- **Ex:**

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \underline{\underline{R}}$$

- row-space $(\underline{A}) \stackrel{\Delta}{=} \langle\langle [1\,1\,1\,1\,1\,2], [1\,1\,1\,2\,2\,3], [1\,1\,1\,2\,3\,2] \rangle\rangle$ → l.i. (obvious)

  $= \langle\langle [1\,1\,1\,0\,0\,1], [0\,0\,0\,1\,0\,2], [0\,0\,0\,0\,1\,-1] \rangle\rangle$

  $\underline{\underline{A}}$ and $\underline{\underline{R}}$ are row-equivalent (i.e. can be converted into each other via ero's)

- col-space $(\underline{A}) \stackrel{\Delta}{=} \left\langle\left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\2\end{bmatrix} \right\rangle\right\rangle = \left\langle\left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\rangle\right\rangle$ → l.i.

  l.i. (as seen/checked from $\underline{\underline{R}}$)

- A basis for row-space $(\underline{A})$: $\{[1\,1\,1\,0\,0\,1], [0\,0\,0\,1\,0\,2], [0\,0\,0\,0\,1\,-1]\}$

  $\xrightarrow{\Delta}$ dim (row-space $(\underline{A})$) $= 3$ $\xrightarrow{\Delta}$ row-rank$(\underline{A}) = 3$

- A basis for col-space $(\underline{A})$: $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$

  $\xrightarrow{\Delta}$ dim(col-space $(\underline{A})$) $= 3$ $\xrightarrow{\Delta}$ column-rank$(\underline{A}) \stackrel{a}{=}$ col-rank $(\underline{A}) = 3$

rank$(\underline{A}) = 3$

• **Ex:**

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3/5 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underline{\underline{R}}$$

- a basis for row-space $(\underline{\underline{A}}) = \{ [1\ 0\ \tfrac{3}{5}\ 1], [0\ 1\ \tfrac{1}{5}\ 0] \}$
- a basis for col-space $(\underline{\underline{A}}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \right\}$
- rank $(\underline{\underline{A}}) = 2$

• **Ex:**

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 8 \\ 2 & 4 & 26 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} = \underline{\underline{R}}$$

- a basis for row-space $(\underline{\underline{A}})$: $\{ [1\ 0\ 3], [0\ 1\ 5] \}$
- P.S. $\{ [1\ 1\ 8], [2\ 4\ 26] \}$ also serves as a basis.
  In fact, you have many other choices for basis.
- A basis for col-space $(\underline{\underline{A}})$: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ $\longrightarrow$ rank$(\underline{\underline{A}}) = 2$
  (It happens that $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is also a basis for col-space $(\underline{\underline{A}})$)

- Recall

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} \underline{c}_1 & \underline{c}_2 & \cdots & \underline{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \underline{c}_1 + x_2 \cdot \underline{c}_2 + \cdots + x_n \cdot \underline{c}_n$$

l.c. of columns of $\underline{\underline{A}}$, with $x_1, x_2, \cdots x_n$ as the coeffs. in the l.c.

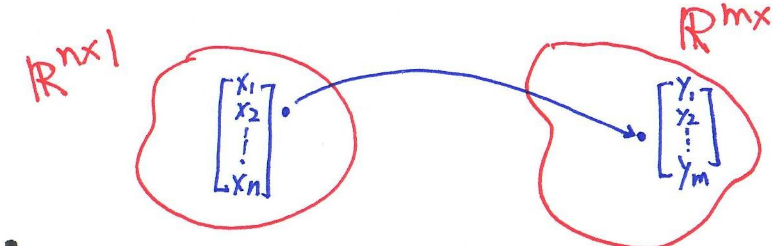
- **Thm** $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is consistent iff $\underline{b} \in \langle\langle \underline{c}_1, \underline{c}_2, \cdots, \underline{c}_n \rangle\rangle$

$$\equiv$$

$\underline{b}$ can be expressed as a l.c. of $\underline{c}_1, \underline{c}_2, \cdots, \underline{c}_n$

- Consider $\underline{\underline{A}} : m \times n$. Let us define $T_{\underline{\underline{A}}} : \mathbb{R}^{n\times 1} \mapsto \mathbb{R}^{m \times 1}$ as

$$T_{\underline{\underline{A}}}(\underbrace{\underline{x}}_{\in \mathbb{R}^{n\times 1}}) \triangleq \underbrace{\underline{\underline{A}}\,\underline{x}}_{\in \mathbb{R}^{m\times 1}}$$



- **Def** $\text{range}(T_{\underline{\underline{A}}}) \triangleq \{ \underline{\underline{A}}\,\underline{x} \mid \underline{x} \in \mathbb{R}^{n\times 1} \} = \{\text{all l.c.'s of columns of } \underline{\underline{A}}\}$

- **Thm** $\text{range}(T_{\underline{\underline{A}}}) = \text{col-space}(\underline{\underline{A}})$

- **Ex** (null space of a matrix)

- Consider $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \stackrel{\circ}{=} \underline{\underline{R}}$

$$\text{null-space}(\underline{\underline{A}}) \stackrel{\Delta}{=} \{ \underline{x} \mid \underline{\underline{A}}\,\underline{x} = \underline{0} \} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \begin{cases} x_1 = -x_2 - x_3 - x_6 \\ x_4 = -2x_6 \\ x_5 = x_6 \end{cases} \quad (\bigstar)$$

$\underline{\underline{R}}$ is obtained from $\underline{\underline{A}}$ via ero's

- x_2, x_3, and x_6 (variables corresponding to columns that do not contain leading 1's) are free variables
- x_1, x_4, and x_5 (variables that correspond to columns that do contain leading 1's) are rapport/matching (配合性的) variables.
- $\#(\text{free variables}) = \dim(\text{null-space}(\underline{\underline{A}})) \triangleq = \text{nullity}(\underline{\underline{A}})$
- $\#(\text{free variables}) + \underbrace{\#(\text{rapport variables})}_{= \text{rank}(\underline{\underline{A}})} = \#(\text{columns of } \underline{\underline{A}})$

- Finding a basis for null-space($\underline{A}$):

From (#) we know

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 - x_6 \\ x_2 \\ x_3 \\ -2x_6 \\ x_6 \\ x_6 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

- A basis for null-space($\underline{A}$) is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

- **Ex** $\underline{A} = \begin{bmatrix} 1 & 1 & 8 \\ 2 & 4 & 26 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} \stackrel{\circ}{=} \underline{R}$

- $\underline{A}\,\underline{x} = \underline{0} \equiv \underbrace{\underline{R}\,\underline{x} = \underline{0}}$ $\Rightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = -5x_3 \end{cases}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -5x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \Rightarrow$

- basis for null-space($\underline{A}$): $\left\{ \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \right\}$

$\Downarrow$

nullity($\underline{A}$) = 1