112-2 Midterm (II) Solution

Chapter: 8-2~8-5 & 8-8

Total: 55 pts

1. Evaluate the following integral. (20 pts)
   a. $\int \ln(x^2+2x+2)\,dx$ (10 pts)
   b. $\int sec^3\theta\, d\theta$. (10 pts)

$a. \int \ln(x^2+2x+2)\,dx$

$\rightarrow u = \ln(x^2+2x+2)\ , dv = dx \rightarrow du = \dfrac{2x+2}{x^2+2x+2}dx\ , v = x$

$$\rightarrow \int \ln(x^2+2x+2)\,dx = x\cdot\ln(x^2+2x+2) - \int x\cdot\frac{2x+2}{x^2+2x+2}dx$$

$$= x\cdot\ln(x^2+2x+2) - \int \frac{2(x^2+2x+2)-2x-4}{x^2+2x+2}dx$$

$$= x\cdot\ln(x^2+2x+2) - \int\left(2 - \frac{2x+2}{x^2+2x+2} - \frac{2}{(x+1)^2+1}\right)dx$$

$$= x\cdot\ln(x^2+2x+2) - [2x - \ln(x^2+2x+2) - 2\tan^{-1}(x+1)] + c$$

$$= (x+1)\ln(x^2+2x+2) + 2\tan^{-1}(x+1) - 2x + c$$

$b. \int sec^3\theta\, d\theta$

$\rightarrow u = \sec\theta\ , dv = sec^2\theta\, d\theta \rightarrow du = sec\theta tan\theta d\theta\ , v = tan\theta$

$$\rightarrow I = \int sec^3\theta\, d\theta = \sec\theta\, tan\theta - \int tan\theta(sec\theta tan\theta)\, d\theta$$

$$= \sec\theta\, tan\theta - \int tan^2\theta sec\theta\, d\theta = \sec\theta\, tan\theta - \int (sec^2\theta - 1)\cdot sec\theta\, d\theta$$

$$= \sec\theta\, tan\theta - \int (sec^3\theta - sec\theta)\, d\theta$$

$$\rightarrow 2I = \sec\theta\, tan\theta + \int sec\theta\, d\theta \rightarrow 2I = \sec\theta\, tan\theta + \ln|sec\theta + tan\theta| + C$$

$$\rightarrow I = \int sec^3\theta\, d\theta = \frac{\sec\theta\, tan\theta + \ln|sec\theta + tan\theta|}{2} + C$$

2. $\int_4^8 sin20° sin35° \sin 45° + cos25° cos45° cos80°\, d\theta$. (10 pts)

   (Hint: $\cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$ might be needed.)

$$\int_4^8 sin20°sin35° \sin 45° + cos25°cos45°cos80° \, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_4^8 (sin20°sin35° + cos25°cos80°)\, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_4^8 \left[\frac{1}{2}(\cos(-15°) - \cos 55°) + \frac{1}{2}(\cos(-55°) + \cos 105°)\right] d\theta$$

$$= \frac{\sqrt{2}}{4}\int_4^8 [\cos 15° + \cos 105°]\, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_4^8 [\cos(45-30)° + \cos(60+45)°]\, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_4^8 [\cos(45-30)° + \cos(60+45)°]\, d\theta$$

$$= \frac{1}{4}\theta|_4^8 = \frac{1}{4}(8-4) = 1$$

3. Evaluate the following integral: $\int \frac{\sqrt{4-x^2}}{x^2} dx$. (10 pts)

$$\int \frac{\sqrt{4-x^2}}{x^2} dx \to x = 2\sin\theta \to dx = 2\cos\theta\, d\theta\,, \theta = \sin^{-1}\frac{x}{2}$$

$$\to \sin\theta = \frac{x}{2}\,, \cos\theta = \frac{\sqrt{4-x^2}}{2}\,, \tan\theta = \frac{x}{\sqrt{4-x^2}} = \frac{1}{\cot\theta}$$

$$\to \int \frac{\sqrt{4-4sin^2\theta}}{4sin^2\theta} 2\cos\theta\, d\theta = \int \frac{\sqrt{4(1-sin^2\theta)}}{4sin^2\theta} 2\cos\theta\, d\theta$$

$$= \int \frac{2\sqrt{cos^2\theta}}{4sin^2\theta} 2\cos\theta\, d\theta = \int \frac{cos^2\theta}{sin^2\theta} d\theta = \int cot^2\theta d\theta = \int (csc^2\theta - 1) d\theta$$

$$\to -\cot\theta - \theta + c = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\frac{x}{2} + c$$

4. Find the integral $\int \frac{x+4}{x^2+5x-6} dx$. (5 pts)

$$\int \frac{x+4}{x^2+5x-6} dx = \int \left(\frac{A}{x+6} + \frac{B}{x-1}\right) dx$$

$$\to \begin{cases} A + B = 1 \\ -A + 6B = 4 \end{cases} \to A = \frac{2}{7}\,, \quad B = \frac{5}{7}$$

$$\to \int \left(\frac{\frac{2}{7}}{x+6} + \frac{\frac{5}{7}}{x-1}\right) dx$$

$$= \frac{2}{7}\ln(x+6) + \frac{5}{7}\ln(x-1) = \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C$$

5. Determine the integral $\int_{-\infty}^{\infty} e^{-3|t|} + 2^{-|t|}\,dt.$ (10 pts)

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} e^{-3|t|} + 2^{-|t|}\,dt.$$

$$= \left[\lim_{a\to-\infty}\int_a^0 e^{-3\cdot(-t)} + 2^{-(-t)}dt\right] + \left[\lim_{b\to\infty}\int_0^b e^{-3\cdot(t)} + 2^{-(t)}dt\right]$$

$$= \left[\lim_{a\to-\infty} \frac{e^{3t}}{3} + \frac{2^t}{\ln 2}\bigg|_a^0\right] + \left[\lim_{b\to\infty} \frac{e^{-3t}}{-3} + \frac{2^t}{-\ln 2}\bigg|_0^b\right]$$

$$= \left[\left(\frac{1}{3} + \frac{1}{\ln 2}\right) - 0\right] + \left[0 - \left(\frac{-1}{3} + \frac{-1}{\ln 2}\right)\right] = \frac{2}{3} + \frac{2}{\ln 2}$$