107 final exam

1. $f(v)=cv^2\exp\left(-\frac{mv^2}{2kT}\right)$ c,k are positive constants $\quad f(v)=cv^2\cdot e^{-\frac{mv^2}{2kT}}$

$$f'(v)=\frac{d}{dv}f(v)=c\cdot\left(2v\cdot e^{-\frac{mv^2}{2kT}}+v^2\cdot e^{-\frac{mv^2}{2kT}}\cdot\frac{-m}{2kT}\cdot 2v\right)$$

$$=c\cdot 2v\cdot e^{-\frac{mv^2}{2kT}}\left(1+\frac{-mv^2}{2kT}\right)$$

$$=2vc\,e^{-\frac{mv^2}{2kT}}\left(\frac{2kT-mv^2}{2kT}\right)$$

$$2kT-mv^2=-(mv^2-2kT)$$
$$=-m\left(v^2-\frac{2kT}{m}\right)$$
$$=-m\left(v+\sqrt{\frac{2kT}{m}}\right)\left(v-\sqrt{\frac{2kT}{m}}\right)$$

Let $f'(v)=0$, $v=0$ or $2kT-mv^2=0$

$v=0$ or $v=\sqrt{\frac{2kT}{m}}$

v	0	$\left(0,\sqrt{\frac{2kT}{m}}\right)$	$\sqrt{\frac{2kT}{m}}$	$\left(\sqrt{\frac{2kT}{m}},+\infty\right)$
$f(v)$	0		$\frac{2kTc}{me}$	
$f'(v)$	0	$+$	0	$-$
		↗	絕對極大值	↘

$$f\left(\sqrt{\frac{2kT}{m}}\right)=c\cdot\frac{2kT}{m}\cdot e^{-\frac{m\cdot\frac{2kT}{m}}{2kT}}$$
$$=c\cdot\frac{2kT}{m}\cdot e^{-1}$$
$$=\frac{2kTc}{m\cdot e}$$

$\therefore f(v)$ has an absolute maximum $\frac{2kTc}{me}$ at $v=\sqrt{\frac{2kT}{m}}$ #

2. Find the derivatives of the following functions

(a) $f(x)=(\sqrt{x})^{\sqrt{x}}$

Sol1:

$$f(x)=\sqrt{x}^{\sqrt{x}}=e^{\ln\sqrt{x}^{\sqrt{x}}}=e^{\sqrt{x}\ln\sqrt{x}}$$

$$f'(x)=\sqrt{x}^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\cdot\ln\sqrt{x}+\cancel{\sqrt{x}}\cdot\frac{\frac{1}{2\sqrt{x}}}{\cancel{\sqrt{x}}}\right)$$

$$=\sqrt{x}^{\sqrt{x}}\left(\frac{\ln\sqrt{x}}{2\sqrt{x}}+\frac{1}{2\sqrt{x}}\right)$$

$$=\sqrt{x}^{\sqrt{x}}\left(\frac{\ln\sqrt{x}+1}{2\sqrt{x}}\right)$$ #

Sol2:

$$y=\sqrt{x}^{\sqrt{x}}$$

$$\xrightarrow{\text{取}\ln}\ \ln y=\ln\sqrt{x}^{\sqrt{x}}=\sqrt{x}\ln\sqrt{x}$$

$$\xrightarrow{\frac{d}{dx}}\ \frac{d}{dx}(\ln y)=\frac{d}{dx}(\sqrt{x}\ln\sqrt{x})$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\left(\frac{1}{2\sqrt{x}}\ln\sqrt{x}+\cancel{\sqrt{x}}\,\frac{\frac{1}{2\sqrt{x}}}{\cancel{\sqrt{x}}}\right)$$

$$\frac{dy}{dx}=y\cdot\left(\frac{\ln\sqrt{x}}{2\sqrt{x}}+\frac{1}{2\sqrt{x}}\right)$$

$$=\sqrt{x}^{\sqrt{x}}\left(\frac{\ln\sqrt{x}+1}{2\sqrt{x}}\right)$$ #

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(b) $g(x)=\int_{\ln x}^{2^x}\sqrt{3+\cos^2 t}\,dt$

$$g'(x)=\frac{d}{dx}\int_{\ln x}^{2^x}\sqrt{3+\cos^2 t}\,dt=\frac{d}{dx}\left[\int_0^{2^x}\sqrt{3+\cos^2 t}\,dt+\int_{\ln x}^{0}\sqrt{3+\cos^2 t}\,dt\right]=\frac{d}{dx}\left[\int_0^{2^x}\sqrt{3+\cos^2 t}\,dt-\int_0^{\ln x}\sqrt{3+\cos^2 t}\,dt\right]$$

$$=\frac{d}{dx}\int_0^{2^x}\sqrt{3+\cos^2 t}\,dt-\frac{d}{dx}\int_0^{\ln x}\sqrt{3+\cos^2 t}\,dt=\left(\frac{d}{du}\int_0^{u}\sqrt{3+\cos^2 t}\,dt\right)\cdot\frac{du}{dx}-\left(\frac{d}{dv}\int_0^{v}\sqrt{3+\cos^2 t}\,dt\right)\frac{dv}{dx}$$

令 $u=2^x$ 令 $v=\ln x$

$$=\sqrt{3+\cos^2 u}\cdot\frac{d}{dx}(2^x)-\sqrt{3+\cos^2 v}\cdot\frac{d}{dx}(\ln x)=\sqrt{3+\cos^2(2^x)}\cdot 2^x\cdot\ln 2-\sqrt{3+\cos^2(\ln x)}\cdot\frac{1}{x}$$ ※

c) $h(x)=\ln\left|\log_2(\sec x+\tan x)\right|$

$$=\ln\left|\frac{\ln(\sec x+\tan x)}{\ln 2}\right|$$

Sol1:

$$h'(x)=\frac{\frac{1}{\ln 2}\times\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}}{\frac{\ln(\sec x+\tan x)}{\ln 2}}$$

$$=\frac{1}{\cancel{\ln 2}}\times\frac{\sec x\cancel{(\sec x+\tan x)}}{\cancel{\sec x+\tan x}}\times\cancel{\ln 2}\times\frac{1}{\ln(\sec x+\tan x)}$$

$$=\frac{\sec x}{\ln(\sec x+\tan x)}$$ ※

Sol2.

$$h(x)=\ln|\ln(\sec x+\tan x)|-\ln|\ln 2|$$

$$h'(x)=\frac{\frac{\sec x\cancel{(\sec x+\tan x)}}{\cancel{\sec x+\tan x}}}{\ln(\sec x+\tan x)}-0$$

$$=\frac{\sec x}{\ln(\sec x+\tan x)}$$ ※

3. Find the integrals of the following functions

(a) $\int_0^{\frac{\pi}{2}}\frac{\cos x\sin x}{\cos^2 x+1}\,dx$ 令 $u=\cos x$, $du=-\sin x\,dx$, $-du=\sin x\,dx$ $\begin{cases}x=\frac{\pi}{2},\ u=0\\ x=0,\ u=1\end{cases}$

$$=-\int_1^0\frac{u}{u^2+1}\,du$$

$$=-\frac{1}{2}\ln|u^2+1|\Big|_1^0$$

$$=-\frac{1}{2}[\ln 1-\ln 2]$$

$$=\frac{1}{2}\ln 2$$ ※

(b) $\int_0^{\frac{\pi}{8}} \tan 2x\,dx$ 令 $u=2x$, $du=2dx$, $\frac{1}{2}du=dx$ $\begin{cases} x=\frac{\pi}{8},\ u=\frac{\pi}{4} \\ x=0,\ u=0 \end{cases}$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{\cos x} d\cos x$
$= -\ln|\cos x| + c$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan u\,du$

$= \frac{1}{2}\cdot -\ln|\cos x|\Big|_0^{\frac{\pi}{4}} = -\frac{1}{2}\left[\ln\frac{\sqrt{2}}{2} - \ln 1\right] = -\frac{1}{2}\ln\frac{\sqrt{2}}{2}$ ∦

(c) $\int_1^2 \frac{dx}{x - x\ln x}$ 令 $u = 1-\ln x$ $du = -\frac{1}{x}dx$, $-du = \frac{1}{x}dx$

$= \int_1^2 \frac{1}{x(1-\ln x)}\,dx$ $\begin{cases} x=2,\ u=1-\ln 2 \\ x=1,\ u=1 \end{cases}$

$= -\int_1^{1-\ln 2} \frac{1}{u}du = -\ln|u|\Big|_1^{1-\ln 2} = -\ln|1-\ln 2|$ ∦

4. $f(x) = e^x + \ln(x+1)$, $x > -1$

(a) show that f has inverse (b) Find $(f^{-1})'(1)$

(a) $f'(x) = e^x + \frac{1}{x+1} > 0$

$f(x)$ 恆遞增

$\Rightarrow$ "Horizontal Line Test"

$\Rightarrow f(x)$ is one-to-one function

$\Rightarrow f(x)$ has inverse ∦

(b) $f(f^{-1}(x)) = x$

$\frac{d}{dx}f(f^{-1}(x)) = \frac{d}{dx}(x)$

$f'(f^{-1}(x))\cdot(f^{-1})'(x) = 1$

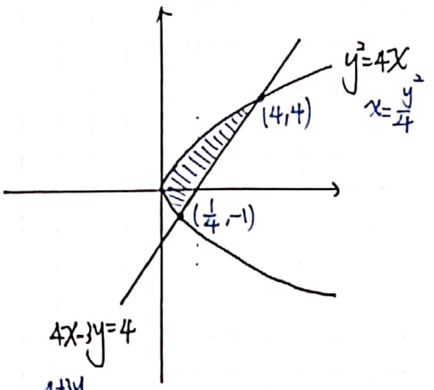
$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$\therefore (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$ $\because f(0)=1$ $\therefore f^{-1}(1)=0$

$= \frac{1}{f'(0)}$

$= \frac{1}{2}$ ∦

5. Find the area of the region between $y^2=4x$ and $4x-3y=4$ by integration with respect to y.



$x = \frac{4+3y}{4}$

x	1	0
y	0	$-\frac{4}{3}$

$y^2-3y-4=0$

$(y-4)(y+1)=0$

$y=-1$ or 4

$x=\frac{1}{4}$ or 4

$A = \int_{-1}^{4} 1 + \frac{3}{4}y - \frac{1}{4}y^2\,dy$

$= \left[y + \frac{3}{8}y^2 - \frac{1}{12}y^3\right]\Big|_{-1}^{4}$

$= 5 + \frac{45}{8} - \frac{65}{12}$

$= \frac{120+135-130}{24}$

$= \frac{125}{24}$ ∦

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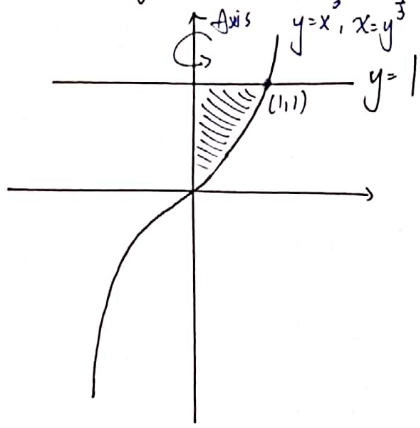
6. Find the Volume of the solid generated by revolving the region bounded by the curve $y=x^3$, the y-axis and the line $y=1$ about y-axis (a) by disc method (b) by cylindrical shell method



(a) disc method,

$$V=\int_0^1 \pi\cdot[R(y)]^2\,dy = \int_0^1 \pi\cdot y^{\frac{2}{3}}\,dy = \pi\cdot\frac{3}{5}y^{\frac{5}{3}}\Big|_0^1$$

$$=\frac{3\pi}{5}\ ※$$

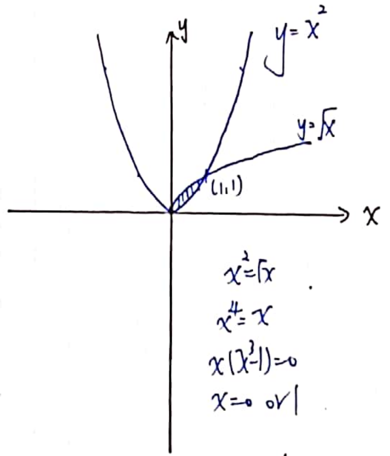
(b) cylindrical shell method:

$$V=\int_0^1 2\pi\cdot(x-0)(1-x^3)\,dx = 2\pi\int_0^1 x-x^4\,dx = 2\pi\cdot\left[\frac{1}{2}x^2-\frac{1}{5}x^5\Big|_0^1\right]$$

$$=2\pi\cdot\frac{3}{10}$$

$$=\frac{3\pi}{5}\ ※$$

7. Let the base of a solid be the first quadrant plane region bounded by $y=\sqrt{x}$ and $y=x^2$. Suppose that the cross sections perpendicular to the x-axis are squares. Find the Volume of the solid.



$x^2=\sqrt{x}$

$x^4=x$

$x(x^3-1)=0$

$x=0$ or 1

$$V=\int_0^1 (x^2-\sqrt{x})^2\,dx = \int_0^1 x^4-2x^2\sqrt{x}+x\,dx = \left[\frac{1}{5}x^5-\frac{4}{7}x^{\frac{7}{2}}+\frac{1}{2}x^2\right]\Big|_0^1$$

$$=\frac{1}{5}-\frac{4}{7}+\frac{1}{2}=\frac{14-40+35}{70}=\frac{9}{70}\ ※$$

8. Find the length of the curve $y=(4-\sqrt[3]{x^2})^{\frac{3}{2}}$ between $x=1$ and $x=8$

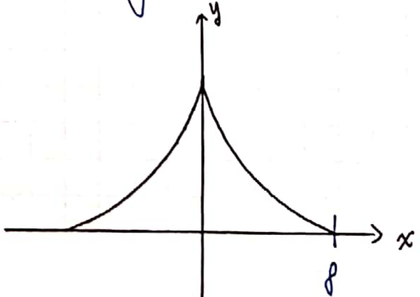
$y^{\frac{2}{3}}=4-x^{\frac{2}{3}}$

$x^{\frac{2}{3}}+y^{\frac{2}{3}}=4=8^{\frac{2}{3}}$

$$y'=\frac{dy}{dx}=\frac{3}{2}(4-x^{\frac{2}{3}})^{\frac{1}{2}}\cdot\left(\frac{-2}{3}\cdot x^{-\frac{1}{3}}\right)$$

$$=-\frac{\sqrt{4-x^{\frac{2}{3}}}}{x^{\frac{1}{3}}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{4-x^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{4}{x^{\frac{2}{3}}}$$



$$\text{length of the curve}=\int_1^8\sqrt{\frac{4}{x^{\frac{2}{3}}}}\,dx=\int_1^8 2\cdot x^{-\frac{1}{3}}\,dx$$

$$=2\cdot\frac{3}{2}x^{\frac{2}{3}}\Big|_1^8=3\cdot(4-1)=9\ ※$$

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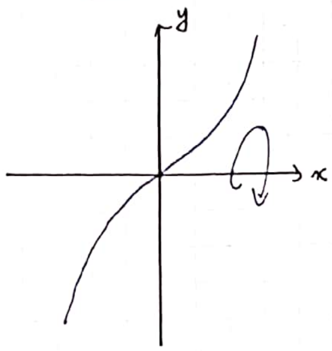
9. Find the area of the surface generated by revolving the curve $y=\frac{x^3}{3}$, $1 \le x \le \sqrt{7}$ about the x-axis



$$\text{area of the surface} = \int_1^{\sqrt{7}} 2\pi y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^{\sqrt{7}} 2\pi \cdot \frac{x^3}{3}\sqrt{1+x^4}\, dx$$

$\frac{1}{2}$ $u = 1+x^4$
$du = 4x^3 dx$
$\frac{1}{6}du = \frac{2}{3}x^3 dx$

$\begin{cases} x=\sqrt{7},\ u=50 \\ x=1,\ u=2 \end{cases}$

$$= \int_2^{50} \frac{\pi}{6} \cdot u^{\frac{1}{2}}\, du$$

$$= \frac{\pi}{6} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}}\Big|_2^{50}$$

$$= \frac{\pi}{9} \cdot \left(50\sqrt{50} - 2\sqrt{2}\right) = \frac{\pi}{9}\left(250\sqrt{2} - 2\sqrt{2}\right) = \underline{\frac{248\sqrt{2}}{9}\pi} \ ※$$

$\frac{dy}{dx} = x^2$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^4$