Engineering mathematics II

Midterm exam., 4/9/2024

This is an open-book test. Moreover, access to the internet is allowed. Interaction with another person/human, however, is absolutely prohobited. The total score is 110 points. In your solution, you need to show your computations.

1. Consider the system of homogeneous linear equations

$$\begin{cases} 2x + (1-t)\cdot z = 0 \\ y + t\cdot z = 0 \\ t\cdot x + y + z = 0 \end{cases}$$

(a). (5%) If the system is to have nontrivial solutions, what is the value(s) of t ?

(b). (10%) Continued from the preceeding subproblem, find the nontrivial solutions.

2. Consider the matrix below:

b 3e

$$\begin{matrix} 7 & 0 & -2 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{matrix} \qquad \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix} \qquad 3 \times 2$$

(a). (5%) Solve the system of linear equations: $\underline{\underline{A}}\,\underline{x} = [-1\ 3\ 2]^{\mathrm{T}}$.

(b). (5%) Find the inverse of $\underline{\underline{A}}^2$.

(c). (10%) Please express $\underline{\underline{A}}$ as a product of some elementary matrices.

3.(10%) It is known that when two rows of a square matrix are swapped, then the determinant of the resultant matrix is equal to the negative of the original matrix's determinant. Based on this fact, show that if a matrix has two identical rows, then its determinant is equal to 0.
<Hint:> What does the resultant matrix look like, as compared to the original matrix, when the two identical rows are swapped?

4. Consider the matrix below:

$$\underline{\underline{A}} = \begin{bmatrix} -1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & -17 & 18 & 19 & 20 \\ -21 & 22 & 23 & 24 & -25 \end{bmatrix}$$

It is known that the determinant of $\underline{\underline{A}}$ is 17000.

(a). (5%) Let us express $\det(\underline{\underline{A}})$ as a sum of K products, where each product is a product of five elements in $\underline{\underline{A}}$. Then $K=$?

(b). (5%) For this SOP (sum-of-product) expression of $\det(\underline{\underline{A}})$, find the signs (+1 or −1) before the following products: (i). $3\cdot 7\cdot 15\cdot 19\cdot(-21)$, (ii). $4\cdot 8\cdot 12\cdot 16\cdot(-25)$.

(c). (5%) Find the $(4,2)$-th minor of $\underline{\underline{A}}$.

(d). (5%) Find the $(4,2)$-th cofactor of $\underline{\underline{A}}$.

(e). (5%) $\det(2 \cdot \underline{\underline{A}}) = ?$

(f). (5%) $\det(\underline{\underline{A}}^{-1}) = ?$

(g). (5%) Show that if $\underline{\underline{A}}$ is the augmented matrix of a system of linear equations, then this system of linear equations is inconsistent.

5. Consider the $\mathcal{R}^3$ vector space, with the standard vector addition and standard scalar multiplication. Consider the set of vectors: $B = \{(3,2,1),(1,-1,4),(2,3,5)\}$.

(a). (5%) Show that B is a linearly independent set (i.e. the vectors in B are linearly independent), by applying the definition of linear independence.

(b). (5%) Show that B can span $\mathcal{R}^3$.

(c). (5%) Is B a basis for $\mathcal{R}^3$?

(d). (5%) Given the values of x, y, and z, if we express (x,y,z) as: $(x,y,z) = \alpha \cdot (3,2,1) + \beta \cdot (1,-1,4) + \gamma \cdot (2,3,5)$. Then $\alpha = ?$

6. (10%) Consider the matrix below:

$$\underline{\underline{A}} = \begin{array}{c} \begin{matrix} C_1 & \overset{C_2}{k_1} & \overset{C_3}{k_2} & \overset{C_4}{k_3} & \overset{C_5}{k_4} \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix} \end{array}$$

$\det(A) = 0$

It is known that the determinant of $\underline{\underline{A}}$ is 0. Let us denote the five columns of $\underline{\underline{A}}$ as: $\underset{\sim}{c}_1$ (the left-most column), $\underset{\sim}{c}_2$, $\underset{\sim}{c}_3$, $\underset{\sim}{c}_4$, and $\underset{\sim}{c}_5$ (the right-most column), respectively. Please express $\underset{\sim}{c}_1$ as a linear combination of the other columns. More specifically speaking, you need to find the coefficients in this particular linear combination.