Engineering mathematics II   Final exam., June 4, 2024

This is an open-book test. Moreover, access to the internet is allowed. Interaction with another person/human, however, is absolutely prohobited. The total score is 105 points. In your solution, you need to show your computations.

1. Consider the matrix below:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

(a). (5%) Let $\underline{\underline{R}}$ denote the rref of $\underline{\underline{A}}$. Find $\underline{\underline{R}}$. And then, express $\underline{\underline{R}}$ as $\underline{\underline{R}} = \underline{\underline{E}}_M \cdots \underline{\underline{E}}_2 \underline{\underline{E}}_1 \underline{\underline{A}}$, where $\underline{\underline{E}}_1, \underline{\underline{E}}_2, \cdots, \underline{\underline{E}}_M$ are elementary matrices. Please make M (i.e. the number of elementary matrices in your expression) as small as possible.

(b). (5%) Verify that your $\underline{\underline{E}}_M \cdots \underline{\underline{E}}_2 \underline{\underline{E}}_1 \underline{\underline{A}}$ is indeed equal to $\underline{\underline{R}}$ by carrying out matrix multiplications (two matrices at a time).

2. Let $\mathcal{P}_n$ denote the set of polynomials whose degrees are equal to or less than n. Consider a transformation $T : \mathcal{P}_2 \mapsto \mathcal{P}_1$, which is defind by: $T(a \cdot x^2 + b \cdot x + c) = (3a + 2b + c) \cdot x + (2a - b - c)$.

(a). (10%) Show that T is onto by showing that for any vector $\underline{y}$ in $\mathcal{P}_1$ (i.e. any polynomial of the form: $\alpha \cdot x + \beta$), there exists at least one vector $\underline{x}$ in $\mathcal{P}_2$ (i.e. a polynomial of the form: $A \cdot x^2 + B \cdot x + C$) such that $T(\underline{x}) = \underline{y}$.

(b). (5%) Continued from the preceeding subproblem, show that actually there are more than one vectors in $\mathcal{P}_2$ that can be transformed into a vector $\underline{y}$ in $\mathcal{P}_1$ (and thus T is not a 1-1 transformation).

(c). (5%) Let us adopt the ordered basis $B = \{1, x, x^2\}$ for $\mathcal{P}_2$(notice that 1 is the first basis vector, x is the second, and x^2 is the third). Let us adopt $D = \{1, x\}$ for $\mathcal{P}_1$ (notice that 1 is the first basis vector, and x is the second). Find the matrix of T with respect to B and D.

(d). (10%) Let us adopt the ordered basis $B = \{1, x, x^2\}$ for $\mathcal{P}_2$(notice that 1 is the first basis vector, x is the second, and x^2 is the third). Let us adopt $E = \{1, x + 1\}$ for $\mathcal{P}_1$ (notice that 1 is the first basis vector, and $x + 1$ is the second). Find the matrix of T with respect to B and E.

3.(10%) Let $\mathcal{P}_1$ denote the set of polynomials whose degrees are no greater than 1. In other words, $\mathcal{P}_1 = \{a \cdot x + b | a, b \in \mathcal{R}\}$. Consider the linear transformation: $T(a \cdot x + b) = (3a + 2b) \cdot x + (2a - b)$. Let S denote the inverse transformation of T. Then, $S(a \cdot x + b) = ?$

4. Consider the matrix below:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

(a). (5%) Show that $[2, -1]^{\mathrm{T}}$ is not an eigenvector of $\underline{\underline{A}}$, by definition.

(b). (5%) Show that $[1 + \sqrt{5}, 2]^{\mathrm{T}}$ is an eigenvector of $\underline{\underline{A}}$, by definition.

(c). (5%) It is known that $[1-\sqrt{5}\ ,2]^{\mathrm{T}}$ is also an eigenvector of $\underline{\underline{A}}$ (you do not need to show it). By combining this fact with the result from the preceeding subproblem, please find the orthonormal matrix that orthogonally diagonalizes $\underline{\underline{A}}$.

5. Consider the $\mathcal{R}^4$ vector space, with the standard vector addition and standard scalar multiplication. Consider the vector space: $\mathcal{V} = \text{span}((1,1,0,1),(1,2,3,0),(0,1,1,-1))$.

(a). (5%) Find the angle between $(1,1,0,1)$ and $(1,2,3,0)$.

(b). (10%) By applying the Gram-Schmdit orthonormalization process, find an orthonormal basis for $\mathcal{V}$. You need to show your derivations.

(c). (5%) Express the vector $(1,2,3,0)$ as a linear combination of the basis vectors that you had constructed in the preceeding subproblem.

6. It is known that any parabola in the x-y plane can be described by an equation of this form: $a \cdot x^2 + b \cdot x + c \cdot y = 1$, as long as it does not pass the origin (i.e. $(0,0)$), where a, b, and c are some constants. Suppose that we had observed five points on the parabola: $(x,y) = (-2.,0.34),(-1,-0.09),(0,0.26),(1,-1.20),(2,-2.64)$. Notice that the observations can be noisy. Next let us try to find an LSE (least square error) fit of the observations to a parabola.

(a). (5%) Write down the system of equation for solving the unknowns: a, b, and c. Please express your answer in the form of $\underline{\underline{A}}\,\underline{x} = \underline{b}$.

(b). (10%) Continued from the preceeding subproblem, please find the LSE solution for a, b, and c.

(c). (5%) Continued from the preceeding subproblem, what value do we expect the y-value of the point on the parabola to be when x is 3.25 ?