Chapter 4　Linear Transformations – Part I

4.1 Definition and Basic Properties

◎ Transformation:　F: D → C

(**u** = F(**v**), with **v** in D, and **u** in C)

◇ **u** is said to be the image of **v** under (the transformation of) F.

◇ D: domain,　C: codomain, image space

◇ In linear algebra, D and C under consideration are vector spaces.

◇ Equivalent names for "transformation":

transform, function, mapping, filtering, computation, processing, etc.

◎ <u>*Def.*</u> A linear transformation (LT) is a transformation T: V → W, where V and W involve the same field (i.e. set of scalars), that satifies

$$T(a\mathbf{u}+b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

for any scalars a, b and any vectors **u**, **v** in V.

◇ "$T(a\mathbf{u}+b\mathbf{v})=aT(\mathbf{u})+bT(\mathbf{v})$ for any a,b,**u**,**v**" is equivalent to "$T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ and $T(c\mathbf{u})=cT(\mathbf{u})$ for any **u**,**v**,c."

◇ A special case (W = V): A LT T: V → V is called a linear operator (Lop) on V.

◎ Some basic properties of a LT T: V → W:

◇ $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n)$

-- LT preserves linear combination (l.c.).

-- Transformation and combination (more exactly, l.c.) are commutable.

◇ $T(\mathbf{0}) = \mathbf{0}$ (more precisely, $T(\mathbf{0}_V) = \mathbf{0}_W$)

Prf: $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0})+T(\mathbf{0})$

Then, add $-T(\mathbf{0})$ to both sides.

◇ $T(-\mathbf{v}) = -\mathrm{T}(\mathbf{v})$

Prf: $T(-\mathbf{v}) = T(-1*\mathbf{v}) = -1*T(\mathbf{v}) = -T(\mathbf{v})$

◇ $T(\mathbf{v}-\mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$

◎ Finding LT from images of basis:

Consider a LT T: V → W. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Any vector v in V can be written as $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$. Then

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n)$$
$$= c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \ldots + c_n T(\mathbf{v}_n).$$

◎ <u>*Def.*</u> Consider T_1: U → V and T_2: V → W. $(T_2 \circ T_1)(u) = T_2(T_1(u))$ is called the composition of T_2 with T_1.

◇ Composition can be generalized to more than two transformations.

◎ *Thm.* Composition of LT's is still a LT.

4.2 Kernel and Range

◎ *Def.* Consider a LT T: V → W. The set of vectors in V that T maps into $\mathbf{0}$ is called the kernel of T (denoted by ker(T)).

◇ $\ker(T) = \{\mathbf{v} | T(\mathbf{v})=\mathbf{0}, \mathbf{v} \text{ in } V\}$

◇ Kernel is also called null-space.

◇ *<u>Thm.</u>* ker(T) is a subspace of V.

Proof: Apply the subspace test.

◇ *<u>Def.</u>* dim(ker(T)) is called the nullity of T (denoted as nullity(T)).

◎ *<u>Def.</u>* Consider a LT T: V → W. The set {T(**v**)|**v** in V} is called the range of T (denoted as range(T)).

◇ *<u>Thm.</u>* range(T) is a subspace of W.

Proof: Apply the subspace test.

◇ *<u>Def.</u>* dim(ker(T)) is called the rank of T (denoted as rank(T)).

◎ <u>*Thm.*</u> Consider **A**: $m \times n$ matrix. Define $T_{\mathbf{A}}$: $R^{n\times 1} \rightarrow R^{m\times 1}$ as left-multiplication by **A** (i.e. $\mathbf{y} = \mathbf{A}\mathbf{x}$). Then, nullity($T_{\mathbf{A}}$) = nullity(**A**), and rank($T_{\mathbf{A}}$) = rank(**A**).

◎ <u>*Thm.*</u> (rank/dimension theorem of LT) Let T: V → W be a LT, where dim(V) = n. Then, rank(T) + nullity(T) = n .

<u>Claim in proof</u>: Let B={$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$} be a basis for V. If E={$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$} ($r \leq n$) is a basis for ker($T$), then F={T($\mathbf{v}_{r+1}$), T($\mathbf{v}_{r+2}$), …, T($\mathbf{v}_n$)} is a basis for range($T$).