

First Midterm—Chapter 1

Total points: 100 points

2 hours to do the work, Oct. 18, 2018

1. Solve the initial value problem.(10 points)

$$xy' = y + x^2 \cos^2\left(\frac{y}{x}\right); y(1) = \frac{\pi}{4}$$

$$y' = \frac{y}{x} + x \cos^2 \frac{y}{x}$$

$$\text{Let } u = \frac{y}{x}$$

$$y = ux$$

$$y' = u'x + u$$

$$u'x + u = u + x \cos^2 u$$

$$\frac{du}{dx} = \cos^2 u$$

$$\int \sec^2 u \, du = \int dx \quad \rightarrow +5$$

$$\tan u = x + c$$

$$u = \tan^{-1}(x + c)$$

$$y = x \tan^{-1}(x + c) \quad \rightarrow +8$$

$$\frac{\pi}{4} = \tan^{-1}(1 + c)$$

$$1 + c = \tan \frac{\pi}{4} = 1$$

$$c = 0$$

$$\therefore y = x \tan^{-1}(x) \quad \rightarrow +10$$

2. Solve the initial value problem.(10 points)

$$y'x\ln x = y \quad ; \quad y(2) = \ln 16$$

$$\frac{dy}{dx}x\ln x = y$$

$$\int \frac{dy}{y} = \int \frac{1}{x\ln x} dx$$

$$\text{Let } u = \ln x, \quad du = \frac{1}{x} dx$$

$$\int \frac{dy}{y} = \int \frac{1}{u} du \quad \rightarrow +5$$

$$\ln|y| = \ln|u| + c_1 \quad \rightarrow +7$$

$$y = ce^{\ln u} = c\ln x \quad \rightarrow +8$$

$$y(2) = \ln 16 = c\ln 2$$

$$c = \frac{\ln 16}{\ln 2} = \frac{4\ln 2}{\ln 2} = 4 \quad \rightarrow +9$$

$$\therefore y = 4\ln x \quad \rightarrow +10$$

3. Test the **exactness** of the given ODE and solve the problem.(10 points)

$$y' = 3.2y - 10y^2$$

$$\text{Let } u = y^{1-a} = y^{1-2} = y^{-1}$$

$$u' = -y^{-2}y'$$

$$y' = -u'y^2$$

$$\text{原式} = -u'y^2 = 3.2y - 10y^2$$

$$u' + 3.2y^{-1} - 10 = 0$$

$$u' + 3.2u - 10 = 0$$

$$\frac{du}{dx} + 3.2u - 10 = 0$$

$$du + (3.2u - 10)dx = 0$$

$$\begin{cases} \frac{\partial(3.2u - 10)}{\partial u} = 3.2 \\ \frac{\partial 1}{\partial x} = 0 \end{cases}$$

$$\frac{\partial 1}{\partial x} \neq \frac{\partial(3.2u - 10)}{\partial u} \quad \therefore \text{non-exact} \quad \rightarrow +5$$

$$F = e^{\int 3.2dx} = e^{3.2x}$$

$$e^{3.2x}u' + e^{3.2x}3.2u = 10e^{3.2x}$$

$$\frac{d(e^{3.2x}u)}{dx} = 3.2ue^{3.2x} + e^{3.2x}u'$$

$$\int \frac{d(e^{3.2x}u)}{dx} = 10 \int e^{3.2x}dx + c \quad \rightarrow +7$$

$$e^{3.2x}u = \frac{10}{3.2}e^{3.2x} + c \quad \rightarrow +8$$

$$u = \frac{10}{3.2} + ce^{-3.2x}$$

$$\therefore y^{-1} = \frac{10}{3.2} + ce^{-3.2x} \rightarrow +10$$

4. Find an **integrating factor**, use it to find the general solution of the differential equation, and then obtain the solution of the initial value problem. (20 points)

$$\sin(x-y) + \cos(x-y) - \cos(x-y)y' = 0 \quad ; \quad y(0) = \frac{7\pi}{6}$$

$$[\sin(x-y) + \cos(x-y)]dx + [-\cos(x-y)]dy = 0$$

$$\frac{\partial M}{\partial y} = -\cos(x-y) + \sin(x-y) \neq \frac{\partial N}{\partial x} = \sin(x-y) \quad \rightarrow +8$$

$$\mu = e^{\int \frac{1}{-\cos(x-y)} [-\cos(x-y) + \sin(x-y) - \sin(x-y)] dx} = e^{\int dx} = e^x \quad \rightarrow +10$$

$$\{e^x [\sin(x-y) + \cos(x-y)]dx\} + \{-e^x [\cos(x-y)]\} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -e^x \cos(x-y) + e^x \sin(x-y)$$

$$\varphi(x, y) = \int -e^x [\cos(x-y)] dy + k(x) = e^x \sin(x-y) + k(x)$$

$$M = \frac{\partial u}{\partial x} = e^x \sin(x-y) + e^x \cos(x-y) + k'(x) = e^x \sin(x-y) + e^x \cos(x-y)$$

$$k'(x) = 0 \quad k(x) = c_1 \quad \rightarrow +15$$

$$\therefore u = e^x \sin(x-y) = c_1$$

$$\text{G.S. : } e^x \sin(x-y) + c_1 = c_2$$

$$\therefore e^x \sin(x-y) = c_3 \quad \rightarrow +18$$

$$\text{代入 } (0, \frac{7\pi}{6})$$

$$\sin \frac{-7\pi}{6} = c_3 = \sin \frac{5\pi}{6} = \frac{1}{2}$$

$$\therefore e^x \sin(x-y) = \frac{1}{2} \quad \rightarrow +20$$

5. Find the general solution of the given ODE. (20 points)

$$2xyy' + (x - 1)y^2 = x^2e^x$$

$$2xydy + [(x - 1)y^2 - x^2e^x]dx = 0$$

$$\begin{cases} \frac{\partial[(x - 1)y^2 - x^2e^x]}{\partial y} = 2(x - 1)y \\ \frac{\partial(2xy)}{\partial x} = 2y \end{cases}$$

\therefore The given ODE is non-exact $\rightarrow +8$

$$\int \frac{2(x - 1)y - 2y}{2xy} dx = \int \frac{2xy - 4y}{2xy} dx = \int \left(1 - \frac{2}{x}\right) dx = x - 2 \ln |x|$$

$$\therefore F(x) = e^{x-2 \ln |x|} = e^x \cdot e^{\ln |x|^{-2}} = \frac{e^x}{x^2} \rightarrow +10$$

$$\frac{e^x}{x^2} 2xydy + \frac{e^x}{x^2} [(x - 1)y^2 - x^2e^x] dx = 0 \text{ is an exact ODE}$$

$$\therefore \exists u(x, y) \ni \begin{cases} \frac{\partial u}{\partial y} = \frac{e^x}{x^2} 2xy & \dots (1) \\ \frac{\partial u}{\partial x} = \frac{e^x}{x^2} [(x - 1)y^2 - x^2e^x] & \dots (2) \end{cases}$$

$$\text{from (1): } u = \int \frac{e^x}{x^2} 2xydy + k(x) = \frac{e^x}{x} y^2 + f(x) \rightarrow +13$$

$$\frac{\partial u}{\partial x} = \frac{\partial(\frac{e^x}{x^2} xy^2)}{\partial x} + \frac{df(x)}{dx} = \frac{e^x}{x^2} [(x - 1)y^2 - x^2e^x] = \frac{(x-1)}{x^2} e^x y^2 - e^{2x} \rightarrow +15$$

$$\begin{aligned} \frac{\partial(\frac{e^x}{x^2} xy^2)}{\partial x} &= \frac{\partial(e^x x^{-1} y^2)}{\partial x} = y^2(e^x x^{-1} + e^x \cdot (-1) \cdot x^{-2}) = y^2 \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right) \\ &= \frac{(x - 1)}{x^2} e^x y^2 \end{aligned}$$

$$\therefore \frac{df(x)}{dx} = -e^{2x} \rightarrow +16$$

$$f(x) = -\frac{1}{2} e^{2x} + c_1 \rightarrow +18$$

$$\therefore u(x, y) = \frac{e^x}{x} y^2 - \frac{1}{2} e^{2x} + c_1$$

$$\text{The general solution of the ODE is } \frac{e^x}{x} y^2 - \frac{1}{2} e^{2x} = c \rightarrow +20$$

6. Please show that the first order linear differential equation $(\frac{dy}{dx} + p(x)y = r(x))$ has a general solution, $y = e^{-\int p(x)dx}(\int e^{\int p(x)dx} r(x)dx + c)$ by the method of exact differential equation. (20 points)

$$y' + p(x)y = r(x)$$

$$\frac{dy}{dx} + p(x)y - r(x) = 0$$

$$[p(x)y - r(x)]dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x) \neq \frac{\partial N}{\partial x} = 0 \rightarrow +8$$

$$\mu(x) = e^{\int_1^x [p(x)-0]dx} = e^{\int p(x)dx} \rightarrow +10$$

$$\varphi(x, y) = \int \mu N dy = \int e^{\int p(x)dx} dy = ye^{\int p(x)dx} + k(x)$$

$$\frac{\partial \varphi}{\partial x} = p(x)ye^{\int p(x)dx} + k'(x) = \mu M = [p(x)y - r(x)]e^{\int p(x)dx}$$

$$k'(x) = -r(x)e^{\int p(x)dx} \rightarrow +15$$

$$k(x) = \int -r(x)e^{\int p(x)dx} dx + c_1 \rightarrow +17$$

$$\varphi(x, y) = ye^{\int p(x)dx} - \int r(x)e^{\int p(x)dx} dx + c_1 = c_2$$

$$ye^{\int p(x)dx} = \int r(x)e^{\int p(x)dx} dx + c \rightarrow +18$$

$$y = e^{-\int p(x)dx} (\int e^{\int p(x)dx} r(x)dx + c) \rightarrow +20$$

7. Solve the differential equation. (10 points)

$$y' = 6(y - 2.5)\tanh(1.5x)$$

$$\frac{dy}{dx} = 6(y - 2.5)\tanh(1.5x)$$

$$\int \frac{dy}{y - 2.5} = \int 6 \tanh(1.5x) dx$$

$$\text{Let } u = \cosh(1.5x)$$

$$du = 1.5 \sinh(1.5x) dx$$

$$\int \frac{dy}{y - 2.5} = 6 \int \frac{1}{1.5} \frac{1}{u} du \rightarrow +5$$

$$\ln|(y - 2.5)| = 4 \ln|u| = 4 \ln|\cosh(1.5x)| + c_1 \rightarrow +8$$

$$e^{\ln|(y - 2.5)|} = e^{4 \ln|\cosh(1.5x)| + c_1}$$

$$\therefore y - 2.5 = c \cosh^4(1.5x) \rightarrow +10$$