First Midterm—Chapter 1

Total points: 100 points 2hours to do the work, Oct. 18, 2018

1. Solve the initial value problem.(10 points)

$$xy' = y + x^2 cos^2\left(\frac{y}{x}\right)\ ;\ y(1) = \frac{\pi}{4}$$

$y' = \frac{y}{x} + x cos^2 \frac{y}{x}$

$Let\ u = \frac{y}{x}$

$y = ux$

$y' = u'x + u$

$u'x + u = u + x\, cos^2 u$

$\frac{du}{dx} = cos^2 u$

$\int sec^2 u\, du = \int dx$ → **+ 5**

$\tan u = x + c$

$u = \tan^{-1}(x + c)$

$y = x \tan^{-1}(x + c)$ → **+ 8**

$\frac{\pi}{4} = \tan^{-1}(1 + c)$

$1 + \mathrm{c} = \tan\frac{\pi}{4} = 1$

c=0

$\therefore y = x \tan^{-1}(x)$ → **+ 10**

2. Solve the initial value problem.(10 points)

$$y'xlnx = y \quad ; \quad y(2) = ln16$$

$$\frac{dy}{dx}xlnx = y$$

$$\int \frac{dy}{y} = \int \frac{1}{xlnx}dx$$

$$Let\ u = lnx \ , \ du = \frac{1}{x}dx$$

$\int \frac{dy}{y} = \int \frac{1}{u}du$ → **+ 5**

$ln|y| = ln|u| + c_1$ → **+ 7**

$y = ce^{lnu} = clnx$ → **+ 8**

$$\mathrm{y(2) = ln16 = cln2}$$

$c = \frac{ln16}{ln2} = \frac{4ln2}{ln2} = 4$ → **+ 9**

$\therefore y = 4lnx$ → **+ 10**

3. Test the **exactness** of the given ODE and solve the problem.(10 points)

$$y' = 3.2y - 10y^2$$

$Let\ u = y^{1-a} = y^{1-2} = y^{-1}$

$u' = -y^{-2}y'$

$y' = -u'y^2$

$原式 = -u'y^2 = 3.2y - 10y^2$

$u' + 3.2y^{-1} - 10 = 0$

$u' + 3.2u - 10 = 0$

$$\frac{du}{dx} + 3.2u - 10 = 0$$

$du + (3.2u - 10)dx = 0$

$$\begin{cases} \dfrac{\partial(3.2u - 10)}{\partial u} = 3.2 \\ \dfrac{\partial 1}{\partial x} = 0 \end{cases}$$

$\frac{\partial 1}{\partial x} \neq \frac{\partial(3.2u-10)}{\partial u}$ $\quad \therefore \boldsymbol{non-exact}$ $\quad \rightarrow$ **+5**

$F = e^{\int 3.2dx} = e^{3.2x}$

$e^{3.2x}u' + e^{3.2x}3.2u = 10e^{3.2x}$

$$\frac{d(e^{3.2x}u)}{dx} = 3.2ue^{3.2x} + e^{3.2x}u'$$

$\int \frac{d(e^{3.2x}u)}{dx} = 10\int e^{3.2x}dx + c$ $\quad \rightarrow$ **+7**

$e^{3.2x}u = \frac{10}{3.2}e^{3.2x} + c$ $\quad \rightarrow$ **+8**

$$u = \frac{10}{3.2} + ce^{-3.2x}$$

$\therefore y^{-1} = \frac{10}{3.2} + ce^{-3.2x}$ $\rightarrow$ **+10**

4. Find an **integrating factor**, use it to find the general solute on of the differential equation, and then obtain the solution of the initial value problem. (20 points)

$$\sin(x-y)+\cos(x-y)-\cos(x-y)\,y'=0 \quad ; \qquad y(0)=\frac{7\pi}{6}$$

$$[\sin(x-y)+\cos(x-y)]dx+[-\cos(x-y)]dy=0$$

$$\frac{\partial M}{\partial y}=-\cos(x-y)+\sin(x-y)\neq\frac{\partial N}{\partial x}=\sin(x-y)$$ → **+ 8**

$$\mu=e^{\int\frac{1}{-\cos(x-y)}[-\cos(x-y)+\sin(x-y)-\sin(x-y)]dx}=e^{\int dx}=e^{x}$$ → **+ 10**

$$\left\{e^{x}[\sin(x-y)+\cos(x-y)]dx\right\}+\left\{-e^{x}\left[\cos(x-y)\right]\right\}dy=0$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-e^{x}\,cos(x-y)+e^{x}\,sin\,(x-y)$$

$$\varphi(x,y)=\int -e^{x}[\cos(x-y)]dy+k(x)=e^{x}\sin(x-y)+k(x)$$

$$M=\frac{\partial u}{\partial x}=e^{x}\sin(x-y)+e^{x}\cos(x-y)+k^{'}(x)=e^{x}\sin(x-y)+e^{x}\cos(x-y)$$

$$k^{'}(x)=0 \quad k(x)=c_1$$ → **+ 15**

$$\therefore u=e^{x}\sin(x-y)=c_1$$

G.S. : $e^{x}\sin(x-y)+c_1=c_2$

$$\therefore e^{x}\sin(x-y)=c_3$$ → **+ 18**

代入 $(0,\dfrac{7\pi}{6})$

$$\sin\frac{-7\pi}{6}=c_3=\sin\frac{5\pi}{6}=\frac{1}{2}$$

$$\therefore e^{x}\sin(x-y)=\frac{1}{2}$$ → **+ 20**

5. Find the general solution of the given ODE. (20 points)

$$2xyy' + (x-1)y^2 = x^2e^x$$

$$2xydy + [(x-1)y^2 - x^2e^x]dx = 0$$

$$\begin{cases} \dfrac{\partial[(x-1)y^2 - x^2e^x]}{\partial y} = 2(x-1)y \\ \dfrac{\partial(2xy)}{\partial x} = 2y \end{cases}$$

$\therefore$.The given ODE is non-exact $\rightarrow$ **+ 8**

$$\int \frac{2(x-1)y - 2y}{2xy}dx = \int \frac{2xy - 4y}{2xy}dx = \int (1 - \frac{2}{x})dx = x - 2\ln|x|$$

$\therefore F(x) = e^{x-2\ln|x|} = e^x \cdot e^{\ln|x|^{-2}} = \frac{e^x}{x^2}$ $\rightarrow$ **+ 10**

$\frac{e^x}{x^2}2xydy + \frac{e^x}{x^2}[(x-1)y^2 - x^2e^x]dx = 0 \quad \textit{is an exact ODE}$

$$\therefore \exists u(x,y) \ni \begin{cases} \dfrac{\partial u}{\partial y} = \dfrac{e^x}{x^2}2xy & \ldots..(1) \\ \dfrac{\partial u}{\partial x} = \dfrac{e^x}{x^2}[(x-1)y^2 - x^2e^x] & \ldots..(2) \end{cases}$$

$from(1): u = \int \frac{e^x}{x^2}2xydy + k(x) = \frac{e^x}{x}y^2 + f(x)$ $\rightarrow$ **+ 13**

$\frac{\partial u}{\partial x} = \frac{\partial(\frac{e^x}{x^2}xy^2)}{\partial x} + \frac{df(x)}{dx} = \frac{e^x}{x^2}[(x-1)y^2 - x^2e^x] = \frac{(x-1)}{x^2}e^xy^2 - e^{2x}$ $\rightarrow$ **+ 15**

$$\frac{\partial(\frac{e^x}{x^2}xy^2)}{\partial x} = \frac{\partial(e^xx^{-1}y^2)}{\partial x} = y^2(e^xx^{-1} + e^x \cdot (-1) \cdot x^{-2}) = y^2\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

$$= \frac{(x-1)}{x^2}e^xy^2$$

$\therefore \frac{df(x)}{dx} = -e^{2x}$ $\rightarrow$ **+16**

$f(x) = -\frac{1}{2}e^{2x} + c_1$ $\rightarrow$ **+ 18**

$$\therefore u(x,y) = \frac{e^x}{x}y^2 - \frac{1}{2}e^{2x} + c_1$$

The general solution of the ODE is $\frac{e^x}{x}y^2 - \frac{1}{2}e^{2x} = c$ $\rightarrow$ **+ 20**

6. Please show that the first order linear differential equation $(\frac{dy}{dx} + p(x)y = r(x))$ has a general solution, $y = e^{-\int p(x)dx}(\int e^{\int p(x)dx} r(x)dx + c)$ by the method of exact differential equation. (20 points)

$$y' + p(x)y = r(x)$$

$$\frac{dy}{dx} + p(x)y - r(x) = 0$$

$$[p(x)y - r(x)]dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x) \neq \frac{\partial N}{\partial x} = 0 \rightarrow \textbf{+ 8}$$

$$\mu(x) = e^{\int \frac{1}{1}[p(x)-0]dx} = e^{\int p(x)dx} \rightarrow \textbf{+ 10}$$

$$\varphi(x,y) = \int \mu N dy = \int e^{\int p(x)dx} dy = ye^{\int p(x)dx} + k(x)$$

$$\frac{\partial \varphi}{\partial x} = p(x)ye^{\int p(x)dx} + k'(x) = \mu M = [p(x)y - r(x)]e^{\int p(x)dx}$$

$$k'(x) = -r(x)e^{\int p(x)dx} \rightarrow \textbf{+ 15}$$

$$k(x) = \int -r(x)e^{\int p(x)dx} dx + c_1 \rightarrow \textbf{+ 17}$$

$$\varphi(\mathrm{x,y}) = \mathrm{y}e^{\int p(x)dx} - \int r(x)e^{\int p(x)dx} dx + c_1 = c_2$$

$$ye^{\int p(x)dx} = \int r(x)e^{\int p(x)dx} dx + c \rightarrow \textbf{+ 18}$$

$$y = e^{-\int p(x)dx}(\int e^{\int p(x)dx} r(x)dx + c) \rightarrow \textbf{+ 20}$$

7. Solve the differential equation. (10 points)

$$y' = 6(y-2.5)\tanh(1.5x)$$

$$\frac{dy}{dx} = 6(y-2.5)\tanh(1.5x)$$

$$\int \frac{dy}{y-2.5} = \int 6\tanh(1.5x)dx$$

Let $u = \cosh(1.5x)$

$$du = 1.5\sinh(1.5x)dx$$

$$\int \frac{dy}{y-2.5} = 6\int \frac{1}{1.5}\frac{1}{u}du \quad \rightarrow \quad \mathbf{+5}$$

$$\ln|(y-2.5)| = 4\ln|u| = 4\ln|\cosh(1.5x)| + c_1 \quad \rightarrow \quad \mathbf{+8}$$

$$e^{\ln|(y-2.5)|} = e^{4\ln|\cosh(1.5x)|+c_1}$$

$$\therefore \; y-2.5 = c\cosh^4(1.5x) \quad \rightarrow \quad \mathbf{+10}$$