Total: 100 points

1. (**10** points) Investigate the convergence of the following sequences. Find the limit of each convergent sequence.

   (a) $a_n = \dfrac{n!}{3^n \cdot 7^n}$

   (b) $a_n = (3^n + 5^n)^{1/n}$

Solution:

(a) $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0$ for any a. $\Rightarrow \lim\limits_{n\to\infty} \dfrac{n!}{3^n \cdot 7^n} = \dfrac{1}{\lim\limits_{n\to\infty} \dfrac{21^n}{n!}} \to \infty$. The sequence is divergent.

(b) $a_n = (3^n + 5^n)^{1/n}$, $\quad \exp(x) = e^x$, $\quad a^b = e^{\ln(a^b)} = e^{b\ln a} = \exp(b\ln a)$.

$$\lim_{n\to\infty} (3^n + 5^n)^{1/n} = \lim_{n\to\infty} \exp\left[\frac{1}{n}\ln(3^n + 5^n)\right] = \lim_{n\to\infty} \exp\left[\frac{\ln(3^n + 5^n)}{n}\right]$$

To investigate if $\lim\limits_{n\to\infty} \exp\left[\dfrac{\ln(3^n + 5^n)}{n}\right]$ converges or not, one can investigate the limit of the continuous function:

$$\lim_{x\to\infty} \exp\left[\frac{\ln(3^x + 5^x)}{x}\right] = \lim_{x\to\infty} \exp\left[\frac{\ln 3 \cdot 3^x + \ln 5 \cdot 5^x}{3^x + 5^x}\right] = \lim_{x\to\infty} \exp\left[\frac{\ln 3 \cdot \frac{3^x}{5^x} + \ln 5 \cdot 1}{\frac{3^x}{5^x} + 1}\right]$$

$$= \lim_{x\to\infty} \exp\left[\frac{\ln 3 \cdot \left(\frac{3}{5}\right)^x + \ln 5 \cdot 1}{\left(\frac{3}{5}\right)^x + 1}\right] = \exp\left[\frac{0 \cdot \ln 3 + \ln 5}{0 + 1}\right] = e^{\ln 5} = 5$$

Therefore, $\lim\limits_{n\to\infty} \exp\left[\dfrac{\ln(3^n + 5^n)}{n}\right] = 5$. This sequence is convergent.

2. (**20** points) Find the Taylor series generated by f at $x = a$.
   (a) $f(x) = \ln(1+x), \quad a = 0$
   (b) $f(x) = e^x, \quad a = 2$

Solution:

(a) $f(x) = \ln(1+x)$ at $x = 0$:

$$f'(x) = (1+x)^{-1}, f''(x) = (-1)(1+x)^{-2}, f'''(x) = (-1)(-2)(1+x)^{-3}, \cdots$$
$$\Rightarrow f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

Therefore,

$$f(0) = 0, f'(0) = 1, f''(0) = (-1), f'''(0) = (-1)(-2), \cdots, f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Taylor series generated by f at $x = 0$ is

$$\begin{aligned}
&f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= x + \frac{(-1)}{2!}x^2 + \frac{(-1)^2 2!}{3!}x^3 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n + \cdots \\
&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n
\end{aligned}$$

(b) $f(x) = e^x$ at $x = 2$:

$$f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

Therefore,

$$f'(2) = f''(2) = f'''(2) = \cdots = f^{(n)}(2) = e^2.$$

Taylor series generated by f at $x = 2$ is

$$\begin{aligned}
&f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\
&= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \cdots + \frac{e^2}{n!}(x-2)^n + \cdots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n
\end{aligned}$$

3. (**15** points) Find the area under one arch of the cycloid

$$x = 5(t - \sin t), \quad y = 5(1 - \cos t).$$

Solution:

- $\dfrac{dx}{dt} = x'(t) = 5(1 - \cos t)$. For $t = 0 \to 2\pi$, $y(t) \geq 0$ and $x'(t) \geq 0$. Therefore, the area is

$$\begin{aligned}\int_{t=0}^{t=2\pi} y\,dx &= \int_0^{2\pi} y(t)x'(t)\,dt = \int_0^{2\pi} 5(1-\cos t)\cdot 5(1-\cos t)\,dt = 25\int_0^{2\pi}(1-\cos t)^2\,dt\\ &= 25\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt = 25\int_0^{2\pi}\left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)dt\\ &= 25\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi\cdot 25 = 75\pi.\end{aligned}$$

4. (**20** points) A parametric curve $x = f(t) = 2t^2$, $y = g(t) = t^3 - 4t$.
   (a) Find the equation for the line tangent to the curve at the point $Q(2, -3)$.
   (b) At the point $Q(2, -3)$, is the curve concave upward or concave downward?

Solution:

(a) At the point $Q(2, -3)$, $(2t^2, t^3 - 4t) = (2, -3) \Rightarrow t = 1$.

$$\frac{dx}{dt} = 4t,\ \frac{dy}{dt} = 3t^2 - 4, \Rightarrow \frac{dy}{dx} = \frac{3t^2-4}{4t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = -\frac{1}{4}$$

The equation of the tangent line is

$$y + 3 = -\frac{1}{4}(x - 2) \Rightarrow x + 4y + 10 = 0.$$

(b) At the point $Q(2, -3)$, $t = 1$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t^2-4}{4t}\right) = \frac{\frac{d}{dt}\left(\frac{3t^2-4}{4t}\right)}{\frac{dx}{dt}} = \frac{\frac{1}{t^2} + \frac{3}{4}}{4t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{7}{16} > 0$$

At point Q, the curve is concave upward.

5. (**20** points) Find the area of the region that lies inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$.

Solution:

- First step is to find the intersections. $3\cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$, or $\theta = \frac{\pi}{3}$.
- Both curves are symmetric about x-axis. Therefore, the area is

$$\begin{aligned}
A &= 2\int_0^{\pi/3}\left[\frac{1}{2}(3\cos\theta)^2 - \frac{1}{2}(1+\cos\theta)^2\right]d\theta \\
&= \int_0^{\pi/3}\left(8\cos^2\theta - 2\cos\theta - 1\right)d\theta = \int_0^{\pi/3}\left[4(1+\cos 2\theta) - 2\cos\theta - 1\right]d\theta \\
&= \int_0^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta \\
&= \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]\Big|_0^{\pi/3} = \pi + \sqrt{3} - \sqrt{3} = \pi
\end{aligned}$$

6. (**15** points) Find the lengths of the spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$.

Solution:

- Length: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$. $\quad \frac{dr}{d\theta} = 2\theta$.

$$L = \int_0^{\sqrt{5}} \sqrt{\theta^4 + (2\theta)^2}\,d\theta = \int_0^{\sqrt{5}} |\theta|\sqrt{\theta^2+4}\,d\theta = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\,d\theta$$

since $\theta \ge 0$. Use $u = \theta^2 + 4 \Rightarrow du = 2\theta\,d\theta$ to solve the integral. Then one can find that

$$\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\,d\theta = \int_4^9 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Bigg|_4^9 = \frac{19}{3}$$