· **Ex** Euclidean IP on $\mathbb{R}^n$

$\underline{u} = (u_1, u_2, \cdots, u_n)$ $(u_1, u_2, \cdots u_n \in \mathbb{R})$

$\underline{v} = (v_1, v_2, \cdots, v_n)$ $(v_1, v_2, \cdots, v_n \in \mathbb{R})$

$$\langle \underline{u}, \underline{v} \rangle \triangleq u_1v_1 + u_2v_2 + \cdots + u_nv_n \quad (\#1)$$

This is the "dot product" we learned from high school.

on $\mathbb{R}^{n\times 1}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$\langle \underline{x}, \underline{y} \rangle \triangleq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

$\overset{?}{=}$ the (only) element in $\underline{x}^T\underline{y}$ $\overset{\circ}{=} \underline{x}^T\underline{y}$

· **Ex** Weighted Euclidean IP on $\mathbb{R}^2$

Consider $\underline{u} = (u_1, u_2)$, $\underline{v} = (v_1, v_2) \in \mathbb{R}^2$

$$\langle \underline{u}, \underline{v} \rangle \triangleq 3u_1v_1 + 2u_2v_2 \quad (\#2)$$

※ **Check**: (#1) and (#2) satisfy the 4 requirements on "IP".

on $\mathbb{R}^{2\times 1}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\langle \underline{x}, \underline{y} \rangle \triangleq 3x_1y_1 + 2x_2y_2$$

$\overset{?}{=}$ the (only) element in $\underline{x}^T \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \underline{y}$

$\overset{\circ}{=} \underline{x}^T \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \underline{y}$

• Ex An IP on $\mathbb{R}^{2\times 2}$

Consider $\underline{\underline{A}} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$\langle \underline{\underline{A}}, \underline{\underline{B}} \rangle \triangleq a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

$$\left(= \mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}^T \underline{\underline{A}}) \right)$$

trace (i.e. sum of elements on the diagonal)

• Ex An IP on P_2

$P_2 \to \{ c_0 + c_1 x + c_2 x^2 \mid c_0, c_1, c_2 \in \mathbb{R} \}$

(with the usual poly. addition and number-to-poly multiplication)

Consider $\underline{p} = a_0 + a_1 x + a_2 x^2$

$\underline{q} = b_0 + b_1 x + b_2 x^2$

$$\langle \underline{p}, \underline{q} \rangle \triangleq a_0 b_0 + a_1 b_1 + a_2 b_2$$

· Ex An IP on $C[a,b]$ {continuous functions defined over $[a,b]$}

Let $\underline{f} = f(x)$ and $\underline{g} = g(x)$

$$\langle \underline{f}, \underline{g} \rangle \triangleq \int_a^b f(x)\, g(x)\, dx$$

Verification (the 4 conditions/requirements/axioms on IP)

A1. $\langle \underline{f}, \underline{g} \rangle = \int_a^b f(x) g(x) dx = \int_a^b g(x) f(x) dx = \langle \underline{g}, \underline{f} \rangle$

A2. $\langle \underline{f} + \underline{g}, \underline{s} \rangle = \int_a^b (f(x) + g(x)) s(x) dx = \int_a^b f(x) s(x) dx + \int_a^b g(x) s(x) dx$

$= \langle \underline{f}, \underline{s} \rangle + \langle \underline{g}, \underline{s} \rangle$

A3. $\langle k \cdot \underline{f}, \underline{g} \rangle = \int_a^b k f(x) \cdot g(x) dx = k \cdot \int_a^b f(x) g(x) dx = k \cdot \langle \underline{f}, \underline{g} \rangle$

A4. $\langle \underline{f}, \underline{f} \rangle = \int_a^b (f(x))^2 dx \geq 0$ $\because (f(x))^2 \geq 0$

Furthermore, because $f^2(x) \geq 0$ and $f(x)$ is continuous on $[a,b]$,
It follows that $\int_a^b f^2(x) dx = 0$ iff $f(x) = 0$ for all x in $[a,b]$.
Therefore, we have $\langle \underline{f}, \underline{f} \rangle = 0$ iff $\underline{f} = \underline{0}$ ※

* Some Exs that are not IP's :

- $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 y_1 - x_2 y_2$

- $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 y_2 + y_1 x_2$

- $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1^2 \cdot y_1 + x_2 \cdot y_2$

- $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \triangleq \cos(x_1 + y_1) + \sin(x_2 y_2)$

- $\langle \underbrace{f(x)}, \underbrace{g(x)} \rangle \triangleq \int_a^b f^2(x) \cdot \sqrt{|g(x)|}\, dx$

  $\in C[a,b]$

• Ex A complex IP space on $\mathbb{C}^n$

Consider $\underline{u} = (u_1, u_2, \cdots, u_n)$ and $\underline{v} = (v_1, v_2, \cdots, v_n)$,

where $u_k, v_k \in \mathbb{C}$, for $k = 1, 2, \cdots, n$.

set of complex numbers $(= \{ a + b \cdot i \mid i = \sqrt{-1},\ a, b \in \mathbb{R} \})$

$$\langle \underline{u}, \underline{v} \rangle \triangleq \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n$$

complex conjugate (共軛複數)

"complex" version

$$= u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n$$

Check A1: $\langle \underline{v}, \underline{u} \rangle = \bar{v}_1 u_1 + \bar{v}_2 u_2 + \cdots + \bar{v}_n u_n$

$$= (v_1 \bar{u}_1 + v_2 \bar{u}_2 + \cdots + v_n \bar{u}_n)^*$$

$$= (\bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n)^* = \langle \underline{u}, \underline{v} \rangle^*$$

With "=" holds iff $u_1 = u_2 = u_3 = \cdots = u_n = 0$ (i.e. $\underline{u} = \underline{0}$)

Check A4: $\langle \underline{u}, \underline{u} \rangle = \bar{u}_1 u_1 + \bar{u}_2 u_2 + \cdots + \bar{u}_n u_n = |u_1|^2 + |u_2|^2 + \cdots + |u_n|^2 \geq 0$

• Ex A complex IP space on $\mathbb{C}^{n\times 1}$

(Virtually the same example as on P.109-1)

↳ Precisely speaking, "isomorphic" vector spaces

Consider $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, where $x_k, y_k \in \mathbb{C}$

$x_k = x_{kR} + i \cdot x_{kI}$ $(x_{kR}, x_{kI} \in \mathbb{R})$

x_{kR}: real part of x_k

x_{kI}: imaginary part of x_k

$\langle \underline{x}, \underline{y} \rangle \triangleq \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$

$\overset{?}{=}$ (the (only) element in) $\underline{x}^{*T} \cdot \underline{y}$

$*$: complex conjugate ↳ performed element by element

T: transpose

$\overset{\circ}{=} \underline{x}^{*T} \cdot \underline{y}$ $\quad {}^{*T} \overset{\circ}{=} H$ Hermitian

※ The Hermitian operation (on a matrix) $\triangleq$ conjugate transpose $\equiv$ transpose conjugate

$= \underline{x}^H \cdot \underline{y} \quad (= \underline{x}^{CT} \cdot \underline{y})$

$$= \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^* \right)^T \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}^T \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\bar{x}_1 \ \bar{x}_2 \ \cdots \ \bar{x}_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n]$$

↳ $= \langle \underline{x}, \underline{y} \rangle$

· **Ex:** In $\mathbb{R}^n$, with the usual inner product (i.e. the dot product) as the IP:

$$\underline{x} = (x_1, x_2, \cdots, x_n)$$

$$\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Ex: $\|(3,5)\| = \sqrt{3^2+5^2} = \sqrt{34}$

· **Ex:** In $\mathbb{R}^{2\times 1}$

$$\langle \underline{x}, \underline{y} \rangle \triangleq \underline{x}^T \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \underline{y}$$

Ex: $\|(3,5)\| = \sqrt{3\cdot 3^2 + 2\cdot 5^2} = \sqrt{77}$

$$\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \sqrt{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \sqrt{[x_1\ x_2] \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \sqrt{3x_1^2 + 2x_2^2}$$

· **Ex:** In $C[-1,1]$

$$\langle f(x), g(x) \rangle \triangleq \int_{-1}^{1} f(x)\, g(x)\, dx$$

$$\|f(x)\| = \sqrt{\int_{-1}^{1} (f(x))^2 dx}$$

$f(x) \rightarrow f$

Ex: $\|x\| = \sqrt{\int_{-1}^{1} (x)^2 dx} = \sqrt{\frac{1}{3}x^3 \Big|_{-1}^{1}} = \sqrt{\frac{2}{3}}$ #

$$\|\cos(x)\| = \sqrt{\int_{-1}^{1} (\cos^2 x)\, dx} = \sqrt{\frac{1}{2}(\cos x \cdot \sin x + x)\Big|_0^1}$$

$$= \sqrt{\cos(1)\cdot\sin(1) + 1} \approx \sqrt{1.4546} \approx 1.20$$ #

· **Ex** (distance bet. vectors)

$$\langle \underline{u}, \underline{v} \rangle = \langle (u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n) \rangle \triangleq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\mathrm{dist}(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \| = \| (u_1 - v_1, u_2 - v_2, \cdots, u_n - v_n) \|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

· **Ex** In $C[-1, 1]$, with $\langle f(x), g(x) \rangle \triangleq \int_{-1}^{1} f(x) g(x) \, dx$

$$\mathrm{dist}(x, x^2) = \| x - x^2 \| = \sqrt{\langle (x - x^2), (x - x^2) \rangle} = \sqrt{\int_{-1}^{1} (x - x^2)^2 \, dx} \approx 1.8328$$

$$\mathrm{dist}(x^2, \cos x) = \sqrt{\int_{-1}^{1} (x^2 - \cos x)^2 \, dx} \approx .9477$$

$$\mathrm{dist}(\cos x, \sin x) \approx 1.41421356 \quad (\text{exact value: } \sqrt{2})$$

$$\mathrm{dist}(e^x, \sin x) \approx 1.6868$$

$$\mathrm{dist}(e^{-x}, \sin x) \approx 2.345$$

· **Thm** $\langle \underline{0}, \underline{v} \rangle = \langle \underline{v}, \underline{0} \rangle = 0$

Prf $\langle \underline{0}, \underline{v} \rangle = \langle \underline{0}+\underline{0}, \underline{v} \rangle \overset{\text{A2 of IP}}{=} \underbrace{\langle \underline{0}, \underline{v} \rangle}_{t} + \underbrace{\langle \underline{0}, \underline{v} \rangle}_{t} \Rightarrow t = 0$ #

$\langle \underline{0}, \underline{v} \rangle \to$ 은 t

property of $\underline{0}$ in V.S.

· **Thm** $\langle \underline{u}, \underline{v}+\underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

Prf $\langle \underline{u}, \underline{v}+\underline{w} \rangle = \langle \underline{v}+\underline{w}, \underline{u} \rangle^*$ ← conjugate (can be omitted if real IP space is being considered)

$\overset{\text{A2}}{=} \left(\langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle \right)^*$

property of complex numbers

$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

$= \langle \underline{v}, \underline{u} \rangle^* + \langle \underline{w}, \underline{u} \rangle^* \overset{\text{A1' of IP}}{=} \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

"complex" version

· **Thm** $\langle \underline{u}, k\underline{v} \rangle = \overline{k} \langle \underline{u}, \underline{v} \rangle$ ← complex conjugate

Prf $\langle \underline{u}, k\underline{v} \rangle = \langle k\underline{v}, \underline{u} \rangle^* \overset{\text{A3 of IP}}{=} \left(k \langle \underline{v}, \underline{u} \rangle \right)^* = \overline{k} \cdot \langle \underline{v}, \underline{u} \rangle^*$

$= \overline{k} \cdot \langle \underline{u}, \underline{v} \rangle$

property of complex numbers

$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

· **Thm** $\langle \underline{u}-\underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle - \langle \underline{v}, \underline{w} \rangle$

Prf $\langle \underline{u}-\underline{v}, \underline{w} \rangle = \langle \underline{u} + \overbrace{(-\underline{v})}^{= -1 \cdot \underline{v}}, \underline{w} \rangle$ (a theorem from V.S.)

$= \langle \underline{u} + (-1)\cdot\underline{v}, \underline{w} \rangle$

A2 $= \langle \underline{u}, \underline{w} \rangle + \langle -1\cdot\underline{v}, \underline{w} \rangle$

A3 $= \langle \underline{u}, \underline{w} \rangle + (-1)\cdot\langle \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle - \langle \underline{v}, \underline{w} \rangle$

· **Thm** $\langle \underline{u}, \underline{v}-\underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle - \langle \underline{u}, \underline{w} \rangle$

Prf $\langle \underline{u}, \underline{v}-\underline{w} \rangle = \langle \underline{v}-\underline{w}, \underline{u} \rangle^* = \left(\langle \underline{v}, \underline{u} \rangle - \langle \underline{w}, \underline{u} \rangle \right)^*$

by the previous theorem

$\left(\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2} \right)$

$= \langle \underline{v}, \underline{u} \rangle^* - \langle \underline{w}, \underline{u} \rangle^*$

A1' $= \langle \underline{u}, \underline{v} \rangle - \langle \underline{u}, \underline{w} \rangle$