D. E. (Second Midterm—Chapter, slide 41 to Chapter 2, slide 20)

Total points: 100 points 2 hours to solve the problems, Nov. 13, 2024

1. Find the general solution. (30 points)

(a) $y' = \frac{y}{x} + \frac{x}{y}$

(b) $e^{x} y^{2} \frac{dy}{dx} = e^{x-y} + e^{-x-y}$

(c) $y' - \left(\frac{1}{x}\right) y = 3x^2 y^3$

1.(a)

This equation is homogeneous. With $y = xu$, we obtain

$$u + xu' = u + \frac{1}{u}. \quad \boxed{+3}$$

Then

$$x\frac{du}{dx} = \frac{1}{u},$$

a separable equation. Write

$$u\,du = \frac{1}{x}\,dx.$$

Integrate to obtain

$$u^2 = 2\ln|x| + c. \quad \boxed{+8}$$

Then

$$\frac{y^2}{x^2} = 2\ln|x| + c \quad \boxed{+10}$$

implicitly defines the general solution of the original differential equation.

1.(b)

3A $e^x y^2 \frac{dy}{dx} = e^{x-y} + e^{-x-y}$

$\Rightarrow e^x y^2 \frac{dy}{dx} = \frac{e^x + e^{-x}}{e^y}$

$\Rightarrow y^2 e^y \, dy = \frac{e^x + e^{-x}}{e^x}$ $\boxed{+3}$

$\Rightarrow \int y^2 e^y \, dy = \int (1 + e^{-2x}) \, dx$

$\Rightarrow y^2 e^y - \int e^y \cdot 2y \, dy = x - \frac{1}{2} e^{-2x} + c$

$\Rightarrow y^2 e^y - 2[\, y e^y - \int e^y dy \,] = x - \frac{1}{2} e^{-2x} + c$ $\boxed{+8}$

$\Rightarrow y^2 e^y - 2y e^y + 2e^y = x - \frac{1}{2} e^{-2x} + c$ $\boxed{+10}$

1.(c)

Let $u = y^{1-n} = y^{-2}$

$$y = u^{\frac{-1}{2}}, y' = \frac{-1}{2} u^{\frac{-3}{2}} u'$$

代入$y' - \left(\frac{1}{x}\right) y = 3x^2 y^3$

$$\frac{-1}{2} u^{\frac{-3}{2}} u' - \frac{1}{x} u^{\frac{-1}{2}} = 3x^2 u^{\frac{-3}{2}}$$ $\boxed{+3}$

同乘$u^{\frac{3}{2}}$

$$\frac{-1}{2} u' - \frac{1}{x} u = 3x^2$$

$$u' + \frac{2}{x} u = -6x^2$$

同乘$e^{\int \frac{2}{x} dx} = e^{2lnx} = x^2$

$$x^2 u' + xu = -6x^4$$

$$\frac{d}{dx}(x^2 u) = -6x^4$$

對 x 積分

$$x^2u = -\frac{6}{5}x^5 + c \quad , \quad u = -\frac{6}{5}x^3 + \frac{c}{x^2} \quad \boxed{+8}$$

代回$y^{-2} = u$

$$y^{-2} = -\frac{6}{5}x^3 + \frac{c}{x^2} \text{ or } y = \frac{1}{\sqrt{-\frac{6}{5}x^3 + \frac{c}{x^2}}} \quad \boxed{+10}$$

2. Solve the initial value problem. (10 points)

$y'' - 5y' + 12y = 0; y(2) = 0 \; y'(2) = -4$

$\lambda^2 - 5\lambda + 12 = 0$

$\lambda = \frac{5 \pm \sqrt{23}i}{2}$ $\boxed{+3}$

$y(t) = c_1 e^{\frac{5+\sqrt{23}i}{2}(t-2)} + c_2 e^{\frac{5-\sqrt{23}i}{2}(t-2)}$

$= e^{\frac{5}{2}(t-2)}[c_1 \cos\left(\frac{\sqrt{23}}{2}(t-2)\right) + c_2 \sin(\frac{\sqrt{23}}{2}(t-2))]$ $\boxed{+7}$

$y(2) = c_1 = 0$

$y'(2) = \frac{5}{2}c_1 + \frac{\sqrt{23}}{2}c_2 = -4$

$c_2 = \frac{-8}{\sqrt{23}}$ $\boxed{+9}$

$y(t) = e^{\frac{5}{2}(t-2)} \frac{-8}{\sqrt{23}} \sin(\frac{\sqrt{23}}{2}(t-2))$ $\boxed{+10}$

3. Consider the linear differential equation $\frac{dy}{dx} - \frac{y}{x} = x\,cosx$. (20 points)

(a) Find the integrating factor. (5 points)
(b) Find the general solution of the differential equation. (5 points)
(c) Provide the largest interval of the definition of your solution in (b). (5 points)
(d) Verify that your answer in (b) is a solution of the differential equation. (5 points)

(a) $P(x) = -\frac{1}{x}$ and $f(x) = x\cos x$ are continuous on $(0,\infty)$ and $(-\infty, 0)$.

$\int P(x)dx = -\int \frac{1}{x}dx = -\ln|x|$

Consider $(0,\infty)$. The integrating factor is

$e^{\int P(x)dx} = e^{-\ln|x|} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \underline{\frac{1}{x}}$

(b) Multiply both sides of the DE by $\frac{1}{x}$, we have

$\frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = \frac{1}{x}(x\cos x)$

$\frac{d}{dx}\left(\frac{1}{x}y\right) = \cos x$

$\frac{1}{x}y = \int \cos x\,dx = \sin x + c$

$\underline{y = cx + x\sin x}$ (1)

<note>
When consider $\underline{(-\infty,0)}$, the integrating factor is $\underline{-\frac{1}{x}}$. Following the same steps, we have the same solution.

(c) The largest interval of definition is $\underline{(0,\infty)}$

(d) Substituting (1) into the left-hand side of the DE, we have

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{d}{dx}(cx + x\sin x) - \frac{1}{x}(cx + x\sin x)$$

$$= (c + \sin x + x\cos x) - (c + \sin x)$$

$$= x\cos x$$

Since both sides of the DE are $x\cos x$, the solution is verified.

4. Riccati Equation. (20points)

(a) The definition of the Riccati Equation. (10points)

(b) The soluition of the Riccati Equation. (10points)

- The Riccati Equation
  - Definition

$y' = p(x)y^2 + Q(x)y + R(x)$

If $p(x) = 0 \Rightarrow y' = Q(x)y + R(x) \Rightarrow y' - Q(x)y = R(x)$ is linear.

If $p(x) \neq 0 \Rightarrow$ use the change of variable $y = S(x) + \frac{1}{z}$

  - Solution

First, find one solution $S(x)$ of the Riccati equation

Second, use the change of variable $y = S(x) + \frac{1}{z}$ to transform the Riccati equation into a linear equation of $z(x)$

Third, solve for $z(x)$

Fourth, obtain the general solution $y = S(x) + \frac{1}{z}$

- The Riccati Equation
  - Definition

$y' + p(x)y = g(x)$

$y' = P(x)y^2 + Q(x)y + R(x)$

If $P(x) = 0$, $y' = Q(x)y + R(x) \Rightarrow y' - Q(x)y = R(x)$ linear

  - Solution

① find one simple solution $S(x)$

② let $y = S(x) + \frac{1}{z}$ to transform the D.E. into a linear one of z

③ solve for z

5. Solve the initial value problem, using the method of Bernoulli Equation.(20points)

$y' - 2y = 3xy^3,\ y(0) = 2$

5. $y' - 2y = 3xy^3$

$v = y^{-2} \quad y = v^{-\frac{1}{2}} \quad y' = -\frac{1}{2}v^{-\frac{3}{2}} \cdot v'$

$-\frac{1}{2}v^{-\frac{3}{2}} \cdot v' - 2v^{-\frac{1}{2}} = 3xv^{-\frac{3}{2}}$

$\times(-2)v^{\frac{3}{2}} \Rightarrow \quad v' + 4v = -6x$

+5

$e^{\int 4dx} = e^{4x}$

$(e^{4x} \cdot v)' = -6x \cdot e^{4x} \qquad u = -6x \quad dv = e^{4x}dx, \quad du = -6dx \quad v = \frac{1}{4}e^{4x}$

$e^{4x} \cdot v = \int -6x \cdot e^{4x}\,dx \Longrightarrow -\frac{3}{2}x \cdot e^{4x} + \int \frac{3}{2}e^{4x}\,dx = -\frac{3}{2}x \cdot e^{4x} + \frac{3}{8}e^{4x} + C$

+10

$v = -\frac{3}{2}x + \frac{3}{8} + c \cdot e^{-4x}$

$y = v^{-\frac{1}{2}} = \left(-\frac{3}{2}x + \frac{3}{8} + c \cdot e^{-4x}\right)^{-\frac{1}{2}}$

$2 = \left(0 + \frac{3}{8} + c\right)^{-\frac{1}{2}}$

+15

$\frac{1}{4} = \frac{3}{8} + C \Rightarrow C = -\frac{1}{8}$

$y = \left(-\frac{3}{2}x + \frac{3}{8} - \frac{1}{8}e^{-4x}\right)^{-\frac{1}{2}}$

+20