Total: 100 points

1. (**10** points) Evaluate the limit. $\lim_{x\to 0} \dfrac{2\tan^{-1} 3x^2}{7x^2}$

Solution:

$$\lim_{x\to 0} \frac{2\tan^{-1} 3x^2}{7x^2} = \lim_{x\to 0} \frac{2 \cdot \frac{1}{1+(3x^2)^2} \cdot 6x}{14x} = \lim_{x\to 0} \frac{\frac{6}{1+(3x^2)^2}}{7} = \frac{6}{7}$$

2. (**10** points) Evaluate the limit. $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

Solution: Let $y = (\cos x)^{\frac{1}{x^2}} \Rightarrow \ln y = \dfrac{1}{x^2} \ln(\cos x)$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{2x} = \lim_{x\to 0} \frac{-\tan x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

$$\lim_{x\to 0} \ln y = \ln\left(\lim_{x\to 0} y\right) = -\frac{1}{2} \Rightarrow \lim_{x\to 0} y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

3. (**10** points) Evaluate the limit. $\lim_{x\to 0} \dfrac{\tanh x}{\tan x}$

Solution:

$$\lim_{x\to 0} \frac{\tanh x}{\tan x} = \lim_{x\to 0} \frac{\operatorname{sech}^2 x}{\sec^2 x} = \frac{\lim_{x\to 0} \operatorname{sech}^2 x}{\lim_{x\to 0} \sec^2 x} = \frac{1}{1} = 1$$

4. (**10** points) Evaluate the limit. $\lim_{x\to 0} \dfrac{\sinh x - x}{x^3}$

Solution:

$$\lim_{x\to 0} \frac{\sinh x - x}{x^3} = \lim_{x\to 0} \frac{\cosh x - 1}{3x^2} = \lim_{x\to 0} \frac{\sinh x}{6x} = \lim_{x\to 0} \frac{\cosh x}{6} = \frac{1}{6}$$

5. (**20** points) Order the following functions from slowest growing to fastest growing as $x \to \infty$.

a. e^x **b.** x^x **c.** $(\ln x)^x$ **d.** $e^{x/2}$

Solution:

- $e^{x/2}$ and e^x: $\lim\limits_{x\to\infty} \dfrac{e^{x/2}}{e^x} = \lim\limits_{x\to\infty} e^{-x/2} = 0 \quad \Rightarrow$ Therefore, **d.** slower than **a.**
- e^x and $(\ln x)^x$: $\lim\limits_{x\to\infty} \dfrac{e^x}{(\ln x)^x} = \lim\limits_{x\to\infty} \left(\dfrac{e}{\ln x}\right)^x = 0 \quad \Rightarrow$ Therefore, **a.** slower than **c.**
- $(\ln x)^x$ and x^x: $\lim\limits_{x\to\infty} \dfrac{(\ln x)^x}{x^x} = \lim\limits_{x\to\infty} \left(\dfrac{\ln x}{x}\right)^x = 0 \quad \Rightarrow$ Therefore, **c.** slower than **b.**

The order is **d. a. c. b.**

6. (**15** points) Find $\dfrac{dy}{dx}$ if $y = \tanh^{-1}\left(x^3\right)$

Solution: $y = \tanh^{-1}\left(x^3\right) \Rightarrow \tanh y = x^3$. Use the implicit differentiation.

$$\tanh y = x^3 \Rightarrow \operatorname{sech}^2 y \cdot y' = 3x^2 \Rightarrow y' = \frac{3x^2}{\operatorname{sech}^2 y}$$

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow 1 - \frac{\sinh^2 y}{\cosh^2 y} = \frac{1}{\cosh^2 y} \Rightarrow 1 - \tanh^2 y = \operatorname{sech}^2 y$$

$$\Rightarrow y' = \frac{3x^2}{\operatorname{sech}^2 y} = \frac{3x^2}{1 - \tanh^2 y} = \frac{3x^2}{1 - (x^3)^2} = \frac{3x^2}{1 - x^6}$$

7. (**15** points) Use implicit differentiation to find $\dfrac{dy}{dx}$ at the point $P\left(0, \dfrac{1}{2}\right)$ if

$$\sin^{-1}(x+y) + \cos^{-1}(x-y) = \frac{5\pi}{6}$$

Solution: $P(0, 1/2)$ is on the curve $\sin^{-1}(x+y) + \cos^{-1}(x-y) = \dfrac{5\pi}{6}$.

$$\sin^{-1}(x+y) + \cos^{-1}(x-y) = \frac{5\pi}{6} \Rightarrow \frac{1+y'}{\sqrt{1-(x+y)^2}} + \frac{-(1-y')}{\sqrt{1-(x-y)^2}} = 0.$$

The goal is to find y' when $x = 0, y = 1/2$. Therefore,

$$\frac{1+y'}{\sqrt{1-(1/2)^2}} + \frac{-(1-y')}{\sqrt{1-(-1/2)^2}} = 0 \Rightarrow \frac{1+y'}{\sqrt{3/4}} - \frac{1-y'}{\sqrt{3/4}} = 0 \Rightarrow 2y' = 0 \Rightarrow y' = 0.$$