Total: 100 points

1. (**30** points) Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 27 at $x = -2$ and a local minimum value of 0 at $x = 1$.

Solution:

$f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$

At $x = 1$, the function has a local minimum value of 0.

$f(1) = 0 \Rightarrow f(1) = a + b + c + d = 0 \ \cdots (1)$

$f'(1) = 0 \Rightarrow f'(1) = 3a + 2b + c = 0 \ \cdots (2).$

At $x = -2$, the function has a local maximum value of 27.

$f(-2) = 27 \Rightarrow f(-2) = -8a + 4b - 2c + d = 27 \ \cdots (3)$

$f'(-2) = 0 \Rightarrow f'(-2) = 12a - 4b + c = 0 \ \cdots (4).$

From $(2) \times 2 + (4) :\Rightarrow 18a + 3c = 0 \Rightarrow c = -6a \Rightarrow a = -\frac{1}{6}c$

From $(2) \times 4 - (4) :\Rightarrow 12b + 3c = 0 \Rightarrow c = -4b \Rightarrow b = -\frac{1}{4}c$

From $(2) - (1) :\Rightarrow 2a + b - d = 0 \Rightarrow d = 2a + b = -\frac{7}{12}c \Rightarrow d = -\frac{7}{12}c$

Therefore, in $(3) :\Rightarrow -8a + 4b - 2c + d = -\frac{27}{12}c = 27 \Rightarrow c = -12 \Rightarrow a = 2, b = 3, d = 7$

This cubic function is $f(x) = 2x^3 + 3x^2 - 12x + 7$

2. (**20** points) Show that the functions $f(x) = x^3 + \frac{4}{x^2} + 7$ has exactly one zero in the interval $(-\infty, 0)$.

Solution:

$f(x) = x^3 + \frac{4}{x^2} + 7 \Rightarrow f'(x) = 3x^2 - \frac{8}{x^3} > 0$ on $(-\infty, 0)$.

Therefore, $f(x)$ is increasing on $(-\infty, 0)$. Also, $f(x) < 0$ if $x < -2$ and $f(x) > 0$ if $-2 < x < 0$.

Thus, $f(x)$ has exactly one zero in $(-\infty, 0)$.

3. A function is given as $f(x) = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$.
   (a) (**12** points) Identify the coordinates of any local and absolute extreme points.
   (b) (**6** points) Specify on which interval(s), $f(x)$ is increasing, and on which interval(s), $f(x)$ is decreasing.
   (c) (**6** points) Identify the coordinates of inflection points.
   (d) (**6** points) Specify on which interval(s), $f(x)$ is concave up, and on which interval(s), $f(x)$ is concave down.

Solution: $f'(x) = \sqrt{3} + 2\sin x,\ f''(x) = 2\cos x$

(a) Critical points: $f'(x) = 0 \Rightarrow x = \dfrac{4\pi}{3}$ and $x = \dfrac{5\pi}{3}$. Endpoints: $x = 0$ and $x = 2\pi$

$f''\left(\dfrac{4\pi}{3}\right) = -1 < 0 \Rightarrow$ There is a local maximum at $\left(\dfrac{4\pi}{3}, \dfrac{4\sqrt{3}\pi}{3} + 1\right)$.

$f''\left(\dfrac{5\pi}{3}\right) = 1 > 0 \Rightarrow$ There is a local minimum at $\left(\dfrac{5\pi}{3}, \dfrac{5\sqrt{3}\pi}{3} - 1\right)$

At endpoint: $x = 0 \Rightarrow f(0) = -2 \Rightarrow (0, -2)$ local minimum and absolute minimum.

At endpoint: $x = 2\pi \Rightarrow f(2\pi) = 2\sqrt{3}\pi - 2 \Rightarrow \left(2\pi, 2\sqrt{3}\pi - 2\right)$ local maximum and absolute maximum

(b) Increasing: $0 < x < \dfrac{4\pi}{3}$ and $\dfrac{5\pi}{3} < x < 2\pi$. Decreasing: $\dfrac{4\pi}{3} < x < \dfrac{5\pi}{3}$.

(c) $f''(x) = 2\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$

$x < \dfrac{\pi}{2} \Rightarrow f''(x) > 0,\ x > \dfrac{\pi}{2} \Rightarrow f''(x) < 0$ Therefore, $\left(\dfrac{\pi}{2}, \dfrac{\sqrt{3}\pi}{2}\right)$ is an inflection point.

$x < \dfrac{3\pi}{2} \Rightarrow f''(x) < 0,\ x > \dfrac{3\pi}{2} \Rightarrow f''(x) > 0$ Therefore, $\left(\dfrac{3\pi}{2}, \dfrac{3\sqrt{3}\pi}{2}\right)$ is an inflection point.

(d) From the results of (c), one can find that:

$f(x)$ is concave up in $0 < x < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < x < 2\pi$.

$f(x)$ is concave down in $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.

4. (**20** points) Verify that the function $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = \frac{1}{x} \quad \text{interval : } [1, 3]$$

Solution:

$f(x) = \dfrac{1}{x}$ is continuous and differentiable on $(-\infty, 0) \cup (0, \infty)$.

Therefore, $f(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$.

Therefore, the mean value theorem can be satisfied.

Find all points $x = c$ satisfy mean value theorem: $f'(x) = -\dfrac{1}{x^2}$

$f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{1/3 - 1}{2} = -\dfrac{1}{3} \Rightarrow -\dfrac{1}{c^2} = -\dfrac{1}{3} \Rightarrow c = \pm\sqrt{3}$.

In $(1, 3)$, only $c = \sqrt{3}$ satisfies the mean value theorem.