國立臺灣科技大學答案卷

National Taiwan University of Science and Technology Answer Sheet

姓名/Name ______ 學號/Student ID ______ 班級/Class ______

科目/Course title 工程數學 日期/Date 111.12.1

評分 Score	教師簽章 Signature of Lecturer
94	

記分欄

從此處開始寫起。試卷用紙務須節用，非經主試認可不得續用其他紙張作答。/Please write from here.

1. +10

$y''+2y'+4y=0$, $y(0)=2$, $y'(0)=1$

$\xrightarrow{y=e^{\lambda x}} \lambda^2+2\lambda+4=0 \Rightarrow \lambda=\frac{-2\pm\sqrt{4-16}}{2} = -1\pm\sqrt{3}i \Rightarrow y_1=e^{-x}\cos(\sqrt{3}x),\ y_2=e^{-x}\sin(\sqrt{3}x)$

$\Rightarrow y_h=C_1e^{-x}\cos(\sqrt{3}x)+C_2e^{-x}\sin(\sqrt{3}x)$ —(1)

$y_h'=-C_1e^{-x}\cos(\sqrt{3}x)-C_1e^{-x}\cdot\sqrt{3}\cdot\sin(\sqrt{3}x)-C_2e^{-x}\sin(\sqrt{3}x)+C_2e^{-x}\cdot\sqrt{3}\cdot\cos(\sqrt{3}x)$ —(2)

subs. $\begin{cases}x=0\\ y=2\end{cases}$ into (1):

$2=C_1\cdot1\cdot\cos(0)+C_2\cdot1\cdot\sin(0) \Rightarrow C_1=2$

subs. $\begin{cases}x=0\\ y'=1\end{cases}$ into (2):

$1=-C_1+\sqrt{3}C_2 \Rightarrow -C_1+\sqrt{3}C_2=1$

$\Rightarrow C_2=\frac{3}{\sqrt{3}}=\sqrt{3}$

$\Rightarrow$ The solution of the initial problem is:

$$\boxed{y_h=2\cdot e^{-x}\cdot\cos(\sqrt{3}x)+\sqrt{3}\cdot e^{-x}\cdot\sin(\sqrt{3}x)}$$ #1

2.(A) +5

這一項的係數要是1

$2x^2y''-10xy'+27y=0 \Rightarrow x^2y''-5xy'+\frac{27}{2}y=0$

$\xrightarrow{x=e^t}$ ~~$Y''-11Y'+27Y=0$~~ $Y''-6Y'+\frac{27}{2}Y=0$

$\Rightarrow$ ~~$Y''-\frac{11}{2}Y'+\frac{27}{2}Y=0$~~

$\frac{6\pm\sqrt{36-54}}{2}$, $\sqrt{-18}=3\sqrt{2}i$, $=3\pm\frac{3}{2}\sqrt{2}\cdot i$

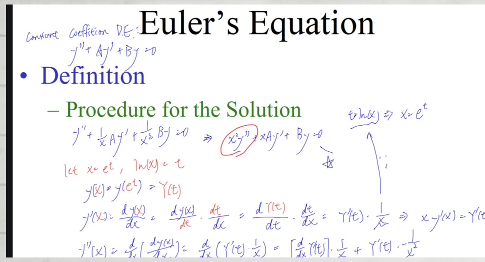
$Y=e^{\lambda t}$

$\Rightarrow \lambda^2-\cancel{11}6\lambda+\frac{27}{2}=0 \Rightarrow \lambda=$ ~~$\frac{\frac{11}{2}\pm\sqrt{\frac{11^2}{4}-\frac{4\cdot27}{2}}}{2}=\frac{11}{4}\pm\frac{1}{2}\sqrt{\frac{95}{4}}i=\frac{11}{4}\pm\frac{\sqrt{95}}{4}i$~~

$\Rightarrow Y_h(t)=C_1e^{3t}\cos(\tfrac{3}{2}\sqrt{2}t)+C_2e^{3t}\sin(\tfrac{3}{2}\sqrt{2}t)$

The general solution is

$$\boxed{y_h(x)=C_1x^{3}\cdot\cos(\tfrac{3}{2}\sqrt{2}\ln x)+C_2x^{3}\cdot\sin(\tfrac{3}{2}\sqrt{2}\ln x)}$$ #2(A)



2(B) +9

$y''+2y'+2y=5e^{6x}$ —(1)

1. Find y_h:

$\xrightarrow{y=e^{\lambda x}} \lambda^2+2\lambda+2=0 \Rightarrow \lambda=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$

$\Rightarrow y_h=C_1\cdot e^{-x}\cos x+C_2e^{-x}\sin x$

2. Find y_p:

Let $y_p=Ae^{6x}$, $y_p'=6Ae^{6x}$, $y_p''=36Ae^{6x}$

subs. $\begin{cases}y_p\\ y_p'\\ y_p''\end{cases}$ into (1):

$36Ae^{6x}+12Ae^{6x}+2Ae^{6x}=5e^{6x} \Rightarrow A=\frac{5}{50}=\frac{1}{10}$

$\Rightarrow y_p=\frac{1}{10}\cdot e^{6x}$

The general solution is:

$$\boxed{y=y_h+y_p=C_1\cdot e^{-x}\cos x+C_2e^{-x}\sin x+\frac{1}{10}e^{6x}}$$ #2(B)

可轉頁再寫。

分欄　轉頁從此開始寫起。

3.(A) $x^2y''-3xy'+4y=0$, $y(1)=5$, $y'(1)=2$

+9

$\xRightarrow{x=e^t} Y''-4Y'+4Y=0 \xRightarrow{Y=e^{\lambda t}} \lambda^2-4\lambda+4=(\lambda-2)^2=0 \Rightarrow \lambda=2$ (重根) $\Rightarrow \begin{cases} y_1=e^{2t} \\ y_2=te^{2t} \end{cases}$ by reduction of order

$\Rightarrow Y_h=c_1e^{2t}+c_2te^{2t} \xRightarrow{t=\ln x} y_h=c_1x^2+c_2(\ln x)x^2 \xrightarrow{\text{subs.}\begin{cases}y=5\\x=1\end{cases}} 5=c_1$

$y_h'=c_1\cdot 2x+c_2\cdot x+2c_2\cdot x\cdot \ln x \xrightarrow[c_1=5]{\text{subs.}\begin{cases}y'=2\\x=1\end{cases}} 2=10+c_2 \Rightarrow c_2=-8$

The solution of the initial problem is:

$\boxed{y=5x^2-8x^2(\ln x)}$ #3(A)

3(B) $y''-2y'+y=e^x\sin 3x$, $y(0)=1$, $y'(0)=1$

+9

1. Find y_h:

$y''-2y'+y=0$

$\xRightarrow{y=e^{\lambda x}} \lambda^2-2\lambda+1=0 \Rightarrow (\lambda-1)^2=0 \Rightarrow \lambda=1$ (重根) $\Rightarrow \begin{cases} y_1=e^x \\ y_2=xe^x \end{cases}$ by reduction of order $\Rightarrow y_h=c_1e^x+c_2xe^x$

2. Find y_p:

$w(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix}=e^{2x}+\cancel{xe^{2x}}-\cancel{xe^{2x}}=e^{2x}$

u_1	v_1
x	$\sin 3x$
1	$-\frac{1}{3}\cos(3x)$
	$\frac{1}{9}\sin(3x)$

let $y_p=uy_1+vy_2$:

$u'=-\frac{y_2f(x)}{w(x)}=\frac{-x\cancel{e^x}\cdot\cancel{e^x}\sin 3x}{\cancel{e^{2x}}}=-x\sin 3x \Rightarrow u=-\int \underbrace{x}_{u_1}\underbrace{\sin 3x\,dx}_{dv_1}=-\left[-\frac{1}{3}\cos(3x)\cdot x+\frac{1}{3}\cancel{\int\cos(3x)dx}\right]$

$v'=\frac{y_1f(x)}{w(x)}=\frac{\cancel{e^x}\cdot\cancel{e^x}\cdot\sin 3x}{\cancel{e^{2x}}}=\sin 3x \Rightarrow v=\int\sin 3x\,dx=-\frac{1}{3}\cos 3x$

$\Rightarrow y_p=\left[\cancel{\frac{1}{3}x\cos(3x)}-\frac{1}{9}\sin(3x)\right]\cdot e^x-\cancel{\frac{1}{3}\cdot\cos(3x)\cdot x\cdot e^x}=-\frac{1}{9}e^x\sin(3x)$

$\Rightarrow y=y_h+y_p=c_1e^x+c_2x\cdot e^x-\frac{1}{9}\cdot e^x\cdot\sin(3x)$ —(1)

$y'=c_1e^x+c_2e^x+c_2xe^x-\frac{1}{9}e^x\sin(3x)-\frac{1}{3}\cdot e^x\cdot\cos(3x)$ —(2)

subs. $\begin{cases}y=1\\x=0\end{cases}$ into (1):

$1=c_1+(-\frac{1}{9})\cdot 0 \Rightarrow c_1=1$

subs. $\begin{cases}y'=1\\x=0\end{cases}$ into (2):

$1=c_1+c_2+(-\frac{1}{3})\cdot 1\cdot 1 \Rightarrow c_1+c_2=\frac{4}{3} \xRightarrow{c_1=1} c_2=\frac{1}{3}$

The solution of the initial problem is:

$\boxed{y=e^x+\frac{1}{3}\cdot x\cdot e^x-\frac{1}{9}\cdot e^x\cdot\sin(3x)}$ #3(B)

4. $y'' + 3y' + 2.25y = -10 \cdot e^{-1.5x}$ —(1)

+25 (A) Find y_h:

$y'' + 3y' + 2.25y = 0 \xRightarrow{y=e^{\lambda x}} \lambda^2 + 3\lambda + (1.5)^2 = 0 \Rightarrow \lambda = -1.5$ (重根) $\Rightarrow \begin{cases} y_1 = e^{-1.5x} \\ y_2 = xe^{-1.5x} \end{cases}$ by reduction of order $\Rightarrow \boxed{y_h = C_1 e^{-1.5x} + C_2 \cdot x \cdot e^{-1.5x}}$ #4(A)

(B) Find y_p using the Method of Undetermined Coefficients:

let $y_p = Ax^2 e^{-1.5x}$:

$y_p' = 2Ax \cdot e^{-1.5x} + A \cdot x^2 \cdot (-1.5) e^{-1.5x} = (2Ax - 1.5Ax^2) e^{-1.5x}$

$y_p'' = (2A - 3Ax) e^{-1.5x} - 1.5(2Ax - 1.5Ax^2) e^{-1.5x}$

subs $\begin{cases} y_p \\ y_p' \\ y_p'' \end{cases}$ into (1):

$-3 \; -3$

$(2A - \cancel{3Ax})\cancel{e^{-1.5x}} + (\cancel{-3Ax + 2.25Ax^2})\cancel{e^{-1.5x}} + (\cancel{6Ax - 4.5Ax^2})\cancel{e^{-1.5x}} + \cancel{2.25Ax^2}\cancel{e^{-1.5x}} = -10 \cdot \cancel{e^{-1.5x}}$

$2A = -10 \Rightarrow A = -5$

$\Rightarrow \boxed{y_p = -5x^2 e^{-1.5x}}$ #4(B)

The general solution is:

$\boxed{y = y_h + y_p = C_1 e^{-1.5x} + C_2 \cdot x \cdot e^{-1.5x} - 5x^2 e^{-1.5x}}$ #4(C)

訂正在P.5

5 $x^2 y'' - 2xy' + 2y = \sin(x^{-1})$ —(2)

+8 (A) Find y_h:

$x^2 y'' - 2xy' + 2y \xRightarrow{x=e^t} Y'' - 3Y' + 2Y = 0 \xRightarrow{Y=e^{\lambda t}} \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1$ or $2 \Rightarrow y_h = C_1 e^t + C_2 e^{2t}$ $\Rightarrow \boxed{y_h = C_1 x + C_2 x^2}$ #5(A)

+9 (B) Find y_p using the Method of Variation of Parameters:

let $y_p = uy_1 + vy_2$:

$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$u' = \dfrac{-y_2 f(x)}{w(x)} = \dfrac{-x^2 \cdot \sin(x^{-1})}{x^2} = -\sin(x^{-1})$

$v' = \dfrac{y_1 f(x)}{w(x)} = \dfrac{x \cdot \sin(x^{-1})}{x^2} = \dfrac{\sin(x^{-1})}{x}$

$\Rightarrow u = -\int \sin(\frac{1}{x}) dx$, $v = \int \frac{1}{x} \cdot \sin(\frac{1}{x}) dx$

$\Rightarrow \boxed{y_p = -x\left[\int \sin(\frac{1}{x}) dx\right] + x^2 \left[\int \frac{1}{x} \cdot \sin(\frac{1}{x}) dx\right]}$ #5(B)

+6 The general solution is:

$\boxed{y = y_h + y_p = C_1 x + C_2 x^2 - x\left[\int \sin(\frac{1}{x}) dx\right] + x^2\left[\int \frac{1}{x} \cdot \sin(\frac{1}{x}) dx\right]}$ #5(C)

6. Given a second order homogeneous differential equation $y'' + Ay' + By = 0$ (1) and one of its solution $y_1 = e^{-\frac{Ax}{2}}$. Suppose $A^2 - 4B = 0$. Show that $y_2 = x \cdot e^{-\frac{Ax}{2}}$ is another solution:

By reduction of order:

Let $y_2 = uy_1 = ue^{-\frac{Ax}{2}}$

$y_2' = u' \cdot e^{-\frac{Ax}{2}} - \frac{A}{2} \cdot u \cdot e^{-\frac{Ax}{2}}$

$y_2'' = u'' \cdot e^{-\frac{Ax}{2}} - \frac{A}{2} \cdot u' \cdot e^{-\frac{Ax}{2}} + u'\left(-\frac{A}{2}\right) \cdot e^{-\frac{Ax}{2}} + \frac{A^2}{4} \cdot u \cdot e^{-\frac{Ax}{2}}$

Subs. $\begin{cases} y_2 \\ y_2' \\ y_2'' \end{cases}$ into (1):

$u'' \cdot \cancel{e^{-\frac{Ax}{2}}} - A \cdot u' \cdot \cancel{e^{-\frac{Ax}{2}}} + \frac{A^2}{4} \cdot u \cdot \cancel{e^{-\frac{Ax}{2}}} + A\left(u' \cdot \cancel{e^{-\frac{Ax}{2}}} - \frac{A}{2} u \cdot \cancel{e^{-\frac{Ax}{2}}}\right) + Bu \cdot \cancel{e^{-\frac{Ax}{2}}} = 0$

$\Rightarrow u'' - \cancel{Au'} + \frac{A^2}{4} u + \cancel{Au'} - \frac{A^2}{2} u + Bu = 0$

$\Rightarrow u'' + \frac{\cancel{4B - A^2}}{4} \cdot u = 0$ $\quad (4B - A^2 = 0)$ $\because$

$\Rightarrow u'' = 0 \Rightarrow \begin{matrix} u' = c \\ u = cx + d \end{matrix} \xrightarrow[d=0]{c=1} u = x$

$\Rightarrow \boxed{y_2 = uy_1 = x \cdot e^{-\frac{Ax}{2}}}$ #6

5 $x^2y'' - 2xy' + 2y = \sin(x^{-1})$

$\xrightarrow{x=e^t} Y'' - 3Y' + 2Y = \sin(e^{-t})$

1. Find Y_h:

$\xrightarrow{Y=e^{\lambda t}} \lambda^2 - 3\lambda + 2 = 0 \xrightarrow{\cdot X^{-1}_{-2}} \lambda = 1 \text{ or } 2 \Rightarrow Y_h = C_1e^t + C_2e^{2t} \Rightarrow \boxed{y_h(x) = C_1x + C_2x^2}$ #5(A)

2. Find Y_p using the Method of Variation of Parameter.

$$w(x) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2\cdot e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t} = e^{3t}$$

let $Y_p = uY_1 + vY_2$;

$$u' = \frac{-e^{2t}\sin(e^{-t})}{e^{3t}} = -e^{-t}\cdot\sin(e^{-t})$$

$$v' = \frac{e^t\cdot\sin(e^{-t})}{e^{3t}} = e^{-2t}\cdot\sin(e^{-t})$$

$$u = \int -e^{-t}\cdot\sin(e^{-t})\,dt \quad \frac{u=e^{-t}}{du=-e^{-t}dt} \quad \int \sin(u)\,du = -\cos(e^{-t})$$

$$v = \int e^{-2t}\cdot\sin(e^{-t})\,dt \quad \frac{u=e^{-t}}{du=-e^{-t}dt} \quad -\int \underbrace{e^{-t}}_{u_1}\cdot\underbrace{\sin(u)\,du}_{dv_1}$$

$$\frac{du_1 = du}{v_1 = -\cos u} \quad -\left[-u\cdot\cos u + \int \cos u\,du\right] = u\cdot\cos u - \sin u = e^{-t}\cdot\cos(e^{-t}) - \sin(e^{-t})$$

$$\Rightarrow Y_p = \cancel{-\cos(e^{-t})\cdot e^t} + (\cancel{e^{-t}\cdot\cos(e^{-t})} - \sin(e^{-t}))\cdot e^{2t} = -e^{2t}\cdot\sin(e^{-t})$$

$\xrightarrow{t=\ln x} \boxed{y_p(x) = -x^2\cdot\sin\left(\frac{1}{x}\right)}$ #5(B)

The general solution is:

$\boxed{y = y_h + y_p = C_1x + C_2x^2 - x^2\cdot\sin\left(\frac{1}{x}\right)}$ #5(C)