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Introduction

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Chapter 1

The Exact Boundary controllability

1.1 Formulation of the control problem

Let Ω be a bounded domain (that is, non-empty open connected set) in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ "Sufficiently smooth", Γ_1 be an open nonempty subset of Γ and $\Gamma_2 = \Gamma\backslash\Gamma_1$. With T a given positive number, we consider the following non-homogeneous wave equation :

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = 0 & \text{for} \quad (x,t) \in \Omega \times (0,T), \quad (1.1.1) \\ y(x,t) = 0 & \text{for} \quad (x,t) \in \Gamma_2 \times (0,T), \quad (1.1.2) \\ y(x,t) = v(x,t) & \text{for} \quad (x,t) \in \Gamma_1 \times (0,T), \quad (1.1.2) \\ y(x,0) = y^0, \quad \dfrac{\partial y}{\partial t}(x,0) = y^1 & \text{for} \quad x \in \Omega. \quad (1.1.3) \end{cases} \tag{1.1}$$

$(1.1.2)$: The boundary conditions.
$(1.1.3)$ The initial conditions.
In (**1.1**), $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, such that $H^{-1}(\Omega)$ is the topological dual space of $H_0^{-1}(\Omega)$ and Δ is the Laplacian operator.
We shall now define the exact boundary controllability for the system (**1.1**).

Definition 1.1.1. *System* (**1.1**) *is controllable in time* $T > 0$ *if for every initial data* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *we can find a control function* $v \in L^2(\Gamma_1 \times (0,T))$ *such that the corresponding solution* (y, y') *of* (**1.1**) *verifies*

$$y(.,T) = y'(.,T) = 0. \tag{1.2}$$

Remark 1.1.1. *For the existence of solution see section* $[1.2]$

Remark 1.1.2. *If the solution of* (**1.1**) *verifies* (**1.2**) *is also said to be null controllable in time* $T > 0$, *for more details* (*see* $[1]$, *page* 100).

The problem that we consider is the following one:
is it possible to find $T > 0$ sufficiently large or optimal and $v \in L^2(\Gamma_1 \times (0,T))$ a boundary

control function such that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we have (1.2). But before that, we have to prove that the system (1.1) admits a solution.

1.2 Existence and uniqueness of solutions

In this section we consider the general case of (1.1) whith v not necessarily null and a bounded domain Ω in $\mathbb{R}^n$.

Definition 1.2.1. *For (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(\Gamma_1 \times [0, +\infty))$ a function $y \in C([0, +\infty), L^2(\Omega)) \bigcap C^1([0, +\infty), H^{-1}(\Omega))$ is called a weak solution of (1.1) if the relation*

$$\begin{aligned} \int_\Omega y(x,t)\varphi(x)dx - \int_\Omega y^0(x)\varphi(x)dx - t < y^1, \varphi >_{-1,1} &= \int_0^t \int_0^s \int_\Omega y(x,\xi)\Delta\varphi(x)dxd\xi ds \\ &- \int_0^t \int_0^s \int_{\Gamma_1} v(x,\xi)\frac{\partial\varphi}{\partial n}(x)d\sigma d\xi ds, \end{aligned} \tag{1.3}$$

holds for every $t \geq 0$ and every $\varphi \in H^2(\Omega) \bigcap H_0^1(\Omega)$.

Remark 1.2.1. *For x in Γ and φ in $H^2(\Omega)$ we have*

$$\frac{\partial\varphi}{\partial n}(x) = \bigtriangledown\varphi(x).\overrightarrow{n}(x),$$

is called the normal derivative, with $\overrightarrow{n}(x)$ is the normal vector at x

Remark 1.2.2. *The normal vector exists because the boundary Γ is sufficiently smooth.*

The main result for the existence of solutions of (1.1) is the following:

Theorem 1.2.1. *For every v in $L^2(\Gamma_1 \times (0, T))$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}$ system (1.1) has a unique weak solution*

$$(y, y^{'}) \in C([0, T], L^2(\Omega) \times H^{-1}),$$

moreover, there exists a constant $C = C(T) > 0$ such that

$$||(y, y^{'})||_{L^\infty([0,T], L^2(\Omega)\times H^{-1})} \leq C \left[||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}} + ||v||_{L^2(\Gamma_1 \times (0,T))} \right]$$

Proof :
The theorem is a consequence of the theory of nonhomogeneous evolution equations.

$*$ Proof with variational method, see [2].
$*$ Proof with Semigroups operator, see [3].

1.3 Controllability by minimization

In this section we will give a necessary and sufficient condition for the exact controllability of (1.1) and also we will transform the controllability problem to a minimization problem, but before that we need to proof the following lemma:

Lemma 1.3.1. *For every (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$ the following wave equation*

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi & = & 0 & & \textit{in} \quad \Omega \times (0,T), \\ \varphi & = & 0 & & \textit{in} \quad \Gamma \times (0,T), \\ \varphi(.,0) & = & \varphi^0, & \dfrac{\partial \varphi}{\partial t}(.,0) = \varphi^1 & \textit{in} \quad \Omega \quad , \end{cases} \tag{1.4}$$

has a unique solution, moreover (1.4) generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$.

Proof:
this lemma is a consequence of the following classic theorem:

Theorem 1.3.1. ***(Stone, 1930)***
Let H be a Hilbert space and A be a linear operator on H with dense domain, then A generates a C_0-group of unitary operators if and only if A is skewadjoint ($A^{'} = -A$)

Let $w(t) = (\varphi(t), \varphi^{'}(t))$ and the state space is $H_0^1(\Omega) \times L^2(\Omega)$,
with the scalar product

$$< (\varphi_1, \varphi_2), (\psi_1, \psi_2) >= \int_\Omega \nabla\varphi_1 \nabla \psi_1 + \int_\Omega \varphi_2 \psi_2,$$

then the corresponding norm on H given by:

$$||(\varphi_1, \varphi_2)||^2 = \int_\Omega || \nabla \varphi_1||^2 + \int_\Omega ||\varphi_2||^2.$$

By the Poincaré inequality, this norm is eqivalent to the usual norm on H.
We define the linear operator A by:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{and}$$

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}.$$

Firstly the domain $D(A)$ is dense in H.
Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D(A)$, then

$$\begin{aligned}
< A(\varphi_1, \varphi_2), (\psi_1, \psi_2) > &=< (\varphi_2, \Delta\varphi_1), (\psi_1, \psi_2) > \\
&= \int_\Omega \bigtriangledown\varphi_2 \bigtriangledown \psi_1 + \int_\Omega \Delta\varphi_1\psi_2 \\
&= \int_\Omega \bigtriangledown\varphi_2 \bigtriangledown \psi_1 - \int_\Omega \bigtriangledown\varphi_1 \bigtriangledown \psi_2 \\
&+ \underbrace{\int_\Gamma \frac{\partial\varphi_1}{\partial n}\psi_2}_{0} \\
&= -\int_\Omega \bigtriangledown\varphi_1 \bigtriangledown \psi_2 - \int_\Omega \varphi_2\Delta\psi_1 \\
&= - < (\varphi_1, \varphi_2), (\psi_2, \Delta\psi_1) > \\
&= - < (\varphi_1, \varphi_2), A(\psi_1, \psi_2) > .
\end{aligned}$$

Which implies that the operator A is skewadjoint on H and thus generates a unitary C_0-group on H by Stone's theorem.
Then if (φ^0, φ^1) in $D(A)$ the system (**1.4**) has a unique strict solution by Semi-group theory, but in the case (φ^0, φ^1) in $H \backslash D(A)$, (**1.4**) has a unique weak solution by follows the proof of lemma.

Proposition 1.3.1. *Assume that $T > 0$ is large enough, then there exists a constant $\hat{C} > 0$ such that:*

$$\int_0^T \int_{\Gamma_1} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\ dt \leq \hat{C}||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.5}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the solution of (**1.4**).

Proof:
The inequality (**1.5**) is a hildden regularity result, that may not be obtained by standard trace results $(see, [.])$.

The key result in this section is the variational characterization of the controllability of (**1.1**) given by the following lemma:

Lemma 1.3.2. *The initial data $(y^0, y^1) \in Ł^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there existes $v \in L^2(\Gamma_1 \times (0, T))$ such that:*

$$\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} v d\sigma dt =< y^1, \varphi^0 >_{-1,1} - \int_\Omega y^0\varphi^1 dx, \tag{1.6}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of the adjoint problem (**1.4**)

Proof:
For $(y^0, y^1) \in D(\Omega) \times D(\Omega)$, $(\varphi^0, \varphi^1) \in D(\Omega) \times D(\Omega)$ and $v \in D(\Gamma_1 \times (0, T))$, then y and φ

are regular solutions. Multiplying the equation (1.1.1) in system (**1.1**) by φ and integrating, we have

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega (y'' - \Delta y)\varphi dx\, dt \\
&= \int_\Omega \int_0^T y''\varphi dt\, dx - \int_0^T \int_\Omega \Delta y\varphi dx\, dt \\
&= \int_\Omega \left[\left[y'\varphi \right]_0^T - \int_0^T y'\varphi' dt \right] dx + \int_0^T \int_\Omega \nabla y \nabla \varphi dx\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx - \int_\Omega \left[y\varphi' \right]_0^T dx + \int_\Omega \int_0^T y\varphi'' dt\, dx \\
&- \int_0^T \int_\Omega \Delta\varphi y dx\, dt + \int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx - \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0))dx + \int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt
\end{aligned}
$$
.

Hence,
$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &- \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0))dx \\
&+ \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0))dx,
\end{aligned}
$$

finally we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &\int_\Omega y^1\varphi^0 dx - \int_\Omega y^0\varphi^1 dx \\
&+ \int_\Omega y(T)\varphi'(T)dx - \int_\Omega y'(T)\varphi(T)dx.
\end{aligned}
$$

We Know that $D(\Omega) \times D(\Omega)$ dense in $L^2(\Omega) \times H^{-1}(\Omega)$ as well as in $H_0^1(\Omega) \times L^2(\Omega)$ and $D(\Gamma_1 \times (0,T))$ dense in $L^2(\Gamma_1 \times (0,T))$. by the inequality (**1.5**), we have $\frac{\partial\varphi}{\partial n}|_{\Gamma_1} \in L^2(\Gamma_1) \times (0,T)$. From a density argument we deduce that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} y\, d\sigma\, dt = &< y^1\varphi^0 >_{-1,1} - \int_\Omega y^0, \varphi^1 dx \\
&+ < (\varphi(T), \varphi'(T)), (y(T), y'(T)) > .
\end{aligned}
$$

With $< .,. >_{-1,1}$ be the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we introduce the duality product

$$
< (\varphi^0, \varphi^1), (y^0, y^1) >= \int_\Omega y^0\varphi^1 - < y^1, \varphi^0 >_{-1,1} .
$$

Such that the wave equation generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$, then (1.6) holds if and anly if (y^0, y^1) is controllable to zero in time $T > 0$. This completes the proof.
From lemme $1.3.2$, the equality (1.6) can be seen as an optimality condition for the minimants of the functional $\mathcal{J} \ : \ H_0^1(\Omega) \times L^2(\Omega) \ \rightarrow \ \mathbb{R}$, defined by

$$\mathcal{J} \ = \ \frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma \ dt + < (\varphi^0, \varphi^1), (y^0, y^1) >, \tag{1.7}$$

where $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the corresponding solution of (1.4).
Before introducing the main theorem in this section we need to proof that the functional $\mathcal{J}$ has a minimant .

Definition 1.3.1. *System (1.4) is said to be observable in time $T > 0$ if there exists a positive a positive constant $C > 0$ such that*

$$C||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \ \leq \ \int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma \ dt, \tag{1.8}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0, φ^1).

In the following we assume that there is a positive time T^* such that for any $T > T^*$ the system (1.4) is observable.
On the other hand, the functional $\mathcal{J}$ is continuous, strictly convex and coercive.
It is easy to see that functional $\mathcal{J}$ is continuous, now let $(\varphi^0, \varphi^1), (\psi^0, \psi^1)$ in $H_0^1(\Omega) \times L^2(\Omega)$ and $\lambda \in]0, 1[$, we have,

$$\begin{aligned}
\mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1)) \ &= \ \frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\lambda\frac{\partial \varphi}{\partial n} + (1-\lambda)\frac{\partial \psi}{\partial n}\right|^2 d\sigma \ dt \\
&+ \ < \lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1), (y^0, y^1) > \\
&= \ \lambda\mathcal{J}((\varphi^0, \varphi^1)) + (1-\lambda)\mathcal{J}((\psi^0, \psi^1)) \\
&- \ \frac{\lambda(1-\lambda)}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n} - \frac{\partial \psi}{\partial n}\right|^2 d\sigma \ dt.
\end{aligned}$$

Using the observability inequation, we obtain,

$$\int_0^T \int_{\Gamma_1} \left|\frac{\partial}{\partial n}(\varphi - \psi)\right|^2 d\sigma \ dt \geq C||(\varphi^0 - \psi^0, \varphi^1 - \psi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)},$$

then if $(\varphi_0, \varphi_1) \neq (\psi_0, \psi_1)$, then,

$$\mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\psi^0, \psi^1)) \ < \ \lambda\mathcal{J}((\varphi^0, \varphi^1)) + (1-\lambda)\mathcal{J}((\psi^0, \psi^1)).$$

Hence $\mathcal{J}$ is strictly convex.
For the coercivity of the the functional $\mathcal{J}$, we have,

$$\begin{aligned}\mathcal{J}((\varphi^0,\varphi^1)) \;\ge\;& \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\,||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}\\ \ge\;& \frac{C}{2}||(\varphi^0,\varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\,||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)},\end{aligned}$$

then $\lim\limits_{||(\varphi^0,\varphi^1)||\to+\infty}\mathcal{J}((\varphi^0,\varphi^1))=\infty$.
We conclude that the functional $\mathcal{J}$ has a unique minimizer $(\hat{\varphi}^0,\hat{\varphi}^1)$ in $H_0^1(\Omega)\times L^2(\Omega)$, we have,

Theorem 1.3.2. *Let (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$ and $(\hat{\varphi}^0,\hat{\varphi}^1)$ in $H_0^1(\Omega)\times L^2(\Omega)$ be the unique minimizer of the functional $\mathcal{J}$, then the function $\hat{v}$ defined on $\Gamma_1\times(0,T)$ by :*

$$\hat{v}(x,t)=\frac{\partial\hat{\varphi}}{\partial n}(x,t),\quad (x,t)\in\Gamma_1\times(0,T),$$

is a control which leads (y^0,y^1) to zero in time $T>0$.

Proof:
The functional $\mathcal{J}$ achieves its minimum at $(\hat{\varphi}^0,\hat{\varphi}^1)$, then

$$\lim_{h\to 0}\frac{1}{h}\left[\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1)+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1))\right]=0, \tag{1.9}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0,φ^1).
On the other hand, we have,

$$\begin{aligned}\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1)+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1)) =& \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n}+h\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt + <(\hat{\varphi}^0,\hat{\varphi}^1)+h(\varphi^0,\varphi^1),(y^0,y^1)>\\ &-\frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n}\right|^2 d\sigma\, dt - <(\hat{\varphi}^0,\hat{\varphi}^1),(y^0,y^1)>,\end{aligned}$$

hence,

$$\begin{aligned}\frac{1}{h}\left[\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1)+h(\varphi^0,\varphi^1))-\mathcal{J}((\hat{\varphi}^0,\hat{\varphi}^1))\right] =& \frac{h}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt+\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\, dt\\ &+<(\varphi^0,\varphi^1),(y^0,y^1)>,\end{aligned}$$

and from (1.8) we deduce that

$$\begin{aligned}\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\, dt =& -<(\varphi^0,\varphi^1),(y^0,y^1)>\\ =& -\int_\Omega y^0\varphi^1 dx + <y^1,\varphi^0>_{-1,1},\end{aligned}$$

for every (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$.

From lemma 1.3.2, it follows that $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ is the control for (**1.1**). This complete the proof.
Now we can find the control of the wave equation by minimization of the functional $\mathcal{J}$, moreover, this control is the control of minimal L^2-norm:

Proposition 1.3.2. *Let $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ be the control given by minimizing the functional $\mathcal{J}$. If $w \in L^2(\Gamma_1 \times (0,T))$ is any other control for (**1.1**), then*

$$||\hat{v}||_{L^2(\Gamma_1 \times (0,T))} \leq ||w||_{L^2(\Gamma_1 \times (0,T))}. \tag{1.10}$$

Proof:
Let $(\hat{\varphi^0}, \hat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega)$ the minimizer of the functional $\mathcal{J}$ and w is a control function of (**1.1**). By taking $(\hat{\varphi^0}, \hat{\varphi^1})$ as initial data for (**1.4**), lemma 1.3.2 gives,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \hat{\varphi}}{\partial n} d\sigma \ dt = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

hence,

$$||\hat{v}||^2_{L^2(\Gamma_1 \times (0,T))} = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1} .$$

On the other hand,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} w \ d\sigma \ dt = -\int_\Omega y^0 \hat{\varphi^1} dx + < y^1, \hat{\varphi^0} >_{-1,1},$$

then,

$$\begin{aligned} ||\hat{v}||^2_{L^2(\Gamma_1 \times (0,T))} &= \int_0^T \int_{\Gamma_1} \hat{v} \ w \ d\sigma \ dt \\ &\leq ||\hat{v}||_{L^2(\Gamma_1 \times (0,T))} ||w||_{L^2(\Gamma_1 \times (0,T))}. \end{aligned}$$

Consequently, (**1.10**) is verified and the proof finishes.

Chapter 2

Conjugate Gradient Algorithm

2.1 Variational Problem

In the previous chapter, we transferred the problem of controllability of (**1.1**) to a problem of minimization:

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \mathcal{J}((\varphi^0,\varphi^1)), \tag{2.1}$$

where the functional $\mathcal{J}$ is defined by (**1.7**).
Problem (**2.1**) can be written as follows:

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \left(\frac{1}{2}a((\varphi^0,\varphi^1),(\varphi^0,\varphi^1)) - L((\varphi^0,\varphi^1))\right), \tag{2.2}$$

where a is defined on $(H_0^1(\Omega)\times L^2(\Omega))\times(H_0^1(\Omega)\times L^2(\Omega))$ by

$$a((\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})) = \int_0^T\int_{\Gamma_1}\frac{\partial\varphi}{\partial n}\frac{\partial\widetilde{\varphi}}{\partial n}d\sigma\, dt, \quad \forall\,(\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega),$$

such that φ,$\widetilde{\varphi}$ respectively the solutions of (**1.4**) with initial data (φ^0,φ^1) and $(\widetilde{\varphi^0},\widetilde{\varphi^1})$, and L is defined on $H_0^1(\Omega)\times L^2(\Omega)$ by

$$\begin{aligned} L((\varphi^0,\varphi^1)) &= -<(\varphi^0,\varphi^1),(y^0,y^1)> \\ &= -\int_\Omega y^0\varphi^1 + <y^1,\varphi^0>_{-1,1}, \end{aligned}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ and (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$. we have,

Lemma 2.1.1. ♦ *The operator a is a bilinear form, continuous and $H_0^1(\Omega)\times L^2(\Omega)$-elliptic.*
♦ *For all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, we have $L\in(H_0^1(\Omega)\times L^2(\Omega))^{'}$.*

Proof:
• it easy to proof that L is linear continuous and a is bilinear
• The continuity of a follows from the inequality (**1.5**) and the coercivity follows from the inequality of observability (**1.8**).
Moreover the bilinear form a is symmetric, by follows and with the theorem of Lax-Milgram

the problem (2.2) reads as follows:

$$\begin{cases} \text{Find } (\widehat{\varphi^0}, \widehat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a((\widehat{\varphi^0}, \widehat{\varphi^1}), (\varphi^0, \varphi^1)) = L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega). \end{cases} \tag{2.3}$$

2.2 General conjugate gradient algorithm

Let H a Hilbert space, a a continuous, symmetric and coercive bilinear form on $H \times H$, and L a continuous linear form on H, the variational problem problem (2.3) is a particular case of the following general variational problem:

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = L(v), \quad \forall v \in H. \end{cases} \tag{2.4}$$

With the above hypotheses, problem (2.4) has a unique solution.
For the numerical solution of (2.4) we need the following algorithm, say the conjugate gradient algorithm:
(1) $u^{(0)}$: any arbitrarily vector in H;

(2) solve

$$\begin{cases} \tilde{u}^{(0)} \in H \\ < \tilde{u}^{(0)}, v >_H = a(u^{(0)}, v) - L(v), \quad \forall v \in H. \end{cases} \tag{2.5}$$

(3) • If $\tilde{u}^{(0)}$ is small ($\frac{||\tilde{u}^{(0)}||}{||u^{(0)}||} < \epsilon$), take $u = u^{(0)}$;
• If not,set $\breve{u}^{(0)} = \tilde{u}^{(0)}$;

Assuming that $u^{(n)}$, $\tilde{u}^{(n)}$, $\breve{u}^{(n)}$ are known, compute $u^{(n+1)}$, $\tilde{u}^{(n+1)}$, $\breve{u}^{(n+1)}$:

(4) $\rho_n = \frac{||\tilde{u}^{(n)}||^2}{a(\breve{u}^{(n)}, \breve{u}^{(n)})}$;

(5) $u^{(n+1)} = u^{(n)} - \rho_n \breve{u}^{(n)}$;

(6) solve

$$\begin{cases} \tilde{u}^{(n+1)} \in H \\ < \tilde{u}^{(n+1)}, v >_H = < \tilde{u}^{(n)}, v >_H - \rho_n a(\breve{u}^{(n)}, v), \quad \forall v \in H. \end{cases} \tag{2.6}$$

(7) • If $\tilde{u}^{(n+1)}$ is small ($\frac{||\tilde{u}^{(n+1)}||}{||\tilde{u}^{(0)}||} < \epsilon$), take $u = u^{(n+1)}$;
• If not,
$\star$ $\gamma_n = \frac{||\tilde{u}^{(n+1)}||^2}{||\tilde{u}^{(n)}||^2}$;

$\star$ $\breve{u}^{(n+1)} = \tilde{u}^{(n+1)} + \gamma_n \breve{u}^{(n)}$;
(8) $n = n + 1$ and go to (4);

2.3 Application of conjugate gradient algorithm

Let us now apply the general conjugate gradient algorithm to the solution of (2.3).
(1) $(\varphi_0^0, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$: Initialization;

(2) solve

$$\begin{cases} (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}) \in H \\ < (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}), (\varphi^0, \varphi^1) >_H = a((\varphi_0^0, \varphi_0^1), (\varphi^0, \varphi^1)) - L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H. \end{cases} \tag{2.7}$$

Consider the following non-homogeneous backward wave equation:

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 &= 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_0 &= \dfrac{\partial \varphi_0}{\partial n} & \text{in} \quad \Gamma_1 \times (0,T), \\ \psi_0 &= 0 & \text{in} \quad \Gamma_2 \times (0,T), \\ \psi_0(.,T) &= 0, & \dfrac{\partial \psi_0}{\partial t}(.,T) = 0 \quad \text{in} \quad \Omega. \end{cases}$$

From the lemma 1.3.2, we have,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \varphi_0}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma \; dt =< \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx,$$

for every $(\varphi^0, \varphi^1) \in H$, then we obtain,

$$\int_\Omega \nabla \widetilde{\varphi_0^0} \nabla \varphi^0 dx + \int_\Omega \widetilde{\varphi_0^1} \varphi^1 dx =< \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx$$
$$+ \int_\Omega y^0 \varphi^1 dx - < y^1, \varphi^0 >_{-1,1} .$$

Hence,

$$< -\Delta \widetilde{\varphi_0^0}, \varphi^0 >_{-1,1} - < \psi_0'(0) - y^1, \varphi^0 >_{-1,1} = \int_\Omega (y^0 - \psi_0(0) - \widetilde{\varphi_0^1}) \varphi^1 dx.$$

Finally,

$$< (\varphi^0, \varphi^1), (y^0 - \psi_0(0) - \widetilde{\varphi_0^1} \; , -\Delta \widetilde{\varphi_0^0} - (\psi_0'(0) - y^1)) >= 0.$$

Its follows: (2)

$$\begin{cases} -\Delta \widetilde{\varphi_0^0} = \psi_0'(0) - y^1, \\ \widetilde{\varphi_0^1} = y^0 - \psi_0(0). \end{cases} \tag{2.8}$$

(3) • If $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$ is small , take $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_0^0, \varphi_0^1)$;

• If not, set $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$.

Assuming that $(\varphi_n^0, \varphi_n^1)$, $(\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1})$, $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ and φ_n, ψ_n are known, compute $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$, $(\check{\varphi}^0_{n+1}, \check{\varphi}^1_{n+1})$, φ_{n+1}, ψ_{n+1}.

we knew that the form linear $(\varphi^0, \varphi^1) \in H \longmapsto a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1))$ is continuous, then by Riesz's theorem there exists unique $(\underline{\varphi_n^0}, \underline{\varphi_n^1})$ in H, such that

$$a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1)) =< (\underline{\varphi_n^0}, \underline{\varphi_n^1}), (\varphi^0, \varphi^1) >, \quad \forall (\varphi^0, \varphi^1) \in H.$$

Like the previous case, we can find $(\underline{\varphi_n^0}, \underline{\varphi_n^1})$ by :

(4) solve

$$\begin{cases} \dfrac{\partial^2 \check{\varphi}_n}{\partial t^2} - \Delta \check{\varphi}_n = 0 & \text{in} \quad \Omega \times (0,T), \\ \check{\varphi}_n = 0 & \text{in} \quad \Gamma \times (0,T), \\ \check{\varphi}_n(.,0) = \check{\varphi}_n^0, \quad \dfrac{\partial \check{\varphi}_n}{\partial t}(.,0) = \check{\varphi}_n^1 & \text{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n = 0 & \text{in} \quad \Omega \times (0,T), \\ \check{\psi}_n = \dfrac{\partial \check{\varphi}_n}{\partial n} & \text{in} \quad \Gamma_1 \times (0,T), \\ \check{\psi}_n = 0 & \text{in} \quad \Gamma_2 \times (0,T), \\ \check{\psi}_n(.,T) = 0, \quad \dfrac{\partial \check{\psi}_n}{\partial t}(.,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

Compute now $(\underline{\varphi_n^0}, \underline{\varphi_n^1}) \in H$ by :

$$\begin{cases} -\Delta \underline{\varphi_n^0} = \check{\psi}_n{}'(0) \quad \text{in } \Omega \\ \\ \underline{\varphi_n^1} = -\check{\psi}_n(0). \end{cases} \tag{2.9}$$

The other steps of the general algorithm are easy to adapt. Now we give the complete algorithm to solve the system (2.3):

(1) $(\varphi_0^1, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$ **are given;**

(2) **solve then**

$$\begin{cases} \dfrac{\partial^2 \varphi_0}{\partial t^2} - \Delta \varphi_0 = 0 & \textbf{in} \quad \Omega \times (0,T), \\ \varphi_0 = 0 & \textbf{in} \quad \Gamma \times (0,T), \\ \varphi_0(\cdot,0) = \varphi_0^0, \quad \dfrac{\partial \varphi_0}{\partial t}(\cdot,0) = \varphi_0^1 & \textbf{in} \quad \Omega, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 = 0 & \textbf{in} \quad \Omega \times (0,T), \\ \psi_0 = \dfrac{\partial \varphi_0}{\partial n} & \textbf{in} \quad \Gamma_1 \times (0,T), \\ \psi_0 = 0 & \textbf{in} \quad \Gamma_2 \times (0,T), \\ \psi_0(\cdot,T) = 0, \quad \dfrac{\partial \psi_0}{\partial t}(\cdot,T) = 0 & \textbf{in} \quad \Omega. \end{cases}$$

(3) **Compute** $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}) \in H$ **by**

$$\begin{cases} -\Delta\widetilde{\varphi_0^0} = \psi_0'(0) - y^1, \\ \widetilde{\varphi_0^0} = 0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\widetilde{\varphi_0^1} = y^0 - \psi_0(0).$$

(4) • **If** $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$ **is small , take** $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_0^0, \varphi_0^1)$**;**

• **If not, set** $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$**.**

Assuming that $(\varphi_n^0, \varphi_n^1)$**,** $(\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1})$**,** $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ **and** φ_n**,** ψ_n **are known, compute** $(\varphi_{n+1}^0, \varphi_{n+1}^1)$**,** $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$**,** $(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1})$**,** φ_{n+1}**,** ψ_{n+1}**.**

<u>*Descent*</u>**:**

(5) **Solve**

$$\begin{cases} \dfrac{\partial^2\check{\varphi}_n}{\partial t^2} - \Delta\check{\varphi}_n &= & 0 & & \textbf{in} \quad \Omega \times (0,T), \\ \check{\varphi}_n &= & 0 & & \textbf{in} \quad \Gamma \times (0,T), \\ \check{\varphi}_n(\cdot,0) &= & \check{\varphi}_n^0, & \dfrac{\partial\check{\varphi}_n}{\partial t}(\cdot,0) = \check{\varphi}_n^1 & \textbf{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2\check{\psi}_n}{\partial t^2} - \Delta\check{\psi}_n &= & 0 & & \textbf{in} \quad \Omega \times (0,T), \\ \check{\psi}_n &= & \dfrac{\partial\check{\varphi}_n}{\partial n} & & \textbf{in} \quad \Gamma_1 \times (0,T), \\ \check{\psi}_n &= & 0 & & \textbf{in} \quad \Gamma_2 \times (0,T), \\ \check{\psi}_n(\cdot,T) &= & 0, & \dfrac{\partial\check{\psi}_n}{\partial t}(\cdot,T) = 0 & \textbf{in} \quad \Omega. \end{cases}$$

(6) **Compute** $(\underline{\varphi_n^0}, \underline{\varphi_n^1}) \in H$ **by : solve**

$$\begin{cases} -\Delta\underline{\varphi_n^0} = \check{\psi}_n{}'(0) \quad \textbf{in } \Omega, \\ \\ \underline{\varphi_n^0} = 0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\underline{\varphi_n^1} = -\check{\psi}_n(0).$$

(7) **Compute** ρ_n **by:**

$$\begin{cases} \rho_n &= \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\check{\varphi}_n^0, \check{\varphi}_n^1))}, \\ &= \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{< \check{\psi}_n{}'(0), \check{\varphi}_n^0 >_{-1,1} - \int_\Omega \check{\psi}_n(0)\check{\varphi}_n^1 dx}, \\ &= \dfrac{\int_\Omega || \nabla \widetilde{\varphi_n^0}||^2 + \int_\Omega ||\widetilde{\varphi_n^1}||^2}{\int_\Omega \nabla \underline{\varphi_n^0} \nabla \check{\varphi}_n^0 dx + \int_\Omega \underline{\varphi_n^1}\check{\varphi}_n^1 dx}. \end{cases}$$

(8) **Once ρ_n is known, compute:**

$\oplus$ $(\varphi_{n+1}^0, \varphi_{n+1}^1) = (\varphi_n^0, \varphi_n^1) - \rho_n(\check{\varphi}_n^0, \check{\varphi}_n^1)$,

$\oplus$ $\varphi_{n+1} = \varphi_n - \rho_n \check{\varphi}_n$,

$\oplus$ $\psi_{n+1} = \psi_n - \rho_n \check{\psi}_n$,

$\oplus$ $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) = (\widetilde{\varphi^0}_n, \widetilde{\varphi^1}_n) - \rho_n(\underline{\varphi_n^0}, \underline{\varphi_n^1})$.

Test of the convergence **and** **construction of the new descent direction.**

(9) **If** $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$ **is small , take** $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_{n+1}^0, \varphi_{n+1}^1)$.
If not, compute

$$\gamma_n = \frac{\int_\Omega || \nabla \widetilde{\varphi^0}_{n+1}||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_{n+1}||^2 dx}{\int_\Omega || \nabla \widetilde{\varphi^0}_n||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_n||^2 dx},$$

and set

$$(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1}) = (\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) + \gamma_n(\check{\varphi}^0{}_n, \check{\varphi}^1{}_n).$$

(10) $n = n + 1$ **and go to** (5)**.**

Chapter 3

Numerical approximation with the finite difference methods

In this chapter we will apply the finite difference methods to control th system (1.1). At the present moment, we shall focus on the case where:

$$\Omega = (0,1), \quad \Gamma_1 = \{1\} \quad \text{and} \quad \Gamma_2 = \{0\}.$$

3.1 Finite difference approximation of the wave equation

In the algorithm at each iteration we need to solve either a homogeneous or a non-homogeneous back-ward wave equation, for this we introduce the general wave system, that it follows:

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \dfrac{\partial^2 \varphi}{\partial x^2} & = & 0 & & \text{in} \quad (0,1) \times (0,T), \\ \varphi(0,\cdot) & = & 0 & & \text{in} \quad (0,T), \\ \varphi(1,\cdot) & = & u(\cdot) & & \text{in} \quad \times (0,T), \\ \varphi(\cdot,0) & = & v(\cdot), & \dfrac{\partial \varphi}{\partial t}(\cdot,0) = w(\cdot) & \text{in} \quad (0,1), \end{cases} \tag{3.1}$$

where $(v,w) \in H_0^1(0,1) \times L^2(0,1)$.

We propose to approach the solution of the system (3.1) by an explicit scheme of finite difference, for this purpose, we have considered a uniform mesh defined by:

$$t_0 = 0 < t_1 = k < \cdots < t_n = nk < \cdots < t_{M+1} = T,$$

and

$$x_0 = 0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

with $k = \dfrac{T}{M+1}$ is the time step and $h = \dfrac{1}{N+1}$ is the space step.

Let φ_j^n be the approximation of $\varphi(x_j, t_n)$, using a central finite differences to approximate the second-order derivative, we obtain :

$$\begin{cases} \dfrac{\varphi_j^{n+1} - 2\varphi_j^n + \varphi_j^{n-1}}{k^2} = \dfrac{\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n}{h^2}, \quad 1 \leq j \leq N, \quad 1 \leq n \leq M, \\ \varphi_0^n = 0, \\ \varphi_{N+1}^n = u(t_n) = u^n, \\ \varphi_j^0 = v(x_j) = v_j, \\ \varphi_j^1 = \varphi_j^0 + kw_j. \end{cases} \tag{3.2}$$

We can prove that the scheme(3.2) is stable if and only if $r = \dfrac{k^2}{h^2} \leq 1$. From (3.2), we have :

$$\varphi^{n+1} = A\,\varphi^n - \varphi^{n-1} + b,$$

where

$$A = \begin{pmatrix} 2(1-r) & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & r & 2(1-r) \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ \vdots \\ ru^n \end{pmatrix}$$

and φ^n the vector of $\mathbb{R}^N$ formed by $\varphi_j^n,\ j \in \{1, \dots, N\}$.

With this discretization we can define a function (in python) which makes it possible to solve the two types of system, for the homogeneous wave equation we take $u \equiv 0$ but for the non-homogeneous back-ward system we have to make a transformation in time.

3.2 Finite difference approximation of the Dirichlet problem

As in the previous section, we need to solve a Dirichlet system on each iteration. We define the general system as follows:

$$\begin{cases} -\dfrac{\partial^2 \phi}{\partial x^2} = \psi'(0) - y \quad \text{in} \quad (0,1), \\ \\ \phi(0) = 0, \\ \\ \phi(1) = 0. \end{cases} \tag{3.3}$$

We approximate the second-order derivative by:

$$\frac{\partial^2 \phi}{\partial x^2}(x_j) \simeq \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2}, \quad 1 \leq j \leq N,$$

and for the second member, we have

$$\psi'(0) = \frac{\psi_j^1 - \psi_j^0}{k}, \quad 0 \leq j \leq N+1.$$

Then the system (3.3) is approximated as follows

$$\begin{cases} -\dfrac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} = \dfrac{\psi_j^1 - \psi_j^0}{k} - y_j \quad 1 \leq j \leq N, \\ \\ \phi(0) = 0, \\ \\ \phi(1) = 0. \end{cases}$$

From this scheme, we obtain

$$B\phi = -\frac{h^2}{k}(\psi^1 - \psi^0) + h^2 Y,$$

with ϕ the vector formed by $\phi_j, \ j \in \{1, \ldots, N\}$ and

$$B = \begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix},$$

$$\psi^k = \begin{pmatrix} \psi_1^k \\ \vdots \\ \psi_N^k \end{pmatrix}; k \in \{0, 1\}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

3.3 Numerical Tests

In this section we present some numerical simulations to control our system, using the general conjugate gradient algorithm and the finite difference discretization method.
For the numerical tests we consider 3 types of initial conditions, and we will discuss the

convergence of the algorithm as well as the controllability of our discrete system.
The conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^0 = 0$, and

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1 \times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1 \times L^2}} < \epsilon,$$

has been used as stopping criterium, with ϵ is a small positive number.
Let's mention also that $||\varphi||_{H^{-1}(0,1)} = ||\psi||_{H_0^1(0,1)}$, where $\psi \in H_0^1(0,1)$ is the solution of the Dirichlet problem:

$$\begin{cases} -\Delta\psi = \varphi & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

and $||\psi||_{H_0^1(0,1)} = \left(\int_0^1 \left(\frac{\partial \psi}{\partial x} \right)^2 dx \right)^{\frac{1}{2}}$. For the approximation of the integrals we used the rectangles method and for the stability of the numerical scheme, we take $r = 1$ (The CFL condition).
Firstly, we consider the following initial conditions associated to the wave equation (**1.1**):

$$\begin{cases} y^0(x) = \exp\left(-5(x - 0.37)^6\right), \\ y^1(x) = 0. \end{cases} \tag{3.4}$$

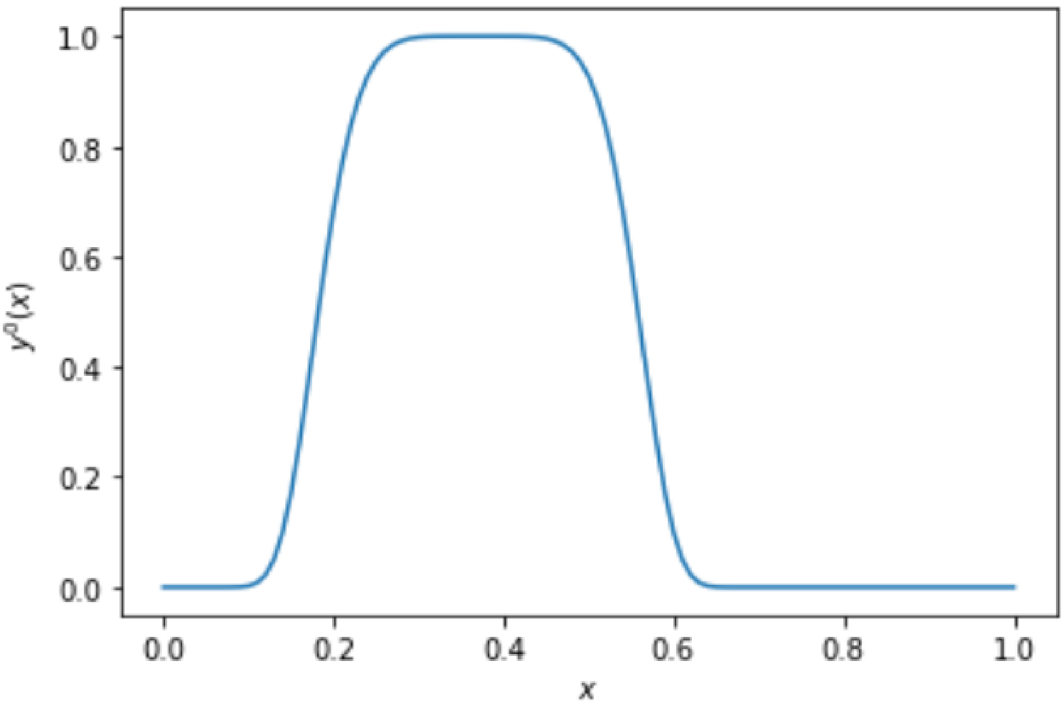

Figure 3.1. The initial position y^0.

N	19	99	499	999
Number of CG iterations	1600	27	7	6
$\|\|y(T)\|\|_{L^2(0,1)}$	9.2857×10^{-11}	1.04928×10^{-10}	1.30932×10^{-11}	3.06843×10^{-12}
$\|\|y'(T)\|\|_{H^{-1}(0,1)}$	5.1037×10^{-11}	4.05311×10^{-11}	1.16793×10^{-11}	2.94205×10^{-12}
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.41628	0.40681	0.40658	0.40657

Table 3.1. Numerical results obtained for different values of $h = \dfrac{1}{N+1}$. $T = 2.2$ and $\epsilon = 10^{-10}$.

For further study of this exemple (3.4), we fix $N = 99$.

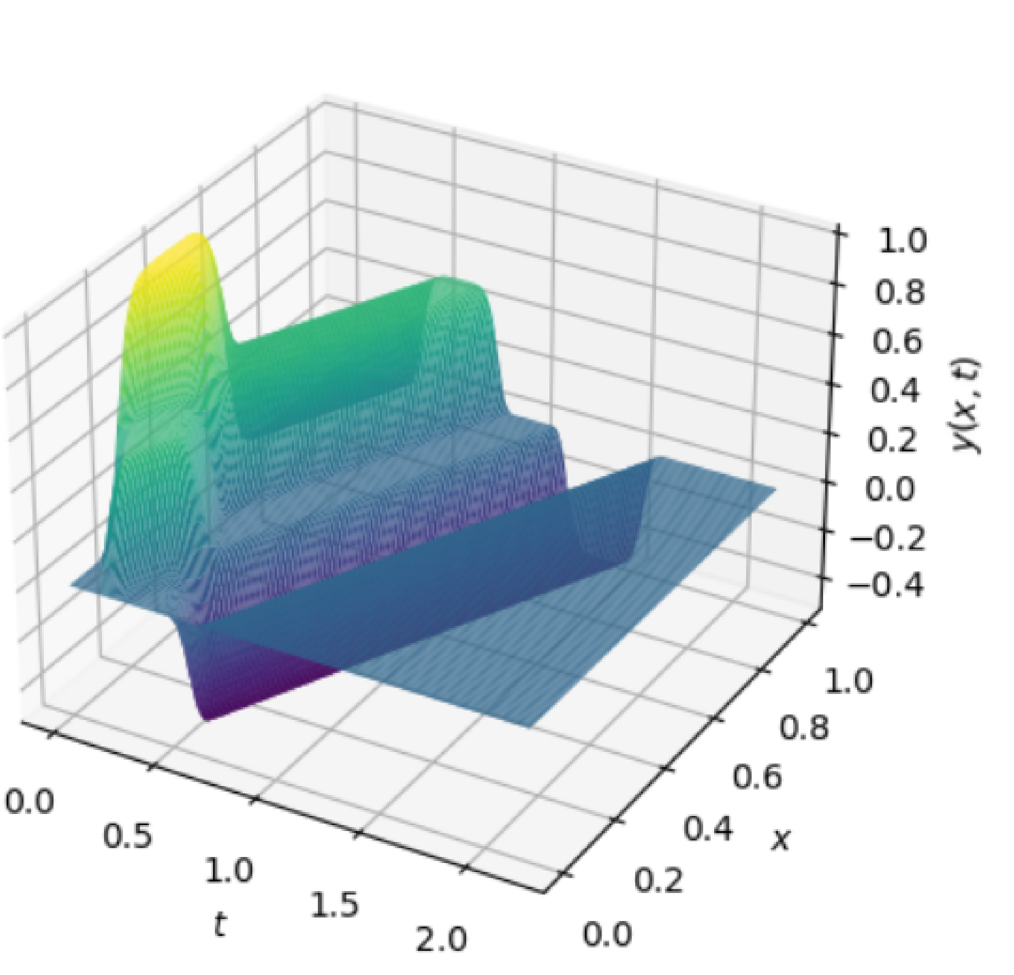

Figure 3.2. The controlled solution y.

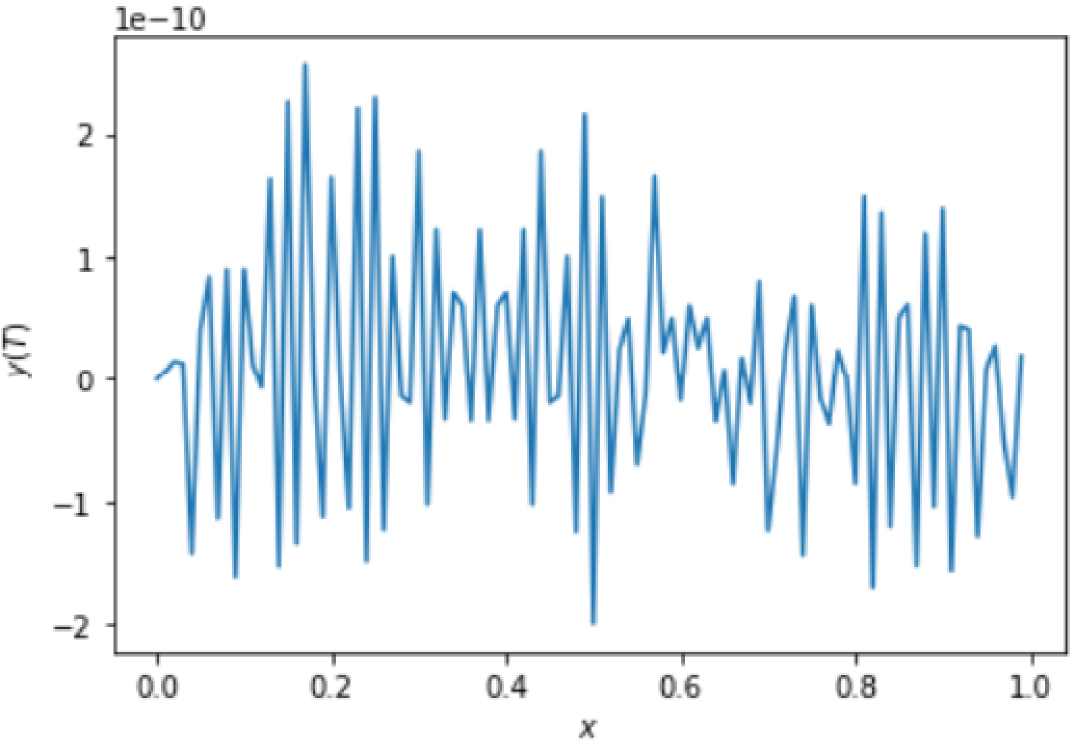

Figure 3.3. The final position $y(T)$.

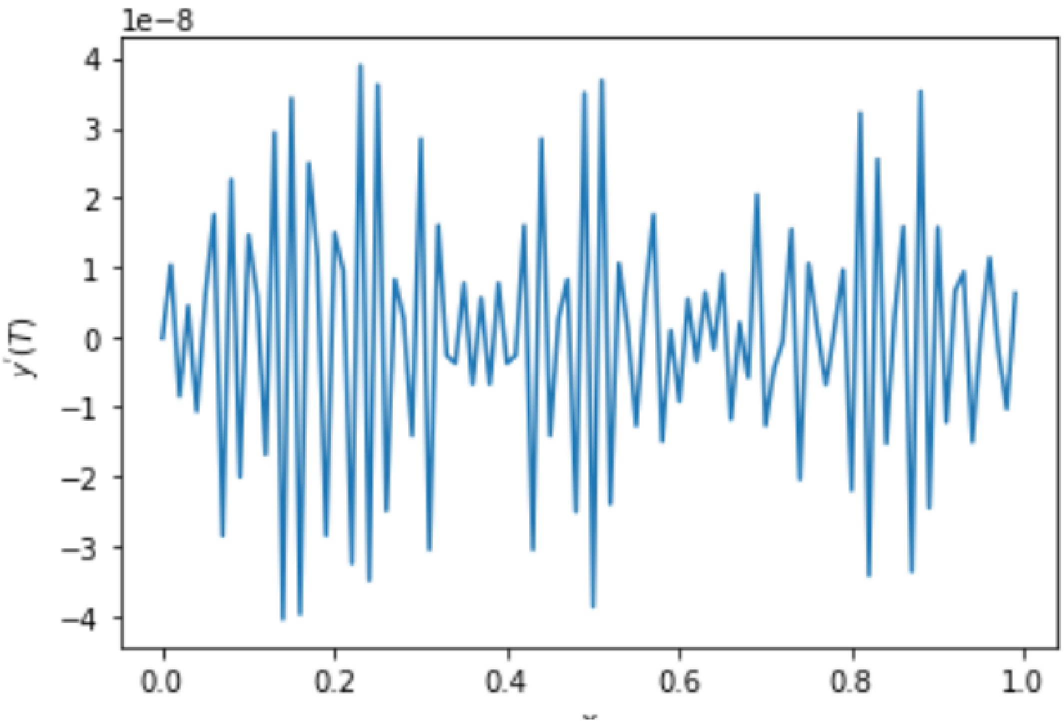

Figure 3.4. The final velocity $y^{'}(T)$.

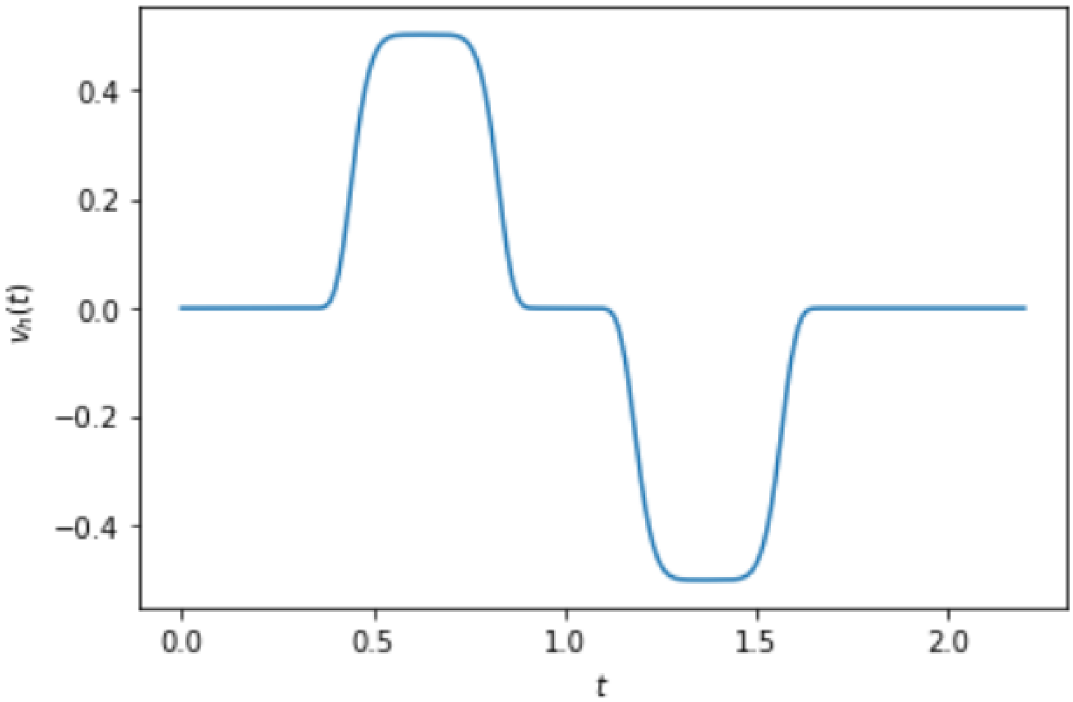

Figure 3.5. The approximation of the control.

From 3.1, we observe that when N gets bigger the number of conjugate gradient iterations decreases, and also we notice that the norme of $y(T)$ in $L^2(0,1)$ and $\dfrac{\partial y}{\partial t}(T)$ in $H^{-1}(0,1)$ converge to 0.
Figures 4.1-3.4 show the controlled solution y_h and the final position, also the final velocity for $T = 2.2$, $N = 99$ and $\epsilon = 10^{-10}$. We remark that the discrete system is controllable, in the figure 3.5 we plot the approximation of the control. Then we can conclude that for the initial conditions (**3.4**), the system is controllable.

Now, we consider the following initial conditions:

$$\begin{cases} y^0(x) = sin(\pi x), \\ y^1(x) = \sqrt{2}\, cos(\pi x). \end{cases} \tag{3.5}$$

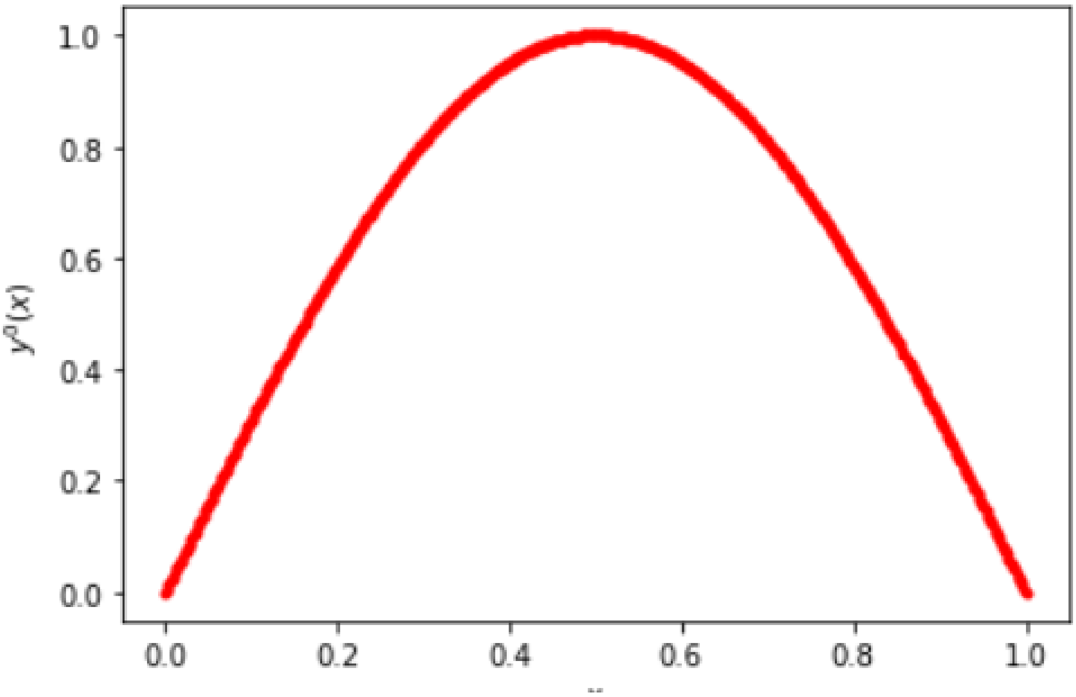

Figure 3.6. The initial position y^0.

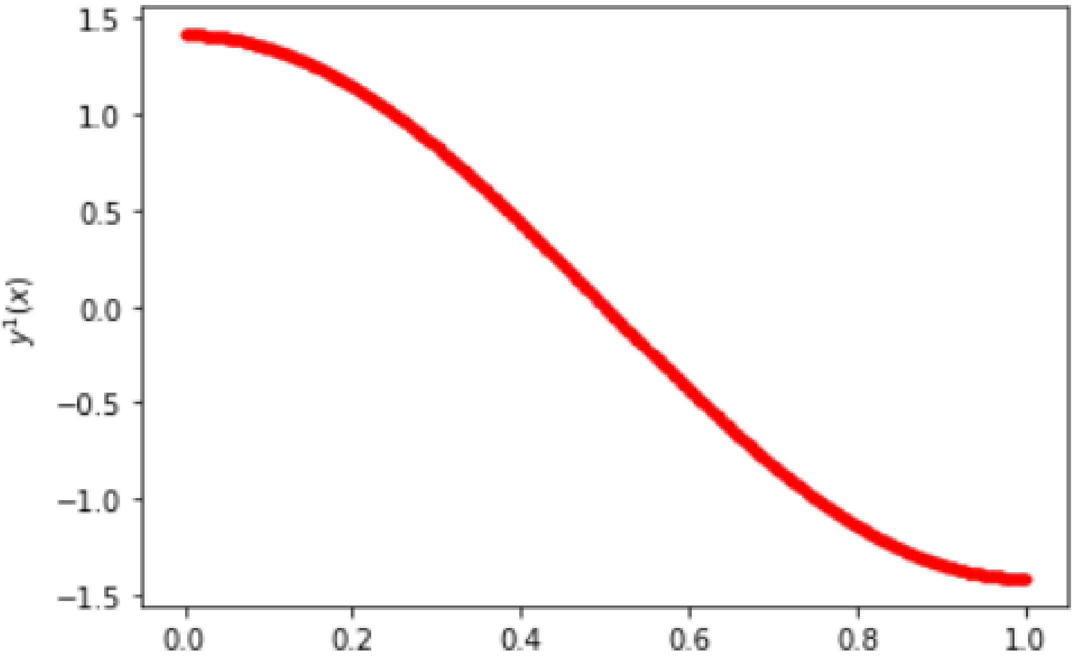

Figure 3.7. The initial velocity y^1.

N	99	499
Number of CG iterations	2905	5003
$\|\|y(T)\|\|_{L^2(0,1)}$	1.2493344460609922$\times 10^{-10}$	1.041012255663964 $\times 10^{-9}$
$\|\|y^{'}(T)\|\|_{H^{-1}(0,1)}$	6.294080645627806$\times 10^{-11}$	5.954733460881318 $\times 10^{-11}$
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.5048527172527582	0.5041223465788962

Table 3.2. Numerical results obtained for different values of $h = \dfrac{1}{N+1}$. $T = 2.2$ and $\epsilon = 10^{-10}$.

For $N = 99$, we obtain:

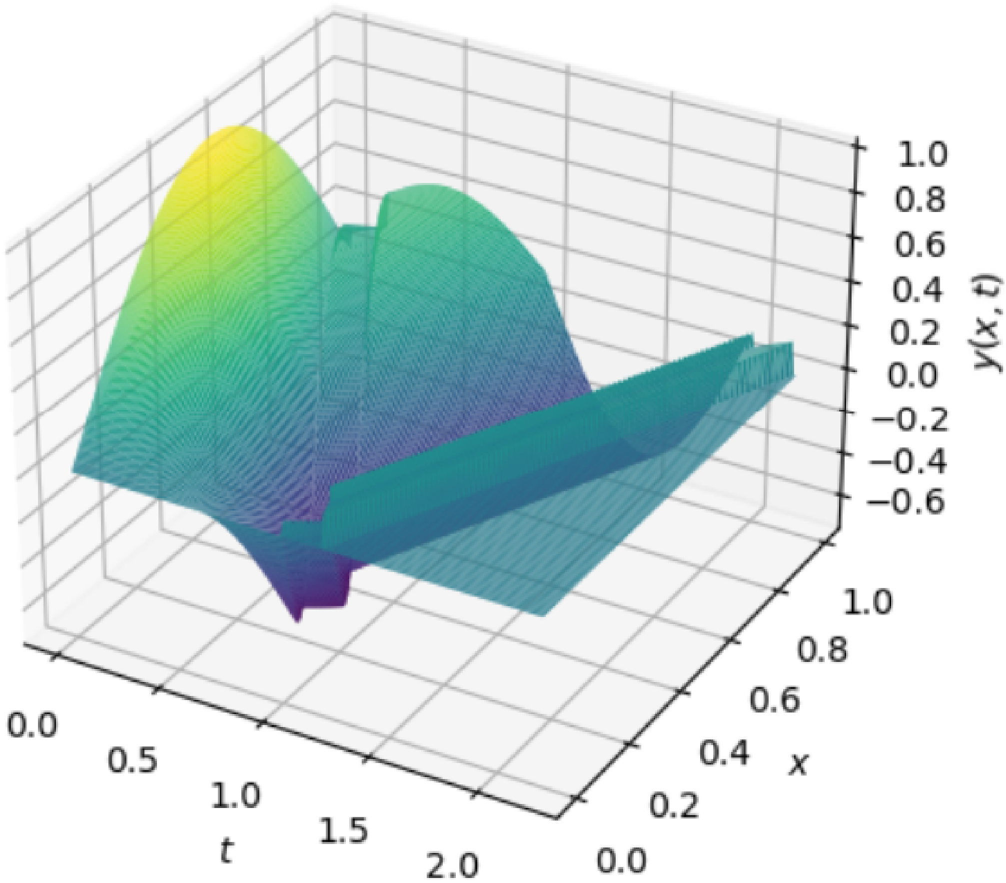

Figure 3.8. The controlled solution y.

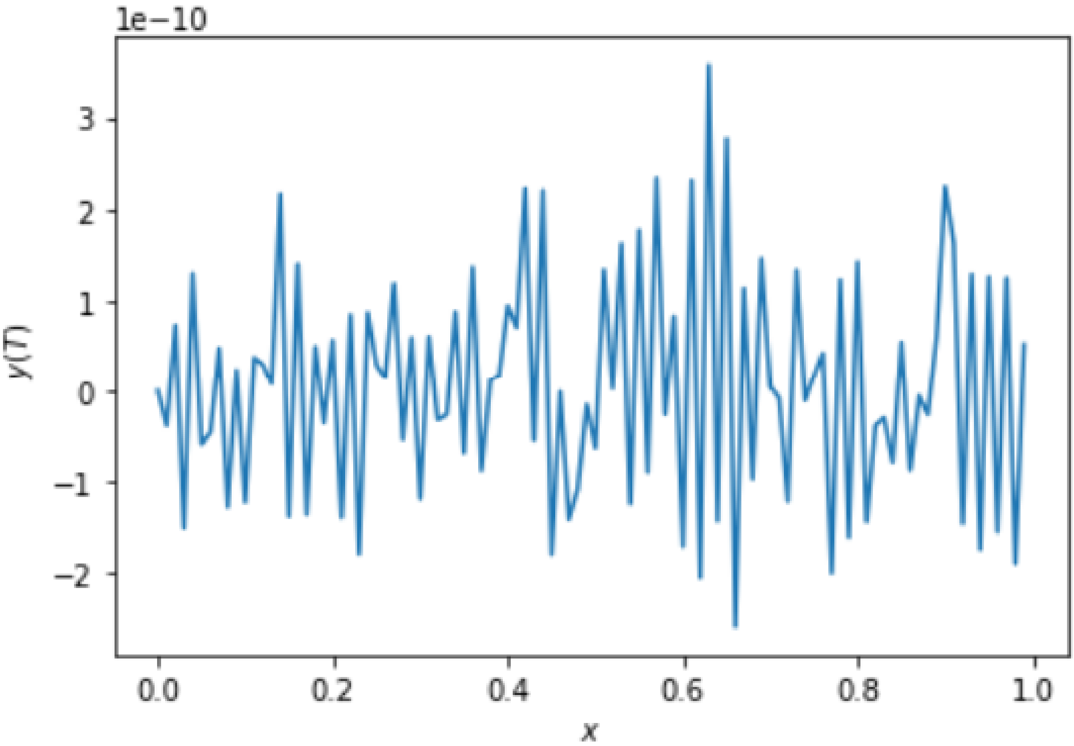

Figure 3.9. The final position $y(T)$.

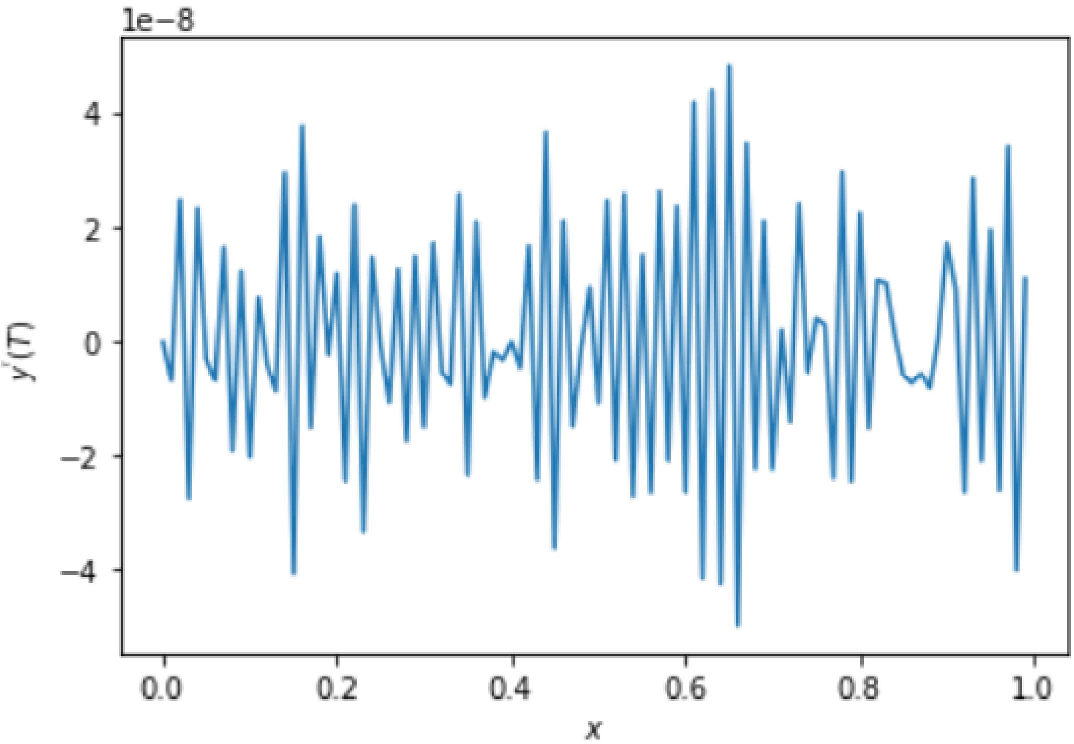

Figure 3.10. The final velocity $y^{'}(T)$.

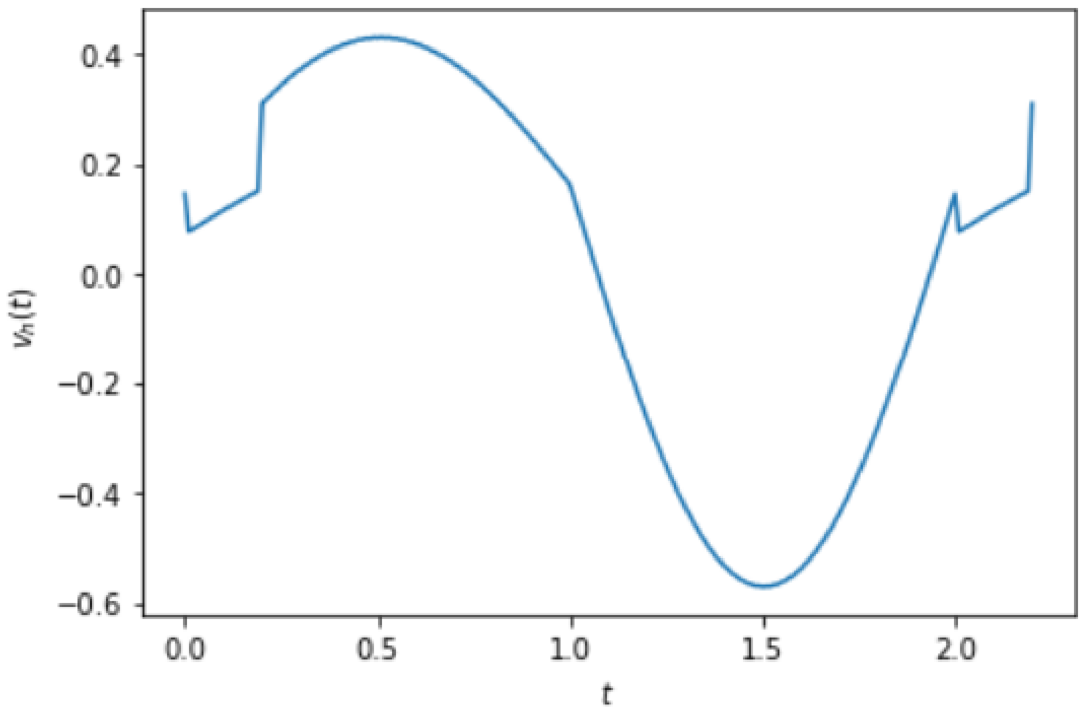

Figure 3.11. The approximation of the control.

From table 3.2 and figures 3.8-3.10, we remark that, the number of iterations for the convergence of the convergence of the algorithm with the initial conditions (**3.5**) is greater compared to (**3.4**), although the system is controllable.

The numerical tests, show that, for $r < 1$ the algorithm does not converge. The problem of the number of iterations or the non convergence is because in the initial conditions there are high frequencies. To solve this problem we filter the initial data, this means that, we are approaching the initial conditions without losing the information and also filtering the high frequencies. For more details (see [..]).

In the last example (**3.6**), we also have the problem of high frequencies and with numerical simulations, we remark that, for $r < 1$ we can't control the system and the algorithm diverges. With $r = 1$, we have the controllability, but the number of iterations is enormous in the case where N small (≤ 100) and the convergence time with python is large in the case where N large. The problem of CPU time can be solved by using Pyccel, Numba or MPI.

finally, we conclude that for $r = 1$, we can control the $1D$ wave equation system using the finite difference method whatever the initial conditions, but the choice of $r = 1$ is not practical, so in this case we need filtration.

$$y^0(x) = \begin{cases} 0, \text{ in } \left[0, \dfrac{1}{2}\right[, \\ \\ 1, \text{ in } \left[\dfrac{1}{2}, 1\right], \end{cases} \tag{3.6}$$

$$y^1(x) = 0.$$

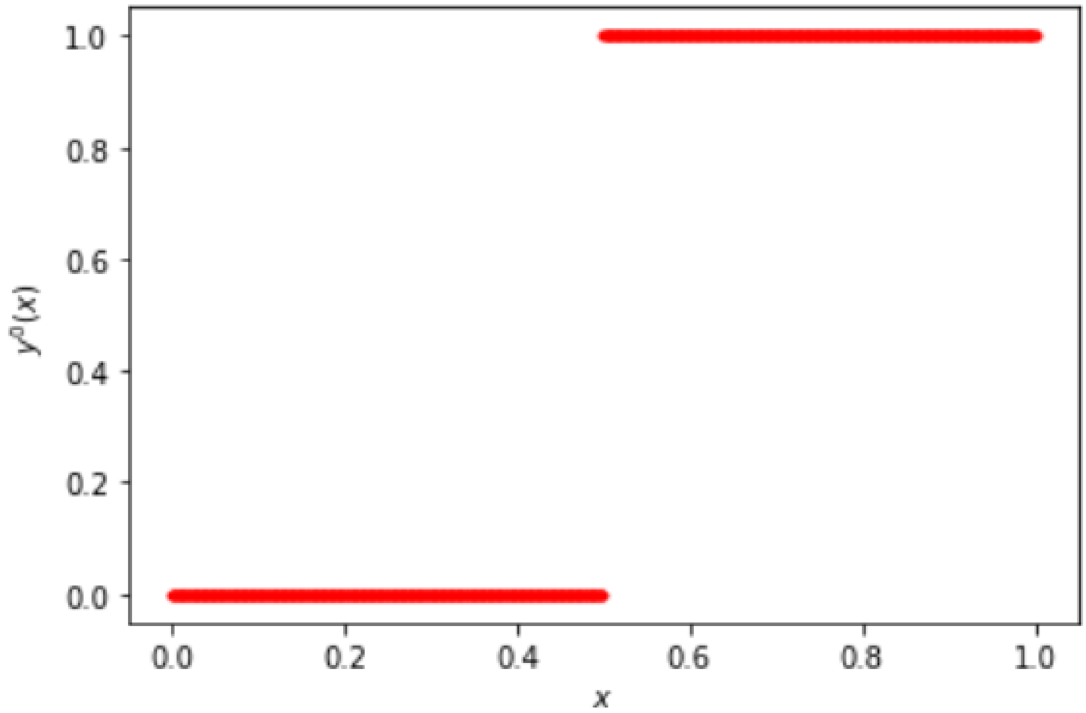

Figure 3.12. The initial position y^0.

We fix $N = 99$, $\epsilon = 10^{-10}$, $r = 1$ and $T = 2.2$, we obtain:
The number of conjugate gradient iterations est 3662.
$||y(T)||_{L^2(0,1)} = 1.87479 \times 10^{-10}$.
$||y^{'}(T)||_{H^{-1}(0,1)} = 5.380761 \times 10^{-11}$.
$||\widehat{v}||_{L^2(0,T)} = 0.58793$.

In the following figures, we plot the controlled solution and the final behavior of our system.

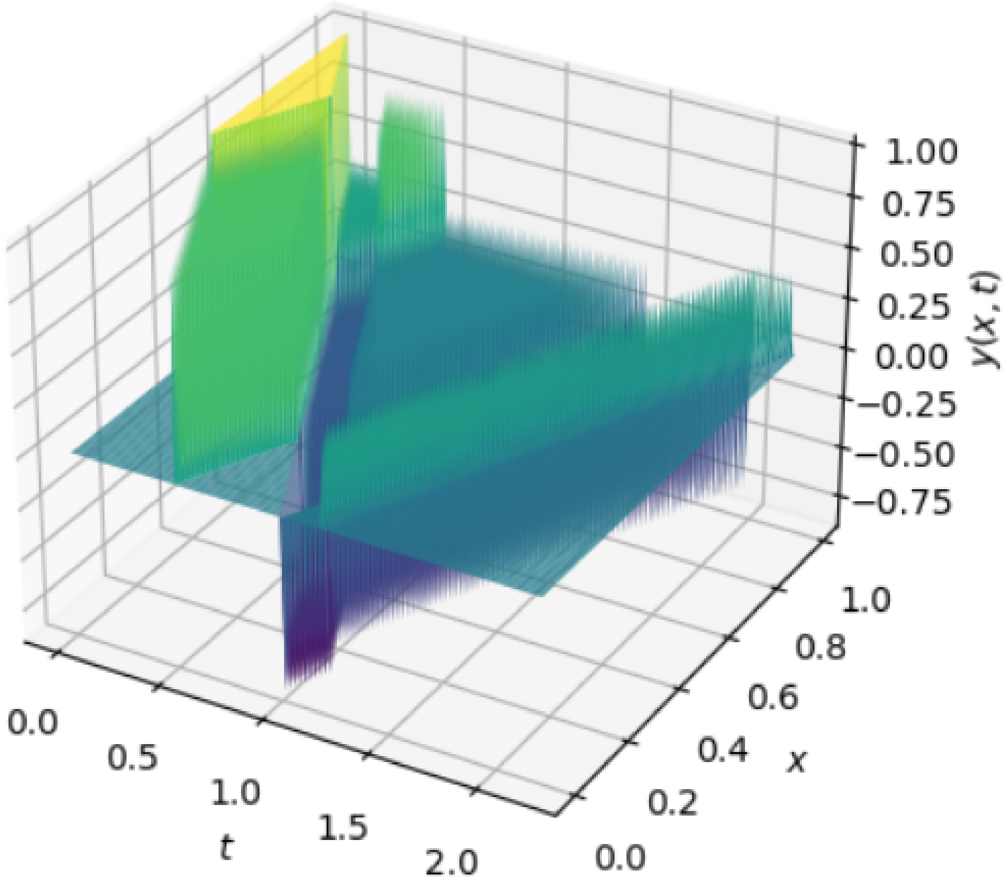

Figure 3.13. The controlled solution y.

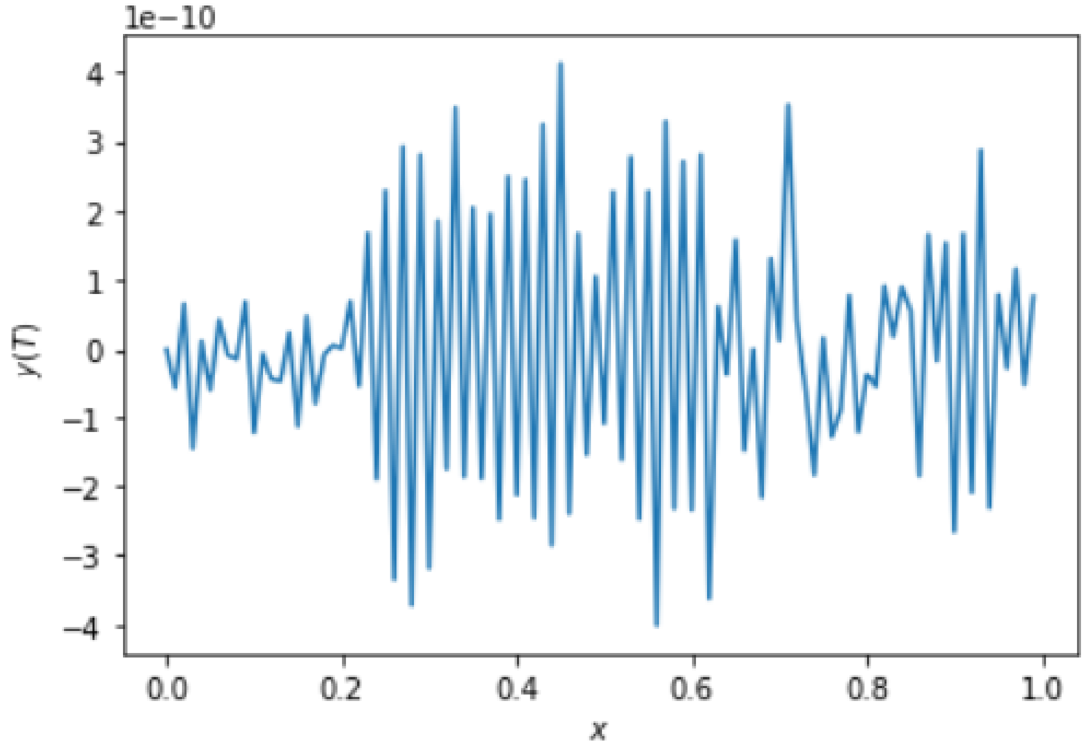

Figure 3.14. The final position $y(T)$.

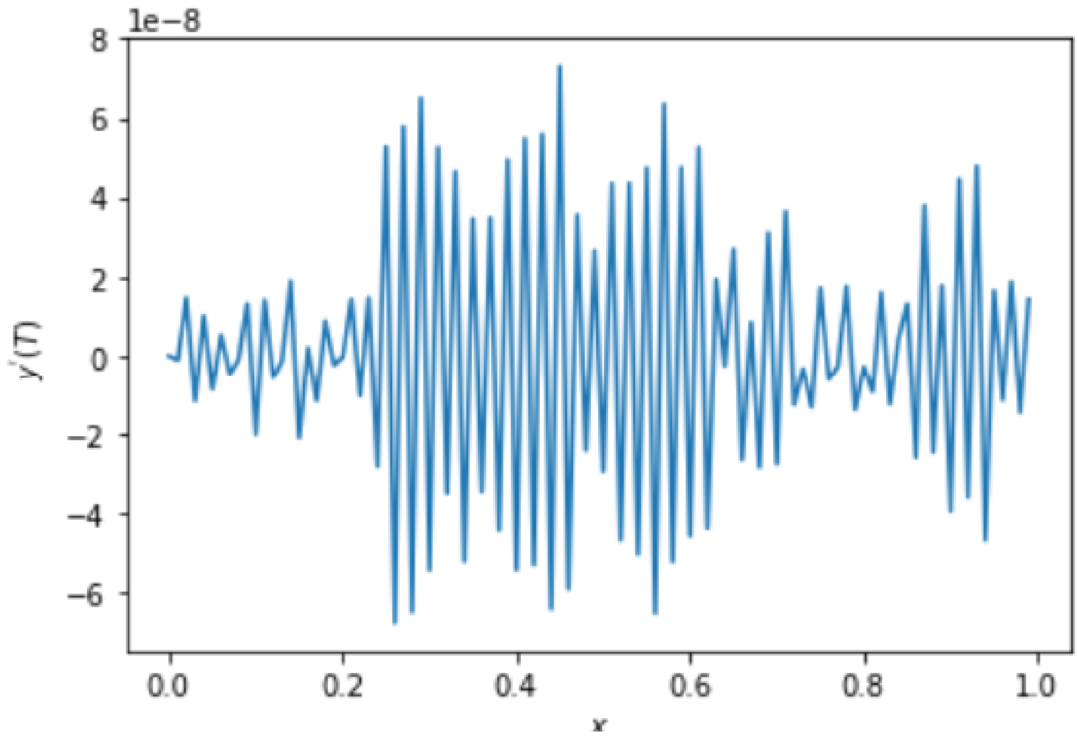

Figure 3.15. The final velocity $y^{'}(T)$.

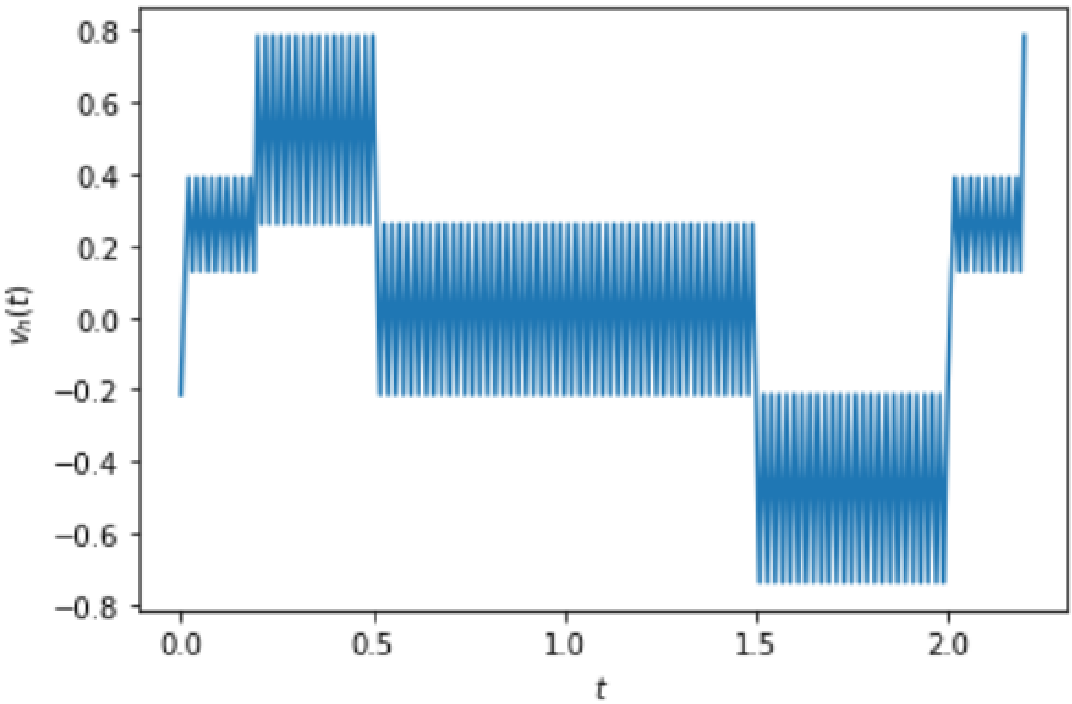

Figure 3.16. The approximation of the control.

Chapter 4

Numerical approximation with the finite difference methods (2D Case)

As the previous chapter, we will apply the finite difference methods to control the system (1.1) in 2D case.
Let

$$\Omega = (0,1) \times (0,1), \quad \Gamma_1 = \Gamma = \partial\Omega.$$

In this case we control the whole boundary.
Note that we can also control a part of the boundary, then it is necessary to verify the geometrical conditions.

4.1 Finite difference approximation of the 2D wave equation

Consider the hyperbolic model problem, with 2D scalar wave equation:

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi = u & \text{in} \quad \Gamma \times (0,T), \\ \varphi(\cdot,0) = v(\cdot), \quad \dfrac{\partial\varphi}{\partial t}(\cdot,0) = w(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $(v,w) \in H_0^1(\Omega) \times L^2(\Omega)$.
We introduce a mesh in time and in space by:

$$t_0 = 0 < t_1 = k < \cdots < t_n = nk < \cdots < t_{M+1} = T,$$

$$x_0 = 0 < x_1 = h_1 < \cdots < x_i = ih_1 < \cdots < x_{N_1+1} = 1,$$

and

$$y_0 = 0 < y_1 = h_2 < \cdots < y_j = jh_2 < \cdots < y_{N_2+1} = 1,$$

with $k = \dfrac{T}{M+1}$, $h_1 = \dfrac{1}{N_1+1}$ and $h_2 = \dfrac{1}{N_2+1}$.
Let $\varphi_{i,j}^n$ be the approximation of $\varphi(x_i, y_j, t_n)$, we obtain:

$$\begin{cases} \dfrac{\varphi_{i,j}^{n+1} - 2\varphi_{i,j}^n + \varphi_{i,j}^{n-1}}{k^2} - \left(\dfrac{\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n}{h_1^2} + \dfrac{\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n}{h_2^2}\right) = 0, \\ 1 \le i \le N_1, 1 \le j \le N_2 \text{ and } \quad 1 \le n \le M, \\ \varphi_{i,j}^n = u_{i,j}^n, \quad \text{for } \; i \in \{0, N_1+1\} \text{ or } j \in \{0, N_2+1\}, \\ \varphi_{i,j}^0 = v_{i,j}, \\ \varphi_{i,j}^1 = \varphi_{i,j}^0 + kw_{i,j}. \end{cases} \tag{4.2}$$

we define $C_1 = \dfrac{k}{h_1}$ and $C_2 = \dfrac{k}{h_2}$, we can prove that for $C = C_1 = C_2$, the explicit scheme is stable if $C \le \dfrac{1}{\sqrt{2}}$, then for the numerical results we use $C_1 = C_2$ and we keep the notations C_1 and C_2 for the modelization of the numerical solution.
From (4.2), we have

$$\varphi_{i,j}^{n+1} - 2\varphi_{i,j}^n + \varphi_{i,j}^{n-1} - C_1^2(\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n) - C_2^2(\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n) = 0,$$

then,

$$\varphi_{i,j}^{n+1} = 2(1 - C_1^2 - C_2^2)\varphi_{i,j}^n - \varphi_{i,j}^{n-1} + C_1^2(\varphi_{i+1,j}^n + \varphi_{i-1,j}^n) + C_2^2(\varphi_{i,j+1}^n + \varphi_{i,j-1}^n)$$

. Let,

$$\lambda = 2(1 - C_1^2 - C_2^2),$$

$$A = \begin{pmatrix} A_1 & A_2 & & 0 \\ A_2 & \ddots & \ddots & \\ & \ddots & \ddots & A_2 \\ 0 & & A_2 & A_1 \end{pmatrix} \text{ and } \varphi^n = \begin{pmatrix} \varphi^n_{1,1} \\ \varphi^n_{1,2} \\ \vdots \\ \varphi^n_{1,N_2} \\ \varphi^n_{2,1} \\ \varphi^n_{2,2} \\ \vdots \\ \varphi^n_{2,N_2} \\ \vdots \\ \vdots \\ \varphi^n_{N_1,1} \\ \varphi^n_{N_1,2} \\ \vdots \\ \varphi^n_{N_1,N_2} \end{pmatrix}.$$

The matrix A is formed by N_1^2 blocks, each of size N_2^2, where A_1 and A_2 defined by:

$$A_1 = \begin{pmatrix} \lambda & C_2^2 & & 0 \\ C_2^2 & \ddots & \ddots & \\ & \ddots & \ddots & C_2^2 \\ 0 & & C_2^2 & \lambda \end{pmatrix}, \; A_2 = \begin{pmatrix} C_1^2 & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & C_1^2 \end{pmatrix}.$$

The system (4.2) is equivalent to:

$$\varphi^{n+1} = A\varphi^n - \varphi^{n-1} + b^n,$$

where b^n is a vector formed by the values of φ^n in the boundary:

$$b^n = \begin{pmatrix} C_1^2\varphi_{0,1}^n + C_2^2\varphi_{1,0}^n \\ C_1^2\varphi_{0,2}^n \\ \vdots \\ C_1^2\varphi_{0,N_2-1}^n \\ C_1^2\varphi_{0,N_2}^n + C_2^2\varphi_{1,N_2+1}^n \\ \\ C_2^2\varphi_{2,0}^n \\ 0 \\ \vdots \\ 0 \\ C_2^2\varphi_{2,N_2+1}^n \\ \\ \vdots \\ \vdots \\ \\ C_1^2\varphi_{N_1+1,1}^n + C_2^2\varphi_{N_1,0}^n \\ C_1^2\varphi_{N_1+1,2}^n \\ \vdots \\ C_1^2\varphi_{N_1+1,N_2}^n + C_2^2\varphi_{N_1,N_2+1}^n \end{pmatrix}.$$

We notice that the matrix A is large if N_1 and N_2 are large, this matrix must be stored in the memory.
The other problem is that the np.dot function in python is numerically unstable and the CPU time to calculate the matrix-vector product is very long, but in our case, the matrix A is tridiagonal and formed by two matrices, one tridiagonal and the other is diagonal, then we can define a python function for calculate the product $A\varphi^n$ and also we can accelerate this product with Numba.

4.2 Finite difference approximation of the 2D Dirichlet problem

In the algorithm (HUM), we need to solve a Dirichlet system on each iteration.
The Dirichlet problem in the algorithm is of the form:

$$\begin{cases} -\left(\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2}\right) = \psi'(0) - y & \text{in} \quad \Omega, \\ \\ \varphi = 0, & \text{in} \quad \Gamma, \end{cases} \tag{4.3}$$

with the finite difference approximation, we obtain:

$$\begin{cases} \dfrac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h_1^2} + \dfrac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{h_2^2} = y_{i,j} - \dfrac{\psi_{i,j}^1 - \psi_{i,j}^0}{k}, \quad 1 \leq i \leq N_1, \quad 1 \leq j \leq N_2, \\ \\ \varphi_{i,j} = 0, \quad \text{for } i \in \{0, N_1 + 1\} \text{ or } j \in \{0, N_2 + 1\}. \end{cases} \tag{4.4}$$

As the previous section, let

$$\varphi = \begin{pmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \vdots \\ \varphi_{1,N_2} \\ \varphi_{2,1} \\ \varphi_{2,2} \\ \vdots \\ \varphi_{2,N_2} \\ \vdots \\ \vdots \\ \varphi_{N_1,1} \\ \varphi_{N_1,2} \\ \vdots \\ \varphi_{N_1,N_2} \end{pmatrix}. \text{ and } \; Y_{i,j} = y_{i,j} - \frac{\psi_{i,j}^1 - \psi_{i,j}^0}{k}, \; (i,j) \in \{1, \ldots, N_1\} \times \{1, \ldots, N_2\}.$$

From (4.4), we have:

$$(**) \quad B\varphi = Y.\text{flatten}(),$$

with B the matrix formed by N_1^2 blocks, each of size N_2^2.

$$B = \begin{pmatrix} B_1 & B_2 & & 0 \\ B_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_2 \\ 0 & & B_2 & B_1 \end{pmatrix},$$

with B_1 is a tridiagonal matrix and B_2 is a diagonal matrix:

$$B = \begin{pmatrix} \frac{-2}{h_1^2} - \frac{2}{h_2^2} & \frac{1}{h_2^2} & & 0 \\ \frac{1}{h_2^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{h_2^2} \\ 0 & & \frac{1}{h_2^2} & \frac{-2}{h_1^2} - \frac{2}{h_2^2} \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{h_1^2} & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \frac{1}{h_1^2} \end{pmatrix}.$$

Like the previous section, the matrix B is large, then for solving the system $(**)$, we need to stored the matrix B and also the function lg.solve in scipy.linalg is numerically unstable if N_1, N_2 are large.
As the matrix B is formed by blocks, we can use the MPI for solving the system $(**)$ and accelerate the time of resolution.
The other solution is to use the package scipy.sparse, in this case we have stored the matrix B in memory, then replace B with pointers in which we keep the terms no nulls (B=coo_matrix(B)).

4.3 Numerical Tests

For the numerical tests we fix $C = C1 = C2$, $T = 3$ (or $T > 2\sqrt{2}$) and we will discuss the convergence of the conjugate gradient algorithm.
As the previous chapter, the conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$ and for the stopping criteria we choose:

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1 \times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1 \times L^2}} < 10^{-10}.$$

The numerical tests show that if $C > \frac{1}{\sqrt{2}}$, the algorithm diverges, so we can't control the initial data using HUM, then does there exist $C \in \left[0, \frac{1}{\sqrt{2}}\right]$ so that all initial data can be controllable.
In case $1D$ we have a condition for C ($C = 1$), for our case there are no theoretical results, by analogy with the previous chapter we choose $C = \frac{1}{\sqrt{2}}$, then we have the stability for the wave equation, but there are initial conditions that we cannot control. For our test we will show that with $C = \frac{1}{\sqrt{2}}$, we can control a set of initial conditions.
Let $y^1 = 0$. We assume that $\exists D \in \Omega$, such as:

$$y^0 = \begin{cases} \approx 0, \text{ in } \Omega \backslash D, \\ \\ y_D^0, \text{ in } D, \end{cases}$$

with D is a compact, convex ($\lambda(D) < \lambda(\Omega)$), we also assume that there are no high frequencies in y_D^0, otherwise we filter y_D^0.
Based on example (3.4) with a modification of the $2D$ Gaussian law, we define :

$$y^0(x,y) = exp\left(-5\left(\frac{r}{2\sigma^2}\right)^6\right),$$

with $r = (x-0.35)^2 + (y-0.35)^2$.
Here $(0.35, 0.35)$ is the centre, $\sigma_x = \sigma_y = \sigma$ are the standard deviations for x and y.

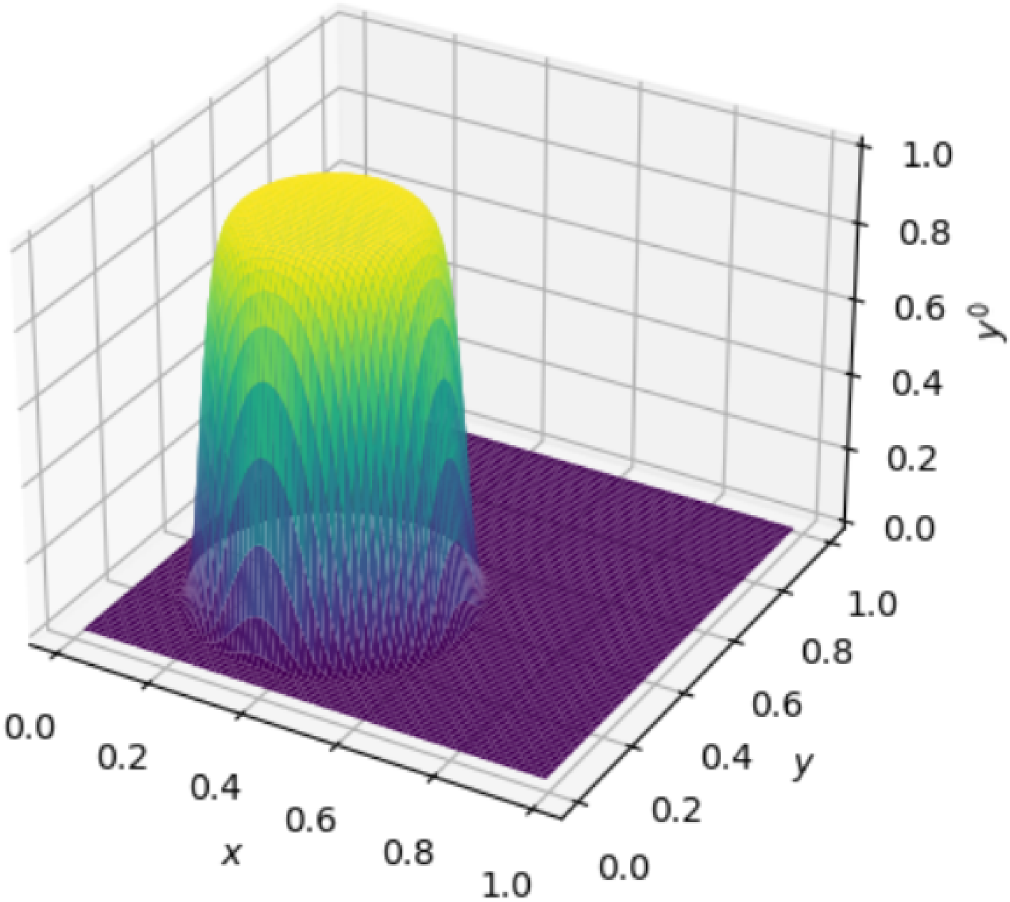

Figure 4.1. The initial position y^0 for $\sigma = 0.2$.

The numerical tests show that for $\sigma \geq 0.3$ the algorithm diverges, moreover when N_1 becomes large the number of iterations increases also the CPU time grow, the other problem that we need a large memory space.

N	15	30	40	50
Number of CG iterations	175	337	547	943
CPU time (secs)	1.625318	9.933084	32.132817	103.455245
$\|\|y(T)\|\|_{L^2(\Omega)}$	0.00295	6.693309×10^{-11}	1.27032×10^{-11}	8.422314×10^{-11}
$\|\|y^{'}(T)\|\|_{H^{-1}(\Omega)}$	5.434475×10^{-11}	6.503109×10^{-11}	1.211519×10^{-10}	8.497743×10^{-11}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.180434	0.167908	0.16638	0.1652834

Table 4.1. Numerical results obtained for $C = \dfrac{1}{\sqrt{2}}$. $T = 3$ and $\sigma = 0.2$.

we fix the maximum number of CG iterations $N_{MAX} = 2000$. tables 4.2-4.4 show the the CPU time, the convergence of CG algorithm and the number of CG iterations for different values of ϵ, in the case of the convergence of the algorithm, we will calculate the norm of $||y(T)||_{L^2(\Omega)}$, $||y^{'}(T)||_{H^{-1}(\Omega)}$ and $||\widehat{v}||_{L^2((0,T)\times\Gamma)}$

N	40	60	100	150
Number of CG iterations	114	21	19	16
CPU time (secs)	10.438298	5.189082	26.279451	99.680222
convergence	CV	CV	CV	CV
$\|\|y(T)\|\|_{L^2(\Omega)}$	0.000366649	0.0003217156	0.0002750246	0.000349270
$\|\|y^{'}(T)\|\|_{H^{-1}(\Omega)}$	0.000372948	0.000300827	0.0002772390	0.000288972
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.166377769	0.165065510	0.1643637404	0.164166635

Table 4.2. $\epsilon = 10^{-3}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	181	465	323	475
CPU time (secs)	13.119077	105.264298	413.225330	2774.186211
convergence	CV	CV	CV	CV
$\|\|y(T)\|\|_{L^2(\Omega)}$	5.56653233×10^{-5}	0.000104486	3.10654843×10^{-5}	3.44585132×10^{-5}
$\|\|y^{'}(T)\|\|_{H^{-1}(\Omega)}$	5.51862416×10^{-5}	0.000106774	3.07522409×10^{-5}	3.40246881×10^{-5}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.16638202	0.165074544	0.16437017	0.164172625

Table 4.3. $\epsilon = 10^{-4}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	218	734	2000	2000
CPU time (secs)	15.639176	170.212021	20320.163	40640.326
convergence	CV	CV	DIV	DIV
$\|\|y(T)\|\|_{L^2(\Omega)}$	1.3622136×10^{-5}	1.497883585×10^{-5}	//	//
$\|\|y^{'}(T)\|\|_{H^{-1}(\Omega)}$	1.3334127×10^{-5}	$1.466640873 \times10^{-5}$	//	//
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.165075143	0.165074544	//	//

Table 4.4. $\epsilon = 10^{-5}$, $\sigma = 0.2$

From table 4.2-4.4 and numerical simulations, we notice that, when $\epsilon \geq 10^{-5}$ and $N > 100$ (or N large) the algorithm does not converge in 2000 iterations, also the CPU time is very

large, and the stop criterion, it had an oscillation around 10^{-5}.
In most cases the approximation with the finite difference method does not work for all the initial data, but we can control some type of initial conditions, in this case we have to accelerate the code and also solve the memory problem, but in general we do not find good approximations $(N \geq 100)$.
in the following chapters we will try to use a finite element approximation and see if we have the controllability.

Chapter 5

Numerical approximation with the finite element method

Conclusion

OK

Bibliography

[1] T. Rossing. Springer Handbook of Acoustics. *Rossing Ed.*, 2007.