1. (**15** points) Evaluate the following integrals. (5 points for each)

(a) $\displaystyle\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$ (b) $\displaystyle\int_9^{64} \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\, dx$ (c) $\displaystyle\int_0^1 x^3\left(1+9x^4\right)^{-3/2}\, dx$

Solution:

(a) Let $u = \sin x \Rightarrow du = \cos x\, dx \quad x : 0 \to \frac{\pi}{2} \Rightarrow u : 0 \to 1$. Therefore,

$$\int_0^{\pi/2} \cos x \sin(\sin x)\, dx = \int_0^1 \sin u\, du = \left[-\cos u\right]\Big|_0^1 = 1 - \cos 1.$$

(b) Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2}\frac{1}{\sqrt{x}}\, dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\, dx \quad x : 9 \to 64 \Rightarrow u : 4 \to 9$. Therefore,

$$\int_9^{64} \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\, dx = \int_4^9 \frac{2}{\sqrt{u}}\, du = 2\int_4^9 u^{-\frac{1}{2}}\, du = \left[4u^{\frac{1}{2}}\right]\Big|_4^9 = 4\cdot(3-2) = 4.$$

(c) Let $u = 1 + 9x^4 \Rightarrow du = 36x^3\, dx \Rightarrow \frac{1}{36}\, du = x^3\, dx \quad x : 0 \to 1 \Rightarrow u : 1 \to 10$. Therefore,

$$\int_0^1 x^3\left(1+9x^4\right)^{-3/2}\, dx = \int_1^{10} \frac{1}{36} u^{-3/2}\, du = \frac{1}{36}\left[-2u^{-1/2}\right]\Big|_1^{10} = \frac{1}{18}\left(1 - \frac{1}{\sqrt{10}}\right) = \frac{10-\sqrt{10}}{180}.$$

2. (**10** points) Find the areas of the region bounded by $y = \sin x$ and $y = \sin^2 x$, between $x = 0$ and $x = \pi/2$.

Solution:

- The area is

$$\begin{aligned} A &= \int_0^{\pi/2}\left(\sin x - \sin^2 x\right) dx = \int_0^{\pi/2}\left(\sin x - \frac{1-\cos 2x}{2}\right) dx \\ &= \int_0^{\pi/2} \sin x\, dx - \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx \\ &= (-\cos x)\Big|_0^{\pi/2} - \left(\frac{1}{2}x\right)\Big|_0^{\pi/2} + \frac{1}{4}(\sin 2x)\Big|_0^{\pi/2} = 1 - \frac{\pi}{4} + 0 = 1 - \frac{\pi}{4}. \end{aligned}$$

3. (**10** points) Air is pumped into a spherical balloon so that to volume increases at a rate of 100 (cm^3/s). How fast is the radius of the balloon increasing when diameter is 50 (cm)?

Solution:

- Based on the statement, $\frac{dV}{dt} = 100(\text{cm}^3/\text{s})$. When diameter is 50 (cm), the radius is $25(\text{cm})$. Therefore,

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

Thus,

$$\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{4\pi \cdot 25^2} \cdot 100 = \frac{1}{25\pi}\ (\text{cm/s}).$$

4. (**10** points) Find the length of the arc of the curve $x^2 = (y-4)^3$ from point $P(1,5)$ to point $Q(8,8)$.

Solution:

- $x^2 = (y-4)^3 \Rightarrow x = (y-4)^{3/2} \Rightarrow \frac{dx}{dy} = \frac{3}{2}(y-4)^{1/2}$. Therefore, the arc length is

$$L = \int_5^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_5^8 \sqrt{1 + \frac{9}{4}(y-4)}\, dy = \int_5^8 \sqrt{\frac{9}{4}y - 8}\, dy$$
$$= \frac{4}{9}\int_{13/4}^{10} u^{1/2} \cdot du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Big|_{13/4}^{10} = \frac{8}{27}\left[10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right].$$

5. (**10** points) Let $0 < a < b$. Use the mean value theorem to show that

$$\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}.$$

(Hint): Use the function $f(x) = \sqrt{x}$.

Solution:

- Let $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$. The function f is continuous and differentiable for all positive real number. Based on mean value theorem, there exists c in (a,b), such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b-a} = \frac{\sqrt{b} - \sqrt{a}}{b-a} \Rightarrow \left(\sqrt{b} - \sqrt{a}\right) = \frac{b-a}{2\sqrt{c}}.$$

Because $a < c < b$,

$$a < c \Rightarrow \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{a}} \Rightarrow \left(\sqrt{b} - \sqrt{a}\right) = \frac{b-a}{2\sqrt{c}} < \frac{b-a}{2\sqrt{a}}.$$

6. A function is defined as

$$f(x) = \int_1^{x^2} \frac{1}{\sqrt{1+t^2}}\, dt.$$

(a) (**2** points) Find $f(1)$.

(b) (**3** points) Find $f'(x)$.

(c) (**5** points) Find the linearization of $f(x)$ at $x = 1$.

(d) (**5** points) At which x the function $f(x)$ has a minimum value.

Solution:

(a) When $x = 1$, one can find that $f(1) = \int_1^1 \frac{1}{\sqrt{1+t^2}}\, dt = 0$.

(b) By Fundamental Theorem of Calculus Part 1,

$$f'(x) = \frac{d}{dx}\int_1^{x^2} \frac{1}{\sqrt{1+t^2}}\, dt = \frac{1}{\sqrt{1+(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1+x^4}}.$$

(c) Because $f(1) = 0$ and $f'(1) = \sqrt{2}$, the linearization of $f(x)$ at $x = 1$ is

$$L(x) = f(1) + f'(1) \cdot (x-1) = 0 + \sqrt{2}(x-1).$$

(d) Find c satisfies $f'(c) = 0$. (critical points)

$$f'(c) = \frac{2c}{\sqrt{1+c^4}} = 0 \Rightarrow c = 0,$$

and $f'(x) > 0$ for $x > 0$, $f'(x) < 0$ for $x < 0$. Therefore, there is a minimum value at $x = 0$.

7. (**10** points) Find the dimensions of the circular cylinder of **greatest** volume that can be inscribed in a cone of base radius R and height H if the base of the cylinder lies in the base of the cone. Please express the radius and height of the cylinder in terms of R and H.

Solution:

- Let the radius and the height of the circular cylinder be r and h. By similar triangles,

$$\frac{h}{R-r} = \frac{H}{R} \Rightarrow h = \frac{H}{R}(R-r).$$

Hence, the volume of the circular cylinder is

$$V(r) = \pi r^2 h = \pi r^2 \frac{H}{R}(R-r) = \pi H\left(r^2 - \frac{1}{R}r^3\right)$$

where $0 \le r \le R$. Since $V(0) = V(R) = 0$, the maximum value of $V(r)$ must be at a critical point. Therefore,

$$\frac{dV}{dr} = \pi H\left(2r - \frac{3}{R}r^2\right) = 0 \Rightarrow r = \frac{2R}{3}.$$

Therefore the cylinder has maximum volume if its radius is $r = \frac{2R}{3}$ and its height is $h = \frac{H}{R}\left(R - \frac{2R}{3}\right) = \frac{H}{R}\cdot\frac{R}{3} = \frac{H}{3}$.

8. (**10** points) Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about the line $y = 2$.

Solution:

- Use washer method. The volume is

$$V = \int_0^2 \pi \left(\left[2 - \left(x^2 - 2x\right)\right]^2 - 2^2 \right) dx = \pi \int_0^2 \left(x^4 - 4x^3 + 8x\right) dx = \pi \left[\frac{1}{5}x^5 - x^4 + 4x^2\right]\Big|_0^2 = \frac{32}{5}\pi.$$

9. (**10** points) Find the exact area of the surface obtained by rotating the curve $x = \frac{1}{3}\left(y^2 + 2\right)^{3/2}$, $1 \le y \le 2$ about the x-axis.

Solution:

- $x = \frac{1}{3}\left(y^2+2\right)^{3/2} \Rightarrow \frac{dx}{dy} = y\sqrt{y^2+2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + y^2\left(y^2+2\right) = \left(y^2+1\right)^2$. The surface area is

$$S = \int_1^2 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_1^2 y\left(y^2+1\right) dy = 2\pi \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]\Big|_1^2 = \frac{21}{2}\pi.$$

10. Let $f(x) = x\sqrt{2-x^2}$.
    (a) (**4** points) Find the domain of the function $f(x)$.
    (b) (**6** points) Find the intervals of increase and decrease.
    (c) (**4** points) Find the intervals of concavity.
    (d) (**4** points) Find the local maximum and minimum values.
    (e) (**2** points) Find the inflection points.

Solution:

(a) Domain: $\left[-\sqrt{2}, \sqrt{2}\right]$.

(b) The first derivative of this function is

$$f'(x) = x \cdot \frac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \frac{2(1+x)(1-x)}{\sqrt{2-x^2}}.$$

One can find that $f'(x) > 0$ when $-1 < x < 1$, and $f'(x) < 0$ when $-\sqrt{2} < x < -1$ and $1 < x < \sqrt{2}$. Therefore, $f(x)$ is increasing on $(-1, 1)$, decreasing on $(-\sqrt{2}, -1)$ and $(1, \sqrt{2})$.

(c) The second derivative of this function is

$$f''(x) = \frac{\sqrt{2-x^2}(-4x) - (2-2x^2)\frac{-x}{\sqrt{2-x^2}}}{2-x^2} = \frac{2x\left(x^2-3\right)}{\left(2-x^2\right)^{3/2}}.$$

Note that $\pm\sqrt{3}$ is not at the domain of $f(x)$. One can find that $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Therefore, $f(x)$ is concave upward on $(-\sqrt{2}, 0)$, concave downward on $(0, \sqrt{2})$.

(d) Based on the result of (b), one can find that $f'(x)$ changes its sign at $x = -1$ and $x = 1$.
Therefore, $f(-1) = -1$ is its local minimum. $f(1) = 1$ is its local maximum.

(e) Based on the result of (c), one can find that $f''(x)$ changes its sign only at $x = 0$.
$f(x)$ is continuous at $x = 0$, too. Therefore, $f(0) = 0$ is its inflection point.