9902 Calculus Midterm Exam (Date: 2011/04/26: 110 minutes)

1. Evaluate

   a. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ (8%)

   b. $\lim_{x \to 0^+} \frac{\int_0^{\sqrt{x}} \sin(t^2)dt}{\sin\left(x^{\frac{3}{2}}\right)}$ (8%)

2. A population grows exponentially. At 10 years, the population is 1,000. At 20 years, it is 2,000. What was the approximate population at 5 years? (10%)

3. Evaluate

   a. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$ (8%),

   b. $\int \frac{\ln(x+1)}{x^2} dx$ (8%)

   c. $\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}$ (8%)

   d. $\int \frac{xdx}{(x-1)^2\sqrt{1+2x-x^2}}$ (8%)

4. Show that

   $\int_1^\infty \frac{\ln x}{x^p} dx = \begin{cases} \text{diverges}, & p \leq 1 \\ \frac{1}{(p-1)^2}, & p > 1 \end{cases}$ (10%)

5. Find the limit of the sequence $\left\{\sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}, \cdots\right\}$. (5%)

6. Determine convergence or divergence.

   a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ (5%)

   b. $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ (5%)

   c. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ (5%)

   d. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\ln(n+1)}$ (5%)

7. Determine the convergence interval of $\sum_{n=2}^{\infty} (-1)^n \frac{(x+1)^n}{n \ln n}$ (10%)

8. Find the Maclaurin series for $\sinh x = \frac{e^x - e^{-x}}{2}$ (8%)

9902 Calculus Midterm Exam [SOLUTION]

1. Evaluate

a. (8%)

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=\lim_{x\to\frac{\pi}{2}}e^{\tan x\ln\sin x}=\lim_{x\to\frac{\pi}{2}}e^{\frac{\ln\sin x}{\cot x}}=\lim_{x\to\frac{\pi}{2}}e^{\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}}=e^0=1$$

b. (8%)

$$\lim_{x\to0^+}\frac{\int_0^{\sqrt{x}}\sin(t^2)\,dt}{\sin\left(x^{\frac{3}{2}}\right)}=\lim_{x\to0^+}\frac{(\sin x)\cdot\frac{1}{2\sqrt{x}}}{\left(\cos x^{\frac{3}{2}}\right)\cdot\frac{3}{2}x^{\frac{1}{2}}}=\frac{1}{3}\lim_{x\to0^+}\frac{1}{\cos x^{\frac{3}{2}}}\cdot\lim_{x\to0^+}\frac{\sin x}{x}$$

$$=\frac{1}{3}\times1\times\lim_{x\to0^+}\frac{\cos x}{1}=\frac{1}{3}$$

2. (10%)

Let $y(t)$ be the population after t years, So that

$$y(t)=ce^{kt}$$

$$\begin{cases}y(10)=ce^{10k}=1000\ ---(1)\\ y(20)=ce^{20k}=2000\ ---(2)\end{cases}$$

$$\frac{(2)}{(1)}e^{10k}=2\quad\Rightarrow\quad k=\frac{\ln 2}{10}\ ,c=500$$

$$y(t)=500e^{\frac{\ln 2}{10}t}\ \ ,\ \ y(5)=500e^{\frac{\ln 2}{2}}=500e^{\ln\sqrt{2}}=500\sqrt{2}\cong 707$$

3. Evaluate

a. (8%)

$$\int\frac{dx}{(x-2)\sqrt{x^2-4x+3}}=\int\frac{dx}{(x-2)\sqrt{x^2-4x+4-1}}$$

$$=\int\frac{dx}{(x-2)\sqrt{(x-2)^2-1}}=\int\frac{d(x-2)}{(x-2)\sqrt{(x-2)^2-1}}$$

$$=\sec^{-1}|x-2|+C$$

b. (8%)

$$\int \frac{\ln(x+1)}{x^2}dx$$

$$u = \ln(x+1) \quad , \quad du = \frac{1}{x+1}dx$$

$$dv = \frac{dx}{x^2} \qquad , \quad v = -\frac{1}{x}$$

$$\int \frac{\ln(x+1)}{x^2}dx = -\frac{\ln(x+1)}{x} + \int \frac{1}{x(x+1)}dx$$

$$= -\frac{\ln(x+1)}{x} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = -\frac{\ln(x+1)}{x} + \ln\left(\frac{x}{x+1}\right) + c$$

c. (8%)

$$\int \frac{dx}{x^5 + x^4 - 2x^3 - 2x^2 + x + 1}$$

$$x^5 + x^4 - 2x^3 - 2x^2 + x + 1 = x^4(x+1) - 2x^2(x+1) + (x+1)$$
$$= (x+1)(x^4 - 2x^2 + 1) = (x+1)^3(x-1)^2$$

$$\frac{1}{x^5 + x^4 - 2x^3 - 2x^2 + x + 1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

$$1 = A(x+1)^2(x-1)^2 + B(x+1)(x-1)^2 + C(x-1)^2 + D(x+1)^3(x-1) + E(x+1)^3$$

$$x = -1 \quad \Rightarrow \quad C = \frac{1}{4} \quad , \quad x = 1 \quad \Rightarrow \quad E = \frac{1}{8}$$

Compare coefficients of x^4 : $A + D = 0$

Compare coefficients of x^3 : $B + 2D + E = 0$

Compare constants : $A + B + C - D + E = 1$

By simultaneous equations : $A = \frac{3}{16}$, $B = \frac{1}{4}$, $D = -\frac{3}{16}$

$$\int \left[\frac{\frac{3}{16}}{x+1} + \frac{\frac{1}{4}}{(x+1)^2} + \frac{\frac{1}{4}}{(x+1)^3} - \frac{\frac{3}{16}}{x-1} + \frac{\frac{1}{8}}{(x-1)^2}\right]dx$$

$$= \frac{3}{16}\ln|x+1| - \frac{1}{4(x+1)} - \frac{1}{8(x+1)^2} - \frac{3}{16}\ln|x-1| - \frac{1}{8(x-1)} + c$$

d. (8%)

$$\int \frac{xdx}{(x-1)^2\sqrt{1+2x-x^2}}$$

$$x - 1 = \sqrt{2}\sin\theta \quad , \quad dx = \sqrt{2}\cos\theta\, d\theta$$

$$\int \frac{xdx}{(x-1)^2\sqrt{1+2x-x^2}} = \int \frac{xdx}{(x-1)^2\sqrt{2-(x-1)^2}}$$

$$= \int \frac{1+\sqrt{2}\sin\theta}{2\sin^2\theta\cdot\sqrt{2}\cos\theta}\cdot\sqrt{2}\cos\theta\, d\theta = \int \frac{1+\sqrt{2}\sin\theta}{2\sin^2\theta}d\theta$$

$$= \int\left(\frac{1}{2}\csc^2\theta + \frac{\sqrt{2}}{2}\csc\theta\right)d\theta = -\frac{1}{2}\cot\theta - \frac{\sqrt{2}}{2}\ln|\csc\theta+\cot\theta| + c$$

$$= -\frac{\sqrt{1+2x-x^2}}{2(x-1)} - \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}+\sqrt{1+2x-x^2}}{x-1}\right| + c$$

4. Show that (10%)

$$\int_1^\infty \frac{\ln x}{x^p}dx = \begin{cases} \text{diverges}, & p \le 1 \\ \dfrac{1}{(p-1)^2}, & p > 1 \end{cases}$$

When $p \neq 1$, $u = \ln x$, $dv = x^{-p}dx,\ du = \frac{dx}{x},\ v = \frac{1}{-p+1}x^{-p+1}$

$$\int_1^\infty \frac{\ln x}{x^p}dx = \left[\frac{1}{-p+1}\frac{\ln x}{x^{p-1}}\right]_1^\infty - \int_1^\infty \frac{1}{-p+1}x^{-p+1}\frac{dx}{x}$$

$$= \left[\frac{1}{-p+1}\frac{\ln x}{x^{p-1}} - \frac{1}{(-p+1)^2}\frac{1}{x^{p-1}}\right]_1^\infty = \begin{cases} \text{diverges}, & p < 1 \\ \dfrac{1}{(p-1)^2}, & p > 1 \end{cases}$$

When $p = 1$,

$$\int_1^\infty \frac{\ln x}{x^p}dx = \frac{1}{2}(\ln x)^2\Big|_1^\infty \ \text{diverges}$$

5. Find the limit of the sequence $\left\{\sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}, \cdots\right\}$. (5%)

$a_1 = \sqrt{2} = 2^{\frac{1}{2}}$, $a_2 = \sqrt{\sqrt{2}} = 2^{\frac{1}{4}}$, $a_n = 2^{\frac{1}{2^n}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{\frac{1}{2^n}} = 2^0 = 1$

6. Determine convergence or divergence.

a. (5%)

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = 1$ diverges.

$$\because \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \text{ (The n} - \text{th Term Test)}$$

b. (5%)

$$\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} \text{ converges}$$

$$\because \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(\cancel{2}n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)}}{\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$$

$$< 1 \text{ (The Ratio Test)}$$

c. (5%)

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \text{ converges.}$$

$$\because \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{n} = \frac{1}{e}$$

$$< 1 \text{(The Root Test)}$$

d. (5%)

$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\ln(n+1)}$ converges.

$$\because \frac{2}{\ln(n+1)} < \frac{2}{\ln(n)} \quad and \quad \lim_{n\to\infty} \frac{2}{\ln n} = 0 \text{(Leibniz's Test)}$$

7. Determine the convergence interval of $\sum_{n=2}^{\infty} (-1)^n \frac{(x+1)^n}{n \ln n}$ (10%)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(x+1)^{n+1}}{(n+1)\ln(n+1)}}{(-1)^n \frac{(x+1)^n}{n \ln n}} \right| = |x+1| < 1$$

when $x = -2$, $\sum_{n=2}^{\infty}(-1)^n \frac{(-2+1)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

when $x = 0$ $\sum_{n=2}^{\infty}(-1)^n \frac{(0+1)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$ converges

The convergence interval is $-2 < x \leq 0$

8. Find the Maclaurin series for $\sinh x = \frac{e^x - e^{-x}}{2}$ (8%)

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right]$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$