# Deriving the Feynman Rules for Quantum Electrodynamics (from the S-matrix expansion)

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## Contents



# 1 Introduction

In this section, I will show you how to derive the Feynman Rules for Quantum Electrodynamics (QED), These rules are one of the most famous parts of Feynman's Legacy. The derivation will begin by expanding the S-Operator to $2^{nd}$ Order. We will then compute the scattering matrix up to $2^{nd}$ Order (Ignoring the zeroth order forward scattering term) for a variety of physical processes by evaluating the corresponding matrix elements of the S-Operator expanded to $2^{nd}$ Order. Once we have the raw matrix elements written out (but before we have evaluated them) I will also show how to interpret each S-matrix expansion term for each physical process. The complete set of physical processes that we will be considering in our effort to Derive the Feynman Rules can be found on the Table of Contents, and in the table below:

| PROCESS | REACTION | S-MATRIX ELEMENT |
|---|---|---|
| $e^+$ Compton Scattering | $\gamma + e^- \to \gamma + e^-$ | $\langle \gamma, e^- \vert S \vert \gamma, e^- \rangle$ |
| $e^-$ Compton Scattering | $\gamma + e^+ \to \gamma + e^+$ | $\langle \gamma, e^+ \vert S \vert \gamma, e^+ \rangle$ |
| Pair Annihilation | $e^- + e^+ \to \gamma + \gamma$ | $\langle e^-, e^+ \vert S \vert \gamma, \gamma \rangle$ |
| Pair Production | $\gamma + \gamma \to e^- + e^+$ | $\langle \gamma, \gamma \vert S \vert e^-, e^+ \rangle$ |
| $e^+$ Moller Scattering | $e^+ + e^+ \to e^+ + e^+$ | $\langle e^+, e^+ \vert S \vert e^+, e^+ \rangle$ |
| $e^-$ Moller Scattering | $e^- + e^- \to e^- + e^-$ | $\langle e^-, e^- \vert S \vert e^-, e^- \rangle$ |
| Bhabha Scattering | $e^- + e^+ \to e^- + e^+$ | $\langle e^-, e^+ \vert S \vert e^-, e^+ \rangle$ |
| Electron Self-Energy | $e^- \to e^-$ | $\langle e^- \vert S \vert e^- \rangle$ |
| Positron Self-Energy | $e^+ \to e^+$ | $\langle e^+ \vert S \vert e^+ \rangle$ |
| Photon Self-Energy | $\gamma \to \gamma$ | $\langle \gamma \vert S \vert \gamma \rangle$ |
| Vacuum Self-Energy | $Vacuum \to Vacuum$ | $\langle 0 \vert S \vert 0 \rangle$ |

When proceeding through this process, it won't be quite as straightforward as I have made it sound so far. There will be lots of little steps to complete as we go. For example, we will have to prove that the first order term doesn't yield a non-vanishing contribution to anything. This will show that we are justified in not including processes that one would expect to have first order contributions in the above table. It is actually easy to see that this must be the case for the listed process above, but I will show that all first order contributions vanish even for matrix elements that, at face value, seem like they ought to have first order contribution.

## 2 Scattering Order to the $2^{nd}$ Order

Obviously, before we expand the S-operator to 2nd order, we need an expression for the S-Operator. This requires us to know what the Interacting Hamiltonian is for the theory.

So let's begin with the Lagrangian density for QED:

$$\mathcal{L}_{\mathcal{QED}} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

Where the U(1) Gauge Covariant Derivative is:

$$D_\mu \psi = [\partial_\mu + ieA_\mu]\psi \tag{2}$$

We can insert this into the Lagrangian density:

$$\mathcal{L}_{\mathcal{QED}} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - e\overline{\psi}\gamma^\mu A_\mu \psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3}$$

So the interaction Lagrangian is:

$$\mathcal{L}_{\mathcal{INT}} = -e\overline{\psi}\not{A}_\mu \psi \tag{4}$$

Therefore in this case the interaction Hamiltonian is:

$$\mathcal{H}_{\mathcal{INT}} = e\overline{\psi}\not{A}_\mu \psi \tag{5}$$

Now we insert this into the S-Operator, and then expand it. For mathematical convenience, I will actually do this in reverse order:

$$S = T\Bigg[exp\bigg(-i\int_{\infty}^{\infty}\mathcal{H}_{\mathcal{INT}}(x)d^4x\bigg)\Bigg] = T\Bigg[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^4x_1 \int \mathcal{H}_{\mathcal{INT}}(x_1)...\mathcal{H}_{\mathcal{INT}}(x_n)d^4x_1\Bigg] = \sum_{n=0}^{\infty}S^{(n)} \tag{6}$$

I will denote a spacetime point argument $(\vec{x}, t) = (x^\mu)$ as $(x)$. $S^0$ is just the identity and therefore just gives the forward scattering part. It can be ignored because all the cross section and decay rate formulas that we might be interested in, ignore the forward scattering part. So, now let's look at the first order term:

$$S^{(1)} = -ie \int T\Big(\overline{\psi}(x) A_\mu(x) \psi(x)\Big) d^4x \tag{7}$$

Wick Expansion is required here, but the wick expansion containts only one term:

$$S^{(1)} = -ie \int : \overline{\psi} A_\mu \psi : d^4x \tag{8}$$

One might think that one would have to account for the contributions made by this term to various S-matrix elements, but it turns out that this term actually only gives a vanishing contribution to physical processes (as mentioned in the intro). This will be shown in section 4 starting from the last expression for $S^{(1)}$, just after I introduce the operation of field operators, which I make use of in that section. Before we get there, let's look at the second term. According to the expansion abive, the second order term is

$$S^{(2)} = \frac{(-ie)^2}{2!} \int T\Big(\overline{\psi}(x_1) A_\mu(x_1) \psi(x_1) \overline{\psi}(x_2) A_\mu(x_2) \psi(x_2)\Big) d^4x_1 d^4x_2 \tag{9}$$

We can use Wick's Theorem to write $S^{(2)}$ in terms of normal ordered products and contracted products of two fields. Here, "contraction" refers to taking vacuum expectation value of the product of two quantum fields, which is just the propagator:

$$A_\mu(x_1) A_\nu(x_2) = \langle 0 | A_\mu(x_1) A_\nu(x_2) | 0 \rangle = i D_F^{\mu\nu}(x_2 - x_1) \tag{10}$$

$$\psi(x_1) \overline{\psi}(x_2) = \langle 0 | \psi(x_1) \overline{\psi}(x_2) | 0 \rangle = i S_F(x_2 - x_1) \tag{11}$$

I will initially denote contraction with highlighting. With this Notation, the Wick expanded second order term of the S-Operator is as follows:

$$S^{(2)} = S_0^{(2)} + S_1^{(2)} + S_2^{(2)} + S_3^{(2)} \tag{12}$$

Where

Zero Contraction Term

$$S_0^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1) \overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{13}$$

One Contraction Term

$$S_{1,1}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1) \overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{14}$$

$$S_{1,2}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1) \overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{15}$$

$$S_{1,3}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1) \overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{16}$$

Two Contraction Term

$$S_{2,1}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{17}$$

$$S_{2,2}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2)\gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{18}$$

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{19}$$

Three Contraction Term

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{20}$$

The various contracted fields are contracted with the field carrying the matching highlighting. We can now insert the propagators. For some of the terms this will require manipulations. I will explain for those that require them here. First, I will discuss $S_{1,3}^{(2)}$. Not only can we manipulate it so that the propagator is convenient to substitute in, we can prove that it is equal to $S_{1,1}^{(2)}$, so that we will work the sum of the two.

$$\begin{aligned} S_{1,1}^{(2)} + S_{1,3}^{(2)} =& \frac{(-ie)^2}{2!} \int \big[ : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2)\psi(x_2) : \\ &+ : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1)\psi(x_1)\overline{\psi}(x_2)\gamma^\nu A_\nu(x_2) \psi(x_2) : \big] d^4x_1 d^4x_2 \end{aligned} \tag{21}$$

In the second term, we will bring the second group of three fields to the left without changing any signs. This is because the fields being commuted or anticommuted are evaluated at different points, so the various necessary commutators and anticommutators vanish, and there are even number of fermion field anticommutations, so all of the minus signs cancel out. This gives:

$$\begin{aligned} S_{1,1}^{(2)} + S_{1,3}^{(2)} =& \frac{(-ie)^2}{2!} \int \big[ : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2)\psi(x_2) : \\ &+ : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2)\psi(x_2) : \big] d^4x_1 d^4x_2 \end{aligned} \tag{22}$$

Because both $x_1$ and $x_2$ are completely integrated over, they are just dummy variables that can be interchanged. I will therefore flip the labels in the second term. We can now see that the two terms are identical. So we can now combine like terms:

$$S_{1,1}^{(2)} + S_{1,3}^{(2)} = -e^2 \int : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2)\psi(x_2) : d^4x_1 d^4x_2 \tag{23}$$

We can now insert the fermion propagator:

$$S_{1,1}^{(2)} + S_{1,3}^{(2)} = -e^2 \int : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2) : d^4x_1 d^4x_2 \tag{24}$$

We essentially find the same situation for $S_{2,1}^{(2)}$ and $S_{2,2}^{(2)}$ as we did for $S_{1,1}^{(2)}$ and $S_{1,3}^{(2)}$, so we will treat the sum of them.

$$\begin{aligned} S_{2,1}^{(2)} + S_{2,2}^{(2)} =& \frac{(-ie)^2}{2!} \int \big[ : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : \\ &+ : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2)\gamma^\nu A_\nu(x_2) \psi(x_2) : \big] d^4x_1 d^4x_2 \end{aligned} \tag{25}$$

In the second term, we can again bring the second group of the three fields to the left without changing any signs. The reason for this is as before:

$$\begin{aligned} S_{2,1}^{(2)} + S_{2,2}^{(2)} =& \frac{(-ie)^2}{2!} \int \big[ : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : \\ & + : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : \big] d^4x_1 d^4x_2 \end{aligned} \tag{26}$$

Both $x_1$ and $x_2$ are dummy variables, so I am free to flip them in the second term. The two terms are now clearly identical:

$$S_{2,1}^{(2)} + S_{2,2}^{(2)} = -e^2 \int : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{27}$$

Now, we can substitute in the propagators:

$$\boxed{S_{2,1}^{(2)} + S_{2,2}^{(2)} = -e^2 \int : \overline{\psi}(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD_F^{\mu\nu}(x_2 - x_1)\psi(x_2) : d^4x_1 d^4x_2} \tag{28}$$

Some special manipulations are also required for the convenient substitution of the propagators into $S_{1,3}^{(2)}$. Namely, we will substitute the propagators in, and we will take the trace of the scalar integrand and make use of the cyclic property of trace:

$$\begin{aligned} S_{2,3}^{(2)} =& \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \\ iS_F(x_2 - x_1) =& \ \psi(x_1)\overline{\psi}(x_2) = \langle 0|\psi(x_1)\overline{\psi}(x_2)|0\rangle \end{aligned} \tag{29}$$

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{30}$$

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[\overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2) \psi(x_2)] : d^4x_1 d^4x_2 \tag{31}$$

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[\psi(x_2) \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)] : d^4x_1 d^4x_2 \tag{32}$$

$$\boxed{S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[iS_F(x_2 - x_1) \gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)] : d^4x_1 d^4x_2} \tag{33}$$

The three contraction term requires similar treatment of the trace:

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu A_\mu(x_1) \psi(x_1)\overline{\psi}(x_2) \gamma^\nu A_\nu(x_2) \psi(x_2) : d^4x_1 d^4x_2 \tag{34}$$

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1) \gamma^\mu S_F(x_2 - x_1)\gamma^\nu A_\nu(x_2) A_\mu(x_1) \psi(x_2) : d^4x_1 d^4x_2 \tag{35}$$

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[\overline{\psi}(x_1) \gamma^\mu S_F(x_2 - x_1)\gamma^\nu iD_F^{\mu\nu}(x_2 - x_1) \psi(x_2)] : d^4x_1 d^4x_2 \tag{36}$$

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : \psi(x_2) \overline{\psi}(x_1) \gamma^\mu S_F(x_2 - x_1)\gamma^\nu iD_F^{\mu\nu}(x_2 - x_1) : d^4x_1 d^4x_2 \tag{37}$$

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[\psi(x_2) \overline{\psi}(x_1) \gamma^\mu S_F(x_2 - x_1)\gamma^\nu iD_F^{\mu\nu}(x_2 - x_1)] : d^4x_1 d^4x_2 \tag{38}$$

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[S_F(x_1 - x_2)\gamma^\mu S_F(x_2 - x_1)\gamma^\nu iD_F^{\mu\nu}(x_2 - x_1)] : d^4x_1 d^4x_2 \tag{39}$$

Substitution of the propagators into the other terms is trivial. The complete list of $2^{nd}$ order S-operator terms of fermion and photon propagators is as follows

$$S^{(2)} = S_0^{(2)} + S_1^{(2)} + S_2^{(2)} + S_3^{(2)} \tag{40}$$

Zero Contraction Term

$$S_0^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1)\not{A}_\mu(x_1)\psi(x_1)\overline{\psi}(x_2)\not{A}_\nu(x_2)\psi(x_2) : d^4x_1 d^4x_2 \tag{41}$$

One Contraction Term

$$S_{1,1}^{(2)} + S_{1,3}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1)\not{A}(x_1)\psi(x_1) iS_F(x_2 - x_1)\not{A}_\mu(x_2)\psi(x_2) : d^4x_1 d^4x_2 \tag{42}$$

$$S_{1,2}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1)\gamma^\mu\psi(x_1) iD_{\mu\nu}^F(x_2 - x_1)\overline{\psi}(x_2)\gamma^\nu\psi(x_2) : d^4x_1 d^4x_2 \tag{43}$$

Two Contraction Term

$$S_{2,1}^{(2)} + S_{2,2}^{(2)} = \frac{(-ie)^2}{2!} \int : \overline{\psi}(x_1)\gamma^\mu\psi(x_1) iS_F(x_2 - x_1)\gamma^\nu iD_{\mu\nu}^F(x_2 - x_1)\psi(x_2) : d^4x_1 d^4x_2 \tag{44}$$

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[iS_F(x_2 - x_1)\gamma^\mu A_\mu iS_F(x_2 - x_1)\gamma^\nu A_\nu] : d^4x_1 d^4x_2 \tag{45}$$

Three Contraction Term

$$S_3^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[iS_F(x_2 - x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu : d^4x_1 d^4x_2 \tag{46}$$

This is the S-operator expansion to the $2^{nd}$ order that we need. Now, we will consider the physical interpretation of field operators and propagators that appear in the S-operator terms. Then we can finally return to the first order term, and prove that its contribution vanishes before moving any further consideration of the second order terms.

# 3 Interpretation of Field Operators and Propagators

Let's now consider the fields themselves. These are interacting quantum fields in the interacting picture, so they have a mathematical form of Heisenberg Free Fields:

$$A_\mu(x) = \int \frac{1}{2\omega}(\epsilon_\mu^\lambda a_\lambda(\vec{k})e^{ikx} + \epsilon_\mu^\lambda a_\lambda^\dagger(\vec{k})e^{-ikx})\frac{d^3k}{(2\pi)^2} \tag{47}$$

$$\psi(x) = \sum_s \int \frac{m}{\omega}[c_s(\vec{p})u_s(\vec{p})e^{ip\cdot x} + d_s^\dagger(\vec{p})\bar{v}_s(\vec{p})e^{ip\cdot x}]\frac{d^3k}{(2\pi)^2} \tag{48}$$

$$\bar{\psi}(x) = \sum_s \int \frac{m}{\omega}[d_s(\vec{p})\bar{v}_s(\vec{p})e^{ip\cdot x} + c_s^\dagger(\vec{p})u_s(\vec{p})e^{ip\cdot x}]\frac{d^3k}{(2\pi)^2} \tag{49}$$

From the free field theory, $c_s^\uparrow(c_s)$ creates(annihilates) an electron with a specific momentum, and $d_s^\uparrow(d_s)$ creates (annihilates) a positron with a specific momentum. In addition, we know that for the photon, $a_s^\uparrow(a_s)$ creates (annihilates) a photon with a specific momentum. Because these states are all particles with a specific momentum, they are all plane wave states. If one were to apply these Fourier coefficient operators times its associated spinor and normalization factor to a vacuum, and then project it onto position space, one would obtain the plane wavefunction for a particle of that momentum.

The quantum field terms simply inverse Fourier transforms (Fourier transform) into position space of these plane wavefunction creating (annihilating) operators. Therefore they have all the same effect as they did before the Fourier transform was applied, but at specific momentum. Putting this into table, we have:

$$\psi = \psi^+ + \psi^- \qquad \bar{\psi} = \bar{\psi}^+ + \bar{\psi}^- \qquad A_\mu = A_\mu^+ + A_\mu^-$$

$$\overbrace{\psi^+ = \sum_s \int \frac{m}{\omega}c_s(\vec{p})u_s(\vec{p})e^{ip\cdot x}\frac{d^3k}{(2\pi)^2}}^{\text{annihilates an electron at x}} \quad \overbrace{\bar{\psi}^+ = \sum_s \int \frac{m}{\omega}d_s(\vec{p})\bar{v}_s(\vec{p})e^{ip\cdot x}\frac{d^3k}{(2\pi)^2}}^{\text{annihilates a positron at x}} \quad \overbrace{A_\mu^+ = \int \frac{1}{2\omega}\epsilon_\mu^\lambda a_\lambda^\dagger(\vec{k})e^{-ikx}\frac{d^3k}{(2\pi)^2}}^{\text{annihilates a photon at x}}$$

$$\overbrace{\psi^- = \sum_s \int \frac{m}{\omega}d_s^\dagger(\vec{p})\bar{v}_s(\vec{p})e^{ip\cdot x}\frac{d^3k}{(2\pi)^2}}^{\text{creates a positron at x}} \quad \overbrace{\bar{\psi}^- = \sum_s \int \frac{m}{\omega}c_s^\dagger(\vec{p})u_s(\vec{p})e^{ip\cdot x}\frac{d^3k}{(2\pi)^2}}^{\text{creates an electron at x}} \quad \overbrace{A_\mu^- = \int \frac{1}{2\omega}\epsilon_\mu^\lambda a_\lambda(\vec{k})e^{ikx}\frac{d^3k}{(2\pi)^2}}^{\text{creates a photon at x}}$$

With these sorted, we can now finally get back to showing that the first order term doesn't yield any nonvanishing contribution.

# 4 Proof that First Order Term Always Vanishes

For all the key processes, we can see quite quickly that the first order term in the S-operator expansion can't contribute to anything nonvanishing. This is because of an inevitable mismatch between the contents of the initial and final states, and the operators in $S^{(1)}$. One of the states would inevitably get annihilated. However, one can conceive of other physical processes for which one might expect the first order term to yield a nonvanishing contribution. I will show in this section that even for those processes, the first order contribution is zero.

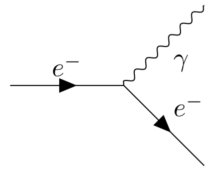
Let's recall the form of $S^{(1)}$ from earlier:

$$S^{(1)} = -ie\int :\bar{\psi}\not{A}\psi: d^4x \tag{50}$$

We can now insert the two break down of the quantum fields into it. Doing that yields the term like following:

$$S_1^{(1)} = -ie\int :\bar{\psi}^-(x)\gamma^\mu A_\mu^-(x)\psi^+(x): \tag{51}$$

We will show that even the most natural matrix element of this part of $S^{(1)}$ vanishes as an example. All the other terms in $S^{(1)}$ can be shown to vanish via essentially identical calculations. Using the field operator interpretations that we have established, we can see that this operator will annihilate an electron at x. This is described by the following Feynman Diagram:




In $S^{(1)}$, $:\bar{\psi}^-(x)\gamma^\mu A^-_\mu(x)\psi^+(x):$ is integrated over all spacetime points to give its complete contribution to the scattering operator, and by extension the scattering matrix. This integration makes sense because this interaction could potentially happen anywhere. Each spacetime point could therefore be expected to contribute to the probability amplitude for any transition that coud possibly mediated by this proccess. Of course, as I have said repeatedly, $S^{(1)}$ actually doesn't give a nonvanishing contribution to anything , in the end. However, without knowing this, the proccess that we would otherwise most expect this part of $S^{(1)}$ to yield a non-zero contribution to, is the following:

$$\begin{aligned} |in\rangle &= |e^-\rangle \\ |out\rangle &= |e^-, \gamma\rangle \end{aligned} \tag{52}$$

$$e^- \rightarrow \gamma e^- \tag{53}$$

So the matrix element of interest is:

$$\langle e^-, \gamma | s_1^{(1)} | e^- \rangle = -ie \int \langle e^-, \gamma | \bar{\psi}^-(x)\gamma^\mu A^-_\mu(x)\psi^+(x) | e^- \rangle \tag{54}$$

In order to show that this matrix element is zero, we need some equations previously presented in this document, and the cannonical commutation and anticommutation relations for the Fourier coefficient operators. The latter are as follows:

$$\begin{aligned} [a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k'})] &= (2\pi)^3 2\omega\eta\lambda\lambda'\delta^3(\vec{k}-\vec{k'}) \\ \{c_r(\vec{p}), c^\dagger_s(\vec{p'})\} &= (2\pi)^3 \frac{\omega}{m}\delta^3(\vec{p}-\vec{p'})\delta_{rs} \\ \{d_r(\vec{p}), d^\dagger_s(\vec{p'})\} &= (2\pi)^3 \frac{\omega}{m}\delta^3(\vec{p}-\vec{p'})\delta_{rs} \end{aligned} \tag{55}$$

All the other femionic operators are zero, all the other bosonic commutators are zero, and all commutators of fermionic operators with bosonic ones are zero. The first step in the calculation is to derive a couple of identities for later use. Namely, we want to evaluate the the following quantities:

$$\psi^+|e^-\rangle \quad A^+_\mu|\gamma\rangle \tag{56}$$

Let's start with the first one. Recall the following two facts:

$$|e^+\rangle = c^\dagger_s(p)|0\rangle \quad \psi^+(x) = \sum_s \int \frac{m}{\omega} c_s(\vec{p}) u_s(\vec{p}) e^{-ip\cdot x} \frac{d^3k}{(2\pi)^2 2\omega} \tag{57}$$

Inserting it into these quantity that we wish to evaluate gives:

$$\begin{aligned} \psi^+(x)|e^-\rangle &= \sum_r \int \frac{m}{\omega} c_r(\vec{k}) u_r(\vec{k}) e^{-ip\cdot x} c^\dagger_s(\vec{p}) \frac{d^3p}{(2\pi)^2 2\omega} \\ &= \int \frac{m}{\omega} e^{-ip\cdot x} \sum_r u_r(\vec{k}) c_r(\vec{k}) c^\dagger_s(\vec{p})|0\rangle \frac{d^3p}{(2\pi)^2 2\omega} \end{aligned} \tag{58}$$

We can now rewrite the operator product in terms of an anticommutator:

$$c_r(\vec{k})c^\dagger_s(\vec{p})|0\rangle = [\{c_r(\vec{k}), c^\dagger_s(\vec{p})\} - c^\dagger_s(\vec{p})c_r(\vec{k})]|0\rangle \tag{59}$$

Annihilation operators acting on a vacuum give zero, so:

$$c_r(\vec{k})c_s^\dagger(\vec{p})|0\rangle = [\{c_r(\vec{k}), c_s^\dagger(\vec{p})\}]|0\rangle \frac{d^3p}{(2\pi)^2 2\omega} \tag{60}$$

Therefore:

$$\psi^+(x)|e^-\rangle = \int \frac{m}{\omega} e^{-ip\cdot x} \sum_r u_r(\vec{k})\{c_r(\vec{k}), c_s^\dagger(\vec{p})\}|0\rangle \tag{61}$$

We can now remember the anticommtation relation from above:

$$\{c_r(\vec{p}), c_s^\dagger(\vec{p})\} = (2\pi)^3 \frac{\omega}{m} \delta^3(\vec{p} - \vec{p'})\delta_{rs} \tag{62}$$

In the k-integration, the time phase factor becomes dependent on the energy associated with $\vec{p}$ instead of that of $\vec{k}$, because the energy is a function of 3-Momentum, which is usually set to $\vec{p}$ by the delta function. Our final result is therefore:

$$\boxed{\psi^+(x)|e^-\rangle = u_s(\vec{p})e^{ip\cdot x}|0\rangle} \tag{63}$$

Now let's evaluate the quantity we wanted an identity for:

$$A_\mu^+(x)|\gamma\rangle = A_\mu^+(x)a_\lambda^\dagger(\vec{k})|0\rangle \tag{64}$$

The formula for $A_\mu^+(x)$ is:

$$A_\mu^+(x) = \int \epsilon_\mu^\lambda a_\lambda(\vec{k})e^{-ik\cdot x} \frac{d^3p}{(2\pi)^2 2\omega} \tag{65}$$

Inserting this gives:

$$A_\mu^+(x)|\gamma\rangle = \int \epsilon_\mu^{\lambda'} a_{\lambda'}(\vec{k})a_\lambda^\dagger(\vec{k})e^{-ik'\cdot x} \frac{d^3p}{(2\pi)^2 2\omega}|0\rangle \tag{66}$$

Using the same rational as in the previous calculation, we can substitute in a commutator:

$$A_\mu^+(x)|\gamma\rangle = \int \epsilon_\mu^{\lambda'} [a_{\lambda'}(\vec{k}), a_\lambda^\dagger(\vec{k})]e^{-ik'\cdot x} \frac{d^3p}{(2\pi)^2 2\omega}|0\rangle \tag{67}$$

$$[a_{\lambda'}(\vec{k}), a_\lambda^\dagger(\vec{k})] = -(2\pi)^3 2\omega \eta_{\lambda\lambda'}\delta^3(\vec{k} - \vec{k'}) \tag{68}$$

$$A_\mu^+(x)|\gamma\rangle = \int \epsilon_\mu^{\lambda'} \eta_{\lambda\lambda'} e^{-ik'\cdot x}\delta^3(\vec{k} - \vec{k'}) \frac{d^3p}{(2\pi)^2 2\omega}|0\rangle = -\epsilon_\mu^{\lambda'}\eta_{\lambda\lambda'}e^{-ik'\cdot x}|0\rangle \tag{69}$$

If we take $|\gamma\rangle$ to be a transverse state, then the longitudinal state, then the longitudinal and temporal annihilation operator terms $\big($ in the sum in $A_\mu^+(x)$ $\big)$ will just annihilate $|\gamma\rangle$, and they therefore vanish from the sum. This reduces the $\lambda'$ to a two-term sum where $\eta_{\lambda\lambda'}$ is replaced with a $-\delta_{\lambda\lambda'}$, where the indices on the Kronecker delta are now two dimensional. This switch to the two dimensional Kronecker delta also means that upper and lower indices are identical, and can be used interchangably. This allows the sum $\lambda'$ to be performed:

$$A_\mu^+(x)|\gamma\rangle = \epsilon_\mu^{\lambda'}\delta_{\lambda\lambda'}e^{-ik'\cdot x}|0\rangle = \epsilon_\mu^{\lambda'}e^{-ik'\cdot x}|0\rangle \tag{70}$$

So, the final result is:

$$
\boxed{A_\mu^+(x)|\gamma\rangle = \epsilon_\mu^\lambda e^{-ik\cdot x}|0\rangle} \tag{71}
$$

To summarize, the two identities that we wanted are:

$$
\begin{aligned}
A_\mu^+(x)|\gamma\rangle &= \epsilon_\mu^\lambda e^{-ik\cdot x}|0\rangle \\
\psi^+(x)|e^-\rangle &= u_s(\vec{p})e^{ip\cdot x}|0\rangle
\end{aligned} \tag{72}
$$

These same identities will also be used for second order matrix evaluations later. So now let's use these identities to evaluate target amplitude:

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
\langle e^-(p',s')\gamma(k',\lambda')|[-ie\int \bar{\psi}^-(x)\gamma^\mu A_\mu^-(x)\psi^+(x)d^4(x)]|e^-(p,s)\rangle \\
-ie\int \langle e^-(p',s')\gamma(k',\lambda')|[\bar{\psi}^-(x)\gamma^\mu A_\mu^-(x)\psi^+(x)]|e^-(p,s)\rangle d^4(x)
\end{aligned} \tag{73}
$$

We can now do a substitution using one of our identities:

$$
\psi^+(x)|e^-\rangle = u_s(\vec{p})e^{-ip\cdot x}|0\rangle \tag{74}
$$

Doing this gives:

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
-ie\int \langle e^-(p',s')\gamma(k',\lambda')|[\bar{\psi}^-(x)\gamma^\mu A_\mu^-(x)u_s(\vec{p})e^{-ip\cdot x}]|0\rangle d^4(x)
\end{aligned} \tag{75}
$$

Now we can deduce another identity from one of the ones that we already derived:

$$
\psi^+(x)|e^-\rangle = u_s(\vec{p})e^{-ip\cdot x}|0\rangle \to \langle e^-|\bar{\psi}^-(x) = \langle 0|e^{ip\cdot x}\bar{u}_s(\vec{p}) \tag{76}
$$

Inserting this gives:

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
-ie\int \langle \gamma(k',\lambda')|[e^{ip'\cdot x}\bar{u}_{s'}(\vec{p})\gamma^\mu A_\mu^-(x)u_s(\vec{p})e^{-ip\cdot x}]|0\rangle d^4(x)
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
-ie\int \langle \gamma(k',\lambda')|[\bar{u}_{s'}(\vec{p})\gamma^\mu A_\mu^-(x)u_s(\vec{p})e^{-i(p-p')\cdot x}]|0\rangle d^4(x)
\end{aligned} \tag{78}
$$

We can similarly derive another helpful identity from the photon field identity we already derived:

$$
A_\mu^+(x)|\gamma\rangle = \epsilon_\mu^\lambda e^{-ik\cdot x}|0\rangle \to \langle\gamma|A_\mu^-(x) = \langle 0|\epsilon_\mu^\lambda e^{ik\cdot x} \tag{79}
$$

Inserting this gives:

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
-ie\int \langle 0|[\epsilon_\mu^\lambda e^{ik'\cdot x}\bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p})e^{-i(p-p')\cdot x}]|0\rangle d^4(x)
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle &= \\
-ie\int \langle 0|[\epsilon_\mu^\lambda \bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p})e^{-i(p-p'-k')\cdot x}]|0\rangle d^4(x)
\end{aligned} \tag{81}
$$

Now, there are no operator left. Everything in the matrix element is a c-number an can be pulled out:

$$\begin{gathered}\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle = \\ -ie\int[\epsilon_\mu^\lambda \bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p})e^{-i(p-p'-k')\cdot x}]\langle 0|0\rangle d^4(x)\end{gathered} \tag{82}$$

We can take the vacuum state to be normalized, as we will do for the rest of the video:

$$\begin{gathered}\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle = \\ -ie\int[\epsilon_\mu^\lambda \bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p})e^{-i(p-p'-k')\cdot x}]d^4(x)\end{gathered} \tag{83}$$

We can recognize the delta function showing up in here. Basically, doing the x integration yields a delta function. One must remember the factor of $1/(2\pi)^4$ that is part of the definition of the delta function:

$$\begin{gathered}\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle = \\ -ie(2\pi)^4\delta^4(p-p'-k)\epsilon_\mu^\lambda \bar{u}_{s'}(\vec{p'})\gamma^\mu u_s(\vec{p})\end{gathered} \tag{84}$$

Now we can identify the Feynman Amplitude:

$$\begin{gathered}\langle e^-(p',s')\gamma(k',\lambda')|s^{(1)}|e^-(p,s)\rangle = \\ -ie(2\pi)^4\delta^4(p-p'-k)\epsilon_\mu^\lambda \mathcal{M}_{fi}\end{gathered} \tag{85}$$

The Feynman Amplitude is

$$\mathcal{M}_{fi} = -ie\bar{u}_{s'}(\vec{p'})\epsilon_\mu^\lambda\gamma^\mu u_s(\vec{p}) \tag{86}$$

Now, the we see this delta function showing up: $\delta^4(p-p'-k)$, we can answer the question of whether or not it's argument can actually be zero. This delta function forces the following constraint:

$$p'_\mu + k'_\mu = p_\mu \tag{87}$$

Which gives the following energy relation:

$$\begin{gathered}p'_0 + k'_0 = p_0 \\ \sqrt{|\vec{p'}|^2+m^2} + |\vec{k}| = \sqrt{|\vec{p'}|^2+m^2}\end{gathered} \tag{88}$$

And the following three-momentum relation:

$$\vec{p'} + \vec{k'} = \vec{p} \tag{89}$$

We can take the rest frame of the incoming electron without loss of generality because the s-matrix is Lorentz Invariant:

$$\begin{gathered}\sqrt{|\vec{p'}|^2+m^2} + |\vec{k'}| = \pm m \\ \vec{p'} + \vec{k'} = \vec{p}\end{gathered} \tag{90}$$

The momentum relation implies:

$$\vec{p'} = \vec{k'} \tag{91}$$

Let's insert this into the energy relation:

$$\sqrt{|\vec{p'}|^2+m^2} + |\vec{p'}| = \pm m \tag{92}$$

The equality is only satisfied when $\vec{p}+\vec{k}=0$. Therefore, the delta function only won't zero the first order contribution tp the scattering matrix if both particles don't leave with zero momentum when in the rest frame of the initial electron. In other words, the emmitted photon doesn't exists, and the initial electron remains unaffected by interaction. Lorentz Invariance of the scattering matrix guarantees this effect if not specific to the rest frame, but it is completely general. Matrix elements of all of the other terms in $s^1$ vanish for the same reason. The calculations of all of them are essentially identical to the one presented here.

Now that we have completed our discussion on the $s^1$ matrix elements, we can move on further discussing the second order terms. The first step is to simplify our discussion by figuring out exactly which second order S-operator terms yield nonvanishing contributions to which matrix elements from the list of physical processes that we care about (see introduction). We can then ignore all of the rest of the S-operator terms. This step comprises of section 5. Once this is done, the following step is to use already discussed interpretations of the field operators to associate Feynman Diagrams with each contributing second order matrix element term (section 6). These terms must then be computed further, to the point where the Feynman amplitudes can't be identified (section 7 and 8), then we compare the Feynman amplitudes to their associated Feynman Diagrams to reveal the Feynman rules (section 9).

# 5 Which scattering Operator Terms Contribute to Each Scattering Matrix Elements

When we take the matrix elements of the S-operator for the value processes, many of the terms in the second-order S-matrix will vanish. The first reason why some S-operator terms will have vanishing matrix elements is that, for a given process, the initial or final state may not be exclusively populated with a set of particles matching the annihilation operators that show up in a particular S-operator term. Annihilation operators zero states not containing their corresponding particle type in the matching quantum state. This is actually by far the most common reason why S-operator terms may yield vanishing matrix element terms. There is, however one other reason. $S_3^{(2)}$ has no operators in it at all, but it yields vanishing contributions to all but the vacuum to vacuum transition. What happens is this: even when the particle matches in the initial and final states, their quantum states won't, because we are ignoring fowrward scattering. Therefore, the matrix element contribution will vanish because of orthogonality.

As it happens, we will find that most terms don't contribute to any of the processes that we are considering. The goal of this section is to figure out which S-operator terms have a chance of giving a nonzero contribution to each matrix element. To begin this process let's consider the following S-operator term:

$$S_{1,1}^{(2)}+S_{1,3}^{(2)}=-e^2\int:\overline{\psi}(x_1)\gamma^{\mu}A_{\mu}(x_1)iS_F(x_2-x_1)\gamma^{\nu}A_{\nu}(x_2)\psi(x_2):d^4x_1d^4x_2 \tag{93}$$

Remember from section 3 that these field operators break apart into creation and annihilation parts that create and annihilate particles that localized at a specific spacetime point. Also, remember that propagators propagate a virtual particle between two space time points. Inspection of Eq. A therefore tells us that this term in the S-operator has the possibility to contribute to any proccess that has two photons in the initial or final state, and two fermions (electrons and positrons) in the initial and final state. Of the proccesses of interest, we therefore expect this term to contribute to the elements describing the Compton Scattering of both types, pair creation and pair annihilation. By the same arguments, we see that Eq. A cannot contribute to any of the other processes. The matrix elements simply vanish because it gets annihilated.

To find out which specific terms from Eq. A contribute to each matrix element, we must insert the fields as a sum of creation and annihilation parts,

$$\psi(x)=\psi^+(x)+\psi^-(x)\quad\bar{\psi}(x)=\bar{\psi}^+(x)+\bar{\psi}^-(x)\quad A_\mu=A_\mu^++A_\mu^- \tag{94}$$

And them multiply them out. Each quantum field is a sum of two parts, so we expect Eq. A to produce 16 terms. We can then decide whether or not a given one of the 16 terms contributes to each matrix element, by looking for annihilation operators, in the S-operator terms that are of a particle type that is not present in the state in which it is being applied. If there is no such case, then the particular S-operator term in question probably contributes to the given matrix element. I say probably because in the case of first order contribution, we found the matrix elements to be still vanishing. This will not turn out to be the case for the second order matrix elements.

In doing this inspection, it is important to remember that creation operators, when acting left, are annihilation operators. For each of the four processes that Eq. A has the potential to contribute to, this inspection yields TABLE A:

TABLE A

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. A Integrand Terms |
|---|---|---|---|
| $e^+$ Compton Scattering | $\gamma + e^- \to \gamma + e^-$ | $\langle \gamma, e^- \vert S \vert \gamma, e^- \rangle$ | $:\overline{\psi}^-(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^+(x_2):$ |
| | | | $:\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| $e^-$ Compton Scattering | $\gamma + e^+ \to \gamma + e^+$ | $\langle \gamma, e^+ \vert S \vert \gamma, e^+ \rangle$ | $:\overline{\psi}^+(x_1)\gamma^\mu A^+_\mu(x_1) iS_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^+(x_2):$ |
| | | | $:\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| pair annihilation | $e^- + e^+ \to \gamma + \gamma$ | $\langle e^-, e^+ \vert S \vert \gamma, \gamma \rangle$ | $:\overline{\psi}^-(x_1) A^-_\mu(x_1) iS_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| pair production | $\gamma + \gamma \to e^- + e^+$ | $\langle \gamma, \gamma \vert S \vert e^-, e^+ \rangle$ | $:\overline{\psi}^+(x_1) A^+_\mu(x_1) iS_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^+(x_2):$ |

If it isn't clear, one can simply insert these integrand terms back in Eq. A to get some S-operator terms which yield a non-vanishing matrix element contribution. See the table in the next section to see this done for all the processes that we are discussing in this video.

The 10 remaining terms in the integrand of Eq. A don't contribute to any of the processes under consideration in this video. This is becase they all contain annihilation operators that would annihilate either the initial or final state in every matrix element we are considering.

These are the 10 terms that follow:

| 3 Particles in, 1 Particle out | 1 Particle in, 3 Particles out |
|---|---|
| $:\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^+(x_2):$ | $:\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| $:\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^+(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| $:\overline{\psi}^+(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^+(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^-(x_2):$ |
| $:\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^+_\mu(x_2)\gamma^\nu \psi^-(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu A^-_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi^+(x_2):$ |
| **0 Particles in, 4 Particles out** | **4 Particles in, 0 Particles out** |
| $:\overline{\psi}(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi(x_2):$ | $:\overline{\psi}(x_1)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1) A^-_\mu(x_2)\gamma^\nu \psi(x_2):$ |

We have identified that $S^{(2)}_{1,1} + S^{(2)}_{1,3}$ contributes to the matrix elements of the processes of TABLE A, but we can go a step further. We can see that there are no other term there is a possibility between matchup between the particle content of the initial and final states, and the term's annihilation operators for any TABLE A proccess. They therefore give vanishing contribution to the matrix element. $S^{(2)}_3$ gives a vanishing contribution because of the orthogonality argument already given. Let's now perform the same inspection analysis with another second order term. Specifically, let's consider $S^{(2)}_{1,2}$:

$$S^{(2)}_{1,2} == \frac{(-ie)^2}{2!} \int :\overline{\psi}(x_1)\gamma^\mu \psi(x_1) D^{\mu\nu}_F(x_2 - x_1)\overline{\psi}(x_2) D\gamma^\nu \psi(x_2): d^4x_1 d^4x_2 \tag{95}$$

This term clearly has the capacity to contribute processes that involve two fermions scattering off to each other, and will yield a vanishing contribution to any other processes we are considering. By again inserting:

$$\psi(x) = \psi^+(x) + \psi^-(x) \quad \bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x) \tag{96}$$

and then remembering the interpretation of individual quantum fields, the inspection analysis yields the results expressed in TABLE B:

TABLE B

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. B Integrand Terms |
|---|---|---|---|
| $e^-$ Moller Scattering | $e^- + e^- \to e^- + e^-$ | $\langle e^-, e^- \vert S \vert e^-, e^- \rangle$ | $:\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)D\gamma^\nu\psi^+(x_2):$ |
| $e^+$ Moller Scattering | $e^+ + e^+ \to e^+ + e^+$ | $\langle e^+, e^+ \vert S \vert e^+, e^+ \rangle$ | $:\overline{\psi}^+(x_1)\gamma^\mu\psi^-(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)D\gamma^\nu\psi^-(x_2):$ |
| | | | $:\overline{\psi}^-(x_1)\gamma^\mu\psi^-(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)D\gamma^\nu\psi^+(x_2):$ |
| Bhabha Scattering | $e^- + e^+ \to e^- + e^+$ | $\langle e^-, e^+ \vert S \vert e^-, e^+ \rangle$ | $:\overline{\psi}^+(x_1)\gamma^\mu\psi^+(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)D\gamma^\nu\psi^-(x_2):$ |
| | | | $:\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)D\gamma^\nu\psi^-(x_2):$ |
| | | | $:\overline{\psi}^+(x_1)\gamma^\mu\psi^-(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)D\gamma^\nu\psi^+(x_2):$ |

It is important to take note that there aren't four distinct contributions to Bhabha Scattering, from Eq. B. When these terms are inserted back into the integral in Eq. B, one obtains 2 pairs of identical terms, because two pairs of integrand terms become identical upon interchanging the dummy integration variables in one term each pair. Remember, the integrations are all over the space, so both integrations are identical. This is why the integration variables are interchangable. One can therefore much more compactly write the Bhabha scattering entry as follows:

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. B Integrand Terms |
|---|---|---|---|
| Bhabha Scattering | $e^- + e^+ \to e^- + e^+$ | $\langle e^-, e^+ \vert S \vert e^-, e^+ \rangle$ | $2:\overline{\psi}^-(x_1)\gamma^\mu\psi^-(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)D\gamma^\nu\psi^+(x_2):$ |
| | | | $2:\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)D_F^{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)D\gamma^\nu\psi^-(x_2):$ |

just like when the first term we analyzed, the 10 remaining terms that come from multiplying out Eq. B don't contribute to any of the processes under consideration in this video. (in fact, they contribute to nothing generally) for exactly the same reason as before. These ten remaining terms are:

| 3 Particles in, 1 Particle out | 1 Particle in, 3 Particles out |
|---|---|
| $:\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\psi^+(x_2)\gamma^\nu\psi^+(x_2):$ | $:\overline{\psi}^+(x_1)\gamma^\mu\overline{\psi}^-(x_1)D^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi^-(x_2):$ |
| $:\overline{\psi}^+(x_1)\gamma^\mu\psi^-(x_1)D^F_{\mu\nu}(x_2 - x_1)\psi^+(x_2)\gamma^\nu\psi^+(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu\overline{\psi}^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi^-(x_2):$ |
| $:\overline{\psi}^+(x_1)\gamma^\mu\psi^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\psi^-(x_2)\gamma^\nu\psi^+(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu\overline{\psi}^-(x_1)D^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^-(x_2):$ |
| $:\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\psi^+(x_2)\gamma^\nu\psi^-(x_2):$ | $:\overline{\psi}^-(x_1)\gamma^\mu\overline{\psi}^-(x_1)D^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi^+(x_2):$ |
| **0 Particles in, 4 Particles out** | **4 Particles in, 0 Particles out** |
| $:\overline{\psi}(x_1)\gamma^\mu\psi^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\psi^-(x_2)\gamma^\nu\psi(x_2):$ | $:\overline{\psi}(x_1)\gamma^\mu\overline{\psi}^+(x_1)D^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi(x_2):$ |

The same type of inspection analysis show that no other S-operator term to $2^{nd}$ order term yields a nonzero matrix element contribution for Moller scattering or Bhabha scattering. The reasons for this are the same as when we considered the contributions of Eq. A.

At this point we have identified all of the second order matrix element contributions for all of the processes of interest except four. We have not yet considered, electron, positron, and photon self-energy at the second order, and we have not yet considered vacuum energy.

Because a self-energy is a self-interaction, we are dealing with a proccess where one of the particle enters or leaves unchanged, but under some sort of self-interaction in-between. So because we have one incoming particle and one outgoing particle, second order contributions should come from the two contraction terms. They are only terms that have the possibility of containing just one creation operator and annihilation operator to create one outgoing particle, and annihilate one incoming particle.

We can work out the specific contributions exactly as we have done it in twice now. We find that $S_{2,1}^{(2)} + S_{2,2}^{(2)}$ gives the only nonzero contribution to the electron and positron self-energy matrix elements at the second-order level.

$$S_{2,1}^{(2)} + S_{2,2}^{(2)} = -e^2 \int : \overline{\psi}(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD_{\mu\nu}^F(x_2 - x_1)\psi(x_2) : d^4x_1 d^4x_2 \tag{97}$$

Of course, there are numerous terms that don't contribute to any of the processes at the beginning. The list is actually comprehensive, so these extraneous terms don't actually contribute to anything. This goes for the extraneous terms in the tables aboe as well. This results from the violations of conservation laws. Think about it carefully. $S_{2,3}^{(2)}$ gives only nonvanishing contriubtion to the photon self-energy matrix element at the second order level:

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. A Integrand Terms |
|---|---|---|---|
| Electron Self-Energy | $e^- \to e^-$ | $\langle e^- \vert S \vert e^- \rangle$ | $: \overline{\psi}^-(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD_{\mu\nu}^F(x_2 - x_1)\psi^-(x_2) :$ |
| Positron Self-Energy | $e^+ \to e^+$ | $\langle e^+ \vert S \vert e^+ \rangle$ | $: \overline{\psi}^+(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD_{\mu\nu}^F(x_2 - x_1)\psi^+(x_2) :$ |

$S_{2,3}^{(2)}$ gives the only nonvanishing contribution to the photon self-energy matrix element at the second order level:

$$S_{2,3}^{(2)} = \frac{(-ie)^2}{2!} \int : Tr[iS_F(x_1 - x_2)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)] : d^4x_1 d^4x_2 d^4x_1 d^4x_2 \tag{98}$$

Specifically, the contributing integrand terms are as follows:

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. A Integrand Terms |
|---|---|---|---|
| Photon Self-Energy | $\gamma \to \gamma$ | $\langle \gamma \vert S \vert \gamma \rangle$ | $: Tr[iS_F(x_1 - x_2)\gamma^\mu A_\mu^+(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu^-(x_2)] :$ |
| | | | $: Tr[iS_F(x_1 - x_2)\gamma^\mu A_\mu^-(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu^+(x_2)] :$ |

These two terms are equivalent under integration and can be combined. Just like the Bhabha scattering, this gives:

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. A Integrand Terms |
|---|---|---|---|
| Photon Self-Energy | $\gamma \to \gamma$ | $\langle \gamma \vert S \vert \gamma \rangle$ | $2 : Tr[iS_F(x_1 - x_2)\gamma^\mu A_\mu^+(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu^-(x_2)] :$ |

Now, for the final proccess, the vacuum energy. $S_3^{(2)}$ is the only term that contains no field operators, and therefore won't annihilate the vacuum states in this matrix element. It is therefore the only term that can yield a non-zero contribution to the vacuum enery matrix element, so we have the following result:

| PROCESS | REACTION | General S-Matrix Element | Contributing Eq. A Integrand Terms |
|---|---|---|---|
| Vacuum Energy | vacuum $\rightarrow$ vacuum | $\langle 0\|S\|0\rangle$ | $S_3^{(2)}$ |

Now on to associating these S-matrix terms with Feynman Diagrams!

# 6 Feynman Diagram Interpretation of Contributing Scattering Operator Terms

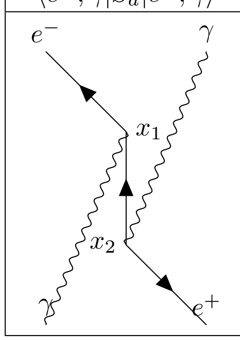
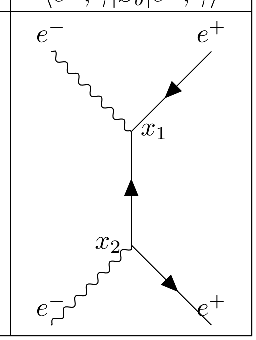
For every creation operator, we know that a particle must exit from the spacetime point that is the argument of the operator. For every annihilation operator, we know that a particle must enter to be annihilated at the spacetime location that is the argument of the annihilation operator, and we know that propagators propagate particles from one spacetime point to another. As is clear from this description, the spacetime arguments tell us how to connect the lines to form vertices. So, we should be able to use the facts to associate each S-matrix term with a Feynman diagram that schematically displays the reaction corresponding term describes. Solid lines represent fermions (with forward arrows, they represent electrons, with backward ones, positrons), and wavy lines represents photons:

$e^-$ Compton Scattering

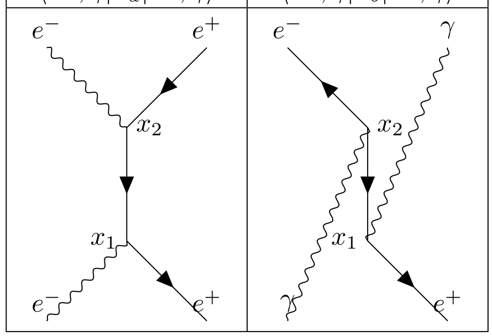
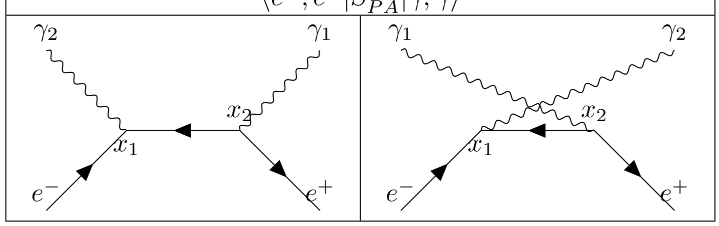
$$\langle e^-,\gamma|S^{(2)}|e^-,\gamma\rangle = \langle e^-,\gamma|S_A|e^-,\gamma\rangle = \langle e^-,\gamma|S_B|e^-,\gamma\rangle \tag{99}$$

$$\langle e^-,\gamma|S_a|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu A_\mu^+(x_1) iS_F(x_2-x_1)\gamma^\nu A_\nu^-(x_2)\psi^+(x_2): d^4x_1 d^4x_2 \tag{100}$$

$$\langle e^-,\gamma|S_b|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu A_\mu^-(x_1) iS_F(x_2-x_1)\gamma^\nu A_\nu^+(x_2)\psi^+(x_2): d^4x_1 d^4x_2 \tag{101}$$

| $\langle e^-,\gamma\|S_a\|e^-,\gamma\rangle$ | $\langle e^-,\gamma\|S_b\|e^-,\gamma\rangle$ |
|---|---|
|   |   |

The integral just accounts for the fact that the vertices could be located anywhere, as we noted with the first order term.

| | |
|---|---|
| $e^+$ Compton Scattering | $$\langle e^-,\gamma\vert S^{(2)}\vert e^-,\gamma\rangle = \langle e^-,\gamma\vert S_A\vert e^-,\gamma\rangle = \langle e^-,\gamma\vert S_B\vert e^-,\gamma\rangle \qquad (102)$$ $$\langle e^-,\gamma\vert S_a\vert e^-,\gamma\rangle = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^+_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^-(x_2): d^4x_1 d^4x_2 \qquad (103)$$ $$\langle e^-,\gamma\vert S_b\vert e^-,\gamma\rangle = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^+_\nu(x_2)\psi^-(x_2): d^4x_1 d^4x_2 \qquad (104)$$ $\langle e^-,\gamma\vert S_a\vert e^-,\gamma\rangle$ $\langle e^-,\gamma\vert S_b\vert e^-,\gamma\rangle$   |
| Pair Annihilation | $$\langle\gamma,\gamma\vert S^{(2)}\vert e^-,e^+\rangle = \langle\gamma,\gamma\vert S^{(2)}_{PA}\vert e^-,e^+\rangle \qquad (105)$$ $$S^{(}2)_{PA} = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^+(x_2): d^4x_1 d^4x_2 \qquad (106)$$ Looking at this amplitude term, we notice something interesting. Because the outgoing particles are identical (both photons), there are two possible Feynman diagrams that we could associate with this term, which differ by an interchange of outgoing photons. When we evaluate these amplitudes further, later on in this video, we will find that $\langle\gamma,\gamma\vert S^{(2)}_{PA}\vert e^-,e^+\rangle$ actually produces two terms. These terms will only differ by an interchange of the photon polarization vectors, and wll correspond to the two different possible Feynman diagrams we have noticed here, which differ by exactly that outgoing photon interchange. $\langle e^-,e^+\vert S^{(2)}_{PA}\vert\gamma,\gamma\rangle$   |

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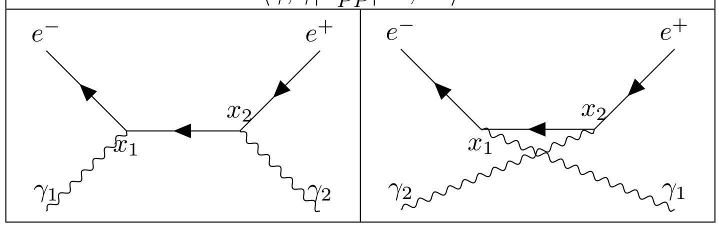
<td>Pair Production</td>
<td>

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle = \langle e^-, e^+|S^{(2)}_{PP}|\gamma,\gamma\rangle \tag{107}$$

$$S^{(}2)_{PP} = -e^2 \int : \overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A^+_\nu(x_2)\psi^-(x_2) : d^4x_1 d^4x_2 \tag{108}$$

Here, we have the same situation we saw with pair annihilation, There are two possible Feynman diagrams that could be associated with this term that, again just differ by the interchange of the identical photons. This happens anytime the incoming or outgoing particles are identical pairs. Just as with pair annihilation, when we evaluate this amplitude further, we will find two terms that differ only by an interchange of the photon polarization vectors.

$\langle\gamma,\gamma|S^{(2)}_{PP}|e^-, e^+\rangle$

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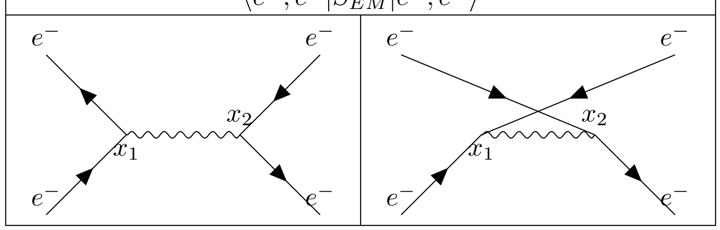
<tr>
<td>$e^-$ Moller Scattering</td>
<td>

$$\langle e^-, e^-|S^{(2)}|e^-, e^-\rangle = \langle e^-, e^-|S_{EM}|e^-, e^-\rangle \tag{109}$$

$$S_{EM} = \frac{-e^2}{2} \int : \overline{\psi}^-(x_1)\gamma^\mu \psi^+(x_1) iD^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^-(x_2)\gamma^\nu \psi^+(x_2) : d^4x_1 d^4x_2 \tag{110}$$

With Moller scattering, we again have a similar situation to what we saw in the last two entries in this table, only this time, it is more extreme. Both the incoming particles and the outgoing particles are identical pairs. Therefore, there are four different Feynman diagrams that we could associate with this term, and when we evaluate the amplitude further, we will find that it does contain four terms. We will also find that two pairs of them are actually identical to the incoming particles this corresponds to the fact that there are only two physically distinct diagrams that could be associated with this matrix. Just like the previous ones, these diagrams differby an interchange of two identical particles. The usual selection for the two physically distinct diagrams is as follows:

$\langle e^-, e^-|S_{EM}|e^-, e^-\rangle$




When we do evaluate this amplitude further, we will find one other thing. The two terms that we do ultimately end up with have opposite signs in addition to the interchanged outgoing electron momenta. This results from the anticommuting property of fermionic creation and annihilation operators. This won't happen for the case of bosons becase their associated operators have commuting properties.

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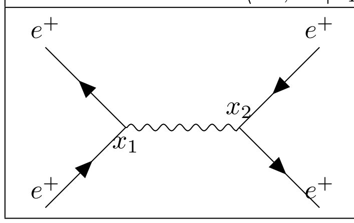
<tr>
<td>$e^+$ Moller Scattering</td>
<td>

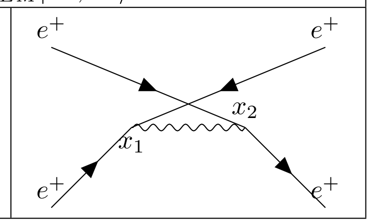
$$\langle e^+, e^+ | S^{(2)} | e^+, e^+ \rangle = \langle e^+, e^+ | S_{EM} | e^+, e^+ \rangle \tag{111}$$

$$S_{PM} = \frac{-e^2}{2} \int : \overline{\psi}^+(x_1)\gamma^\mu \psi^-(x_1) i D^F_{\mu\nu}(x_2 - x_1) \overline{\psi}^+(x_2)\gamma^\nu \psi^-(x_2) : d^4x_1 d^4x_2 \tag{112}$$

The multiplicity of the graphs follows exactly the same description as in the $e^-$ Moller Scattering case.

$\langle e^+, e^+ | S_{EM} | e^+, e^+ \rangle$

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<tr>
<td>Bhabha Scattering</td>
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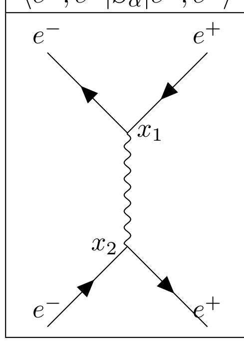
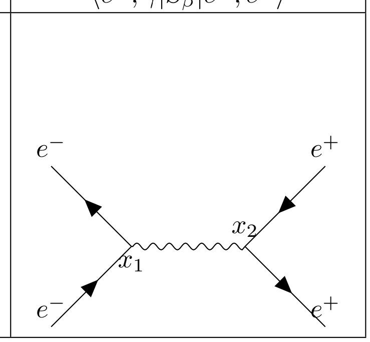
$$\langle e^-, e^+ | S^{(2)} | e^-, e^+ \rangle = \langle e^-, e^+ | S_\alpha | e^-, e^+ \rangle = \langle e^-, \gamma | S_\beta | e^-, e^+ \rangle \tag{113}$$

$$S_\alpha = -e^2 \int : \overline{\psi}^-(x_1)\gamma^\mu \psi^-(x_1) i D^F_{\mu\nu}(x_2 - x_1) \overline{\psi}^+(x_2)\gamma^\nu \psi^+(x_2) : d^4x_1 d^4x_2 \tag{114}$$

$$S_\beta = -e^2 \int : \overline{\psi}^-(x_1)\gamma^\mu \psi^+(x_1) i D^F_{\mu\nu}(x_2 - x_1) \overline{\psi}^+(x_2)\gamma^\nu \psi^-(x_2) : d^4x_1 d^4x_2 \tag{115}$$

| $\langle e^-, e^+ \| S_\alpha \| e^-, e^+ \rangle$ | $\langle e^-, \gamma \| S_\beta \| e^-, e^+ \rangle$ |
|---|---|
|   |   |

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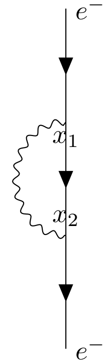
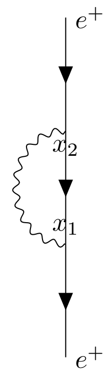
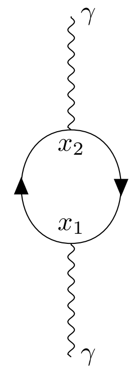
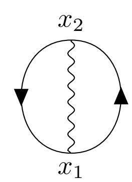
| | |
|---|---|
| Electron Self Energy | $\langle e^-\vert S^{(2)}\vert e^-\rangle = \langle e^-\vert S_{ESE}\vert e^-\rangle$ (116)<br>$S_{ESE} = -e^2 \int : \overline{\psi}^-(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)\psi^+(x_2) : d^4x_1 d^4x_2$ (117)<br> |
| Positron Self Energy | $\langle e^+\vert S^{(2)}\vert e^+\rangle = \langle e^+\vert S_{PSE}\vert e^+\rangle$ (118)<br>$S_{PSE} = -e^2 \int : \overline{\psi}^+(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)\psi^-(x_2) : d^4x_1 d^4x_2$ (119)<br> |

| | |
|---|---|
| Photon Self Energy | $$\langle\gamma\vert S^{(2)}\vert\gamma\rangle = \langle\gamma\vert S_{PhSE}\vert\gamma\rangle \quad (120)$$ <br> $$S_{PhSE} = -e^2 \int : Tr[iS_F(x_1 - x_2)\gamma^\mu A^+_\mu(x_1) S_F(x_2 - x_1)\gamma^\nu A^-_\nu(x_2)] : d^4x_1 d^4x_2 \quad (121)$$ <br>   |
| Vacuum Energy | $$\langle 0\vert S\vert 0\rangle = \langle 0\vert S^{(2)}_3\vert 0\rangle \quad (122)$$ <br> $$S^{(2)}_3 = \frac{(-ie)^2}{2!} \int : Tr[iS_F(x_1 - x_2)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)] : d^4x_1 d^4x_2 \quad (123)$$ <br>   |

Now that we have this Feynman diagram interpretation in place, we can notice another interesting thing or two about the two particle scattering processes (Compton Scattering, Pair Annihilation, Pair Production, Moller Scattering, Bhabha Scattering). All of the diagrams to second order are loop free. Also, with orders any higher than second, the additional vertices would be would force loops into the diagram. Therefore, for these processes, we have not just worked out the complete $2^{nd}$ order S-matrix contribution, but with the complete "tree level" contribution.

Let us now work on computing these amplitudes further. The goal is to simplify them to the point where their Feynman amplitudes can be extracted. In the last section, we will compare these Feynman amplitudes to the Feynman diagrams that we have just associated to their corresponding matrix elements. This comparison will reveal the Feynman rules.

# 7 Direct Computation of Second Order Amplitudes

This is where the calculation starts to get fun, we will see the familiar expressions yielded by Feynman's Rules emerge from direct matrix element calculations.

## 7.1 $e^-$ Compton Scattering

$$\langle e^-, \gamma|S^{(2)}|e^-, \gamma\rangle = \langle e^-, \gamma|S_A|e^-, \gamma\rangle = \langle e^-, \gamma|S_B|e^-, \gamma\rangle \tag{124}$$

$$\langle e^-, \gamma|S_A|e^-, \gamma\rangle = -e^2 \int \langle e^-, \gamma| : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2) : |e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{125}$$

$$\langle e^-, \gamma|S_A|e^-, \gamma\rangle = -e^2 \int \langle e^-, \gamma| : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2) : |e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{126}$$

We can write this without the normal ordering signs as long as the operators are properly normal ordered. From there we can bring it out of strict normal ordered form without destroying the equality as long as only mutually commuting factors are moved past each other:

$$\langle e^-, \gamma|S_A|e^-, \gamma\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{127}$$

$$\langle e^-, \gamma|S_A|e^-, \gamma\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{128}$$

Let's start by computing $\langle e^-, \gamma|S_A|e^-, \gamma\rangle$:

$$\langle f|S_A|i\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}^-(x_1)\gamma^\mu A^-_\nu(x_2) iS_F(x_2 - x_1)\gamma^\nu \psi^+(x_2)|e^-, \gamma A^+_\mu(x_1)\rangle d^4x_1 d^4x_2 \tag{129}$$

$$\begin{aligned} A^+_\mu(x)|\gamma\rangle &= \epsilon^\lambda_\mu e^{-ik\cdot x}|0\rangle \qquad & \langle\gamma|A^-_\mu(x) &= \langle 0|\epsilon^\lambda_\mu e^{ik\cdot x} \\ \psi^+(x)|e^-\rangle &= u_s(\vec{p})e^{-ip\cdot x}|0\rangle \qquad & \langle e^-|\overline{\psi}^-(x) &= \langle 0|e^{ip\cdot x}\overline{u}_s(\vec{p}) \end{aligned} \tag{130}$$

$$\langle f|S_A|i\rangle = -e^2 \int \langle 0|e^{ik\cdot x_2}e^{ip\cdot x_1}\overline{u}_s(\vec{p})\gamma^\mu \epsilon^\lambda_\mu iS_F(x_2 - x_1)\gamma^\nu \epsilon^{\lambda'}_\nu u_s(\vec{p})e^{-ip\cdot x_2}e^{-ik\cdot x_1}|0\rangle d^4x_1 d^4x_2 \tag{131}$$

There are no operators left, only c-numbers, and we will take the vacuum state to be normalized:

$$\langle f|S_A|i\rangle = -e^2 \int e^{ik\cdot x_2}e^{ip\cdot x_1}\overline{u}_s(\vec{p})\gamma^\mu \epsilon^\lambda_\mu iS_F(x_2 - x_1)\gamma^\nu \epsilon^{\lambda'}_\nu u_s(\vec{p})e^{-ip\cdot x_2}e^{-ik\cdot x_1} d^4x_1 d^4x_2 \tag{132}$$

We can make further simplifications and also switch to Feynman slash notation. Also, remember that we are only dealing with transverse polarized photons, so the polarization index can be placed up or down:

$$\langle f|S_A|i\rangle = -e^2 \int \overline{u}_s(\vec{p})\not{\epsilon}^\lambda_\mu iS_F(x_2 - x_1)\not{\epsilon}^{\lambda'}_\nu u_s(\vec{p})e^{-i(p-k')\cdot x_2}e^{-i(k-p')\cdot x_1} d^4x_1 d^4x_2 \tag{133}$$

Now we can write the propagator in terms of its Fourier transform:

$$\langle f|S_A|i\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{134}$$

$$\langle f|S_A|i\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{135}$$

Next we do the $d^4x_1$ and $d^4x_2$ integrations. They yield the delta functions:

$$\langle f|S_A|i\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{136}$$

$$\langle f|S_A|i\rangle = -e^2 \int \langle e^-, \gamma|\overline{\psi}(x_1)\gamma^\mu A_\mu(x_1) iS_F(x_2 - x_1)\gamma^\nu A_\nu(x_2)\psi(x_2)|e^-, \gamma\rangle d^4x_1 d^4x_2 \tag{137}$$

Now recall the following property of the delta function:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{138}$$

Applying that to the product of delta functions in the integral gives:

$$\delta^4(k-p'+q)\delta^4(p-k'-q) = \delta^4(p+k-p'-k')\delta^4(p-k'-q) \tag{139}$$

Inserting this gives:

$$\langle f|S_A|i\rangle = -e^2(2\pi)^4 \int \delta^4(p+k-p'-k')\delta^4(p-k'-q)\overline{u}_{s'}(\vec{p})\not{\epsilon}_\lambda iS_F(q)\not{\epsilon}_{\lambda'}u_s(\vec{p})d^4q \tag{140}$$

Now that only one of the delta functions has q dependence, we can easily do the q integration:

$$\langle f|S_A|i\rangle = -e^2(2\pi)^4\delta^4(p+k-p'-k')\overline{u}_{s'}(\vec{p})\not{\epsilon}_\lambda iS_F(p-k')\not{\epsilon}_{\lambda'}u_s(\vec{p}) \tag{141}$$

So we can finally write:

$$\langle f|S_A|i\rangle = (2\pi)^4\delta^4(p+k-p'-k')\mathcal{M}^!_{fi} \tag{142}$$

$$\mathcal{M}_{fi} = -e^2\overline{u}_{s'}(\vec{p})\not{\epsilon}_\lambda iS_F(p-k')\not{\epsilon}_{\lambda'}u_s(\vec{p}) \tag{143}$$

An extremely calculation gives the follwing result for $\langle f|S_B|i\rangle$:

$$\langle f|S_B|i\rangle = (2\pi)^4\delta^4(p+k-p'-k')\mathcal{M}^B_{fi} \tag{144}$$

$$\mathcal{M}_{fi} = -e^2\overline{u}_{s'}(\vec{p})\not{\epsilon}_\lambda iS_F(p+k)\not{\epsilon}_{\lambda'}u_s(\vec{p}) \tag{145}$$

## 7.2 $e^+$ Compton Scattering

$$\langle e^-,\gamma|S^{(2)}|e^-,\gamma\rangle = \langle e^-,\gamma|S_A|e^-,\gamma\rangle = \langle e^-,\gamma|S_B|e^-,\gamma\rangle \tag{146}$$

$$\langle e^-,\gamma|S_B|e^-,\gamma\rangle = -e^2\int\langle e^-,\gamma| : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A_\nu(x_2)\psi(x_2) : |e^-,\gamma\rangle d^4x_1d^4x_2 \tag{147}$$

$$\langle e^-,\gamma|S_B|e^-,\gamma\rangle = -e^2\int\langle e^-,\gamma| : \overline{\psi}(x_1)\gamma^\mu A_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A_\nu(x_2)\psi(x_2) : |e^-,\gamma\rangle d^4x_1d^4x_2 \tag{148}$$

A calculation essentially identical to $e^-$ Compton scattering yields the same results for $e^+$ Compton Scattering:

$$\mathcal{M}_{fi} = \mathcal{M}^A_{fi} + \mathcal{M}^B_{fi} \tag{149}$$

## 7.3 Pair Annihilation

$$\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = -e^2\int\langle\gamma,\gamma| : \overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^+(x_2) : |e^-,e^+\rangle d^4x_1d^4x_2 \tag{150}$$

$$\psi^+(x) = \sum_s \frac{m}{\omega}\int c_s(\vec{p})u_s(\vec{p})e^{-ip\cdot x}\frac{d^3k}{(2\pi)^3} \qquad \overline{\psi}^+(x) = \sum_s \frac{m}{\omega}\int d_s(\vec{p})\overline{v}_s(\vec{p})e^{-ip\cdot x}\frac{d^3k}{(2\pi)^3} \tag{151}$$

$$A^-_\mu(x) = \int \epsilon^\lambda_\mu a^\dagger_\lambda e^{ik\cdot x}\frac{d^3k}{(2\pi)^3 2\omega} \tag{152}$$

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle\gamma,\gamma| : d_s(\vec{k_1})a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})c_r(\vec{k_2}) : |e^-,e^+\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'} iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{ik_1\cdot x}e^{-ik_2'\cdot x}e^{ik_2\cdot x}e^{-ik_1'\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{153}$$

Now we can do normal ordering:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle\gamma,\gamma|a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})d_s(\vec{k_1})c_r(\vec{k_2})|e^-,e^+\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'} iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{154}$$

We can now pull the operators out of the state:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle 0|a_\lambda(\vec{p_2'})a_{\lambda'}(\vec{p_1'})a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})d_s(\vec{k_2})c_r(\vec{k_1})c_r^\dagger(\vec{p_1})d_s^\dagger(\vec{p_2})|0\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'} iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{155}$$

We expect the k integrations and the spin/helicity sums to yield two terms because there are two different ways of assigning the k values and spin/helicity values that don't result in the states being annihilated

Just like when we proved that the first order term contributes to nothing, the photons are transverse, so the longitudinal and temporal annihilation operator terms h and h' sums to two term sums where $\eta_{\lambda\lambda'}$ is replaced with $\delta_{\lambda\lambda'}$, where the indices on the Kronecker delta are therefore now two dimensional. As with the first order calculation, this also changes the commutation relations to:

$$[a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k'})] = (2\pi)^3 2\omega\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'}) \tag{156}$$

The relation will be used next

We now know that for angular momentum to be conserved, the exciting photons must have opposite helicity, so both photons cannot be exciting in the same quantum state. Therefore, given the commutation relation, we can see that the photon creation operators (in the above matrix element) can fail to commute with at most one of the photon annihilation operators, so we can pick which ones definitely commute in two diffrent ways. Given the particular selection made, one can then pair each creation operator with the annihilation operator that it will not necessary commute with, and replace the pair with its commutator. We can do this because the added term annihilates on the vacuum. The delta functions and the Kronecker deltas in the commutator then force a momentum and spin assignment that represents one of the contributions described in Box A. Because there are two such pairings, there are two contributions to the integral, as mentioned in Box A. The fermion operators that can also be paired by type and replaced by anticommutators. The pairing by type is the only pairing of fermionic operators that has the capacity to allow a nonvanishing contributions to the integral. Doing all these gives:

$$
\begin{aligned}
\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle \\
&\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\
&\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2 \\
&- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle \\
&\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\
&\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2
\end{aligned}
\tag{157}
$$

Now, we can insert the values of these commutators, and anticommutators:

$$
\begin{aligned}
[a_\lambda(\vec{k}),a_{\lambda'}^\dagger(\vec{k'})] &= (2\pi)^3 2\omega\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'}) \\
c_r(\vec{p}),c_s^\dagger(\vec{p'}) &= (2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'}) \\
d_r(\vec{p}),d_s^\dagger(\vec{p'}) &= (2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})
\end{aligned}
\tag{158}
$$

This gives:

$$
\begin{aligned}
&\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = \\
&-e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})(2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})(2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})|0\rangle \\
&\qquad\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\
&\qquad\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2 \\
&-e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})(2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})(2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})|0\rangle \\
&\qquad\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^{h}u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x} \\
&\qquad\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2
\end{aligned}
\tag{159}
$$

We can now do all of the k-integrations, spin-sums, and helicity sums:

$$
\begin{aligned}
&\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = \\
&\qquad -e^2\int\overline{v}_{s_1'}(\vec{p}_1)\not{\epsilon}_\lambda iS_F(p+k)\not{\epsilon}_{\lambda'}u_{s_2'}(\vec{p'}_2)e^{-i(p_1-p_1')\cdot x_2}e^{-i(p_2-p_2')\cdot x_1}d^4q\,d^4x_1d^4x_2 \\
&\qquad -e^2\int\overline{v}_{s_1'}(\vec{p}_1)\not{\epsilon}_\lambda iS_F(p+k)\not{\epsilon}_{\lambda'}u_{s_2'}(\vec{p'}_2)e^{-i(p_1-p_2')\cdot x_2}e^{-i(p_2-p_1')\cdot x_1}d^4q\,d^4x_1d^4x_2
\end{aligned}
\tag{160}
$$

We can rewrite this in terms of Fourier transform of the Fermion Propagator:

$$
\begin{aligned}
&\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = \\
&\qquad -e^2\int\overline{v}_{s_1'}(\vec{p}_1)\not{\epsilon}_\lambda iS_F(p+k)\not{\epsilon}_{\lambda'}u_{s_2'}(\vec{p'}_2)e^{-i(p_2-p_2'+q)\cdot x_1}e^{-i(p_1-p_1'-q)\cdot x_2}d^4x_1d^4x_2 \\
&\qquad -e^2\int\overline{v}_{s_1'}(\vec{p}_1)\not{\epsilon}_\lambda iS_F(p+k)\not{\epsilon}_{\lambda'}u_{s_2'}(\vec{p'}_2)e^{-i(p_2-p_1'+q)\cdot x_1}e^{-i(p_1-p_2'-q)\cdot x_2}d^4x_1d^4x_2
\end{aligned}
\tag{161}
$$

Now, we can do the x integrations:

$$\begin{aligned}
&\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = \\
&\quad -e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)(2\pi)^4 \delta^4(p_1 + p_2 - q)\delta^4(p_1 - p_1' - q) d^4q \\
&\quad -e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)(2\pi)^4 \delta^4(p_1 + p_2 - q)\delta^4(p_1 - p_2' - q) d^4q
\end{aligned} \tag{162}$$

We can then use that same famous delta function identity to get it ready for the q integration:

$$\delta^4(k - p' + q)\delta^4(p - k' - q) = \delta^4(p + k - p' - k')\delta^4(p - k' - q) \tag{163}$$

Applying it gives:

$$\begin{aligned}
&\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = \\
&\quad -e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')\delta^4(p_1 - p_1' - q) d^4q \\
&\quad -e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')\delta^4(p_1 - p_2' - q) d^4q
\end{aligned} \tag{164}$$

$$\begin{aligned}
\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = {} & (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \\
& [-e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)\delta^4(p_1 - p_1' - q) d^4q \\
& -e^2 \int \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2)\delta^4(p_1 - p_2' - q) d^4q]
\end{aligned} \tag{165}$$

Now we can finish this off by doing the q integration using the delta function:

$$\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')[-e^2 \overline{u}_{s'}(\vec{p}) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_s(\vec{p}) - e^2 \overline{u}_{s'}(\vec{p}) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_s(\vec{p})] \tag{166}$$

$$\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \mathcal{M}_{fi} \tag{167}$$

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^1 + \mathcal{M}_{fi}^2 \tag{168}$$

$$\mathcal{M}_{fi}^1 = -e^2 \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2) \tag{169}$$

$$\mathcal{M}_{fi}^2 = -e^2 \overline{v}_{s_1'}(\vec{p}_1) \not{\epsilon}_\lambda i S_F(p+k) \not{\epsilon}_{\lambda'} u_{s_2'}(\vec{p'}_2) \tag{170}$$

## 7.4 Pair Production

$$\langle \gamma, \gamma | S^{(2)} | e^-, e^+ \rangle = -e^2 \int \langle \gamma, \gamma | : \overline{\psi}^+(x_1) \gamma^\mu A_\mu^-(x_1) i S_F(x_2 - x_1) \gamma^\nu A_\nu^-(x_2) \psi^+(x_2) : | e^-, e^+ \rangle d^4x_1 d^4x_2 \tag{171}$$

$$\psi^+(x) = \sum_s \frac{m}{\omega} \int c_s(\vec{p}) u_s(\vec{p}) e^{-ip\cdot x} \frac{d^3k}{(2\pi)^3} \qquad \overline{\psi}^+(x) = \sum_s \frac{m}{\omega} \int d_s(\vec{p}) \overline{v}_s(\vec{p}) e^{-ip\cdot x} \frac{d^3k}{(2\pi)^3} \tag{172}$$

$$A^-_\mu(x) = \int \epsilon^\lambda_\mu a^\dagger_\lambda e^{ik\cdot x}\frac{d^3k}{(2\pi)^3 2\omega} \tag{173}$$

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &-e^2\sum_{rs}\int\langle\gamma,\gamma|:d_s(\vec{k_1})a^\dagger_h(\vec{k'_1})a^\dagger_{h'}(\vec{k'_2})c_r(\vec{k_2}):|e^-,e^+\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{ik_1\cdot x}e^{-ik'_2\cdot x}e^{ik_2\cdot x}e^{-ik'_1\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}d^4x_1 d^4x_2\end{aligned} \tag{174}$$

Now we can do the normal ordering:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &-e^2\sum_{rs}\int\langle\gamma,\gamma|a^\dagger_h(\vec{k'_1})a^\dagger_{h'}(\vec{k'_2})d_s(\vec{k_1})c_r(\vec{k_2})|e^-,e^+\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k'_2)\cdot x}e^{-i(k_2-k'_1)\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}d^4x_1 d^4x_2\end{aligned} \tag{175}$$

Now we can pull the operators out of the states:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &-e^2\sum_{rs}\int\langle 0|a_\lambda(\vec{p'_2})a_{\lambda'}(\vec{p'_1})a^\dagger_h(\vec{k'_1})a^\dagger_{h'}(\vec{k'_2})d_s(\vec{k_2})c_r(\vec{k_1})c^\dagger_r(\vec{p_1})d^\dagger_s(\vec{p_2})|0\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k'_2)\cdot x}e^{-i(k_2-k'_1)\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}d^4x_1 d^4x_2\end{aligned} \tag{176}$$

In the exact same way and for all of the same reasons as with pair annihilation case, the k integration and spin/helicity sums have two contributing terms, once the operators have been paired and replaced with commutators/anticommutators. Also just like in the pair annihilation case, the photons must have opposite helicity to conserve angular momentum. This means that each photon annihilation operator can fail to commute with at most one photon creation operator. This makes the necessary operator reordering as possible. So, we have:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &-e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p'_2}),a^\dagger_h(\vec{k'_1}][a_{\lambda'}(\vec{p'_1}),a^\dagger_{h'}(\vec{k'_2})]\{c_r(\vec{k_1}),c^\dagger_r(\vec{p_1})\}\{d_s(\vec{k_2}),d^\dagger_s(\vec{p_2})\}|0\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k'_2)\cdot x}e^{-i(k_2-k'_1)\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}d^4x_1 d^4x_2 \\ &-e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p'_2}),a^\dagger_h(\vec{k'_1}][a_{\lambda'}(\vec{p'_1}),a^\dagger_{h'}(\vec{k'_2})]\{c_r(\vec{k_1}),c^\dagger_r(\vec{p_1})\}\{d_s(\vec{k_2}),d^\dagger_s(\vec{p_2})\}|0\rangle \\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k'_2)\cdot x}e^{-i(k_2-k'_1)\cdot x} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}d^4x_1 d^4x_2\end{aligned} \tag{177}$$

Next, we can insert values of all of these commutators and anticommutators:

$$\begin{aligned}[a_\lambda(\vec{k}),a^\dagger_{\lambda'}(\vec{k'})] &= (2\pi)^3 2\omega\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'}) \\ c_r(\vec{p}),c^\dagger_s(\vec{p'}) &= (2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'}) \\ d_r(\vec{p}),d^\dagger_s(\vec{p'}) &= (2\pi)^3(2\omega)^3\frac{\omega}{m}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})\end{aligned} \tag{178}$$

Inserting these gives:

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle \tag{179}$$

Now, for some simplifications, the spin and helicity sums, and the k integration:

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle \tag{180}$$

We can then rewrite it this in terms of the momentum space propagator:

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle \tag{181}$$

Now, the x integration can be done to yield delta functions:

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle \tag{182}$$

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^A + \mathcal{M}_{fi}^B \tag{183}$$

We can use that delta function identity again:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{184}$$

This gives:

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle \tag{185}$$

The q integrations can now be done:

$$\langle\gamma,\gamma|S^{(2)}|e^-, e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\mathcal{M}_{fi} \tag{186}$$

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^1 + \mathcal{M}_{fi}^2 \tag{187}$$

$$\mathcal{M}_{fi}^1 = -e^2\overline{v}_{s_1'}(\vec{p}_1)\not\epsilon_\lambda iS_F(p+k)\not\epsilon_{\lambda'}u_{s_2'}(\vec{p'}_2) \tag{188}$$

$$\mathcal{M}_{fi}^2 = -e^2\overline{v}_{s_1'}(\vec{p}_1)\not\epsilon_\lambda iS_F(p+k)\not\epsilon_{\lambda'}u_{s_2'}(\vec{p'}_2) \tag{189}$$

## 7.5 $e^-$ Moller Scattering

$$\langle\gamma,\gamma|S^{(2)}|e^-, e^+\rangle = -e^2\int\langle\gamma,\gamma| : \overline{\psi}^+(x_1)\gamma^\mu A_\mu^-(x_1)iS_F(x_2-x_1)\gamma^\nu A_\nu^-(x_2)\psi^+(x_2) : |e^-, e^+\rangle d^4x_1d^4x_2 \tag{190}$$

$$\psi^+(x) = \sum_s\frac{m}{\omega}\int c_s(\vec{p})u_s(\vec{p})e^{-ip\cdot x}\frac{d^3k}{(2\pi)^3} \qquad \overline{\psi}^+(x) = \sum_s\frac{m}{\omega}\int d_s(\vec{p})\overline{v}_s(\vec{p})e^{-ip\cdot x}\frac{d^3k}{(2\pi)^3} \tag{191}$$

$$A_\mu^-(x) = \int\epsilon_\mu^\lambda a_\lambda^\dagger e^{ik\cdot x}\frac{d^3k}{(2\pi)^3 2\omega} \tag{192}$$

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-, e^+\rangle = &-e^2\sum_{rs}\int\langle\gamma,\gamma|a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})d_s(\vec{k_1})c_r(\vec{k_2})|e^-, e^+\rangle\\ &\overline{v}_s(\vec{k}_2)\not\epsilon^{h'}iS_F(x_2-x_1)\not\epsilon^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{193}$$

Now let's insert the momentum state propagator, and do the normal ordering:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle\gamma,\gamma|a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})d_s(\vec{k_1})c_r(\vec{k_2})|e^-,e^+\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{194}$$

We expect the k integrations to yield four terms because there are four different ways of assigning the k values and spin/helicity values that don't result in the states being annihilated.

We can pull the creation operators out of the states to get:

$$\begin{aligned}\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = &- e^2\sum_{rs}\int\langle 0|a_\lambda(\vec{p_2'})a_{\lambda'}(\vec{p_1'})a_h^\dagger(\vec{k_1'})a_{h'}^\dagger(\vec{k_2'})d_s(\vec{k_2})c_r(\vec{k_1})c_r^\dagger(\vec{p_1})d_s^\dagger(\vec{p_2})|0\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{195}$$

Given that we are dealing with fermions which can't be in the same quantum state, we can do some anticommuting that is useful, because any one annihilation operator can't possibly correspond to the quantum state of more than one of the creation operators. We can use this to pair up creation and annihilation operators. Once creation and annihilation operators are paired up, each of the operators can be replaced by its anticommutator. The anticommutation relations satisfied by the operators tell us that these anticommutators contain delta functions and Kronecker deltas that will cause the k integrations and spin sums to fix momentum and spin assignments of all dummy variables according to the particular pairing. Because there are four pairings, the integration of four terms, As mentioned in Box A:

$$\begin{aligned}\langle e^-,e^-|S_{EM}|e^-,e^-\rangle = &- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\\ &- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\\ &- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\\ &- e^2\sum_{rs}\int\langle 0|[a_\lambda(\vec{p_2'}),a_h^\dagger(\vec{k_1'}][a_{\lambda'}(\vec{p_1'}),a_{h'}^\dagger(\vec{k_2'})]\{c_r(\vec{k_1}),c_r^\dagger(\vec{p_1})\}\{d_s(\vec{k_2}),d_s^\dagger(\vec{p_2})\}|0\rangle\\ &\overline{v}_s(\vec{k}_2)\not{\epsilon}^{h'}iS_F(x_2-x_1)\not{\epsilon}^h u_r(\vec{k}_1)e^{-i(k_1-k_2')\cdot x}e^{-i(k_2-k_1')\cdot x}\\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3 2\omega_1'}\frac{d^3k_2'}{(2\pi)^3 2\omega_2'}d^4x_1d^4x_2\end{aligned} \tag{196}$$

Now we can insert the values of the commutators:

$$\{c_r(\vec{p}), c_s^\dagger(\vec{p'})\} = (2\pi)^3 \frac{\omega}{m} \delta^3(\vec{p} - \vec{p'})\delta_{rs} \tag{197}$$

Inserting it gives:

$$
\begin{aligned}
&\langle e^-, e^- | S_{EM} | e^-, e^- \rangle = \\
&-e^2 \sum_{rs} \int \langle 0 | [a_\lambda(\vec{p_2'}), a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'}) | 0 \rangle \\
&\qquad \overline{v}_s(\vec{k}_2) \not{\epsilon}^{h'} i S_F(x_2 - x_1) \not{\epsilon}^h u_r(\vec{k}_1) e^{-i(k_1 - k_2') \cdot x} e^{-i(k_2 - k_1') \cdot x} \\
&\qquad \frac{d^3 k_1}{(2\pi)^3} \frac{m}{\omega_1} \frac{d^3 k_2}{(2\pi)^3} \frac{m}{\omega_2} \frac{d^3 k_1'}{(2\pi)^3 2\omega_1'} \frac{d^3 k_2'}{(2\pi)^3 2\omega_2'} d^4 x_1 d^4 x_2 \\
&-e^2 \sum_{rs} \int \langle 0 | [a_\lambda(\vec{p_2'}), a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'}) | 0 \rangle \\
&\qquad \overline{v}_s(\vec{k}_2) \not{\epsilon}^{h'} i S_F(x_2 - x_1) \not{\epsilon}^h u_r(\vec{k}_1) e^{-i(k_1 - k_2') \cdot x} e^{-i(k_2 - k_1') \cdot x} \\
&\qquad \frac{d^3 k_1}{(2\pi)^3} \frac{m}{\omega_1} \frac{d^3 k_2}{(2\pi)^3} \frac{m}{\omega_2} \frac{d^3 k_1'}{(2\pi)^3 2\omega_1'} \frac{d^3 k_2'}{(2\pi)^3 2\omega_2'} \\
&-e^2 \sum_{rs} \int \langle 0 | [a_\lambda(\vec{p_2'}), a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'}) | 0 \rangle \\
&\qquad \overline{v}_s(\vec{k}_2) \not{\epsilon}^{h'} i S_F(x_2 - x_1) \not{\epsilon}^h u_r(\vec{k}_1) e^{-i(k_1 - k_2') \cdot x} e^{-i(k_2 - k_1') \cdot x} \\
&\qquad \frac{d^3 k_1}{(2\pi)^3} \frac{m}{\omega_1} \frac{d^3 k_2}{(2\pi)^3} \frac{m}{\omega_2} \frac{d^3 k_1'}{(2\pi)^3 2\omega_1'} \frac{d^3 k_2'}{(2\pi)^3 2\omega_2'} d^4 x_1 d^4 x_2 \\
&-e^2 \sum_{rs} \int \langle 0 | [a_\lambda(\vec{p_2'}), a_h^\dagger(\vec{k_1'}](2\pi)^3 2\omega \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})(2\pi)^3 (2\omega)^3 \frac{\omega}{m} \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'}) | 0 \rangle \\
&\qquad \overline{v}_s(\vec{k}_2) \not{\epsilon}^{h'} i S_F(x_2 - x_1) \not{\epsilon}^h u_r(\vec{k}_1) e^{-i(k_1 - k_2') \cdot x} e^{-i(k_2 - k_1') \cdot x} \\
&\qquad \frac{d^3 k_1}{(2\pi)^3} \frac{m}{\omega_1} \frac{d^3 k_2}{(2\pi)^3} \frac{m}{\omega_2} \frac{d^3 k_1'}{(2\pi)^3 2\omega_1'} \frac{d^3 k_2'}{(2\pi)^3 2\omega_2'}
\end{aligned}
\tag{198}
$$

All of the k-integrations can finally be done now, as well as spin sums. Also, all that is within the vacuum expectation value is not an operator, and can be pulled out. We can then assume the vacuum state to be normalized. This yields:

$$
\begin{aligned}
&\langle e^-, e^- | S_{EM} | e^-, e^- \rangle = \\
&\qquad \frac{e^2}{2} \int \overline{u}_{s_1'}(\vec{p_1'}) \gamma^\mu u_{s_2'}(\vec{p}) i D^F_{\mu\nu}(q) \overline{u}_{s_2'}(\vec{p}_2) \gamma^\nu u_{s_1}(\vec{p}_1) e^{-i \cdot (p_1 - p_1' + q) \cdot x_1} e{-}i \cdot (p_2 - p_2' - q) \cdot x_2 d^4 q \\
&\qquad + \frac{e^2}{2} \int \overline{u}_{s_1'}(\vec{p_1'}) \gamma^\mu u_{s_2'}(\vec{p}) i D^F_{\mu\nu}(q) \overline{u}_{s_2'}(\vec{p}_2) \gamma^\nu u_{s_1}(\vec{p}_1) e^{-i \cdot (-p_2' + p_2 + q) \cdot x_1} e{-}i \cdot (p_1 - p_1' - q) \cdot x_2 d^4 q \\
&\qquad \frac{e^2}{2} \int \overline{u}_{s_2'}(\vec{p_2'}) \gamma^\mu u_{s_2'}(\vec{p}) i D^F_{\mu\nu}(q) \overline{u}_{s_1'}(\vec{p_1'}) \gamma^\nu u_{s_1}(\vec{p}_1) e^{-i \cdot (p_1 + p_2 + q) \cdot x_1} e{-}i \cdot (p_1 - p'2 - q) \cdot x_2 d^4 q \\
&\qquad + \frac{e^2}{2} \int \overline{u}_{s_2'}(\vec{p_2'}) \gamma^\mu u_{s_2'}(\vec{p}) i D^F_{\mu\nu}(q) \overline{u}_{s_1'}(\vec{p_1'}) \gamma^\nu u_{s_1}(\vec{p}_1) e^{-i \cdot (p_1 - p_2' + q) \cdot x_1} e{-}i \cdot (p_2 - p_1' - q) \cdot x_2 d^4 q
\end{aligned}
\tag{199}
$$

Now the x integrations can be done to yield the delta functions:

$$\langle e^-, e^-|S_{EM}|e^-, e^-\rangle =$$
$$\begin{aligned}
&\frac{e^2}{2}\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_1' + q)\delta^4(p_2 - p_2' - q)d^4q \\
&+ \frac{e^2}{2}\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(-p_2' + p_2 + q)\delta^4(p_1 - p_1' - q)d^4q \\
&\frac{e^2}{2}\int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 + p_2 + q)\delta^4(p_1 - p'2 - q)d^4q \\
&+ \frac{e^2}{2}\int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_2' + q)\delta^4(p_2 - p_1' - q)d^4q
\end{aligned} \tag{200}$$

Now, just like above, we can apply the following identity repeatedly on the products of delta functions:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{201}$$

Doing this gives:

$$\langle e^-, e^-|S_{EM}|e^-, e^-\rangle =$$
$$\begin{aligned}
&\frac{e^2}{2}\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_1' + p_2 - p_2')\delta^4(p_2 - p_2' - q)d^4q \\
&+ \frac{e^2}{2}\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_2' + p_2 - p_1')\delta^4(p_1 - p_1' - q)d^4q \\
&\frac{e^2}{2}\int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_1' + p_2 - p_2')\delta^4(p_1 - p'2 - q)d^4q \\
&+ \frac{e^2}{2}\int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1 - p_2' + p_2 - p_1')\delta^4(p_2 - p_1' - q)d^4q
\end{aligned} \tag{202}$$

We can do some factoring:

$$\begin{gathered}
\langle e^-, e^-|S_{EM}|e^-, e^-\rangle = \\
\frac{e^2}{2}[\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)d^4q + \int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)d^4q \\
\int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)d^4q + \int \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(q)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)d^4q] \\
(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')
\end{gathered} \tag{203}$$

Then we can do the q integrations:

$$\begin{gathered}
\langle e^-, e^-|S_{EM}|e^-, e^-\rangle = \\
\frac{e^2}{2}[\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1 - p_2)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1) + \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1 - p_2)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1) \\
\overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1' - p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1) + \overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1' - p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)] \\
(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')
\end{gathered} \tag{204}$$

The remaining delta function enforces some momentum conservation:

$$p_1 + p_2 = p_1' + p_2' \tag{205}$$

We can see that the first two terms are identical and that the last two terms are identical. We can therefore combine them:

$$\langle e^-, e^-|S_{EM}|e^-, e^-\rangle = e^2[\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1-p_2)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)+\overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1'-p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)] (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') \tag{206}$$

We can therefore identify the Feynman Amplitude, which I will denote with the usual $\mathcal{M}_{fi}$:

$$\mathcal{M}_{fi} = e^2[\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1-p_2)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)+\overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1'-p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)] \tag{207}$$

$$\langle e^-, e^-|S^{(2)}|e^-, e^-\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\mathcal{M}_{fi} \tag{208}$$

$$\mathcal{M}_{fi} = \mathcal{M}^1_{fi} + \mathcal{M}^2_{fi} \tag{209}$$

$$\begin{aligned}\mathcal{M}^1_{fi} &= e^2\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1-p_2)\overline{u}_{s_2'}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\\ \mathcal{M}^2_{fi} &= e^2\overline{u}_{s_2'}(\vec{p}_2')\gamma^\mu u_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1'-p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\end{aligned} \tag{210}$$

### 7.6 $e^+$ Moller Scattering

$$\langle e^-, e^+|S^{(2)}|e^-, e^+\rangle = -\frac{e^2}{2}\langle e^-, e^+| : \overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)iD^F_{\mu\nu}(x_2-x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi^+(x_2) : |e^-, e^+\rangle \tag{211}$$

An essentially identical calculation to that of $e^-$ Moller scattering yields the following results for $e^+$ Moller scattering:

$$\langle e^-, e^+|S^{(2)}|e^-, e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\mathcal{M}_{fi} \tag{212}$$

$$\mathcal{M}_{fi} = \mathcal{M}^1_{fi} + \mathcal{M}^2_{fi} \tag{213}$$

$$\begin{aligned}\mathcal{M}^1_{fi} &= e^2\overline{v}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1-p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu v_{s_1}(\vec{p}_1)\\ \mathcal{M}^2_{fi} &= e^2\overline{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1'-p_1)\overline{v}_{s_1'}(\vec{p}_1')\gamma^\nu v_{s_1}(\vec{p}_1)\end{aligned} \tag{214}$$

# 8 Direct Order of Second Order Amplitudes Continued

## 8.1 Bhabha Scattering

$$\langle e^-, e^+|S^{(2)}|e^-, e^+\rangle = \langle e^-, e^+|S_A|e^-, e^+\rangle = \langle e^-, \gamma|S_B|e^-, e^+\rangle \tag{215}$$

$$\langle e^-, e^+|S_\alpha|e^-, e^+\rangle = -e^2\int\langle e^-, e^+| : \overline{\psi}^-(x_1)\gamma^\mu\psi^-(x_1)iS_F(x_2-x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^+(x_2) : |e^-, e^+\rangle d^4x_1d^4x_2 \tag{216}$$

$$\langle e^-, e^+|S_\beta|e^-, e^+\rangle = -e^2\int\langle e^-, e^+| : \overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1)iS_F(x_2-x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^-(x_2) : |e^-, e^+\rangle d^4x_1d^4x_2 \tag{217}$$

Let's consider $\langle e^-, e^+|S^{(2)}|e^-, e^+\rangle$ first:

$$\langle e^-, e^+|S_\alpha|e^-, e^+\rangle = -e^2 \int \langle e^-, e^+| : \overline{\psi}^-(x_1)\gamma^\mu \psi^-(x_1) iS_F(x_2 - x_1)\overline{\psi}^+(x_2)\gamma^\nu \psi^+(x_2) : |e^-, e^+\rangle d^4x_1 d^4x_2 \tag{218}$$

$$\begin{aligned} \psi^+(x) &= \sum_s \int \frac{m}{\omega} c_s(\vec{p}) u_s(\vec{p}) e^{ip\cdot x} \frac{d^3p}{(2\pi)^3} \qquad & \overline{\psi}^+(x) &= \sum_s \int \frac{m}{\omega} d_s(\vec{p}) \overline{v}_s(\vec{p}) e^{ip\cdot x} \frac{d^3p}{(2\pi)^3} \\ \psi^-(x) &= \sum_s \int \frac{m}{\omega} d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{ip\cdot x} \frac{d^3p}{(2\pi)^3} \qquad & \overline{\psi}^-(x) &= \sum_s \int \frac{m}{\omega} c_s^\dagger(\vec{p}) \overline{u}_s(\vec{p}) e^{ip\cdot x} \frac{d^3p}{(2\pi)^3} \end{aligned} \tag{219}$$

The operators already happen to be normal ordering can be dropped:

$$\begin{aligned} \langle \gamma, \gamma|S^{(2)}|e^-, e^+\rangle = &- e^2 \sum_{mnrs} \int \langle 0|c_r^\dagger(\vec{k}_1') d_s^\dagger(\vec{k}_2') d_m(\vec{k}_2') c_n(\vec{k}_1')|0\rangle \\ &e^2 \int \overline{u}_r(\vec{p}_1')\gamma^\mu v_s(\vec{p}) iD^F_{\mu\nu}(x_2 - x_1)\overline{v}_m(\vec{p}_2)\gamma^\nu u_n(\vec{p}_1)(2\pi)^4 e^{-ik_1'\cdot x_1} e^{-ik_2'\cdot x_1} e^{-ik_2\cdot x_2} e^{-ik_1\cdot x_2} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2'}{(2\pi)^3}\frac{m}{\omega_2} d^4x_1 d^4x_2 \end{aligned} \tag{220}$$

Since the incoming particles are not identical, and the outgoing particles are not identical, there is only one contribution to all the k integrations and spin sums, and that is where the momenta and spins of particles in the initial and final states match annihilation operators of the corresponding type of the matrix element (remember that creation operators acting to the left are annihilation operators). This is in contrast with Moller Scattering, where the identical nature of particles causes there to be four such momentum assignments, and therefore four contributions to the integrals.

We can now pull out the creation operators from the states:

$$\begin{aligned} \langle \gamma, \gamma|S^{(2)}|e^-, e^+\rangle = &- e^2 \sum_{mnrs} \int \langle 0|c_{s_1'}(\vec{p}_1') d_{s_2'}(\vec{p}_2') c_r^\dagger(\vec{k}_1') d_s^\dagger(\vec{k}_2') d_m(\vec{k}_2') c_n(\vec{k}_1') c_{s_1}^\dagger(\vec{p}_1') d_{s_2}^\dagger(\vec{p}_2')|0\rangle \\ &e^2 \int \overline{u}_r(\vec{p}_1')\gamma^\mu v_s(\vec{p}) iD^F_{\mu\nu}(x_2 - x_1)\overline{v}_m(\vec{p}_2)\gamma^\nu u_n(\vec{p}_1)(2\pi)^4 e^{-ik_1'\cdot x_1} e^{-ik_2'\cdot x_1} e^{-ik_2\cdot x_2} e^{-ik_1\cdot x_2} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2'}{(2\pi)^3}\frac{m}{\omega_2} d^4x_1 d^4x_2 \end{aligned} \tag{221}$$

Because there is only one dummy variable assignment that contributes to the k integrals and spin sums, there is only one way to pair up the creation and annihilation operators to yield a nonzero contribution once the operator pairs have been replaced by anticommutators. Otherwise, the process is similar to the Moller Scattering Case:

$$\begin{aligned} \langle \gamma, \gamma|S^{(2)}|e^-, e^+\rangle = &- e^2 \sum_{mnrs} \int \langle 0|\{c_{s_1'}(\vec{p}_1') c_r^\dagger(\vec{k}_1')\}\{d_{s_2'}(\vec{p}_2') d_s^\dagger(\vec{k}_2')\}\{c_n(\vec{k}_1') c_{s_1}^\dagger(\vec{p}_1')\}\{d_m(\vec{k}_2') d_{s_2}^\dagger(\vec{p}_2')\}|0\rangle \\ &e^2 \int \overline{u}_r(\vec{p}_1')\gamma^\mu v_s(\vec{p}) iD^F_{\mu\nu}(x_2 - x_1)\overline{v}_m(\vec{p}_2)\gamma^\nu u_n(\vec{p}_1)(2\pi)^4 e^{-ik_1'\cdot x_1} e^{-ik_2'\cdot x_1} e^{-ik_2\cdot x_2} e^{-ik_1\cdot x_2} \\ &\frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2'}{(2\pi)^3}\frac{m}{\omega_2} d^4x_1 d^4x_2 \end{aligned} \tag{222}$$

Now we can insert the anticommutator values:

$$\begin{aligned} \{c_r(\vec{p}), c_s^\dagger(\vec{p'})\} &= (2\pi)^3 \frac{\omega}{m}\delta^3(\vec{p} - \vec{p'})\delta_{rs} \\ \{d_r(\vec{p}), d_s^\dagger(\vec{p'})\} &= (2\pi)^3 \frac{\omega}{m}\delta^3(\vec{p} - \vec{p'})\delta_{rs} \end{aligned} \tag{223}$$

Doing this gives:

$$\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = -e^2\sum_{mnrs}\int \langle 0|(2\pi)^3\frac{\omega_1'}{m}\delta^3(\vec{p}_1'-\vec{k}_1')\delta_{s_1'r}(2\pi)^3\frac{\omega_2'}{m}\delta^3(\vec{p}_2'-\vec{k}_2')\delta_{s_2's}(2\pi)^3\frac{\omega_1}{m}\delta^3(\vec{k}_1-\vec{p}_1)\delta_{ns_1}(2\pi)^3\frac{\omega_2}{m}\delta^3(\vec{p}_2-\vec{k}_2)\delta_{ms_2}|0\rangle \\ e^2\int \overline{u}_r(\vec{p}_1')\gamma^\mu v_s(\vec{p})iD^F_{\mu\nu}(x_2-x_1)\overline{v}_m(\vec{p}_2)\gamma^\nu u_n(\vec{p}_1)(2\pi)^4 e^{-ik_1'\cdot x_1}e^{-ik_2'\cdot x_1}e^{-ik_2\cdot x_2}e^{-ik_1\cdot x_2} \\ \frac{d^3k_1}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2}{(2\pi)^3}\frac{m}{\omega_2}\frac{d^3k_1'}{(2\pi)^3}\frac{m}{\omega_1}\frac{d^3k_2'}{(2\pi)^3}\frac{m}{\omega_2}d^4x_1d^4x_2 \tag{224}$$

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = e^2\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(x_2-x_1)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)(2\pi)^4 e^{-i(-p_1'-p_2')\cdot x_1}e^{-i(p_1+p_2)\cdot x_2} \tag{225}$$

Now let's insert the momentum space propagators:

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = e^2\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1+p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)(2\pi)^4 e^{-i(-p_1'-p_2'-q)\cdot x_1}e^{-i(p_1+p_2-q)\cdot x_2} \tag{226}$$

We can do the x integrations to yield delta functions:

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = e^2\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1+p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)(2\pi)^4\delta^4(-p_1'-p_2'-q)\delta^4(p_1+p_2-q) \tag{227}$$

We can now apply the usual identity to the delta functions:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{228}$$

Doing this gives:

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = e^2\int \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1+p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\delta^4(p_1+p_2-q) \tag{229}$$

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\int e^2\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1+p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1)\delta^4(p_1+p_2-q) \tag{230}$$

Now the q integration can be done:

$$\langle e^-,e^+|S^{(2)}_\alpha|e^-,e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')e^2\overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1+p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1) \tag{231}$$

A similar calculation can be used to yield the other term:

$$\langle e^-,e^+|S^{(2)}_\beta|e^-,e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')e^2\overline{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{s_2'}(\vec{p})iD^F_{\mu\nu}(p_1'-p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1) \tag{232}$$

$$\langle e^-,e^+|S^{(2)}|e^-,e^+\rangle = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\mathcal{M}_{fi} \tag{233}$$

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^A + \mathcal{M}_{fi}^B \tag{234}$$

$$\begin{aligned}\mathcal{M}_{fi}^A &= e^2 \overline{u}_{s_1'}(\vec{p}_1')\gamma^\mu v_{s_2'}(\vec{p}) i D_{\mu\nu}^F(p_1 + p_2)\overline{v}_{s_2}(\vec{p}_2)\gamma^\nu u_{s_1}(\vec{p}_1) \\ \mathcal{M}_{fi}^B &= e^2 \overline{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{s_2'}(\vec{p}) i D_{\mu\nu}^F(p_1' - p_1)\overline{u}_{s_1'}(\vec{p}_1')\gamma^\nu u_{s_1}(\vec{p}_1)\end{aligned} \tag{235}$$

## 8.2 Electron Self-Energy

$$\langle e^-|S^{(2)}| : e^-\rangle = -e^2 \int \langle e^-| : \overline{\psi}^-(x)\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)\psi^+(x) : |e^-\rangle d^4x_1 d^4x_2 \tag{236}$$

The operators in the matrix are already normal ordered. We can therefore drop the normal ordering notation:

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \langle e^-|\overline{\psi}^-(x)\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)\psi^+(x)|e^-\rangle d^4x_1 d^4x_2 \tag{237}$$

Remember from previously:

$$\begin{aligned}\psi^+(x)|e^-\rangle &= u_s(\vec{p})e^{-ip\cdot x}|0\rangle \\ \langle e^-|\overline{\psi}^-(x) &= \langle 0|e^{ip\cdot x}\overline{u}_s(\vec{p})\end{aligned} \tag{238}$$

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \langle 0|e^{ip'\cdot x_1}\vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})e^{-ip\cdot x_2}|0\rangle d^4x_1 d^4x_2 \tag{239}$$

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})e^{ip'\cdot x_1}e^{-ip\cdot x_2} d^4x_1 d^4x_2 \tag{240}$$

We can now write this in terms of momentum space propagators:

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})e^{-i(-p'+q+k)\cdot x_1}e^{-i(p-q-k)\cdot x_2} d^4x_1 d^4x_2 \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \tag{241}$$

Now we can do the x integration to yield the delta functions:

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})\delta^4(p' + q + p - q)\delta^4(p - q - k)d^4q d^4k \tag{242}$$

We can now use the same delta function identity that we have been using:

$$\delta(x - a)f(x) = \delta(x - a)f(a) \tag{243}$$

This yields:

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int k\vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})\delta^4(p' + q + p - q)\delta^4(p - q - k)d^4q d^4k \tag{244}$$

$$\langle e^-|S^{(2)}|e^-\rangle = -\delta^4(p - p')e^2 \int \vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)u_s(\vec{p})\delta^4(p - q - k)d^4q d^4k \tag{245}$$

Now we can do the k-integration:

$$\langle e^-|S^{(2)}|e^-\rangle = -e^2 \int \vec{u}_s(\vec{p})\gamma^\mu i S_F(q)\gamma^\nu i D_{\mu\nu}^F(p' - q)v_s(\vec{p})\frac{d^4q}{(2\pi)^4} \tag{246}$$

$$\langle e^-|S^{(2)}|e^-\rangle = (2\pi)^4\delta^4(p-p')\mathcal{M}_{fi} \tag{247}$$

$$\mathcal{M}_{fi} = -e^2\int \vec{v}_s(\vec{p})\gamma^\mu iS_F(q)\gamma^\nu iD^F_{\mu\nu}(p'-q)v_s(\vec{p})\frac{d^4q}{(2\pi)^4} \tag{248}$$

## 8.3 Positron Self-Energy

$$\langle e^+|S^{(2)}|e^+\rangle = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu iD^F_{\mu\nu}(x_2-x_1)\psi^-(x_2): d^4x_1d^4x_2 \tag{249}$$

A calculation essentially identical to the one above for the electron self energy:

$$\langle e^+|S^{(2)}|e^+\rangle = (2\pi)^4\delta^4(p-p')\mathcal{M}_{fi} \tag{250}$$

$$\mathcal{M}_{fi} = -e^2\int \vec{v}_s(\vec{p})\gamma^\mu iS_F(q)\gamma^\nu iD^F_{\mu\nu}(p'-q)v_s(\vec{p})\frac{d^4q}{(2\pi)^4} \tag{251}$$

## 8.4 Photon Self-Energy

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int\langle 0|:Tr[iS_F(x_2;x_1)\gamma^\mu A^-_\mu(x)iS_F(x_2-x_1)\gamma^\nu A^+_\nu(x)]:|0\rangle d^4x_1d^4x_2 \tag{252}$$

We can pull off a trace of non-operators from the vacuum expectation value:

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(x_2;x_1)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu]\langle 0|:A^-_\nu(x)A^+_\mu(x):|0\rangle d^4x_1d^4x_2 \tag{253}$$

Now, we can take care of normal ordering:]

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(x_2;x_1)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu]\langle 0|A^-_\nu(x)A^+_\mu(x)|0\rangle d^4x_1d^4x_2 \tag{254}$$

$$A^+_\mu(x)|\gamma\rangle = \epsilon^\lambda_\mu e^{-ik\cdot x}|0\rangle \quad \langle\gamma|A^-_\mu(x) = \langle 0|\epsilon^\lambda_\mu e^{ik'\cdot x} \tag{255}$$

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(x_2;x_1)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu]\langle 0|\epsilon^\lambda_\nu e^{-ik\cdot x_1}\epsilon^\lambda_\mu e^{ik'\cdot x_2}|0\rangle d^4x_1d^4x_2 \tag{256}$$

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(x_2;x_1)\not\epsilon_\lambda iS_F(x_2-x_1)\not\epsilon_\lambda]e^{-ik\cdot x_1}e^{ik'\cdot x_2}d^4x_1d^4x_2 \tag{257}$$

Next, we can write this in terms of momentum space propagators:

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(q)\not\epsilon_\lambda iS_F(q)\not\epsilon_\lambda]e^{-i(k+q-p)\cdot x_1}e^{-i(-k'-q+p)\cdot x_2}d^4x_1d^4x_2\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{258}$$

Then we can do the x-integrations to yield the delta functions:

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2\int Tr[iS_F(q)\not\epsilon_\lambda iS_F(q)\not\epsilon_\lambda]\delta^4(k+q-p)\delta^4(-k'-q+q)d^4pd^4q \tag{259}$$

We can now use the same delta function property:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{260}$$

Doing this yields:

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2 \int Tr[iS_F(q)\not{\epsilon}_\lambda iS_F(q)\not{\epsilon}_\lambda]\delta^4(k+q-p)\delta^4(-k'-q+k+q)d^4p d^4q \tag{261}$$

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2(2\pi)^{(4)}\delta^4(k-k') \int Tr[iS_F(q)\not{\epsilon}_\lambda iS_F(q)\not{\epsilon}_\lambda]\delta^4(k+q-p)d^4p\frac{d^4q}{(2\pi)^4} \tag{262}$$

Now we are ready to do the p integration:

$$\langle\gamma|S^{(2)}|\gamma\rangle = -e^2(2\pi)^{(4)}\delta^4(k-k') \int Tr[iS_F(k+q)\not{\epsilon}_\lambda iS_F(q)\not{\epsilon}_\lambda]\frac{d^4q}{(2\pi)^4} \tag{263}$$

$$\langle\gamma|S^{(2)}|\gamma\rangle = (2\pi)^{(4)}\delta^4(k-k')\mathcal{M}_{fi} \tag{264}$$

$$M_{fi} = -e^2 \int Tr[iS_F(k+q)\not{\epsilon}_\lambda iS_F(q)\not{\epsilon}_\lambda]\frac{d^4q}{(2\pi)^4} \tag{265}$$

## 8.5 Vacuum Self-Energy

There are no operators, so the normal ordering notation can be dropped, and the vacuum is normalized, so the inner product of vacuum states yields unity:

$$\langle 0|S^{(2)}|0\rangle = \frac{-e^2}{2!}\int \langle 0| : Tr[iS_F(x_1-x_2)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu iD^F_{\mu\nu}(x_2-x_1)] : |0\rangle d^4x_1 d^4x_2 \tag{266}$$

Now we can write this in terms of momentum space propagators:

$$S_3^{(2)} = \frac{-e^2}{2!}\int Tr[iS_F(x_1-x_2)\gamma^\mu iS_F(x_2-x_1)\gamma^\nu iD^F_{\mu\nu}(x_2-x_1)]d^4x_1 d^4x_2 \tag{267}$$

Now we can do the x integration to yield delta functions:

$$S_3^{(2)} = \frac{-e^2}{2!}\int Tr[iS_F(k)\gamma^\mu iS_F(p)\gamma^\nu iD^F_{\mu\nu}(q)]\delta^4(-k+p+k-p)\delta^4(k-p-q)d^4p d^4q\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{268}$$

We can now apply the usual delta function identity:

$$\delta(x-a)f(x) = \delta(x-a)f(a) \tag{269}$$

This gives:

$$S_3^{(2)} = \frac{-e^2}{2!}\delta^4(0)\int Tr[iS_F(k)\gamma^\mu iS_F(p)\gamma^\nu iD^F_{\mu\nu}(q)]\delta^4(-k+p+k-p)\delta^4(k-p-q)d^4p d^4q\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{270}$$

$$S_3^{(2)} = \frac{-e^2}{2!}\delta^4(0)\int Tr[iS_F(k)\gamma^\mu iS_F(p)\gamma^\nu iD^F_{\mu\nu}(q)]\delta^4(k-p-q)d^4p\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{271}$$

$$S_3^{(2)} = \frac{-e^2}{2!}\delta^4(0)\int Tr[iS_F(k)\gamma^\mu iS_F(k-q)\gamma^\nu iD^F_{\mu\nu}(q)]\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{272}$$

$$\langle 0|S^{(2)}|0\rangle = (2\pi)^4\delta^4(0)\mathcal{M}_{fi} \tag{273}$$

$$\mathcal{M}_{fi} = \frac{-e^2}{2!}\int Tr[iS_F(k)\gamma^\mu iS_F(k-q)\gamma^\nu iD^F_{\mu\nu}(q)]\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \tag{274}$$

Now that we have computed these Feynman amplitudes to their Feynman diagrams, to yield the Feynman rules.

# 9 Derivation of Feynman Rules by Inspection of Feynman Diagrams and Calculated Amplitudes

A little review: In Section, we associated the various second order terms in the S-matrix expansion to Feynman Diagrams. Then in section 7 and 8, we further evaluated these amplitudes to a simplified form where we could identify the Feynman Rules. We will now insert these further evaluated S-matrix terms back into the table from Section 6, and then compare the two and read off the Feynman rules for QED. The updated table is as follows:

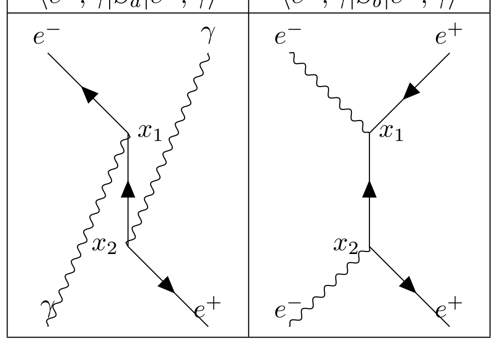
<table>
<tr>
<td>$e^-$ Compton Scattering</td>
<td>

$$\langle e^-,\gamma|S^{(2)}|e^-,\gamma\rangle = \langle e^-,\gamma|S_A|e^-,\gamma\rangle = \langle e^-,\gamma|S_B|e^-,\gamma\rangle \tag{275}$$

$$\langle e^-,\gamma|S_a|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^+(x_2): d^4x_1d^4x_2 \tag{276}$$

$$\langle e^-,\gamma|S_b|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu A^-_\mu(x_1)iS_F(x_2-x_1)\gamma^\nu A^+_\nu(x_2)\psi^+(x_2): d^4x_1d^4x_2 \tag{277}$$

| $\langle e^-,\gamma\|S_a\|e^-,\gamma\rangle$ | $\langle e^-,\gamma\|S_b\|e^-,\gamma\rangle$ |
|---|---|




The integral just accounts for the fact that the vertices could be located anywhere, as we noted with the first order term.
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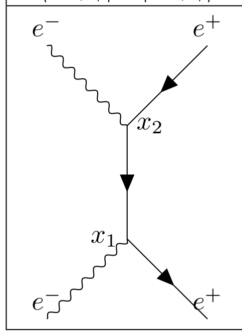
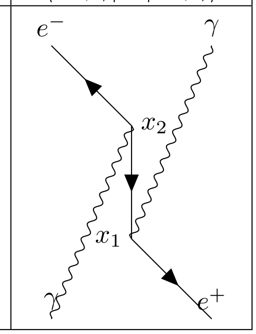
<tr>
<td>$e^+$ Compton Scattering</td>
<td>

$$\langle e^-,\gamma|S^{(2)}|e^-,\gamma\rangle = \langle e^-,\gamma|S_A|e^-,\gamma\rangle = \langle e^-,\gamma|S_B|e^-,\gamma\rangle \tag{278}$$

$$\langle e^-,\gamma|S_a|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^+_\mu(x_1) iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^-(x_2): d^4x_1 d^4x_2 \tag{279}$$

$$\langle e^-,\gamma|S_b|e^-,\gamma\rangle = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1) iS_F(x_2-x_1)\gamma^\nu A^+_\nu(x_2)\psi^-(x_2): d^4x_1 d^4x_2 \tag{280}$$

<table>
<tr><th>$\langle e^-,\gamma|S_a|e^-,\gamma\rangle$</th><th>$\langle e^-,\gamma|S_b|e^-,\gamma\rangle$</th></tr>
<tr><td>
</td><td>
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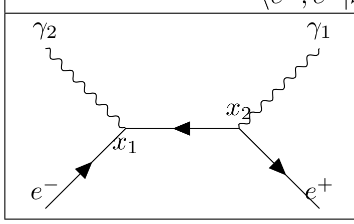
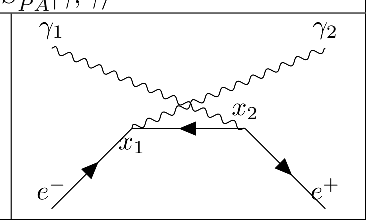
<tr>
<td>Pair Annihilation</td>
<td>

$$\langle\gamma,\gamma|S^{(2)}|e^-,e^+\rangle = \langle\gamma,\gamma|S^{(2)}_{PA}|e^-,e^+\rangle \tag{281}$$

$$S^(2)_{PA} = -e^2\int :\overline{\psi}^+(x_1)\gamma^\mu A^-_\mu(x_1) iS_F(x_2-x_1)\gamma^\nu A^-_\nu(x_2)\psi^+(x_2): d^4x_1 d^4x_2 \tag{282}$$

Looking at this amplitude term, we notice something interesting. Because the outgoing particles are identical (both photons), there are two possible Feynman diagrams that we could associate with this term, which differ by an interchange of outgoing photons. When we evaluate these amplitudes further, later on in this video, we will find that $\langle\gamma,\gamma|S^{(2)}_{PA}|e^-,e^+\rangle$ actually produces two terms. These terms will only differ by an interchange of the photon polarization vectors, and wll correspond to the two different possible Feynman diagrams we have noticed here, which differ by exactly that outgoing photon interchange.

<table>
<tr><th colspan="2">$\langle e^-,e^+|S^{(2)}_{PA}|\gamma,\gamma\rangle$</th></tr>
<tr><td>
</td><td>
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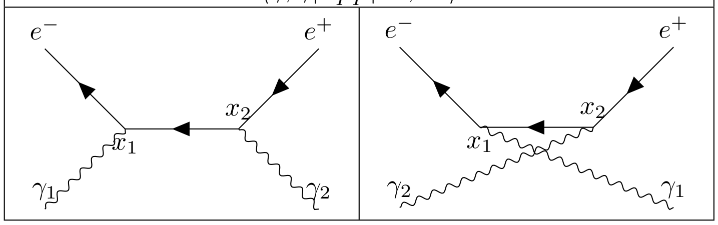
**Pair Production**

$$\langle e^-, e^+|S^{(2)}|\gamma,\gamma\rangle = \langle e^-, e^+|S^{(2)}_{PP}|\gamma,\gamma\rangle \tag{283}$$

$$S^{(2)}_{PP} = -e^2 \int :\overline{\psi}^-(x_1)\gamma^\mu A^+_\mu(x_1) iS_F(x_2-x_1)\gamma^\nu A^+_\nu(x_2)\psi^-(x_2): d^4x_1 d^4x_2 \tag{284}$$

Here, we have the same situation we saw with pair annihilation, There are two possible Feynman diagrams that could be associated with this term that, again just differ by the interchange of the identical photons. This happens anytime the incoming or outgoing particles are identical pairs. Just as with pair annihilation, when we evaluate this amplitude further, we will find two terms that differ only by an interchange of the photon polarization vectors.

$\langle \gamma,\gamma|S^{(2)}_{PP}|e^-, e^+\rangle$



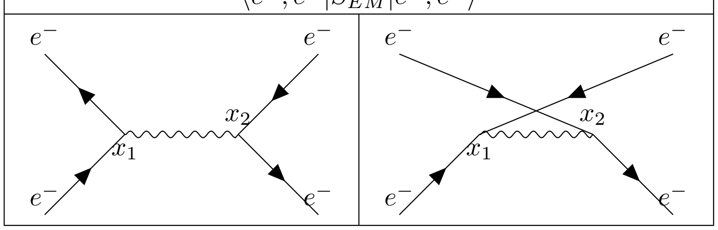

**$e^-$ Moller Scattering**

$$\langle e^-, e^-|S^{(2)}|e^-, e^-\rangle = \langle e^-, e^-|S_{EM}|e^-, e^-\rangle \tag{285}$$

$$S_{EM} = \frac{-e^2}{2}\int :\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1) iD^F_{\mu\nu}(x_2-x_1)\overline{\psi}^-(x_2)\gamma^\nu\psi^+(x_2): d^4x_1 d^4x_2 \tag{286}$$

With Moller scattering, we again have a similar situation to what we saw in the last two entries in this table, only this time, it is more extreme. Both the incoming particles and the outgoing particles are identical pairs. Therefore, there are four different Feynman diagrams that we could associate with this term, and when we evaluate the amplitude further, we will find that it does contain four terms. We will also find that two pairs of them are actually identical to the incoming particles this corresponds to the fact that there are only two physically distinct diagrams that could be associated with this matrix. Just like the previous ones, these diagrams differby an interchange of two identical particles. The usual selection for the two physically distinct diagrams is as follows:

$\langle e^-, e^-|S_{EM}|e^-, e^-\rangle$




When we do evaluate this amplitude further, we will find one other thing. The two terms that we do ultimately end up with have opposite signs in addition to the interchanged outgoing electron momenta. This results from the anticommuting property of fermionic creation and annihilation operators. This won't happen for the case of bosons becase their associated operators have commuting properties.

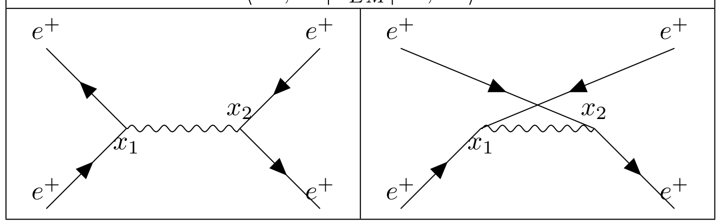
**$e^+$ Moller Scattering**

$$\langle e^+, e^+|S^{(2)}|e^+, e^+\rangle = \langle e^+, e^+|S_{EM}|e^+, e^+\rangle \tag{287}$$

$$S_{PM} = \frac{-e^2}{2}\int :\overline{\psi}^+(x_1)\gamma^\mu\psi^-(x_1) iD^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^-(x_2) : d^4x_1 d^4x_2 \tag{288}$$

The multiplicity of the graphs follows exactly the same description as in the $e^-$ Moller Scattering case.

$\langle e^+, e^+|S_{EM}|e^+, e^+\rangle$




**Bhabha Scattering**

$$\langle e^-, e^+|S^{(2)}|e^-, e^+\rangle = \langle e^-, e^+|S_\alpha|e^-, e^+\rangle = \langle e^-, \gamma|S_\beta|e^-, e^+\rangle \tag{289}$$

$$S_\alpha = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu\psi^-(x_1) iD^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^+(x_2) : d^4x_1 d^4x_2 \tag{290}$$

$$S_\beta = -e^2\int :\overline{\psi}^-(x_1)\gamma^\mu\psi^+(x_1) iD^F_{\mu\nu}(x_2 - x_1)\overline{\psi}^+(x_2)\gamma^\nu\psi^-(x_2) : d^4x_1 d^4x_2 \tag{291}$$

| $\langle e^-, e^+\|S_\alpha\|e^-, e^+\rangle$ | $\langle e^-, \gamma\|S_\beta\|e^-, e^+\rangle$ |
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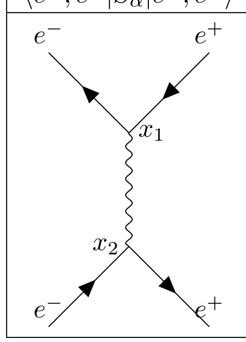
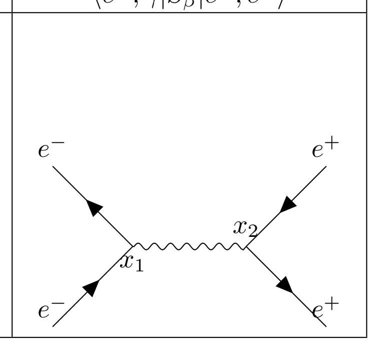
|  |  |

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<td>Electron Self Energy</td>
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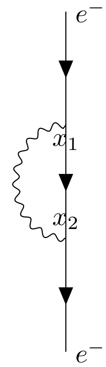
$$\langle e^-|S^{(2)}|e^-\rangle = \langle e^-|S_{ESE}|e^-\rangle \tag{292}$$

$$S_{ESE} = -e^2 \int :\overline{\psi}^-(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)\psi^+(x_2) : d^4x_1 d^4x_2 \tag{293}$$

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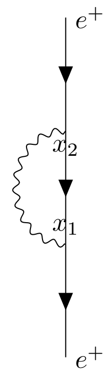
<tr>
<td>Positron Self Energy</td>
<td>

$$\langle e^+|S^{(2)}|e^+\rangle = \langle e^+|S_{PSE}|e^+\rangle \tag{294}$$

$$S_{PSE} = -e^2 \int :\overline{\psi}^+(x_1)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)\psi^-(x_2) : d^4x_1 d^4x_2 \tag{295}$$

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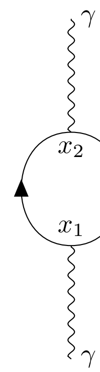
<table>
<tr><td>Photon Self Energy</td><td>

$$\langle\gamma|S^{(2)}|\gamma\rangle = \langle\gamma|S_{PhSE}|\gamma\rangle \tag{296}$$

$$S_{PhSE} = -e^2\int : Tr[iS_F(x_1 - x_2)\gamma^\mu A^+_\mu(x_1)S_F(x_2 - x_1)\gamma^\nu A^-_\nu(x_2)] : d^4x_1d^4x_2 \tag{297}$$

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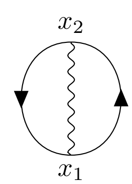
<tr><td>Vacuum Energy</td><td>

$$\langle 0|S|0\rangle = \langle 0|S^{(2)}_3|0\rangle \tag{298}$$

$$S^{(2)}_3 = \frac{(-ie)^2}{2!}\int : Tr[iS_F(x_1 - x_2)\gamma^\mu iS_F(x_2 - x_1)\gamma^\nu iD^F_{\mu\nu}(x_2 - x_1)] : d^4x_1d^4x_2 \tag{299}$$

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After carefully scrutinizing each Feynman amplitude and its Associated Feynman diagram, one concludes that the Associated Feynman amplitude can be generated from its diagram with the following Feynman rules:

| **The Feynman Rules for QED** | |
|---|---|
| 1 | The first step in the use of Feynman Rules in QED is to write out all topologically distinct Feynman Diagrams containing at most a number of vertices equal to which one wishes to study the S-matrix. These diagrams can contain external and internal photon lines, external electron lines, external positron lines, internal fermion lines, and two fermion one photon vertices. These diagrams must also conserve charge, and respect conservation of angular momentum. |

| | | |
|---|---|---|
| 2 | Factor Ordering | 1. Pick a fermion line and travel along it. Add the factos as described in the following entries of this table based on what features are encountered in the diagram.<br>2. As one completes this step, each new factor must be placed from right to left (opposite to how reading normally works)<br>3. Do this for all non-directly connected fermion lines |
| 3 | Momentum conservation | Many of the various factors that show up in the amplitude terms are momentum dependent. The momentum values that they are evaluated at are determined by the incoming and outgoing momentum values (the sums are equal due to overall conservation), and by **momentum conservation at each vertex within the corresponding diagram.** |
| 4 | Two-Fermion One-Photon vertices | For every such vertex, insert the following factor: $-ie\gamma^{\mu}$ |
| 5 | Incoming external Electron lines | For Incoming external Electron lines, insert the following factor: $u_s(p)$ |
| 6 | Outgoing external Electron lines | For Outgoing external Electron lines, insert the following factor: $\overline{u}_s(p)$ |
| 7 | Incoming external Positron lines | For Incoming external Positron lines, insert the following factor: $v_s(p)$ |
| 8 | Outgoing external Positron lines | For Outgoing external Positron lines, insert the following factor: $\overline{v}_s(p)$ |
| 9 | Interchanging external fermion lines | When one encounters a process with two contributing diagrams that are identical apart from interchanged external fermion lines, the associated terms have opposite signs. |
| 10 | External Photon lines | Contract the Lorentz index on the photon polarization $\epsilon_{\mu}$ vector with the one on the associated vertex function. |
| 11 | Internal Fermion lines | For internal fermion lines, insert the following factor (momentum space Dirac fermion propagator):<br>$iS_F(p) = \frac{i}{\not{p} - m + i(\epsilon = 0)} = \frac{\not{p} + m}{p^2 - m^2}$ (300) |
| 12 | Internal Photon lines | For internal photon lines, insert the following factor (momentum space photon propagator):<br>$iD^F_{\mu\nu}(p) = -\frac{ig_{\mu\nu}}{q^2 + i(\epsilon = 0)}$ (301) |
| 13 | Loop integrations | Integrate over all loop momentum variables that aren't fixed by momentum conservation at vertices. When doing this, use the following integration measure:<br>$\frac{d^4q}{(2\pi)^4}$ (302) |

| | | |
|---|---|---|
| 14 | Loop Integration Coefficient | Divide each term by the factoria of the number of loops in its corresponding diagram (remember that the factoral of zero is one). |
| 15 | Trace rule | Take a trace anytime that applying the previous rules leads to any uncontracted spinor indices. |

Inspection analysis has already shown that these rules can generate the second order terms in the perturbation series, but we have not yet shown that these rules can be used to generate the entire perturbative expansion of the s-matrix to all orders of a perturbation theory. This is however, our claim. Specifically, the claim is that one can draw out all the topologically distinct Feynman diagrams with the allowed vertex and particle lines up to a given order (number of vertices) and then use the rest of the Feynman rules to convert them to every term that shows up in the Perturbation series to that order, without missing anything.

It is however, not to hard to see that it is true

First, we know that the form of the Interacting Hamiltonian will control the ordering of the factors such that the two-fermion-one-photon vertex is the only one possible. Second, Wick expansion ensures that all possible combinations of propagators (internal lines) and spinors and photon polarization vectors (external lines) at each vertex shows up in the perturbation series. Third, because each quantum field contains a creation and annihilation part, we can see that regardless of perturbative order, multiplying out a given term will ensure that every sensical temporal orientation of each vertex will show up in the perturbation series, and with every sensical combinations of external positrons and electrons.

From the considerations laid out in the last paragraph, we can see that to all orders in perturbation theory, the Feynman Rules yield every kind of term that will show up in the perturbation series, and nothing more. The only remaining question is: does it do so with the right multiplicity? In the second order example, that we have so far treated, for multiple different reasons, every distinct term showed up twice, (except for vacuum energy term). This factor of two canceled against the factor of two factorial in the denominator that arose from the Taylor expansion of the exponential. The Feynman Rules don't require the inclusion of such a factor for most of the terms generated from various Feynman Diagrams (the only exception to this is for loop diagrams). Therefore, if the Feynman rules are correct as they currently stand (to which they are), then expanding the S-matrix must yield each type of term with multiplicity equal to the factorial of its order divided by the factorial of the loop number, thus canceling the multiplicative factors not included by the Feynman rules. One can see that this is the case via the following inspection analysis:

When one inserts (into the S-matrix term), the creation and annihilation two-term expansion of the quantum field, and multiplies everything out, the complete space-time integration over each factor of the Interacting Hamiltonian will ensure that all terms within somme number of interchanges of Hamiltonian factors of each other are identical. Therefore, all terms with nonidentical incoming external legs, and nonidentical outgoing external legs, we will have the right multiplicity. Fewer identical terms can of course be, generated this way for term types where some of the quantum fields have been contracted into propagators. This is already accounted for in the Feynman rules (rule 14). When one has identical particles in the incoming and outgoing set, things are slightly different. As we saw in the second order case, different pairings between the creation and annihilation operators yield terms that are topologically distinct Feynman diagrams, and may also yield additional multiplicity on top of that. Multiplicity is only yielded if some number of identical pairs of particles shows up in both the incoming set, and the outgoing set (the only example that we saw of this was Moller scattering). In order for such situation to occur, the original wick expansion term must contain the same product of quantum fields at least twice, each evaluated at different dummy variables. This is however, exactly the situation that inhibits multiplicity via the first mechanism (multiple terms proving identical under dummy variable interchange) to exactly the extent that the second mechanism yields multiplicity. Thus the perturbation series yields each type of term with the correct multiplicity to match what is produced by the listed Feynman Rules to all order in Perturbation theory.