Probability: Multivariate Models

1.

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]
\end{aligned}
$$

Because $X \sim U(-1,1)$, therefore $\mathbb{E}[X] = 0, \mathbb{E}[X^3] = 0$

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] \\
&= 0
\end{aligned}
$$

therefore

$$
\begin{aligned}
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\
&= 0
\end{aligned}
$$

Although X and Y are uncorrelated, it is definite that Y is dependent on X.

2.

To make the correlation coefficient meaningful, there must be $\mathbb{V}[X] > 0$ and $\mathbb{V}[Y] > 0$. Therefore, considering a random variable $Z = aX + Y$.

$$
\begin{aligned}
\mathbb{V}[Z] &= \mathbb{V}[aX + Y] \\
&= \mathbb{V}[X]a^2 + 2\mathrm{Cov}[X,Y]a + \mathbb{V}[Y]
\end{aligned}
$$

Because $\mathbb{V}[Z] \geq 0$ for all a. Therefore,

$$
\begin{aligned}
\Delta &\leq 0 \\
4\mathrm{Cov}^2[X,Y] - 4\mathbb{V}[X]\mathbb{V}[Y] &\leq 0 \\
\frac{\mathrm{Cov}^2[X,Y]}{\mathbb{V}[X]\mathbb{V}[Y]} &\leq 1
\end{aligned}
$$

So, there is $\rho^2 \leq 1$. Therefore, $-1 \leq \rho \leq 1$.

3.

$$\begin{aligned}\mathrm{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[aX^2 + bX] - \mathbb{E}[X]\mathbb{E}[aX+b] \\ &= a\mathbb{E}[X^2] + b\mathbb{E}[X] - \mathbb{E}[X](a\mathbb{E}[X]+b) \\ &= a\mathbb{V}[X]\end{aligned}$$

$$\begin{aligned}\mathbb{V}[X]\mathbb{V}[Y] &= \mathbb{V}[X]\mathbb{V}[aX+b] \\ &= a^2\mathbb{V}^2[X]\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(X,Y) &= \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} \\ &= \frac{a}{|a|}\end{aligned}$$

Therefore, if $a > 0$, then $\rho(X,Y) = 1$. If $a < 0$, then $\rho(X,Y) = -1$.

4.

a.

$$\begin{aligned}\mathrm{Cov}[Ax] &= \mathbb{E}[(Ax - \mathbb{E}[Ax])(Ax - \mathbb{E}[Ax])^T] \\ &= \mathbb{E}[(Ax - A\mathbb{E}[x])(x^T A^T - \mathbb{E}^T[x]A^T)] \\ &= \mathbb{E}[A(x - \mathbb{E}[x])(x^T - \mathbb{E}^T[x])A^T] \\ &= A\mathbb{E}[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^T]A^T \\ &= A\Sigma A^T\end{aligned}$$

b.

If $C = AB$, then

$$\begin{aligned}\mathrm{tr}[AB] &= \Sigma_k c_{kk} \\ &= \Sigma_k \Sigma_i a_{ki} b_{ik} \\ &= \Sigma_k \Sigma_i b_{ik} a_{ki} \\ &= \Sigma_i \Sigma_k b_{ik} a_{ki} \\ &= \mathrm{tr}[BA]\end{aligned}$$

c.

$$\begin{aligned}
\mathbb{E}[x^T A x] &= \mathbb{E}[\mathrm{tr}(x^T A x)] \\
&= \mathbb{E}[\mathrm{tr}(A x x^T)] \\
&= \mathrm{tr}(A \mathbb{E}[x x^T]) \\
&= \mathrm{tr}(A(\Sigma + m m^T)) \\
&= \mathrm{tr}(A\Sigma) + m^T A m
\end{aligned}$$

5.
a.

$$\begin{aligned}
p(Y \le y) &= p(WX \le y) \\
&= p(X \le y | W = 1) p(W = 1) + p(X \ge -y | W = -1) p(W = -1) \\
&= 0.5 p(X \le y) + 0.5 p(X \le y) \\
&= p(X \le y)
\end{aligned}$$

Because there is $X \sim \mathcal{N}(0, 1)$, therefore, $Y \sim \mathcal{N}(0, 1)$.

b.

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[WX^2] \\
&= \mathbb{E}[\mathbb{E}[WX^2 | W]] \\
&= p(W = 1)\mathbb{E}[X^2] + p(W = -1)\mathbb{E}[-X^2] \\
&= 0.5\mathbb{E}[X^2] - 0.5\mathbb{E}[X^2] \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Cov}[X, Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= 0
\end{aligned}$$