Probability: Multivariate Models

1.

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]$$

Because $X \sim U(-1, 1)$, therefore $\mathbb{E}[X] = 0, \mathbb{E}[X^3] = 0$

$$Cov(X, Y) = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]$$
$$= 0$$

therefore

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}}$$
$$= 0$$

Although X and Y are uncorrelated, it is definite that Y is dependent on X.

2.

To make the correlation coefficient meaningful, there must be $\mathbb{V}[X] > 0$ and $\mathbb{V}[Y] > 0$. Therefore, considering a random variable $Z = aX + Y$.

$$\mathbb{V}[Z] = \mathbb{V}[aX + Y]$$
$$= \mathbb{V}[X]a^2 + 2Cov[X, Y]a + \mathbb{V}[Y]$$

Because $\mathbb{V}[Z] \geq 0$ for all a. Therefore,

$$\Delta \leq 0$$
$$4\text{Cov}^2[X, Y] - 4\mathbb{V}[X]\mathbb{V}[Y] \leq 0$$
$$\frac{\text{Cov}^2[X, Y]}{\mathbb{V}[X]\mathbb{V}[Y]} \leq 1$$

So, there is $\rho^2 \leq 1$. Therefore, $-1 \leq \rho \leq 1$.

3.

$$\begin{aligned}
\mathrm{Cov}[X, Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[aX^2 + bX] - \mathbb{E}[X]\mathbb{E}[aX + b] \\
&= a\mathbb{E}[X^2] + b\mathbb{E}[X] - \mathbb{E}[X](a\mathbb{E}[X] + b) \\
&= a\mathbb{V}[X]
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}[X]\mathbb{V}[Y] &= \mathbb{V}[X]\mathbb{V}[aX + b] \\
&= a^2\mathbb{V}^2[X]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(X, Y) &= \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\
&= \frac{a}{|a|}
\end{aligned}$$

Therefore, if $a > 0$, then $\rho(X, Y) = 1$. If $a < 0$, then $\rho(X, Y) = -1$.

4.

a.

$$\begin{aligned}
\mathrm{Cov}[Ax] &= \mathbb{E}[(Ax - \mathbb{E}[Ax])(Ax - \mathbb{E}[Ax])^T] \\
&= \mathbb{E}[(Ax - A\mathbb{E}[x])(x^T A^T - \mathbb{E}^T[x]A^T)] \\
&= \mathbb{E}[A(x - \mathbb{E}[x])(x^T - \mathbb{E}^T[x])A^T] \\
&= A\mathbb{E}[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^T]A^T \\
&= A\Sigma A^T
\end{aligned}$$

b.

If $C = AB$, then

$$\begin{aligned}
\mathrm{tr}[AB] &= \Sigma_k c_{kk} \\
&= \Sigma_k \Sigma_i a_{ki} b_{ik} \\
&= \Sigma_k \Sigma_i b_{ik} a_{ki} \\
&= \Sigma_i \Sigma_k b_{ik} a_{ki} \\
&= \mathrm{tr}[BA]
\end{aligned}$$

c.

$$\begin{aligned}
\mathbb{E}[x^T A x] &= \mathbb{E}[\text{tr}(x^T A x)] \\
&= \mathbb{E}[\text{tr}(A x x^T)] \\
&= \text{tr}(A \mathbb{E}[x x^T]) \\
&= \text{tr}(A(\Sigma + m m^T)) \\
&= \text{tr}(A\Sigma) + m^T A m
\end{aligned}$$