Total: 100 points

1. (**20** points) Let $f(x,y) = e^x \sin y$:
   (a) Find ∇f.
   (b) Find the directional derivative of f at the point $(0, \pi/3)$ in the direction of $\vec{\mathbf{v}} = -6\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$.
   (c) In which direction does f **decrease most rapidly** at $(0, \pi/3)$.
   (d) In which direction does f have **zero change** at $(0, \pi/3)$.

Solution:

(a) $f(x,y) = e^x \sin y \Rightarrow \dfrac{\partial f}{\partial x} = e^x \sin y,\ \dfrac{\partial f}{\partial y} = e^x \cos y \quad \Rightarrow \quad \nabla f = (e^x \sin y)\,\hat{\mathbf{i}} + (e^x \cos y)\,\hat{\mathbf{j}}.$

(b) At the point $(0, \pi/3)$, the gradient vector is

$$\nabla f\Big|_{(0,\pi/3)} = \frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}.$$

Unit vector of the direction is

$$\hat{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = -\frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}.$$

Therefore, the directional derivative is

$$D_{\hat{\mathbf{u}}} f(0, \pi/3) = \left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right) \cdot \left(-\frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}\right) = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4 - 3\sqrt{3}}{10}.$$

(c) The direction is $-\nabla f\Big|_{(0,\pi/3)} = -\dfrac{\sqrt{3}}{2}\hat{\mathbf{i}} - \dfrac{1}{2}\hat{\mathbf{j}}.$

(d) Assume the direction of zero change is $\hat{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}$ where $u_1^2 + u_2^2 = 1$. Then,

$$D_{\hat{\mathbf{u}}} f(0, \pi/3) = \left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right) \cdot (u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}) = 0 \Rightarrow \frac{\sqrt{3}}{2}u_1 + \frac{1}{2}u_2 = 0 \Rightarrow u_2 = -\sqrt{3}u_1.$$

Because $u_1^2 + u_2^2 = 1$, then one can find that $u_1 = \pm\dfrac{1}{2}$. Therefore,

$$u_1 = \frac{1}{2} \Rightarrow u_2 = -\frac{\sqrt{3}}{2} \Rightarrow \hat{\mathbf{u}} = \frac{1}{2}\hat{\mathbf{i}} - \frac{\sqrt{3}}{2}\hat{\mathbf{j}}$$
$$u_1 = -\frac{1}{2} \Rightarrow u_2 = \frac{\sqrt{3}}{2} \Rightarrow \hat{\mathbf{u}} = -\frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}$$

2. (**20** points) Find all the local maxima, local minima, and saddle point(s) of the function $f(x,y) = x^3 + y^3 - 3xy$.

Solution:

- $f(x,y) = x^3 + y^3 - 3xy \Rightarrow f_x = 3x^2 - 3y, f_y = 3y^2 - 3x$.

$$f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y,$$
$$f_y = 3y^2 - 3x = 0 \Rightarrow y^2 = x.$$

Therefore, one can find that $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x(x-1)(x^2+x+1) = 0 \Rightarrow x = 0, x = 1$. The critical points are $(0,0)$ and $(1,1)$. Next step is to use second partial derivative test.

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3$$

For $(0,0)$:

$$f_{xx}(0,0) = 0, f_{yy}(0,0) = 0, f_{xy} = -3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$$

The point $(0,0)$ is a saddle point of f.
For $(1,1)$:

$$f_{xx}(1,1) = 6 > 0, f_{yy}(1,1) = 6, f_{xy} = -3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$$

The point $(1,1)$ is a local minimum of f.

3. (**20** points) Find an equation for the plane tangent to the level surface $f(x,y,z) = \ln(x - 2y) - z = 0$ at the point $P_0(3,1,0)$. Also, find parametric equations for the line that is normal to the surface at P_0.

Solution:

- Find the gradient vector for $f(x,y,z)$ at P_0:

$$f_x = \frac{1}{x - 2y} \Rightarrow f_x(3,1,0) = 1$$
$$f_y = \frac{-2}{x - 2y} \Rightarrow f_y(3,1,0) = -2$$
$$f_z = -1 \Rightarrow f_z(3,1,0) = -1$$

The tangent plane is $1 \cdot (x-3) + (-2) \cdot (y-1) + (-1) \cdot (z-0) = 0 \Rightarrow x - 2y - z = 1$.
The parametric equations for the normal line is $x = 3 + t$, $y = 1 - 2t$, $z = -t$.

4. (**10** points) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $e^z - xyz = 0$.

Solution:

- Use implicit differentiation. $F(x, y, z) = e^z - xyz = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}$$

5. (**10** points) Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if $z = (x-y)^5$ and $\quad x = s^2 t, \quad y = st^2.$

Solution:

- The chain rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 5(x-y)^4 \cdot 2st - 5(x-y)^4 \cdot t^2 = 5(x-y)^4(2st - t^2)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 5(x-y)^4 \cdot s^2 - 5(x-y)^4 \cdot 2st = 5(x-y)^4(s^2 - 2st)$$

6. (**20** points) A function $f(x, y) = xe^{y+x^2}$.
   (a) Find the linearization $L(x, y)$ of the function $f(x, y)$ at the point $(2, -4)$.
   (b) Utilize the result in (a) to estimate the value of $f(x, y)$ when $x = 2.05, y = -3.92$.

Solution:

(a) Linear approximation at $(2, -4)$:

$$\begin{aligned} f(x, y) &= xe^{y+x^2} \Rightarrow f(2, -4) = 2 \\ f_x(x, y) &= e^{y+x^2} + 2x^2e^{y+x^2} = e^{y+x^2}(1 + 2x^2) \Rightarrow f_x(2, -4) = 9 \\ f_y(x, y) &= xe^{y+x^2} \Rightarrow f_y(2, -4) = 2 \end{aligned}$$

Thus, the linearization of $f(x, y)$ at $(2, -4)$ is

$$L(x, y) = f(2, -4) + f_x(2, -4)(x - 2) + f_y(2, -4)(y + 4) = 2 + 9(x - 2) + 2(y + 4)$$

(b) Use (a) to estimate $f(2.05, -3.92)$:

$$\begin{aligned} f(2.05, -3.92) \approx L(2.05, -3.92) &= f(2, -4) + f_x(2, -4)(2.05 - 2) + f_y(2, -4)(-3.92 + 4) \\ &= 2 + 9(2.05 - 2) + 2(-3.92 + 4) \\ &= 2 + 9 \cdot 0.05 + 2 \cdot 0.08 = 2.61 \end{aligned}$$