112-1 Calculus Quiz 2 Solution

1. Use the Mean Value Theorem to find $\lim_{x\to\infty}\{\sin\sqrt{x+4} - sin\sqrt{x}\}$. (10 pts)

1.
Let $f(x) = \sin\sqrt{x}$, $\Rightarrow f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}} \Rightarrow f'(c) = \frac{\cos\sqrt{c}}{2\sqrt{c}}$

$\begin{cases} a = x \\ b = x+4 \end{cases}$ $\Rightarrow f'(c)(b-a) = f(b) - f(a)$

$x < c < x+4$

$$\Rightarrow \frac{\cos\sqrt{c}}{2\sqrt{c}}\left[(x+4)-(x)\right] = \sin\sqrt{x+4} - \sin\sqrt{x} \quad (x<c<x+4)$$

$$\lim_{x\to\infty}\left[\sin\sqrt{x+4} - \sin\sqrt{x}\right] = \lim_{x\to\infty}\frac{2\cos\sqrt{c}}{\sqrt{c}}$$

$$\Rightarrow \lim_{x\to\infty}\left[\sin\sqrt{x+4} - \sin\sqrt{x}\right] = \lim_{c\to\infty}\frac{2\cos\sqrt{c}}{\sqrt{c}}$$

$$\frac{-2}{\sqrt{c}} \le \frac{2\cos\sqrt{c}}{\sqrt{c}} \le \frac{2}{\sqrt{c}}$$

$$\Rightarrow \lim_{c\to\infty}\frac{-2}{\sqrt{c}} \le \lim_{c\to\infty}\frac{2\cos\sqrt{c}}{\sqrt{c}} \le \lim_{c\to\infty}\frac{2}{\sqrt{c}}$$

$$\Rightarrow 0 \le \lim_{c\to 0}\frac{2\cos\sqrt{c}}{\sqrt{c}} \le 0$$

$$\therefore \boxed{\lim_{c\to\infty}\frac{2\cos\sqrt{c}}{\sqrt{c}} = \lim_{x\to 0}\left[\sin\sqrt{x+4} - \sin\sqrt{x}\right] = 0} \quad ※$$

2. $f(x) = x^{\frac{4}{3}}|x-1|, x \in R$, Find the local two(relative) extrema and points of inflection of $f(x)$. (10 pts)

$$f(x) = x^{\frac{4}{3}}|x-1| = \begin{cases} x^{\frac{4}{3}}(x-1) & , x \ge 1 \\ x^{\frac{4}{3}}(1-x) & , x < 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \frac{7}{3}x^{\frac{4}{3}} - \frac{4}{3}x^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}}(7x-4), & x > 1 \\ \frac{4}{3}x^{\frac{1}{3}} - \frac{7}{3}x^{\frac{4}{3}} = \frac{1}{3}x^{\frac{1}{3}}(4-7x), & x < 1 \\ \text{不存在} & , x = 1 \end{cases}$$

$\hookrightarrow \because f(1^-) \ne f(1^+)$

$$f''(x) = \begin{cases} \frac{28}{9}x^{\frac{1}{3}} - \frac{4}{9}x^{\frac{-2}{3}} = \frac{4}{9}x^{-\frac{2}{3}}(7x-1) & , x > 1 \\ \frac{4}{9}x^{-\frac{2}{3}}(1-7x), & x < 1 \\ \text{不存在} & , x = 1 \end{cases}$$

① $f'(x) = 0 \Rightarrow x = 0, x = \frac{4}{7}$

$f'(x)$ 不存在 $\Rightarrow x = 1$

$f'(0^-) < 0$, $f'(0^+) > 0 \Rightarrow$ $x=0$ 處有相對極小值 $f(0)=0$

$f(\frac{4}{7}^-) > 0$, $f'(\frac{4}{7}^+) < 0 \Rightarrow$ $x=\frac{4}{7}$ 處有相對極大值 $f(\frac{4}{7}) = \frac{3}{7}(\frac{4}{7})^{\frac{4}{3}}$

$f'(1^-) < 0$, $f'(1^+) > 0 \Rightarrow$ $x=1$ 處有相對極小值 $f(1)=0$

※

② $f''(x) = 0 \Rightarrow x = \frac{1}{7}$, $f''(\frac{1}{7}^-) < 0$, $f''(\frac{1}{7}^+) > 0 \Rightarrow f''(\frac{1}{7}^-)\cdot f''(\frac{1}{7}^+) < 0$

$f''(x)$ 不存在 $\Rightarrow x = 1$, $f''(1^-) < 0$, $f''(1^+) > 0 \Rightarrow f''(1^-)f''(1) < 0$

反曲點：$(\frac{1}{7}, f(\frac{1}{7})) = (\frac{1}{7}, \frac{6}{7}(\frac{1}{7})^{\frac{4}{3}})$

$(1, f(1)) = (1, 0)$

※

3. Estimate the approximation of $\frac{\sqrt{4.02}}{2+\sqrt{9.02}}$ (approximate to at least four decimal place). (10 pts)

3.

Let $f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2}\frac{1}{\sqrt{x}}$，由 $f(x+h) \approx f(x)+f'(x)\cdot h$

① 取 $x=4$，$h=0.02 \Rightarrow \sqrt{4.02}=2+\frac{1}{2}(4)^{-\frac{1}{2}}\cdot(0.02)=2.005$

② 取 $x=9$，$h=0.02 \Rightarrow \sqrt{9.02}=3+\frac{1}{2}(9)^{-\frac{1}{2}}\cdot(0.02)=3.0033$

$\therefore \frac{\sqrt{4.02}}{2+\sqrt{9.02}} \approx \frac{2.005}{2+3.0033}=\boxed{0.4007}$

4. The radius of an inflating balloon A spherical balloon is inflated with helium at the rate of $100\pi\ \mathrm{ft}^3/\mathrm{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 ft ? How fast is the surface area increasing? (10 pts)

4.

$V=\frac{4}{3}\pi r^3$，$r=5$ and $\frac{dV}{dt}=100\pi\ \mathrm{ft}^3/\mathrm{min}$

then $\frac{dV}{dt}=4\pi r^2\cdot\left(\frac{dr}{dt}\right) \Rightarrow \frac{dr}{dt}=1\ \mathrm{ft}/\mathrm{min}$

Then $S=4\pi r^2 \Rightarrow \frac{dS}{dt}=8\pi r\left(\frac{dr}{dt}\right)=8\pi(5)(1)=40\pi\ \mathrm{ft}^2/\mathrm{min}$

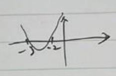
$\boxed{\therefore \frac{dr}{dt}=1\ \mathrm{ft}/\mathrm{min},\quad \frac{dS}{dt}=40\pi\ \mathrm{ft}^2/\mathrm{min}}$

5. Let $f(x)=\frac{1}{x(x+2)}$ solve $f(x)=-x$ by Newton's method (approximate to at least one decimal place). (10 pts)

5.

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

(1) $f(x)=\frac{1}{x(x+2)}$ 且 $f(x)=-x \Rightarrow \frac{1}{x(x+2)}=-x$



$\Rightarrow -x^2(x+2)=1 \Rightarrow x^3+2x^2+1=0 \Rightarrow g(x)$

$g'(x)=3x^2+4x$ 且 $g(-2)=1$，$g(-3)=-8$

(2). 由勘根定理可知 $\Rightarrow g(x)$ 至少一實根 $r\in(-3,-2)$

用 New-ton method，取起始近似根 $x_1=-2$

$x_2=x_1-\frac{g(x_1)}{g'(x_1)}=-2-\frac{x_1^3+2x_1^2+1}{3x_1^2+4x_1}=-2-\frac{1}{4}=-2.25$

$x_3=x_2-\frac{g(x_2)}{g'(x_2)}=-2.25-\frac{(-2.25)^3+2\cdot(2.25)^2+1}{3(-2.25)^2+4(-2.25)} \doteqdot -2.207$

$\boxed{\because |x_3-x_2|<0.1 \quad \therefore r \doteqdot -2.21 \text{ 為題解}}$

6. Find the following integrals (20 pts)

a. $\int \frac{dx}{(1-sin^2x)\sqrt{1+tanx}}$ (10 pts)

b. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}dr$ (10 pts)

6. (a).

$u = 1+\tan x \Rightarrow du = \sec^2 x\, dx \Rightarrow dx = (\cos^2 x) du$

$$\int \frac{dx}{(1-\sin^2 x)\cdot\sqrt{1+\tan x}} \Rightarrow \int \frac{du}{\sqrt{u}} \cdot \left(\frac{\cos^2 x}{\cos^2 x}\right) = 2\sqrt{u} + C$$

$$= \boxed{2\sqrt{1+\tan x} + C}$$ #

6. (b).

$$\int \frac{(2r-1)\cdot\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\, dr$$

① Let $3(2r-1)^2+6 = u \Rightarrow du = 6(2r-1)\times 2\, dr \Rightarrow$ $du = 12(2r-1)dr$

$$\int \frac{\cancel{(2r-1)}\cdot\cos\sqrt{u}}{\sqrt{u}} \cdot \frac{du}{12\cancel{(2r-1)}} = \frac{1}{12}\int \frac{\cos\sqrt{u}}{\sqrt{u}}\, du$$

② Let $t = \sqrt{u} \Rightarrow \frac{1}{12}\int \frac{\cos t}{t}\cdot(2t)\, dt = \int \frac{2}{12}\cos t\, dt$

$dt = \frac{1}{2\sqrt{u}} du$

$= \frac{1}{6}\int \cos t\, dt$

$\Rightarrow du = (2\sqrt{u})dt$

$\Rightarrow du = 2t\, dt$

$= \frac{\sin(t)}{6} + C$

$$= \boxed{\frac{\sin\left[\sqrt{3(2r-1)^2+6}\right]}{6} + C}$$ #

7. Find the area of the region enclosed by parabola $y = -x^2 + 4x - 3$ and its two tangents at the points (0,-3) and (4,-3). (10 pts)

7.

$\frac{dy}{dx} = -2x+4 = m$

$\left.\frac{dy}{dx}\right|_{(x=0,y=-3)} = m_1 = 4$

$\left.\frac{dy}{dx}\right|_{(x=4,y=-3)} = m_2 = (-2)\times 4 + 4 = -4$

tangent line: $\begin{cases} y-(-3) = 4(x-0) \Rightarrow y = 4x-3 \\ y-(-3) = -4(x-4) \Rightarrow y = -4x+13 \end{cases} \rightarrow$ intersection: $(2,5)$

$\int_0^2 [(4x-3)-(-x^2+4x-3)]\,dx + \int_2^4 [(-4x+13)-(-x^2+4x-3)]\,dx$

$= \int_0^2 (x^2)\,dx + \int_2^4 (x^2-8x+16)\,dx = \frac{1}{3}x^3\Big|_0^2 + \left[\frac{1}{3}x^3 - 4x^2 + 16x\right]\Big|_2^4$

$= \frac{8}{3} + \left[\frac{64}{3} - 64 + 64 - \left(\frac{8}{3} - 16 + 32\right)\right]$

$= \frac{8}{3} + \frac{8}{3} = \boxed{\frac{16}{3}}$ ※

8. What values of a and b maximize the value of $\int_a^b (x - x^2)dx$? Explain your answer. (10 pts)

8.

To find where $x - x^2 \geq 0$, let $x - x^2 = 0 \Rightarrow x(1-x) = 0$
$\Rightarrow x = 0$ or $x = 1$

If $0 < x < 1$, then $0 < x - x^2 \Rightarrow a = 0, b = 1$ maximize the integral

$\boxed{\int_0^1 (x-x^2)\,dx \text{ has the max value. } (a=0, b=1)}$ ※

9. Find the linearization of $f(x) = 2 - \int_2^{x+1} \frac{9}{1+t} dt$ at $x = 1$. (10 pts)

9.

$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt$

$\Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2}$

$\Rightarrow f'(1) = -3$

$f(1) = 2 - \int_2^{(1)+1} \frac{9}{1+t}\,dt = 2 - 0 = 2$

$L(x) = -3(x-1) + f(1) = -3(x-1) + 2 = -3x+5$

$\boxed{L(x) = -3x+5}$ ※