

MOHAMMED VI
POLYTECHNIC
UNIVERSITY

Ingénierie des Systèmes Complexes et Systèmes Humains

Master Modélisation Hybride Avancée et Calcul Scientifique

M9: Méthodes Numériques

Notes de cours sur la Méthode des Éléments Finis

Années 2019-2020

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Introduction

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Chapter 1

The Exact Boundary controllability

1.1 Formulation of the control problem

Let Ω be a bounded domain (that is, non-empty open connected set) in \mathbb{R}^n with boundary $\Gamma = \partial\Omega$ "Sufficiently smooth", Γ_1 be an open nonempty subset of Γ and $\Gamma_2 = \Gamma \setminus \Gamma_1$. With T a given positive number, we consider the following non-homogeneous wave equation :

$$\left\{ \begin{array}{ll} \frac{\partial^2 y}{\partial t^2} - \Delta y = 0 & \text{for } (x, t) \in \Omega \times (0, T), \quad (1.1.1) \\ y(x, t) = 0 & \text{for } (x, t) \in \Gamma_2 \times (0, T), \quad (1.1.2) \\ y(x, t) = v(x, t) & \text{for } (x, t) \in \Gamma_1 \times (0, T), \quad (1.1.2) \\ y(x, 0) = y^0, \quad \frac{\partial y}{\partial t}(x, 0) = y^1 & \text{for } x \in \Omega. \quad (1.1.3) \end{array} \right. \quad (1.1)$$

(1.1.2): The boundary conditions.

(1.1.3) The initial conditions.

In (1.1), $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, such that $H^{-1}(\Omega)$ is the topological dual space of $H_0^{-1}(\Omega)$ and Δ is the Laplacian operator.

We shall now define the exact boundary controllability for the system (1.1).

Definition 1.1.1. System (1.1) is controllable in time $T > 0$ if for every initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we can find a control function $v \in L^2(\Gamma_1 \times (0, T))$ such that the corresponding solution (y, y') of (1.1) verifies

$$y(., T) = y'(., T) = 0. \quad (1.2)$$

Remark 1.1.1. For the existence of solution see section [1.2]

Remark 1.1.2. If the solution of (1.1) verifies (1.2) is also said to be null controllable in time $T > 0$, for more details (see [1], page 100).

The problem that we consider is the following one:
is it possible to find $T > 0$ sufficiently large or optimal and $v \in L^2(\Gamma_1 \times (0, T))$ a boundary

control function such that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we have (1.2).

But before that, we have to prove that the system (1.1) admits a solution.

1.2 Existence and uniqueness of solutions

In this section we consider the general case of (1.1) with v not necessarily null and a bounded domain Ω in \mathbb{R}^n .

Definition 1.2.1. For (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(\Gamma_1 \times [0, +\infty))$ a function $y \in C([0, +\infty), L^2(\Omega)) \cap C^1([0, +\infty), H^{-1}(\Omega))$ is called a weak solution of (1.1) if the relation

$$\begin{aligned} \int_{\Omega} y(x, t) \varphi(x) dx - \int_{\Omega} y^0(x) \varphi(x) dx - t \langle y^1, \varphi \rangle_{-1,1} &= \int_0^t \int_0^s \int_{\Omega} y(x, \xi) \Delta \varphi(x) dx d\xi ds \\ &- \int_0^t \int_0^s \int_{\Gamma_1} v(x, \xi) \frac{\partial \varphi}{\partial n}(x) d\sigma d\xi ds, \end{aligned} \quad (1.3)$$

holds for every $t \geq 0$ and every $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 1.2.1. For x in Γ and φ in $H^2(\Omega)$ we have

$$\frac{\partial \varphi}{\partial n}(x) = \nabla \varphi(x) \cdot \vec{n}(x),$$

is called the normal derivative, with $\vec{n}(x)$ is the normal vector at x

Remark 1.2.2. The normal vector exists because the boundary Γ is sufficiently smooth.

The main result for the existence of solutions of (1.1) is the following:

Theorem 1.2.1. For every v in $L^2(\Gamma_1 \times (0, T))$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}$ system (1.1) has a unique weak solution

$$(y, y') \in C([0, T], L^2(\Omega) \times H^{-1}),$$

moreover, there exists a constant $C = C(T) > 0$ such that

$$\|(y, y')\|_{L^\infty([0, T], L^2(\Omega) \times H^{-1})} \leq C \left[\|(y^0, y^1)\|_{L^2(\Omega) \times H^{-1}} + \|v\|_{L^2(\Gamma_1 \times (0, T))} \right]$$

Proof :

The theorem is a consequence of the theory of nonhomogeneous evolution equations.

- * Proof with variational method, see [2].
- * Proof with Semigroups operator, see [3].

1.3 Controllability by minimization

In this section we will give a necessary and sufficient condition for the exact controllability of (1.1) and also we will transform the controllability problem to a minimization problem, but before that we need to prove the following lemma:

Lemma 1.3.1. *For every (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$ the following wave equation*

$$\begin{cases} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{in } \Gamma \times (0, T), \\ \varphi(\cdot, 0) = \varphi^0, \quad \frac{\partial \varphi}{\partial t}(\cdot, 0) = \varphi^1 & \text{in } \Omega, \end{cases} \quad (1.4)$$

has a unique solution, moreover (1.4) generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$.

Proof:

this lemma is a consequence of the following classic theorem:

Theorem 1.3.1. (Stone, 1930)

Let H be a Hilbert space and A be a linear operator on H with dense domain, then A generates a C_0 -group of unitary operators if and only if A is skewadjoint ($A' = -A$)

Let $w(t) = (\varphi(t), \varphi'(t))$ and the state space is $H_0^1(\Omega) \times L^2(\Omega)$, with the scalar product

$$\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle = \int_{\Omega} \nabla \varphi_1 \nabla \psi_1 + \int_{\Omega} \varphi_2 \psi_2,$$

then the corresponding norm on H given by:

$$\|(\varphi_1, \varphi_2)\|^2 = \int_{\Omega} \|\nabla \varphi_1\|^2 + \int_{\Omega} \|\varphi_2\|^2.$$

By the Poincaré inequality, this norm is equivalent to the usual norm on H .

We define the linear operator A by:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{and}$$

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}.$$

Firstly the domain $D(A)$ is dense in H .

Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D(A)$, then

$$\begin{aligned}
 \langle A(\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle &= \langle (\varphi_2, \Delta\varphi_1), (\psi_1, \psi_2) \rangle \\
 &= \int_{\Omega} \nabla\varphi_2 \nabla\psi_1 + \int_{\Omega} \Delta\varphi_1 \psi_2 \\
 &= \int_{\Omega} \nabla\varphi_2 \nabla\psi_1 - \int_{\Omega} \nabla\varphi_1 \nabla\psi_2 \\
 &\quad + \underbrace{\int_{\Gamma} \frac{\partial\varphi_1}{\partial n} \psi_2}_0 \\
 &= - \int_{\Omega} \nabla\varphi_1 \nabla\psi_2 - \int_{\Omega} \varphi_2 \Delta\psi_1 \\
 &= - \langle (\varphi_1, \varphi_2), (\psi_2, \Delta\psi_1) \rangle \\
 &= - \langle (\varphi_1, \varphi_2), A(\psi_1, \psi_2) \rangle.
 \end{aligned}$$

Which implies that the operator A is skewadjoint on H and thus generates a unitary C_0 -group on H by Stone's theorem.

Then if $(\varphi^0, \varphi^1) \in D(A)$ the system (1.4) has a unique strict solution by Semi-group theory, but in the case $(\varphi^0, \varphi^1) \in H \setminus D(A)$, (1.4) has a unique weak solution by follows the proof of lemma.

Proposition 1.3.1. *Assume that $T > 0$ is large enough, then there exists a constant $\hat{C} > 0$ such that:*

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial\varphi}{\partial n} \right|^2 d\sigma dt \leq \hat{C} \|(\varphi^0, \varphi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2, \quad (1.5)$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the solution of (1.4).

Proof:

The inequality (1.5) is a hidden regularity result, that may not be obtained by standard trace results (see, [.]).

The key result in this section is the variational characterization of the controllability of (1.1) given by the following lemma:

Lemma 1.3.2. *The initial data $(y^0, y^1) \in \mathbb{L}^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there exists $v \in L^2(\Gamma_1 \times (0, T))$ such that:*

$$\int_0^T \int_{\Gamma_1} \frac{\partial\varphi}{\partial n} v d\sigma dt = \langle y^1, \varphi^0 \rangle_{-1,1} - \int_{\Omega} y^0 \varphi^1 dx, \quad (1.6)$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of the adjoint problem (1.4)

Proof:

For $(y^0, y^1) \in D(\Omega) \times D(\Omega)$, $(\varphi^0, \varphi^1) \in D(\Omega) \times D(\Omega)$ and $v \in D(\Gamma_1 \times (0, T))$, then y and φ

are regular solutions. Multiplying the equation (1.1.1) in system (1.1) by φ and integrating, we have

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega} (y'' - \Delta y) \varphi dx dt \\
&= \int_{\Omega} \int_0^T y'' \varphi dt dx - \int_0^T \int_{\Omega} \Delta y \varphi dx dt \\
&= \int_{\Omega} \left[[y' \varphi]_0^T - \int_0^T y' \varphi' dt \right] dx + \int_0^T \int_{\Omega} \nabla y \nabla \varphi dx dt \\
&= \int_{\Omega} (y'(T) \varphi(T) - y'(0) \varphi(0)) dx - \int_{\Omega} [y \varphi']_0^T dx + \int_{\Omega} \int_0^T y \varphi'' dt dx \\
&\quad - \int_0^T \int_{\Omega} \Delta \varphi y dx dt + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y d\sigma dt \\
&= \int_{\Omega} (y'(T) \varphi(T) - y'(0) \varphi(0)) dx - \int_{\Omega} (y(T) \varphi'(T) - y(0) \varphi'(0)) dx + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y d\sigma dt .
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y d\sigma dt &= - \int_{\Omega} (y'(T) \varphi(T) - y'(0) \varphi(0)) dx \\
&\quad + \int_{\Omega} (y(T) \varphi'(T) - y(0) \varphi'(0)) dx,
\end{aligned}$$

finally we have,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y d\sigma dt &= \int_{\Omega} y^1 \varphi^0 dx - \int_{\Omega} y^0 \varphi^1 dx \\
&\quad + \int_{\Omega} y(T) \varphi'(T) dx - \int_{\Omega} y'(T) \varphi(T) dx.
\end{aligned}$$

We Know that $D(\Omega) \times D(\Omega)$ dense in $L^2(\Omega) \times H^{-1}(\Omega)$ as well as in $H_0^1(\Omega) \times L^2(\Omega)$ and $D(\Gamma_1 \times (0, T))$ dense in $L^2(\Gamma_1 \times (0, T))$. by the inequality (1.5), we have $\frac{\partial \varphi}{\partial n}|_{\Gamma_1} \in L^2(\Gamma_1) \times (0, T)$. From a density argument we deduce that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ we have,

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y d\sigma dt &= \langle y^1 \varphi^0 \rangle_{-1,1} - \int_{\Omega} y^0, \varphi^1 dx \\
&\quad + \langle (\varphi(T), \varphi'(T)), (y(T), y'(T)) \rangle .
\end{aligned}$$

With $\langle ., . \rangle_{-1,1}$ be the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we introduce the duality product

$$\langle (\varphi^0, \varphi^1), (y^0, y^1) \rangle = \int_{\Omega} y^0 \varphi^1 - \langle y^1, \varphi^0 \rangle_{-1,1} .$$

Such that the wave equation generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$, then (1.6) holds if and only if (y^0, y^1) is controllable to zero in time $T > 0$. This completes the proof. From lemme 1.3.2, the equality (1.6) can be seen as an optimality condition for the minimants of the functional $\mathcal{J} : H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$, defined by

$$\mathcal{J} = \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt + \langle (\varphi^0, \varphi^1), (y^0, y^1) \rangle, \quad (1.7)$$

where $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ is the corresponding solution of (1.4).

Before introducing the main theorem in this section we need to proof that the functional \mathcal{J} has a minimant.

Definition 1.3.1. System (1.4) is said to be observable in time $T > 0$ if there exists a positive a positive constant $C > 0$ such that

$$C \|(\varphi^0, \varphi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt, \quad (1.8)$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0, φ^1) .

In the following we assume that there is a positive time T^* such that for any $T > T^*$ the system (1.4) is observable.

On the other hand, the functional \mathcal{J} is continuous, strictly convex and coercive.

It is easy to see that functional \mathcal{J} is continuous, now let $(\varphi^0, \varphi^1), (\psi^0, \psi^1)$ in $H_0^1(\Omega) \times L^2(\Omega)$ and $\lambda \in]0, 1[$, we have,

$$\begin{aligned} \mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1 - \lambda)(\psi^0, \psi^1)) &= \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \lambda \frac{\partial \varphi}{\partial n} + (1 - \lambda) \frac{\partial \psi}{\partial n} \right|^2 d\sigma dt \\ &+ \langle \lambda(\varphi^0, \varphi^1) + (1 - \lambda)(\psi^0, \psi^1), (y^0, y^1) \rangle \\ &= \lambda \mathcal{J}((\varphi^0, \varphi^1)) + (1 - \lambda) \mathcal{J}((\psi^0, \psi^1)) \\ &- \frac{\lambda(1 - \lambda)}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} - \frac{\partial \psi}{\partial n} \right|^2 d\sigma dt. \end{aligned}$$

Using the observability inequation, we obtain,

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial}{\partial n}(\varphi - \psi) \right|^2 d\sigma dt \geq C \|(\varphi^0 - \psi^0, \varphi^1 - \psi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2,$$

then if $(\varphi_0, \varphi_1) \neq (\psi_0, \psi_1)$, then,

$$\mathcal{J}(\lambda(\varphi^0, \varphi^1) + (1 - \lambda)(\psi^0, \psi^1)) < \lambda \mathcal{J}((\varphi^0, \varphi^1)) + (1 - \lambda) \mathcal{J}((\psi^0, \psi^1)).$$

Hence \mathcal{J} is strictly convex.

For the coercivity of the the functional \mathcal{J} , we have,

$$\begin{aligned}
\mathcal{J}((\varphi^0, \varphi^1)) &\geq \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt - \|(\varphi^0, \varphi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)} \|(y^0, y^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \\
&\geq \frac{C}{2} \|(\varphi^0, \varphi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 - \|(\varphi^0, \varphi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)} \|(y^0, y^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)},
\end{aligned}$$

then $\lim_{\|(\varphi^0, \varphi^1)\| \rightarrow +\infty} \mathcal{J}((\varphi^0, \varphi^1)) = \infty$.

We conclude that the functional \mathcal{J} has a unique minimizer $(\hat{\varphi}^0, \hat{\varphi}^1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, we have,

Theorem 1.3.2. *Let (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and $(\hat{\varphi}^0, \hat{\varphi}^1)$ in $H_0^1(\Omega) \times L^2(\Omega)$ be the unique minimizer of the functional \mathcal{J} , then the function \hat{v} defined on $\Gamma_1 \times (0, T)$ by :*

$$\hat{v}(x, t) = \frac{\partial \hat{\varphi}}{\partial n}(x, t), \quad (x, t) \in \Gamma_1 \times (0, T),$$

is a control which leads (y^0, y^1) to zero in time $T > 0$.

Proof:

The functional \mathcal{J} achieves its minimum at $(\hat{\varphi}^0, \hat{\varphi}^1)$, then

$$\lim_{h \rightarrow 0} \frac{1}{h} [\mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1))] = 0, \quad (1.9)$$

for all (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (1.4) with initial data (φ^0, φ^1) . On the other hand, we have,

$$\begin{aligned}
\mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1)) &= \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \hat{\varphi}}{\partial n} + h \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt + \langle (\hat{\varphi}^0, \hat{\varphi}^1) + h(\varphi^0, \varphi^1), (y^0, y^1) \rangle \\
&\quad - \frac{1}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \hat{\varphi}}{\partial n} \right|^2 d\sigma dt - \langle (\hat{\varphi}^0, \hat{\varphi}^1), (y^0, y^1) \rangle,
\end{aligned}$$

hence,

$$\begin{aligned}
\frac{1}{h} [\mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}((\hat{\varphi}^0, \hat{\varphi}^1))] &= \frac{h}{2} \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt + \int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma dt \\
&\quad + \langle (\varphi^0, \varphi^1), (y^0, y^1) \rangle,
\end{aligned}$$

and from (1.8) we deduce that

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma dt &= - \langle (\varphi^0, \varphi^1), (y^0, y^1) \rangle \\
&= - \int_{\Omega} y^0 \varphi^1 dx + \langle y^1, \varphi^0 \rangle_{-1,1},
\end{aligned}$$

for every (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$.

From lemma 1.3.2, it follows that $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ is the control for (1.1). This complete the proof.

Now we can find the control of the wave equation by minimization of the functional \mathcal{J} , moreover, this control is the control of minimal L^2 -norm:

Proposition 1.3.2. *Let $\hat{v} = \frac{\partial \hat{\varphi}}{\partial n}|_{\Gamma_1}$ be the control given by minimizing the functional \mathcal{J} . If $w \in L^2(\Gamma_1 \times (0, T))$ is any other control for (1.1), then*

$$\|\hat{v}\|_{L^2(\Gamma_1 \times (0, T))} \leq \|w\|_{L^2(\Gamma_1 \times (0, T))}. \quad (1.10)$$

Proof:

Let $(\hat{\varphi}^0, \hat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ the minimizer of the functional \mathcal{J} and w is a control function of (1.1). By taking $(\hat{\varphi}^0, \hat{\varphi}^1)$ as initial data for (1.4), lemma 1.3.2 gives,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} \frac{\partial \hat{\varphi}}{\partial n} d\sigma dt = - \int_{\Omega} y^0 \hat{\varphi}^1 dx + \langle y^1, \hat{\varphi}^0 \rangle_{-1,1},$$

hence,

$$\|\hat{v}\|_{L^2(\Gamma_1 \times (0, T))}^2 = - \int_{\Omega} y^0 \hat{\varphi}^1 dx + \langle y^1, \hat{\varphi}^0 \rangle_{-1,1}.$$

On the other hand,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \hat{\varphi}}{\partial n} w d\sigma dt = - \int_{\Omega} y^0 \hat{\varphi}^1 dx + \langle y^1, \hat{\varphi}^0 \rangle_{-1,1},$$

then,

$$\begin{aligned} \|\hat{v}\|_{L^2(\Gamma_1 \times (0, T))}^2 &= \int_0^T \int_{\Gamma_1} \hat{v} w d\sigma dt \\ &\leq \|\hat{v}\|_{L^2(\Gamma_1 \times (0, T))} \|w\|_{L^2(\Gamma_1 \times (0, T))}. \end{aligned}$$

Consequently, (1.10) is verified and the proof finishes.

Chapter 2

Conjugate Gradient Algorithm

2.1 Variational Problem

In the previous chapter, we transferred the problem of controllability of (1.1) to a problem of minimization:

$$\min_{(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)} \mathcal{J}((\varphi^0, \varphi^1)), \quad (2.1)$$

where the functional \mathcal{J} is defined by (1.7).

Problem (2.1) can be written as follows:

$$\min_{(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)} \left(\frac{1}{2} a((\varphi^0, \varphi^1), (\varphi^0, \varphi^1)) - L((\varphi^0, \varphi^1)) \right), \quad (2.2)$$

where a is defined on $(H_0^1(\Omega) \times L^2(\Omega)) \times (H_0^1(\Omega) \times L^2(\Omega))$ by

$$a((\varphi^0, \varphi^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1)) = \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} \frac{\partial \widetilde{\varphi}}{\partial n} d\sigma dt, \quad \forall (\varphi^0, \varphi^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega),$$

such that $\varphi, \widetilde{\varphi}$ respectively the solutions of (1.4) with initial data (φ^0, φ^1) and $(\widetilde{\varphi}^0, \widetilde{\varphi}^1)$, and L is defined on $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\begin{aligned} L((\varphi^0, \varphi^1)) &= - \langle (\varphi^0, \varphi^1), (y^0, y^1) \rangle \\ &= - \int_{\Omega} y^0 \varphi^1 + \langle y^1, \varphi^0 \rangle_{-1,1}, \end{aligned}$$

for all (φ^0, φ^1) in $H_0^1(\Omega) \times L^2(\Omega)$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$. we have,

Lemma 2.1.1. ♦ *The operator a is a bilinear form, continuous and $H_0^1(\Omega) \times L^2(\Omega)$ -elliptic. ♦ For all $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we have $L \in (H_0^1(\Omega) \times L^2(\Omega))'$.*

Proof:

- it easy to proof that L is linear continuous and a is bilinear
- The continuity of a follows from the inequality (1.5) and the coercivity follows from the inequality of observability (1.8).

Moreover the bilinear form a is symmetric, by follows and with the theorem of Lax-Milgram

the problem (2.2) reads as follows:

$$\begin{cases} \text{Find } (\widehat{\varphi^0}, \widehat{\varphi^1}) \in H_0^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a((\widehat{\varphi^0}, \widehat{\varphi^1}), (\varphi^0, \varphi^1)) = L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega). \end{cases} \quad (2.3)$$

2.2 General conjugate gradient algorithm

Let H a Hilbert space, a a continuous, symmetric and coercive bilinear form on $H \times H$, and L a continuous linear form on H , the variational problem problem (2.3) is a particular case of the following general variational problem:

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u, v) = L(v), \quad \forall v \in H. \end{cases} \quad (2.4)$$

With the above hypotheses, problem (2.4) has a unique solution.

For the numerical solution of (2.4) we need the following algorithm, say the conjugate gradient algorithm:

(1) $u^{(0)}$: any arbitrarily vector in H ;

(2) solve

$$\begin{cases} \tilde{u}^{(0)} \in H \\ \langle \tilde{u}^{(0)}, v \rangle_H = a(u^{(0)}, v) - L(v), \quad \forall v \in H. \end{cases} \quad (2.5)$$

(3) • If $\tilde{u}^{(0)}$ is small ($\frac{\|\tilde{u}^{(0)}\|}{\|u^{(0)}\|} < \epsilon$), take $u = u^{(0)}$;
 • If not, set $\check{u}^{(0)} = \tilde{u}^{(0)}$;

Assuming that $u^{(n)}$, $\tilde{u}^{(n)}$, $\check{u}^{(n)}$ are known, compute $u^{(n+1)}$, $\tilde{u}^{(n+1)}$, $\check{u}^{(n+1)}$:

(4) $\rho_n = \frac{\|\tilde{u}^{(n)}\|^2}{a(\check{u}^{(n)}, \check{u}^{(n)})}$;

(5) $u^{(n+1)} = u^{(n)} - \rho_n \check{u}^{(n)}$;

(6) solve

$$\begin{cases} \tilde{u}^{(n+1)} \in H \\ \langle \tilde{u}^{(n+1)}, v \rangle_H = \langle \tilde{u}^{(n)}, v \rangle_H - \rho_n a(\check{u}^{(n)}, v), \quad \forall v \in H. \end{cases} \quad (2.6)$$

(7) • If $\tilde{u}^{(n+1)}$ is small ($\frac{\|\tilde{u}^{(n+1)}\|}{\|\tilde{u}^{(0)}\|} < \epsilon$), take $u = u^{(n+1)}$;

• If not,

★ $\gamma_n = \frac{\|\tilde{u}^{(n+1)}\|^2}{\|\tilde{u}^{(n)}\|^2}$;

★ $\check{u}^{(n+1)} = \tilde{u}^{(n+1)} + \gamma_n \check{u}^{(n)}$;

(8) $n = n + 1$ and go to (4);

2.3 Application of conjugate gradient algorithm

Let us now apply the general conjugate gradient algorithm to the solution of (2.3).

(1) $(\varphi_0^0, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$: Initialization;

(2) solve

$$\begin{cases} (\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1) \in H \\ \langle (\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1), (\varphi^0, \varphi^1) \rangle_H = a((\varphi_0^0, \varphi_0^1), (\varphi^0, \varphi^1)) - L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H. \end{cases} \quad (2.7)$$

Consider the following non-homogeneous backward wave equation:

$$\begin{cases} \frac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 = 0 & \text{in } \Omega \times (0, T), \\ \psi_0 = \frac{\partial \varphi_0}{\partial n} & \text{in } \Gamma_1 \times (0, T), \\ \psi_0 = 0 & \text{in } \Gamma_2 \times (0, T), \\ \psi_0(., T) = 0, \quad \frac{\partial \psi_0}{\partial t}(., T) = 0 & \text{in } \Omega. \end{cases}$$

From the lemma 1.3.2, we have,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \varphi_0}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma dt = \langle \psi_0'(0), \varphi^0 \rangle_{-1,1} - \int_{\Omega} \psi_0(0) \varphi^1 dx,$$

for every $(\varphi^0, \varphi^1) \in H$, then we obtain,

$$\begin{aligned} \int_{\Omega} \nabla \widetilde{\varphi}_0^0 \nabla \varphi^0 dx + \int_{\Omega} \widetilde{\varphi}_0^1 \varphi^1 dx &= \langle \psi_0'(0), \varphi^0 \rangle_{-1,1} - \int_{\Omega} \psi_0(0) \varphi^1 dx \\ &+ \int_{\Omega} y^0 \varphi^1 dx - \langle y^1, \varphi^0 \rangle_{-1,1}. \end{aligned}$$

Hence,

$$\langle -\Delta \widetilde{\varphi}_0^0, \varphi^0 \rangle_{-1,1} - \langle \psi_0'(0) - y^1, \varphi^0 \rangle_{-1,1} = \int_{\Omega} (y^0 - \psi_0(0) - \widetilde{\varphi}_0^1) \varphi^1 dx.$$

Finally,

$$\langle (\varphi^0, \varphi^1), (y^0 - \psi_0(0) - \widetilde{\varphi}_0^1, -\Delta \widetilde{\varphi}_0^0 - (\psi_0'(0) - y^1)) \rangle = 0.$$

Its follows: (2)

$$\begin{cases} -\Delta \widetilde{\varphi}_0^0 = \psi_0'(0) - y^1, \\ \widetilde{\varphi}_0^1 = y^0 - \psi_0(0). \end{cases} \quad (2.8)$$

(3) • If $(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)$ is small, take $(\widehat{\varphi}^0, \widehat{\varphi}^1) = (\varphi_0^0, \varphi_0^1)$;

• If not, set $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)$.

Assuming that $(\varphi_n^0, \varphi_n^1)$, $(\widetilde{\varphi}_n^0, \widetilde{\varphi}_n^1)$, $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ and φ_n , ψ_n are known, compute $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, $(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)$, $(\check{\varphi}_{n+1}^0, \check{\varphi}_{n+1}^1)$, φ_{n+1} , ψ_{n+1} .

we knew that the form linear $(\varphi^0, \varphi^1) \in H \mapsto a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1))$ is continuous, then by Riesz's theorem there exists unique $(\underline{\varphi}_n^0, \underline{\varphi}_n^1)$ in H , such that

$$a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1)) = \langle (\underline{\varphi}_n^0, \underline{\varphi}_n^1), (\varphi^0, \varphi^1) \rangle, \quad \forall (\varphi^0, \varphi^1) \in H.$$

Like the previous case, we can find $(\underline{\varphi}_n^0, \underline{\varphi}_n^1)$ by :

(4) solve

$$\begin{cases} \frac{\partial^2 \check{\varphi}_n}{\partial t^2} - \Delta \check{\varphi}_n = 0 & \text{in } \Omega \times (0, T), \\ \check{\varphi}_n = 0 & \text{in } \Gamma \times (0, T), \\ \check{\varphi}_n(\cdot, 0) = \check{\varphi}_n^0, \quad \frac{\partial \check{\varphi}_n}{\partial t}(\cdot, 0) = \check{\varphi}_n^1 & \text{in } \Omega, \end{cases}$$

and then

$$\begin{cases} \frac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n = 0 & \text{in } \Omega \times (0, T), \\ \check{\psi}_n = \frac{\partial \check{\varphi}_n}{\partial n} & \text{in } \Gamma_1 \times (0, T), \\ \check{\psi}_n = 0 & \text{in } \Gamma_2 \times (0, T), \\ \check{\psi}_n(\cdot, T) = 0, \quad \frac{\partial \check{\psi}_n}{\partial t}(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

Compute now $(\underline{\varphi}_n^0, \underline{\varphi}_n^1) \in H$ by :

$$\begin{cases} -\Delta \underline{\varphi}_n^0 = \check{\psi}_n'(0) & \text{in } \Omega \\ \underline{\varphi}_n^1 = -\check{\psi}_n(0). \end{cases} \quad (2.9)$$

The other steps of the general algorithm are easy to adapt. Now we give the complete algorithm to solve the system (2.3):

(1) $(\varphi_0^1, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$ **are given;**

(2) **solve then**

$$\begin{cases} \frac{\partial^2 \varphi_0}{\partial t^2} - \Delta \varphi_0 = 0 & \text{in } \Omega \times (0, T), \\ \varphi_0 = 0 & \text{in } \Gamma \times (0, T), \\ \varphi_0(\cdot, 0) = \varphi_0^0, \quad \frac{\partial \varphi_0}{\partial t}(\cdot, 0) = \varphi_0^1 & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} \frac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 = 0 & \text{in } \Omega \times (0, T), \\ \psi_0 = \frac{\partial \varphi_0}{\partial n} & \text{in } \Gamma_1 \times (0, T), \\ \psi_0 = 0 & \text{in } \Gamma_2 \times (0, T), \\ \psi_0(\cdot, T) = 0, \quad \frac{\partial \psi_0}{\partial t}(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

(3) **Compute** $(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1) \in H$ **by**

$$\begin{cases} -\Delta \widetilde{\varphi}_0^0 = \psi_0'(0) - y^1, \\ \widetilde{\varphi}_0^0 = 0 \quad \text{in } \Gamma, \end{cases}$$

and

$$\widetilde{\varphi}_0^1 = y^0 - \psi_0(0).$$

(4) • **If** $(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)$ **is small** , **take** $(\widehat{\varphi}^0, \widehat{\varphi}^1) = (\varphi_0^0, \varphi_0^1)$;

• **If not**, **set** $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)$.

Assuming that $(\varphi_n^0, \varphi_n^1)$, $(\widetilde{\varphi}_n^0, \widetilde{\varphi}_n^1)$, $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ **and** φ_n , ψ_n **are known, compute** $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, $(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)$, $(\check{\varphi}_{n+1}^0, \check{\varphi}_{n+1}^1)$, φ_{n+1} , ψ_{n+1} .

Descent:

(5) **Solve**

$$\begin{cases} \frac{\partial^2 \check{\varphi}_n}{\partial t^2} - \Delta \check{\varphi}_n = 0 & \text{in } \Omega \times (0, T), \\ \check{\varphi}_n = 0 & \text{in } \Gamma \times (0, T), \\ \check{\varphi}_n(\cdot, 0) = \check{\varphi}_n^0, \quad \frac{\partial \check{\varphi}_n}{\partial t}(\cdot, 0) = \check{\varphi}_n^1 & \text{in } \Omega, \end{cases}$$

and then

$$\begin{cases} \frac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n = 0 & \text{in } \Omega \times (0, T), \\ \check{\psi}_n = \frac{\partial \check{\varphi}_n}{\partial n} & \text{in } \Gamma_1 \times (0, T), \\ \check{\psi}_n = 0 & \text{in } \Gamma_2 \times (0, T), \\ \check{\psi}_n(\cdot, T) = 0, \quad \frac{\partial \check{\psi}_n}{\partial t}(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

(6) **Compute** $(\underline{\varphi}_n^0, \underline{\varphi}_n^1) \in H$ **by : solve**

$$\begin{cases} -\Delta \underline{\varphi}_n^0 = \check{\psi}_n'(0) & \text{in } \Omega, \\ \underline{\varphi}_n^0 = 0 & \text{in } \Gamma, \end{cases}$$

and

$$\underline{\varphi}_n^1 = -\check{\psi}_n(0).$$

(7) **Compute** ρ_n **by:**

$$\left\{ \begin{aligned} \rho_n &= \frac{\|(\widetilde{\varphi}_n^0, \widetilde{\varphi}_n^1)\|^2}{a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\check{\varphi}_n^0, \check{\varphi}_n^1))}, \\ &= \frac{\|(\widetilde{\varphi}_n^0, \widetilde{\varphi}_n^1)\|^2}{\langle \check{\psi}_n'(0), \check{\varphi}_n^0 \rangle_{-1,1} - \int_{\Omega} \check{\psi}_n(0) \check{\varphi}_n^1 dx}, \\ &= \frac{\int_{\Omega} \|\nabla \widetilde{\varphi}_n^0\|^2 + \int_{\Omega} \|\widetilde{\varphi}_n^1\|^2}{\int_{\Omega} \nabla \underline{\varphi}_n^0 \nabla \check{\varphi}_n^0 dx + \int_{\Omega} \underline{\varphi}_n^1 \check{\varphi}_n^1 dx}. \end{aligned} \right.$$

(8) **Once ρ_n is known, compute:**

$$\oplus (\varphi_{n+1}^0, \varphi_{n+1}^1) = (\varphi_n^0, \varphi_n^1) - \rho_n(\check{\varphi}_n^0, \check{\varphi}_n^1),$$

$$\oplus \varphi_{n+1} = \varphi_n - \rho_n \check{\varphi}_n,$$

$$\oplus \psi_{n+1} = \psi_n - \rho_n \check{\psi}_n,$$

$$\oplus (\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1) = (\widetilde{\varphi}_n^0, \widetilde{\varphi}_n^1) - \rho_n(\underline{\varphi}_n^0, \underline{\varphi}_n^1).$$

Test of the convergence and construction of the new descent direction.

(9) **If $(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)$ is small , take $(\widehat{\varphi}^0, \widehat{\varphi}^1) = (\varphi_{n+1}^0, \varphi_{n+1}^1)$.**

If not, compute

$$\gamma_n = \frac{\int_{\Omega} \|\nabla \widetilde{\varphi}_{n+1}^0\|^2 dx + \int_{\Omega} \|\widetilde{\varphi}_{n+1}^1\|^2 dx}{\int_{\Omega} \|\nabla \widetilde{\varphi}_n^0\|^2 dx + \int_{\Omega} \|\widetilde{\varphi}_n^1\|^2 dx},$$

and set

$$(\check{\varphi}_{n+1}^0, \check{\varphi}_{n+1}^1) = (\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1) + \gamma_n(\check{\varphi}_n^0, \check{\varphi}_n^1).$$

(10) $n = n + 1$ **and go to (5).**

Chapter 3

Numerical approximation with the finite difference methods

In this chapter we will apply the finite difference methods to control the system (1.1). At the present moment, we shall focus on the case where:

$$\Omega = (0, 1), \quad \Gamma_1 = \{1\} \quad \text{and} \quad \Gamma_2 = \{0\}.$$

3.1 Finite difference approximation of the wave equation

In the algorithm at each iteration we need to solve either a homogeneous or a non-homogeneous back-ward wave equation, for this we introduce the general wave system, that it follows:

$$\left\{ \begin{array}{ll} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0 & \text{in } (0, 1) \times (0, T), \\ \varphi(0, \cdot) = 0 & \text{in } (0, T), \\ \varphi(1, \cdot) = u(\cdot) & \text{in } \times (0, T), \\ \varphi(\cdot, 0) = v(\cdot), \quad \frac{\partial \varphi}{\partial t}(\cdot, 0) = w(\cdot) & \text{in } (0, 1), \end{array} \right. \quad (3.1)$$

where $(v, w) \in H_0^1(0, 1) \times L^2(0, 1)$.

We propose to approach the solution of the system (3.1) by an explicit scheme of finite difference, for this purpose, we have considered a uniform mesh defined by:

$$t_0 = 0 < t_1 = k < \dots < t_n = nk < \dots < t_{M+1} = T,$$

and

$$x_0 = 0 < x_1 = h < \dots < x_j = jh < \dots < x_{N+1} = 1,$$

with $k = \frac{T}{M+1}$ is the time step and $h = \frac{1}{N+1}$ is the space step.

Let φ_j^n be the approximation of $\varphi(x_j, t_n)$, using a central finite differences to approximate the

second-order derivative, we obtain :

$$\left\{ \begin{array}{l} \frac{\varphi_j^{n+1} - 2\varphi_j^n + \varphi_j^{n-1}}{k^2} = \frac{\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n}{h^2}, \quad 1 \leq j \leq N, \quad 1 \leq n \leq M, \\ \varphi_0^n = 0, \\ \varphi_{N+1}^n = u(t_n) = u^n, \\ \varphi_j^0 = v(x_j) = v_j, \\ \varphi_j^1 = \varphi_j^0 + kw_j. \end{array} \right. \quad (3.2)$$

We can prove that the scheme(3.2) is stable if and only if $r = \frac{k^2}{h^2} \leq 1$. From (3.2), we have :

$$\varphi^{n+1} = A \varphi^n - \varphi^{n-1} + b,$$

where

$$A = \begin{pmatrix} 2(1-r) & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & r & 2(1-r) \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ \vdots \\ ru^n \end{pmatrix}$$

and φ^n the vector of \mathbb{R}^N formed by φ_j^n , $j \in \{1, \dots, N\}$.

With this discretization we can define a function (in python) which makes it possible to solve the two types of system, for the homogeneous wave equation we take $u \equiv 0$ but for the non-homogeneous back-ward system we have to make a transformation in time.

3.2 Finite difference approximation of the Dirichlet problem

As in the previous section, we need to solve a Dirichlet system on each iteration. We define the general system as follows:

$$\begin{cases} -\frac{\partial^2 \phi}{\partial x^2} = \psi'(0) - y & \text{in } (0, 1), \\ \phi(0) = 0, \\ \phi(1) = 0. \end{cases} \quad (3.3)$$

We approximate the second-order derivative by:

$$\frac{\partial^2 \phi}{\partial x^2}(x_j) \simeq \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2}, \quad 1 \leq j \leq N,$$

and for the second member, we have

$$\psi'(0) = \frac{\psi_j^1 - \psi_j^0}{k}, \quad 0 \leq j \leq N+1.$$

Then the system (3.3) is approximated as follows

$$\begin{cases} -\frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} = \frac{\psi_j^1 - \psi_j^0}{k} - y_j & 1 \leq j \leq N, \\ \phi(0) = 0, \\ \phi(1) = 0. \end{cases}$$

From this scheme, we obtain

$$B\phi = -\frac{h^2}{k}(\psi^1 - \psi^0) + h^2 Y,$$

with ϕ the vector formed by ϕ_j , $j \in \{1, \dots, N\}$ and

$$B = \begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix},$$

$$\psi^k = \begin{pmatrix} \psi_1^k \\ \vdots \\ \psi_N^k \end{pmatrix}; k \in \{0, 1\}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

3.3 Numerical Tests

In this section we present some numerical simulations to control our system, using the general conjugate gradient algorithm and the finite difference discretization method.

For the numerical tests we consider 3 types of initial conditions, and we will discuss the

convergence of the algorithm as well as the controllability of our discrete system.

The conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$, and

$$\frac{\|(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)\|_{H_0^1 \times L^2}}{\|(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)\|_{H_0^1 \times L^2}} < \epsilon,$$

has been used as stopping criterium, with ϵ is a small positive number.

Let's mention also that $\|\varphi\|_{H^{-1}(0,1)} = \|\psi\|_{H_0^1(0,1)}$, where $\psi \in H_0^1(0,1)$ is the solution of the Dirichlet problem:

$$\begin{cases} -\Delta\psi = \varphi & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

and $\|\psi\|_{H_0^1(0,1)} = \left(\int_0^1 \left(\frac{\partial\psi}{\partial x} \right)^2 dx \right)^{\frac{1}{2}}$. For the approximation of the integrals we used the rectangles method and for the stability of the numerical scheme, we take $r = 1$ (The CFL condition).

Firstly, we consider the following initial conditions associated to the wave equation (1.1):

$$\begin{cases} y^0(x) = \exp(-5(x - 0.37)^6), \\ y^1(x) = 0. \end{cases} \quad (3.4)$$

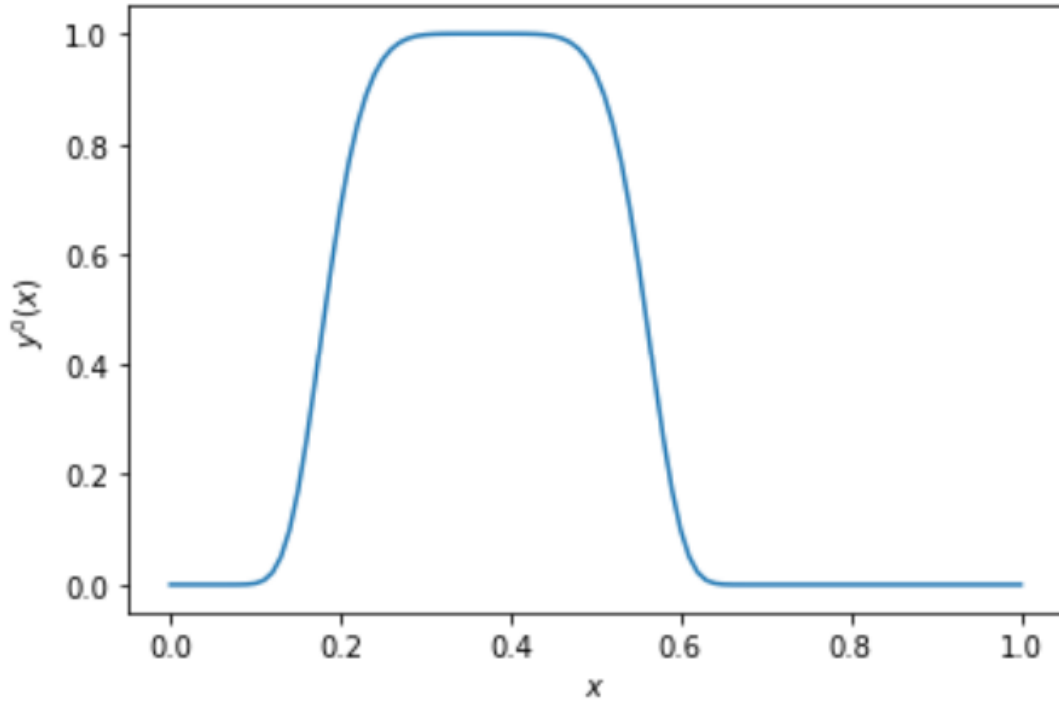


Figure 3.1. The initial position y^0 .

N	19	99	499	999
Number of CG iterations	1600	27	7	6
$\ y(T)\ _{L^2(0,1)}$	9.2857×10^{-11}	1.04928×10^{-10}	1.30932×10^{-11}	3.06843×10^{-12}
$\ y'(T)\ _{H^{-1}(0,1)}$	5.1037×10^{-11}	4.05311×10^{-11}	1.16793×10^{-11}	2.94205×10^{-12}
$\ \hat{v}\ _{L^2(0,T)}$	0.41628	0.40681	0.40658	0.40657

Table 3.1. Numerical results obtained for different values of $h = \frac{1}{N+1}$. $T = 2.2$ and $\epsilon = 10^{-10}$.

For further study of this exemple (3.4), we fix $N = 99$.

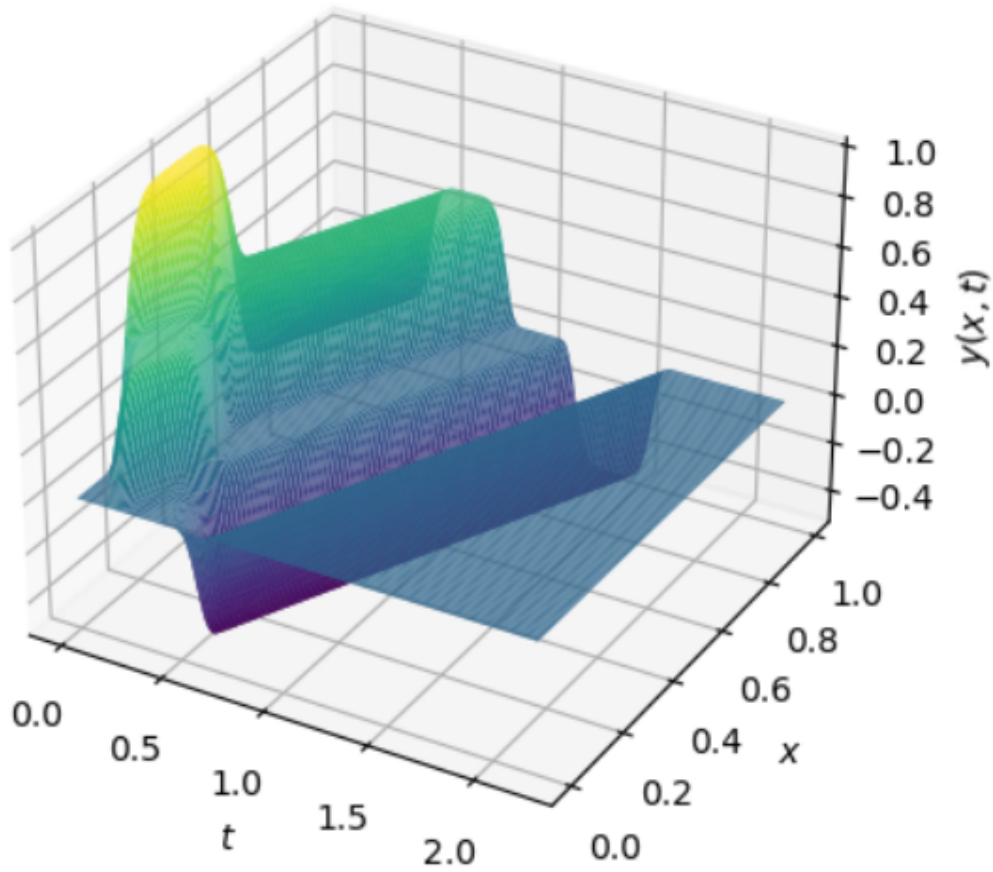


Figure 3.2. The controlled solution y .

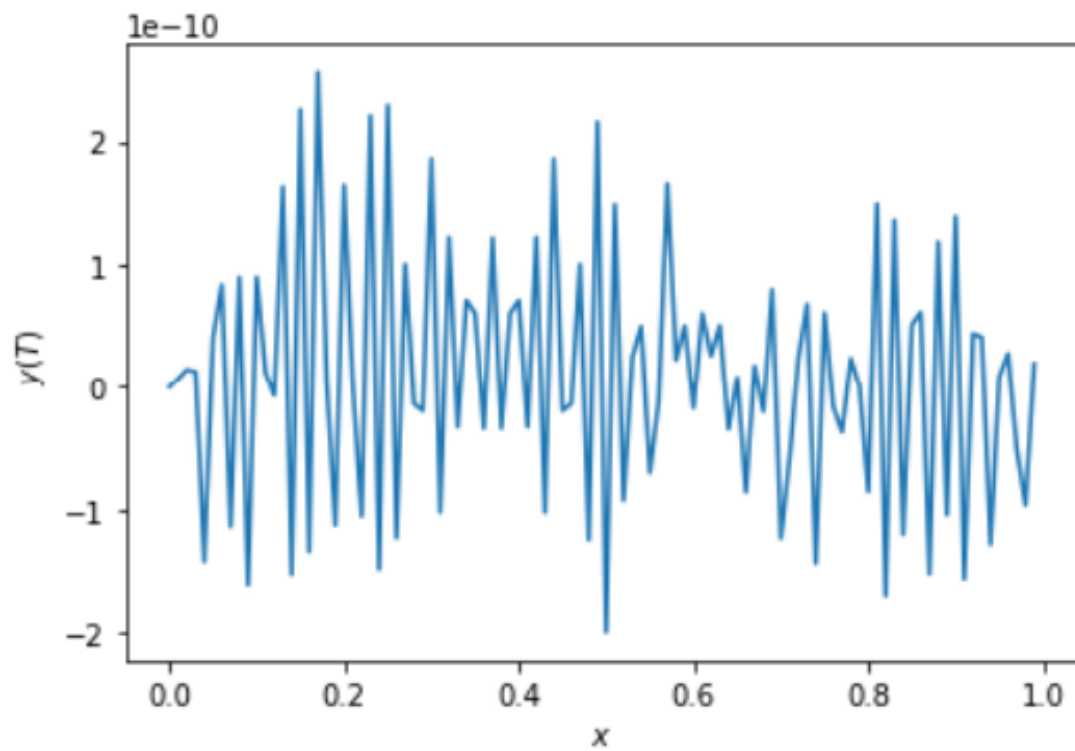


Figure 3.3. The final position $y(T)$.

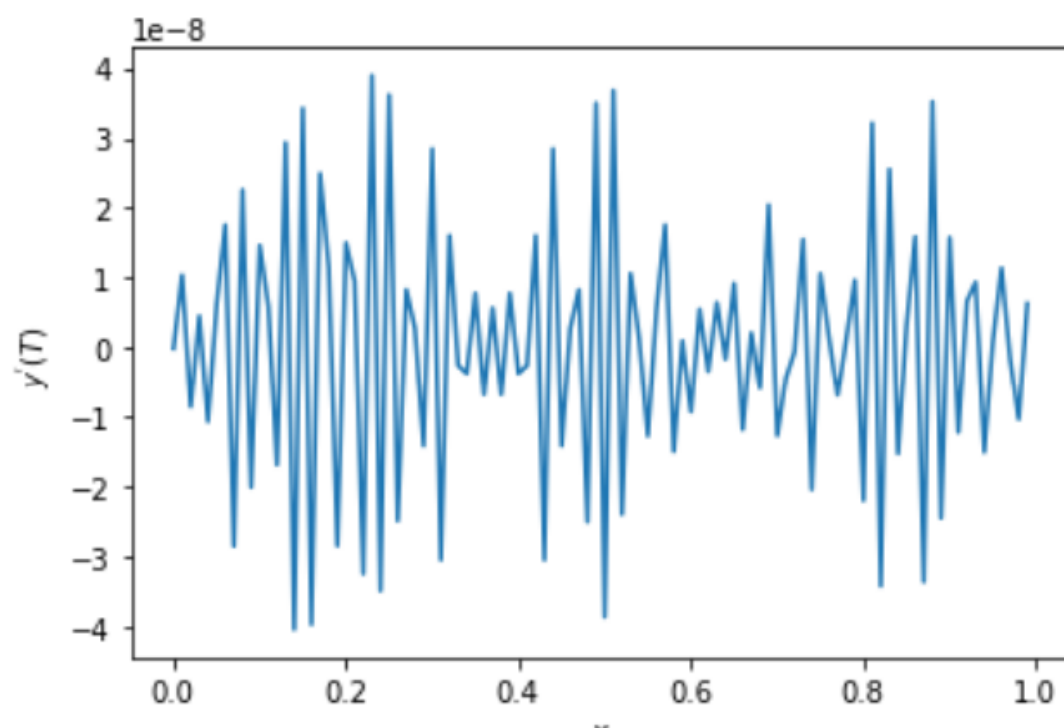


Figure 3.4. The final velocity $y'(T)$.

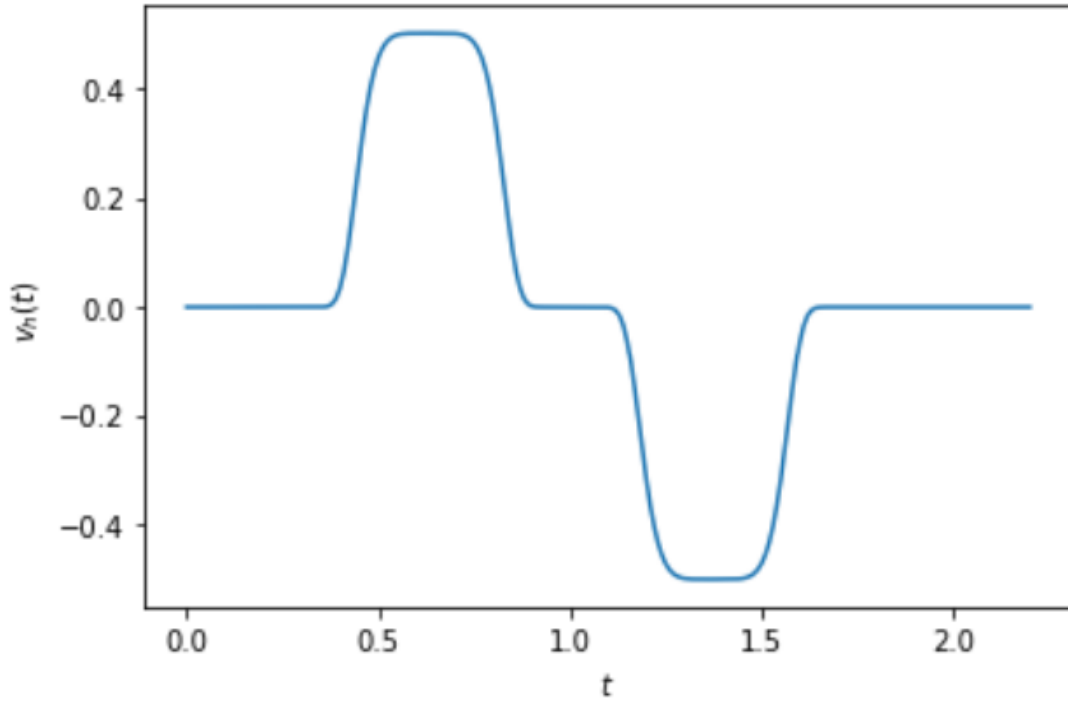


Figure 3.5. The approximation of the control.

From 3.1, we observe that when N gets bigger the number of conjugate gradient iterations decreases, and also we notice that the norm of $y(T)$ in $L^2(0, 1)$ and $\frac{\partial y}{\partial t}(T)$ in $H^{-1}(0, 1)$ converge to 0.

Figures 4.1-3.4 show the controlled solution y_h and the final position, also the final velocity for $T = 2.2$, $N = 99$ and $\epsilon = 10^{-10}$. We remark that the discrete system is controllable, in the figure 3.5 we plot the approximation of the control. Then we can conclude that for the initial conditions (3.4), the system is controllable.

Now, we consider the following initial conditions:

$$\begin{cases} y^0(x) = \sin(\pi x), \\ y^1(x) = \sqrt{2} \cos(\pi x). \end{cases} \quad (3.5)$$

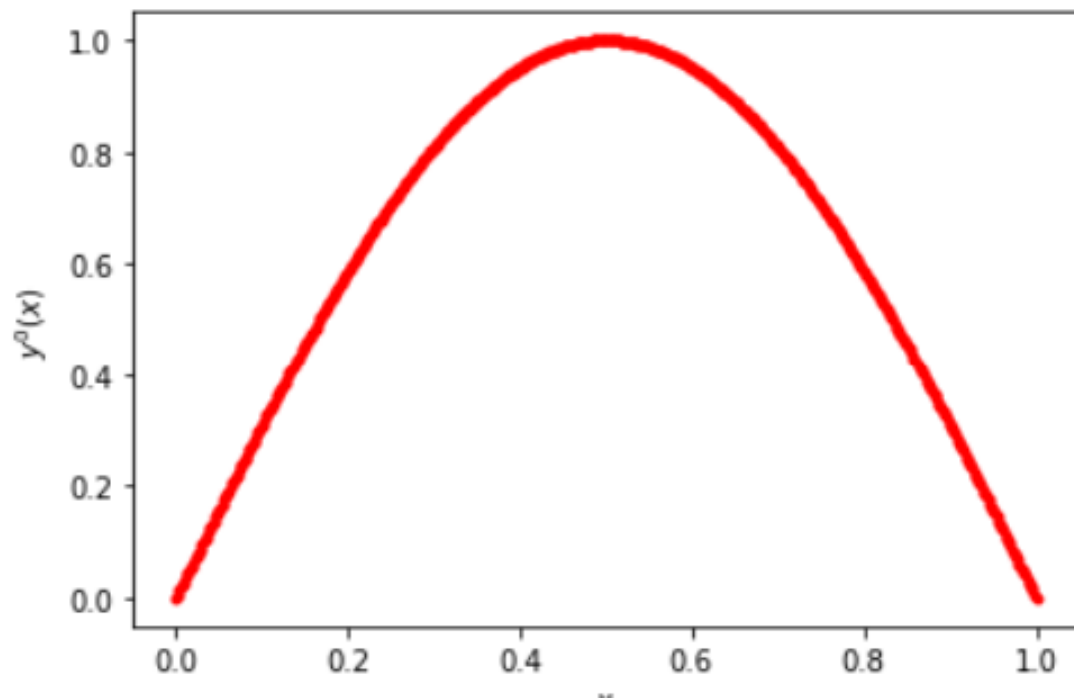


Figure 3.6. The initial position y^0 .

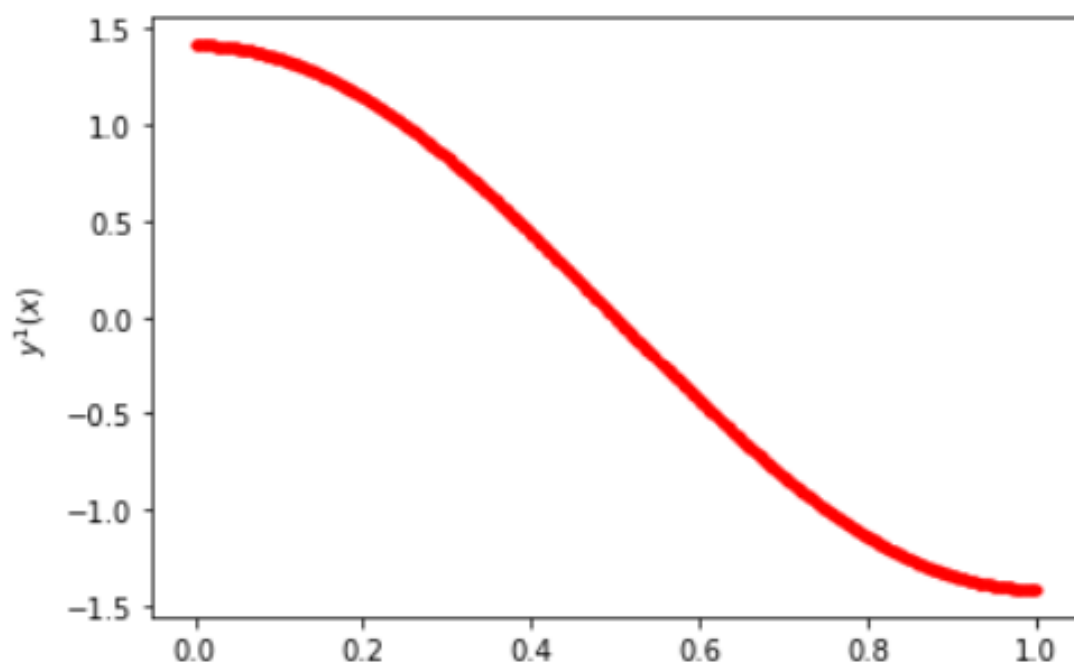


Figure 3.7. The initial velocity y^1 .

N	99	499
Number of CG iterations	2905	5003
$\ y(T)\ _{L^2(0,1)}$	$1.2493344460609922 \times 10^{-10}$	$1.041012255663964 \times 10^{-9}$
$\ y'(T)\ _{H^{-1}(0,1)}$	$6.294080645627806 \times 10^{-11}$	$5.954733460881318 \times 10^{-11}$
$\ \hat{v}\ _{L^2(0,T)}$	0.5048527172527582	0.5041223465788962

Table 3.2. Numerical results obtained for different values of $h = \frac{1}{N+1}$. $T = 2.2$ and $\epsilon = 10^{-10}$.

For $N = 99$, we obtain:

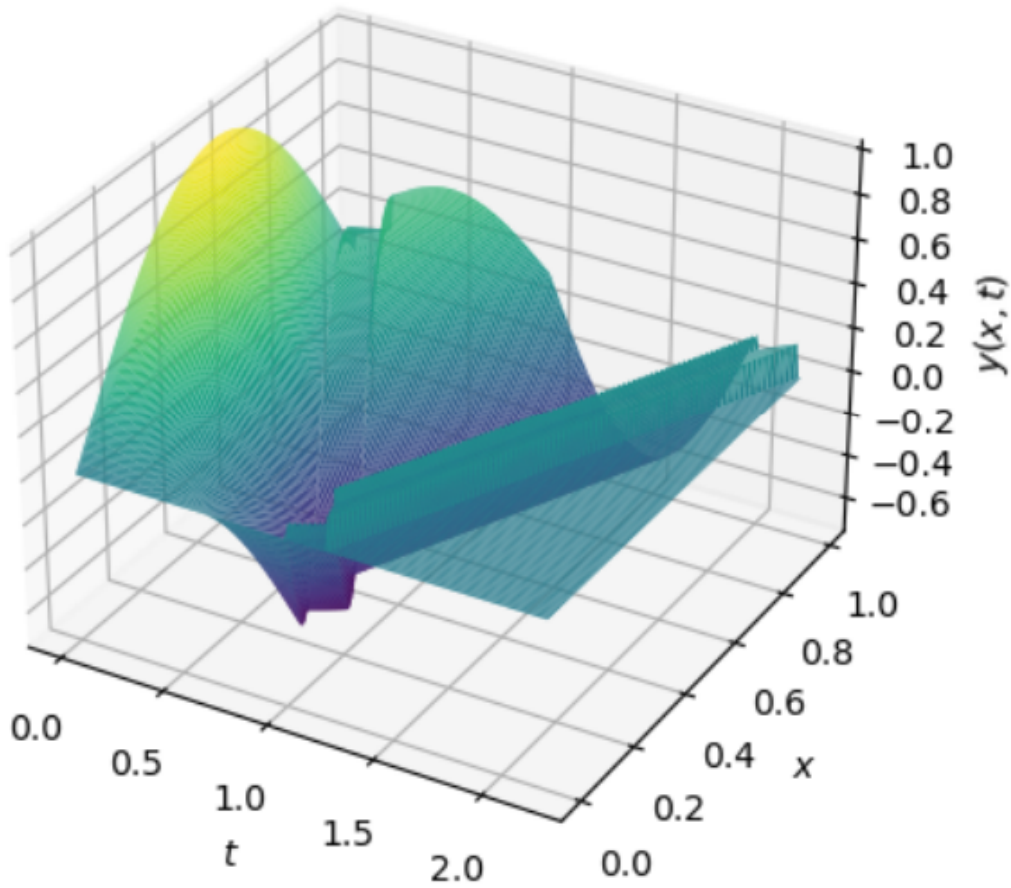


Figure 3.8. The controlled solution y .

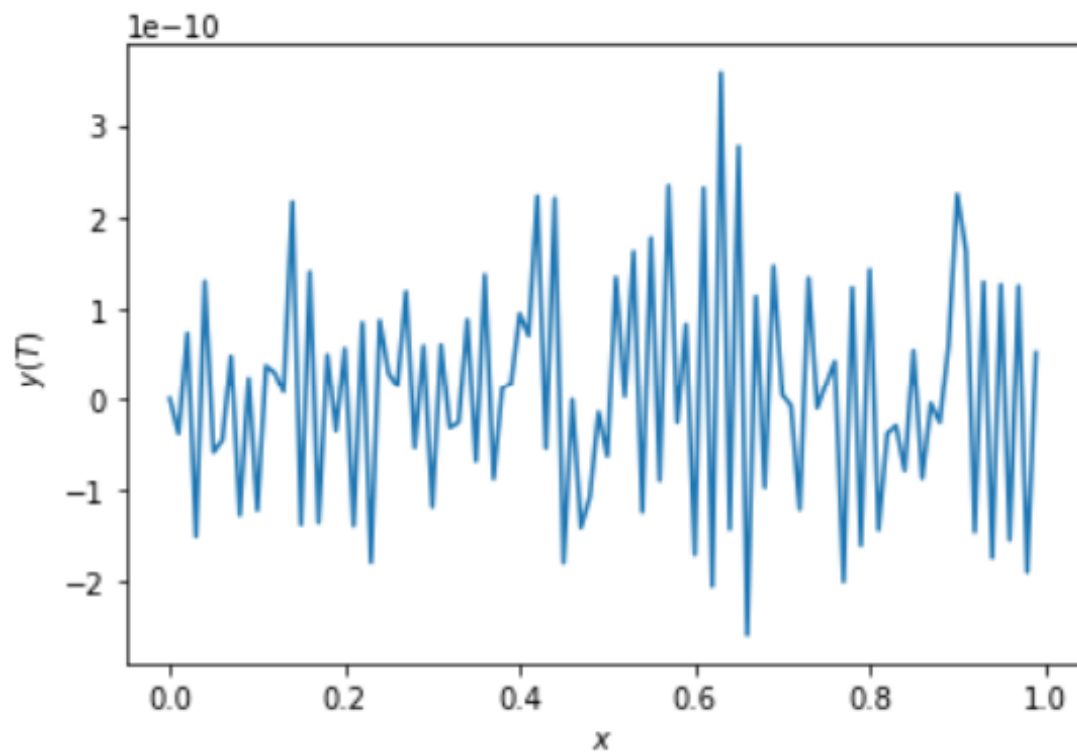


Figure 3.9. The final position $y(T)$.

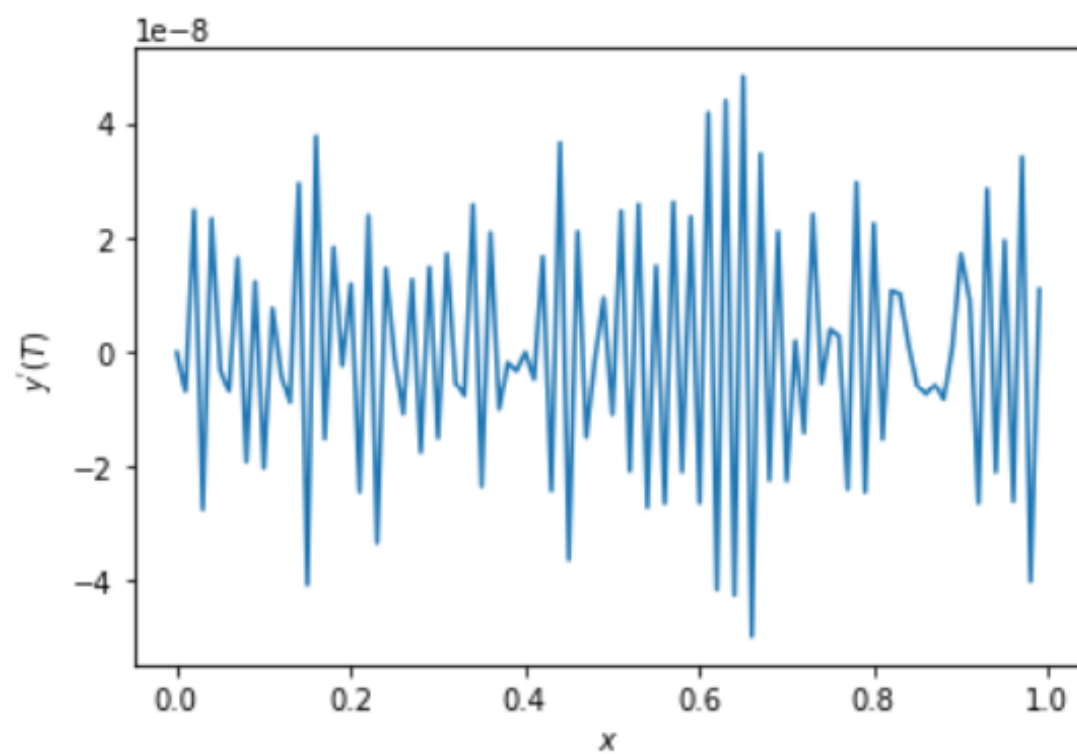


Figure 3.10. The final velocity $y'(T)$.

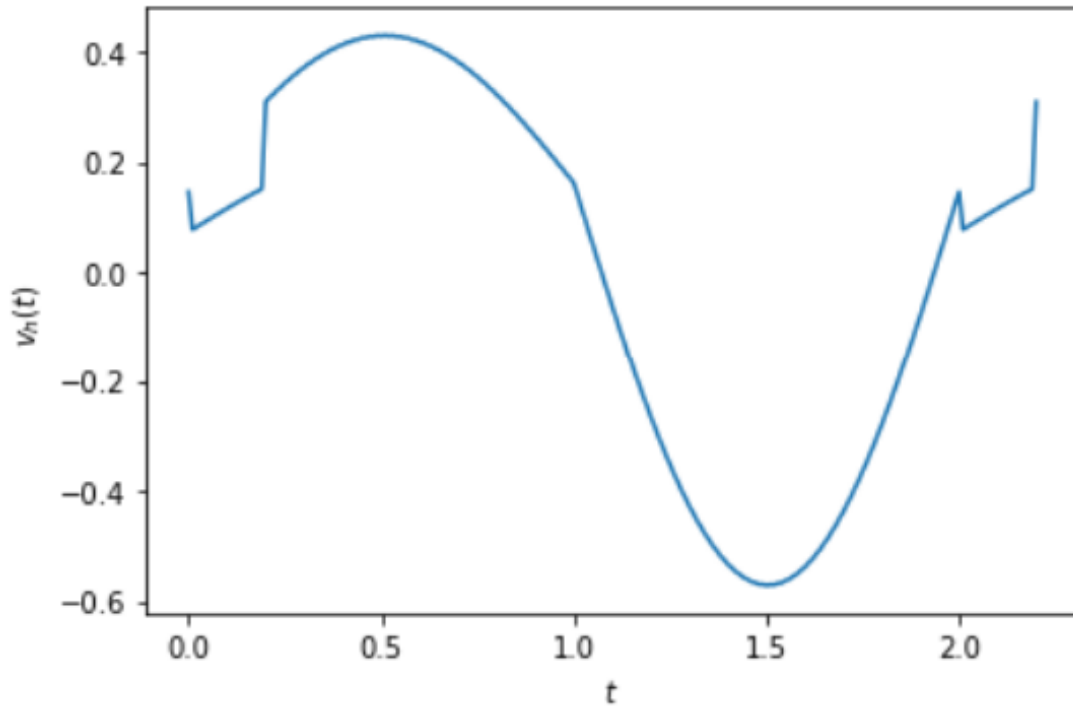


Figure 3.11. The approximation of the control.

From table 3.2 and figures 3.8-3.10, we remark that, the number of iterations for the convergence of the algorithm with the initial conditions (3.5) is greater compared to (3.4), although the system is controllable.

The numerical tests, show that, for $r < 1$ the algorithm does not converge. The problem of the number of iterations or the non convergence is because in the initial conditions there are high frequencies. To solve this problem we filter the initial data, this means that, we are approaching the initial conditions without losing the information and also filtering the high frequencies. For more details (see [..]).

In the last example (3.6), we also have the problem of high frequencies and with numerical simulations, we remark that, for $r < 1$ we can't control the system and the algorithm diverges. With $r = 1$, we have the controllability, but the number of iterations is enormous in the case where N small (≤ 100) and the convergence time with python is large in the case where N large. The problem of CPU time can be solved by using Pyccl, Numba or MPI.

finally, we conclude that for $r = 1$, we can control the $1D$ wave equation system using the finite difference method whatever the initial conditions, but the choice of $r = 1$ is not practical, so in this case we need filtration.

$$y^0(x) = \begin{cases} 0, & \text{in } \left[0, \frac{1}{2}\right], \\ 1, & \text{in } \left[\frac{1}{2}, 1\right], \end{cases} \quad (3.6)$$

$$y^1(x) = 0.$$

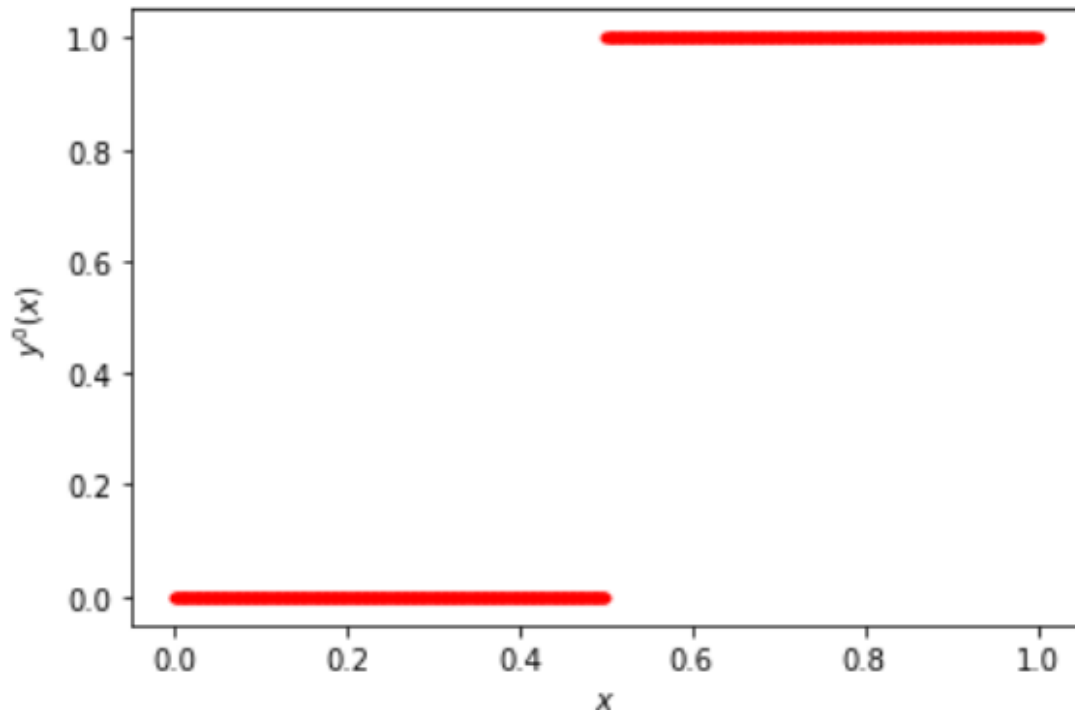


Figure 3.12. The initial position y^0 .

We fix $N = 99$, $\epsilon = 10^{-10}$, $r = 1$ and $T = 2.2$, we obtain:

The number of conjugate gradient iterations est 3662.

$$\|y(T)\|_{L^2(0,1)} = 1.87479 \times 10^{-10}.$$

$$\|y'(T)\|_{H^{-1}(0,1)} = 5.380761 \times 10^{-11}.$$

$$\|\hat{v}\|_{L^2(0,T)} = 0.58793.$$

In the following figures, we plot the controlled solution and the final behavior of our system.

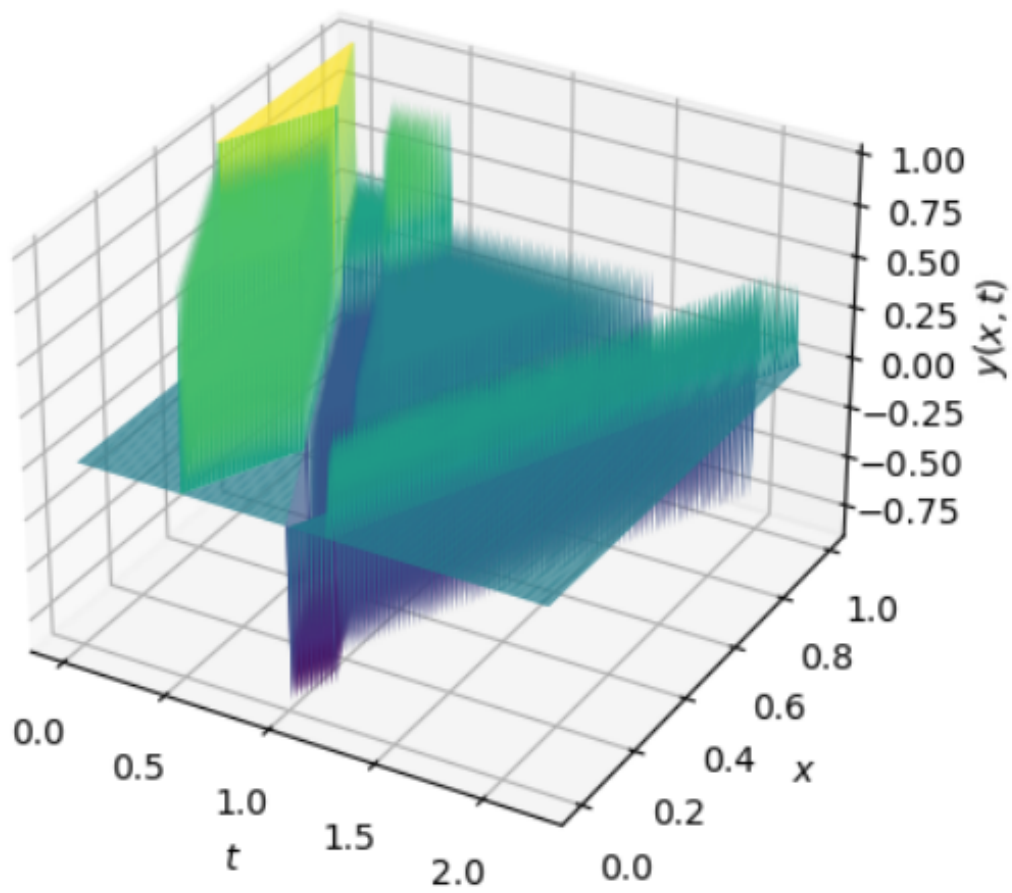


Figure 3.13. The controlled solution y .

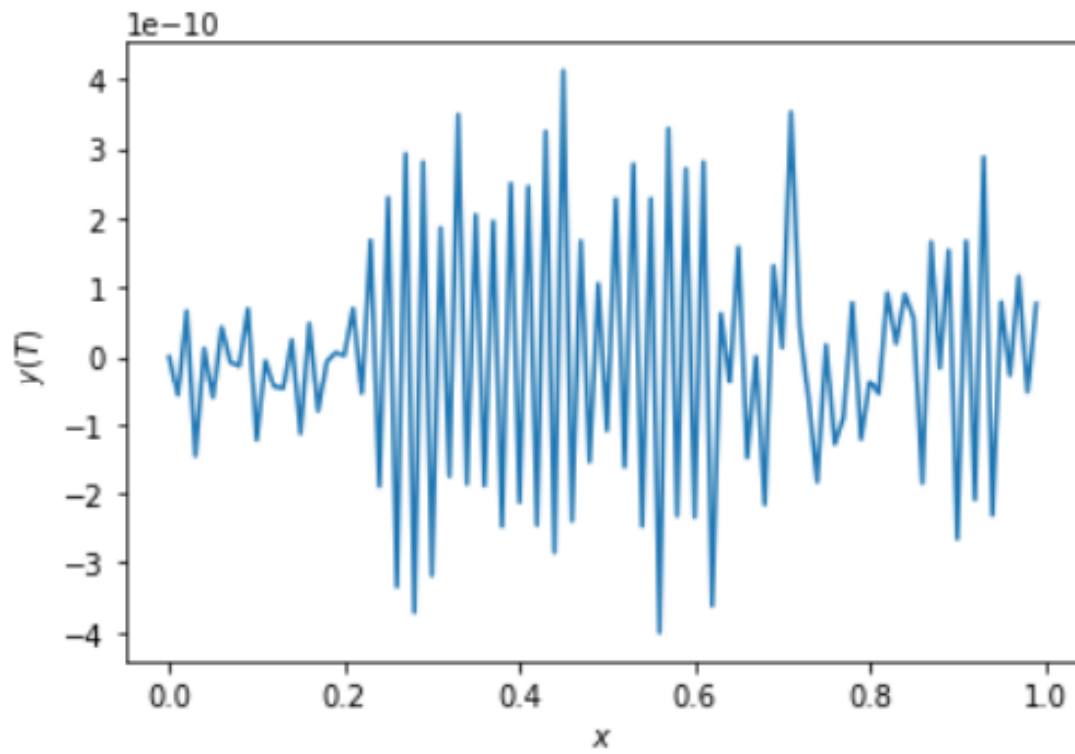


Figure 3.14. The final position $y(T)$.

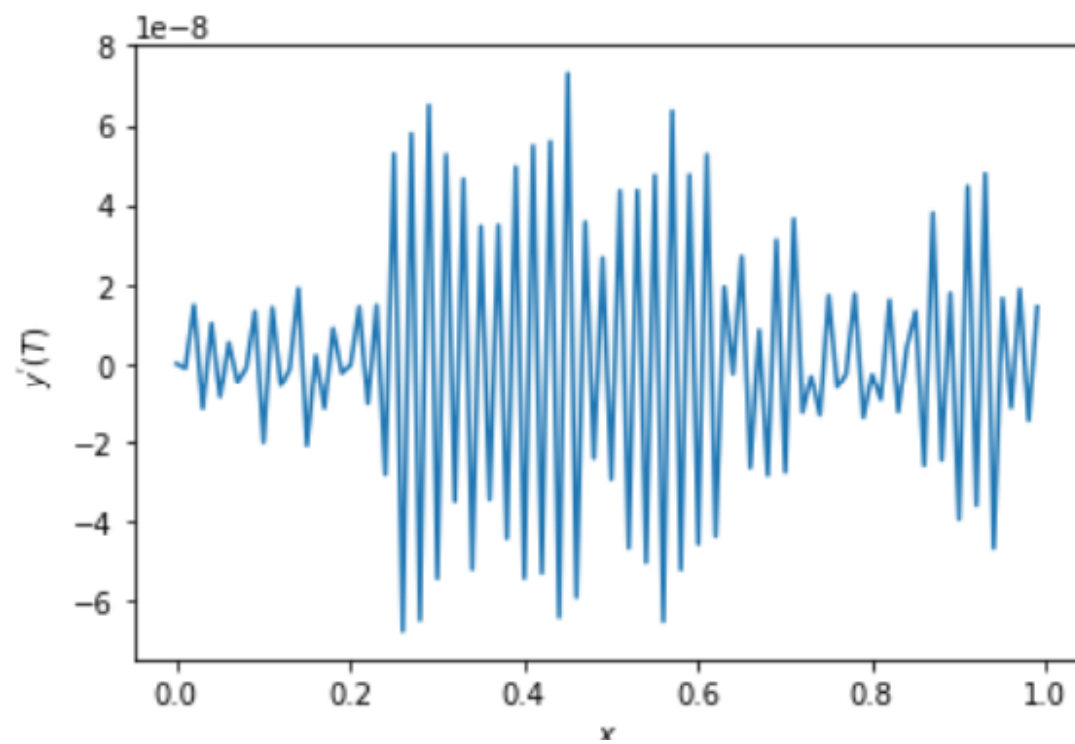


Figure 3.15. The final velocity $y'(T)$.

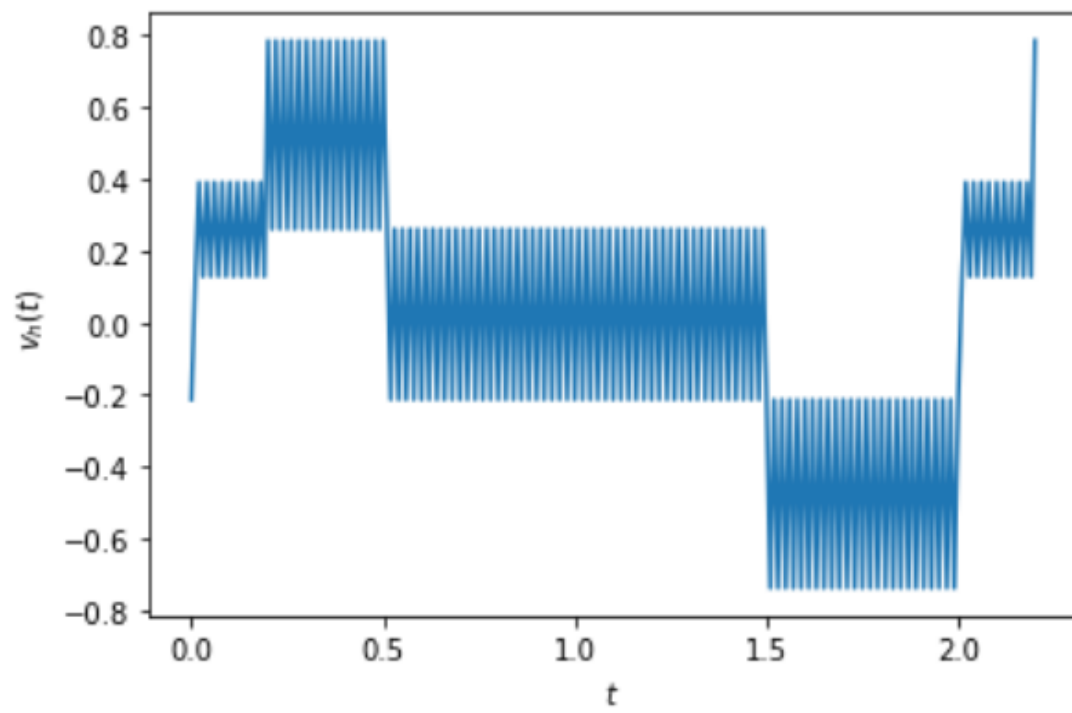


Figure 3.16. The approximation of the control.

Chapter 4

Numerical approximation with the finite difference methods (2D Case)

As the previous chapter, we will apply the finite difference methods to control the system (1.1) in 2D case.

Let

$$\Omega = (0, 1) \times (0, 1), \quad \Gamma_1 = \Gamma = \partial\Omega.$$

In this case we control the whole boundary.

Note that we can also control a part of the boundary, then it is necessary to verify the geometrical conditions.

4.1 Finite difference approximation of the 2D wave equation

Consider the hyperbolic model problem, with 2D scalar wave equation:

$$\begin{cases} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = u & \text{in } \Gamma \times (0, T), \\ \varphi(\cdot, 0) = v(\cdot), \quad \frac{\partial \varphi}{\partial t}(\cdot, 0) = w(\cdot) & \text{in } \Omega, \end{cases} \quad (4.1)$$

where $(v, w) \in H_0^1(\Omega) \times L^2(\Omega)$.

We introduce a mesh in time and in space by:

$$t_0 = 0 < t_1 = k < \dots < t_n = nk < \dots < t_{M+1} = T,$$

$$x_0 = 0 < x_1 = h_1 < \dots < x_i = ih_1 < \dots < x_{N_1+1} = 1,$$

and

$$y_0 = 0 < y_1 = h_2 < \cdots < y_j = jh_2 < \cdots < y_{N_2+1} = 1,$$

with $k = \frac{T}{M+1}$, $h_1 = \frac{1}{N_1+1}$ and $h_2 = \frac{1}{N_2+1}$.

Let $\varphi_{i,j}^n$ be the approximation of $\varphi(x_i, y_j, t_n)$, we obtain:

$$\left\{ \begin{array}{l} \frac{\varphi_{i,j}^{n+1} - 2\varphi_{i,j}^n + \varphi_{i,j}^{n-1}}{k^2} - \left(\frac{\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n}{h_1^2} + \frac{\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n}{h_2^2} \right) = 0, \\ 1 \leq i \leq N_1, 1 \leq j \leq N_2 \text{ and } 1 \leq n \leq M, \\ \varphi_{i,j}^n = u_{i,j}^n, \text{ for } i \in \{0, N_1 + 1\} \text{ or } j \in \{0, N_2 + 1\}, \\ \varphi_{i,j}^0 = v_{i,j}, \\ \varphi_{i,j}^1 = \varphi_{i,j}^0 + kw_{i,j}. \end{array} \right. \quad (4.2)$$

we define $C_1 = \frac{k}{h_1}$ and $C_2 = \frac{k}{h_2}$, we can prove that for $C = C_1 = C_2$, the explicit scheme is stable if $C \leq \frac{1}{\sqrt{2}}$, then for the numerical results we use $C_1 = C_2$ and we keep the notations C_1 and C_2 for the modelization of the numerical solution.

From (4.2), we have

$$\varphi_{i,j}^{n+1} - 2\varphi_{i,j}^n + \varphi_{i,j}^{n-1} - C_1^2(\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n) - C_2^2(\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n) = 0,$$

then,

$$\varphi_{i,j}^{n+1} = 2(1 - C_1^2 - C_2^2)\varphi_{i,j}^n - \varphi_{i,j}^{n-1} + C_1^2(\varphi_{i+1,j}^n + \varphi_{i-1,j}^n) + C_2^2(\varphi_{i,j+1}^n + \varphi_{i,j-1}^n)$$

. Let,

$$\lambda = 2(1 - C_1^2 - C_2^2),$$

$$A = \begin{pmatrix} A_1 & A_2 & & 0 \\ A_2 & \ddots & \ddots & \\ & \ddots & \ddots & A_2 \\ 0 & & A_2 & A_1 \end{pmatrix} \text{ and } \varphi^n = \begin{pmatrix} \varphi_{1,1}^n \\ \varphi_{1,2}^n \\ \vdots \\ \varphi_{1,N_2}^n \\ \varphi_{2,1}^n \\ \varphi_{2,2}^n \\ \vdots \\ \varphi_{2,N_2}^n \\ \vdots \\ \vdots \\ \varphi_{N_1,1}^n \\ \varphi_{N_1,2}^n \\ \vdots \\ \varphi_{N_1,N_2}^n \end{pmatrix}.$$

The matrix A is formed by N_1^2 blocks, each of size N_2^2 , where A_1 and A_2 defined by:

$$A_1 = \begin{pmatrix} \lambda & C_2^2 & & 0 \\ C_2^2 & \ddots & \ddots & \\ & \ddots & \ddots & C_2^2 \\ 0 & & C_2^2 & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} C_1^2 & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & C_1^2 \end{pmatrix}.$$

The system (4.2) is equivalent to:

$$\varphi^{n+1} = A\varphi^n - \varphi^{n-1} + b^n,$$

where b^n is a vector formed by the values of φ^n in the boundary:

$$b^n = \begin{pmatrix} C_1^2 \varphi_{0,1}^n + C_2^2 \varphi_{1,0}^n \\ C_1^2 \varphi_{0,2}^n \\ \vdots \\ C_1^2 \varphi_{0,N_2-1}^n \\ C_1^2 \varphi_{0,N_2}^n + C_2^2 \varphi_{1,N_2+1}^n \\ \\ C_2^2 \varphi_{2,0}^n \\ 0 \\ \vdots \\ 0 \\ C_2^2 \varphi_{2,N_2+1}^n \\ \\ \vdots \\ \vdots \\ \\ C_1^2 \varphi_{N_1+1,1}^n + C_2^2 \varphi_{N_1,0}^n \\ C_1^2 \varphi_{N_1+1,2}^n \\ \vdots \\ C_1^2 \varphi_{N_1+1,N_2}^n + C_2^2 \varphi_{N_1,N_2+1}^n \end{pmatrix}.$$

We notice that the matrix A is large if N_1 and N_2 are large, this matrix must be stored in the memory.

The other problem is that the `np.dot` function in python is numerically unstable and the CPU time to calculate the matrix-vector product is very long, but in our case, the matrix A is tridiagonal and formed by two matrices, one tridiagonal and the other is diagonal, then we can define a python function for calculate the product $A\varphi^n$ and also we can accelerate this product with Numba.

4.2 Finite difference approximation of the 2D Dirichlet problem

In the algorithm (HUM), we need to solve a Dirichlet system on each iteration.

The Dirichlet problem in the algorithm is of the form:

$$\begin{cases} -\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = \psi'(0) - y & \text{in } \Omega, \\ \varphi = 0, & \text{in } \Gamma, \end{cases} \quad (4.3)$$

with the finite difference approximation, we obtain:

$$\begin{cases} \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h_1^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{h_2^2} = y_{i,j} - \frac{\psi_{i,j}^1 - \psi_{i,j}^0}{k}, & 1 \leq i \leq N_1, \quad 1 \leq j \leq N_2, \\ \varphi_{i,j} = 0, & \text{for } i \in \{0, N_1 + 1\} \text{ or } j \in \{0, N_2 + 1\}. \end{cases} \quad (4.4)$$

As the previous section, let

$$\varphi = \begin{pmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \vdots \\ \varphi_{1,N_2} \\ \varphi_{2,1} \\ \varphi_{2,2} \\ \vdots \\ \varphi_{2,N_2} \\ \vdots \\ \vdots \\ \varphi_{N_1,1} \\ \varphi_{N_1,2} \\ \vdots \\ \varphi_{N_1,N_2} \end{pmatrix} \text{ and } Y_{i,j} = y_{i,j} - \frac{\psi_{i,j}^1 - \psi_{i,j}^0}{k}, \quad (i,j) \in \{1, \dots, N_1\} \times \{1, \dots, N_2\}.$$

From (4.4), we have:

$$(**) \quad B\varphi = Y.\text{flatten}(),$$

with B the matrix formed by N_1^2 blocks, each of size N_2^2 .

$$B = \begin{pmatrix} B_1 & B_2 & & 0 \\ B_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_2 \\ 0 & & B_2 & B_1 \end{pmatrix},$$

with B_1 is a tridiagonal matrix and B_2 is a diagonal matrix:

$$B = \begin{pmatrix} \frac{-2}{h_1^2} - \frac{2}{h_2^2} & \frac{1}{h_2^2} & & 0 \\ \frac{1}{h_2^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{h_2^2} \\ 0 & & \frac{1}{h_2^2} & \frac{-2}{h_1^2} - \frac{2}{h_2^2} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{h_1^2} & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \frac{1}{h_1^2} \end{pmatrix}.$$

Like the previous section, the matrix B is large, then for solving the system (**), we need to stored the matrix B and also the function `lg.solve` in `scipy.linalg` is numerically unstable if N_1, N_2 are large.

As the matrix B is formed by blocks, we can use the MPI for solving the system (**) and accelerate the time of resolution.

The other solution is to use the package `scipy.sparse`, in this case we have stored the matrix B in memory, then replace B with pointers in which we keep the terms no nulls (`B=coo_matrix(B)`).

4.3 Numerical Tests

For the numerical tests we fix $C = C1 = C2$, $T = 3$ (or $T > 2\sqrt{2}$) and we will discuss the convergence of the conjugate gradient algorithm.

As the previous chapter, the conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$ and for the stopping criteria we choose:

$$\frac{\|(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)\|_{H_0^1 \times L^2}}{\|(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)\|_{H_0^1 \times L^2}} < 10^{-10}.$$

The numerical tests show that if $C > \frac{1}{\sqrt{2}}$, the algorithm diverges, so we can't control the initial data using HUM, then does there exist $C \in \left[0, \frac{1}{\sqrt{2}}\right]$ so that all initial data can be controllable.

In case 1D we have a condition for C ($C = 1$), for our case there are no theoretical results, by analogy with the previous chapter we choose $C = \frac{1}{\sqrt{2}}$, then we have the stability for the wave equation, but there are initial conditions that we cannot control. For our test we will show that with $C = \frac{1}{\sqrt{2}}$, we can control a set of initial conditions.

Let $y^1 = 0$. We assume that $\exists D \in \Omega$, such as:

$$y^0 = \begin{cases} \approx 0, & \text{in } \Omega \setminus D, \\ y_D^0, & \text{in } D, \end{cases}$$

with D is a compact, convex ($\lambda(D) < \lambda(\Omega)$), we also assume that there are no high frequencies in y_D^0 , otherwise we filter y_D^0 .

Based on example (3.4) with a modification of the 2D Gaussian law, we define :

$$y^0(x, y) = \exp\left(-5 \left(\frac{r}{2\sigma^2}\right)^6\right),$$

with $r = (x - 0.35)^2 + (y - 0.35)^2$.

Here $(0.35, 0.35)$ is the centre, $\sigma_x = \sigma_y = \sigma$ are the standard deviations for x and y .

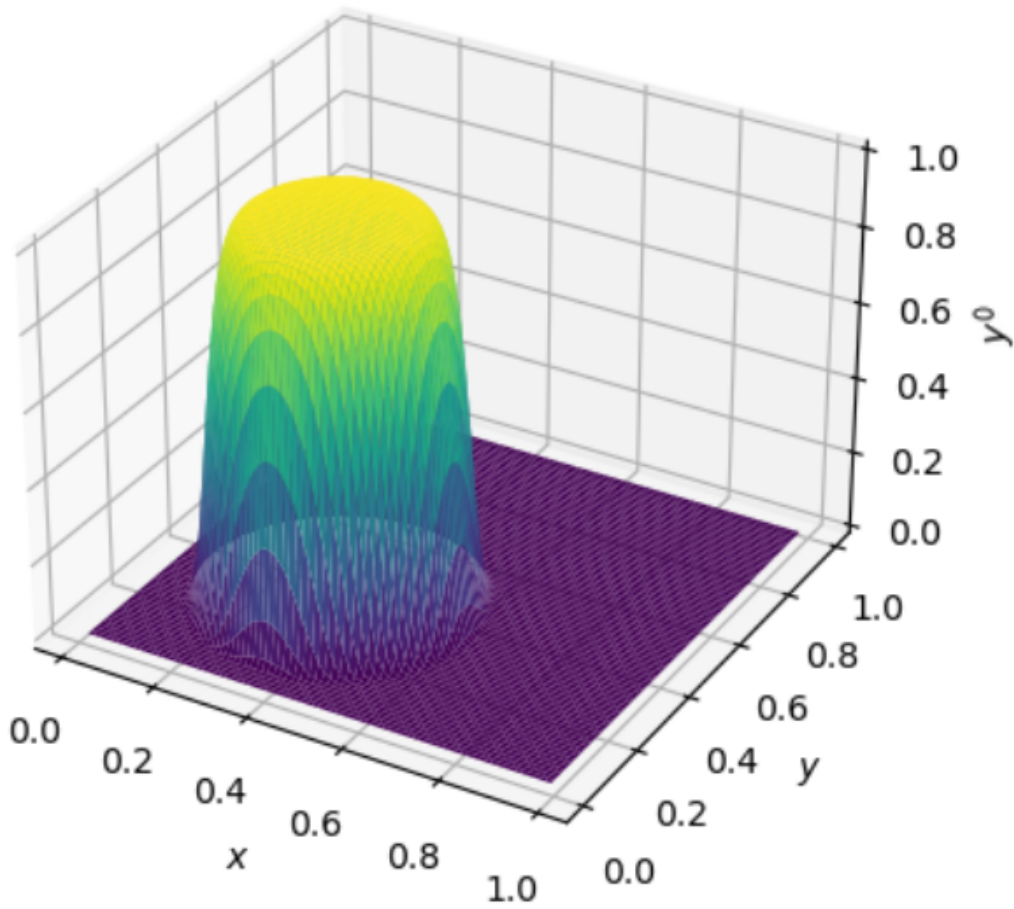


Figure 4.1. The initial position y^0 for $\sigma = 0.2$.

The numerical tests show that for $\sigma \geq 0.3$ the algorithm diverges, moreover when N_1 becomes large the number of iterations increases also the CPU time grow, the other problem that we need a large memory space.

N	15	30	40	50
Number of CG iterations	175	337	547	943
CPU time (secs)	1.625318	9.933084	32.132817	103.455245
$\ y(T)\ _{L^2(\Omega)}$	0.00295	6.693309×10^{-11}	1.27032×10^{-11}	8.422314×10^{-11}
$\ y'(T)\ _{H^{-1}(\Omega)}$	5.434475×10^{-11}	6.503109×10^{-11}	1.211519×10^{-10}	8.497743×10^{-11}
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	0.180434	0.167908	0.16638	0.1652834

Table 4.1. Numerical results obtained for $C = \frac{1}{\sqrt{2}}$, $T = 3$ and $\sigma = 0.2$.

we fix the maximum number of CG iterations $N_{MAX} = 2000$. tables 4.2-4.4 show the the CPU time, the convergence of CG algorithm and the number of CG iterations for different values of ϵ , in the case of the convergence of the algorithm, we will calculate the norm of $\|y(T)\|_{L^2(\Omega)}$, $\|y'(T)\|_{H^{-1}(\Omega)}$ and $\|\hat{v}\|_{L^2((0,T) \times \Gamma)}$

N	40	60	100	150
Number of CG iterations	114	21	19	16
CPU time (secs)	10.438298	5.189082	26.279451	99.680222
convergence	CV	CV	CV	CV
$\ y(T)\ _{L^2(\Omega)}$	0.000366649	0.0003217156	0.0002750246	0.000349270
$\ y'(T)\ _{H^{-1}(\Omega)}$	0.000372948	0.000300827	0.0002772390	0.000288972
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	0.166377769	0.165065510	0.1643637404	0.164166635

Table 4.2. $\epsilon = 10^{-3}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	181	465	323	475
CPU time (secs)	13.119077	105.264298	413.225330	2774.186211
convergence	CV	CV	CV	CV
$\ y(T)\ _{L^2(\Omega)}$	$5.56653233 \times 10^{-5}$	0.000104486	$3.10654843 \times 10^{-5}$	$3.44585132 \times 10^{-5}$
$\ y'(T)\ _{H^{-1}(\Omega)}$	$5.51862416 \times 10^{-5}$	0.000106774	$3.07522409 \times 10^{-5}$	$3.40246881 \times 10^{-5}$
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	0.16638202	0.165074544	0.16437017	0.164172625

Table 4.3. $\epsilon = 10^{-4}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	218	734	2000	2000
CPU time (secs)	15.639176	170.212021	20320.163	40640.326
convergence	CV	CV	DIV	DIV
$\ y(T)\ _{L^2(\Omega)}$	1.3622136×10^{-5}	$1.497883585 \times 10^{-5}$	//	//
$\ y'(T)\ _{H^{-1}(\Omega)}$	1.3334127×10^{-5}	$1.466640873 \times 10^{-5}$	//	//
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	0.165075143	0.165074544	//	//

Table 4.4. $\epsilon = 10^{-5}$, $\sigma = 0.2$

From table 4.2-4.4 and numerical simulations, we notice that, when $\epsilon \geq 10^{-5}$ and $N > 100$ (or N large) the algorithm does not converge in 2000 iterations, also the CPU time is very

large, and the stop criterion, it had an oscillation around 10^{-5} .

In most cases the approximation with the finite difference method does not work for all the initial data, but we can control some type of initial conditions, in this case we have to accelerate the code and also solve the memory problem, but in general we do not find good approximations ($N \geq 100$).

in the following chapters we will try to use a finite element approximation and see if we have the controllability.

Chapter 5

Numerical approximation with the finite element method

Now, we will use the finite element method to solve the wave and Dirichlet problems and control the system 1.1.

As the chapter 3, let

$$\Omega = (0, 1), \quad \Gamma_1 = \{1\} \quad \text{and} \quad \Gamma_2 = \{0\}.$$

5.1 Finite element approximation of the wave equation

As chapter 3, we define the general wave equation as following:

$$\left\{ \begin{array}{ll} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0 & \text{in } (0, 1) \times (0, T), \\ \varphi(0, \cdot) = 0 & \text{in } (0, T), \\ \varphi(1, \cdot) = u(\cdot) & \text{in } \times (0, T), \\ \varphi(\cdot, 0) = v(\cdot), \quad \frac{\partial \varphi}{\partial t}(\cdot, 0) = w(\cdot) & \text{in } (0, 1), \end{array} \right. \quad (5.1)$$

where $(v, w) \in H_0^1(0, 1) \times L^2(0, 1)$.

5.1.1 Variational formulation

Let $V(t) = \{\varphi \in H^1(0, 1) / \varphi(0) = 0, \varphi(1) = u(t)\}$ and $U = H_0^1(0, 1)$.

We fix $\varphi \in H^2(0, 1)$, solution of 5.1. By multiplying the wave equation with $\psi \in U$ and integrating it over the unit interval we get following expression:

$$\int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx - \int_{\Omega} \frac{\partial^2 \varphi}{\partial x^2} \psi dx = 0,$$

using integration by parts, we obtain:

$$\int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx + \int_{\Omega} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} dx = 0.$$

Then the weak formulation of 5.1, is :

$$\forall t \in (0, T), \left\{ \begin{array}{l} \text{Find } \varphi(., t) \in V(t), \text{ such that :} \\ a(\varphi(., t), \psi(.)) = 0, \forall \psi \in U, \\ \varphi(., 0) = v(.), \frac{\partial \varphi}{\partial t}(., 0) = w(.), \end{array} \right. \quad (5.2)$$

with

$$a(\varphi(., t), \psi(.)) = \int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx + \int_{\Omega} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} dx$$

. Before the numerical discretization, we must prove that, if φ a solution of , 5.2 then is a solution of 5.1, for that, we fix $\varphi(., t) \in V(t)$, such that, $a(\varphi(., t), \psi(.)) = 0, \forall \psi \in U$.

Let $\psi \in D(0, 1)$, then

$$\begin{aligned} \int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx + \int_{\Omega} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} dx &= 0, \\ \int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx + \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \psi}{\partial x} \right\rangle &= 0, \\ \int_{\Omega} \frac{\partial^2 \varphi}{\partial t^2} \psi dx - \left\langle \psi, \frac{\partial^2 \varphi}{\partial x^2} \right\rangle &= 0, \end{aligned}$$

finally,

$$\left\langle \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2}, \psi \right\rangle = 0, \quad \forall \psi \in D(0, 1),$$

hence, $\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0$, almost everywhere.

We have proved that, the system 5.1 is equivalent to 5.2, therefore system 5.2 admits a unique solution, because system 5.1 has only one solution. The problem here that we cannot prove that 5.2 admits a unique solution using the Stampacchia theorem or the Lax-Milgram theorem (see [...]), because u is usually only L2, in this case $V(t)$ is an affine space.

5.1.2 Discrete variational problem

We define:

$$x_0 = 0 < x_1 = h < \dots < x_j = jh < \dots < x_{N+1} = 1,$$

be partition of $[0, 1]$ and $h = \frac{1}{N+1}$ is the space step.

We approximate the $V(t)$ and U spaces by:

$$V_h(t) = \{\varphi_h \in C^0(0, 1) / \varphi_h|_{[x_j, x_{j+1}]} \in \mathbb{P}_1, \varphi_h(0) = 0, \varphi_h(1) = u(t), j \in \{0, \dots, N\}\},$$

$$U_h = \{\psi_h \in C^0(0,1) / \psi_h|_{[x_j, x_{j+1}]} \in \mathbb{P}_1, \psi_h(0) = 0, \psi_h(1) = 0, j \in \{0, \dots, N\}\},$$

respectively, with \mathbb{P}_1 is the space of polynomials of degree 1,
then the discrete variational problem, written as follows:

$$\forall t \in (0, T), \begin{cases} \text{Find } \varphi_h(., t) \in V_h(t), \text{ such that :} \\ a(\varphi_h(., t), \psi_h(.)) = 0, \forall \psi_h \in U_h, \\ \varphi_h(., 0) = v_h(.), \frac{\partial \varphi_h}{\partial t}(., 0) = w_h(.), \end{cases} \quad (5.3)$$

moreover U_h is a vector space of dimension N and $\{\phi_1, \dots, \phi_N\}$ is a basis of U_h , with ϕ_1, \dots, ϕ_N are the shape functions, defined by:

$$\phi_i = \begin{cases} 1 + \frac{x - x_i}{h}, & \text{in } [x_{i-1}, x_i], \\ 1 - \frac{x - x_i}{h}, & \text{in } [x_i, x_{i+1}], \\ 0, & \text{Otherwise.} \end{cases}$$

we define the last shape function ϕ_{N+1} , as follows

$$\phi_{N+1} = \begin{cases} 1 + \frac{x - x_{N+1}}{h}, & \text{in } [x_N, x_{N+1}], \\ 0, & \text{Otherwise,} \end{cases}$$

then for all $\varphi_h(., t) \in V_h(t)$, φ_h can be written as:

$$\varphi_h(x, t) = \sum_{j=1}^N \varphi_j(t) \phi_j(x) + u(t) \phi_{N+1}(x).$$

The last discrete formulation of our problem is the following:

$$\begin{cases} \text{Find } \varphi_j(t), \quad j \in \{1, \dots, N\}, \text{ such that :} \\ a \left(\sum_{j=1}^N \varphi_j(t) \phi_j(\cdot) + u(t) \phi_{N+1}(\cdot), \phi_i(\cdot) \right) = 0, \quad \forall i \in \{1, \dots, N\}. \end{cases} \quad (5.4)$$

For each shape function ϕ_i with $i = 1, \dots, N$, we can write

$$\begin{aligned}
 0 &= a \left(\sum_{j=1}^N \varphi_j(t) \phi_j(\cdot) + u(t) \phi_{N+1}(\cdot), \phi_i(\cdot) \right) \\
 &= \sum_{j=1}^N \int_0^1 \frac{\partial^2 \varphi_j}{\partial t^2}(t) \phi_j(x) \phi_i(x) dx + \int_0^1 \frac{\partial^2 u}{\partial t^2}(t) \phi_{N+1}(x) \phi_i(x) dx \\
 &\quad + \sum_{j=1}^N \int_0^1 \varphi_j(t) \frac{\partial \phi_j(x)}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx + \int_0^1 u(t) \frac{\partial \phi_{N+1}(x)}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx \\
 &= \sum_{j=1}^N \frac{\partial^2 \varphi_j}{\partial t^2}(t) \int_0^1 \phi_j(x) \phi_i(x) dx + \sum_{j=1}^N \varphi_j(t) \int_0^1 \frac{\partial \phi_j(x)}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx \\
 &\quad + \frac{\partial^2 u}{\partial t^2}(t) \int_0^1 \phi_{N+1}(x) \phi_i(x) dx + u(t) \int_0^1 \frac{\partial \phi_{N+1}(x)}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx.
 \end{aligned}$$

Let $A_{ij} = \int_0^1 \phi_j \phi_i dx$, $B_{ij} = \int_0^1 \frac{\partial \phi_j(x)}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx$, for $i, j \in \{1, \dots, N\}$,
and $\varphi(t) = (\varphi_1(t), \dots, \varphi_N(t))$.

the system 5.4 is equivalent to :

$$(***) \quad A \frac{\partial^2 \varphi(t)}{\partial t^2} + B \varphi(t) = C(t),$$

such as :

$$A = \begin{pmatrix} \frac{2h}{3} & \frac{h}{6} & 0 \\ \frac{h}{6} & \ddots & \ddots \\ & \ddots & \ddots & \frac{h}{6} \\ 0 & & \frac{h}{6} & \frac{2h}{3} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{2}{h} & \frac{-1}{h} & 0 \\ \frac{-1}{h} & \ddots & \ddots \\ & \ddots & \ddots & \frac{-1}{h} \\ 0 & & \frac{-1}{h} & \frac{2}{h} \end{pmatrix}$$

and

$$C(t) = \begin{pmatrix} 0 \\ \vdots \\ \frac{1}{h} u(t) - \frac{h}{6} \frac{\partial^2 u(t)}{\partial t^2} \end{pmatrix}.$$

To solve (***), we will use a second order approximation for the second derivative (FDM).
The final numerical scheme is as follows:

$$\left\{ \begin{array}{l} \text{Find } \varphi^n, \text{ such that :} \\ A_0 \varphi^{n+1} = (2A_0 - 6C^2 B_0) \varphi^n - A_0 \varphi^{n-1} + C^n, \\ \varphi^0 = v_h, \quad \varphi^1 = \varphi^0 + k w_h, \end{array} \right. \quad (5.5)$$

here, $A_0 = \frac{6}{h}A$, $B_0 = hB$, $C = \frac{k}{h}$ and

$$C^m = \begin{pmatrix} 0 \\ \vdots \\ (6C^2 + 2)u(t_n) - (u(t_{n+1}) + u(t_{n-1})) \end{pmatrix}.$$

As in the previous chapter, we will define a function for the matrix-vector product and for the resolution of the linear system we will use the COO scheme.

For the stability of the scheme 5.5, we can prove mathematically and numerically that 5.5 is stable if $C \leq \frac{1}{\sqrt{3}}$.

5.2 Finite element approximation of the Dirichlet problem

The Poisson equation that we have to solve in each iteration is defined as follows:

$$\begin{cases} -\frac{\partial^2 \varphi}{\partial x^2} = \psi'(0) - y & \text{in } (0, 1), \\ \varphi(0) = 0, \\ \varphi(1) = 0. \end{cases} \quad (5.6)$$

Let $U = H_0^1(0, 1)$. The variational formulation for the problem 5.6 is written as follows:

$$\begin{cases} \text{Find } \varphi \in U, \text{ such that :} \\ A(\varphi, \Psi) = L(\Psi), \quad \forall \Psi \in U, \end{cases} \quad (5.7)$$

with

$$A(\varphi, \Psi) = \int_0^1 \frac{\partial \varphi}{\partial x} \frac{\partial \Psi}{\partial x} dx, \quad \forall \varphi, \Psi \in U,$$

$$L(\Psi) = \int_0^1 (\psi'(0) - y) \Psi dx, \quad \forall \Psi \in U.$$

We can easily show the 5.6 is equivalent to 5.7, moreover the Lax-Milgram theorem shows that the system 5.7 admits a unique solution.

As in the previous section, we discretize the variational problem, we obtain:

$$\begin{cases} \text{Find } \varphi_h \in U_h, \text{ such that :} \\ A(\varphi_h, \Psi_h) = L(\Psi_h), \quad \forall \Psi_h \in U_h, \end{cases} \quad (5.8)$$

here $U_h = \{\Psi_h \in C^0(0,1) / \Psi_h|_{[x_j, x_{j+1}]} \in \mathbb{P}_1, \Psi_h(0) = 0, \Psi_h(1) = 0, j \in \{0, \dots, N\}\}$ is a finite dimensional ($\dim(U_h) = N$) subspace of U .

since A is a bilinear form and L is a linear form, then 5.8 is equivalent to:

$$\begin{cases} \text{Find } \varphi_h \in U_h, \text{ such that :} \\ A(\varphi_h, \phi_i) = L(\phi_i), i \in \{1, \dots, N\}. \end{cases} \quad (5.9)$$

We know that $\varphi_h = \sum_{j=1}^N \varphi_j \phi_j$, where $\{\phi_1, \dots, \phi_N\}$ are the shape functions that define a basis of U_h , but the ψ in 5.6 is a solution of a wave equation and y is an initial condition for the wave equation that we have to control, then $\psi = \sum_{j=1}^{N+1} \psi_j \phi_j$ and $y = \sum_{j=0}^{N+1} y_j \phi_j$.

The first shape function ϕ_0 defined as follows:

$$\phi_0 = \begin{cases} 1 - \frac{x - x_0}{h}, & \text{in } [x_0, x_1], \\ 0, & \text{Otherwise.} \end{cases}$$

We approximation the first derivative in time by finite difference and we replace in 5.9, we obtain:

$$\begin{aligned} \sum_{j=1}^N \varphi_j \int_0^1 \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} dx &= \int_0^1 \left[\sum_{j=1}^{N+1} \frac{\psi_j^1 - \psi_j^0}{k} \phi_j - \sum_{j=0}^{N+1} y_j \phi_j \right] \phi_i dx \\ &= \sum_{j=1}^N \left(\frac{\psi_j^1 - \psi_j^0}{k} - y_j \right) \int_0^1 \phi_i \phi_j dx + \left(\frac{\psi_{N+1}^1 - \psi_{N+1}^0}{k} - y_{N+1} \right) \int_0^1 \phi_{N+1} \phi_i \\ &\quad - y_0 \int_0^1 \phi_0 \phi_i dx. \end{aligned}$$

we set $\varphi = (\varphi_1, \dots, \varphi_N)$, with the notation of the previous section, the final scheme is as follows:

$$B_0 \varphi = \frac{h^2}{6} A_0 Y + b,$$

where, $Y = \left(\frac{\psi_j^1 - \psi_j^0}{k} - y_j \right)_{1 \leq j \leq N}$ and

$$b = \frac{h^2}{6} \begin{pmatrix} -y_0 \\ 0 \\ \vdots \\ 0 \\ \frac{\psi_{N+1}^1 - \psi_{N+1}^0}{k} - y_{N+1} \end{pmatrix}$$

5.3 Numerical Tests

In the present section, we apply the methods of sections 5.1 and 5.2, and the conjugate gradient algorithm, to control our system, and also, we will compare this method with the finite difference method in section 3.3.

As section 3.3, the conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$ and for the stopping criteria we choose:

$$\frac{\|(\widetilde{\varphi}_{n+1}^0, \widetilde{\varphi}_{n+1}^1)\|_{H_0^1 \times L^2}}{\|(\widetilde{\varphi}_0^0, \widetilde{\varphi}_0^1)\|_{H_0^1 \times L^2}} < \epsilon.$$

For the numerical tests, we approximate $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H_0^1}$ using the following approximation:

$$\int_0^1 \nabla \varphi \nabla \psi dx = \frac{1}{h} \sum_{i=1}^N \varphi_i \psi_i + \frac{1}{h} \sum_{i=0}^{N+1} \varphi_i \psi_i - \frac{1}{h} \sum_{i=0}^N (\varphi_{i+1} \psi_i + \varphi_i \psi_{i+1}),$$

$$\int_0^1 \varphi \psi dx = \frac{h}{3} \sum_{i=1}^N \varphi_i \psi_i + \frac{h}{3} \sum_{i=0}^{N+1} \varphi_i \psi_i + \frac{h}{6} \sum_{i=0}^N (\varphi_{i+1} \psi_i + \varphi_i \psi_{i+1}).$$

We know that we have the stability for the solver of the wave equation if we take $C = \frac{1}{\sqrt{3}}$, then for the moment, we fix $C = \frac{1}{\sqrt{3}}$, $T = 2.2$ and the maximum number of iterations equal to 2000, we will see if we have the controllability and also the convergence of the CG for the initial conditions

$$\begin{cases} y^0(x) = \exp(-5(x - 0.35)^6), \\ y^1(x) = 0. \end{cases} \quad (5.10)$$

From tables 5.1-5.2, we remark that, when ϵ becomes small, we must increase the N to get the convergence, also if N large, the number of iterations to have convergence becomes small.

The numerical tests, show that, we can have convergence for N small ($N \leq 50$), but it is necessary to choose an $\epsilon \leq 10^{-4}$.

N	15	30	69	70	100
Number of CG iterations	2000	2000	2000	8	6
Convergence	no conv	no conv	no conv	conv	conv
$\ y(T)\ _{L^2(\Omega)}$	//	//	//	0.0001907	4.0131002×10^{-5}
$\ y'(T)\ _{H^{-1}(\Omega)}$	//	//	//	0.0001203	1.8678941×10^{-5}
$\ \widehat{v}\ _{L^2((0,T) \times \Gamma)}$	//	//	//	0.4063041	0.4064398

Table 5.1. Numerical results obtained for $C = \frac{1}{\sqrt{3}}$, $T = 2.2$ and $\epsilon = 10^{-5}$.

N	70	109	110	200	300
Number of CG iterations	2000	2000	20	5	4
Convergence	no conv	no conv	conv	conv	conv
$\ y(T)\ _{L^2(\Omega)}$	//	//	6.106698×10^{-5}	1.878805×10^{-7}	5.3617078×10^{-8}
$\ y'(T)\ _{H^{-1}(\Omega)}$	//	//	8.0236302×10^{-6}	1.755967×10^{-7}	6.3979525×10^{-8}
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	//	//	0.4064625	0.406538	0.4065570

Table 5.2. Numerical results obtained for $C = \frac{1}{\sqrt{3}}$. $T = 2.2$ and $\epsilon = 10^{-6}$.

Now we fix $\epsilon = 10^{-10}$, from Table 5.3 we observe that when N becomes bigger, the norm of $y(T)$ in $L^2(0, 1)$ and $\frac{\partial y}{\partial t}(T)$ in $H^{-1}(0, 1)$ close to 0.

Figure 5.2, show the approximation of the control function for $N = 1000$, in figure 5.1 we plot the controlled solution y_h for $N = 1000$.

Then we conclude that for the initial data 5.10, the system is controllable.

N	350	400	500	1000
Number of CG iterations	7	6	6	5
CPU Time (secs)	4.322983	6.488691	11.949790	78.507318
$\ y(T)\ _{L^2(\Omega)}$	1.296216×10^{-10}	$3.6776111 \times 10^{-11}$	$7.5309572 \times 10^{-12}$	$7.40331232 \times 10^{-12}$
$\ y'(T)\ _{H^{-1}(\Omega)}$	3.597765×10^{-11}	$3.4716749 \times 10^{-11}$	$7.1260334 \times 10^{-12}$	$6.31366379 \times 10^{-12}$
$\ \hat{v}\ _{L^2((0,T) \times \Gamma)}$	0.4065610	0.4065635	0.4065666	0.40657065

Table 5.3. Numerical results obtained for $C = \frac{1}{\sqrt{3}}$. $T = 2.2$ and $\epsilon = 10^{-10}$.

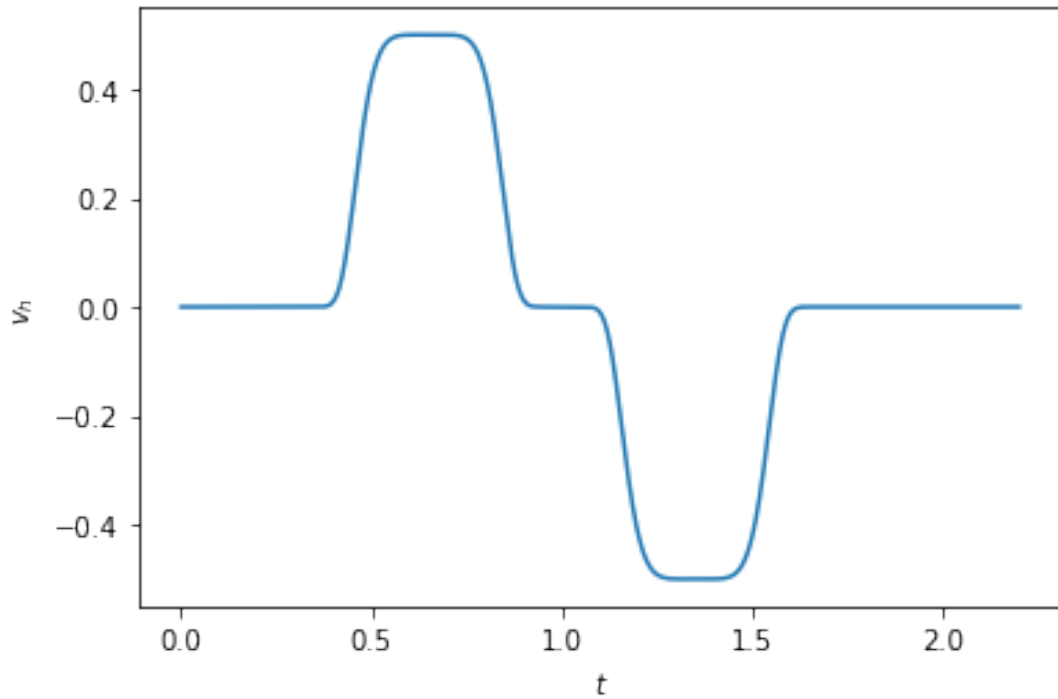


Figure 5.1. The approximation of the control.

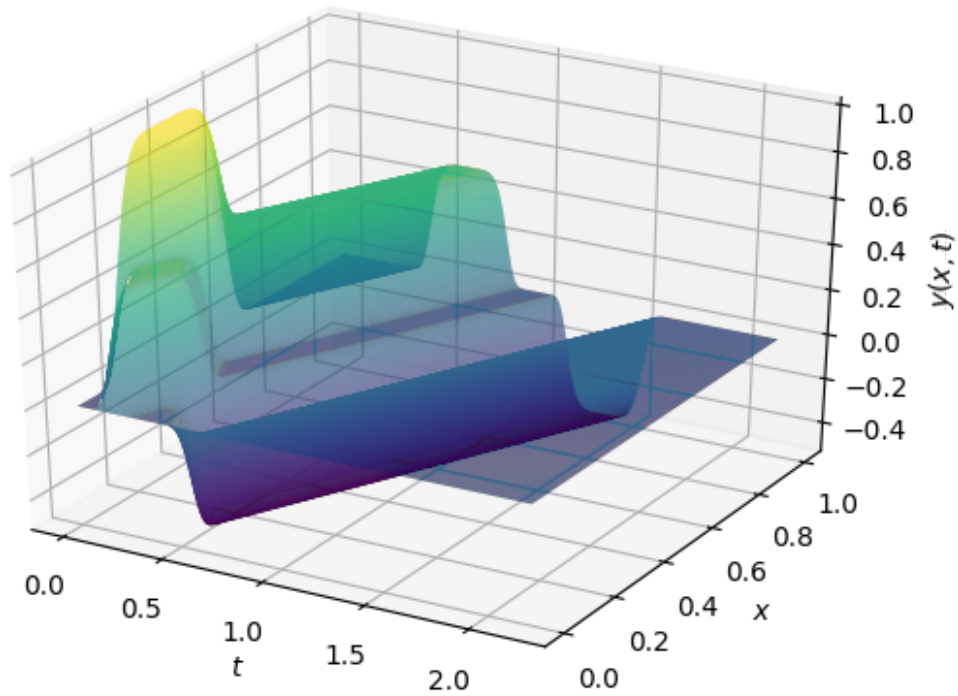


Figure 5.2. The controlled solution y .

With numerical tests, we can show that, for C in $\left]0, \frac{1}{\sqrt{3}} + 2 \times 10^{-5}\right]$, we have the convergence of the CG algorithm, we also obtain convergence with certain values in $\left[\frac{1}{\sqrt{3}} + 2 \times 10^{-5}, \frac{1}{\sqrt{3}} + 10^{-4}\right]$, but if $C \geq \frac{1}{\sqrt{3}} + 10^{-4}$ the algorithm diverge.

As in chapter 3, we cannot control all the initial conditions. and the high frequency problem increases the initial data set that can be controlled, but the method works well for some types of initial conditions, moreover with the finite difference method we have a mathematical criterion to choose C to control all the initial data, but this choice limits the domain of application of our code for other controllable systems.

the common point between the two methods is that we cannot control all the initial data with different choices of C , for example, in our case of finite elements, the initial conditions:

$$\begin{cases} y^0(x) = x, \\ y^1(x) = 0, \end{cases}$$

are not controllable. We can show that if we choose $N = 350$ and C in $\{0.1, 0.2, 0.3, 0.4, 0.5, \frac{1}{\sqrt{3}}\}$, and we can do numerous tests, but in general the algorithm diverges. Even if we find a C with which we can control all the initial data and filtering, the the method is not practical.

Chapter 6

Numerical approximation with the finite element method (2D Case)

Conclusion

OK

Bibliography

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