陳柏辰

Total: **120** points

Note: To get full points, you should write down the procedure **in detail**.

1. Find the Taylor series centered at $x = a$ for the following function:

   (a) (**5** points) $f(x) = \dfrac{1}{1-x}$ at $a = 2$

   (b) (**5** points) $f(x) = 3^x$ at $a = 1$

   $e^{x\ln 3}$

2. A infinite geometric series

$$\sum_{n=2}^{\infty}(1+c)^{-n} = \frac{1}{(1+c)^2} + \frac{1}{(1+c)^3} + \cdots + \frac{1}{(1+c)^n} + \cdots$$

   (a) (**5** points) If this geometric series converges, what is the range of c?

   (b) (**5** points) If $\sum\limits_{n=2}^{\infty}(1+c)^{-n} = 2$, what is the value of c?

3. (**10** points) Find the length of the parametric curve $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

4. (**10** points) Find the area of the region common to the two regions bounded by the curves

$$r = -6\cos\theta \quad \text{and} \quad r = 2 - 2\cos\theta.$$

5. (**10** points) Let $f(x,y) = \sqrt{x^2+y^2}$. Find

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

6. (**10** points) Evaluate the following integrals. (5 points for each)

   Hint: You can change the order of integration if necessary.

   (a) $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx$ (b) $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\, dx\, dy$

(Questions on both sides of the paper!)

7. (**15** points) Find all the local maxima, local minima, and saddle point(s) of the function

$$f(x,y) = x^3 - 3x^2 + 6y^2 + 5$$

8. In order to find the absolute extreme value of $f(x,y) = x^2 + 3y^2 + 2y$ on the disk $x^2 + y^2 \leq 1$, one can solve this problem by answering the following questions:
   (a) (**5** points) Find the extreme value located at the interior of the disk by **finding the critical points of** $f(x,y)$ inside the disk.
   (b) (**10** points) Find the extreme value of $f(x,y)$ on the circle $g(x,y) = x^2 + y^2 - 1 = 0$.
   (c) (**5** points) Based on the results of (a) and (b), find the absolute maximum and minimum of $f(x,y)$ on the disk $x^2 + y^2 \leq 1$.

9. (**25** points) Let $f(x,y) = \dfrac{1}{\sqrt{x^2 + y^2}}$. (5 points for each)
   (a) Find the gradient of f.
   (b) Find the directional derivative of f at the point $A(1,1)$ in the direction toward the point $B(3,3)$.
   (c) Find the maximum increasing rate of change of f at the point $A(1,1)$. Which is the direction of the maximum increasing rate of change?
   (d) Find the tangent plane of $z = f(x,y)$ at the point $(1, 1, \frac{1}{\sqrt{2}})$.
   (e) Use linear approximation of $f(x,y)$ at $(1,1)$ to estimate the value of $f(1.01, 0.99)$.