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MASTER'S THESIS

Exact boundary controllability of the wave equation : Numerical Approach

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ABSTRACT

In this thesis we develop a systematic method that provides a constructive algorithm to build controls, and we discuss the numerical implementation of this method for the exact boundary controllability of the wave equation. The numerical methods considered here consist of explicit finite difference approximation in time-space and a combination of finite element approximation for the space and an explicit finite difference schemes for the time discretization in the cases 1D and 2D. We discuss also the convergence of the adapted algorithm and we give some techniques to solve the problem of divergence and to accelerate the codes.

Keywords: Wave Equation, Exact Boundary Controllability, Uniform Boundary Controllability/Observability, Observability, Conjugate Gradient, Finite Elements, Finite Differences, Acceleration, Filtering.

CONTENTS

Acronyms

HUM	Hilbert Uniqueness Method
CG	Conjugate Gradient
CFL	Richard Courant, Kurt Friedrichs, Hans Lewy: Stability Condition

LIST OF FIGURES

INTRODUCTION

The control theory is the study of the possibility of acting on a system so that it functions for a desired goal. Going back in time, we see that the philosophy of this theory is already adapted by the Romans and ancient Egypt in practical life [For Further historical back ground, we refer to [19], page 18].
Now, the control theory has becomes one of the most active mathematics's areas and it allowed us to solve a lot of problems in large variety of fields, including: Physics, Biology, Engineering,... . At the present moment with technology, a new branch is being developed on the control theory and the Machine learning "Learning by doing" (CML) and the connections between the both.
It is natural that between ancient Egypt and the present moment, especially after 1868 with J.C. Maxwell who made the first mathematical analysis for controlled systems in finite dimension, and the intersection of control theory with a great variety of domains, many notions and definitions of controllability have been introduced to give this notion the ability to deals with different situations and problems. In our study, of the wave equation, we discuss some of these notions.

The wave equation is a partial differential equation that models many Physical phenomena, for examples : The propagation of a disturbance along a string and the sound wave. The controllability problem associated with this equation belongs to the linear evolution equations, written in the formal form as follows:

$$\begin{cases} Y^{'}(t) = \mathscr{A}Y(t) + \mathscr{B}v(t), \quad t \in [0,T], \\ \\ Y(0) = Y^0, \end{cases} \tag{1}$$

with Y^0 is the initial data given in an appropriate functional space (The state space), $\mathscr{A}$ is a differential unbounded linear operator, v is the control function and $\mathscr{B}$ is an operator defined from the space of controls to the state space. the system 1 is said to be exactly controllable in time T, if for all Y_T (final data), we can find a control function such that $Y(T) = Y_T$. The exact controllability is too strict, sometimes we replace it by the approximate controllability, that is the final position $Y(T)$ is close to Y_T, which implies that the approximate controllability depends on the choice of the topology and the state space, differently to the exact controllabilty which depends only on the choice of the state space. In some cases, as in the wave system the exact controllability is equivalent to the what is called null controllability, where we want to droves the system to 0 (equilibrium) in time T.
The control operator $\mathscr{B}$ models how the control v acts on the system, in our study of the wave equation, the control v acts on a part of the boundary and in this case the operator $\mathscr{B}$ is unbounded, what we call the boundary controllability. But in some cases the control acts on a part of the domain (inside), then the operator $\mathscr{B}$ is bounded and we are in the context of the internal controllability.
The study of the controllability problems naturally leads to the notion of observability in the sense that system 1 is observable in time T if, and only if its adjoint system (dual system) is observable (duality Controllabiliy/Observability). The observability of the an evolution equation is the possibility of estimating the total energy of the solution, in a given time interval, in terms of the energy concentrated on the boundary.
On this thesis we focus on the boundary controllability proprety of the wave system. A systematic

approach is called HUM method introduced by Jacques-Luis Lions in 1980 [13], provides a constructive method for constructing controls (with minimal L^2-norm), applies not only to the particular hyperbolic problem considered here, but also for more general class of distributed systems.
In this work, we have analyzed the two notions (exact controllabiliy/observability) in the continuous case as well as in discrete case, to do this we have separated this work into 7 chapters that can be also separated into 2 parts: Numerical analysis: The focus will be on the one and the two-dimensional wave equation.
Mathematical analysis: to show the numerical result mathematically and also study the boundary (oberavbility / controllability) in the continuous case.

Chapter 1:
In the first chapter we introduce the control problem and we discuss the existence of the solutions of the wave equation, after that we transform the exact boundary controllability problem to a minimization problem by transfering the exact boundary controllability to an obsarvability problem and using the hidden regularity and the observability inequality.

Chapter 2:
In the second chapter, we introduce the general conjugate gradient algorithm which can be used to solve the continuous problem, and we adapt this algorithm to our problem of minimization by introducing a non-homogeneous backward wave equation.

Chapter 3:
In this chapter we focus on the $1D$ wave equation and we discretize the continuous algorithm using the finite difference approximation in space and time, and we end this chapter by analyzing two types of initial conditions and we discuss the problem of high frequencies.

Chapter 4:

This chapter is based on the numerical results of **Chapter 3:**, we introduce two notions of controllability and observability associated with the space semi-discret wave equation, we show that the divergence of the controls (Non-uniform controllability) is due to the fact that the scheme introduces high frequencies, and to maintain the uniformity (uniform Boundary observability), filtering mechanisms were used to exclude the high frequency components. Finally we apply the method known as initial condition filtering to obtain uniform controllability.

Chapters 5, 6, 7:
In these chapters we make tests on different approximations of the continuous problem: Finite element $1D$, $2D$ and finite difference $2D$, that is to say different approximation of the Laplacian operator. In each case we analyze the results obtained and we discuss the problem of high frequencies. See more

about the theoretical aspect of these methods and the analysis of convergence in [16].

CHAPTER 1

THE EXACT BOUNDARY CONTROLLABILITY

1.1 Formulation of the control problem

Let Ω be a bounded domain (that is, non-empty open connected set) in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ "Sufficiently smooth", Γ_1 be an open nonempty subset of Γ and $\Gamma_2 = \Gamma\backslash\Gamma_1$. With T a given positive number, we consider the following non-homogeneous wave equation:

$$\left\{\begin{array}{rcl} \dfrac{\partial^2 y}{\partial t^2} - \Delta y &=& 0 \qquad \text{for} \quad (x,t) \in \Omega \times (0,T), \quad (1.1.1) \\ y(x,t) &=& 0 \qquad \text{for} \quad (x,t) \in \Gamma_2 \times (0,T), \quad (1.1.2) \\ y(x,t) &=& v(x,t) \qquad \text{for} \quad (x,t) \in \Gamma_1 \times (0,T), \quad (1.1.2) \\ y(x,0) &=& y^0, \quad \dfrac{\partial y}{\partial t}(x,0) = y^1 \quad \text{for} \quad x \in \Omega. \quad (1.1.3) \end{array}\right. , \qquad (1.1)$$

where (1.1), $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, and $H^{-1}(\Omega)$ is the topological dual space of $H_0^1(\Omega)$ and Δ is the Laplacian operator.

In this section we consider the general case where $v \in L^2(\Gamma_1 \times (0,T))$, not necessarily null, and $\Omega \subseteq \mathbb{R}^n$ (the multidimensional case).

Remark 1.1.1
- *The normal vector exists because the boundary Γ is sufficiently smooth.*
- *For x in Γ and φ in $H^2(\Omega)$, the normal derivative is given by*

$$\frac{\partial\varphi}{\partial n}(x) = \nabla\varphi(x).\overrightarrow{n}(x),$$

where $\overrightarrow{n}(x)$ is the normal vector at x.

The main result for the existence of solutions of (1.1) is the following:

Theorem 1.1.1 *For every v in $L^2(\Gamma_1 \times (0,T))$ and (y^0, y^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ system* (1.1) *has a unique weak solution*

$$(y, y') \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega)),$$

moreover, there exists a constant $C = C(T) > 0$ such that

$$||(y, y')||_{L^\infty([0,T],L^2(\Omega)\times H^{-1})} \leq C\left[||(y^0, y^1)||_{L^2(\Omega)\times H^{-1}} + ||v||_{L^2(\Gamma_1\times(0,T))}\right].$$

Proof :
The theorem is a consequence of the theory of nonhomogeneous evolution equations. For more details see [12].

We shall now define the exact boundary controllability for the system (1.1).

Definition 1.1.1 *System* (1.1) *is controllable in time* $T > 0$ *if for every initial data* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, we can find a control function* $v \in L^2(\Gamma_1 \times (0,T))$ *such that the corresponding solution* (y, y') *of* (1.1) *verifies*

$$y(\cdot, T) \;=\; y'(\cdot, T) = 0. \tag{1.2}$$

Remark 1.1.2 *If a solution of* (1.1) *verifies* (1.2) *is also said to be null controllable in time* $T > 0$*, for more details* (*see* [19], *page* 100).

The problem that we consider is the following : *is it possible to find a time* $T > 0$*, sufficiently large or optimal, and a boundary control function* $v \in L^2(\Gamma_1 \times (0,T))$*, such that for any* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *we have* (1.2)*.*

1.2 Controllability by minimization

In this section we will give a necessary and sufficient condition for the exact controllability of (1.1), and also we will transform the controllability problem to a minimization problem, but before that we need to the following result:

Lemma 1.2.1 *For every* (φ^0, φ^1) *in* $H_0^1(\Omega) \times L^2(\Omega)$ *the following wave equation*

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi \;=\; 0 & in \quad \Omega \times (0,T), \\ \varphi \;=\; 0 & in \quad \Gamma \times (0,T), \\ \varphi(.,0) \;=\; \varphi^0, \quad \dfrac{\partial \varphi}{\partial t}(.,0) = \varphi^1 & in \quad \Omega, \end{cases} \tag{1.3}$$

has a unique solution, moreover (1.3) *generates a group of isometries in* $H_0^1(\Omega) \times L^2(\Omega)$*.*

Proof:
This lemma is a consequence of the following classic theorem:

Theorem 1.2.1 ***(Stone, 1930)***
Let H *be a Hilbert space and* A *be a linear operator on* H *with dense domain, then* A *generates a* C_0*-group of unitary operators if and only if* A *is skewadjoint* ($A' = -A$)*.*

see more details on the C_0–semigroup theory in [8] and also the proof of Stone's theorem in [8] ,page 44.

Let $w(t) = (\varphi(t), \varphi^{'}(t))$ and the state space is $H = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the scalar product:

$$< (\varphi_1, \varphi_2), (\psi_1, \psi_2) >= \int_\Omega \nabla \varphi_1 \nabla \psi_1 + \int_\Omega \varphi_2 \psi_2,$$

then, the corresponding norm on H is given by:

$$||(\varphi_1, \varphi_2)||^2 = \int_\Omega ||\nabla \varphi_1||^2 + \int_\Omega ||\varphi_2||^2.$$

Using Poincaré inequality, we can show that this norm is equivalent to the usual norm on H.
We define the linear operator A by:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{and}$$

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}.$$

We mention that the domain $D(A)$ is dense in H and $w^{'}(t) = Aw(t)$.
In order to show that A is skewadjoint, let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D(A)$, then

$$\begin{aligned}
< A(\varphi_1, \varphi_2), (\psi_1, \psi_2) > &=< (\varphi_2, \Delta\varphi_1), (\psi_1, \psi_2) > \\
&= \int_\Omega \nabla \varphi_2 \nabla \psi_1 + \int_\Omega \Delta \varphi_1 \psi_2 \\
&= \int_\Omega \nabla \varphi_2 \nabla \psi_1 - \int_\Omega \nabla \varphi_1 \nabla \psi_2 \\
&\quad + \underbrace{\int_\Gamma \frac{\partial \varphi_1}{\partial n} \psi_2}_{0} \\
&= - \int_\Omega \nabla \varphi_1 \nabla \psi_2 - \int_\Omega \varphi_2 \Delta \psi_1 \\
&= - < (\varphi_1, \varphi_2), (\psi_2, \Delta \psi_1) > \\
&= - < (\varphi_1, \varphi_2), A(\psi_1, \psi_2) > .
\end{aligned}$$

Which implies that, the operator A is skewadjoint on H and thus generates a unitary C_0-group on H by Stone's theorem.
Then if (φ^0, φ^1) in $D(A)$ the system (1.3) has a unique strict solution, but in the case where (φ^0, φ^1) is in $H \backslash D(A)$, (1.3) has a unique weak solution defined by:

$$< \varphi''(t), \psi >_{-1,1} + \int_\Omega \nabla \varphi(t) \cdot \nabla \psi dx = 0, \quad \forall t \in (0, T), \, \forall \psi \in H_0^1(\Omega),$$

here, $< \cdot, \cdot >_{-1,1}$ denotes the duality between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, (See also the proof with the Fourier method in [13]).
Now let's come back to our goal, which is to transform the controllability problem into an optimiza-

tion problem. The following proposition is important to achieve this goal.

Proposition 1.2.1 *Assume that* $T > 0$ *is large enough, then there exists a constant* $\hat{C} > 0$ *such that:*

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma\, dt \leq \hat{C} ||(\varphi^0, \varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.4}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* φ *is the solution of* (1.3).

Proof: See [11].
The inequality (1.4) is a hidden regularity result, that may not be obtained by standard trace results (*see*, [11]). Moreover, the existence of T is ensured by the geometric conditions (CCG) (*see*, [1]) .

The key result in this section is the variational characterization of the controllability of (1.1) given by the following lemma:

Lemma 1.2.2 *The initial data* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *is controllable to zero if and only if there exists* $v \in L^2(\Gamma_1 \times (0,T))$ *such that:*

$$\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} v d\sigma dt = < y^1, \varphi^0 >_{-1,1} - \int_\Omega y^0 \varphi^1 dx, \tag{1.5}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and where* φ *is the solution of the adjoint problem* (1.3).

Proof:
Let $(y^0, y^1) \in D(\Omega) \times D(\Omega)$, $(\varphi^0, \varphi^1) \in D(\Omega) \times D(\Omega)$ and $v \in D(\Gamma_1 \times (0,T))$, then y and φ are regular solutions. Multiplying the equation (1.1.1) in system (1.1) by φ and integrating, we have

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega (y'' - \Delta y)\varphi dx\, dt \\
&= \int_\Omega \int_0^T y'' \varphi dt\, dx - \int_0^T \int_\Omega \Delta y \varphi dx\, dt \\
&= \int_\Omega \left[\left[y' \varphi \right]_0^T - \int_0^T y' \varphi' dt \right] dx + \int_0^T \int_\Omega \nabla y \nabla \varphi dx\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0)) dx - \int_\Omega \left[y\varphi' \right]_0^T dx + \int_\Omega \int_0^T y \varphi'' dt\, dx \\
&\quad - \int_0^T \int_\Omega \Delta \varphi y dx\, dt + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt \\
&= \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0)) dx - \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0)) dx + \int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt,
\end{aligned}
$$

hence,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &- \int_\Omega (y'(T)\varphi(T) - y'(0)\varphi(0)) dx \\
&+ \int_\Omega (y(T)\varphi'(T) - y(0)\varphi'(0)) dx.
\end{aligned}
$$

Finally we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &\int_\Omega y^1 \varphi^0 dx - \int_\Omega y^0 \varphi^1 dx \\
&+ \int_\Omega y(T)\varphi'(T) dx - \int_\Omega y'(T)\varphi(T) dx.
\end{aligned}
$$

We know that $D(\Omega) \times D(\Omega)$ is dense in $L^2(\Omega) \times H^{-1}(\Omega)$ and also in $H_0^1(\Omega) \times L^2(\Omega)$, and that $D(\Gamma_1 \times (0,T))$ is dense in $L^2(\Gamma_1 \times (0,T))$ and by the inequality (1.4), we have $\frac{\partial \varphi}{\partial n}|_{\Gamma_1} \in L^2(\Gamma_1) \times (0,T)$. Hence, using a density argument, we deduce that for any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ we have,

$$
\begin{aligned}
\int_0^T \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} y\, d\sigma\, dt = &< y^1 \varphi^0 >_{-1,1} - \int_\Omega y^0, \varphi^1 dx \\
&+ < (\varphi(T), \varphi'(T)), (y(T), y'(T)) > .
\end{aligned}
$$

With $< \cdot,\cdot >_{-1,1}$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and $< \cdot,\cdot >$ denote the duality product between $H_0^1(\Omega)\times L^2(\Omega)$ and $L^2(\Omega)\times H^{-1}(\Omega)$, defined by:

$$< (\varphi^0,\varphi^1),(y^0,y^1) >= \int_\Omega y^0\varphi^1 - < y^1,\varphi^0 >_{-1,1},$$

for all $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$, $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$.
Taking into account that the wave equation generates a group of isometries in $H_0^1(\Omega)\times L^2(\Omega)$, then (1.5) holds if and only if (y^0,y^1) is controllable to zero in time $T>0$. This completes the proof.

From lemme 1.3.2, the equality (1.5) can be seen as an optimality condition for the minimum of the functional $\mathscr{J}\ :\ H_0^1(\Omega)\times L^2(\Omega)\ \rightarrow\ \mathbb{R}$, defined by

$$\mathscr{J} \;=\; \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt + < (\varphi^0,\varphi^1),(y^0,y^1) >, \tag{1.6}$$

where $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$ and φ is the corresponding solution of (1.3).
Before introducing the main theorem in this section we need to prove that the functional $\mathscr{J}$ has a minimum .

Definition 1.2.1 *System* (1.3) *is said to be observable in time* $T>0$ *if there exists a positive constant* $C>0$ *such that*

$$C||(\varphi^0,\varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \;\le\; \int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt, \tag{1.7}$$

for all $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$ *where* φ *is the solution of* (1.3) *with initial data* (φ^0,φ^1).

Note that, the inequality 1.7 represents the observability inequality in the continuous case, with which we can estimate the total energy of the system by the energy concentrated in Γ_1.

In the following we assume that there exists a positive time T^* such that for any $T>T^*$ the system (1.3) is observable (see [7]).
On the other hand, the functional $\mathscr{J}$ is continuous, strictly convex and coercive.
It is easy to see that functional $\mathscr{J}$ is continuous, now let $(\varphi^0,\varphi^1),(\psi^0,\psi^1)$ in $H_0^1(\Omega)\times L^2(\Omega)$ and $\lambda\in]0,1[$, we have,

$$\begin{aligned}
\mathscr{J}(\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1)) \;&=\; \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\lambda\frac{\partial\varphi}{\partial n}+(1-\lambda)\frac{\partial\psi}{\partial n}\right|^2 d\sigma\, dt\\
&+\; < \lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1),(y^0,y^1) >\\
&=\; \lambda\mathscr{J}((\varphi^0,\varphi^1))+(1-\lambda)\mathscr{J}((\psi^0,\psi^1))\\
&-\; \frac{\lambda(1-\lambda)}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}-\frac{\partial\psi}{\partial n}\right|^2 d\sigma\, dt.
\end{aligned}$$

Using the observability inequality, we obtain

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial}{\partial n}(\varphi - \psi) \right|^2 d\sigma\, dt \geq C||(\varphi^0 - \psi^0, \varphi^1 - \psi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)},$$

then if $(\varphi_0, \varphi_1) \neq (\psi_0, \psi_1)$, we get

$$\mathscr{J}(\lambda(\varphi^0,\varphi^1) + (1-\lambda)(\psi^0,\psi^1)) \;\; < \;\; \lambda\mathscr{J}((\varphi^0,\varphi^1)) + (1-\lambda)\mathscr{J}((\psi^0,\psi^1)).$$

Hence $\mathscr{J}$ is strictly convex.
For the coercivity of the the functional $\mathscr{J}$, we have

$$\begin{aligned}\mathscr{J}((\varphi^0,\varphi^1)) \;\; &\geq \;\; \frac{1}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma\, dt - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} \\ &\geq \;\; \frac{C}{2}||(\varphi^0,\varphi^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} - ||(\varphi^0,\varphi^1)||_{H_0^1(\Omega)\times L^2(\Omega)}\, ||(y^0,y^1)||_{L^2(\Omega)\times H^{-1}(\Omega)},\end{aligned}$$

then $\lim\limits_{||(\varphi^0,\varphi^1)||\to+\infty} \mathscr{J}((\varphi^0,\varphi^1)) = \infty$.
We conclude that the functional $\mathscr{J}$ has a unique minimizer $(\hat{\varphi^0}, \hat{\varphi^1})$ in $H_0^1(\Omega)\times L^2(\Omega)$, and we have the following result

Theorem 1.2.2 *Let (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$ and $(\hat{\varphi^0},\hat{\varphi^1})$ in $H_0^1(\Omega)\times L^2(\Omega)$ be the unique minimizer of the functional $\mathscr{J}$, then the function $\hat{v}$ defined on $\Gamma_1\times(0,T)$ by :*

$$\hat{v}(x,t) = \frac{\partial\hat{\varphi}}{\partial n}(x,t), \quad (x,t)\in\Gamma_1\times(0,T),$$

is a control which leads (y^0,y^1) to zero in time $T>0$.

Proof:
The functional $\mathscr{J}$ achieves its minimum at $(\hat{\varphi^0},\hat{\varphi^1})$, then

$$\lim_{h\to 0}\frac{1}{h}\left[\mathscr{J}((\hat{\varphi^0},\hat{\varphi^1}) + h(\varphi^0,\varphi^1)) - \mathscr{J}((\hat{\varphi^0},\hat{\varphi^1}))\right] = 0, \tag{1.8}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ where φ is the solution of (1.3) with initial data (φ^0,φ^1).
On the other hand, we have,

$$\begin{aligned}\mathscr{J}((\hat{\varphi^0},\hat{\varphi^1}) + h(\varphi^0,\varphi^1)) - \mathscr{J}((\hat{\varphi^0},\hat{\varphi^1})) &= \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n} + h\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\, dt + <(\hat{\varphi^0},\hat{\varphi^1}) + h(\varphi^0,\varphi^1),(y^0,y^1)> \\ &\quad - \frac{1}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\hat{\varphi}}{\partial n}\right|^2 d\sigma\, dt - <(\hat{\varphi^0},\hat{\varphi^1}),(y^0,y^1)>,\end{aligned}$$

hence,

$$\frac{1}{h}\left[\mathscr{J}((\hat{\varphi^0},\hat{\varphi^1})+h(\varphi^0,\varphi^1))-\mathscr{J}((\hat{\varphi^0},\hat{\varphi^1}))\right]=\frac{h}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma\,dt+\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\,dt$$
$$+<(\varphi^0,\varphi^1),(y^0,y^1)>,$$

and from (1.7) we deduce that

$$\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\varphi}{\partial n}d\sigma\,dt=-<(\varphi^0,\varphi^1),(y^0,y^1)>$$
$$=-\int_\Omega y^0\varphi^1dx+<y^1,\varphi^0>_{-1,1},$$

for every (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$.
From lemma 1.3.2, it follows that $\hat{v}=\frac{\partial\hat{\varphi}}{\partial n}|_{\Gamma_1}$ is a control for (1.1). This complete the proof.
Now we can find the control of the wave equation by minimization of the functional $\mathscr{J}$, moreover, this control is the control of minimal L^2-norm:

Proposition 1.2.2 *Let* $\hat{v}=\frac{\partial\hat{\varphi}}{\partial n}|_{\Gamma_1}$ *be the control given by minimizing the functional* $\mathscr{J}$. *If* $w\in L^2(\Gamma_1\times(0,T))$ *is any other control for* (1.1)*, then*

$$||\hat{v}||_{L^2(\Gamma_1\times(0,T))}\leq||w||_{L^2(\Gamma_1\times(0,T))}. \tag{1.9}$$

Proof:
Let $(\hat{\varphi^0},\hat{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega)$ the minimizer of the functional $\mathscr{J}$ and w is a control function of (1.1).
By taking $(\hat{\varphi^0},\hat{\varphi^1})$ as initial data for (1.3), lemma 1.3.2 gives,

$$\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}\frac{\partial\hat{\varphi}}{\partial n}d\sigma\,dt=-\int_\Omega y^0\hat{\varphi^1}dx+<y^1,\hat{\varphi^0}>_{-1,1},$$

hence,

$$||\hat{v}||^2_{L^2(\Gamma_1\times(0,T))}=-\int_\Omega y^0\hat{\varphi^1}dx+<y^1,\hat{\varphi^0}>_{-1,1}.$$

On the other hand,

$$\int_0^T\int_{\Gamma_1}\frac{\partial\hat{\varphi}}{\partial n}w\,d\sigma\,dt=-\int_\Omega y^0\hat{\varphi^1}dx+<y^1,\hat{\varphi^0}>_{-1,1},$$

then,

$$\begin{aligned} ||\hat{v}||^2_{L^2(\Gamma_1\times(0,T))} &= \int_0^T \int_{\Gamma_1} \hat{v}\, w\, d\sigma\, dt \\ &\leq ||\hat{v}||_{L^2(\Gamma_1\times(0,T))} ||w||_{L^2(\Gamma_1\times(0,T))}. \end{aligned}$$

Consequently, (1.9) is verified and the proof finishes.

Chapter 2

Conjugate Gradient Algorithm

2.1 Variational Problem

In the previous chapter, we transferred the problem of controllability of (1.1) to a problem of minimization:

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \mathscr{J}((\varphi^0,\varphi^1)), \tag{2.1}$$

where the functional $\mathscr{J}$ is defined by (1.6).
Problem (2.1) can be written as follows:

$$\min_{(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)} \left(\frac{1}{2}a((\varphi^0,\varphi^1),(\varphi^0,\varphi^1)) - L((\varphi^0,\varphi^1))\right), \tag{2.2}$$

where a is defined on $(H_0^1(\Omega)\times L^2(\Omega))\times(H_0^1(\Omega)\times L^2(\Omega))$ by

$$a((\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})) = \int_0^T\int_{\Gamma_1}\frac{\partial\varphi}{\partial n}\frac{\partial\widetilde{\varphi}}{\partial n}d\sigma\, dt, \quad \forall\, (\varphi^0,\varphi^1),(\widetilde{\varphi^0},\widetilde{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega),$$

such that φ,$\widetilde{\varphi}$ represent the solutions of (1.3) with initial data (φ^0,φ^1) and $(\widetilde{\varphi^0},\widetilde{\varphi^1})$ respectively. While L is the linear operator defined on $H_0^1(\Omega)\times L^2(\Omega)$ by

$$\begin{aligned} L((\varphi^0,\varphi^1)) &= -<(\varphi^0,\varphi^1),(y^0,y^1)> \\ &= -\int_\Omega y^0\varphi^1 + <y^1,\varphi^0>_{-1,1}, \end{aligned}$$

for all (φ^0,φ^1) in $H_0^1(\Omega)\times L^2(\Omega)$ and (y^0,y^1) in $L^2(\Omega)\times H^{-1}(\Omega)$.

Lemma 2.1.1 ♦ *The operator a is a bilinear form, continuous and $H_0^1(\Omega)\times L^2(\Omega)$-elliptic.*
♦ *For all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, we have $L\in\left(H_0^1(\Omega)\times L^2(\Omega)\right)'$.*

Proof:
• it easy to proof that L is linear continuous and a is bilinear.
• The continuity of a follows from the inequality (1.4) and the coercivity follows from the inequality of observability (1.7).
Moreover the bilinear form a is symmetric.
With these notation, problem (2.2) reads as follows:

$$\begin{cases} \text{Find } (\widehat{\varphi^0},\widehat{\varphi^1})\in H_0^1(\Omega)\times L^2(\Omega) \text{ such that} \\ a((\widehat{\varphi^0},\widehat{\varphi^1}),(\varphi^0,\varphi^1)) = L((\varphi^0,\varphi^1)), \quad \forall(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega). \end{cases} \tag{2.3}$$

which has a unique solution, using Lax-Milgram theorem.

2.2 General conjugate gradient algorithm

Let H be a Hilbert space, a a continuous, symmetric and coercive bilinear form on $H \times H$, and L a continuous linear form on H, the variational problem problem (2.3) is a particular case of the following general variational problem:

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = L(v), \quad \forall v \in H. \end{cases} \tag{2.4}$$

With the above hypotheses, problem (2.4) has a unique solution.
For the numerical solution of (2.4) we recall the conjugate gradient algorithm:
(1) $u^{(0)}$: any arbitrarily vector in H;

(2) solve

$$\begin{cases} \widetilde{u}^{(0)} \in H \\ < \widetilde{u}^{(0)}, v >_H = a(u^{(0)}, v) - L(v), \quad \forall v \in H. \end{cases} \tag{2.5}$$

(3) • If $\widetilde{u}^{(0)}$ is small ($\dfrac{||\widetilde{u}^{(0)}||}{||u^{(0)}||} < \varepsilon$), take $u = u^{(0)}$;
• If not,set $\breve{u}^{(0)} = \widetilde{u}^{(0)}$;

Assuming that $u^{(n)}$, $\widetilde{u}^{(n)}$, $\breve{u}^{(n)}$ are known, compute $u^{(n+1)}$, $\widetilde{u}^{(n+1)}$, $\breve{u}^{(n+1)}$:

(4) $\rho_n = \dfrac{||\widetilde{u}^{(n)}||^2}{a(\breve{u}^{(n)}, \breve{u}^{(n)})}$;

(5) $u^{(n+1)} = u^{(n)} - \rho_n \breve{u}^{(n)}$;

(6) solve

$$\begin{cases} \widetilde{u}^{(n+1)} \in H \\ < \widetilde{u}^{(n+1)}, v >_H = < \widetilde{u}^{(n)}, v >_H - \rho_n a(\breve{u}^{(n)}, v), \quad \forall v \in H. \end{cases} \tag{2.6}$$

(7) • If $\widetilde{u}^{(n+1)}$ is small ($\dfrac{||\widetilde{u}^{(n+1)}||}{||\widetilde{u}^{(0)}||} < \varepsilon$), take $u = u^{(n+1)}$;
• If not,
$\star$ $\gamma_n = \dfrac{||\widetilde{u}^{(n+1)}||^2}{||\widetilde{u}^{(n)}||^2}$;
$\star$ $\breve{u}^{(n+1)} = \widetilde{u}^{(n+1)} + \gamma_n \breve{u}^{(n)}$;
(8) $n = n+1$ and go to (4);
See more about this algorithm and also the convergence in [3] and [5].

2.3 Application of conjugate gradient algorithm

Let us now apply the general conjugate gradient algorithm to the solution of (2.3).
(1) $(\varphi_0^0, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$: Initialization;

(2) solve

$$\begin{cases} (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}) \in H \\ < (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1}), (\varphi^0, \varphi^1) >_H = a((\varphi_0^0, \varphi_0^1), (\varphi^0, \varphi^1)) - L((\varphi^0, \varphi^1)), \quad \forall (\varphi^0, \varphi^1) \in H. \end{cases} \tag{2.7}$$

Consider the following non-homogeneous backward wave equation:

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} - \Delta \psi_0 &= & 0 & & \text{in} \quad \Omega \times (0,T), \\ \psi_0 &= & \dfrac{\partial \varphi_0}{\partial n} & & \text{in} \quad \Gamma_1 \times (0,T), \\ \psi_0 &= & 0 & & \text{in} \quad \Gamma_2 \times (0,T), \\ \psi_0(\cdot, T) &= & 0, & \dfrac{\partial \psi_0}{\partial t}(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases}$$

From the lemma 1.3.2, we have,

$$\int_0^T \int_{\Gamma_1} \frac{\partial \varphi_0}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma \, dt = < \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx,$$

for every $(\varphi^0, \varphi^1) \in H$, then we obtain,

$$\begin{aligned} \int_\Omega \nabla \widetilde{\varphi_0^0} \nabla \varphi^0 dx + \int_\Omega \widetilde{\varphi_0^1} \varphi^1 dx = & < \psi_0'(0), \varphi^0 >_{-1,1} - \int_\Omega \psi_0(0) \varphi^1 dx \\ & + \int_\Omega y^0 \varphi^1 dx - < y^1, \varphi^0 >_{-1,1} . \end{aligned}$$

Hence,

$$< -\Delta \widetilde{\varphi_0^0}, \varphi^0 >_{-1,1} - < \psi_0'(0) - y^1, \varphi^0 >_{-1,1} = \int_\Omega (y^0 - \psi_0(0) - \widetilde{\varphi_0^1}) \varphi^1 dx.$$

Finally,

$$< (\varphi^0, \varphi^1), (y^0 - \psi_0(0) - \widetilde{\varphi_0^1} \, , -\Delta \widetilde{\varphi_0^0} - (\psi_0'(0) - y^1)) >= 0.$$

Its follows: (2)

$$\begin{cases} -\Delta \widetilde{\varphi_0^0} = \psi_0'(0) - y^1, \\ \widetilde{\varphi_0^1} = y^0 - \psi_0(0). \end{cases} \tag{2.8}$$

(3) • If $(\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$ is small , take $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi_0^0, \varphi_0^1)$;

• If not, set $(\check{\varphi}_0^0, \check{\varphi}_0^1) = (\widetilde{\varphi_0^0}, \widetilde{\varphi_0^1})$.

Assuming that $(\varphi_n^0, \varphi_n^1)$, $(\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1})$, $(\check{\varphi}_n^0, \check{\varphi}_n^1)$ and φ_n, ψ_n are known, compute $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$, $(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1})$, φ_{n+1}, ψ_{n+1}.
we knew that the form linear $(\varphi^0, \varphi^1) \in H \longmapsto a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1))$ is continuous, then by Riesz's theorem there exists unique $(\underline{\varphi_n^0}, \underline{\varphi_n^1})$ in H, such that

$$a((\check{\varphi}_n^0, \check{\varphi}_n^1), (\varphi^0, \varphi^1)) = < (\underline{\varphi_n^0}, \underline{\varphi_n^1}), (\varphi^0, \varphi^1) >, \quad \forall (\varphi^0, \varphi^1) \in H.$$

Like the previous case, we can find $(\underline{\varphi_n^0}, \underline{\varphi_n^1})$ by :

(4) solve

$$\begin{cases} \dfrac{\partial^2 \check{\varphi}_n}{\partial t^2} - \Delta \check{\varphi}_n = 0 & \text{in} \quad \Omega \times (0,T), \\ \check{\varphi}_n = 0 & \text{in} \quad \Gamma \times (0,T), \\ \check{\varphi}_n(\cdot,0) = \check{\varphi}_n^0, \quad \dfrac{\partial \check{\varphi}_n}{\partial t}(\cdot,0) = \check{\varphi}_n^1 & \text{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n = 0 & \text{in} \quad \Omega \times (0,T), \\ \check{\psi}_n = \dfrac{\partial \check{\varphi}_n}{\partial n} & \text{in} \quad \Gamma_1 \times (0,T), \\ \check{\psi}_n = 0 & \text{in} \quad \Gamma_2 \times (0,T), \\ \check{\psi}_n(\cdot,T) = 0, \quad \dfrac{\partial \check{\psi}_n}{\partial t}(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

Compute now $(\underline{\varphi_n^0}, \underline{\varphi_n^1}) \in H$ by :

$$\begin{cases} -\Delta \underline{\varphi_n^0} = \check{\psi}_n{}'(0) \quad \text{in } \Omega \\ \\ \underline{\varphi_n^1} = -\check{\psi}_n(0). \end{cases} \tag{2.9}$$

The other steps of the general algorithm are easy to adapt. Now we give the complete algorithm to solve the system (2.3):
(1) $(\varphi_0^1, \varphi_0^1) \in H_0^1(\Omega) \times L^2(\Omega) = H$ **are given;**

(2) **solve then**

$$\begin{cases} \dfrac{\partial^2\varphi_0}{\partial t^2}-\Delta\varphi_0 = 0 & \textbf{in} \quad \Omega\times(0,T), \\ \varphi_0 = 0 & \textbf{in} \quad \Gamma\times(0,T), \\ \varphi_0(\cdot,0) = \varphi_0^0, \quad \dfrac{\partial\varphi_0}{\partial t}(\cdot,0)=\varphi_0^1 & \textbf{in} \quad \Omega, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2\psi_0}{\partial t^2}-\Delta\psi_0 = 0 & \textbf{in} \quad \Omega\times(0,T), \\ \psi_0 = \dfrac{\partial\varphi_0}{\partial n} & \textbf{in} \quad \Gamma_1\times(0,T), \\ \psi_0 = 0 & \textbf{in} \quad \Gamma_2\times(0,T), \\ \psi_0(\cdot,T) = 0, \quad \dfrac{\partial\psi_0}{\partial t}(\cdot,T)=0 & \textbf{in} \quad \Omega. \end{cases}$$

(3) **Compute** $(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})\in H$ **by**

$$\begin{cases} -\Delta\widetilde{\varphi_0^0}=\psi_0'(0)-y^1, \\ \widetilde{\varphi_0^0}=0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\widetilde{\varphi_0^1}=y^0-\psi_0(0).$$

(4) • **If** $(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})$ **is small , take** $(\widehat{\varphi^0},\widehat{\varphi^1})=(\varphi_0^0,\varphi_0^1)$**;**

• **If not, set** $(\check{\varphi}_0^0,\check{\varphi}_0^1)=(\widetilde{\varphi_0^0},\widetilde{\varphi_0^1})$**.**

Assuming that $(\varphi_n^0,\varphi_n^1)$**,** $(\widetilde{\varphi_n^0},\widetilde{\varphi_n^1})$**,** $(\check{\varphi}_n^0,\check{\varphi}_n^1)$ **and** φ_n**,** ψ_n **are known, compute** $(\varphi_{n+1}^0,\varphi_{n+1}^1)$**,** $(\widetilde{\varphi^0}_{n+1},\widetilde{\varphi^1}_{n+1})$**,** $(\check{\varphi}^0{}_{n+1},\check{\varphi}^1{}_{n+1})$**,** φ_{n+1}**,** ψ_{n+1}**.**

<u>*Descent*</u>**:**

(5) **Solve**

$$\begin{cases} \dfrac{\partial^2\check{\varphi}_n}{\partial t^2}-\Delta\check{\varphi}_n = 0 & \textbf{in} \quad \Omega\times(0,T), \\ \check{\varphi}_n = 0 & \textbf{in} \quad \Gamma\times(0,T), \\ \check{\varphi}_n(\cdot,0) = \check{\varphi}_n^0, \quad \dfrac{\partial\check{\varphi}_n}{\partial t}(\cdot,0)=\check{\varphi}_n^1 & \textbf{in} \quad \Omega \quad , \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial^2 \check{\psi}_n}{\partial t^2} - \Delta \check{\psi}_n = 0 & \textbf{in} \quad \Omega \times (0,T), \\ \check{\psi}_n = \dfrac{\partial \check{\varphi}_n}{\partial n} & \textbf{in} \quad \Gamma_1 \times (0,T), \\ \check{\psi}_n = 0 & \textbf{in} \quad \Gamma_2 \times (0,T), \\ \check{\psi}_n(\cdot,T) = 0, \quad \dfrac{\partial \check{\psi}_n}{\partial t}(\cdot,T) = 0 & \textbf{in} \quad \Omega. \end{cases}$$

(6) **Compute** $(\underline{\varphi_n^0}, \underline{\varphi_n^1}) \in H$ **by : solve**

$$\begin{cases} -\Delta \underline{\varphi_n^0} = \check{\psi}_n{}'(0) \quad \textbf{in } \Omega, \\ \underline{\varphi_n^0} = 0 \quad \textbf{in} \quad \Gamma, \end{cases}$$

and

$$\underline{\varphi_n^1} = -\check{\psi}_n(0).$$

(7) **Compute** ρ_n **by:**

$$\begin{cases} \rho_n = \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{a((\check{\varphi}_n^0, \check{\varphi}_n^1),(\check{\varphi}_n^0, \check{\varphi}_n^1))}, \\ \quad = \dfrac{||((\widetilde{\varphi_n^0}, \widetilde{\varphi_n^1}))||^2}{< \check{\psi}_n{}'(0), \check{\varphi}_n^0 >_{-1,1} - \displaystyle\int_\Omega \check{\psi}_n(0)\check{\varphi}_n^1 dx}, \\ \quad = \dfrac{\displaystyle\int_\Omega ||\nabla \widetilde{\varphi_n^0}||^2 + \int_\Omega ||\widetilde{\varphi_n^1}||^2}{\displaystyle\int_\Omega \nabla \underline{\varphi_n^0} \nabla \check{\varphi}_n^0 dx + \int_\Omega \underline{\varphi_n^1} \check{\varphi}_n^1 dx}. \end{cases}$$

(8) **Once** ρ_n **is known, compute:**

$\oplus$ $(\varphi_{n+1}^0, \varphi_{n+1}^1) = (\varphi_n^0, \varphi_n^1) - \rho_n(\check{\varphi}_n^0, \check{\varphi}_n^1)$,

$\oplus$ $\varphi_{n+1} = \varphi_n - \rho_n \check{\varphi}_n$,

$\oplus$ $\psi_{n+1} = \psi_n - \rho_n \check{\psi}_n$,

$\oplus$ $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) = (\widetilde{\varphi^0}_n, \widetilde{\varphi^1}_n) - \rho_n(\underline{\varphi_n^0}, \underline{\varphi_n^1})$.

Test of the convergence **and** **construction of the new descent direction.**

(9) **If** $(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})$ **is small , take** $(\widehat{\varphi^0}, \widehat{\varphi^1}) = (\varphi^0_{n+1}, \varphi^1_{n+1})$.
If not, compute

$$\gamma_n = \frac{\displaystyle\int_\Omega ||\nabla \widetilde{\varphi^0}_{n+1}||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_{n+1}||^2 dx}{\displaystyle\int_\Omega ||\nabla \widetilde{\varphi^0}_{n}||^2 dx + \int_\Omega ||\widetilde{\varphi^1}_{n}||^2 dx},$$

and set

$$(\check{\varphi}^0{}_{n+1}, \check{\varphi}^1{}_{n+1}) = (\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1}) + \gamma_n (\check{\varphi}^0{}_{n}, \check{\varphi}^1{}_{n}).$$

(10) $n = n+1$ **and go to** (5)**.**

Chapter 3

Numerical approximation with the finite difference methods (1D Case)

In this chapter we will apply the finite difference methods to control system (1.1). At the present moment, we shall focus on the case where:

$$\Omega = (0,1), \quad \Gamma_1 = \{1\} \quad \text{and} \quad \Gamma_2 = \{0\}.$$

3.1 Finite difference approximation of the wave equation

In the algorithm at each iteration we need to solve either a homogeneous or a non-homogeneous back-ward wave equation, for this we introduce the general wave system

$$\left\{\begin{array}{rcll} \dfrac{\partial^2\varphi}{\partial t^2} - \dfrac{\partial^2\varphi}{\partial x^2} & = & 0 & \text{in} \quad (0,1)\times(0,T), \\ \varphi(0,\cdot) & = & 0 & \text{in} \quad (0,T), \\ \varphi(1,\cdot) & = & u(\cdot) & \text{in} \quad \times(0,T), \\ \varphi(\cdot,0) & = & v(\cdot), & \dfrac{\partial\varphi}{\partial t}(\cdot,0) = w(\cdot) \quad \text{in} \quad (0,1), \end{array}\right. \tag{3.1}$$

where $(v,w) \in H_0^1(0,1)\times L^2(0,1)$.
We propose to approach the solution of the system (3.1) by an explicit scheme of finite difference, for this purpose, we have considered a uniform mesh defined by:

$$t_0 = 0 < t_1 = k < \cdots < t_n = nk < \cdots < t_{M+1} = T,$$

and

$$x_0 = 0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

with $k = \dfrac{T}{M+1}$ is the time step and $h = \dfrac{1}{N+1}$ is the space step.
Let φ_j^n be the approximation of $\varphi(x_j,t_n)$, using a central finite differences to approximate the second-order derivative, we obtain :

$$\left\{\begin{array}{l} \dfrac{\varphi_j^{n+1} - 2\varphi_j^n + \varphi_j^{n-1}}{k^2} = \dfrac{\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n}{h^2}, \quad 1 \le j \le N, \quad 1 \le n \le M, \\ \varphi_0^n = 0, \\ \varphi_{N+1}^n = u(t_n) = u^n, \\ \varphi_j^0 = v(x_j) = v_j, \\ \varphi_j^1 = \varphi_j^0 + k w_j. \end{array}\right. \tag{3.2}$$

We can prove that the scheme(3.2) is stable if and only if $r = \dfrac{k^2}{h^2} \leq 1$. From (3.2), we have :

$$\varphi^{n+1} = A\,\varphi^n - \varphi^{n-1} + b,$$

where

$$A = \begin{pmatrix} 2(1-r) & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & r & 2(1-r) \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ \vdots \\ ru^n \end{pmatrix}$$

and φ^n the vector of $\mathbb{R}^N$ formed by $\varphi_j^n,\ j \in \{1,\ldots,N\}$.

With this discretization we can define a function (in python) which makes it possible to solve the two types of system, for the homogeneous wave equation we take $u \equiv 0$ but for the non-homogeneous back-ward system we have to make a transformation in time.

3.2 Finite difference approximation of the Dirichlet problem

As in the previous section, we need to solve a Dirichlet system on each iteration. We define the general system as follows:

$$\begin{cases} -\dfrac{\partial^2\phi}{\partial x^2} = \psi^{'}(0) - y \quad \text{in} \quad (0,1), \\ \phi(0) = 0, \\ \phi(1) = 0. \end{cases} \tag{3.3}$$

We approximate the second-order derivative by:

$$\frac{\partial^2\phi}{\partial x^2}(x_j) \simeq \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2}, \quad 1 \leq j \leq N,$$

and for the second member, we have

$$\psi^{'}(0) = \frac{\psi_j^1 - \psi_j^0}{k}, \quad 0 \leq j \leq N+1.$$

Then the system (3.3) is approximated as follows

$$\begin{cases} -\dfrac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{h^2} = \dfrac{\psi_j^1-\psi_j^0}{k} - y_j \quad 1 \le j \le N, \\ \phi(0) = 0, \\ \phi(1) = 0. \end{cases}$$

From this scheme, we obtain

$$B\phi = -\frac{h^2}{k}(\psi^1-\psi^0) + h^2 Y,$$

with ϕ the vector formed by $\phi_j,\ j \in \{1,\dots,N\}$ and

$$B = \begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix},$$

$$\psi^k = \begin{pmatrix} \psi_1^k \\ \vdots \\ \psi_N^k \end{pmatrix}; k \in \{0,1\}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

3.3 Numerical Tests

In this section we present some numerical simulations to control our system, using the general conjugate gradient algorithm and the finite difference discretization method.
For the numerical tests we consider 3 types of initial conditions, and we will discuss the convergence of the algorithm as well as the controllability of our discrete system.
The conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^0 = 0$, and

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1\times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1\times L^2}} < \varepsilon,$$

has been used as stopping criterium, with ε is a small positive number.
Let's mention also that $||\varphi||_{H^{-1}(0,1)} = ||\psi||_{H_0^1(0,1)}$, where $\psi \in H_0^1(0,1)$ is the solution of the Dirichlet problem:

$$\begin{cases} -\Delta\psi = \varphi & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

and $||\psi||_{H_0^1(0,1)} = \left(\int_0^1 \left(\frac{\partial\psi}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}$. For the approximation of the integrals we used the rectangles method and for the stability of the numerical scheme, we take $r = 1$ (The CFL condition).
Firstly, we consider the following initial conditions associated to the wave equation (1.1):

$$\begin{cases} y^0(x) = \exp(-5(x-0.37)^6), \\ y^1(x) = 0. \end{cases} \tag{3.4}$$

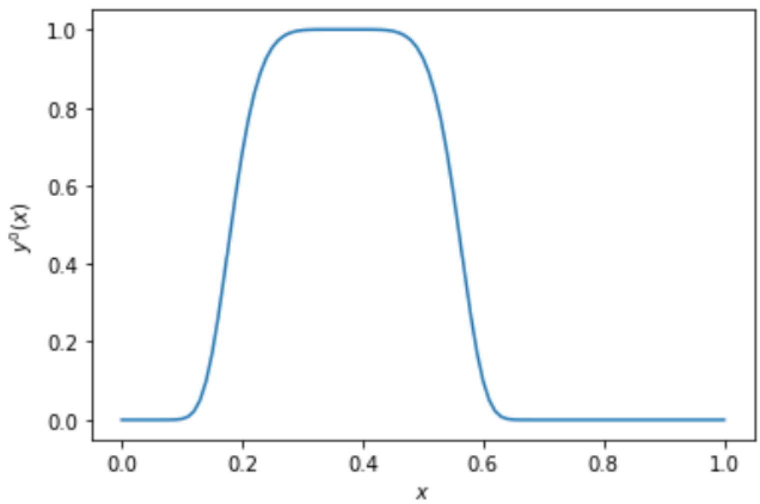

Figure 3.1: The initial position y^0.

N	19	99	499	999
Number of CG iterations	1600	27	7	6
$\|\|y(T)\|\|_{L^2(0,1)}$	9.2857×10^{-11}	1.04928×10^{-10}	1.30932×10^{-11}	3.06843×10^{-12}
$\|\|y'(T)\|\|_{H^{-1}(0,1)}$	5.1037×10^{-11}	4.05311×10^{-11}	1.16793×10^{-11}	2.94205×10^{-12}
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.41628	0.40681	0.40658	0.40657

Table 3.1: Numerical results obtained for different values of $h = \dfrac{1}{N+1}$. $T = 2.2$ and $\varepsilon = 10^{-10}$.

For further study of this exemple (3.4), we fix $N = 99$.

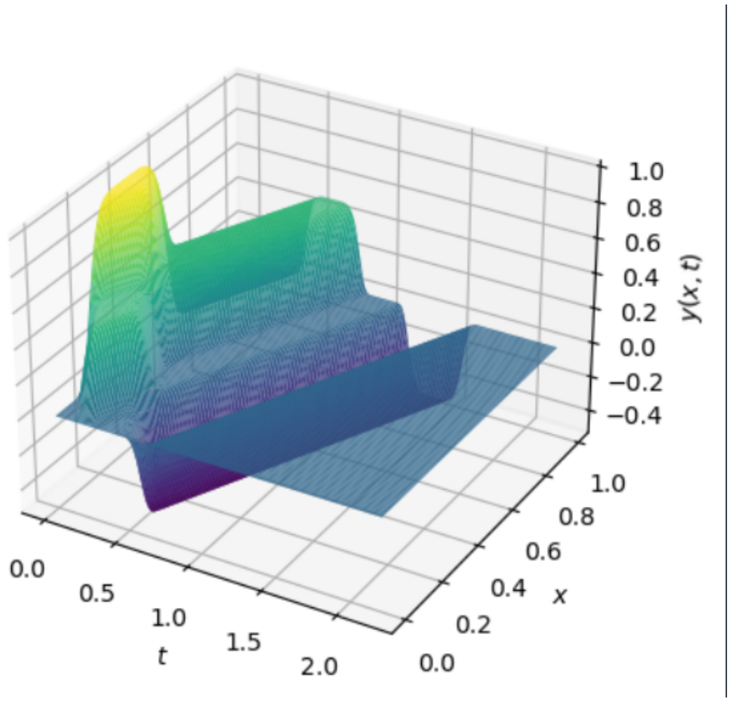

Figure 3.2: The controlled solution y.

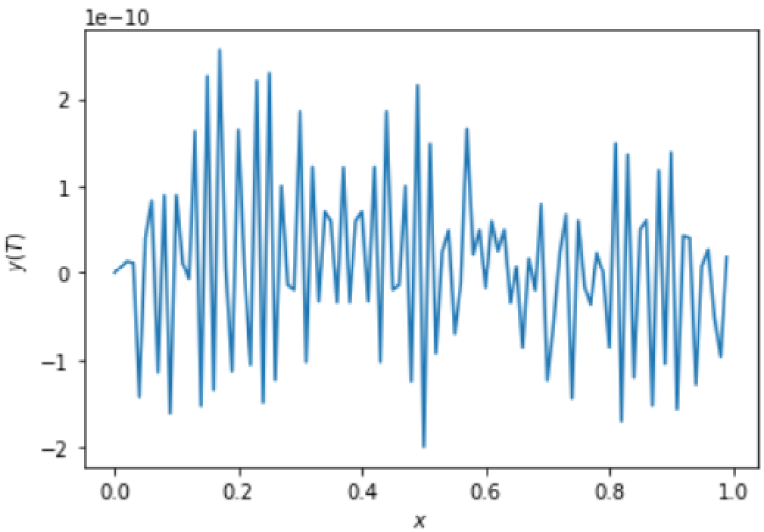

Figure 3.3: The final position $y(T)$.

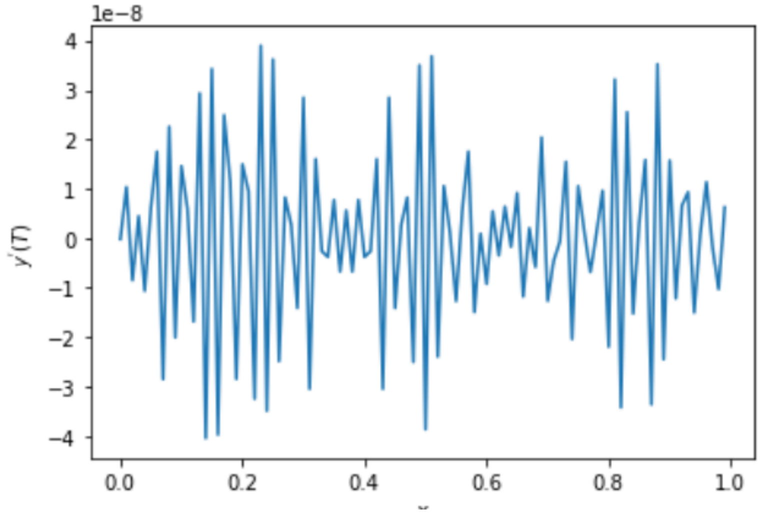

Figure 3.4: The final velocity $y^{'}(T)$.

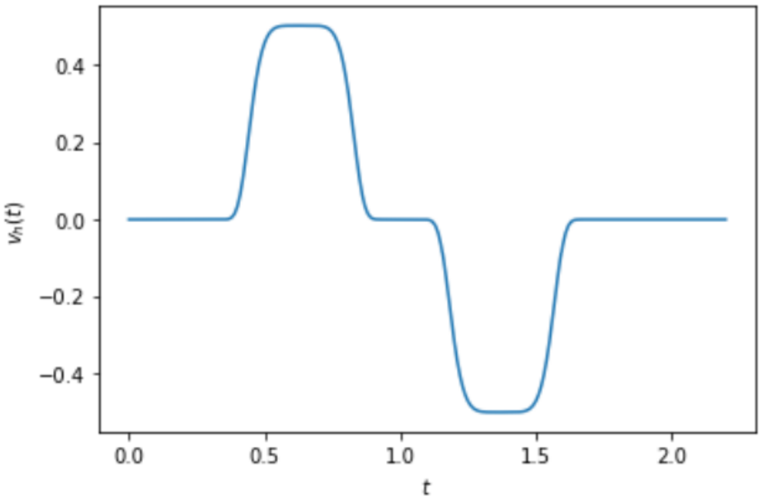

Figure 3.5: The approximation of the control.

Table 3.1 shows that when N gets bigger the number of conjugate gradient iterations decreases, and also we notice that the norm of $y(T)$ in $L^2(0,1)$ and $\dfrac{\partial y}{\partial t}(T)$ in $H^{-1}(0,1)$ converge to 0.
Figures 5.1-3.4 show the controlled solution y_h and the final position, also the final velocity for $T = 2.2$, $N = 99$ and $\varepsilon = 10^{-10}$. We remark that the discrete system is controllable, in the figure 3.5 we plot the approximation of the control. Then we can conclude that for the initial conditions (3.4), the control is well approximated.

Now, we consider the following initial conditions:

$$\begin{cases} y^0(x) = sin(\pi x), \\ y^1(x) = \sqrt{2}\, cos(\pi x). \end{cases} \tag{3.5}$$

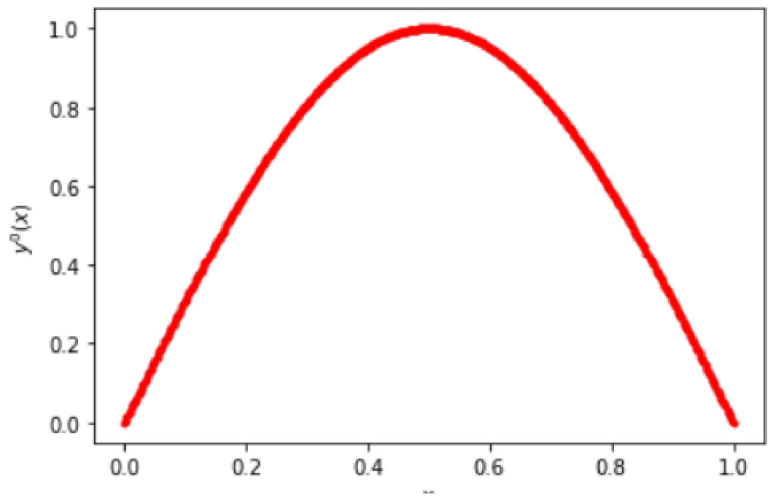

Figure 3.6: The initial position y^0.

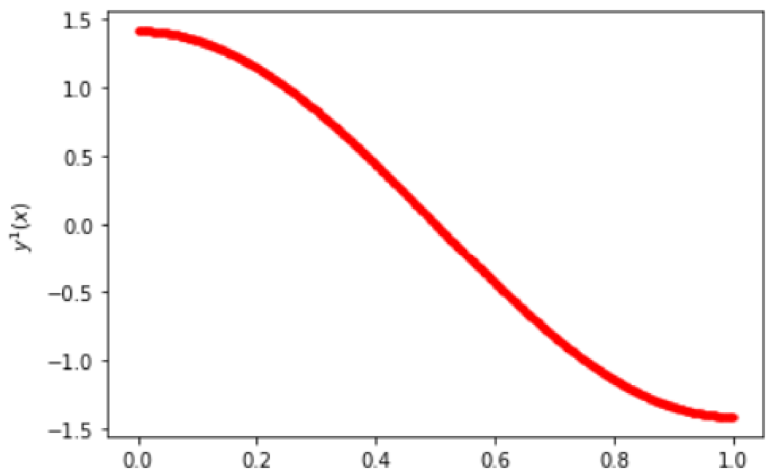

Figure 3.7: The initial velocity y^1.

N	99	499
Number of CG iterations	2905	5003
$\|\|y(T)\|\|_{L^2(0,1)}$	$1.2493344460609922\times 10^{-10}$	$1.041012255663964\times 10^{-9}$
$\|\|y'(T)\|\|_{H^{-1}(0,1)}$	$6.294080645627806\times 10^{-11}$	$5.954733460881318\times 10^{-11}$
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.5048527172527582	0.5041223465788962

Table 3.2: Numerical results obtained for different values of $h = \dfrac{1}{N+1}$. $T = 2.2$ and $\varepsilon = 10^{-10}$.

For $N = 99$, we obtain:

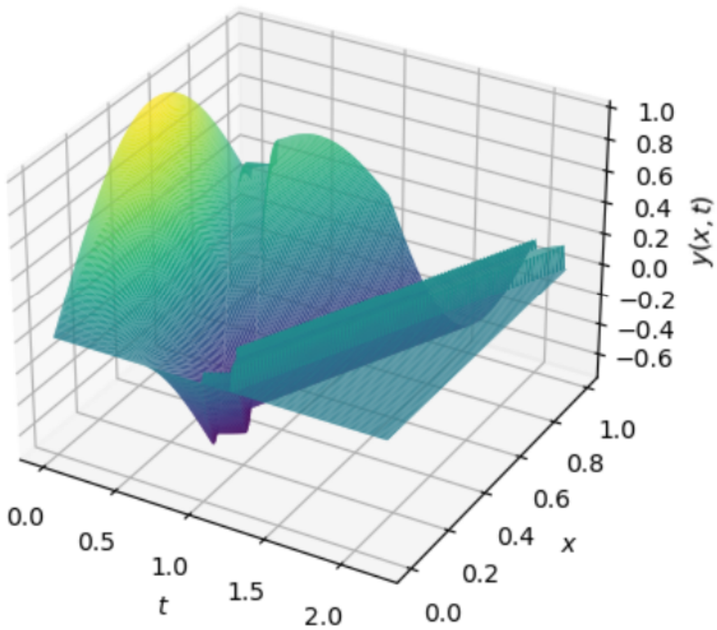

Figure 3.8: The controlled solution y.

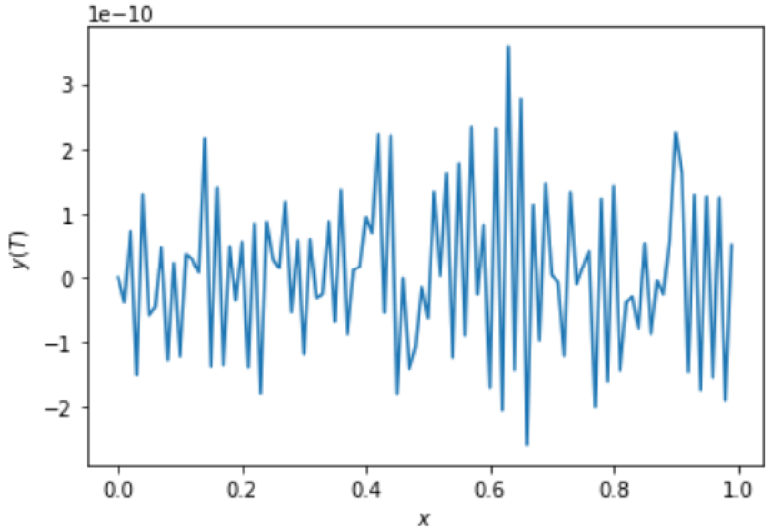

Figure 3.9: The final position $y(T)$.

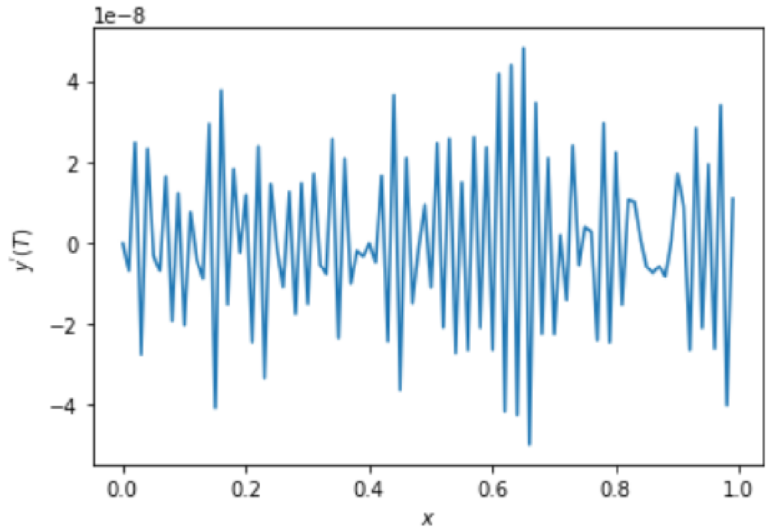

Figure 3.10: The final velocity $y^{'}(T)$.

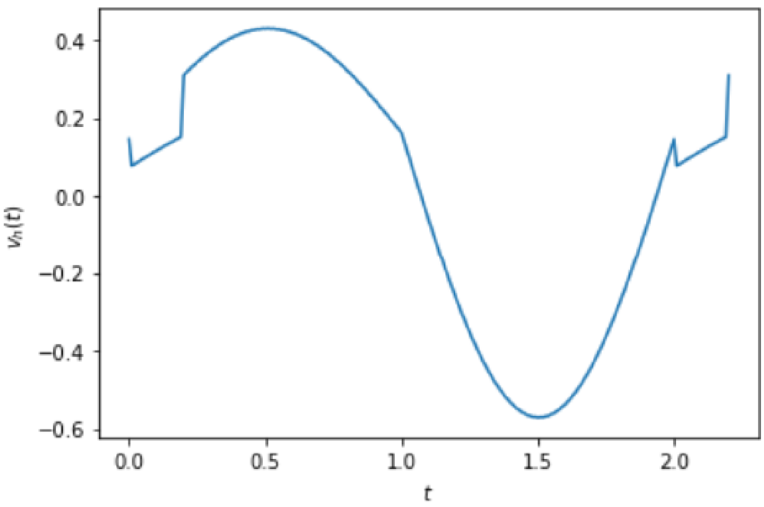

Figure 3.11: The approximation of the control.

From table 4.2 and figures 3.8-3.10, we remark that, the number of iterations for the convergence of the algorithm with the initial conditions (3.5) is greater compared to (3.4), although the system is controllable.
The numerical tests, show that, for $r < 1$ the algorithm does not converge. The problem of the number of iterations or the non convergence is related to the high frequencies. To solve this problem we filter the initial data, this means that, we are approaching the initial conditions without losing the information and also filtering the high frequencies. For more details (see [17]).
In the last example (3.6), we also have the problem of high frequencies and with numerical simulations, we remark that, for $r < 1$ we can't control the system and the algorithm diverges.
With $r = 1$, we have the controllability, but the number of iterations is enormous in the case where N is small (≤ 100) and the convergence time with python is large in the case where N is large. The problem of CPU time can be solved by using Pyccel, Numba or MPI.
finally, we conclude that for $r = 1$, we can control the $1D$ wave equation system using the finite difference method whatever the initial conditions, but the choice of $r = 1$ is not practical, so in this case we need filtration.

$$y^0(x) = \begin{cases} 0, \text{ in } \left[0, \dfrac{1}{3}\right[, \\ 1, \text{ in } \left[\dfrac{1}{3}, \dfrac{2}{3}\right[, \\ 0, \text{ in } \left[\dfrac{2}{3}, 1\right]. \end{cases} \tag{3.6}$$
$$y^1(x) = 0.$$

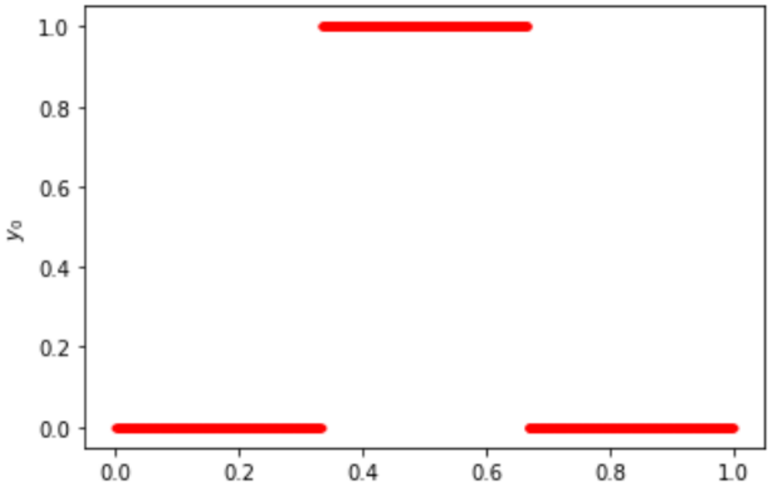

Figure 3.12: The initial position y^0.

We fix $N = 99$, $\varepsilon = 10^{-10}$, $r = 1$ and $T = 2.2$, we obtain:
The number of conjugate gradient iterations : 4586.
$||y(T)||_{L^2(0,1)} = 2.21816584783467 \times 10^{-10}$.
$||y^{'}(T)||_{H^{-1}(0,1)} = 4.91396980603545 \times 10^{-11}$.
$||\widehat{v}||_{L^2(0,T)} = 0.69123306$.

In the following figures, we plot the controlled solution ,the final behavior of our system and the control function .

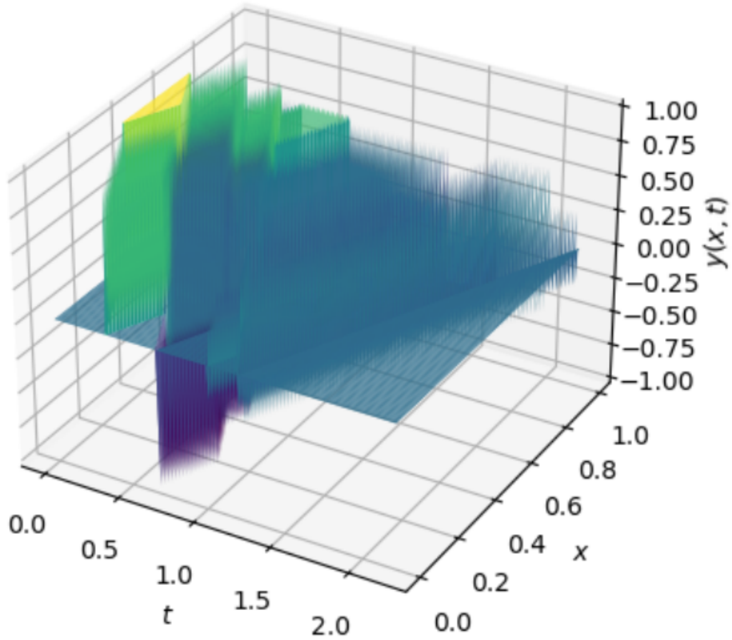

Figure 3.13: The controlled solution y.

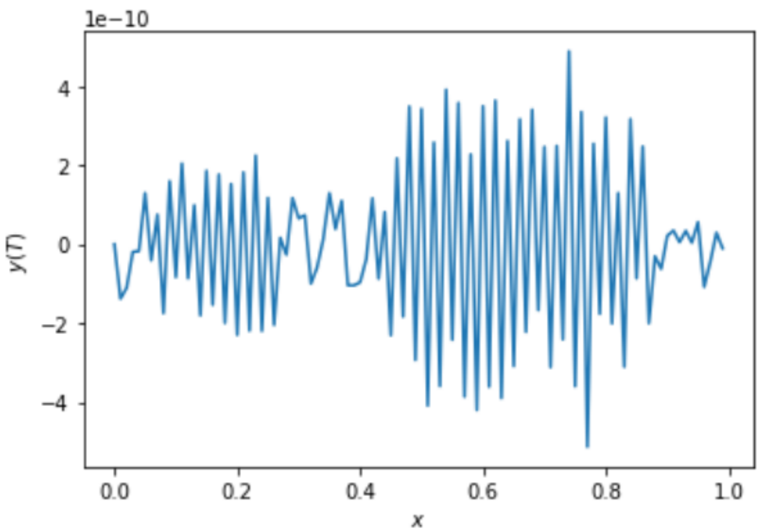

Figure 3.14: The final position $y(T)$.

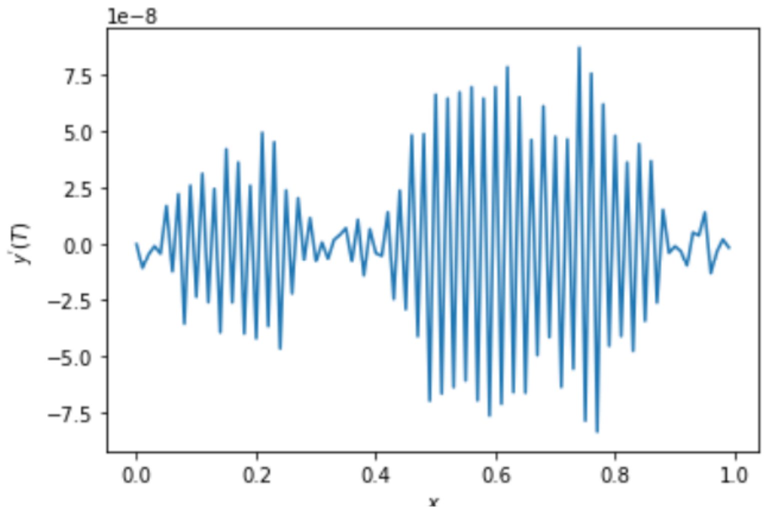

Figure 3.15: The final velocity $y^{'}(T)$.

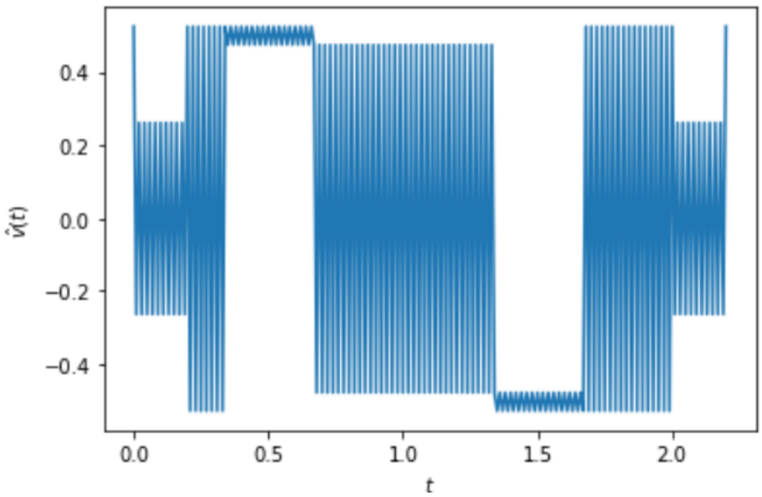

Figure 3.16: The approximation of the control.

CHAPTER 4

BOUNDARY OBSERVABILITY: CONVERGENCE ANALYSIS

From The numerical tests we can see that there are high frequencies in the final solution also in the control function, and that it influences the convergence of the CG algorithm and also the number of iterations. R.GLowinski has already observed this phenomenon, see [17], [6] and [5].
In this chapter we will see this problem mathematically.

4.1 Observability and controllability

With the notation of the previous chapter, we introduce the finite difference semi-discretizations of the $1D$ wave system 1.1:

$$\begin{cases} \dfrac{\partial^2 y_j(t)}{\partial t^2} = \dfrac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2}, & 1 \le j \le N, \\ y_0(t) = 0, & 1 \le t \le T, \\ y_{N+1}(t) = v(t), & 1 \le t \le T, \\ y_j(0) = y_j^0, & 0 \le j \le N+1, \\ \dfrac{\partial y_j}{\partial t}(0) = y_j^1, & 0 \le j \le N+1. \end{cases} \tag{4.1}$$

Let $Y_h(t) = \begin{pmatrix} y(t) \\ y^{'}(t) \end{pmatrix}$, then 4.1 is equivalent to :

$$\begin{cases} Y_h^{'}(t) + \mathscr{A}_h Y_h(t) = \mathscr{B}_h v(t), \\ Y_h(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in \mathbb{R}^{2N}, \end{cases} \tag{4.2}$$

such as

$$\mathscr{A}_h = \begin{pmatrix} O & I \\ A_h & O \end{pmatrix}, \text{ with } \quad A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix},$$

and

$$\mathscr{B}_h = \begin{pmatrix} 0 \\ \vdots \\ B_h \end{pmatrix}, \text{ such that } B_h = \begin{pmatrix} 0 \\ \vdots \\ \dfrac{1}{h^2} \end{pmatrix} \in \mathbb{R}^N.$$

By the Kalman criterion, we have that: $[\mathscr{A}_h, \mathscr{B}_h, -]$ is controllable if and only if, $rang([\mathscr{A}_h : \mathscr{B}_h] = 2N$.
On the other hand,

$$rang([\mathscr{A}_h : \mathscr{B}_h] = range([\mathscr{B}_h \ \mathscr{A}_h\mathscr{B}_h \ldots \mathscr{A}_h^{2N-1}\mathscr{B}_h]).$$

It's easy to check that:

$$\mathscr{A}_h^{2n} = \begin{pmatrix} A_h^n & O \\ O & A_h^n \end{pmatrix}, \text{ and } \quad \mathscr{A}_h^{2n-1} = \begin{pmatrix} O & A_h^{n-1} \\ A_h^n & O \end{pmatrix},$$

for all n in $\{1,\ldots,N\}$.
Thus,

$$\begin{aligned} rang([\mathscr{A}_h : \mathscr{B}_h] &= rang \begin{bmatrix} O & B_h & 0 & \ldots & O & A_h^{N-1}B_h \\ B_h & 0 & A_hB_h & \ldots & A_h^{N-1}B_h & O \end{bmatrix} \\ &= 2\, rang \begin{bmatrix} B_h & A_hB_h & \ldots & A_h^{N-1}B_h \end{bmatrix} \\ &= 2\, rang[A_h : B_h]. \end{aligned}$$

Note that, the matrix A_h is tridiagonal and $B_h = \dfrac{1}{h^2}e_N$, with this remark we can show that, $rang([A_h : B_h]) = N$.
Finally, the system 4.2 is controllable, that is, there exists a $v_h \in L^2(0,1)$, such that $y_h(T) = y^{'}(T) = 0$. Moreover the existence of v_h is independent of T, now the question is whether the sequence $(v_h)_h$ converges to the control function of 1.1, but from our numerical tests, the convergence of the numerical controls is not verified for all initial data, (see [14], theorem 1.1).

We start the study of the controllability properties of 1.1 by the duality between controllability and observability.
considering the following adjoint system:

$$\begin{cases} \dfrac{\partial^2 u_j(t)}{\partial t^2} = \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2}, & 1 \le j \le N, \\ u_0(t) = 0, & 1 \le t \le T, \\ u_{N+1}(t) = 0, & 1 \le t \le T, \\ u_j(0) = u_j^0, & 0 \le j \le N+1, \\ \dfrac{\partial u_j}{\partial t}(0) = u_j^1, & 0 \le j \le N+1. \end{cases} \tag{4.3}$$

Let $U_h(t) = \begin{pmatrix} u_h^{'}(t) \\ u_h(t) \end{pmatrix}$, we can show that 4.3 is equivalent to :

$$\begin{cases} U_h^{'}(t) + \mathscr{A}_h^* U_h(t) = 0, \\ U_h(0) = \begin{pmatrix} u_h^1 \\ u_h^0 \end{pmatrix} \in \mathbb{R}^{2N}. \end{cases} \tag{4.4}$$

The energy of system 4.4, is given by:

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[|u_j^{'}(t)|^2 + \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2 \right], \forall t \in [0,T],$$

which is a discretization with finite difference of the continuous energy E defined by :

$$E(t) = \frac{1}{2} \int_0^1 \left[\left| \frac{\partial u}{\partial t}(x,t) \right|^2 + \left| \frac{\partial u}{\partial x}(x,t) \right|^2 \right] dx.$$

As for the continuous energy, E_h is conserved along time, that is :

$$E_h(t) = E_h(0), \quad \forall t \in (0,T). \tag{4.5}$$

Indeed, multiplying 4.3 by $u_j^{'}$ and adding for $j = 1, \ldots, N$, we obtain :

$$\sum_{j=1}^{N} u_j^{'} u_j^{''} = \frac{1}{h^2} \sum_{j=1}^{N} (u_{j+1} - 2u_j + u_{j-1}) u_j^{'}.$$

In addition,

$$\sum_{j=1}^{N} u_j^{'} u_j^{''} = \frac{1}{2} \frac{d}{dt} \sum_{j=1}^{N} |u_j^{'}|^2.$$

On the other hand,

$$-\frac{1}{h^2} \sum_{j=1}^{N} (u_{j+1} - 2u_j + u_{j-1}) u_j^{'} = \frac{1}{2h^2} \frac{d}{dt} \sum_{j=0}^{N} |u_j - u_{j+1}|^2,$$

finally, we get :

$$\frac{d}{dt} \sum_{j=0}^{N} \left[|u_j^{'}|^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right] = 0.$$

We can also prove the result by derivation of $E_h(t)$, [see [4], page 9].
We have already shown that the system 4.1 is controllable, we add the duality between controllability

and observability, we obtain:

$$\exists C_0(T,h) > 0,\ \forall U_h \text{ solution of } 4.4:\ ||U_h(0)||_2^2 \leq C_0(T,h)\int_0^T ||\mathscr{B}_h^* U_h(t)||_2^2 dt.$$

let $||\cdot||_h$ the norm of $\mathbb{R}^{2N}$ defined by:

$$\forall w \in \mathbb{R}^{2N},\ ||w||_h^2 = \sum_{j=1}^{N} |w_j|^2 + \sum_{j=N}^{2N} \left|\frac{w_{j+1} - w_j}{h}\right|^2,$$

then, there exists $C_h \in \mathbb{R}^{*+}$, such as:

$$\forall w \in \mathbb{R}^{2N},\ \ ||w||_h^2 \leq C_h ||w||_2^2,$$

hence,

$$\frac{1}{C_h}||U_h(0)||_h^2 \leq C_0(T,h)\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt,$$

then,

$$E_h(0) \leq \frac{C_0(T,h)C_h}{h}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt.$$

Let $C(T,h) = \dfrac{C_0(T,h)C_h}{h}$, finally we obtain :

$$E_h(0) \leq C(T,h)\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt. \tag{4.6}$$

this inequality 4.6, defines the observability of the system 4.4 and also represents the discrete case of the observability inequality in the continuous case.

In what follows, we will analyze the inequality 4.4.

4.2 Spectral analysis

Our goal is to see if we can find a T^* such that, for any $T > T^*$, $C(T,h)$ is independent of h and the inequality 4.6 holds for every solution of system 4.3, which amounts to saying that the inequality 4.6 is uniform.

First of all, we will start by analyzing the eigenvalues associates with the discrete system:

$$\begin{cases} -\dfrac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} = \lambda\varphi_j, \quad 1 \le j \le N, \\ \\ \varphi_0 = \varphi_{N+1} = 0, \end{cases} \tag{4.7}$$

which is equivalent to analyzing the spectrum of A_h.
Let V be a eigenvector of A_h associated to the eigenvalue λ, that is , $A_hV = \lambda V$, such that $V = (V_j)_{1\le j\le N}$, we add the condition $V_0 = V_{N+1} = 0$, we get:

$$-V_{j+1} + (2-\lambda h^2)V_j - V_{j-1} = 0, \quad \forall j \in \{1,\dots,N\}.$$

This is a three-level recurrence relation, then we search for solutions in the form $V_j = V^j$, with this formula, we find the following equation:

$$r^2 + (2-\lambda h^2)r + 1 = 0,$$

we assume that , this equation admits two roots, noted r_1, r_2, then :

$$r_1 + r_2 = 2-\lambda h^2, \;\; r_1r_2 = 1.$$

Therefore, there are two components α_1 and α_2 such that:

$$V_j = \alpha_1 r_1^j + \alpha_2 r_2^j.$$

We have, $V_0 = 0$ and $V_{N+1} = 0$, then

$$\alpha_1 = -\alpha_2, \;\; \alpha_1(r_1^{N+1} - r_2^{N+1}) = 0,$$

hence,

$$V_j = \alpha_1(r_1^j - r_2^j), \;\; \alpha_1(r_1^{N+1} - r_2^{N+1}) = 0,$$

which implies,

$$\alpha_1 \neq 0, \text{ and } \left(\frac{r_1}{r_2}\right)^{N+1} = 1.$$

on the other hand $r_1 = \dfrac{1}{r_2}$,which gives

$$r_{1,k}^2 = \exp\left(\frac{2ik\pi}{N+1}\right), \text{ for } k \in \{0,\dots,N\}.$$

Now, fix $k \in \{0,\dots,N\}$, we obtain:

$$\begin{aligned}\lambda_k &= \frac{1}{h^2}(2-(r_{1,k}+r_{2,k}))\\ &= \frac{1}{h^2}\left(2-2\cos\left(\frac{\pi k}{N+1}\right)\right)\\ &= \frac{1}{h^2}\,4\sin^2\left(\frac{k\pi}{2(N+1)}\right),\end{aligned}$$

and,

$$\begin{aligned}V_j^k &= \alpha_1(r_{1,k}^j+r_{2,k}^j)\\ &= \alpha_1\left(2i\ \sin\left(j\frac{k\pi}{N+1}\right)\right).\end{aligned}$$

For $k=0$, we have $\lambda_k=0$, which gives $V_j=0$, for all $j\in\{1,\dots,N\}$.
Finally, the eigenvalues and eigenvectors of 4.7 written as follows:

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2(N+1)}\right),\ \ k\in\{1,\dots,N\},$$

and

$$\varphi_j^k = \sin\left(j\frac{k\pi}{N+1}\right),\ \ k,j\in\{1,\dots,N\}.$$

Moreover,

$$0<\lambda_1(h)<\dots<\lambda_N(h).$$

The family $(\varphi^k)_{1\le k\le N}$ define a basis of $\mathbb{R}^N$, then every solution $u_h = \begin{pmatrix} u_1\\ \vdots\\ u_N\end{pmatrix}$ of system 4.3, admit a Fourier developement on this basis.
Using this remark we show that:

$$u_h(t) = \sum_{k=1}^{N}\left[a_k\sin\left(\sqrt{\lambda_k(h)}\,t\right)+b_k\cos\left(\sqrt{\lambda_k(h)}\,t\right)\right]\varphi^k.$$

With $a_k, b_k\in\mathbb{R}$, $k=1,\dots,N$, than can be computed explicitly in terms of the initial conditions in 4.3.
In the following lemma, we introduce two relations for the eigenvectors of system 4.7:

Lemma 4.2.1 *$i)$ For any eigenvector φ of system 4.7 associated to eigenvalue λ, we have:*

$$\sum_{j=0}^{N}\left|\frac{\varphi_j-\varphi_{j+1}}{h}\right|^2=\lambda\sum_{j=1}^{N}|\varphi_j|^2. \tag{4.8}$$

$ii)$ Let φ^k and φ^l two eigenvectors of 4.7, associated to eigenvalues $\lambda_k\neq\lambda_l$, we have:

$$\sum_{j=0}^{N}(\varphi_j^k-\varphi_{j+1}^k)(\varphi_j^l-\varphi_{j+1}^l)=0. \tag{4.9}$$

Proof :
$i)$ Multiplying in 4.7, by φ_j and adding for $j=1,\ldots,N$, we have :

$$-\sum_{j=1}^{N}\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2}\varphi_j=\sum_{j=1}^{N}\lambda|\varphi_j|^2.$$

On the other hand,

$$\begin{aligned}
-\sum_{j=1}^{N}\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2}\varphi_j &= -\frac{1}{h^2}\sum_{j=1}^{N}\left[\varphi_{j+1}\varphi_j-2\varphi_j^2+\varphi_{j-1}\varphi_j\right]\\
&=\frac{1}{h^2}\left[\sum_{j=1}^{N}\varphi_j^2+\sum_{j=1}^{N}\varphi_j^2-\sum_{j=1}^{N}\varphi_j\varphi_{j+1}-\sum_{j=1}^{N}\varphi_j\varphi_{j-1}\right]\\
&=\frac{1}{h^2}\left[\sum_{j=0}^{N}\varphi_{j+1}^2+\sum_{j=0}^{N}\varphi_j^2-\sum_{j=0}^{N}\varphi_{j+1}\varphi_j-\sum_{j=0}^{N}\varphi_j\varphi_{j+1}\right],\ (\text{ add } \varphi_0=\varphi_{N+1}=0)\\
&=\frac{1}{h^2}\sum_{j=0}^{N}|\varphi_{j+1}-\varphi_j|^2,
\end{aligned}$$

finally,

$$\sum_{j=0}^{N}\left|\frac{\varphi_j-\varphi_{j+1}}{h}\right|^2=\lambda\sum_{j=1}^{N}|\varphi_j|^2.$$

$ii)$ Let φ^k associated to λ_k and φ^l associated to λ_l, two eigenvectors, such that $\lambda_k\neq\lambda_l$, we have,

$$\begin{aligned}
\sum_{j=0}^{N}(\varphi_j^k-\varphi_{j+1}^k)(\varphi_j^l-\varphi_{j+1}^l) &= \sum_{j=0}^{N}\varphi_j^k\varphi_j^l-\varphi_j^k\varphi_{j+1}^l-\varphi_{j+1}^k\varphi_j^l+\varphi_{j+1}^k\varphi_{j+1}^l\\
&=2<\varphi^k,\varphi^l>-\sum_{j=0}^{N}\varphi_j^k\varphi_{j+1}^l+\varphi_{j+1}^k\varphi_j^l.
\end{aligned}$$

We hnow that A_h is symmetric, then $< \varphi^k, \varphi^l >= 0$ and $< A_h\varphi^k, \varphi^l >= 0$,
then ,

$$\sum_{j=1}^{N}(\varphi_{j+1}^k - 2\varphi_j^k + \varphi_j^k)\varphi_j^l = 0,$$

implies,

$$\sum_{j=0}^{N}\varphi_j^k\varphi_{j+1}^l + \varphi_{j+1}^k\varphi_j^l = 0,$$

finally,

$$\sum_{j=0}^{N}(\varphi_j^k - \varphi_{j+1}^k)(\varphi_j^l - \varphi_{j+1}^l) = 0.$$

4.3 Boundary observability of eigenvectors

Recall that our objective is to analyze the observability of the solutions of the system 4.3, but before that we need to analyze the observability of each eigenvector.

Lemma 4.3.1 *For any eigenvector $\varphi = (\varphi_1, \dots, \varphi_N)$ of system 4.7 the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{\varphi_{j+1} - \varphi_j}{h}\right|^2 = \frac{2}{4 - \lambda h^2}\left|\frac{\varphi_N}{h}\right|^2. \tag{4.10}$$

Proof :
Let φ an eigenvector of system 4.7 associated to λ.
Firstly, we normalize φ such that:

$$h\sum_{j=0}^{N}|\varphi_j|^2 = 1, \tag{4.11}$$

then, from lemma 4.2.1 4.8, we get :

$$h\sum_{j=0}^{N}\left|\frac{\varphi_j - \varphi_{j+1}}{h}\right|^2 = \lambda. \tag{4.12}$$

Thus,

$$\lambda = \frac{1}{h}\sum_{j=0}^{N}\left[|\varphi_j|^2 + |\varphi_{j+1}|^2 - 2\varphi_{j+1}\varphi_j\right]$$
$$= \frac{2}{h}\sum_{j=0}^{N}|\varphi_j|^2 - \frac{2}{h}\sum_{j=0}^{N}\varphi_{j+1}\varphi_j.$$

But φ is normalized 4.11, then:

$$\sum_{j=0}^{N}\varphi_{j+1}\varphi_j = \frac{1}{h} - \frac{\lambda h}{2}. \tag{4.13}$$

We multiply in 4.7 by $j\dfrac{\varphi_{j+1}-\varphi_{j-1}}{2}$ and add for $j = 1,\ldots,N$, we get :

$$-\frac{1}{h^2}\sum_{j=1}^{N}\left[\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}\right]\, j\frac{(\varphi_{j+1}-\varphi_{j-1})}{2} = \lambda\sum_{j=1}^{N} j\varphi_j\frac{(\varphi_{j+1}-\varphi_{j-1})}{2}.$$

We obtain on the left hand side:

$$-\frac{1}{h^2}\sum_{j=1}^{N}\left[\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}\right]\, j\frac{(\varphi_{j+1}-\varphi_{j-1})}{2} = -\frac{1}{h^2}\sum_{j=1}^{N}\frac{j}{2}\left[|\varphi_{j+1}|^2 - |\varphi_{j-1}|^2 - 2\varphi_j(\varphi_{j+1}-\varphi_{j-1})\right]$$
$$= -\frac{1}{h^2}\sum_{j=1}^{N}\frac{j}{2}\left[|\varphi_{j+1}|^2 - |\varphi_{j-1}|^2\right] + \frac{1}{h^2}\sum_{j=1}^{N} j\varphi_j(\varphi_{j+1}-\varphi_{j-1})$$
$$= \frac{-1}{2h^2}\left[\sum_{j=1}^{N} j|\varphi_{j+1}|^2 - \sum_{j=1}^{N} j|\varphi_{j-1}|^2\right] + \frac{1}{h^2}\left[\sum_{j=1}^{N} j\varphi_j\varphi_{j+1} - j\varphi_j\varphi_{j-1}\right.$$
$$= \frac{-1}{2h^2}\left[\sum_{j=2}^{N}(j-1)|\varphi_j|^2 - \sum_{j=0}^{N-1}(j+1)|\varphi_j|^2\right] + \frac{1}{h^2}\Big(\sum_{j=1}^{N} j\varphi_j\varphi_{j+1}$$
$$-\sum_{j=0}^{N} j\varphi_{j+1}\varphi_j\Big)$$
$$= \frac{1}{h^2}\sum_{j=1}^{N}|\varphi_j|^2 - \frac{N+1}{2h^2}|\varphi_N|^2 - \frac{1}{h^2}\sum_{j=1}^{N}\varphi_j\varphi_{j+1}$$
$$= \frac{\lambda}{2h} - \frac{N+1}{2}\left|\frac{\varphi_N}{h}\right|^2,$$

by virtue of 4.11 and 4.13. On the right hand side, we have:

$$\lambda \sum_{j=1}^{N} j\varphi_j \left(\frac{\varphi_{j+1}-\varphi_{j-1}}{2}\right) = -\frac{\lambda}{2}\sum_{j=1}^{N}\varphi_j\varphi_{j+1}$$
$$= -\frac{\lambda}{2}\left(\frac{1}{h}-\frac{\lambda h}{2}\right),$$

finally, we get

$$\frac{\lambda}{2h} - \frac{N+1}{2}\left|\frac{\varphi_N}{h}\right|^2 = -\frac{\lambda}{2}\left(\frac{1}{h}-\frac{\lambda h}{2}\right).$$

In others words,

$$\frac{1}{2}\left|\frac{\varphi_N}{h}\right|^2 = \frac{(N+1)h}{2}\left|\frac{\varphi_N}{h}\right|^2 = \frac{\lambda}{2}+\frac{\lambda}{2}-\frac{\lambda^2h^2}{4} \tag{4.14}$$
$$= \lambda(1-\frac{\lambda h^2}{4}). \tag{4.15}$$

Combining 4.12 and 4.14, we get :

$$h\sum_{j=0}^{N}\left|\frac{\varphi_j-\varphi_{j+1}}{h}\right|^2 = \frac{2}{4-\lambda h^2}\left|\frac{\varphi_N}{h}\right|^2.$$

The equality 4.10 provides an explicit relation between the total energy of the eigenvectors and the energy concentrated on the extreme $x=1$, wich is represented by the quality $\left|\frac{\varphi_N}{h}\right|^2$. On the other hand, it is easy to check that $\lambda h^2 < 4$, for all $h>0$ and all eigenvalues of 4.7, but this inequality does not exclud the blow up of the constant in the right hand side of 4.10. In fact, we can check that $\lambda_N(h)h^2 \longrightarrow 4$ as $h \longrightarrow 0$, we can see that if we develop $\lambda_N(h)h^2$.

4.4 Non-uniform observability

The following theorem shows that, we cannot find a T^*, such that, for any $T>T^*$, the $C(T,h)$ is independent of h, this negative result is a consequence of the non-uniform observability of the eigenvectors of the system 4.7.

Theorem 4.4.1 *for any $T>0$, we have :*

$$\sup_{u_h \text{ solution of } 4.3} \frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt} \longrightarrow \infty \text{ as } h \longrightarrow 0. \tag{4.16}$$

Proof :
To prove this theorem it was enough to use the lemma 4.3.1 4.10.
Let u_h be the solution of 4.3, such as:

$$u_h(t) = \cos(\sqrt{\lambda_N(h)}\, t)\varphi^N.$$

Thus, the energy of system 4.3 is:

$$\begin{aligned} E_h(0) &= \frac{h}{2}\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}^N - \varphi_j^N}{h}\right|^2 \\ &= \frac{1}{2}\frac{2}{4-\lambda_N(h)h^2}\left|\frac{\varphi_N^N}{h}\right|^2 \\ &= \frac{1}{4-\lambda_N(h)h^2}\left|\frac{\varphi_N^N}{h}\right|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt &= \left|\frac{\varphi_N^N}{h}\right|^2 \int_0^T \cos^2(\sqrt{\lambda_N(h)}\, t)dt \\ &\leq \left|\frac{\varphi_N^N}{h}\right|^2 T, \end{aligned}$$

which implies,

$$\frac{1}{\displaystyle\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt} \geq \frac{1}{\left|\frac{\varphi_N^N}{h}\right|^2 T},$$

hence,

$$\frac{E_h(0)}{\displaystyle\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt} \geq \frac{1}{T(4-\lambda_N(h)h^2)}.$$

Finally

$$\lim_{h\longrightarrow 0^+} \frac{E_h(0)}{\displaystyle\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt} = +\infty.$$

4.5 Filtering technique and uniform observability

We have shown that our discritization scheme introduced high frequencies in the approximation of the solution of system 4.1, moreover, with these frequencies one cannot get the uniform observability. In order to obtain a positive connterpart, we consider a class of solution of 4.3 generated by the eigenvectors of the system 4.7 which have a low frequencies.
Let γ in $]0,4[$, we introduce the class $\mathscr{C}_h(\gamma)$ of solutions of 4.3 generated by eigenvectors of 4.7 associated with eigenvalues such that

$$\lambda_k(h)h^2 \leq \gamma. \tag{4.17}$$

More precisely:

$$\mathscr{C}_h(\gamma) = \left\{ u_h = \sum_{h^2\lambda_k(h)\leq\gamma} \left(a_k \cos(\sqrt{\lambda_k(h)}\, t) + b_k \sin(\sqrt{\lambda_k(h)}\, t) \right) \varphi^k, \;\; a_k, b_k \in \mathbb{R} \right\}. \tag{4.18}$$

The construction of this space is based on the fact that, the energy of every eigenvector of system 4.7 such that the associated eigenvalue verify 4.17, can be estimated uniformly in terms of the energy concentrated on $x = 1$.
The following theorem show that, for all solutions of 4.3 in the class $\mathscr{C}_h(\gamma)$we can get the uniform observability.

Theorem 4.5.1 *For all $\gamma \in]0,4[$, there exists $T(\gamma) \leq 2$ such that, for all $T > T(\gamma)$ there exists $C = C(T,\gamma)$, such that:*

$$\sup_{u_h \in \mathscr{C}_h(\gamma)} \frac{E_h(0)}{\displaystyle\int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt} \leq C, \;\; \forall h \in]0,1[. \tag{4.19}$$

Moreover
i) $T(\gamma) \longrightarrow \infty$ as $\gamma \longrightarrow 4$ and $T(\gamma) \longrightarrow 2$ as $\gamma \longrightarrow 0$.
ii) $C(T,\gamma) \longrightarrow \dfrac{1}{2(T-2)}$ as $\gamma \longrightarrow 0$.

Proof:
This theorem it allows us to get the uniform observability if we filter the solution of 4.3 and then we can also get the convergence of our algorithm (HUM).
To prove the theorem it was enough to apply the Ingham inequality which represents a result of the theory of non-harmonic Fourier series [10].

Theorem 4.5.2 (Ingham's theorem) *Let $(\mu)_{k\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$, such that:*

$$\mu_{k+1} - \mu_k \geq \gamma, \;\; \forall k \in \mathbb{Z},$$

then, for all $T > \dfrac{2\pi}{\gamma}$, there exists $C_1(T,\gamma) > 0$ and $C_2(T,\gamma) > 0$, such that:

$$\begin{aligned} C_1(T,\gamma) \sum_{k\in\mathbb{Z}} |b_k|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}} b_k e^{i\mu_k t} \right|^2 dt \\ \leq C_2(T,\gamma) \sum_{k\in\mathbb{Z}} |b_k|^2, \end{aligned}$$

for any sequence of complex numbers $(b_k)_{k\in\mathbb{Z}} \in l^2_{\mathbb{C}}$.

The following lemma, it will help us to adapt the previous theorem to prove 4.19:

Lemma 4.5.1 *Let $\gamma \in]0,4[$ and assume that:*

$$\gamma = 4\ \sin^2\left(\frac{\pi\varepsilon}{2}\right), \tag{4.20}$$

for some $0 \leq \varepsilon < 1$, then :

$$\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} \geq \pi \cos\left(\frac{\pi\varepsilon}{2}\right), \tag{4.21}$$

for all eigenvalues verify 4.17.

Proof
The equality 4.20 is aconsequence of the fact that the function sin is bijective from $\left[0, \dfrac{\pi}{2}\right]$ into $[0,1]$.
Now, let λ_k, λ_{k+1} two eigenvalues in range of 4.19,
then,

$$\begin{aligned} \sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} &= \frac{2}{h}\left(\sin\left(\frac{(k+1)\pi}{2}h\right) - \sin\left(\frac{k\pi}{2}h\right)\right) \\ &= \pi\cos(\eta), \end{aligned}$$

for some $\eta \in \left[\frac{\pi k}{2}h, \frac{\pi(k+1)}{2}h\right]$ (Finite increment theorem). On the other hand, we have,

$$\lambda_{k+1}(h)h^2 \leq \gamma \Longleftrightarrow 4\sin\left(\frac{(k+1)\pi}{2}h\right) \leq 4\ \sin\left(\frac{\pi\varepsilon}{2}\right) \tag{4.22}$$

$$\Longleftrightarrow (k+1)h \leq \varepsilon. \tag{4.23}$$

In view of 4.22, $0 \leq \eta \leq \frac{\pi\varepsilon}{2} < \frac{\pi}{2}$ and therefore, $\cos(\eta) \geq \cos\left(\frac{\pi\varepsilon}{2}\right)$,
finally, we obtain

$$\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} \geq \pi\cos\left(\frac{\pi\varepsilon}{2}\right).$$

We return to the proof of 4.19, for that, let u_h solution of 4.3 in the class $\mathscr{C}_h(\gamma)$. It can be written as :

$$u_h = \sum_{h^2\lambda_k(h)\leq\gamma} \tilde{a}_k e^{i\sqrt{\lambda_k(h)}\,t}\varphi^k,$$

such that $(\tilde{a}_k)_k \in l^2_{\mathbb{C}}$.
Let denote $\mu_k = \sqrt{\lambda_k(h)}$, we have:

$$\begin{aligned}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt &= \frac{1}{h^2}\int_0^T \left|\sum_{\mu_k h\leq\sqrt{\gamma}} \tilde{a}_k e^{i\mu_k\,t}\varphi_N^k\right| \\ &\geq C_1(T,\gamma)\sum_{\mu_k h\leq\sqrt{\gamma}} |\tilde{a}_k| \left|\frac{\varphi_N^k}{h}\right|^2.\end{aligned}$$

On the other hand, we know from 4.10 thet:

$$\begin{aligned}h\sum_{j=0}^{N}\left|\frac{\varphi_j^k - \varphi_{j+1}^k}{h}\right|^2 &= \frac{2}{4-\lambda_k h^2}\left|\frac{\varphi_N^k}{h}\right|^2 \\ &\leq \frac{2}{4-\gamma}\left|\frac{\varphi_N^k}{h}\right|^2.\end{aligned}$$

Thus,

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \geq \frac{h(4-\gamma)C_1(T,\gamma)}{2}\sum_{\mu_k h\leq\sqrt{\gamma}} |\tilde{a}_k| \sum_{j=0}^{N}\left|\frac{\varphi_j^k - \varphi_{j+1}^k}{h}\right|^2. \tag{4.24}$$

Moreover

$$
\begin{aligned}
E_h(0) &= \frac{h}{2}\sum_{j=0}^{N}\left(|u_j^{'}|^2 + \left|\frac{u_{j+1}-u_j}{h}\right|^2\right) \\
&= \frac{h}{2}\sum_{\mu_k h\leq\sqrt{\gamma}}|\tilde{a}_k|^2\lambda_k(h)\sum_{j=0}^{N}|\varphi_j^k|^2 + \frac{h}{2}\sum_{\mu_k h\leq\sqrt{\gamma}}|\tilde{a}_k|^2\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}^k-\varphi_j^k}{h}\right|^2.
\end{aligned}
$$

Using the equality 4.8, we can show that:

$$
E_h(0) = h\sum_{\mu_k h\leq\sqrt{\gamma}}|\tilde{a}_k|^2\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}^k-\varphi_j^k}{h}\right|^2. \tag{4.25}
$$

By combining 4.24 and 4.25, we obtain:

$$
E_h(0) \leq \frac{2}{(4-\gamma)C_1(T,\gamma)}\int_0^T\left|\frac{u_N}{h}\right|^2 dt.
$$

The Theorem 4.4.2 holds with:

$$
T(\gamma) = \frac{2}{\sqrt{1-\frac{\gamma}{4}}},
$$

and

$$
C(T,\gamma) = \frac{2}{(4-\gamma)C_1(T,\gamma)}.
$$

4.6 Filtration of initial conditions

In this part, we introduce the notion of filtering of the initial conditions, to obtain the controllabilty as well as the convergence of the algorithm in the case $r<1$. Note that, in this chapter we have presented the filtering of the solution, but what we can do in practice is to filter the initial data.
Now, we consider the example 3.6, from the section 4.2,we can show that:

$$
Y_0 = \sum_{k=1}^{N}\alpha_k\varphi^k,
$$

with Y_0 is the projection of y_0 on the discretization of x, that is:

$$Y_0 = \begin{pmatrix} y_0(h) \\ y_0(2h) \\ \vdots \\ y_0(Nh) \end{pmatrix}.$$

The idea of filtration is instead of sum until N, we limit at $f(N)$, such that: $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ and

$$\begin{cases} f(N) \leq N, \ \forall N \in \mathbb{N}^*, \\ \\ \lim_{N\to\infty} f(N) = +\infty, \\ \\ \lim_{N\to\infty} \sup \dfrac{f(N)}{N} < 1. \end{cases}$$

For more details see [14], see also optimal filtration in [15].
First of all, we apply our filter on the initial data 3.4, to show that, under these conditions, there are no high frequencies in the scheme, after that, we will aplly the filter on 3.6.
Let $f(N) = N^{\beta}$, with $\beta \in]0,1[$, we can show that f verify the condition in [14]. For the following numerical tests, we choose $\beta = \dfrac{3}{4}$ and $r = 0.5$.

N	299	499	1000
Number of CG iterations	7	6	5
$\|\|y(T)\|\|_{L^2(0,1)}$	$3.383249061\times 10^{-11}$	2.881917×10^{-12}	3.594377×10^{-12}
$\|\|y'(T)\|\|_{H^{-1}(0,1)}$	$3.333827592\times 10^{-11}$	$3.0510117 \times 10^{-12}$	2.9039126×10^{-12}
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.4065889	0.4065780	0.40657352

Table 4.1: Numerical results obtained for different values of $h = \dfrac{1}{N+1}$, for 3.4 with $T = 2.2$, $\varepsilon = 10^{-10}$ and $r = 0.5$.

Now, we return to 3.6, we remember that we don't have the convergence with this initial data in the case $r < 1$. The following table show that, we can get the controllability and also the convergence of the algorithm:

N	299	499	2000
Number of CG iterations	8	8	5
$\|\|y(T)\|\|_{L^2(0,1)}$	4.608712×10^{-6}	4.5380172×10^{-6}	8.5339918×10^{-7}
$\|\|y^{'}(T)\|\|_{H^{-1}(0,1)}$	3.1613970×10^{-6}	3.242659×10^{-6}	6.02109142×10^{-7}
$\|\|\widehat{v}\|\|_{L^2(0,T)}$	0.4067637	0.4076172	0.40755088

Table 4.2: Numerical results obtained for different values of $h = \dfrac{1}{N+1}$, for 3.6 with $T = 2.2$, $\varepsilon = 10^{-5}$ and $r = 0.5$.

In figures 4.1-4.3 we plot the y_0 filtering, the controlled solution and the control function, for $r = 0.5$, $\beta = \dfrac{3}{4}$, $T = 2.2$ and $N = 2000$.

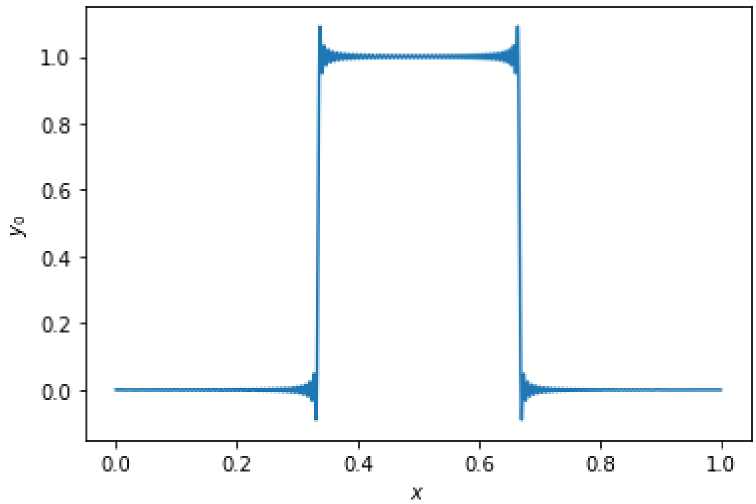

Figure 4.1: y_0 filtered .

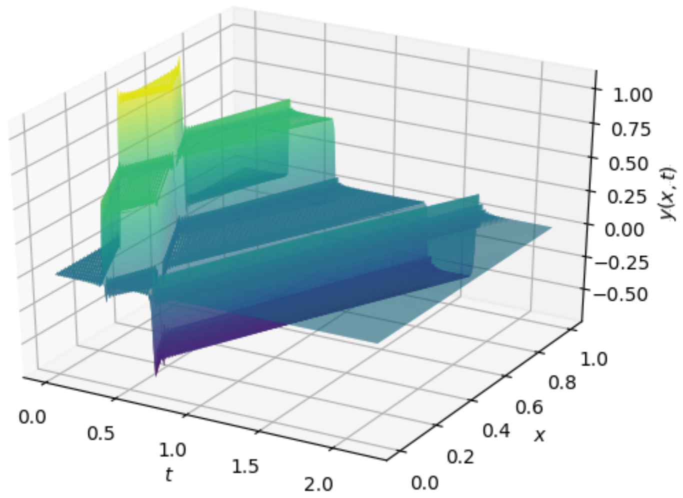

Figure 4.2: The controlled solution y.

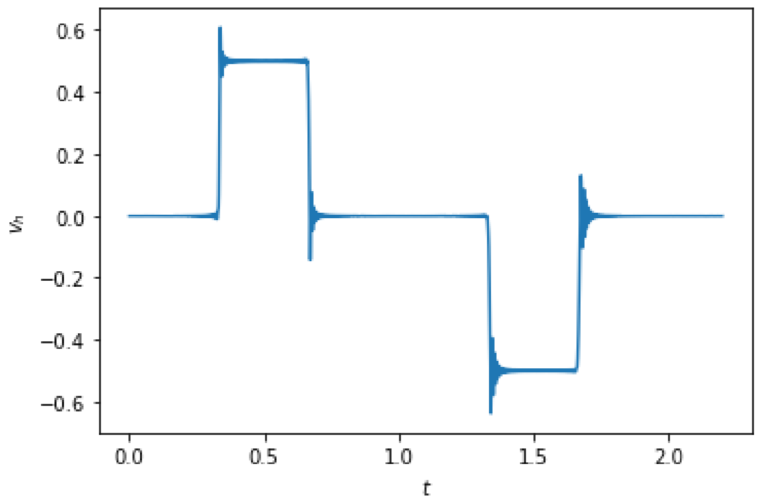

Figure 4.3: The approximation of the control function $\hat{v}_h$.

CHAPTER 5

NUMERICAL APPROXIMATION WITH THE FINITE DIFFERENCE METHODS (2D CASE)

As the previous chapter, we will apply the finite difference methods to control the system (1.1) in 2D case.
Let

$$\Omega = (0,1)\times(0,1), \quad \Gamma_1 = \Gamma = \partial\Omega.$$

In this case we control the whole boundary.
Note that we can also control a part of the boundary, then it is necessary to verify the geometrical conditions [1].

5.1 Finite difference approximation of the 2D wave equation

Consider the hyperbolic model problem, with 2D scalar wave equation:

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2} - \Delta\varphi & = & 0 & & \text{in} \quad \Omega\times(0,T), \\ \varphi & = & u & & \text{in} \quad \Gamma\times(0,T), \\ \varphi(\cdot,0) & = & v(\cdot), & \dfrac{\partial\varphi}{\partial t}(\cdot,0) = w(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

where $(v,w) \in H_0^1(\Omega)\times L^2(\Omega)$.
We introduce a mesh in time and in space by:

$$t_0 = 0 < t_1 = k < \cdots < t_n = nk < \cdots < t_{M+1} = T,$$

$$x_0 = 0 < x_1 = h_1 < \cdots < x_i = ih_1 < \cdots < x_{N_1+1} = 1,$$

and

$$y_0 = 0 < y_1 = h_2 < \cdots < y_j = jh_2 < \cdots < y_{N_2+1} = 1,$$

with $k = \dfrac{T}{M+1}$, $h_1 = \dfrac{1}{N_1+1}$ and $h_2 = \dfrac{1}{N_2+1}$.
Let $\varphi_{i,j}^n$ be the approximation of $\varphi(x_i,y_j,t_n)$, we obtain:

$$
\begin{cases}
\dfrac{\varphi_{i,j}^{n+1}-2\varphi_{i,j}^{n}+\varphi_{i,j}^{n-1}}{k^2}-\left(\dfrac{\varphi_{i+1,j}^{n}-2\varphi_{i,j}^{n}+\varphi_{i-1,j}^{n}}{h_1^2}+\dfrac{\varphi_{i,j+1}^{n}-2\varphi_{i,j}^{n}+\varphi_{i,j-1}^{n}}{h_2^2}\right)=0,\\
1\le i\le N_1, 1\le j\le N_2 \ \text{ and } \ 1\le n\le M,\\
\varphi_{i,j}^{n}=u_{i,j}^{n}, \quad \text{for } \ i\in\{0,N_1+1\} \text{ or } j\in\{0,N_2+1\},\\
\varphi_{i,j}^{0}=v_{i,j},\\
\varphi_{i,j}^{1}=\varphi_{i,j}^{0}+kw_{i,j}.
\end{cases}
\tag{5.2}
$$

we define $C_1=\dfrac{k}{h_1}$ and $C_2=\dfrac{k}{h_2}$, we can prove that for $C=C_1=C_2$, the explicit scheme is stable if $C\le\dfrac{1}{\sqrt{2}}$, then for the numerical results we use $C_1=C_2$ and we keep the notations C_1 and C_2 for the modelization of the numerical solution.
From (5.2), we have

$$\varphi_{i,j}^{n+1}-2\varphi_{i,j}^{n}+\varphi_{i,j}^{n-1}-C_1^2(\varphi_{i+1,j}^{n}-2\varphi_{i,j}^{n}+\varphi_{i-1,j}^{n})-C_2^2(\varphi_{i,j+1}^{n}-2\varphi_{i,j}^{n}+\varphi_{i,j-1}^{n})=0,$$

then,

$$\varphi_{i,j}^{n+1}=2(1-C_1^2-C_2^2)\varphi_{i,j}^{n}-\varphi_{i,j}^{n-1}+C_1^2(\varphi_{i+1,j}^{n}+\varphi_{i-1,j}^{n})+C_2^2(\varphi_{i,j+1}^{n}+\varphi_{i,j-1}^{n})$$

. Let,

$$\lambda=2(1-C_1^2-C_2^2),$$

$$
A=\begin{pmatrix} A_1 & A_2 & & 0\\ A_2 & \ddots & \ddots & \\ & \ddots & \ddots & A_2\\ 0 & & A_2 & A_1\end{pmatrix} \text{ and } \varphi^n=\begin{pmatrix}\varphi_{1,1}^n\\ \varphi_{1,2}^n\\ \vdots\\ \varphi_{1,N_2}^n\\ \varphi_{2,1}^n\\ \varphi_{2,2}^n\\ \vdots\\ \varphi_{2,N_2}^n\\ \vdots\\ \vdots\\ \varphi_{N_1,1}^n\\ \varphi_{N_1,2}^n\\ \vdots\\ \varphi_{N_1,N_2}^n\end{pmatrix}.
$$

The matrix A is formed by N_1^2 blocks, each of size N_2^2, where A_1 and A_2 defined by:

$$A_1 = \begin{pmatrix} \lambda & C_2^2 & & 0 \\ C_2^2 & \ddots & \ddots & \\ & \ddots & \ddots & C_2^2 \\ 0 & & C_2^2 & \lambda \end{pmatrix}, \ A_2 = \begin{pmatrix} C_1^2 & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & C_1^2 \end{pmatrix}.$$

The system (5.2) is equivalent to:

$$\varphi^{n+1} = A\varphi^n - \varphi^{n-1} + b^n,$$

where b^n is a vector formed by the values of φ^n in the boundary:

$$b^n = \begin{pmatrix} C_1^2\varphi_{0,1}^n + C_2^2\varphi_{1,0}^n \\ C_1^2\varphi_{0,2}^n \\ \vdots \\ C_1^2\varphi_{0,N_2-1}^n \\ C_1^2\varphi_{0,N_2}^n + C_2^2\varphi_{1,N_2+1}^n \\ \\ C_2^2\varphi_{2,0}^n \\ 0 \\ \vdots \\ 0 \\ C_2^2\varphi_{2,N_2+1}^n \\ \\ \vdots \\ \vdots \\ \\ C_1^2\varphi_{N_1+1,1}^n + C_2^2\varphi_{N_1,0}^n \\ C_1^2\varphi_{N_1+1,2}^n \\ \vdots \\ C_1^2\varphi_{N_1+1,N_2}^n + C_2^2\varphi_{N_1,N_2+1}^n \end{pmatrix}.$$

We notice that the matrix A is large if N_1 and N_2 are large, this matrix must be stored in the memory. The other problem is that the np.dot function in python is numerically unstable and the CPU time to calculate the matrix-vector product is very long, but in our case, the matrix A is tridiagonal and formed by two matrices, one tridiagonal and the other is diagonal, then we can define a python function for calculate the product $A\varphi^n$ and also we can accelerate this product with Numba.

5.2 Finite difference approximation of the 2D Dirichlet problem

In the algorithm (HUM), we need to solve a Dirichlet system on each iteration.
The Dirichlet problem in the algorithm is of the form:

$$\begin{cases} -\left(\dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\partial^2\varphi}{\partial y^2}\right)=\psi'(0)-y & \text{in} \quad \Omega, \\ \\ \varphi=0, & \text{in} \quad \Gamma, \end{cases} \tag{5.3}$$

with the finite difference approximation, we obtain:

$$\begin{cases} \dfrac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{h_1^2}+\dfrac{\varphi_{i,j+1}-2\varphi_{i,j}+\varphi_{i,j-1}}{h_2^2}=y_{i,j}-\dfrac{\psi^1_{i,j}-\psi^0_{i,j}}{k}, \quad 1\le i\le N_1, \quad 1\le j\le N_2, \\ \\ \varphi_{i,j}=0, \quad \text{for } i\in\{0,N_1+1\} \text{ or } j\in\{0,N_2+1\}. \end{cases} \tag{5.4}$$

As the previous section, let

$$\varphi=\begin{pmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \vdots \\ \varphi_{1,N_2} \\ \varphi_{2,1} \\ \varphi_{2,2} \\ \vdots \\ \varphi_{2,N_2} \\ \vdots \\ \vdots \\ \varphi_{N_1,1} \\ \varphi_{N_1,2} \\ \vdots \\ \varphi_{N_1,N_2} \end{pmatrix}. \text{ and } Y_{i,j}=y_{i,j}-\frac{\psi^1_{i,j}-\psi^0_{i,j}}{k}, \ (i,j)\in\{1,\dots,N_1\}\times\{1,\dots,N_2\}.$$

From (5.4), we have:

$$(**) \quad B\varphi=Y.\text{flatten}(),$$

with B the matrix formed by N_1^2 blocks, each of size N_2^2.

$$B = \begin{pmatrix} B_1 & B_2 & & 0 \\ B_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_2 \\ 0 & & B_2 & B_1 \end{pmatrix},$$

with B_1 is a tridiagonal matrix and B_2 is a diagonal matrix:

$$B = \begin{pmatrix} \frac{-2}{h_1^2} - \frac{2}{h_2^2} & \frac{1}{h_2^2} & & 0 \\ \frac{1}{h_2^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{h_2^2} \\ 0 & & \frac{1}{h_2^2} & \frac{-2}{h_1^2} - \frac{2}{h_2^2} \end{pmatrix}, B = \begin{pmatrix} \frac{1}{h_1^2} & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \frac{1}{h_1^2} \end{pmatrix}.$$

Like the previous section, the matrix B is large, then for solving the system $(**)$, we need to stored the matrix B and also the function lg.solve in scipy.linalg is numerically unstable if N_1, N_2 are large.
As the matrix B is formed by blocks, we can use the MPI for solving the system $(**)$ and accelerate the time of resolution.
The other solution is to use the package scipy.sparse, in this case we have stored the matrix B in memory, then replace B with pointers in which we keep the terms no nulls (B=coo_matrix(B)).

5.3 Numerical Tests

For the numerical tests we fix $C = C1 = C2$, $T = 3$ (or $T > 2\sqrt{2}$) and we will discuss the convergence of the conjugate gradient algorithm.
As the previous chapter, the conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$ and for the stopping criteria we choose:

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1 \times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1 \times L^2}} < 10^{-10}.$$

The numerical tests show that if $C > \frac{1}{\sqrt{2}}$, the algorithm diverges, so we can't control the initial data using HUM, then does there exist $C \in \left[0, \frac{1}{\sqrt{2}}\right]$ so that all initial data can be controllable.
In case $1D$ we have a condition for C ($C = 1$), for our case there are no theoretical results, by analogy

with the previous chapter we choose $C = \frac{1}{\sqrt{2}}$, then we have the stability for the wave equation, but there are initial conditions that we cannot control. For our test we will show that with $C = \frac{1}{\sqrt{2}}$, we can control a set of initial conditions.
Let $y^1 = 0$. We assume that $\exists D \in \Omega$, such as:

$$y^0 = \begin{cases} \approx 0, \text{ in } \Omega \backslash D, \\ \\ y^0_D, \text{ in } D, \end{cases}$$

with D is a compact, convex $(\lambda(D) < \lambda(\Omega))$, we also assume that there are no high frequencies in y^0_D, otherwise we filter y^0_D.
Based on example (3.4) with a modification of the $2D$ Gaussian law, we define :

$$y^0(x,y) = exp\left(-5\left(\frac{r}{2\sigma^2}\right)^6\right), \tag{5.5}$$

with $r = (x-0.35)^2 + (y-0.35)^2$.
Here $(0.35, 0.35)$ is the centre, $\sigma_x = \sigma_y = \sigma$ are the standard deviations for x and y.

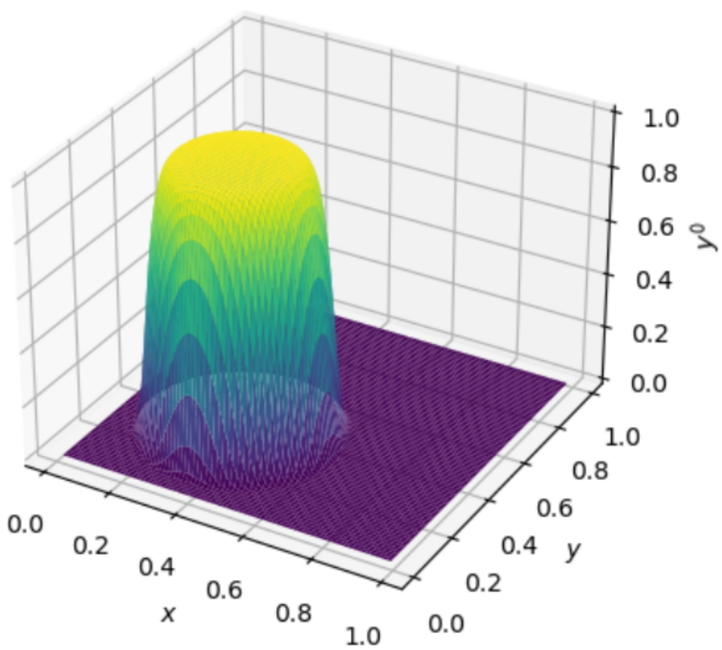

Figure 5.1: The initial position y^0 for $\sigma = 0.2$.

The numerical tests show that for $\sigma \geq 0.3$ the algorithm diverges, moreover when N_1 becomes large the number of iterations increases also the CPU time grow, the other problem that we need a large memory space.

N	15	30	40	50
Number of CG iterations	175	337	547	943
CPU time (secs)	1.625318	9.933084	32.132817	103.455245
$\|\|y(T)\|\|_{L^2(\Omega)}$	0.00295	6.693309×10^{-11}	1.27032×10^{-11}	8.422314×10^{-11}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	5.434475×10^{-11}	6.503109×10^{-11}	1.211519×10^{-10}	8.497743×10^{-11}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.180434	0.167908	0.16638	0.1652834

Table 5.1: Numerical results obtained for $C = \dfrac{1}{\sqrt{2}}$. $T = 3$ and $\sigma = 0.2$.

we fix the maximum number of CG iterations $N_{MAX} = 2000$. tables 5.2-5.4 show the the CPU time, the convergence of CG algorithm and the number of CG iterations for different values of ε, in the case of the convergence of the algorithm, we will calculate the norm of $||y(T)||_{L^2(\Omega)}$, $||y'(T)||_{H^{-1}(\Omega)}$ and $||\widehat{v}||_{L^2((0,T)\times\Gamma)}$

N	40	60	100	150
Number of CG iterations	114	21	19	16
CPU time (secs)	10.438298	5.189082	26.279451	99.680222
convergence	CV	CV	CV	CV
$\|\|y(T)\|\|_{L^2(\Omega)}$	0.000366649	0.0003217156	0.0002750246	0.000349270
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	0.000372948	0.000300827	0.0002772390	0.000288972
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.166377769	0.165065510	0.1643637404	0.164166635

Table 5.2: $\varepsilon = 10^{-3}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	181	465	323	475
CPU time (secs)	13.119077	105.264298	413.225330	2774.186211
convergence	CV	CV	CV	CV
$\|\|y(T)\|\|_{L^2(\Omega)}$	5.56653233×10^{-5}	0.000104486	3.10654843×10^{-5}	3.44585132×10^{-5}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	5.51862416×10^{-5}	0.000106774	3.07522409×10^{-5}	3.40246881×10^{-5}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.16638202	0.165074544	0.16437017	0.164172625

Table 5.3: $\varepsilon = 10^{-4}$, $\sigma = 0.2$

N	40	60	100	150
Number of CG iterations	218	734	2000	2000
CPU time (secs)	15.639176	170.212021	20320.163	40640.326
convergence	CV	CV	DIV	DIV
$\|\|y(T)\|\|_{L^2(\Omega)}$	1.3622136×10^{-5}	1.497883585×10^{-5}	//	//
$\|\|y^{'}(T)\|\|_{H^{-1}(\Omega)}$	1.3334127×10^{-5}	1.466640873×10^{-5}	//	//
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.165075143	0.165074544	//	//

Table 5.4: $\varepsilon = 10^{-5}$, $\sigma = 0.2$

From table 5.2-5.4 and numerical simulations, we notice that, when $\varepsilon \geq 10^{-5}$ and $N > 100$ (or N large) the algorithm does not converge in 2000 iterations, also the CPU time is very large, and the stop criterion, it had an oscillation around 10^{-5}.

In most cases the approximation with the finite difference method does not work for all the initial data, but we can control some type of initial conditions, in this case we have to accelerate the code and also solve the memory problem, but in general we do not find good approximations ($N \geq 100$).

As in the $1-d$ finite difference approximation, we can mathematically prove that the divergence of the algorithm is due to the fact that there are high frequencies in the solution and also in the initial conditions, see [18].

In the following figures, we plot the controlled solution at different times and the norm of the control function for $\varepsilon = 10^{-4}$ and $N = 100$:

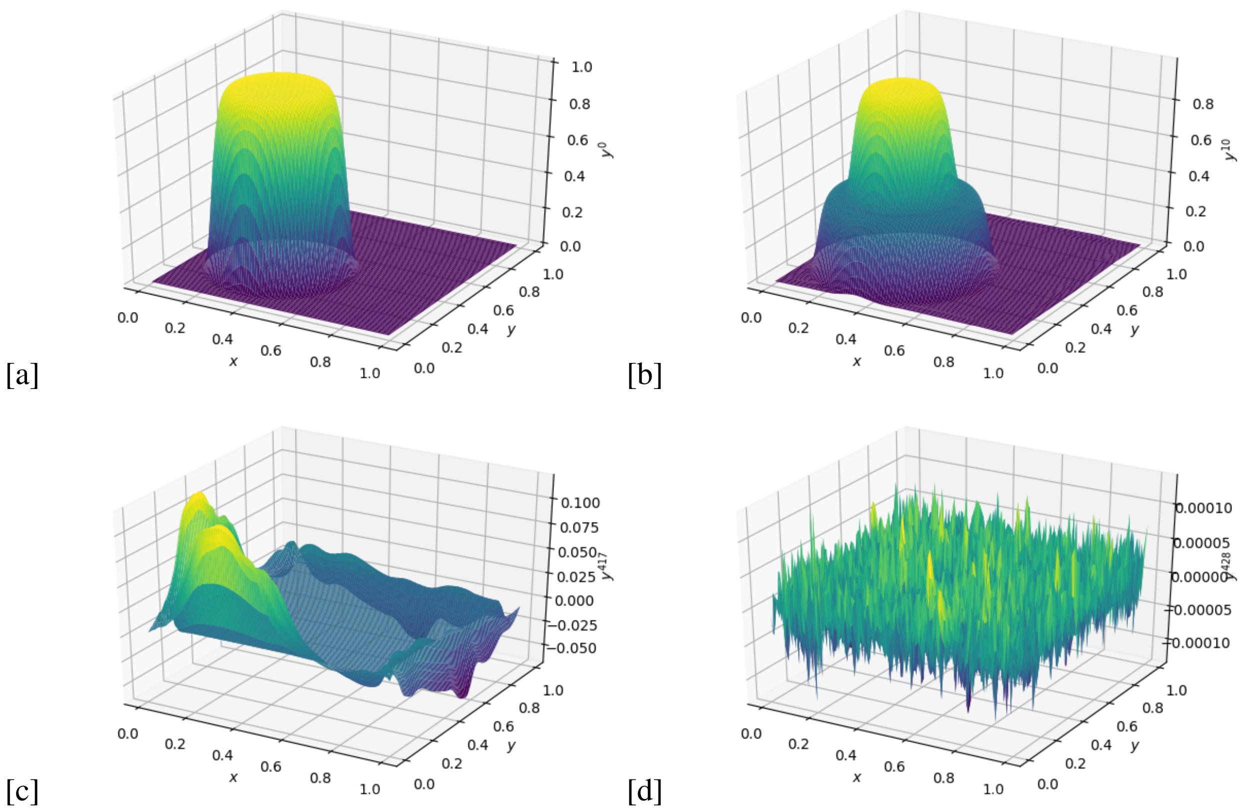

Figure 5.2: Controlled solution at 0, 10, $M-10$ and $M+1$.

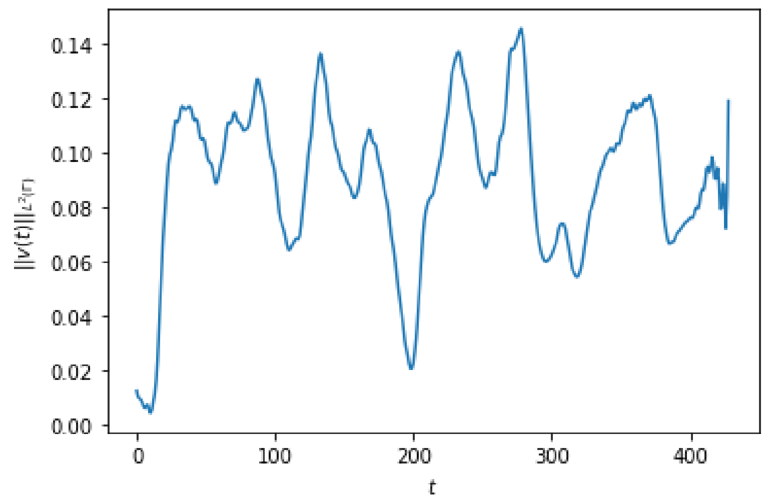

Figure 5.3: The norm of the approximation of control function $\hat{v}$.

In the following chapters we will try to use a finite element approximation and see if we have the controllability

Chapter 6

Numerical approximation with the finite element method (1D Case)

Now, we will use the finite element method to solve the wave and Dirichlet problems and control the system 1.1.
As the chapter 4, let

$$\Omega = (0,1), \quad \Gamma_1 = \{1\} \quad \text{and} \quad \Gamma_2 = \{0\}.$$

6.1 Finite element approximation of the wave equation

As chapter 3, we define the general wave equation as following:

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2} - \dfrac{\partial^2\varphi}{\partial x^2} = 0 & \text{in} \quad (0,1)\times(0,T), \\ \varphi(0,\cdot) = 0 & \text{in} \quad (0,T), \\ \varphi(1,\cdot) = u(\cdot) & \text{in} \quad \times(0,T), \\ \varphi(\cdot,0) = v(\cdot), \quad \dfrac{\partial\varphi}{\partial t}(\cdot,0) = w(\cdot) & \text{in} \quad (0,1), \end{cases} \tag{6.1}$$

where $(v,w) \in H_0^1(0,1)\times L^2(0,1)$.

6.1.1 Variational formulation

Let $V(t) = \{\varphi \in H^1(0,1)/\ \varphi(0) = 0,\ \varphi(1) = u(t)\}$ and $U = H_0^1(0,1)$.
We fix $\varphi \in H^2(0,1)$, solution of 6.1. By multiplying the wave equation with $\psi \in U$ and integrating it over the unit interval we get following expression:

$$\int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx - \int_\Omega \frac{\partial^2\varphi}{\partial x^2}\ \psi dx = 0,$$

using integration by parts, we obtain:

$$\int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx + \int_\Omega \frac{\partial\varphi}{\partial x}\ \frac{\partial\psi}{\partial x} dx = 0.$$

Then the weak formulation of 6.1, is :

$$\forall t \in (0,T), \begin{cases} \text{Find } \ \varphi(.,t) \in V(t), \text{such that } : \\ a(\varphi(.,t), \psi(.)) = 0,\ \forall\psi \in U, \\ \varphi(.,0) = v(.),\ \dfrac{\partial\varphi}{\partial t}(.,0) = w(.), \end{cases} \tag{6.2}$$

with

$$a(\varphi(.,t),\psi(.)) = \int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx + \int_\Omega \frac{\partial\varphi}{\partial x}\ \frac{\partial\psi}{\partial x}dx$$

. Before the numerical discretization, we must prove that, if φ a solution of , 6.2 then is a solution of 6.1, for that, we fix $\varphi(.,t) \in V(t)$, such that, $a(\varphi(.,t),\psi(.)) = 0$, $\forall\psi \in U$.
Let $\psi \in D(0,1)$, then

$$\int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx + \int_\Omega \frac{\partial\varphi}{\partial x}\ \frac{\partial\psi}{\partial x}dx = 0,$$

$$\int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx + < \frac{\partial\varphi}{\partial x}, \frac{\partial\psi}{\partial x} >= 0,$$

$$\int_\Omega \frac{\partial^2\varphi}{\partial t^2}\ \psi dx - < \psi, \frac{\partial^2\varphi}{\partial x^2} >= 0,$$

finally,

$$< \frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}, \psi >= 0,\ \ \forall\psi \in D(0,1),$$

hence, $\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0$, almost everywhere.
We have proved that, the system 6.1 is equivalent to 6.2, therefore system 6.2 admits a unique solution, because system 6.1 has only one solution. The problem here that we cannot prove that 6.2 admits a unique solution using the Stampacchia theorem or the Lax-Milgram theorem , because u is usually only L^2 and in this case $V(t)$ is an affine space.

6.1.2 Discrete variational problem

We define:

$$x_0 = 0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

be partition of $[0,1]$ and $h = \frac{1}{N+1}$ is the space step.
We approximate the $V(t)$ and U spaces by:

$$V_h(t) = \{\varphi_h \in C^0(0,1)/\ \varphi_h|_{[x_j,x_{j+1}]} \in \mathbb{P}_1,\ \varphi_h(0) = 0,\ \varphi_h(1) = u(t),\ j \in \{0,\ldots,N\}\},$$

$$U_h = \{\psi_h \in C^0(0,1)/\ \psi_h|_{[x_j,x_{j+1}]} \in \mathbb{P}_1,\ \psi_h(0) = 0,\ \psi_h(1) = 0,\ j \in \{0,\ldots,N\}\},$$

respectively, with $\mathbb{P}_1$ is the space of polynomials of degree 1,
then the discrete variational problem, written as follows:

$$\forall t \in (0,T), \begin{cases} \text{Find } \varphi_h(.,t) \in V_h(t), \text{such that :} \\ \\ a(\varphi_h(.,t), \psi_h(.)) = 0, \ \forall \psi_h \in U_h, \\ \\ \varphi_h(.,0) = v_h(.), \ \dfrac{\partial \varphi_h}{\partial t}(.,0) = w_h(.), \end{cases} \tag{6.3}$$

moreover U_h is a vector space of dimension N and $\{\phi_1, \dots, \phi_N\}$ is a basis of U_h, with $\phi_1, \dots, \phi_N$ are the shape functions, defined by:

$$\phi_i = \begin{cases} 1 + \dfrac{x - x_i}{h}, \text{ in } [x_{i-1}, x_i], \\ \\ 1 - \dfrac{x - x_i}{h}, \text{ in } [x_i, x_{i+1}], \\ \\ 0, \quad \text{Otherwise.} \end{cases}$$

we define the last shape function ϕ_{N+1}, as follows

$$\phi_{N+1} = \begin{cases} 1 + \dfrac{x - x_{N+1}}{h}, \text{ in } [x_N, x_{N+1}], \\ \\ 0, \quad \text{Otherwise,} \end{cases}$$

then for all $\varphi_h(.,t) \in V_h(t)$, φ_h can be written as:

$$\varphi_h(x,t) = \sum_{j=1}^{N} \varphi_j(t)\phi_j(x) + u(t)\phi_{N+1}(x).$$

The last discrete formulation of our problem is the following:

$$\begin{cases} \text{Find } \varphi_j(t), \quad j \in \{1, \dots, N\}, \text{ such that :} \\ \\ a\left(\displaystyle\sum_{j=1}^{N} \varphi_j(t)\phi_j(\cdot) + u(t)\phi_{N+1}(\cdot) \ , \ \phi_i(\cdot) \right) = 0, \ \forall i \in \{1, \dots, N\}. \end{cases} \tag{6.4}$$

For each shape function ϕ_i with $i = 1,\dots,N$, we can write

$$
\begin{aligned}
0 &= a\left(\sum_{j=1}^{N} \varphi_j(t)\phi_j(\cdot) + u(t)\phi_{N+1}(\cdot)\ ,\ \phi_i(\cdot)\right)\\
&= \sum_{j=1}^{N}\int_0^1 \frac{\partial^2\varphi_j}{\partial t^2}(t)\ \phi_j(x)\phi_i(x)dx + \int_0^1 \frac{\partial^2 u}{\partial t^2}(t)\ \phi_{N+1}(x)\phi_i(x)dx\\
&\quad + \sum_{j=1}^{N}\int_0^1 \varphi_j(t)\ \frac{\partial\phi_j(x)}{\partial x}\frac{\partial\phi_i(x)}{\partial x}dx + \int_0^1 u(t)\frac{\partial\phi_{N+1}(x)}{\partial x}\ \frac{\partial\phi_i(x)}{\partial x}dx\\
&= \sum_{j=1}^{N}\frac{\partial^2\varphi_j}{\partial t^2}(t)\int_0^1 \phi_j(x)\phi_i(x)dx + \sum_{j=1}^{N}\varphi_j(t)\int_0^1 \frac{\partial\phi_j(x)}{\partial x}\frac{\partial\phi_i(x)}{\partial x}dx\\
&\quad + \frac{\partial^2 u}{\partial t^2}(t)\int_0^1 \phi_{N+1}(x)\phi_i(x)dx + u(t)\int_0^1 \frac{\partial\phi_{N+1}(x)}{\partial x}\ \frac{\partial\phi_i(x)}{\partial x}dx.
\end{aligned}
$$

Let $A_{ij} = \displaystyle\int_0^1 \phi_j\phi_i\,dx$, $B_{ij} = \displaystyle\int_0^1 \frac{\partial\phi_j(x)}{\partial x}\frac{\partial\phi_i(x)}{\partial x}dx$, for $i,j \in \{1,\dots,N\}$,
and $\varphi(t) = (\varphi_1(t),\dots,\varphi_N(t))$.
the system 6.4 is equivalent to :

$$
(***)\quad A\frac{\partial^2\varphi(t)}{\partial t^2} + B\varphi(t) = C(t),
$$

such as :

$$
A = \begin{pmatrix}
\frac{2h}{3} & \frac{h}{6} & & 0\\
\frac{h}{6} & \ddots & \ddots & \\
 & \ddots & \ddots & \frac{h}{6}\\
0 & & \frac{h}{6} & \frac{2h}{3}
\end{pmatrix},\
B = \begin{pmatrix}
\frac{2}{h} & \frac{-1}{h} & & 0\\
\frac{-1}{h} & \ddots & \ddots & \\
 & \ddots & \ddots & \frac{-1}{h}\\
0 & & \frac{-1}{h} & \frac{2}{h}
\end{pmatrix}
$$

and

$$
C(t) = \begin{pmatrix}
0\\
\vdots\\
\frac{1}{h}u(t) - \frac{h}{6}\frac{\partial^2 u(t)}{\partial t^2}
\end{pmatrix}.
$$

To solve $(***)$, we will use a second order approximation for the second derivative (FDM). The final numerical scheme is as follows:

$$\begin{cases} \text{Find } \ \varphi^n, \text{such that } : \\ A_0\varphi^{n+1} = (2A_0 - 6C^2B_0)\varphi^n - A_0\varphi^{n-1} + C^n, \\ \varphi^0 = v_h, \ \varphi^1 = \varphi^0 + kw_h, \end{cases} \tag{6.5}$$

here, $A_0 = \frac{6}{h}A$, $B_0 = hB$, $C = \frac{k}{h}$ and

$$C^n = \begin{pmatrix} 0 \\ \vdots \\ (6C^2+2)u(t_n) - (u(t_{n+1} + u(t_{n-1}) \end{pmatrix}.$$

As in the previous chapter, we will define a function for the matrix-vector product and for the resolution of the linear system we will use the COO scheme.
For the stability of the scheme 6.5, we can prove mathematically and numerically that 6.5 is stable if $C \leq \frac{1}{\sqrt{3}}$.

6.2 Finite element approximation of the Dirichlet problem

The Poisson equation that we have to solve in each iteration is defined as follows:

$$\begin{cases} -\frac{\partial^2\varphi}{\partial x^2} = \psi^{'}(0) - y \quad \text{in} \quad (0,1), \\ \varphi(0) = 0, \\ \varphi(1) = 0. \end{cases} \tag{6.6}$$

Let $U = H_0^1(0,1)$. The variational formulation for the problem 6.6 is written as follows:

$$\begin{cases} \text{Find } \ \varphi \in U, \text{such that } : \\ A(\varphi,\Psi) = L(\Psi), \ \forall\Psi \in U, \end{cases} \tag{6.7}$$

with

$$A(\varphi,\Psi) = \int_0^1 \frac{\partial\varphi}{\partial x}\frac{\partial\Psi}{\partial x}dx, \quad \forall\varphi,\Psi \in U,$$

$$L(\Psi) = \int_0^1 \left(\psi^{'}(0) - y\right)\Psi dx, \quad \forall\Psi \in U.$$

We can easily show the 6.6 is equivalent to 7.4, moreover the Lax-Milgram theorem shows that the system 7.4 admits a unique solution.

As in the previous section, we discretize the variational problem, we obtain:

$$\begin{cases} \text{Find } \varphi_h \in U_h, \text{such that :} \\ \\ A(\varphi_h, \Psi_h) = L(\Psi_h), \ \forall \Psi_h \in U_h, \end{cases} \tag{6.8}$$

here $U_h = \{\Psi_h \in C^0(0,1) /\ \Psi_h|_{[x_j,x_{j+1}]} \in \mathbb{P}_1,\ \Psi_h(0) = 0,\ \Psi_h(1) = 0,\ j \in \{0,\dots,N\}\}$ is a finite dimensional ($\dim(U_h) = N$) subspace of U.
since A is a bilinear form and L is a linear form, then 6.8 is equivalent to:

$$\begin{cases} \text{Find } \varphi_h \in U_h, \text{such that :} \\ \\ A(\varphi_h, \phi_i) = L(\phi_i),\ i \in \{1,\dots,N\}. \end{cases} \tag{6.9}$$

We know that $\varphi_h = \sum_{j=1}^{N} \varphi_j \phi_j$, where $\{\phi_1,\dots,\phi_N\}$ are the shape functions that define a basis of U_h, but the ψ in 6.6 is a solution of a wave equation and y is an initial condition for the wave equation that we have to control, then $\psi = \sum_{j=1}^{N+1} \psi_j \phi_j$ and $y = \sum_{j=0}^{N+1} y_j \phi_j$.
The first shape function ϕ_0 defined as follows:

$$\phi_0 = \begin{cases} 1 - \dfrac{x - x_0}{h}, \text{ in } [x_0, x_1], \\ \\ 0, \quad \text{Otherwise.} \end{cases}$$

We approximation the first derivative in time by finite difference and we replace in 6.9, we obtain:

$$\begin{aligned} \sum_{j=1}^{N} \varphi_j \int_0^1 \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} dx &= \int_0^1 \left[\sum_{j=1}^{N+1} \frac{\psi_j^1 - \psi_j^0}{k} \phi_j - \sum_{j=0}^{N+1} y_j \phi_j \right] \phi_i dx \\ &= \sum_{j=1}^{N} \left(\frac{\psi_j^1 - \psi_j^0}{k} - y_j \right) \int_0^1 \phi_i \phi_j dx + \left(\frac{\psi_{N+1}^1 - \psi_{N+1}^0}{k} - y_{N+1} \right) \int_0^1 \phi_{N+1} \phi_i \\ &\quad - y_0 \int_0^1 \phi_0 \phi_i dx. \end{aligned}$$

we set $\varphi = (\varphi_1, \dots, \varphi_N)$, with the notation of the previous section, the final scheme is as follows:

$$B_0 \varphi = \frac{h^2}{6} A_0 Y + b,$$

where, $Y = \left(\dfrac{\psi_j^1 - \psi_j^0}{k} - y_j \right)_{1 \le j \le N}$ and

$$b = \frac{h^2}{6} \begin{pmatrix} -y_0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{\psi_{N+1}^1 - \psi_{N+1}^0}{k} - y_{N+1} \end{pmatrix}$$

6.3 Numerical Tests

In the present section, we apply the methods of sections 5.1 and 5.2, and the conjugate gradient algorithm, to control our system, and also, we will compare this method with the finite difference method in section 3.3.
As section 3.3, the conjugate gradient algorithm has been initialized with $\varphi_0^0 = \varphi_0^1 = 0$ and for the stopping criteria we choose:

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1 \times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1 \times L^2}} < \varepsilon.$$

For the numerical tests, we approximate $||\cdot||_{L^2}$ and $||\cdot||_{H_0^1}$ using the following approximation:

$$\int_0^1 \nabla\varphi \nabla\psi dx = \frac{1}{h}\sum_{i=1}^{N} \varphi_i \psi_i + \frac{1}{h}\sum_{i=0}^{N+1} \varphi_i \psi_i - \frac{1}{h}\sum_{i=0}^{N} (\varphi_{i+1}\psi_i + \varphi_i \psi_{i+1}),$$

$$\int_0^1 \varphi\psi dx = \frac{h}{3}\sum_{i=1}^{N} \varphi_i \psi_i + \frac{h}{3}\sum_{i=0}^{N+1} \varphi_i \psi_i + \frac{h}{6}\sum_{i=0}^{N} (\varphi_{i+1}\psi_i + \varphi_i \psi_{i+1}).$$

We know that we have the stability for the solver of the wave equation if we take $C = \dfrac{1}{\sqrt{3}}$, then for the moment, we fix $C = \dfrac{1}{\sqrt{3}}$, $T = 2.2$ and the maximum number of iterations equal to 2000, we will see if we have the controllability and also the convergence of the CG for the initial conditions

$$\begin{cases} y^0(x) = \exp(-5(x-0.35)^6), \\ y^1(x) = 0. \end{cases} \tag{6.10}$$

.

From tables 6.1-6.2, we remark that, when ε becomes small, we must increase the N to get the

convergence, also if N large, the number of iterations to have convergence becomes small.
The numerical tests, show that, we can have convergence for N small ($N \leq 50$), but it is necessary to choose an $\varepsilon \leq 10^{-4}$.

N	15	30	69	70	100
Number of CG iterations	2000	2000	2000	8	6
Convergence	no conv	no conv	no conv	conv	conv
$\|\|y(T)\|\|_{L^2(\Omega)}$	//	//	//	0.0001907	4.0131002×10^{-5}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	//	//	//	0.0001203	1.8678941×10^{-5}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	//	//	//	0.4063041	0.4064398

Table 6.1: Numerical results obtained for $C = \dfrac{1}{\sqrt{3}}$. $T = 2.2$ and $\varepsilon = 10^{-5}$.

N	70	109	110	200	300
Number of CG iterations	2000	2000	20	5	4
Convergence	no conv	no conv	conv	conv	conv
$\|\|y(T)\|\|_{L^2(\Omega)}$	//	//	6.106698×10^{-5}	1.878805×10^{-7}	5.3617078×10^{-8}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	//	//	8.0236302×10^{-6}	1.755967×10^{-7}	6.3979525×10^{-8}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	//	//	0.4064625	0.406538	0.4065570

Table 6.2: Numerical results obtained for $C = \dfrac{1}{\sqrt{3}}$. $T = 2.2$ and $\varepsilon = 10^{-6}$.

Now we fix $\varepsilon = 10^{-10}$, from Table 6.3 we observe that when N becomes bigger, the norm of $y(T)$ in $L^2(0,1)$ and $\dfrac{\partial y}{\partial t}(T)$ in $H^{-1}(0,1)$ close to 0.
Figure 6.2, show the approximation of the control function for $N = 1000$, in figure 6.1 we plot the controlled solution y_h for $N = 1000$.
Then we conclude that for the initial data 6.10, the system is controllable.

N	350	400	500	1000
Number of CG iterations	7	6	6	5
CPU Time (secs)	4.322983	6.488691	11.949790	78.507318
$\|\|y(T)\|\|_{L^2(\Omega)}$	1.296216×10^{-10}	3.6776111×10^{-11}	7.5309572×10^{-12}	7.40331232×10^{-12}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	3.597765×10^{-11}	3.4716749×10^{-11}	7.1260334×10^{-12}	6.31366379×10^{-12}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.4065610	0.4065635	0.4065666	0.40657065

Table 6.3: Numerical results obtained for $C = \dfrac{1}{\sqrt{3}}$. $T = 2.2$ and $\varepsilon = 10^{-10}$.

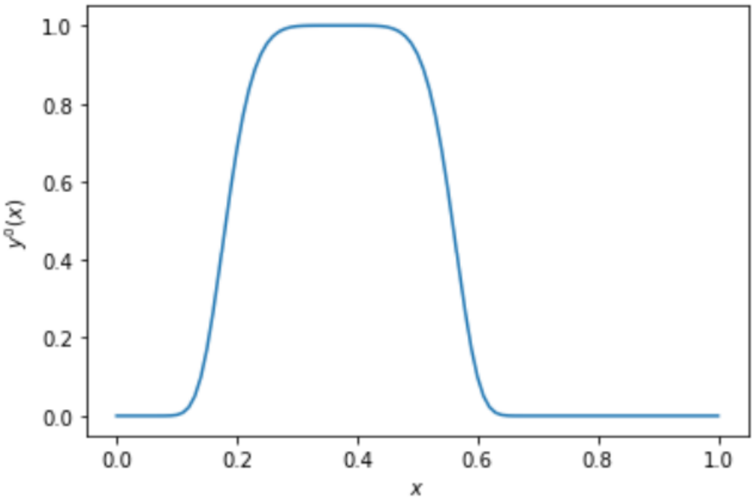

Figure 6.1: The approximation of the control.

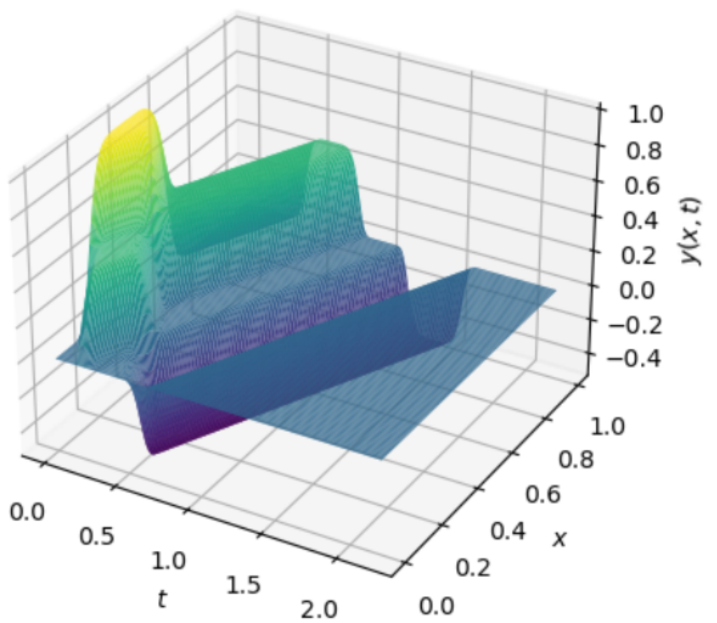

Figure 6.2: The controlled solution y.

With numerical tests, we can show that, for C in $\left]0, \dfrac{1}{\sqrt{3}} + 2 \times 10^{-5}\right[$, we have the convergence of the CG algorithm,we also obtain convergence with certain values in $\left[\dfrac{1}{\sqrt{3}} + 2 \times 10^{-5}, \dfrac{1}{\sqrt{3}} + 10^{-4}\right[$, but if $C \geq \dfrac{1}{\sqrt{3}} + 10^{-4}$ the algorithm diverge.

As in chapter 4, we cannot control all the initial conditions. and the high frequency problem increases the initial data set that can be controlled, but the method works well for some types of initial conditions, moreover with the finite difference method we have a mathematical criterion to choose C to control all the initial data, but this choice limits the domain of application of our code for other controllable systems.
the common point between the two methods is that we cannot control all the initial data with different choices of C, for example, in our case of finite elements, the initial conditions 3.6:

$$y^0(x) = \begin{cases} 0, \text{ in } \left[0, \frac{1}{3}\right[, \\ 1, \text{ in } \left[\frac{1}{3}, \frac{2}{3}\right[, \\ 0, \text{ in } \left[\frac{2}{3}, 1\right]. \end{cases}$$
$$y^1(x) = 0.$$

are not controllable. We can show that if we choose $N = 350$ and C in $\{0.1, 0.2, 0.3, 0.4, 0.5, \frac{1}{\sqrt{3}}\}$, and we can do numerous tests, but in general the algorithm diverges. Even if we find a C with which we can control all the initial, the the method is not practical.
As in chapter 4, J.A INFANTE and E. ZUAZUA proves that we do not have the uniform observability due to the high frequencies observed by R.GLOWINSKI, but if we filter the solutions we obtain the uniformity, see [9].
In the following, we will try to filter 3.6 as in chapter 4, Note that also with the finite element method in the case $1D$, we have the same eigenvectors, but the eigenvalues have changed because we changed the Laplacian approximation. For that we fix $C = \frac{1}{\sqrt{3}}$, $T = 2.2$, $\beta = \frac{3}{4}$ and $\varepsilon = 10^{-4}$.
The following table 6.4, show that we have the controllability and the convergence of the algorithm, and in figure 6.4, we plot the The controlled solution and the approximation of the control function.

N	299	499	799	1599
Number of CG iterations	6	6	4	4
$\|\|y(T)\|\|_{L^2(\Omega)}$	$5.24750786 \times 10^{-5}$	7.386422×10^{-4}	2.390×10^{-4}	4.7520114×10^{-5}
$\|\|y'(T)\|\|_{H^{-1}(\Omega)}$	2.434480×10^{-4}	6.6170897×10^{-5}	4.230181×10^{-5}	1.4736214×10^{-5}
$\|\|\widehat{v}\|\|_{L^2((0,T)\times\Gamma)}$	0.4064211	0.4074540	0.4076571	0.40762141

Table 6.4: Numerical results obtained for $C = \frac{1}{\sqrt{3}}$. $T = 2.2$ and $\varepsilon = 10^{-4}$.

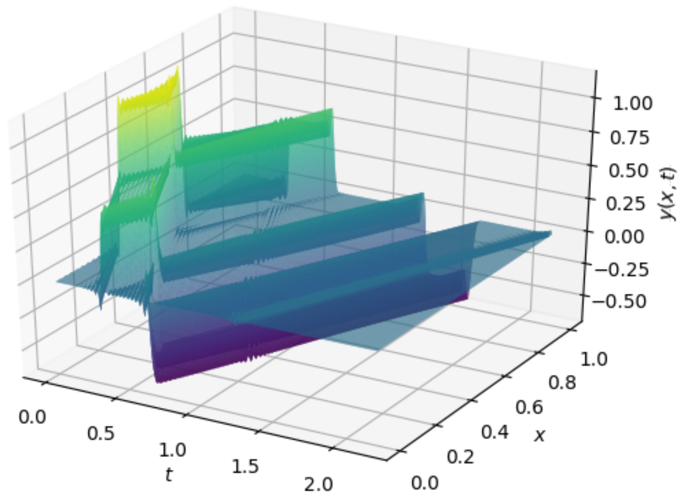

Figure 6.3: The controlled solution y.

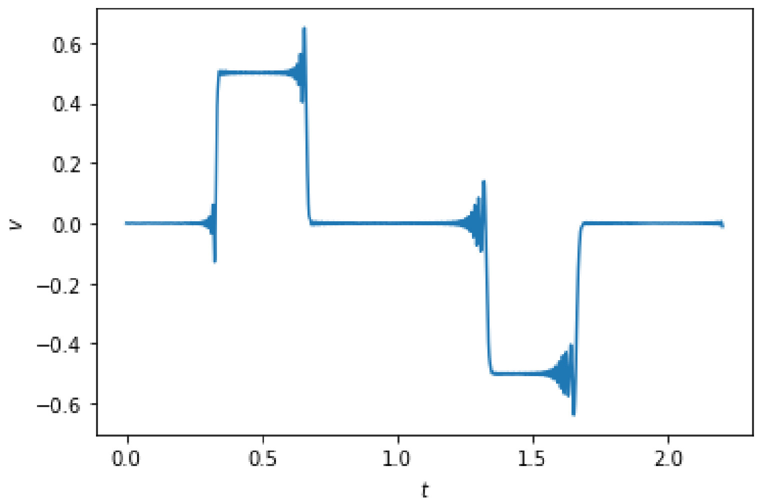

Figure 6.4: The approximation of the control function $\hat{v}_h$.

Chapter 7

Numerical approximation with the finite element method (2D Case)

The last chapter concerns an application of the finite elements method in $2D$ case, to control the wave equation, and we discuss the convergence of the GC algorithm, as well as the problem of high frequencies.
As in the chapter 4, we set

$$\Omega = (0,1)\times(0,1), \quad \Gamma_1 = \Gamma = \partial\Omega.$$

7.1 Mesh and finite element

We see in the previous chapter, that we approach the variational formula by another discrete formulation by using a mesh.
We define :

$$(R_k) = \{(x,y);\ ih_1 \le x \le (i+1)h_1,\ jh_2 \le y \le (j+1)h_2,\ i = 0,1,\dots,N_1,\ j = 0,1,\dots,N_2\},$$

be rectangular mesh on Ω, with $h_1 = \dfrac{1}{N_1+1}$, $h_2 = \dfrac{1}{N_2+1}$ and $(N_1,N_2) \in \mathbb{N}^{*2}$.
We can show that, there are $N_\tau = (N_1+1)(N_2+1)$ elements, the mesh nodes are :

$$(ih_1, jh_2),\ (i,j) \in \{0,\dots,N_1+1\}\times\{0,\dots,N_2+1\}.$$

We denote the number total of nodes by $N_T = (N_1+2)(N_2+2)$, including $N_B = 2(N_1+N_2)+4$ boundary nodes and $N_I = N_1N_2$ internal nodes.
For coding, it is necessary to define an order to traverse the elements of the mesh also the nodes, more we need to separate the internal nodes and the nodes of the boundary. In our code, we traverse the nodes along y with a translation along the x axis, and each element in the apposite direction of the trigonometric direction.
Note that this order that we have defined is possible becouse the geometry of Ω is simple, otherwise we can use Gmsh, in this case we don't need to define any code to generate the mesh.
For all that follows, we denote by I the ensemble formed by internal nodes, and B the ensemble formed by the boundary nodes, then $I\cup B$ is the set which includes all the nodes of the mesh. The order in $I\cup B$ is important, i.e we start with the nodes in I then the nodes in B. The following figure 7.1, we give an example of the mesh with $N1 = 3$ and $N2 = 2$.

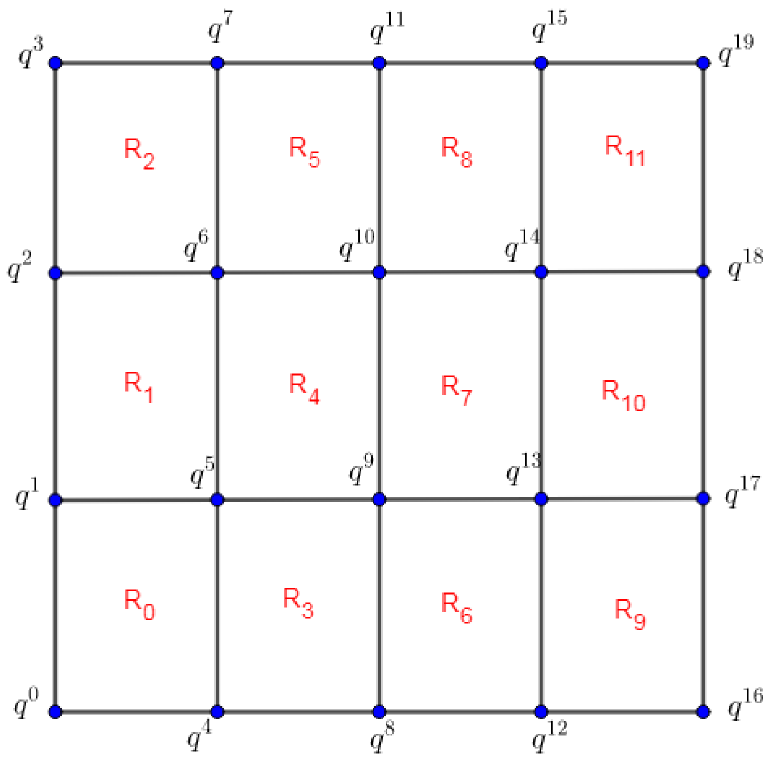

Figure 7.1: Mesh for $N1 = 3$ and $N2 = 2$.

For the moment the mesh is generated, now for each element we need to define a finite element.

Let $\mathbb{Q}_1$ the space of polynomials of partial degree less or equal to 1, we can show that $\mathbb{P}_1 \subset \mathbb{Q}_1$ and $\dim(\mathbb{Q}_1) = 4$, and we know that $\mathbb{Q}_1$ polynomials are affine on any segment in the mesh (R_k).
Now we fix R be a rectangle of the triangulation, we define the reference element $\hat{R} = [0,1] \times [0,1]$, such that $R = q^0q^1q^2q^3$ and $\hat{R} = \hat{q}^0\hat{q}^1\hat{q}^2\hat{q}^3$.
We first not that, in one dimension, the basis polynomials for $\mathbb{P}_1$ Lagrange interpolation on $[0,1]$ are

$$w_1(x_1) = 1 - x_1, \quad w_2(x_1) = x_1,$$

we can prove that $\mathbb{Q}_1(\hat{R}) = \mathrm{vect}(\mathbb{P}_1([0,1]) \otimes \mathbb{P}_1([0,1]))$, then the shape function on $\hat{R}$ defined by:

$$\hat{p}_0(\hat{X}) = (w_1 \otimes w_1)(\hat{X}),$$

$$\hat{p}_1(\hat{X}) = (w_2 \otimes w_1)(\hat{X}),$$
$$\hat{p}_2(\hat{X}) = (w_2 \otimes w_2)(\hat{X}),$$
$$\hat{p}_3(\hat{X}) = (w_1 \otimes w_2)(\hat{X}),$$

with $w \otimes v$ is the tensor product of w and v.
Finally, we can show that $(\hat{R}, \mathbb{Q}_1(\hat{R}), \{p(\hat{q}^0), p(\hat{q}^1), p(\hat{q}^2), p(\hat{q}^3)\})$ is a unisolvent finite element.

Now, let F be the unique affine bijective mapping, from the reference element to R, defined by :

$$F(\hat{X}) = \begin{pmatrix} x_0 + h_1\hat{x} \\ y_0 + h_2\hat{y} \end{pmatrix},$$

we can show that ,

$$F^{-1}(X) = \begin{pmatrix} \dfrac{x-x_0}{h_1} \\ \dfrac{y-y_0}{h_2} \end{pmatrix},$$

$$\nabla F(\hat{X}) = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$$

The idea behind the definition of F, that we can constrict a finite element on R, using F and the finite element defined on $\hat{R}$, then the shape function on R are defined by:

$$p_i(X) = \hat{p}_i(F^{-1}(X)), \quad i \in \{0,1,2,3\}.$$

7.2 Finite element approximation of the wave equation

In $2D$ case, the variational formulation is the same as in the $1D$ case, it suffices to change the integration by part by the Green formula.
Let $V(t) = \{\varphi \in H^1(\Omega)/\ \varphi|_\Gamma = u(t)\}$ and $U = H_0^1(\Omega)$.
the variational formulation of the wave system in $2D$, is written as follows:

$$\forall t \in (0,T), \begin{cases} \text{Find } \ \varphi(\cdot,t) \in V(t), \text{such that } : \\ \\ a(\varphi(\cdot,t), \psi(\cdot)) = 0,\ \forall \psi \in U, \\ \\ \varphi(\cdot,0) = v(\cdot),\ \dfrac{\partial \varphi}{\partial t}(\cdot,0) = w(\cdot), \end{cases} \tag{7.1}$$

with

$$a(\varphi(.,t), \psi(.)) = \int_\Omega \frac{\partial^2 \varphi}{\partial t^2}\ \psi\, dX + \int_\Omega \nabla\varphi\ \nabla\psi\, dX.$$

Now, we introduce the approximation spaces of $V(t)$ and U by :

$$V_h(t) = \{\varphi_h \in C^0(\bar{\Omega})/\ \varphi_h|_{R_k} \in \mathbb{Q}_1,\ \varphi_h|_\Gamma = u(t),\ k \in \{0,\ldots,N_\tau - 1\}\},$$

$$U_h = \{\psi_h \in C^0(\bar{\Omega})/\ \psi_h|_{R_k} \in \mathbb{Q}_1,\ \psi_h|_\Gamma = 0,\ k \in \{0,\ldots,N_\tau - 1\}\},$$

then, the discrete variational formulation, written as follows:

$$\forall t \in (0,T), \begin{cases} \text{Find } \varphi_h(\cdot,t) \in V_h(t), \text{such that :} \\ \\ a(\varphi_h(\cdot,t), \psi_h(\cdot)) = 0,\ \forall \psi_h \in U_h, \\ \\ \varphi_h(\cdot,0) = v_h(\cdot),\ \dfrac{\partial \varphi_h}{\partial t}(\cdot,0) = w_h(\cdot). \end{cases} \tag{7.2}$$

As in the previous chapter, we need to find a basis of U_h and $V_h(t)$.
Using the remark that, $\mathbb{Q}_1$ polynomials are affine on any segment in the mesh R_k, this is possible, becouse each segment of the mesh is parallel to one of the two axes x or y. we show that, there exist a unique family $(\phi_i)_{i\in B\cup I}$, such that $\phi_i(q^i) = \delta_{ij}$, $\forall i, j \in B\cup I$, and $(\phi_i)_{i\in I}$ define a basis of U_h, moreover, for all φ_h in $V_h(t)$, we have:

$$\varphi_h(X,t) = \sum_{j\in I} \varphi_j(t)\phi_j(X) + \sum_{j\in B} u_{h,j}(t)\phi_j(X).$$

Here $(q^j)_{j\in B\cup I}$ is the set of all nodes.
From 7.2, we get ,

$$a(\varphi_h(\cdot,t), \phi_i(\cdot)) = 0,\ \forall i \in I,$$

we fix $i \in I$, we obtain :

$$a\left(\sum_{j\in I} \varphi_j(t)\phi_j(\cdot) + \sum_{j\in B} u_h(t)\phi_j(\cdot), \phi_i(\cdot)\right) = 0,$$

develop this formula, as in the previous chapter, we obtain:

$$\sum_{j\in I}\int_\Omega \frac{\partial^2\varphi_j(t)}{\partial t^2}\phi_j\phi_i dX + \sum_{j\in B}\frac{\partial^2 u_{h,j}(t)}{\partial t^2}\int_\Omega \phi_j\phi_i dX + \sum_{j\in I}\int_\Omega \varphi_j(t)\nabla\phi_j\nabla\phi_i dX + \sum_{j\in B}\int_\Omega u_{h,j}(t)\nabla\phi_j\nabla\phi_i dX = 0,$$

hence,

$$\sum_{j\in I}\frac{\partial^2\varphi_j(t)}{\partial t^2}\int_\Omega \phi_j\phi_i dX + \sum_{j\in I}\varphi_j(t)\int_\Omega \nabla\phi_j\nabla\phi_i dX + \sum_{j\in B}\frac{\partial^2 u_{h,j}(t)}{\partial t^2}\int_\Omega \phi_j\phi_i dX + \sum_{j\in B} u_{h,j}(t)\int_\Omega \nabla\phi_j\nabla\phi_i dX = 0.$$

Let $A_{ij} = \int_\Omega \phi_j\phi_i dX$, $B_{ij} = \int_\Omega \nabla\phi_j\nabla\phi_i dX$, for $i, j \in I\cup B$.
In the last equation, we see that there are two parts, one that we need to calculate the mass matrix and the stiffness matrix only for the internal shape functions, in the other part we have to calculate the elements of each matrix, for each internal shape function with all the edge shape functions.
For the moment, one calculates the global matrices, i.e for all the shape functions, for that we fix an i

and j in $I \cup B$, we have

$$A_{i,j} = \sum_{k=0}^{N_\tau - 1} \int_{R_k} \phi_j \phi_i dX = \sum_{k=0}^{N_\tau - 1} A_{ij}(R_k),$$

$$B_{i,j} = \sum_{k=0}^{N_\tau - 1} \int_{R_k} \nabla\phi_j \nabla\phi_i dX = \sum_{k=0}^{N_\tau - 1} B_{ij}(R_k),$$

with

$$A_{ij}(R_k) = \int_{R_k} \phi_j \phi_i dX, \quad \text{and} \quad B_{ij}(R_k) = \int_{R_k} \nabla\phi_j \nabla\phi_i dX.$$

The strong point of this formulation, it is that, if we fixes a rectangle R_k, only the indices i and j associated with the nodes of the rectangle R_k, intervene in the effective computation of A_{ij} and B_{ij}, more, this idea, facilitates the assembly of matrices.

First, we have to form the local matrices $A(R_k)$ and $B(R_k)$, for that, let $R \in (R_k)$, and

$$A_{ls}(R) = \int_R \phi_s \phi_l dX, \text{ and } \quad B_{ls}(R) = \int_R \nabla\phi_s \nabla\phi_l dX, \quad l, s \in \{0, 1, 2, 3\}.$$

We will calculate the element of each matrix in the case of $l = s = 3$, for the other indices the method is the same.

For $l = s = 3$, we have $B_{33}(R) = \displaystyle\int_R \nabla p_3(X) \nabla p_3(X) dX$, and we know that

$p_3(X) = \hat{P}_3(F^{-1}(X))$, then $p_3(X) = \left(1 - \dfrac{x - x_0}{h_1}\right)\left(\dfrac{y - y_0}{h_2}\right)$,

hence,

$$\nabla p_3(X) == \begin{pmatrix} -\dfrac{1}{h_1}\dfrac{y - y_0}{h_2} \\ \\ \dfrac{1}{h_2}\left(1 - \dfrac{x - x_0}{h_1}\right) \end{pmatrix},$$

then,

$$B_{33}(R) = \int_R \left(\frac{1}{h_1}\right)^2 \left(\frac{y - y_0}{h_2}\right)^2 + \left(\frac{1}{h_2}\right)^2 \left(1 - \frac{x - x_0}{h_1}\right)^2 dxdy$$

$$= h_1 h_2 \int_{\hat{R}} \left(\frac{1}{h_1}\right)^2 \hat{y}^2 + \left(\frac{1}{h_2}\right)^2 (1 - \hat{x})^2 d\hat{x} d\hat{y},$$

finally,

$$B_{33}(R) = \frac{h_1 h_2}{3}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right).$$

For the mass matrix, it is easy to calculate the indices, we have

$$\begin{aligned} A_{33}(R) &= \int_R p_3(X)p_3(X)dX \\ &= h_1h_2\int_{\hat{R}} \hat{p}_3(\hat{X})\hat{p}_3(\hat{X})d\hat{X} \\ &= h_1h_2\int_{\hat{R}} (1-\hat{x})^2\hat{y}^2 d\hat{x}d\hat{y}, \end{aligned}$$

finally,

$$A_{33}(R) = \frac{h_1h_2}{9}.$$

The matrices $A(R)$ and $B(R)$, are written as follows:

$$A(R) = \begin{pmatrix} \frac{h_1h_2}{9} & \frac{h_1h_2}{18} & \frac{h_1h_2}{36} & \frac{h_1h_2}{18} \\ . & \frac{h_1h_2}{9} & \frac{h_1h_2}{18} & \frac{h_1h_2}{36} \\ . & . & \frac{h_1h_2}{9} & \frac{h_1h_2}{18} \\ . & . & . & \frac{h_1h_2}{9} \end{pmatrix},$$

$$B(R) = \begin{pmatrix} \frac{h_1h_2}{3}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & \frac{h_1h_2}{3}\left(\frac{-1}{h_1^2}+\frac{1}{2h_2^2}\right) & \frac{h_1h_2}{6}\left(\frac{-1}{h_1^2}+\frac{-1}{h_2^2}\right) & \frac{h_1h_2}{3}\left(\frac{1}{2h_1^2}+\frac{-1}{h_2^2}\right) \\ . & \frac{h_1h_2}{3}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & \frac{h_1h_2}{3}\left(\frac{1}{2h_1^2}+\frac{-1}{h_2^2}\right) & \frac{h_1h_2}{6}\left(\frac{-1}{h_1^2}+\frac{-1}{h_2^2}\right) \\ . & . & \frac{h_1h_2}{3}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & \frac{h_1h_2}{3}\left(\frac{-1}{h_1^2}+\frac{1}{2h_2^2}\right) \\ . & . & . & \frac{h_1h_2}{3}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) \end{pmatrix}.$$

We suppose that, we have assembled the global stiffness matrix and also the global mass matrix. We denote by A_I the internal mass matrix and B_I the internal stiffness matrix, we have :

$$A_I\frac{\partial^2\varphi(t)}{\partial t^2} + B_I\varphi(t) = C(t),$$

with $\varphi = (\varphi_i)_{i\in I}$ and $C(t)$ is a vector of size $N_I = N_1N_2$, defined by:

$$C(t)_i = -\left[\sum_{j\in B}\frac{\partial^2 u_{h,j}(t)}{\partial t^2}A_{ij} + \sum_{j\in B}u_{h,j}(t)B_{ij}\right], \quad \forall i \in I.$$

As in the previous chapter, we will use a second order approximation for the second derivative in time, we get:

$$A_I\left(\frac{\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}}{\Delta t^2}\right) + B_I\varphi^n = C^n, \quad (****)$$

such as,

$$C_i^n = -\left[\sum_{j\in B}\frac{u_{h,j}^{n+1} - 2u_{h,j}^n + u_{h,j}^{n-1}}{\Delta t^2}A_{ij} + \sum_{j\in B}u_{h,j}^nB_{ij}\right], \quad \forall i \in I,$$

and $1 \le n \le M$, $\Delta t = \dfrac{T}{M+1}$ (mesh for time). We can simplify $(****)$ and also optimize the number of flops, if we change $A(R)$ and $B(R)$ by :

$$A(R) = \frac{h_1h_2}{9}A^0(R), \quad B(R) = \frac{h_1h_2}{3}B^0(R),$$

then,

$$A = \frac{h_1h_2}{9}A^0, \quad B = \frac{h_1h_2}{3}B^0,$$

finally, we obtain,

$$A_I^0\varphi^{n+1} = (2A_I^0 - 3\Delta t^2B_I^0)\varphi^n - A_I^0\varphi^{n-1} + C_0^n,$$

with

$$C_{0i}^n = -\left[\sum_{j\in B}(u_{h,j}^{n+1} - 2u_{h,j}^n + u_{h,j}^{n-1})A_{ij}^0 + 3\Delta t^2u_{h,j}^nB_{ij}^0\right], \quad \forall i \in I,$$

7.3 Finite element approximation of 2D Dirichlet problem

In the algorithm, we have to solve the following Poisson equation, at each iteration:

$$\begin{cases} -\Delta\varphi = \psi'(0) - y \quad \text{in} \quad \Omega, \\ \\ \varphi = 0, \quad \text{in } .\Gamma \end{cases} \tag{7.3}$$

As in the case $1D$, the variational formulation, is the following:

$$\begin{cases} \text{Find } \varphi \in U, \text{such that :} \\ \\ A(\varphi,\Psi) = L(\Psi),\ \forall\Psi \in U, \end{cases} \tag{7.4}$$

here, $U = H_0^1(\Omega)$,

$$A(\varphi,\Psi) = \int_\Omega \nabla\varphi\nabla\Psi dX, \quad \forall\varphi,\Psi \in U,$$

and

$$L(\Psi) = \int_\Omega \left(\psi'(0) - y\right)\Psi dX, \quad \forall\Psi \in U.$$

Now, we discretize the varitional problem, we obtain:

$$\begin{cases} \text{Find } \ \varphi_h \in U_h, \text{such that } : \\ \\ A(\varphi_h,\Psi_h) = L(\Psi_h), \ \forall\Psi_h \in U_h, \end{cases} \tag{7.5}$$

with,

$$U_h = \{\psi_h \in C^0(\bar\Omega)/\ \psi_h|_{R_k} \in \mathbb{Q}_1,\ \psi_h|_\Gamma = 0,\ k \in \{0,\dots,N_\tau - 1\}\}.$$

From the previous section, we know that, the $(\phi)_{i\in I}$ family, formed a basis of U_h.
Then 7.5 equivalent to:

$$\begin{cases} \text{Find } \ \varphi_h \in U_h, \text{such that } : \\ \\ A(\varphi_h,\phi_i) = L(\phi_i), \ \forall i \in I. \end{cases} \tag{7.6}$$

Let $i \in I$, we have $\varphi_h = \sum_{j\in I} \varphi_j\phi_i$, $\psi(X,t) = \sum_{j\in I\cup B} \psi_j(t)\phi_j(X)$ and $y = \sum_{j\in I\cup B} y_j\phi_j$.
We develop equation 7.6, we find:

$$\begin{aligned} \sum_{j\in I}\varphi_j\int_\Omega \nabla\phi_j\nabla\phi_i &= \sum_{j\in I\cup B}\int_\Omega\left(\frac{\psi_j^1-\psi_j^0}{\Delta t} - y_j\right)\phi_j\phi_i \\ &= \sum_{j\in I}\int_\Omega\left(\frac{\psi_j^1-\psi_j^0}{\Delta t} - y_j\right)\phi_j\phi_i + \sum_{j\in B}\int_\Omega\left(\frac{\psi_j^1-\psi_j^0}{\Delta t} - y_j\right)\phi_j\phi_i. \end{aligned}$$

With the notation used in the previous sections, we have:

$$B_I\varphi = A_I Y + C_L,$$

such as: $\varphi = (\varphi_j)_{j\in I}$, $Y = \left(\frac{\psi_j^1-\psi_j^0}{\Delta t} - y_j\right)_{j\in I}$, and

$$(C_L)_i = \sum_{j\in B}\left(\frac{\psi_j^1-\psi_j^0}{\Delta t} - y_j\right)A_{ij}, \quad \forall i \in I.$$

Finally,

$$3B_I^0\varphi = A_I^0 Y + C_L^0,$$

with,

$$(C_L^0)_i = \sum_{j\in B} \left(\frac{\psi_j^1 - \psi_j^0}{\Delta t} - y_j \right) A_{ij}^0, \quad \forall i \in I.$$

7.4 Numerical Tests

In this part, we will use the two solvers that we defined in the previous sections in the conjugate gradient algorithm, to control our system in $2D$ case.
For that, we fix $T = 3$, $N_1 = N_2$ and we define $C = \frac{\Delta t}{h}$, with $h = h_1 = h_2$, and we initializes the conjugate gradient algorithm by $\varphi_0^0 = \varphi_0^1 = 0$, for the stopping criteria we choose:

$$\frac{||(\widetilde{\varphi^0}_{n+1}, \widetilde{\varphi^1}_{n+1})||_{H_0^1\times L^2}}{||(\widetilde{\varphi^0}_0, \widetilde{\varphi^1}_0)||_{H_0^1\times L^2}} < \varepsilon,$$

with ε is a small positive number.
Moreover, in the CG algorithm, we have to approximate the norm L^2 and the norm H_0^1, also the normal derivative.
for the normal derivative, we know that:

$$\frac{\partial \varphi}{\partial n}|_\Gamma = \nabla\varphi|_\Gamma \cdot \overrightarrow{n} = \begin{pmatrix} \frac{\partial \varphi}{\partial x}|_\Gamma(X) \\ \\ \frac{\partial \varphi}{\partial y}|_\Gamma(X) \end{pmatrix} \cdot \overrightarrow{n}(X).$$

on another side, Γ is defined by four parts, such that in each part, $\overrightarrow{n}$ constant, then for approximate $\frac{\partial \varphi}{\partial n}$ and respect the order of mesh, we traverses the four edges respectively:

$$\Gamma_0 = \{(0,y),\ y \in [0,1]\}, \quad \Gamma_1 = \{(x,0),\ x \in]0,1[\},$$

$$\Gamma_2 = \{(x,1),\ x \in]0,1[\}, \quad \Gamma_3 = \{(1,y),\ y \in [0,1]\}.$$

Note that φ is zero on Γ, then in each edge we approximate the derivative using the values of φ on the closest parallel edge that contains internal nodes.
We get the approximation of norms using the following two approximation of integrals:

$$\int_\Omega \varphi\psi dX \simeq \sum_{i,j\in I\cup B} \varphi_i \psi_j A_{ij},$$

$$\int_\Omega \nabla\varphi\nabla\psi dX \simeq \sum_{i,j\in I\cup B} \varphi_i \psi_j B_{ij}.$$

With the numerical test on 5.5 with $\rho = 0.2$, one can show that one does not have the convergence of the algorithm in the case of $\varepsilon > 10^{-3}$ and for $\varepsilon \leq 10^{-3}$ the norm of $y(T)$ and $y^{'}(T)$ is around of $o(10^{-3})$. we can compare these results with the finite element approximation with a triangular mesh [5], we can see that we do not have the convergence, due to the existence of high frequencies, Also note that we have no mathematical proof of the divergence in the case of the approximation with $Q1$, but with numerical tests, one can show that this approximation does not work, then it is necessary to filter the solutions or the initial conditions.
The tables 7.1-7.2show the L^2 error, made by the solver of wave equation and the solver of Dirichlet problem in the case of the following exact solutions:
$\star$ Exact solution of the wave system:

$$\varphi(x,y,t) = txy + x + 1.$$

$\star$ Exact solution of Dirichlet problem in HUM algorithm :

$$\varphi(x,y) = \frac{1}{\pi}\sin(\pi x)\sin(\pi y),$$

$$Y(x,y) = -\sin(\pi x)\sin(\pi y),$$

and

$$\psi(x,y,t) = \sin(\pi t)\sin(\pi x)\sin(\pi y).$$

N	15	30	50	100
L^2 Error	5.526412×10^{-4}	1.481362×10^{-4}	5.481034747×10^{-5}	1.39839×10^{-5}

Table 7.1: L^2 error of Dirichlet solver for different values of $h = \dfrac{1}{N+1}$.

N	15	30	50	100
L^2 Error	9.4170786×10^{-15}	1.979523×10^{-14}	2.5622188×10^{-14}	7.623461×10^{-14}

Table 7.2: L^2 error of Wave solver for different values of $h = \dfrac{1}{N+1}$ and $C = 0.3$.

CONCLUSION

With the HUM method, we transformed the controllability problem into a minimization problem, and indirectly we transformed the controllability problem into an observability problem by using the duality between the two notions. Finally we get an algorithm with which we can build controls numerically, such that these controls have a minimum L^2-norm.
To build the HUM-controls, we made the numerical simulations in $1D$ and $2D$ by using the finite element and the finite difference methods. We have shown that in general we cannot control all the initial conditions, except the $1D$ case with a finite difference discritization with $r = 1$ (CFL) or initial data which do not contain high frequencies.
The divergence of the algorithm is due to the fact that the discretization schemes that we used introduce high frequencies in the approximation of solutions, what we call non-uniform observability/Controllability. We have also shown that to obtain the uniform observability/Controllability, the solutions had to be filtered. But this method is not practical, that's why we have applied the method of filtration of initial conditions, and with the numerical simulation we show that we can control all the initial data.
In 2005, Carlos Castro and Sorin Micu [2], proved mathematically in $1D$ that with the mixed finite element method, we can obtain the uniform boundary observability/controllability without filtering, and the discrete control of HUM converge. There are still several unsolved problems, for instance a mathematical proof of the divergence of the semi-discretization with finite element method $\geq 2D$.
In sense of the difficulties, we have to make a mixed finite element code in the cases $\geq 1D$, to accelerate this code by Pyccel or Numba, also to manage the memory with MPI. In addition we also have to prove mathematically the convergence of this discretization.

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