Chapter 4 Linear Transformations – Part I

4.3 Inverse Linear Transformations

◎ *Def.* A LT T: V → W is said to be one-to-one (abbreviated as 1-1) iff T maps distinct vectors in V into distinct vectors in W.

- ◇ That is, if $\mathbf{v}_1 \neq \mathbf{v}_2$, then $T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$.
- ◇ *Thm.* T is 1-1 iff $\ker(T)=\{\mathbf{0}\}$ (and thus, $\text{nullity}(T)=0$).

◎ *Thm.* A Lop T: V → V is 1-1 iff $\text{range}(T)=V$.

◎ *Def.* A LT T: V → W is said to be onto iff every vector in W is an image of at least one vector in V under T.

- ◇ That is, for any vector **w** in W, there exists a vector **v** in V such that $\mathbf{w}=T(\mathbf{v})$.
- ◇ In other words, W is included in range(T).
- ◇ *Thm.* T: V → W is onto iff range(T)=W.

◎ *Def.* If a LT T: V → W is 1-1 and onto, then any vector **w** in W can be regarded the image of some vector **v** in V (because of "onto"), and it is unique (because of "1-1"). The mapping from **w** back to **v** is called the inverse of T (denoted as T^{-1}).

- ◇ *Thm.* If T: V → W is 1-1 and onto, then T^{-1}: W → V is also 1-1 and onto.
- ◇ *Thm.* $(T^{-1})^{-1}=T$
- ◇ *Def.* When T^{-1} exists (and thus $(T^{-1})^{-1}=T$ exists), we say that T is invertible.

◎ *Def.* An isomorphism between vector spaces V and W is an invertible LT between them (i.e. from V to W, or from W to V).

- ◇ *Ex.* The transformation from a vector **v** in an n-dim vector space V to its coordinate vector (wrt to some o.b. B), denoted as C_B: $V \rightarrow R^{n\times 1}$ (i.e. $C_B(\mathbf{v}) = [\mathbf{v}]_B$), is an isomorphism.

◇ *<u>Def.</u>* When there exists an isomorphism between V and W, we say that V and W are isomorphic to each other.

◎ *<u>Thm.</u>* All vector spaces (say, V_1, V_2, V_3,…) of the same finite dimension (say, n) are isomorphic.

◇ Key idea: Use $C_{B,k}$: $V_k \rightarrow R^{n\times 1}$ ($k=1,2,3,\ldots$) and their inverses as the isomorphisms to (uniquely) map a vector in V_i into another vector in V_j.

◎ *Thm.* Consider a Lop T: V → V. The following statements are equivalent:

◇1 T is invertible.

◇2 T is 1-1 and onto.

◇3 T is 1-1 or onto.

◇4 ker(T)={**0**}.

◇5 nullity(T)=0.

◇6 range(T)=V.

◇7 **rank**(T)=dim(V).

◎ *Thm.* Let T_1: U → V and T_2: V → W be two invertible LT's. Then their composition ($T_2 \circ T_1$) is invertible, and $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

◇ This theorem can be easily generalized to composition of more than two LT's.

4.4 Matrices of Linear Transformations

◎ Recall: coordinate vector

◇ Given o.b. B = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and

if $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$

then, $[\mathbf{v}]_B = [k_1, k_2, \ldots, k_n]^T$.

◎ Change of basis:

◇ $D = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is another o.b.

◇ $[\mathbf{v}]_D = \Phi_{D,B}\,[\mathbf{v}]_B$, where

$\Phi_{D,B} = [[\mathbf{v}_1]_D, [\mathbf{v}_2]_D, \ldots, [\mathbf{v}_n]_D]$

◇ $\Phi_{D,B}$ is called the change-of-basis (COB) matrix from B to D.

◇ $[\mathbf{v}]_D = \Phi_{D,B}[\mathbf{v}]_B$, $[\mathbf{v}]_B = \Phi_{B,D}[\mathbf{v}]_D$,

and $\Phi_{B,D} = \Phi_{D,B}^{-1}$, $\Phi_{D,B} = \Phi_{B,D}^{-1}$.

◎ By working with coordinate vectors, any LT T: V → W, where dim(V)=n and dim(W)=m, can be handled by a LT $T_{\mathbf{A}}$: $R^{n\times 1} \rightarrow R^{m\times 1}$.

- Suppose that we want to find $\mathbf{w} = T(\mathbf{v})$.
- First, we need to select an o.b. $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V, and an o.b. $D = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for W.
- $\mathbf{v}$ can be equivalently expressed as $[\mathbf{v}]_B$.
- $\mathbf{w}$ can be equivalently expressed as $[\mathbf{w}]_D$.
- We can construct an m×n matrix $\mathbf{A}$ such that $[\mathbf{w}]_D = \mathbf{A}\,[\mathbf{v}]_B$. This $\mathbf{A}$ is called the matrix of T wrt bases B and D. It is denoted as $[T]_{D,B}$.
- Construction of $[T]_{D,B}$:

$[T]_{D,B} = [[T(\mathbf{v}_1)]_D\ [T(\mathbf{v}_2)]_D\ \ldots\ [T(\mathbf{v}_n)]_D]$

◇ Summary: $[T(\mathbf{v})]_D = [T]_{D,B}\ [\mathbf{v}]_B$

◎ Effect of change of bases on the matrix of T:

Let B, C be o.b.'s of V (domain), and D, E be o.b.'s of W (codomain).

(B: old basis for V, D: old basis for W;
C: new basis for V, E: new basis for W)

◇ $[T(\mathbf{v})]_D=[T]_{D,B}[\mathbf{v}]_B$,

$[T(\mathbf{v})]_E=[T]_{E,C}[\mathbf{v}]_C$

◇ $[T(\mathbf{v})]_E = \Phi_{E,D}[T(\mathbf{v})]_D$,

$[\mathbf{v}]_B=\Phi_{B,C}[\mathbf{v}]_C$

◇ $[T(\mathbf{v})]_E = \Phi_{E,D}[T(\mathbf{v})]_D$

$= \Phi_{E,D}[T]_{D,B}[\mathbf{v}]_B$

$= \Phi_{E,D}[T]_{D,B}\Phi_{B,C}[\mathbf{v}]_C$

$\to [T]_{E,C} = \Phi_{E,D}[T]_{D,B}\Phi_{B,C}$ -- (#)

◎ Matrix of Lop T: V → V :

◇ This is just a special case for the matrix of a LT.

◇ Usually the same basis is used for the domain and the codomain: $[T]_{B,B}=[T]_B$.

◇ Effect of change of basis: Apply (#) with B ← B, D ← B ; C ← D, E ← D :

$$[T]_{D,D} = \Phi_{D,B}[T]_{B,B}\Phi_{B,D}$$

$$(\text{or, } [T]_D = \Phi_{D,B}[T]_B\Phi_{B,D} = \Phi_{B,D}^{-1}\,[T]_B\Phi_{B,D})$$

◎ <u>*Thm*.</u> Let T: V → V be a Lop, and B be an o.b. of V. Then T is invertible iff $[T]_B$ is invertible. Moreover, when T is invertible, $[T^{-1}]_B=[T]_B^{-1}$.

◎ <u>*Thm*.</u> (Matrix of composition of LT's):

Let T_1: U → V and T_2: V → W be LT's. Let B, C, D be o.b.'s of U, V, and W, respectively. Then, $[T_2 \circ T_1]_{D,B} = [T_2]_{D,C}\,[T_1]_{C,B}$

◇ This theorem can be easily generalized to composition of more than two LT's.

4.5 Similar Linear Transformations

◎ Previously, we see that $[T]_D = \Phi_{D,B}[T]_B \Phi_{B,D} = \Phi_{B,D}^{-1} [T]_B \Phi_{B,D}$. We wonder if we can make $[T]_D$ look simple (e.g. diagonal or triangular) by some proper choice of the o.b. D.

◎ *Def.* Two square matrices **A** and **B** are said to be similar (to each other) iff there exists an invertible matrix **P** such that $\mathbf{B}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

◇ *Def.* A property of square matrices is said to be a similarity invariant (or invariant under similarity) if that property is shared by any two similar matrices.

- *Ex.* det($\mathbf{P}^{-1}\mathbf{AP}$)=det($\mathbf{A}$). So, determinant is a similarity-invariant property.
- Some other similarity-invariant properties: invertibility, rank, nullity, trace.

◎ *Thm.* Let T: V → V be a Lop. The matrix of T wrt an o.b. of V is similar to the matrix of T wrt any other o.b. of V.

- *Prf.* $[T]_D = \Phi_{D,B}[T]_B \Phi_{B,D}$

  $= \Phi_{B,D}^{-1}\,[T]_B \Phi_{B,D}$ (cf. $\mathbf{B}=\mathbf{P}^{-1}\mathbf{AP}$)

◎ <u>*Thm.*</u> Let T: V → V be a Lop. Let B be any o.b. of V. Then, the properties below hold:

◇ $\text{rank}(T) = \text{rank}([T]_B)$

◇ $\text{nullity}(T) = \text{nullity}([T]_B)$

◇ T is invertible iff $[T]_B$ is invertible.

◎ <u>*Def.*</u> Let T: V → V be a Lop. The determinant of T (denoted as $\det(T)$) is defined to be $\det([T]_B)$, where B is any o.b. of V.

◎ To find a B so that $[T]_B$ looks simple for a Lop T, we need the concept of eigenvalues and eigenvectors.