Note: To get full points, you should write down the procedure **in detail**.

1. (15 points) Find the derivative of the following functions. (5 points for each)
   (a) $f(x) = a^{a^x} + x^{a^a} + a^{x^a}$
   (b) $f(x) = \ln[\ln[\ln x]]$
   (c) $f(x) = \log_x |\ln x|$

Solution:

(a)

$\dfrac{d}{dx}(a^u) = \ln a \cdot a^u \cdot \dfrac{du}{dx}$

$(a^{a^x})' = \ln a \cdot a^{a^x} \cdot \dfrac{d}{dx}(a^x) = \ln a \cdot a^{a^x} \cdot \ln a \cdot a^x$

$(x^{a^a})' = a^a \cdot x^{a^a - 1}$

$(a^{x^a})' = \ln a \cdot a^{x^a} \cdot \dfrac{d}{dx}(x^a) = \ln a \cdot a^{x^a} \cdot ax^{a-1}$

Therefore, the derivative of $f(x)$ is $f'(x) = (\ln a)^2 a^{(a^x + x)} + a^a x^{(a^a - 1)} + (\ln a) a^{(x^a + 1)} x^{(a-1)}$

(b)

$f'(x) = \dfrac{1}{\ln[\ln x]} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$

(c)

$f(x) = \log_x |\ln x| = \dfrac{\ln|\ln x|}{\ln x}$

$f'(x) = \dfrac{\frac{1}{\ln x} \cdot \frac{1}{x} \cdot \ln x - \frac{1}{x}\ln|\ln x|}{(\ln x)^2} = \dfrac{1 - \ln|\ln x|}{x(\ln x)^2}$

2. (25 points) Evaluate the following integrals. (5 points for each)

(a) $\displaystyle\int \frac{\tan(\ln x)}{x}dx$ (b) $\displaystyle\int \frac{dx}{x\cos^2(\log_2 x)}$ (c) $\displaystyle\int \cos\theta\sin(\sin\theta)d\theta$

(d) $\displaystyle\int \frac{e^{-1/x^2}}{x^3}dx$ (e) $\displaystyle\int \frac{1}{x-x^{1/3}}dx$

Solution:

(a)

Let $u=\ln x$, $\quad du=\frac{1}{x}dx \qquad \Rightarrow \qquad \displaystyle\int \frac{\tan(\ln x)}{x}dx=\int \tan u du=\ln\big|\sec u\big|+C$

Therefore $\displaystyle\int \frac{\tan(\ln x)}{x}dx=\ln\big|\sec(\ln x)\big|+C=-\ln\big|\cos(\ln x)\big|+C$

(b)

Let $u=\log_2 x$, $\quad du=\frac{1}{x\ln 2}dx \qquad \Rightarrow \qquad \displaystyle\int \frac{dx}{x\cos^2(\log_2 x)}=\ln 2\int \frac{du}{\cos^2 u}=\ln 2\int \sec^2 u du=\ln 2\tan u+C$

Therefore $\displaystyle\int \frac{dx}{x\cos^2(\log_2 x)}=\ln 2\tan(\log_2 x)+C$

(c)

Let $u=\sin\theta$, $\quad du=\cos\theta d\theta \qquad \Rightarrow \qquad \displaystyle\int \cos\theta\sin(\sin\theta)d\theta=\int \sin u du=-\cos u+C$

Therefore $\displaystyle\int \cos\theta\sin(\sin\theta)d\theta=-\cos(\sin u)+C$

(d)

Let $u=-1/x^2$, $\quad du=2/x^3dx \qquad \Rightarrow \qquad \displaystyle\int \frac{e^{-1/x^2}}{x^3}dx=\frac{1}{2}\int e^u du=\frac{1}{2}e^u+C$

Therefore $\displaystyle\int \frac{e^{-1/x^2}}{x^3}dx=\frac{1}{2}e^{-1/x^2}+C$

(e)

$$\int \frac{1}{x-x^{1/3}}dx=\int \frac{1}{x^{1/3}(x^{2/3}-1)}dx$$

Let $u=x^{2/3}-1$, $\quad du=\frac{2}{3}\frac{1}{x^{1/3}}dx \qquad \Rightarrow \qquad \displaystyle\int \frac{1}{x^{1/3}(x^{2/3}-1)}dx=\frac{3}{2}\int \frac{1}{u}du=\frac{3}{2}\ln|u|+C$

Therefore $\displaystyle\int \frac{1}{x-x^{1/3}}dx=\frac{3}{2}\ln\left|x^{2/3}-1\right|+C$

3. Consider the function $f(x) = \dfrac{\ln x}{x}$.
   (a) (5 points) Find all critical points, inflection points, and the maximum value for $f(x)$
   (b) (5 points) Use the result of (a) to verify that $e^\pi > \pi^e$.

Solution:

The domain of $f(x)$ is $\{x \in \mathbb{R} | x > 0\}$

(a)

$f'(x) = \dfrac{1 - \ln x}{x^2}$, $f''(x) = \dfrac{2\ln x - 3}{x^3}$

Critical points: $f'(x) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$

When $0 < x < e$, $f'(x) = \dfrac{1 - \ln x}{x^2} > 0$ (increasing)

When $x > e$, $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ (decreasing)

Therefore, the critical point at $x = e$ is the maximum value of $f(x)$.

Maximum value occurs at $x = e$, the maximum value of $f(x)$ is $1/e$

Inflection points: $f''(x) = 0 \Rightarrow \ln x = 3/2 \Rightarrow x = e^{3/2}$ ■

(b)

Because the maximum value of $f(x)$ is at $x = e$, thus $f(e) \geq f(x)$ for all x. Therefore, $f(e) > f(\pi)$.

$\dfrac{\ln e}{e} > \dfrac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi \Rightarrow \ln e^\pi > \ln \pi^e$

Therefore $e^\pi > \pi^e$ ■

4. (10 points) Find the exact length of the curve for $f(x) = \ln(\sec x), 0 \leq x \leq \pi/4$.

Solution:

$f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

The arc length is $L = \displaystyle\int_0^{\pi/4} \sqrt{1 + (f'(x))^2}dx$

Thus, $L = \displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}dx = \int_0^{\pi/4} \sec x dx = \Big[\ln \big|\sec x + \tan x\big|\Big]_0^{\pi/4} = \ln(\sqrt{2} + 1)$ ■

5. (10 points) Find the total area of the region between the curve $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 3\pi/2$

Solution:

Note: $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $\sin(\frac{5\pi}{4}) = \cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

In $0 < x < \pi/4$, $\cos x > \sin x$. Thus, the area between $\cos x$ and $\sin x$ in $0 < x < \pi/4$ is

$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx = \Big[\sin x + \cos x\Big]_0^{\pi/4} = \sqrt{2} - 1$$

In $\pi/4 < x < 5\pi/4$, $\cos x < \sin x$. Thus, the area between $\cos x$ and $\sin x$ in $\pi/4 < x < 5\pi/4$ is

$$A_2 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4} = 2\sqrt{2}$$

In $5\pi/4 < x < 3\pi/2$, $\cos x > \sin x$. Thus, the area between $\cos x$ and $\sin x$ in $5\pi/4 < x < 3\pi/2$ is

$$A_3 = \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x) dx = \Big[\sin x + \cos x\Big]_{5\pi/4}^{3\pi/2} = \sqrt{2} - 1$$

The total area of the region is $A = A_1 + A_2 + A_3 = 4\sqrt{2} - 2$. ■

6. If $f(x)$ is continuous, and $\int_0^{2x} f(t)dt = -7 + 2x^2 + k\cos x$, then
   (a) (5 points) Find $k =$?
   (b) (5 points) Find $f(\frac{\pi}{3}) =$?

Solution:
(a)
Because $\int_0^0 f(t)dt = 0$, to substitute $x = 0$ into the equation $\int_0^{2x} f(t)dt = -7 + 2x^2 + k\cos x$
will have $\int_0^0 f(t)dt = -7 + k\cos 0 = 0 \Rightarrow -7 + k = 0 \Rightarrow k = 7$ ■
(b)
To differentiate both sides of the equation $\int_0^{2x} f(t)dt = -7 + 2x^2 + k\cos x$:
$\frac{d}{dx}\int_0^{2x} f(t)dt = 2f(2x)$ (Chain rule & Fundamental Theorem of Calculus)
$\frac{d}{dx}(-7 + 2x^2 + 7\cos x) = 4x - 7\sin x$
$\Rightarrow 2f(2x) = 4x - 7\sin x \Rightarrow f(2x) = 2x - \frac{7}{2}\sin x$
Let $u = 2x$
$f(u) = u - \frac{7}{2}\sin(\frac{u}{2}) \Rightarrow f(\frac{\pi}{3}) = \frac{\pi}{3} - \frac{7}{2}\sin(\frac{\pi}{6}) = \frac{\pi}{3} - \frac{7}{4}$ ■

7. (10 points) If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2R}{(R+r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

Solution:
$P(R) = \frac{E^2R}{(R+r)^2} \Rightarrow P'(R) = \frac{dP}{dR} = \frac{E^2(R+r)^2 - E^2 \cdot 2R(R=r)}{(R+r)^4} = \frac{E^2(r-R)}{(R+r)^3}$
$P'(R) = 0 \Rightarrow R = r$ is the extrema of $P(R)$.
When $R < r$, $P'(R) > 0$ (increasing)
When $R > r$, $P'(R) < 0$ (decreasing)
Therefore, $P(R)$ have maximum value at $R = r$. The maximum value of the power is $P(r) = \frac{E^2}{4r}$. ■

8. (10 points) The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid. Please find the volume of the solid.

Solution:
$V = \int_{\frac{1}{4}}^{4} \pi \Big[f(x)\Big]^2 dx = \int_{\frac{1}{4}}^{4} \pi \Big[\frac{1}{2\sqrt{x}}\Big]^2 dx = \frac{\pi}{4}\int_{\frac{1}{4}}^{4} \frac{1}{x} dx = \frac{\pi}{4}(\ln 4 - \ln\frac{1}{4})$

By the properties of logarithm, $\frac{\pi}{4}(\ln 4 - \ln\frac{1}{4}) = \frac{\pi}{4}\ln\frac{4}{1/4} = \frac{\pi}{4}(\ln 16) = \frac{\pi}{4}(\ln 2^4) = \frac{\pi}{4}(4\ln 2) = \pi \ln 2.$ ■

9. (10 points) If $f(x) = \sqrt{x-2}$, find $(f^{-1})'(2) =?$

Solution:
$f'(x) = \frac{df}{dx} = \frac{1}{2\sqrt{x-2}}$

Find $a =?$ if $f(a) = 2$.

$f(a) = \sqrt{a-2} = 2 \Rightarrow a = 6$. Therefore, $f(6) = 2$ and $f^{-1}(2) = 6$.

Therefore, $(f^{-1})'(2) = \frac{df^{-1}}{dx}\Big|_{x=2} = \frac{1}{\frac{df}{dx}\big|_{x=f^{-1}(2)}} = \frac{1}{\frac{df}{dx}\big|_{x=6}} = \frac{1}{\frac{1}{2\sqrt{6-2}}} = 4$. ■

10. (10 points) Assume $0 \le a \le 1$. Find the value of a such that $\int_0^1 |x^2 - ax| dx$ achieves its maximum.

Solution:

$$|x^2 - ax| = x|x-a| = \begin{cases} -x^2 + ax & (0 \le x \le a) \\ x^2 - ax & (a \le x \le 1) \end{cases}$$

$$\int_0^1 |x^2 - ax| dx = \int_0^a (-x^2 + ax)dx + \int_a^1 (x^2 - ax)dx = \big(-\frac{1}{3}x^3 + \frac{1}{2}ax^2\big)\Big|_0^a + \big(\frac{1}{3}x^3 - \frac{1}{2}ax^2\big)\Big|_a^1 = \frac{1}{3}a^3 - \frac{1}{2}a + \frac{1}{3}$$

Let $f(a) = \int_0^1 |x^2 - ax| dx = \frac{1}{3}a^3 - \frac{1}{2}a + \frac{1}{3}$.

The integral $f(a)$ has its maximum at the end points or $f'(c) = 0$ for $c \in (0,1)$.

Find critical points:

$$\frac{df}{da} = f'(a) = a^2 - \frac{1}{2} = 0 \Rightarrow a = \frac{1}{\sqrt{2}} \qquad (a > 0, \therefore a \ne -\frac{1}{\sqrt{2}})$$

At end points, $f(0) = \frac{1}{3}, f(1) = \frac{1}{6}$.

At critical point, $f(\frac{1}{\sqrt{2}}) = \frac{1}{3} - \frac{1}{3\sqrt{2}}$

Therefore, the maximum of the integral is at $a = 0$. ■