Lana’s Better CALC II Lecture Notes

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Contents

Chapter 1

Linear operators and diagonalization

Let us assume $\mathbb{F}$ is always either the field $\mathbb{R}$ or the field $\mathbb{C}$. We may use the word *scalar* to mean "element of $\mathbb{F}$". This chapter revolves around the concepts of eigenvalues and eigenvectors of linear operators

$$T : V \to V \tag{1.0.1}$$

from a finite-dimensional vector space V over $\mathbb{F}$ to itself, or (equivalently) matrices

$$A : \mathbb{F}^n \to \mathbb{F}^n. \tag{1.0.2}$$

We shall also learn a procedure for "diagonalising" some square matrices, which is of extreme importance in many applications.

> *Note that in Chapter 1 we assume all matrices to be square and all linear operators to be from V to V (as opposed to going from V to a different vector space W).*

1.1 Linear Operators: Introduction and Review.

We begin by recalling that for any n-dimensional $\mathbb{F}$-vector space V, a choice of a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ determines an isomorphism $V \longrightarrow \mathbb{F}^n$. Namely, the isomorphism is defined by

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.1.1}$$

This is the map sending $\mathbf{v}$ to the column vector made up of the coefficients $x_i \in \mathbb{F}$ of the unique representation of $\mathbf{v}$ as linear combination $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ in the basis elements.

This chapter is concerned with linear operators on a vector space V.

Definition 1.1.1. Let V be a vector space. A linear transformation $T : V \to V$ is called a **linear operator** on V. The set of linear operators on V is denoted $\mathrm{End}(V)$.[1]

Linear transformations $T : \mathbb{F}^n \to \mathbb{F}^n$ can be represented by $n \times n$ matrices, as was explained in Linear Algebra and Geometry I (LAG-I). We introduce the following notation.

Definition 1.1.2. Let $M_n(\mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F}$. To summarize:

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> Suppose V is an n-dimensional vector space over $\mathbb{F}$ *with a fixed basis* $\mathcal{B}$. Then
>
> - V can be identified with $\mathbb{F}^n$ by the isomorphism described in (**??**), and
> - the set $\mathrm{End}(V)$ of linear operators on V is identified with the set $M_n(\mathbb{F})$.
>
> Both of the identifications above depend on the choice of basis $\mathcal{B}$.

Example 1.1.3. *Recall that* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F})$ *represents the linear operator on* $\mathbb{F}^2$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}. \tag{1.1.2}$$

*If V is a 2-dimensional vector space over $\mathbb{F}$ with basis $\mathcal{B} = \{v_1, v_2\}$, then we have an isomorphism $V \longrightarrow \mathbb{F}^2$ defined as in (**??**) that sends*

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.1.3}$$

The matrix A therefore determines a linear operator T on V sending $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ to the vector

$$T(\mathbf{v}) = (ax_1 + bx_2)\mathbf{v}_1 + (cx_1 + dx_2)\mathbf{v}_2. \tag{1.1.4}$$

This construction, which turns $A \in M_2(\mathbb{F})$ into the linear operator $T \in \mathrm{End}(V)$, describes

[1] Note that linear operators on V would, in more general categorical language, be called *endomorphisms* of V, hence the notation $\mathrm{End}(V)$.

the identification between $M_2(\mathbb{F})$ *and* $\mathrm{End}(V)$. *Observe how the definition of the linear operator* $T : V \to V$ *from the matrix* A *relies on the choice of basis* $\mathcal{B}$.

Remark 1.1.4.