Total: 100 points

1. (**30** points) Use implicit differentiation to find an equation of the tangent line to the curve at the given point P.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4, \quad P(-3\sqrt{3}, 1)$$

Solution:

$x^{2/3} + y^{2/3} = 4 \;\Rightarrow\; \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \;\Rightarrow\; \frac{1}{\sqrt[3]{x}} + \frac{y'}{\sqrt[3]{y}} = 0 \;\Rightarrow\; y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}.$

When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\frac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\frac{1}{(-3^{3/2})^{1/3}} = \frac{1}{3^{1/2}} = \frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

2. (**20** points) Find dy if $y = 2\cot\left(\frac{1}{\sqrt{x}}\right)$

Solution: $dy = -2\csc^2(x^{-1/2}) \cdot \left(-\frac{1}{2}\right)x^{-3/2}dx \Rightarrow dy = \frac{1}{\sqrt{x^3}}\csc^2\left(\frac{1}{\sqrt{x}}\right)dx$

3. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
   (a) (**10** points) Use differentials to estimate the maximum error in the calculated area of the disk.

   (b) (**10** points) What is the relative error and the percentage error?

Solution:

(a) $A = \pi r^2 \Rightarrow dA = 2\pi r\, dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\, dr}{\pi r^2} = \dfrac{2\, dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error = relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

4. (**30** points) A balloon is rising vertically above a level, straight road at a constant rate of 0.5 (m/s). Just when the balloon is 18.5 (m) above the ground, a bicycle moving at a constant rate of 5 (m/s) passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing **3 (s) later**?

Solution:

Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is

$$s^2 = h^2 + x^2.$$

After 3 seconds, $h = 18.5 + 3 \cdot 0.5 = 20$ (m), and $x = 0 + 3 \cdot 5 = 15$ (m).
$\Rightarrow s = \sqrt{20^2 + 15^2} = 25$ (m). Thus,

$$\frac{ds}{dt} = \frac{1}{s}\left(h\frac{dh}{dt} + x\frac{dx}{dt}\right) = \frac{1}{25}(20 \cdot 0.5 + 15 \cdot 5) = \frac{17}{5} = 3.4(\text{m/s})$$