100).

The problem that we consider is the following one :
is it possible to find $T > 0$ sufficiently large or optimal and $v \in L^2(\Gamma_1 \times (0,T))$ a boundary

control function such that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we have (1.2). But before that, we have to prove that the system (1.1) admits a solution.

1.2 Existence and uniqueness of solutions

In this section we consider the general case of (1.1) whith v not necessarily null and a bounded domain Ω in $\mathbb{R}^n$.

Definition 1.2.1. *For (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(\Gamma_1 \times [0, +\infty))$ a function $y \in C([0, +\infty), L^2(\Omega)) \bigcap C^1([0, +\infty), H^{-1}(\Omega))$ is called a weak solution of (1.1) if the relation*

$$\begin{aligned} \int_\Omega y(x,t)\varphi(x)dx - \int_\Omega y^0(x)\varphi(x)dx - t < y^1, \varphi >_{-1,1} &= \int_0^t \int_0^s \int_\Omega y(x,\xi)\Delta\varphi(x)dxd\xi ds \\ &- \int_0^t \int_0^s \int_{\Gamma_1} v(x,\xi)\frac{\partial \varphi}{\partial n}(x)d\sigma d\xi ds, \end{aligned} \tag{1.3}$$

holds for every $t \geq 0$ and every $\varphi \in H^2(\Omega) \bigcap H_0^1(\Omega)$.

Remark 1.2.1. *For x in Γ and φ in $H^2(\Omega)$ we have*

$$\frac{\partial \varphi}{\partial n}(x) = \bigtriangledown \varphi(x).\overrightarrow{n}(x),$$

is called the normal derivative, with $\overrightarrow{n}(x)$ is the normal vector at x

Remark 1.2.2. *The normal vector exists because the boundary Γ is sufficiently smooth.*

The main result for the existence of solutions of (1.1) is the following :

Theorem 1.2.1. *For every v in $L^2(\Gamma_1 \times (0, T))$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}$ system (1.1) has a unique weak solution*

$$(y, y^{'}) \in C([0, T], L^2(\Omega) \times H^{-1}),$$

moreover, there exists a constant $C = C(T) > 0$ such that

$$||(y, y^{'})||_{L^\infty([0,T], L^2(\Omega) \times H^{-1})} \leq C \left[||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}} + ||v||_{L^2(\Gamma_1 \times (0,T))} \right]$$

Proof :
The theorem is a consequence of the theory of nonhomogeneous evolution equations.

$\ast$ Proof with variational method, see [2].
$\ast$ Proof with Semigroups operator, see [3].

1.3 Controllability by minimization

In this section we will give a necessary and sufficient condition for the exact controllability of (1.1) and also we will transform the controllability problem to a minimization problem, but before that we need to proof the following lemma :

Lemma 1.3.1. *For every* (φ^0, φ^1) *in* $H_0^1(\Omega) \times L^2(\Omega)$ *the following wave equation*

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi &= & 0 & & \text{in} \quad \Omega \times (0,T), \\ \varphi &= & 0 & & \text{in} \quad \Gamma \times (0,T), \\ \varphi(.,0) &= & \varphi^0, & \dfrac{\partial \varphi}{\partial t}(.,0) = \varphi^1 & \text{in} \quad \Omega \quad , \end{cases} \tag{1.4}$$

has a unique solution, moreover (1.4) *generates a group of isometries in* $H_0^1(\Omega) \times L^2(\Omega)$.

Proof :
this lemma is a consequence of the following classic theorem :

Theorem 1.3.1. ***(Stone, 1930)***
Let H *be a Hilbert space and* A *be a linear operator on* H *with dense domain, then* A *generates a* C_0*-group of unitary operators if and only if* A *is skewadjoint* ($A^{'} = -A$)

Let $w(t) = (\varphi(t), \varphi^{'}(t))$ and the state space is $H_0^1(\Omega) \times L^2(\Omega)$,
with the scalar product

$$< (\varphi_1, \varphi_2), (\psi_1, \psi_2) >= \int_\Omega \nabla\varphi_1 \nabla \psi_1 + \int_\Omega \varphi_2 \psi_2,$$

then the corresponding norm on H given by :

$$||(\varphi_1, \varphi_2)||^2 = \int_\Omega || \nabla \varphi_1||^2 + \int_\Omega ||\varphi_2||^2.$$

By the Poincaré inequality, this norm is eqivalent to the usual norm on H.
We define the linear operator A by :

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{and}$$

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}.$$

Firstly the domain $D(A)$ is dense in H.
Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D(A)$, then

$$
\begin{aligned}
< A(\varphi_1, \varphi_2), (\psi_1, \psi_2) > &=< (\varphi_2, \Delta\varphi_1), (\psi_1, \psi_2) > \\
&= \int_\Omega \nabla\varphi_2 \nabla \psi_1 + \int_\Omega \Delta\varphi_1\psi_2 \\
&= \int_\Omega \nabla\varphi_2 \nabla \psi_1 - \int_\Omega \nabla\varphi_1 \nabla \psi_2 \\
&+ \underbrace{\int_\Gamma \frac{\partial\varphi_1}{\partial n}\psi_2}_{0} \\
&= -\int_\Omega \nabla\varphi_1 \nabla \psi_2 - \int_\Omega \varphi_2\Delta\psi_1 \\
&= - < (\varphi_1, \varphi_2), (\psi_2, \Delta\psi_1) > \\
&= - < (\varphi_1, \varphi_2), A(\psi_1, \psi_2) > .
\end{aligned}
$$

Which implies that the operator A is skewadjoint on H and thus generates a unitary C_0-group on H by Stone's theorem.
Then if (φ^0, φ^1) in $D(A)$ the system (**1.4**) has a unique strict solution by Semi-group theory, but in the case (φ^0, φ^1) in $H\backslash D(A)$, (**1.4**) has a unique weak solution by follows the proof of lemma.

Proposition 1.3.1. *Assume that $T > 0$ is large enough, then there exists a constant $\hat{C} > 0$ such that :*

$$\int_0^T \int_{\Gamma_1} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt \leq \hat{C}||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.5}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the solution of (**1.4**).

Proof :
The inequality (**1.5**) is a hildden regularity result, that may not be obtained by standard trace results $(see, [.])$.

The key result in this section is the variational characterization of the controllability of (**1.1**) given by the following lemma :

Lemma 1.3.2. *The initial data $(y^0, y^1) \in Ł^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there existes $v \in L^2(\Gamma_1 \times (0, T))$ such that :*

$$\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} v d\sigma dt =< y^1, \varphi^0 >_{-1,1} - \int_\Omega y^0\varphi^1 dx, \tag{1.6}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of the adjoint problem (**1.4**)

Proof :
For $(y^0, y^1) \in D(\Omega) \times D(\Omega)$, $(\varphi^0, \varphi^1) \in D(\Omega) \times D(\Omega)$ and $v \in D(\Gamma_1 \times (0, T))$, then y and

φ are regular solutions. Multiplying the equation (1.1.1) in system (**1.1**) by φ and integrating, we have

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega (y'' - \Delta y)\varphi dx\, dt \\
&= \int_\Omega \int_0^T y''\varphi dt\, dx - \int_0^T \int_\Omega \Delta y \varphi dx\, dt \\
&= \int_\Omega \left[\left[y'\varphi\right]_0^T - \int_0^T y'\varphi' dt \right] dx + \int_0^T \int_\Omega \nabla y \nabla \varphi dx\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx - \int_\Omega \left[y\varphi'\right]_0^T dx + \int_\Omega \int_0^T y\varphi'' dt\, dx \\
&- \int_0^T \int_\Omega \Delta\varphi y dx\, dt + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx - \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0))dx + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt
\end{aligned}
$$
.

Hence,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &- \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx \\
&+ \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0))dx,
\end{aligned}
$$

finally we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &\int_\Omega y^1\varphi^0 dx - \int_\Omega y^0\varphi^1 dx \\
&+ \int_\Omega y(T)\varphi'(T)dx - \int_\Omega y'(T)\varphi(T)dx.
\end{aligned}
$$

We Know that $D(\Omega) \times D(\Omega)$ dense in $L^2(\Omega) \times H^{-1}(\Omega)$ as well as in $H_0^1(\Omega) \times L^2(\Omega)$ and $D(\Gamma_1 \times (0,T))$ dense in $L^2(\Gamma_1 \times (0,T))$. by the inequality (**1.5**), we have $\frac{\partial \varphi}{\partial n}|_{\Gamma_1} \in L^2(\Gamma_1) \times (0,T)$. From a density argument we deduce that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &< y^1\varphi^0 >_{-1,1} - \int_\Omega y^0, \varphi^1 dx \\
&+ < (\varphi(T), \varphi'(T)), (y(T), y'(T)) > .
\end{aligned}
$$

With $< .,. >_{-1,1}$ be the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we introduce the duality product

$$
< (\varphi^0, \varphi^1), (y^0, y^1) >= \int_\Omega y^0\varphi^1 - < y^1, \varphi^0 >_{-1,1} .
$$

Such that the wave equation generates a group of isometries in $H_0^1(\Omega)\times L^2(\Omega)$, then (**1.6**) holds if and anly if (y^0,y^1) is controllable to zero in time $T>0$. This completes the proof.
From lemme $1.3.2$, the equality (**1.6**) can be seen as an optimality condition for the minimants of the functional $\mathcal{J} \;:\; H_0^1(\Omega)\times L^2(\Omega) \;\to\; \mathbb{R}$, defined by

$$\mathcal{J} \;=\; \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\; dt+ <(\varphi^0,\varphi^1),(y^0,y^1)>, \tag{1.7}$$

where $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$ and φ is the corresponding solution of (**1.4**).
Before introducing the main theorem in this section we need to proof that the functional $\mathcal{J}$ has a minimant .

Definition 1.3.1. *System* (**1.4**) *is said to be observable in time* $T>0$ *if there exists a positive a positive constant* $C>0$ *such that*

$$C||(\varphi^0,\varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \;\leq\; \int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\; dt, \tag{1.8}$$

for all $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$ *where* φ *is the solution of* (**1.4**) *with initial data* (φ^0,φ^1).

In the following we assume that there is a positive time T^* such that for any $T>T^*$ the system (**1.4**) is observable.
On the other hand, the functional $\mathcal{J}$ is continuous, strictly convex and coercive.
It is easy to see that functional $\mathcal{J}$ is continuous, now let $(\varphi^0,\varphi^1),(\psi^0,\psi^1)$ in $H_0^1(\Omega)\times L^2(\Omega)$ and $\lambda\in]0,1[$, we have,

$$\begin{aligned}
\mathcal{J}(\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1)) \;&=\; \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\lambda\frac{\partial\varphi}{\partial n}+(1-\lambda)\frac{\partial\psi}{\partial n}\right|^2 d\sigma\; dt\\
&+\; <\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1),(y^0,y^1)>\\
&=\; \lambda\mathcal{J}((\varphi^0,\varphi^1))+(1-\lambda)\mathcal{J}((\psi^0,\psi^1))\\
&-\; \frac{\lambda(1-\lambda)}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}-\frac{\partial\psi}{\partial n}\right|^2 d\sigma\; dt.
\end{aligned}$$

Using the observability inequation, we obtain,

$$\int_0^T\int_{\Gamma_1}\left|\frac{\partial}{\partial n}(\varphi-\psi)\right|^2 d\sigma\; dt\geq C||(\varphi^0-\psi^0,\varphi^1-\psi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)},$$

then if $(\varphi_0,\varphi_1)\neq(\psi_0,\psi_1)$, then,

$$\mathcal{J}(\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1)) \;<\; \lambda\mathcal{J}((\varphi^0,\varphi^1))+(1-\lambda)\mathcal{J}((\psi^0,\psi^1)).$$

Hence $\mathcal{J}$ is strictly convex.
For the coercivity of the the functional $\mathcal{J}$, we have,

$$\begin{aligned}
\mathcal{J}((\varphi^0,\varphi^1)) &\geq \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} \\
&\geq \frac{C}{2}||(\varphi^0,\varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)},
\end{aligned}$$

then $\lim\limits_{||(\varphi^0,\varphi^1)||\to+\infty}\mathcal{J}((\varphi^0,\varphi^1))=\infty$.
We conclude that the functional $\mathcal{J}$ has a unique minimizer $(\hat{\varphi^0},\hat{\varphi^1})$ in $H_0^1(\Omega)\times L^2(\Omega)$, we have,

Theorem 1.3.2. *Let (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$ and $(\hat{\varphi^0},\hat{\varphi^1})$ in $H_0^1(\Omega)\times L^2(\Omega)$ be the unique minimizer of the functional $\mathcal{J}$, then the function $\hat{v}$ defined on $\Gamma_1\times(0,T)$ by :*

$$\hat{v}(x,t)=\frac{\partial\hat{\varphi}}{\partial n}(x,t),\quad (x,t)\in\Gamma_1\times(0,T),$$

is a control which leads (y^0,y^1) to zero in time $T>0$.

Proof :
The functional $\mathcal{J}$ achieves its minimum at $(\hat{\varphi^0},\hat{\varphi^1})$, then

$$\lim_{h\to 0}\frac{1}{h}\left[\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1})+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1}))\right]=0, \tag{1.9}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0,φ^1). On the other hand, we have,

$$\begin{aligned}
\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1})+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1})) &= \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n}+h\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt + < (\hat{\varphi^0},\hat{\varphi^1})+h(\varphi^0,\varphi^1),(y^0,y^1) > \\
&\quad -\frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n}\right|^2 d\sigma\, dt - < (\hat{\varphi^0},\hat{\varphi^1}),(y^0,y^1) >,
\end{aligned}$$

hence,

$$\begin{aligned}
\frac{1}{h}\left[\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1})+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi^0},\hat{\varphi^1}))\right] &= \frac{h}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt + \int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\, dt \\
&\quad + < (\varphi^0,\varphi^1),(y^0,y^1) >,
\end{aligned}$$

and from (1.8) we deduce that

$$\begin{aligned}
\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\, dt &= - < (\varphi^0,\varphi^1),(y^0,y^1) > \\
&= -\int_\Omega y^0\varphi^1 dx + < y^1,\varphi^0 >_{-1,1},
\end{aligned}$$

for every (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$.

From lemma 1.3.2, it follows that $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ is the control for (**1.1**). This complete the proof. Now we can find the control of the wave equation by minimization of the functional $\mathcal{J}$, moreover, this control is the control of minimal L^2-norm :

Proposition 1.3.2. *Let $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ be the control given by minimizing the functional $\mathcal{J}$. If $w \in L^2(\Gamma_1 \times (0,T))$ is any other control for (**1.1**), then*

$$||\hat{v}||_{L^2(\Gamma_1\times(0,T))} \leq ||w||_{L^2(\Gamma_1\times(0,T))}. \tag{1.10}$$

Proof :
Let $(\hat{\varphi^0}, \hat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega)$ the minimizer of the functional $\mathcal{J}$ and w is a control function of (**1.1**). By taking $(\hat{\varphi^0}, \hat{\varphi^1})$ as initial data for (**1.4**), lemma 1.3.2 gives,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n}\frac{\partial \hat{\varphi}}{\partial n} d\sigma\ dt = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

hence,

$$||\hat{v}||^2_{L^2(\Gamma_1\times(0,T))} = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1} .$$

On the other hand,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} w\ d\sigma\ dt = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

then,

$$\begin{aligned} ||\hat{v}||^2_{L^2(\Gamma_1\times(0,T))} &= \int_0^T \int_{\Gamma_1} \hat{v}\ w\ d\sigma\ dt \\ &\leq ||\hat{v}||_{L^2(\Gamma_1\times(0,T))} ||w||_{L^2(\Gamma_1\times(0,T))}. \end{aligned}$$

Consequently, (**1.10**) is verified and the proof finishes.

Chapitre 2

Conjugate Gradient Algorithm

2.1 Variational Problem

In the previous chapter, we transferred the problem of controllability of (**1.1**) to a problem of minimization :

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \mathcal{J}((\varphi^0,\varphi^1)), \tag{2.1}$$

where the functional $\mathcal{J}$ is defined by (**1.7**).
Problem (**2.1**) can be written as follows :

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \left(\frac{1}{2}a((\varphi^0,\varphi^1),(\varphi^0,\varphi^1)) - L((\varphi^0,\varphi^1))\right), \tag{2.2}$$

where a is defined on $(H_0^1(\Omega)\times L^2(\Omega))\times(H_0^1(\Omega)\times L^2(\Omega))$ by

$$a((\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})) = \int_0^T\int_{\Gamma_1}\frac{\partial\varphi}{\partial n}\frac{\partial\widetilde{\varphi}}{\partial n}d\sigma\, dt, \quad \forall\,(\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega),$$

such that φ,$\widetilde{\varphi}$ respectively the solutions of (**1.4**) with initial data (φ^0,φ^1) and $(\widetilde{\varphi^0},\widetilde{\varphi^1})$, and L is defined on $H_0^1(\Omega)\times L^2(\Omega)$ by

$$\begin{aligned} L((\varphi^0,\varphi^1)) &= -<(\varphi^0,\varphi^1),(y^0,y^1)> \\ &= -\int_\Omega y^0\varphi^1 + <y^1,\varphi^0>_{-1,1}, \end{aligned}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ and (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$. we have,

Lemma 2.1.1. ♦ *The operator a is a bilinear form, continuous and $H_0^1(\Omega)\times L^2(\Omega)$-elliptic.* ♦ *For all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, we have $L\in(H_0^1(\Omega)\times L^2(\Omega))'$.*

Proof :
• it easy to proof that L is linear continuous and a is bilinear
• The continuity of a follows from the inequality (**1.5**) and the coercivity follows from the inequality of observability (**1.8**).
Moreover the bilinear form a is symmetric, by follows and with the theorem of Lax-Milgram

the problem (2.2) reads as follows :

$$\begin{cases} \text{Find } (\widehat{\varphi^0}, \widehat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a((\widehat{\varphi^0}, \widehat{\varphi^1}), (\varphi^0, \varphi^1)) = L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega). \end{cases} \tag{2.3}$$

2.2 General conjugate gradient algorithm

Let H a Hilbert space, a a continuous, symmetric and coercive bilinear form on $H \times H$, and L a continuous linear form on H, the variational problem problem (2.3) is a particular case of the following general variational problem :

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u, v) = L(v), \quad \forall v \in H. \end{cases} \tag{2.4}$$

With the above hypotheses, problem (2.4) has a unique solution.
For the numerical solution of (2.4) we need the following algorithm, say the conjugate gradient algorithm :
(1) $u^{(0)}$: any arbitrarily vector in H ;

(2) solve

$$\begin{cases} \widetilde{u}^{(0)} \in H \\ <\widetilde{u}^{(0)}, v>_H = a(u^{(0)}, v) - L(v), \quad \forall v \in H. \end{cases} \tag{2.5}$$

(3) • If $\widetilde{u}^{(0)}$ is small ($\dfrac{||\widetilde{u}^{(0)}||}{||u^{(0)}||} < \epsilon$), take $u = u^{(0)}$;
• If not,set $\breve{u}^{(0)} = \widetilde{u}^{(0)}$;

Assuming that $u^{(n)}$, $\widetilde{u}^{(n)}$, $\breve{u}^{(n)}$ are known, compute $u^{(n+1)}$, $\widetilde{u}^{(n+1)}$, $\breve{u}^{(n+1)}$:

(4) $\rho_n = \dfrac{||\widetilde{u}^{(n)}||^2}{a(\breve{u}^{(n)}, \breve{u}^{(n)})}$;

(5) $u^{(n+1)} = u^{(n)} - \rho_n \breve{u}^{(n)}$;

(6) solve

$$\begin{cases} \widetilde{u}^{(n+1)} \in H \\ <\widetilde{u}^{(n+1)}, v>_H = <\widetilde{u}^{(n)}, v>_H - \rho_n a(\breve{u}^{(n)}, v), \quad \forall v \in H. \end{cases} \tag{2.6}$$

(7) • If $\widetilde{u}^{(n+1)}$ is small ($\dfrac{||\widetilde{u}^{(n+1)}||}{||\widetilde{u}^{(0)}||} < \epsilon$), take $u = u^{(n+1)}$;
• If not,
$\star$ $\gamma_n = \dfrac{||\widetilde{u}^{(n+1)}||^2}{||\widetilde{u}^{(n)}||^2}$;

$\star$ $\breve{u}^{(n+1)} = \widetilde{u}^{(n+1)} + \gamma_n \breve{u}^{(n)}$;
(8) $n = n + 1$ and go to (4);

2.3 Application of conjugate gradient algorithm

Let us now apply the general conjugate gradient algorithm to the solution of (2.3).
(1) $(\varphi_0^0,\varphi_0^1)\in H_0^1(\Omega)\times L^2(\Omega)=H$: Initialization ;

(2) solve

$$\begin{cases} (\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})\in H \\ <(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1}),(\varphi^0,\varphi^1)>_H = a((\varphi_0^0,\varphi_0^1),(\varphi^0,\varphi^1))-L((\varphi^0,\varphi^1)), \quad \forall(\varphi^0,\varphi^1)\in H. \end{cases} \tag{2.7}$$

Consider the following non-homogeneous backward wave equation :

$$\begin{cases} \dfrac{\partial^2\psi_0}{\partial t^2}-\Delta\psi_0 = 0 & \text{in} \quad \Omega\times(0,T), \\ \psi_0 = \dfrac{\partial\varphi_0}{\partial n} & \text{in} \quad \Gamma_1\times(0,T), \\ \psi_0 = 0 & \text{in} \quad \Gamma_2\times(0,T), \\ \psi_0(.,T) = 0, \quad \dfrac{\partial\psi_0}{\partial t}(.,T)=0 & \text{in} \quad \Omega. \end{cases}$$

From the lemma 1.3.2, we have,

$$\int_0^T\int_{\Gamma_1}\frac{\partial\varphi_0}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\,dt=<\psi_0'(0),\varphi^0>_{-1,1}-\int_\Omega\psi_0(0)\varphi^1dx,$$

for every $(\varphi^0,\varphi^1)\in H$, then we obtain,

$$\begin{aligned}\int_\Omega\nabla\widetilde{\varphi_0^0}\nabla\varphi^0dx+\int_\Omega\widetilde{\varphi_0^1}\varphi^1dx=<\psi_0'(0),\varphi^0>_{-1,1}-\int_\Omega\psi_0(0)\varphi^1dx \\ +\int_\Omega y^0\varphi^1dx-<y^1,\varphi^0>_{-1,1}.\end{aligned}$$

Hence,

$$<-\Delta\widetilde{\varphi_0^0},\varphi^0>_{-1,1}-<\psi_0'(0)-y^1,\varphi^0>_{-1,1}=\int_\Omega(y^0-\psi_0(0)-\widetilde{\varphi_0^1})\varphi^1dx.$$

Finally,

$$<(\varphi^0,\varphi^1),(y^0-\psi_0(0)-\widetilde{\varphi_0^1}\ ,-\Delta\widetilde{\varphi_0^0}-(\psi_0'(0)-y^1))>=0.$$

Its follows : (2)

$$\begin{cases} -\Delta\widetilde{\varphi_0^0}=\psi_0'(0)-y^1, \\ \widetilde{\varphi_0^1}=y^0-\psi_0(0). \end{cases} \tag{2.8}$$

(3) • If $(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})$ is small , take $(\widehat{\varphi^0},\widehat{\varphi^1})=(\varphi_0^0,\varphi_0^1)$;

• If not, set $(\check{\varphi}_0^0,\check{\varphi}_0^1)=(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})$.

Assuming that $(\varphi_n^0,\varphi_n^1)$, $(\widetilde{\varphi_n^0},\widetilde{\varphi_n^1})$, $(\check{\varphi}_n^0,\check{\varphi}_n^1)$ and φ_n, ψ_n are known, compute $(\varphi_{n+1}^0,\varphi_{n+1}^1)$, $(\widetilde{\varphi^0}_{n+1},\widetilde{\varphi^1}_{n+1})$, $(\check{\varphi}^0_{n+1},\check{\varphi}^1_{n+1})$, φ_{n+1}, ψ_{n+1}.

we knew that the form linear $(\varphi^0,\varphi^1)\in H \longmapsto a((\check{\varphi}_n^0,\check{\varphi}_n^1),(\varphi^0,\varphi^1))$ is continuous, then by Riesz's theorem there exists unique $(\underline{\varphi_n^0},\underline{\varphi_n^1})$ in H, such that

$$a((\check{\varphi}_n^0,\check{\varphi}_n^1),(\varphi^0,\varphi^1)) =< (\underline{\varphi_n^0},\underline{\varphi_n^1}),(\varphi^0,\varphi^1) >, \quad \forall(\varphi^0,\varphi^1)\in H.$$

Like the previous case, we can find $(\underline{\varphi_n^0},\underline{\varphi_n^1})$ by :

(4) solve

$$\begin{cases} \dfrac{\partial^2\check{\varphi}_n}{\partial t^2}-\Delta\check{\varphi}_n &= 0 & \text{in} \quad \Omega\times(0,T), \\ \check{\varphi}_n &= 0 & \text{in} \quad \Gamma\times(0,T), \\ \check{\varphi}_n(.,0) &= \check{\varphi}_n^0, \quad \dfrac{\partial\check{\varphi}_n}{\partial t}(.,0)=\check{\varphi}_n^1 & \text{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2\check{\psi}_n}{\partial t^2}-\Delta\check{\psi}_n &= 0 & \text{in} \quad \Omega\times(0,T), \\ \check{\psi}_n &= \dfrac{\partial\check{\varphi}_n}{\partial n} & \text{in} \quad \Gamma_1\times(0,T), \\ \check{\psi}_n &= 0 & \text{in} \quad \Gamma_2\times(0,T), \\ \check{\psi}_n(.,T) &= 0, \quad \dfrac{\partial\check{\psi}_n}{\partial t}(.,T)=0 & \text{in} \quad \Omega. \end{cases}$$

Compute now $(\underline{\varphi_n^0},\underline{\varphi_n^1})\in H$ by :

$$\begin{cases} -\Delta\underline{\varphi_n^0}=\check{\psi}_n{}'(0) \quad \text{in } \Omega \\ \underline{\varphi_n^1}=-\check{\psi}_n(0). \end{cases} \tag{2.9}$$

The other steps of the general algorithm are easy to adapt. Now we give the complete algorithm to solve the system (2.3) :

(1) $(\varphi_0^1,\varphi_0^1)\in H_0^1(\Omega)\times L^2(\Omega)=H$ **are given ;**

(2) **solve then**

$$\begin{cases} \dfrac{\partial^2\varphi_0}{\partial t^2}-\Delta\varphi_0 &= 0 & \textbf{in} \quad \Omega\times(0,T), \\ \varphi_0 &= 0 & \textbf{in} \quad \Gamma\times(0,T), \\ \varphi_0(\cdot,T) &= 0, \quad \dfrac{\partial\varphi_0}{\partial t}(\cdot,T)=0 & \textbf{in} \quad \Omega, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2\psi_0}{\partial t^2}-\Delta\psi_0 &= 0 & \textbf{in} \quad \Omega\times(0,T), \\ \psi_0 &= \dfrac{\partial\varphi_0}{\partial n} & \textbf{in} \quad \Gamma_1\times(0,T), \\ \psi_0 &= 0 & \textbf{in} \quad \Gamma_2\times(0,T), \\ \psi_0(\cdot,T) &= 0, \quad \dfrac{\partial\psi_0}{\partial t}(\cdot,T)=0 & \textbf{in} \quad \Omega. \end{cases}$$

(3) **Compute** $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}) \in H$ **by**

$$\begin{cases} -\Delta \widetilde{\varphi_0^0} = \psi_0'(0) - y^1, \\ \widetilde{\varphi_0^0} = 0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\widetilde{\varphi_0^1} = y^0 - \psi_0(0).$$

(4) • **If** $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$ **is small , take** $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_0^0, \varphi_0^1)$ **;**

• **If not, set** $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$**.**

Assuming that $(\varphi_n^0, \varphi_n^1)$**,** $(\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1})$**,** $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ **and** φ_n**,** ψ_n **are known, compute** $(\varphi_{n+1}^0, \varphi_{n+1}^1)$**,** $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$**,** $(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1})$**,** φ_{n+1}**,** ψ_{n+1}**.**

<u>*Descent*</u> **:**

(5) **Solve**

$$\begin{cases} \dfrac{\partial^2 \check{\varphi}_n}{\partial t^2} - \Delta \check{\varphi}_n & = & 0 & & \textbf{in} \quad \Omega \times (0,T), \\ \check{\varphi}_n & = & 0 & & \textbf{in} \quad \Gamma \times (0,T), \\ \check{\varphi}_n(\cdot,0) & = & \check{\varphi}_n^0, & \dfrac{\partial \check{\varphi}_n}{\partial t}(\cdot,0) = \check{\varphi}_n^1 & \textbf{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n & = & 0 & & \textbf{in} \quad \Omega \times (0,T), \\ \check{\psi}_n & = & \dfrac{\partial \check{\varphi}_n}{\partial n} & & \textbf{in} \quad \Gamma_1 \times (0,T), \\ \check{\psi}_n & = & 0 & & \textbf{in} \quad \Gamma_2 \times (0,T), \\ \check{\psi}_n(\cdot,T) & = & 0, & \dfrac{\partial \check{\psi}_n}{\partial t}(\cdot,T) = 0 & \textbf{in} \quad \Omega. \end{cases}$$

(6) **Compute** $(\underline{\varphi_n^0}, \underline{\varphi_n^1}) \in H$ **by : solve**

$$\begin{cases} -\Delta \underline{\varphi_n^0} = \check{\psi}_n{}'(0) \quad \textbf{in } \Omega, \\ \\ \underline{\varphi_n^0} = 0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\underline{\varphi_n^1} = -\check{\psi}_n(0).$$

(7) **Compute** ρ_n **by :**

$$\begin{cases} \rho_n & = \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\check{\varphi}_n^0, \check{\varphi}_n^1))}, \\ & = \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{< \check{\psi}_n{}'(0), \check{\varphi}_n^0 >_{-1,1} - \displaystyle\int_\Omega \check{\psi}_n(0)\check{\varphi}_n^1 dx}, \\ & = \dfrac{\displaystyle\int_\Omega || \nabla \widetilde{\varphi_n^0}||^2 + \int_\Omega ||\widetilde{\varphi_n^1}||^2}{\displaystyle\int_\Omega \nabla \underline{\varphi_n^0} \nabla \check{\varphi}_n^0 dx + \int_\Omega \underline{\varphi_n^1}\check{\varphi}_n^1 dx}. \end{cases}$$

(8) **Once ρ_n is known, compute :**

$\oplus$ $(\varphi_{n+1}^0, \varphi_{n+1}^1) = (\varphi_n^0, \varphi_n^1) - \rho_n(\check{\varphi}_n^0, \check{\varphi}_n^1)$,

$\oplus$ $\varphi_{n+1} = \varphi_n - \rho_n \check{\varphi}_n$,

$\oplus$ $\psi_{n+1} = \psi_n - \rho_n \check{\psi}_n$,

$\oplus$ $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) = (\widetilde{\varphi^0}_n, \widetilde{\varphi^1}_n) - \rho_n(\underline{\varphi_n^0}, \underline{\varphi_n^1})$.

Test of the convergence **and** **construction of the new descent direction.**

(9) **If** $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$ **is small , take** $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_{n+1}^0, \varphi_{n+1}^1)$.
If not, compute

$$\gamma_n = \frac{\displaystyle\int_\Omega || \nabla \widetilde{\varphi^0}_{n+1}||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_{n+1}||^2 dx}{\displaystyle\int_\Omega || \nabla \widetilde{\varphi^0}_n||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_n||^2 dx},$$

and set

$$(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1}) = (\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) + \gamma_n(\check{\varphi}^0{}_n, \check{\varphi}^1{}_n).$$

(10) $n = n+1$ **and go to** (5).

Conclusion

OK

Bibliographie

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