<table>
<tr><td>1.</td><td>
a.

$$\lim_{x\to 0}\frac{x-\tan^{-1}x}{\sin^{-1}x-x}=\lim_{x\to 0}\frac{1-\dfrac{1}{1+x^2}}{\dfrac{1}{\sqrt{1-x^2}}-1}=\lim_{x\to 0}\frac{(1+x^2)^{-2}(2x)}{(\frac{-1}{2})(1-x^2)^{-\frac{3}{2}}(-2x)}=2$$

b.

$$\lim_{n\to\infty}(\ln n)^{\frac{1}{n}}=\lim_{n\to\infty}e^{\frac{1}{n}\ln(\ln n)}=e^{\lim_{n\to\infty}\frac{\ln(\ln n)}{n}}\quad(\tfrac{\infty}{\infty})=e^{\lim_{n\to\infty}\frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1}}=e^{\frac{1}{\infty}}=e^0=1$$
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<tr><td>2.</td><td>
a.

令 $u=\ln x\,,dv=xdx\quad\Longrightarrow\quad du=\frac{1}{x}dx,v=\frac{1}{2}x^2$

原式 $=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\cdot\frac{1}{x}dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int xdx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c$

b.

$$\int\sin^3x\cos^2x\,d\sin x=\int\sin^3x\,(1-\sin^2x)d\sin x=\frac{\sin^4x}{4}-\frac{\sin^6x}{6}+c$$

c.

$$\int\frac{1}{(x^2+a^2)^{\frac{3}{2}}}dx=\int\frac{1}{(a^2\sec^2t)^{\frac{3}{2}}}a\sec^2t\,dt=\frac{\sin t}{a^2}+c=\frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}+c$$

d.

(1) $\frac{1}{x^4+x^2-2}=\frac{1}{(x^2-1)(x^2+2)}=\frac{1}{(x-1)(x+1)(x^2+2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+2}$

$A(\mathrm{x}+1)(x^2+2)+B(\mathrm{x}-1)(x^2+2)+(C\mathrm{x}+D)(x-1)(x+1)=1$

令 $x=1\quad\Rightarrow\quad 6A=1\quad\Rightarrow\quad A=\frac{1}{6}$

$x=-1\quad\Rightarrow\quad -6B=1\quad\Rightarrow\quad \mathrm{B}=-\frac{1}{6}$

比較 x^4 的係數可知 $C=0$

比較常數項的係數可知 $2A-2B-D=1\quad\Longrightarrow\quad \mathrm{D}=-\frac{1}{3}$

(2) $\therefore\frac{1}{x^4+2x^2-3}=\frac{\frac{1}{6}}{x-1}+\frac{-\frac{1}{6}}{x+1}+\frac{-\frac{1}{3}}{x^2+2}$

$$\Longrightarrow\int\frac{dx}{x^4+2x^2-3}dx=\frac{1}{6}\int\frac{1}{x-1}dx-\frac{1}{6}\int\frac{1}{x+1}dx-\frac{1}{3}\int\frac{1}{x^2+2}dx$$

$$=\frac{1}{6}\ln|x-1|-\frac{1}{6}\ln|x+1|-\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c$$
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<tr><td>3.</td><td>
a. Prove that $\Gamma(1)=1$

$$\Gamma(1)=\int_0^\infty t^{1-1}e^{-t}dt=\lim_{m\to\infty}\int_0^m t^0e^{-t}dt=\lim_{m\to\infty}(-e^{-t}|_0^m)=1\qquad \mathrm{Q.E.D}$$

b. Prove that $\Gamma(x+1)=x\Gamma(x)$

$$\Gamma(x+1)=\int_0^\infty t^xe^{-t}dt=\lim_{m\to\infty}\int_0^m t^xe^{-t}dt$$

$u=t^x,\quad dv=e^{-t}dt,\quad du=xt^{x-1}dt,\quad v=-e^{-t}$
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	$\lim\limits_{m\to\infty}\int_0^m t^x e^{-t}dt = \lim\limits_{m\to\infty}\left(-t^x e^{-t}\|_0^m + \int_0^m x t^{x-1}e^{-t}dt\right)$ $= -\lim\limits_{m\to\infty}\frac{m^x}{e^m} + x\lim\limits_{m\to\infty}\int_0^m t^{x-1}e^{-t}dt = -0 + x\Gamma(x) = x\Gamma(x) \qquad \mathrm{Q.E.D}$
4.	a. 由 $\int_1^\infty \frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^\infty = 1$，故 $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^2}$ 收斂, by Integral Test b. $\lim\limits_{n\to\infty} n\sin\frac{1}{n} = \lim\limits_{n\to\infty}\frac{\sin x}{x} = 1 \neq 0$，故發散, by nth − Term Test c. $\frac{1}{\ln n} > \frac{1}{n}$ ，已知 $\sum_{n=1}^{\infty}\frac{1}{n}$ 發散，故 $\sum_{n=1}^{\infty}\frac{1}{\ln n}$ 發散。 d. $\because \lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \lim\limits_{n\to\infty}\frac{1}{\sqrt{(n+1)}} = 0 < 1$，故收斂, by Ratio Test e. $\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1$，故收斂, by Root Test f. $\because \frac{2}{\ln(n+1)} < \frac{2}{\ln(n)}$ ，且 $\lim\limits_{n\to\infty}\frac{2}{\ln(n+1)} = 0$ 故由交錯級數審練法知原級數為收斂, by Leibniz's Test
5.	$\lim\limits_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \lim\limits_{n\to\infty}\left\|\frac{\frac{(x-3)^{n+1}}{(n+2)2^{n+1}}}{\frac{(x-3)^n}{(n+1)2^n}}\right\| = \frac{\|x-3\|}{2}\lim\limits_{n\to\infty}\frac{n+1}{n+2} = \frac{\|x-3\|}{2} < 1$ $\therefore \|x-3\| < 2 \longrightarrow 1 < x < 5$，現在討論端點如下: $(1) x = 5$ 時，$\sum\limits_{n=2}^{\infty}\frac{1}{n+1}$ 為發散 $\quad (2) x = 1$ 時，$\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n+1}$ 為收斂 故得收斂區間為 $1 \leq x < 5$
6.	$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots$ $1 - e^{-x} = \ x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + (-1)^{n+1}\frac{x^n}{n!} + \cdots$ $f(x) = \frac{1-e^{-x}}{x} = 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!}\cdots + (-1)^{n+1}\frac{x^{n-1}}{n!} + \cdots$

7.	$f(x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \cdots + \frac{2^n x^n}{n} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$ $\Rightarrow f'(x) = 2 + 4x + 8x^2 + \cdots + 2^n x^{n-1} + \cdots$ $= 2[1 + 2x + (2x)^2 + \cdots + (2x)^{n-1} + \cdots] = 2\left(\frac{1}{1-2x}\right)$ $\left(\textit{when } \ |2x| < 1 \ \Rightarrow |x| < \frac{1}{2} \ \Rightarrow x \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ $\therefore f(x) = \int 2\left(\frac{1}{1-2x}\right) dx = -\ln\|1-2x\| + c$ $f(0) = 0, \qquad \therefore \mathrm{c} = 0 \Rightarrow f(x) = -\ln\|1-2x\| = -\ln(1-2x), \forall x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$