Probability: Multivariate Models

1.

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]
\end{aligned}
$$

Because $X \sim U(-1,1)$, therefore $\mathbb{E}[X] = 0, \mathbb{E}[X^3] = 0$

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] \\
&= 0
\end{aligned}
$$

therefore

$$
\begin{aligned}
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\
&= 0
\end{aligned}
$$

Although X and Y are uncorrelated, it is definite that Y is dependent on X.

2.

To make the correlation coefficient meaningful, there must be $\mathbb{V}[X] > 0$ and $\mathbb{V}[Y] > 0$. Therefore, considering a random variable $Z = aX + Y$.

$$
\begin{aligned}
\mathbb{V}[Z] &= \mathbb{V}[aX + Y] \\
&= \mathbb{V}[X]a^2 + 2\mathrm{Cov}[X,Y]a + \mathbb{V}[Y]
\end{aligned}
$$

Because $\mathbb{V}[Z] \geq 0$ for all a. Therefore,

$$
\begin{aligned}
\Delta &\leq 0 \\
4\mathrm{Cov}^2[X,Y] - 4\mathbb{V}[X]\mathbb{V}[Y] &\leq 0 \\
\frac{\mathrm{Cov}^2[X,Y]}{\mathbb{V}[X]\mathbb{V}[Y]} &\leq 1
\end{aligned}
$$

So, there is $\rho^2 \leq 1$. Therefore, $-1 \leq \rho \leq 1$.