Engineering mathematics I   Midterm exam., 10/27/2022

This is an open-book test. Please show your computations. The total score is 110 points.

1. For the range $x > 0$, let us consider the differential equation:

$$y' - \frac{4}{x} \cdot y = x^5 \cdot e^x \ .$$

(a). (5%) Show that this is a first-order linear differential equation.

(b). (5%) Find an integrating factor for this differential equation.

(c). (5%) Solve for $y(x)$. Please write down its general form.

(d). (5%) Assume that, in addition to the differential equation, we also have the initial condition: $y(1) = 3$. Then, $y(x) =$?

2.(10%) Consider the first-order differential equation (DE): $y' = x \cdot \sqrt{y}$. Notice that it is a seperable differential equation. Also notice that $y = 0$ is a solution – a trivial solution. What we are really interested is to find some non-trivial solutions. Now, in addition to the DE, let us assume that we have the intial condition: $y(0) = 0$. Then, $y(3) =$?

3. Consider this first-order differential equation:

$$(x^2y^3) + (x^3y^2 + 1) \cdot \frac{dy}{dx} = 0 \ .$$

(a). (5%) Show that it is an exact differential equation.

(b). (10%) Find the general form of the solution to this differential equation.

(c). (5%) In addition to the exact differential equation, let us assume that we also have the initial condition: $y(1) = 1$. Find the exact solution to this initial-value problem.

4. Consider the differential equation:

$$y'' + 3y' + 2y = 3x^2 + e^{-x}$$

(a). (5%) Find the homogeneous solutions to this differential equation.

(b). (10%) Find a particular solution to this differential equation.

(c). (5%) Find the general solutions to this differential equation.

(d). (5%) In addition to the differential equation, let us assume that we also have the initial condition: $y(0) = 1$, $y'(0) = 0$. Find the exact solution to this initial-value problem.

5. Consider the system of two first-order differential equations:

$$\begin{cases} x_1'(t) = x_1(t) + 3\,x_2(t) \\ x_2'(t) = 5\,x_1(t) + 3\,x_2(t) + e^{-t} \end{cases}$$

together with initial conditions:

$$x_1(0) = -1 \text{ , and } x_2(0) = 1 \quad .$$

(a). (5%) Transform this system of differential equations into the s-domain, via the Laplace transform.

(b). (10%) Let $X_1(s)$ denote the Laplace transform of $x_1(t)$. Then, $X_1(s) =?$

(c). (5%) $x_1(t) =?$

6. Let us consider a high-order initial value problem:

$$\begin{cases} \textbf{DE:} & y'''' + 2y''' + \cos(y)\cdot y'' - 3\cdot(y')^2 + \sqrt{y}\cdot y = x^2 + 1 \\ \textbf{IC:} & y(0) = -1,\ y'(0) = 3,\ y''(0) = -5,\ y'''(0) = -2 \end{cases}$$

(a). (10%) Please reduce this problem to a system of first-order differential equations. Notice that the associated initial conditions must also be clearly specified.

(b). (5%) What is the main use/advantage in reducing a high-order differential equation into a system of first-order differential equations?