Ex $\underline{\underline{A}} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$: real and symmetric

- characteristic poly : $|\lambda \underline{\underline{I}} - \underline{\underline{A}}| = \begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = (\lambda-2)^2 \cdot (\lambda-8)$

  $\Rightarrow$ eigenvalues : 2, 2, 8 : real-valued

- With respect to $\lambda = 8 \longrightarrow$ eigenvector : $t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \overset{\diamond}{=} \underline{P}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

  $\left[8 \cdot \underline{\underline{I}} - \underline{\underline{A}}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

  $t \overset{\diamond}{=} \frac{1}{\sqrt{3}}$ (for normalization)

- With respect to $\lambda = 2 \longrightarrow$

  eigenvectors : $t_1 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $t_2 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ eigenspace wrt $\lambda = 2$

  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_2(\underline{\underline{A}})$ orthonormal

  Gram-Schmidt process $\longrightarrow$ $\left\{ \underbrace{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}_{\overset{\diamond}{=} \underline{P}_1}, \underbrace{\begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}}_{\overset{\diamond}{=} \underline{P}_2} \right\}$ is an o.n. basis for $E_2(\underline{\underline{A}})$

  orthogonal and normal

- $$\underline{\underline{P}} \triangleq [\underline{p}_1\ \underline{p}_2\ \underline{p}_3] = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

- Check:
$$\underline{\underline{P}}^T \underline{\underline{P}} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \#$$

- Check:
$$\underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}} = \underline{\underline{P}}^T \underline{\underline{A}}\, \underline{\underline{P}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \#$$

$$\underline{\underline{P}}^{-1} = \underline{\underline{P}}^T$$

$\underline{\underline{P}}$ is an orthogonal matrix

In the teacher's opinion, "orthonormal" would have been a better term.

IP

- Inner product space is needed for "orthogonal", "o.n.", Gram-Schmidt process, etc.