ENGINEERING MATHEMATICS (II): LINEAR ALGEBRA FINAL

Winter 2022

Note: Provide clear derivations or explanations of your answers. You will not receive full credits if only the final results are given.

PROBLEM 1 (15 pts)

Consider a linear transformation T given by

$$T\ : \mathcal{R}^3 \to \mathcal{R}^4$$

with

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_1 - a_2 \\ a_2 - a_3 \\ a_3 - a_1 \end{bmatrix}$$

(a) (10 pts) Determine the image and kernel of T, and their dimensions.

(b) (5 pts) Is T one-to-one? Is T onto?

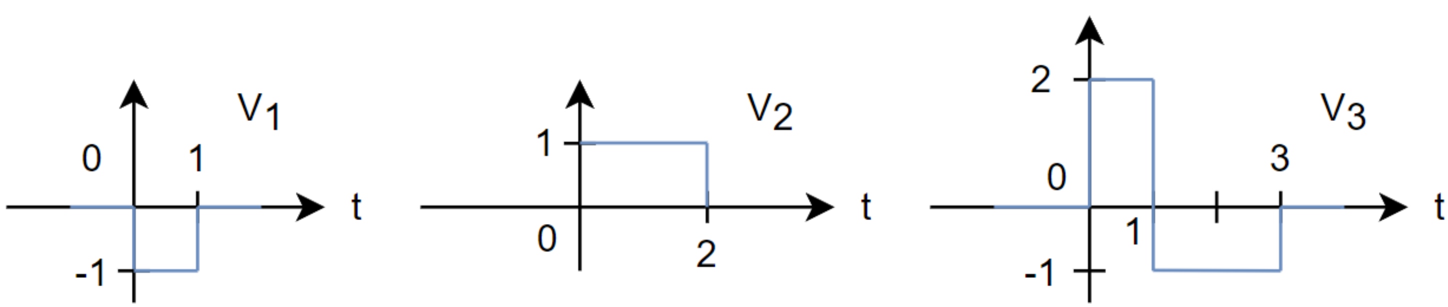
PROBLEM 2 (10 pts)

Find the *standard matrix representation* for the following linear transformation from $\mathcal{R}^2$ to $\mathcal{R}^2$:

double the length of each vector, rotate it 60^0 in the clockwise direction, and then reflect it about the x-axis.

PROBLEM 3 (30 pts)

Consider the vector space of real-valued piecewise continuous functions with inner product and norm defined by $< f(t), g(t) >= \int_{-3}^{3} f(t)g(t)\, dt$ and $\|f\| = \sqrt{< f, f >}$, respectively. Now consider three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ given below.



(a) (5 pts) Find $< \mathbf{v}_1, \mathbf{v}_2 >$.

(b) (5 pts) Find $\|\mathbf{v}_1\|$.

(c) (20 pts) Transform $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ into an orthogonal set of functions and **plot these functions**.

PROBLEM 4 (15 pts)

Suppose that $\mathbf{A} = \begin{bmatrix} 5 & \alpha \\ -2 & -2 \end{bmatrix}$

(a) (5 pts) Determine α such that $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$. **Hint**: definition of eigenvector.

(b) (5 pts) (a) continued. Determine the eigenvalues of $(2\mathbf{A} + \mathbf{I})^{-1}$.

(c) (5 pts) (a) continued. Determine $det\Big((2\mathbf{A} + \mathbf{I})^{-1}\Big)$.

PROBLEM 5 (20 pts)

Assume that in a sequence $\{a_0, a_1, \ldots\}$ each number is the average of the two previous numbers, *i.e.* $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$. If $a_0 = -1$ and $a_1 = 1$, find a formula for a_n using the diagonalization approach.

PROBLEM 6 (10 pts)

Determine β to minimize

$$\|f(t) - \beta g(t)\|$$

where $\|f\| = \sqrt{< f, f >}$, in which $< f(t), g(t) >= \int_{-1}^{1} f(t)g(t)\, dt$, and $f(t)$ and $g(t)$ are given below.