Calculus Quiz 1 Solution

1.	$\lim\limits_{t\to 0}\frac{t(1-\cos t)}{t-\sin t} = \lim\limits_{t\to 0}\frac{(1-\cos t)+t(\sin t)}{1-\cos t} = \lim\limits_{t\to 0}\frac{\sin t+(\sin t+t\cos t)}{\sin t} = \lim\limits_{t\to 0}\frac{\cos t+\cos t+\cos t-t\sin t}{\cos t} = \frac{1+1+1-0}{1} = 3$
2.	The limit leads to the indeterminate form 1^{∞}. Let $f(x) = \left(\frac{x+2}{x-1}\right)^x \Rightarrow \ln f(x) = \ln\left(\frac{x+2}{x-1}\right)^x = x\ln\left(\frac{x+2}{x-1}\right) \Rightarrow \lim\limits_{x\to\infty}\ln f(x)$ $= \lim\limits_{x\to\infty} x\ln\left(\frac{x+2}{x-1}\right) = \lim\limits_{x\to\infty}\left(\frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty}\left(\frac{\ln(x+2)-\ln(x-1)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty}\left(\frac{\frac{1}{x+2}-\frac{1}{x-1}}{-\frac{1}{x^2}}\right) = \lim\limits_{x\to\infty}\left(\frac{\frac{-3}{(x+2)(x-1)}}{-\frac{1}{x^2}}\right)$ $= \lim\limits_{x\to\infty}\left(\frac{3x^2}{(x+2)(x-1)}\right) = \lim\limits_{x\to\infty}\left(\frac{6x}{2x+1}\right) = \lim\limits_{x\to\infty}\left(\frac{6}{2}\right) = 3$. Therefore, $\lim\limits_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$
3.	$\int\frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2}\int\frac{du}{u\sqrt{u^2-4}}$, where $u = 2x-1$ and $du = 2\,dx$ $= \frac{1}{2}\cdot\frac{1}{2}\sec^{-1}\left\|\frac{u}{2}\right\| + C = \frac{1}{4}\sec^{-1}\left\|\frac{2x-1}{2}\right\| + C$
4.	$\int_{\sqrt{2}}^{2}\frac{\sec^2(\sec^{-1}x)}{x\sqrt{x^2-1}}dx = \int_{\pi/4}^{\pi/3}\sec^2 u\,du$, where $u = \sec^{-1}x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$ $= [\tan u]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \sqrt{3}-1$
5.	$\int\frac{\operatorname{csch}(\ln t)\cosh(\ln t)}{t}dt \quad u = \ln t \quad du = \frac{dt}{t}$ $= \int\operatorname{csch}u\cosh u\,du$ $= \int\frac{\cosh u}{\sinh u}du \quad k = \sinh u \quad \frac{dk}{du} = \cosh u\,du$ $\Rightarrow \int\frac{1}{k}dk = \ln k + C$ $= \ln(\sinh u) + c$ $= \ln(\sinh(\ln t)) + C$
6.	$\int_0^{\pi}\frac{\cos x}{\sqrt{1+\sin^2 x}}dx = \int_0^0\frac{1}{\sqrt{1+u^2}}du = [\sinh^{-1}u]_0^0 = \sinh^{-1}0 - \sinh^{-1}0 = 0$, where $u = \sin x$, $du = \cos x\,dx$
7.	(a) true; $\frac{\left(\frac{1}{x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x}{x+3} < 1$ if $x > 1$ (or sufficiently large) (b) false; $\lim\limits_{x\to\infty}\frac{\left(\frac{1}{x}-\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right) = 1$ (c) true; $\frac{e^x+x}{e^x} = 1+\frac{x}{e^x}$ and $\frac{x}{e^x} \to 0$ as $x \to \infty \Rightarrow 1+\frac{x}{e^x} < 2$ if x is sufficiently large (d) true; $\lim\limits_{x\to\infty}\frac{x\ln x}{x^2} = \lim\limits_{x\to\infty}\frac{\ln x}{x} = \lim\limits_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{1} = 0$

Calculus Quiz 1 Solution

8.	$u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\,dx,\ v = \frac{x^4}{4};$ $\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$
9.	$I = \int e^{-y}\cos y\,dy;\ [u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$ $\Rightarrow I = -e^{-y}\cos y - \int(-e^{-y})(-\sin y)\ dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy;\ [u = \sin y,\ du = \cos y\,dy;$ $dv = e^{-y}\,dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int(-e^{y})\cos y\,dy\right) = -e^{-y}\cos y + e^{-y}\sin y - I + C'$ $\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant
10.	$\int e^x \sin e^x\,dx\ \left[\text{Let } u = e^x,\ du = e^x\,dx\right] \rightarrow \int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u + C = -\cos e^x + C$
11.	$\int \sin^5 x\,dx = \int(\sin^2 x)^2 \sin x\,dx = \int(1-\cos^2 x)^2 \sin x\,dx = \int(1 - 2\cos^2 x + \cos^4 x)\sin x\,dx$ $= \int \sin x\,dx - \int 2\cos^2 x \sin x\,dx + \int \cos^4 x \sin x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$
12.	$L(x) = L_0 e^{-kx} \Rightarrow \frac{L_0}{2} = L_0 e^{-18k} \Rightarrow \ln\frac{1}{2} = -18k \Rightarrow k = \frac{\ln 2}{18} \approx 0.0385 \Rightarrow L(x) = L_0 e^{-0.0385x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0 e^{-0.0385x} \Rightarrow \ln 10 = 0.0385x \Rightarrow x \approx 59.8$ ft