Total: 100 points

1. Answer the following questions about the functions $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi$

   (a) (10 points) Find the intervals of increase or decrease.

   (b) (5 points) Find the local extrema values on $0 < \theta < 2\pi$.

   (c) (15 points) Find the intervals of concavity and the inflection points.

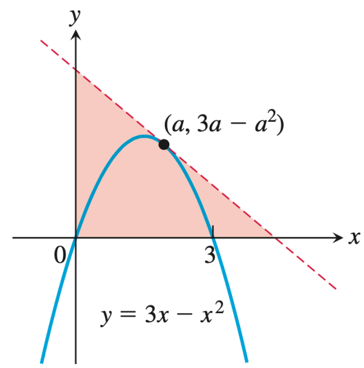
Solution:

(a) $f'(\theta) = -2\sin\theta\,(1+\cos\theta)$. $\quad f'(\theta) = 0 \iff \theta = 0, \pi, 2\pi$.
Therefore, When $0 < \theta < \pi, f'(\theta) < 0. \Longrightarrow f$ is decreasing on $0 < \theta < \pi$.
When $\pi < \theta < 2\pi, f'(\theta) > 0. \Longrightarrow f$ is increasing on $\pi < \theta < 2\pi$.

(b) On $0 < \theta < 2\pi$, When $\theta = \pi, f'(\pi) = 0$. At $\theta = \pi, f(\theta)$ has a local minimum. $f(\pi) = -1$.

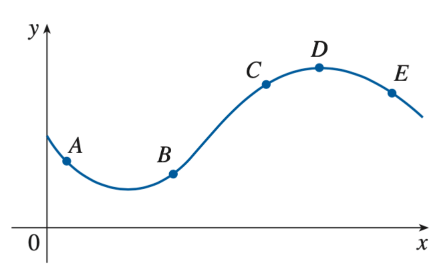
(c) $f''(\theta) = -4\cos^2\theta - 2\cos\theta + 2 = -2(2\cos\theta - 1)(\cos\theta + 1)$.
When $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}, f''(\theta) = 0$. But $\theta = \pi$ is not a point of inflection.
Therefore, the inflection points are $\left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, \frac{5}{4}\right)$.
Interval of concave downward: $0 < \theta < \frac{\pi}{3}$ and $\frac{5\pi}{3} < \theta < 2\pi$.
Interval of concave upward: $\frac{\pi}{3} < \theta < \frac{5\pi}{3}$.

2. (20 points) Among all triangles in the first quadrant formed by the x-axis, the y-axis and tangent lines to the graph of $y = 3x - x^2$, what is the smallest possible area?



Solution:
$y' = 3 - 2x$. Therefore, the slope of the tangent line at $x = a$ is $3 - 2a$.
The equation of the tangent line at $x = a$ is $y = (3a - a^2) + (3 - 2a)(x - a)$.
If $x = 0, y = a^2$. If $y = 0, x = \frac{a^2}{2a-3}$.
The area of the described triangle is $A = \frac{1}{2} \cdot a^2 \cdot \frac{a^2}{2a-3} = \frac{a^4}{4a-6} \Longrightarrow \frac{dA}{da} = \frac{12a^3(a-2)}{(4a-6)^2}$
Therefore, the critical points of $A(a)$ are $a = 0, a = \frac{3}{2}, a = 2$.
But $a = 0$ and $a = \frac{3}{2}$ are not in the domain. Due to $A''(2) > 0$, $A(a)$ has a minimum at $a = 2$.
The minimum area is $A(2) = 8$.

3. (15 points) The graph of a function $y = f(x)$ is shown. At which point(s) are the following true?
(a) y' and y'' are both positive. (b) y' and y'' are both negative. (c) $y' < 0$ but $y'' > 0$.



Solution:
(a) B. (b) E. (c) A.

4. (20 points) Find $f(\theta)$ if $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3$, $f'(0) = 4$.

Solution:
$f''(\theta) = \sin\theta + \cos\theta \quad \Rightarrow \quad f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \quad \Rightarrow \quad C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \quad \Rightarrow \quad D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

5. (15 points) Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

Solution:
$f(x) = ax^3 + bx^2 + cx + d \Longrightarrow f'(x) = 3ax^2 + 2bx + c$.
$f(0) = 0 \Longrightarrow d = 0. \quad f(1) = -1 \Longrightarrow a + b + c + d = -1$
$f'(0) = 0 \Longrightarrow c = 0. \quad f'(1) = 0 \Longrightarrow 3a + 2b + c = 0$.
Therefore, $a = 2, b = -3 \Longrightarrow a = 2, b = -3, c = 0, d = 0$.