1. 試繪 $\exp(-x^2)$ 　　106 哲質②

$f(x)=e^{-x^2}=\dfrac{1}{e^{x^2}}$

① Domain: $\{x \mid x\in R\}$

② $f(x)$ is continuous on R

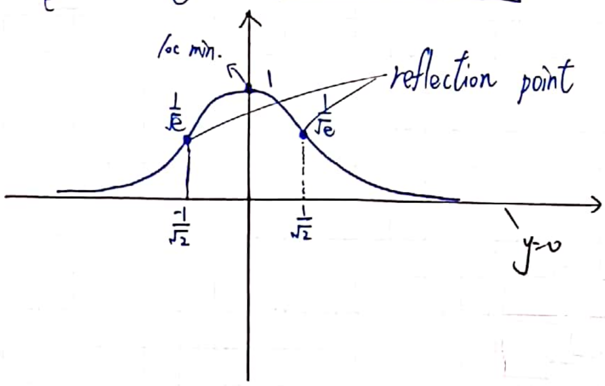
③
$$\left.\begin{aligned}\lim_{x\to+\infty}\frac{1}{e^{x^2}}=0\\ \lim_{x\to-\infty}\frac{1}{e^{x^2}}=0\end{aligned}\right\}\ y=0 \text{ is a horizontal asymptote of } f(x)$$

④ $f(-x)=e^{-(-x)^2}=e^{-x^2}=f(x) \Rightarrow f(x)$ 對稱於 y 軸（偶函數）

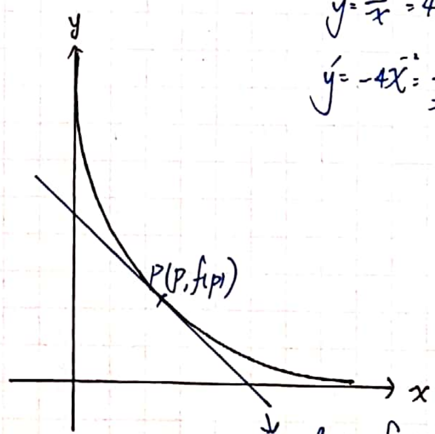
$f'(x)=e^{-x^2}\cdot(-2x)=\dfrac{-2x}{e^{x^2}} \Rightarrow \boxed{\text{Let } f'(x)=0,\ x=0}$

$f''(x)=-2\cdot e^{-x^2}+(-2x)\cdot e^{-x^2}\cdot(-2x)=-2e^{-x^2}+4x^2e^{-x^2}=\dfrac{4x^2-2}{e^{x^2}}=\dfrac{4(x+\frac{1}{\sqrt2})(x-\frac{1}{\sqrt2})}{e^{x^2}} \Rightarrow \boxed{\text{Let } f''(x)=0,\ x=\pm\frac{1}{\sqrt2}}$

x	$(-\infty,-\frac{1}{\sqrt2})$	$-\frac{1}{\sqrt2}$	$(-\frac{1}{\sqrt2},0)$	0	$(0,\frac{1}{\sqrt2})$	$\frac{1}{\sqrt2}$	$(\frac{1}{\sqrt2},+\infty)$
$f(x)$		$\frac{1}{\sqrt e}$		1		$\frac{1}{\sqrt e}$	
$f'(x)$	+	+	+	0	−	−	−
$f''(x)$	+	0	−	−	−	0	+
graph	↗	反曲點	↗	相對極大值	↘	反曲點	↘



2. 在第一象限中，從曲線 $xy=4$ 上的任一點 P 做切線，分別交於 x 軸與 y 軸於點 A, B，求 P 點的座標使 $\overline{AB}^2$ 為最小值.



$y=\frac{4}{x}=4x^{-1}$

$y'=-4x^{-2}=\frac{-4}{x^2}$

$L(x)=f(p)+f'(p)(x-p)$

$y=\frac{4}{p}+\frac{-4}{p^2}(x-p)$

$p^2y=4p-4x+4p$

$4x+p^2y=8p$

x	0	$2p$
y	$\frac{8}{p}$	0

$A(2p,0)$

$B(0,\frac{8}{p})$

$\overline{AB}^2=\left(\sqrt{4p^2+\frac{64}{p^2}}\right)^2$

$=4p^2+\frac{64}{p^2}$

$=4p^2+64p^{-2}$

$\frac{d}{dp}(\overline{AB}^2)=8p-128p^{-3}=8p+\frac{-128}{p^3}$

$=\frac{8p^4-128}{p^3}=\frac{8(p^4-16)}{p^3}=0$

$p=\pm2$（負不合）　Ans: P 點座標 $(2,2)$ #

3. 求下列函數的導函數

ⓐ $\int_{\ln x}^{e^x}\sqrt{1+(\ln t)^2}\,dt$

$\frac{d}{dx}\int_{\ln x}^{e^x}\sqrt{1+(\ln t)^2}\,dt = \frac{d}{dx}\left[\int_0^{e^x}\sqrt{1+(\ln t)^2}\,dt + \int_{\ln x}^{0}\sqrt{1+(\ln t)^2}\,dt\right] = \frac{d}{dx}\left\{\int_0^{e^x}\sqrt{1+(\ln t)^2}\,dt - \int_0^{\ln x}\sqrt{1+(\ln t)^2}\,dt\right]$

$= \underset{\text{令 } u=e^x}{\frac{d}{dx}\int_0^{e^x}\sqrt{1+(\ln t)^2}\,dt} - \underset{\text{令 } v=\ln x}{\frac{d}{dx}\int_0^{\ln x}\sqrt{1+(\ln t)^2}\,dt} = \left(\frac{d}{du}\int_0^{u}\sqrt{1+(\ln t)^2}\,dt\right)\cdot\frac{du}{dx} - \left(\frac{d}{dv}\int_0^{v}\sqrt{1+(\ln t)^2}\,dt\right)\cdot\frac{dv}{dx}$

$= \sqrt{1+(\ln e^x)^2}\cdot e^x - \sqrt{1+(\ln(\ln x))^2}\cdot\frac{1}{x} = \underline{\sqrt{1+x^2}\cdot e^x - \sqrt{1+[\ln(\ln x)]^2}\cdot\frac{1}{x}}$ #

ⓑ $g(x)=(\ln x)^x$

Sol1:

$y=(\ln x)^x$

$\xrightarrow{\text{取}\ln}\ \ln y = \ln[(\ln x)^x]$

$= x\cdot\ln(\ln x)$

$\xrightarrow{\text{微分}}\ \frac{1}{y}\cdot\frac{dy}{dx} = \ln(\ln x) + x\cdot\frac{\frac{1}{x}}{\ln x}$

$\frac{dy}{dx} = y\cdot\left[\ln(\ln x)+\frac{1}{\ln x}\right]$

$= \underline{(\ln x)^x\left[\ln(\ln x)+\frac{1}{\ln x}\right]}$ #

Sol2:

$y=(\ln x)^x = e^{\ln(\ln x)^x} = e^{x\cdot\ln(\ln x)}$

$y' = e^{x\cdot\ln(\ln x)}\cdot\left(1\cdot\ln(\ln x) + x\cdot\frac{\frac{1}{x}}{\ln x}\right)$

$= \underline{(\ln x)^x\left[\ln(\ln x)+\frac{1}{\ln x}\right]}$ #

4. $h(t)=e^{-kt}\sin(wt+\delta)$, k, w, δ 皆為常數，求 $h''(t)$

$h'(t) = e^{-kt}\cdot(-k)\cdot\sin(wt+\delta) + e^{-kt}\cdot\cos(wt+\delta)\cdot w$

$h''(t) = -k\left[e^{-kt}\cdot(-k)\cdot\sin(wt+\delta) + e^{-kt}\cos(wt+\delta)\cdot w\right] + w\left[e^{-kt}\cdot(-k)\cos(wt+\delta) + e^{-kt}\cdot -\sin(wt+\delta)\cdot w\right]$

$= e^{-kt}\left[k^2\sin(wt+\delta) - kw\cos(wt+\delta)\right] + e^{-kt}\left[-kw\cos(wt+\delta) - w^2\sin(wt+\delta)\right]$

$= \underline{e^{-kt}\left[(k^2-w^2)\sin(wt+\delta) - 2kw\cos(wt+\delta)\right]}$ #

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5. 求下列積分

ⓐ $\int_0^1 e^x\sqrt{1+e^x}\,dx$ 令 $1+e^x=u$ $\begin{cases} x=1, u=1+e \\ x=0, u=2 \end{cases}$

$du=e^x dx$

$=\int_2^{1+e} u^{\frac{1}{2}}\,du=\frac{2}{3}u^{\frac{3}{2}}\Big|_2^{1+e}=\underline{\frac{2}{3}\left[(1+e)^{\frac{3}{2}}-2^{\frac{3}{2}}\right]}$ #

ⓑ $\int_1^e \frac{\sqrt[3]{\ln x}}{x}\,dx$ 令 $u=\ln x$ $\begin{cases} x=e \Rightarrow u=1 \\ x=1 \Rightarrow u=0 \end{cases}$

$du=\frac{1}{x}dx$

$=\int_0^1 u^{\frac{1}{3}}\,du$

$=\frac{3}{4}u^{\frac{4}{3}}\Big|_0^1=\frac{3}{4}[1-0]=\underline{\frac{3}{4}}$ #

ⓒ $\int_0^1 \frac{x^2}{\sqrt{1+3x}}\,dx$ 令 $u=1+3x$ $\begin{cases} x=1, u=4 \\ x=0, u=1 \end{cases}$ $x^2=\frac{(u-1)^2}{9}=\frac{u^2-2u+1}{9}$

$x=\frac{u-1}{3}$

$du=3\,dx$

$=\int_1^4 \frac{u^2-2u+1}{9\cdot u^{\frac{1}{2}}}\cdot\frac{1}{3}du$

$=\frac{1}{27}\int_1^4 u^{\frac{3}{2}}-2u^{\frac{1}{2}}+u^{-\frac{1}{2}}\,du=\frac{1}{27}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{4}{3}u^{\frac{3}{2}}+2u^{\frac{1}{2}}\Big|_1^4\right]=\frac{1}{27}\left[\frac{2}{5}\cdot 31-\frac{4}{3}\cdot 7+2\cdot 1\right]=\frac{1}{27}\cdot\left(\frac{62}{5}-\frac{28}{3}+2\right)$

$=\frac{1}{27}\cdot\frac{76}{15}=\underline{\frac{76}{405}}$ #

6. $f(x)=3+x+e^x$ (a) 試證 f 為 1-1 函數 (b) 求 $(f^{-1})'(4)$

ⓐ $f'(x)=1+e^x>0$

$\Rightarrow f(x)$ 為恆遞增函數

$\Rightarrow$ "Horizontal Line Test"

$\Rightarrow f(x)$ 是一對一函數

ⓑ $f(f^{-1}(x))=x$

$f'(f^{-1}(x))\cdot(f^{-1})'(x)=1$

$\therefore (f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(4)=\frac{1}{f'(f^{-1}(4))}=\frac{1}{f'(0)}=\frac{1}{1+e^0}=\underline{\frac{1}{2}}$ #

$\because f(0)=3+0+e^0=4$

$\therefore f^{-1}(4)=0$

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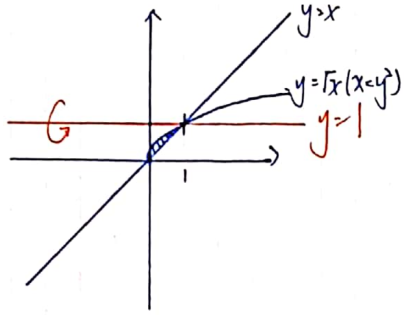
7. $y=x$ 和 $y=\sqrt{x}$ 所圍成之區域繞 $y=1$ 旋轉所旋轉體的體積



$x=\sqrt{x}$

$x^2=x$

$x(x-1)=0$

$x=0$ or 1

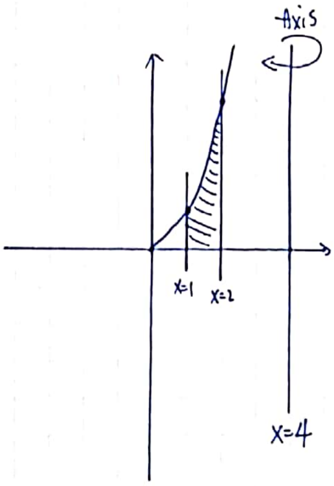
Sol1: (shell)

$$V=\int_0^1 2\pi(1-y)(y-y^2)\,dy$$
$$=2\pi\int_0^1 y-y^2-y^2+y^3\,dy$$
$$=2\pi\int_0^1 y^3-2y^2+y\,dy$$
$$=2\pi\left[\frac{1}{4}y^4-\frac{2}{3}y^3+\frac{1}{2}y^2\Big|_0^1\right]$$
$$=2\pi\left[\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right]$$
$$=2\pi\cdot\frac{3-8+6}{12}$$
$$=\frac{\pi}{6}\ \#$$

Sol2: (washer)

$$V=\int_0^1 \pi\left[R(x)^2-r(x)^2\right]dx$$
$$=\pi\int_0^1\left[(1-x)^2-(1-\sqrt{x})^2\right]dx$$
$$=\pi\int_0^1 x^2-2x+1-(x-2\sqrt{x}+1)\,dx$$
$$=\pi\int_0^1 x^2-2x+\cancel{1}-x+2\sqrt{x}-\cancel{1}\,dx$$
$$=\pi\int_0^1 x^2-3x+2x^{\frac{1}{2}}\,dx$$
$$=\pi\cdot\left[\frac{1}{3}x^3-\frac{3}{2}x^2+\frac{4}{3}x^{\frac{3}{2}}\Big|_0^1\right]$$
$$=\pi\left(\frac{1}{3}-\frac{3}{2}+\frac{4}{3}\right)$$
$$=\pi\cdot\frac{2-9+8}{6}$$
$$=\frac{\pi}{6}\ \#$$

8. 利用圓柱殼法求 $y=x^2$, $y=0$, $x=1$, $x=2$ 所圍成之區域繞 $x=4$ 旋轉所成旋轉體的體積



$$V=\int_1^2 2\pi(4-x)(x^2-0)\,dx$$
$$=2\pi\int_1^2 4x^2-x^3\,dx$$
$$=2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\Big|_1^2\right]$$
$$=2\pi\left[\frac{28}{3}-\frac{15}{4}\right]$$
$$=2\pi\cdot\frac{112-45}{12}$$
$$=\frac{67}{6}\pi\ \#$$

9. 求 $y^2=x^3$ 之圖形自 $(0,0)$ 至點 $(5,5\sqrt{5})$ 的弧長

$y=x^{\frac{3}{2}}$

$y'=\frac{3}{2}x^{\frac{1}{2}}$

$(y')^2=\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2=\frac{9}{4}x$

$$S=\int_0^5\sqrt{1+[f'(x)]^2}\,dx=\int_0^5\sqrt{1+\frac{9}{4}x}\,dx$$

令 $u=1+\frac{9}{4}x$, $du=\frac{9}{4}dx$, $dx=\frac{4}{9}du$

$\begin{cases} x=5,\ u=1+\frac{45}{4}=\frac{49}{4} \\ x=0,\ u=1+0=1 \end{cases}$

$$\Rightarrow S=\int_1^{\frac{49}{4}}u^{\frac{1}{2}}\cdot\frac{4}{9}\,du=\frac{4}{9}\cdot\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{\frac{49}{4}}\right]$$
$$=\frac{4}{9}\cdot\frac{2}{3}\cdot\left[\left(\frac{7}{2}\right)^3-1\right]=\frac{8}{27}\cdot\left(\frac{343}{8}-1\right)=\frac{\cancel{8}}{27}\times\frac{335}{\cancel{8}}=\frac{335}{27}\ \#$$

Ïhý uú

10. 求曲線 $y=\sqrt{4-x^2}$, $-1\le x\le 1$ 繞 x 軸旋轉所成旋轉曲面的面積

$$A=\int_{-1}^{1}2\pi\cdot y\cdot\sqrt{1+[f'(x)]^2}\,dx$$

$$=2\pi\int_{-1}^{1}\left(\cancel{\sqrt{4-x^2}}\right)\cdot\sqrt{\frac{4}{\cancel{4-x^2}}}\,dx$$

$$=2\pi\int_{-1}^{1}2\,dx$$

$$=2\pi\cdot 2x\Big|_{-1}^{1}$$

$$=2\pi\cdot 4$$

$$=\underline{8\pi}\ \#$$

$$y'=\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$=\frac{-x}{\sqrt{4-x^2}}$$

$$1+(y')^2=1+\frac{x^2}{4-x^2}=\frac{4}{4-x^2}$$