Chapter 5 Eigenvalues and Eigenvectors

5.1 Definition and basic properties

◎ *Def.* Let **A** be an $n \times n$ matrix. For a given scalar λ, if there exists a nonzero vector **x** in $C^{n\times 1}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$, then λ is said to be an eigenvalue of **A**.

◇ Eigenvalue is also called characteristic value.

◎ *Def.* If **A** is an $n \times n$ matrix, then a nonzero vector **x** in $C^{n\times 1}$ is called an eigenvector of **A** if $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

- $\mathbf{x}$ is said to be an eigenvector of $\mathbf{A}$ wrt λ.
- Eigenvector is also known as characteristic vector.

◎ Finding eigenvalues and eigenvectors:

- $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} \quad \rightarrow \quad (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$.
- Nontrivial solution (i.e. nonzero $\mathbf{x}$) exists iff $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.
- $\det(\lambda\mathbf{I} - \mathbf{A})$, seen as a polynomial in λ, is called the characteristic polynomial of $\mathbf{A}$.

◇ $\det(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{0}$ is called the characteristic equation of $\mathbf{A}$.

◎ *Thm.* The statements below are equivalent:

◇1 λ is an eigenvalue of $\mathbf{A}$.

◇2 The system of linear equations $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

◇3 There exists a nonzero column vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

◇4 λ is a solution of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

◎ *Thm.* Let λ be an eigenvalue of an $n \times n$ matrix **A**. The set of all eigenvectors wrt λ together with the zero vector is called the eigenspace of **A** wrt λ (denoted as $E_\lambda(\mathbf{A})$).

◇ $E_\lambda(\mathbf{A})=\{\mathbf{x} \mid \mathbf{Ax} = \lambda\mathbf{x}\}$.

◇ $E_\lambda(\mathbf{A})$ is a subspace of $R^{n\times 1}$.

◎ *Thm.* A square matrix **A** is invertible iff $\lambda = 0$ is not an eigenvalue of **A**.

Prf. Combine the facts:

(1). eigenvector: $\mathbf{Ax}=\lambda\mathbf{x}$ for some $\mathbf{x}\neq\mathbf{0}$

(2). $\mathbf{Ax} = \lambda\mathbf{x} = \mathbf{0}$, if $\lambda = 0$

(3). **A** is invertible iff $\mathbf{Ax} = \mathbf{0}$ has the trivial solution only.

5.2 Diagonalization of Matrices

◎ *Def.* A square matrix **A** is said to be diagonalizable if there is an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix (i.e., $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$). In other words, **A** is similar to a diagonal matrix. The matrix **P** is said to diagonalize **A**.

◇ If **A** is diagonalizable, then $\mathbf{AP} = \mathbf{PD}$.

◇ $\mathbf{PD} = \mathbf{P}\ \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$

$= [\lambda_1\mathbf{p}_1, \lambda_2\mathbf{p}_2,\ldots, \lambda_n\mathbf{p}_n]$

where $\mathbf{p}_k$ is the k^{th} column of **P**.

◇ $\mathbf{AP} = [\mathbf{Ap}_1, \mathbf{Ap}_2,\ldots, \mathbf{Ap}_n]]$

◇ $\mathbf{AP} = \mathbf{PD} \rightarrow \mathbf{Ap}_k = \lambda_k\mathbf{p}_k$

◇ The diagonalization matrix **P** consists of the eigienvectors of **A**, and the diagonal matrix matrix **D** consists of the eigenvalues of **A**.

◎ *Thm.* An n×n matrix **A** is diagonalizable iff it has n linearly independent (l.i.) eigenvectors.

◎ Procedure for diagonalizing a matrix: If an $n \times n$ matrix $\mathbf{A}$ has n l.i. eigenvectors, then stack those eigenvectors into an $n \times n$ matrix $\mathbf{P}$. This $\mathbf{P}$ diagonalizes $\mathbf{A}$.

◇ The matrix $\mathbf{P}^{-1}\mathbf{AP}$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_k is the eigenvalue corresponding to $\mathbf{p}_k$ (k^{th} column of $\mathbf{P}$), for $k = 1, 2, \ldots, n$.

◎ <u>*Thm.*</u> If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, are eigenvectors of $\mathbf{A}$ corresponding to distinct eigenvalues λ_1,

$\lambda_2, \ldots, \lambda_k$, then { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ } is a l.i. set.

<u>*Prf.*</u> Consider $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ **--** (#)

$\mathbf{A}$ ((#)): $k_1\lambda_1\mathbf{v}_1 + k_2\lambda_2\mathbf{v}_2 = \mathbf{0}$ **--** ($)

$\lambda_2 \cdot$(#) $-$ ($): $k_1(\lambda_2 - \lambda_1)\mathbf{v}_1 = \mathbf{0} \rightarrow k_1 = 0$

(because $\lambda_2 - \lambda_1 \neq 0$ and $\mathbf{v}_1 \neq \mathbf{0}$)

Similarly, it can be shown $k_2 = 0$

Q: How do we continue to and beyond 3 eigenvalues?

◎ <u>*Thm.*</u> If an $n \times n$ matrix $\mathbf{A}$ is has n distinct eigenvalues, then $\mathbf{A}$ is diagonalizable.

<u>*Prf.*</u> n distinct eigenvalues $\rightarrow$ n l.i.

eigenvectors (so, we have enough l.i. eigenvectors for constructing **P**)

◎ <u>*Thm.*</u> Assume that an n×n matrix **A** is has m distinct eigenvalues, and their corresponding eigenspaces are $E_{\lambda 1}(\mathbf{A})$, $E_{\lambda 2}(\mathbf{A})$, ... , $E_{\lambda m}(\mathbf{A})$. Then, **A** is diagonalizable iff $\dim(E_{\lambda 1}(\mathbf{A}))+\dim(E_{\lambda 2}(\mathbf{A}))+\ldots+\dim(E_{\lambda m}(\mathbf{A}))=n$.

◇ $\dim(E_{\lambda k}(\mathbf{A}))$ is called the geometric multiplicity of λ_k .

◇ The number of times that $\lambda - \lambda_k$ appears as a factor in the characteristic poly. of **A** is called the algebraic multiplicity of **A** .

◇ For every eigenvalue of **A**, the geometric multiplicity is less than or equal to the algebraic multiplicity.

◇ **A** is diagonalizable iff the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.

5.3 Orthogonal Diagonalization

◎ ***Def.*** A square matrix **A** is said to be an orthogonal matrix iff $\mathbf{A}^{-1}=\mathbf{A}^{T}$.

◎ ***Def.*** A square matrix **A** is said to be orthogonally diagonalizable iff there exists an orthogonal matrix P such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{T}\mathbf{A}\mathbf{P}$ is a didgonal matrix.

◎ ***Thm.*** If **A** is an n×n matrix, then the following statements are equivalent.

◇ **A** is orthogonally diagonalizable.

◇ **A** has an o.n. (orthonormal) set of n eigenvectors.

◇ **A** is symmetric.

◎ To talk about "orthonormal" or "orthogonal", the concept of inner-product space is needed.

◎ *Thm.* If **A** is a symmetric matrix, then:

◇ The eigenvalues of **A** are real numbers.

◇ Eigenvectors from different eigenspaces are orthogonal.

◎ Procedure for orthogonally diagonalizing an $n \times n$ symmetric matrix:

Step1: Find a basis for each eigenspace.

Step2: Apply the Gram-Schmidt process to find an o.n. basis for each eigenspace.

Step3: Form the matrix **P** whose columns are the basis vectors constructed in Step2; this matrix orthogonally diagonalizes **A**.

◎ Let us study inner-product space next.