Probability: Multivariate Models

1.

$$
\begin{aligned}
Cov(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]
\end{aligned}
$$

Because $X \sim U(-1, 1)$, therefore $\mathbb{E}[X] = 0, \mathbb{E}[X^3] = 0$

$$
\begin{aligned}
Cov(X, Y) &= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] \\
&= 0
\end{aligned}
$$

therefore

$$
\begin{aligned}
\rho(X, Y) &= \frac{Cov(X, Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\
&= 0
\end{aligned}
$$

Although X and Y are uncorrelated, it is definite that Y is dependent on X.

2.

To make the correlation coefficient meaningful, there must be $\mathbb{V}[X] > 0$ and $\mathbb{V}[Y] > 0$. Therefore, considering a random variable $Z = aX + Y$.

$$
\begin{aligned}
\mathbb{V}[Z] &= \mathbb{V}[aX + Y] \\
&= \mathbb{V}[X]a^2 + 2\text{Cov}[X, Y]a + \mathbb{V}[Y]
\end{aligned}
$$

Because $\mathbb{V}[Z] \geq 0$ for all a. Therefore,

$$
\begin{aligned}
\Delta &\leq 0 \\
4\text{Cov}^2[X, Y] - 4\mathbb{V}[X]\mathbb{V}[Y] &\leq 0 \\
\frac{\text{Cov}^2[X, Y]}{\mathbb{V}[X]\mathbb{V}[Y]} &\leq 1
\end{aligned}
$$

So, there is $\rho^2 \leq 1$. Therefore, $-1 \leq \rho \leq 1$.

3.

$$\begin{aligned}
\mathrm{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[aX^2 + bX] - \mathbb{E}[X]\mathbb{E}[aX+b] \\
&= a\mathbb{E}[X^2] + b\mathbb{E}[X] - \mathbb{E}[X](a\mathbb{E}[X]+b) \\
&= a\mathbb{V}[X]
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}[X]\mathbb{V}[Y] &= \mathbb{V}[X]\mathbb{V}[aX+b] \\
&= a^2\mathbb{V}^2[X]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(X,Y) &= \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\
&= \frac{a}{|a|}
\end{aligned}$$

Therefore, if $a > 0$, then $\rho(X,Y) = 1$. If $a < 0$, then $\rho(X,Y) = -1$.