國立臺灣科技大學答案卷

National Taiwan University of Science and Technology Answer Sheet

評分 Score	教師簽章 Signature of Lecturer

姓名/Name ________ 學號/Student ID ________ 班級/Class ________

科目/Course title ________ 日期/Date ________

記分欄

從此處開始寫起。試卷用紙務須節用，非經主試認可不得續用其他紙張作答。/Please write from here.

1.

(a) $\lim_{x\to 0} \frac{\tan^{-1}(ax)}{\tan^{-1}(bx)}$

$= \lim_{x\to 0} \frac{\frac{1}{(ax)^2+1}\cdot a}{\frac{1}{(bx)^2+1}\cdot b}$

$= \lim_{x\to 0} \frac{(b^2x^2+1)\,a}{(a^2x^2+1)\,b} = \frac{a}{b}$ ※

(b) $\lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}$

Let $f(x) = (x+e^x)^{\frac{2}{x}}$

$\ln f(x) = \frac{2}{x}\ln(x+e^x)$

$\lim_{x\to\infty} \frac{2}{x}\ln(x+e^x)$

$= \lim_{x\to\infty} \frac{2\ln(x+e^x)}{x}$

$= \lim_{x\to\infty} \frac{2\cdot\frac{1}{x+e^x}(1+e^x)}{1}$

$= \lim_{x\to\infty} \frac{2(1+e^x)}{x+e^x}$

$= \lim_{x\to\infty} \frac{2e^x/e^x}{(1+e^x)/e^x}$

$= \lim_{x\to\infty} \frac{2}{e^{-x}+1} = 2$

$\therefore \lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}$

$= e^2$ ※

2.

$x^{\log x}$ vs. π^x

$\log x \ln x$ vs. $x\ln\pi$

$\lim_{x\to\infty} \frac{\log x \ln x}{x\ln\pi} = \lim_{x\to\infty} \frac{\frac{\ln x}{x\ln 10} + \frac{\ln x}{x\ln 10}}{\ln\pi} = \lim_{x\to\infty} \frac{2\ln x}{x\ln 10\ln\pi} = \cdots = 0$

$f_2 > f_3 > f_1$

3. (a)

$\int \cot^4 x\,dx$

$= \int (\csc^2 x - 1)\cot^2 x\,dx$

$= \int (\cot^2 x\csc^2 x - \cot^2 x)\,dx$

$= \int [\cot^2 x\csc^2 x - (\csc^2 x - 1)]\,dx$

Let $u = \cot x \Rightarrow du = -\csc^2 x\,dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + C$ ※

(b) $\int e^{-x}\sin x\,dx$

Let $u = e^{-x}$ $dv = \sin x\,dx$

$du = -e^{-x}dx$ $v = -\cos x$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$

Let $u = e^{-x}$ $dv = \cos x\,dx$

$du = -e^{-x}dx$ $v = \sin x$

$= -e^{-x}\cos x + \left(e^{-x}\sin x + \int e^{-x}\sin x\,dx\right)$

$\int e^{-x}\sin x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C$

$\int_0^\infty e^{-x}\sin x\,dx = \lim_{t\to\infty}\frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^t$

$= \frac{1}{2}\left(0 - (-1)\right) = \frac{1}{2}$ ※

(c) $\int_1^2 \frac{x}{x^2-2x+2}\,dx$

$\frac{x}{x^2-2x+2} = \frac{x}{(x^2-2x+1)+2-1}$

$= \frac{x}{(x-1)^2+1}$

Let $u = x-1 \Rightarrow du = dx$

$x\in(1,2) \Rightarrow u\in(0,1)$

$\int_0^1 \frac{u+1}{u^2+1}\,du$

$= \int_0^1 \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right)du$

$= \frac{1}{2}\ln(u^2+1) + \tan^{-1}u \Big|_0^1$

$= \left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right) - (0+0)$

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$ ※

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(d) $\int \sin(\ln x)\,dx$

Let $u = \ln x \Rightarrow x = e^u$, $dx = e^u du$

$= \int \sin u \cdot e^u du$

Let $u_1 = e^u$ $\quad dv_1 = \sin u\,du$

$du_1 = e^u du$ $\quad v_1 = -\cos u$

$= -e^u \cos u + \int e^u \cos u\,du$

Let $u_2 = e^u$ $\quad dv_2 = \cos u\,du$

$du_2 = e^u du$ $\quad v_2 = \sin u$

$= -e^u \cos u + e^u \sin u - \int e^u \sin u\,du$

$\Rightarrow \int e^u \sin u\,du = \frac{1}{2} e^u (\sin u - \cos u) + C$

$\int \sin(\ln x)\,dx = \frac{1}{2} x(\sin(\ln x) - \cos(\ln x)) + C$ ※

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$

$a_n = \frac{1}{\sqrt[n]{n}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}$

$= \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = 1$ ※

$\ln a_n = \frac{1}{n} \ln n$

$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln n}{n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\therefore \lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1$

$\because \lim_{n\to\infty} a_n \neq 0$

$\therefore$ The series diverges

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+1}$

$u_n = \frac{\sqrt{n}}{n+1} > 0$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{2} n^{-\frac{1}{2}}}{1} = 0$

Let $f(x) = \frac{\sqrt{x}}{x+1} = \sqrt{x}(x+1)^{-1}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} (x+1)^{-1} + (-1)\sqrt{x}(x+1)^{-2} = -x^{\frac{1}{2}}(x+1)^{2}$, $x > 1$

$\frac{1}{2} x^{-\frac{1}{2}} (x+1)^{-1} > 0$, $(-1)\sqrt{x}(x+1)^{-2} < 0$

$\therefore f$ is decreasing $\forall x > x_0, \exists x_0$.

By the alternating series test, the series converges

7. $\log(n!) = O(n \log n)$

$\log(n!) \leq \log(n^n)$

$= n \log n$

4. (a)

$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$a_n = \left(\frac{n}{n+1}\right)^{n^2} > 0$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

Let $f(n) = \left(\frac{n}{n+1}\right)^n$

$\ln f(n) = n \ln\left(\frac{n}{n+1}\right)$ $\quad (0 \cdot \infty)$

$\lim_{n\to\infty} \ln f(n) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{-1}{n^2}}$

$= \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-n^2}{n^2+n} = \lim_{n\to\infty} \frac{-1}{1+\frac{1}{n}} = -1$

$\therefore \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1} < 1$

By the root test, the series converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ $\quad a_n = \frac{1}{\sqrt{n!}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{\sqrt{(n+1)!}}}{\frac{1}{\sqrt{n!}}}\right| = \lim_{n\to\infty} \left|\sqrt{\frac{n!}{(n+1)!}}\right| = \lim_{n\to\infty} \sqrt{\frac{1}{n+1}} = 0 < 1$

By the ratio test, the series converges

5. $\sum_{n=2}^{\infty} \frac{(x-3)^n}{(n+1)2^n}$ $\quad a_n = \frac{1}{n+1} \cdot \frac{(x-3)^n}{2^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n+2} \cdot \frac{(x-3)^{n+1}}{2^{n+1}}}{\frac{1}{n+1} \cdot \frac{(x-3)^n}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n+2} \cdot \frac{(x-3)}{2}\right|$

$= \left|\frac{x-3}{2}\right|$

if the series converges $\Rightarrow \left|\frac{x-3}{2}\right| < 1$

$\Rightarrow |x-3| < 2$

$x = 1$: $\sum_{n=2}^{\infty} \frac{(1-3)^n}{(n+1)2^n}$ converges $\quad \Rightarrow -2 < x-3 < 2$

$x = 5$: $\sum_{n=2}^{\infty} \frac{(5-3)^n}{(n+1)2^n}$ diverges $\quad \Rightarrow 1 < x < 5$

The interval for convergence is $[1, 5)$

6. $\cosh x = \frac{e^x + e^{-x}}{2}$

Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

.... e^{-x} is $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

$\cosh x = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots\right)\right]$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$