國立臺灣科技大學答案卷

National Taiwan University of Science and Technology Answer Sheet

評分 Score	教師簽章 Signature of Lecturer

姓名/Name ______ 學號/Student ID ______ 班級/Class ______

科目/Course title Calculus Ⅱ 日期/Date 2021/3/29

記分欄 從此處開始寫起。試卷用紙務須節用，非經主試認可不得續用其他紙張作答。/Please write from here.

1. Show that $\lim\limits_{k\to\infty}(1+\frac{r}{k})^k = e^r$

Let $f(k) = (1+\frac{r}{k})^k$

$\ln f(k) = k\ln(1+\frac{r}{k})$

$\because \lim\limits_{k\to\infty} f(k) = \lim\limits_{k\to\infty} e^{\ln f(k)}$

$\therefore$ We calculate $\lim\limits_{k\to\infty}\ln f(k)$ first.

$\lim\limits_{k\to\infty} k\ln(1+\frac{r}{k})$

$= \lim\limits_{k\to\infty} \dfrac{\ln(1+\frac{r}{k})}{k^{-1}}$

$= \lim\limits_{k\to\infty} \dfrac{\frac{1}{1+\frac{r}{k}}(-r\cdot k^{-2})}{-k^{-2}} = \lim\limits_{k\to\infty} \dfrac{r}{1+\frac{r}{k}} = r$

$\Rightarrow \lim\limits_{k\to\infty} f(k) = \lim\limits_{k\to\infty} e^{\ln f(k)} = e^r$ #

2. $\lim\limits_{x\to0^+} x^2\ln x$

$= \lim\limits_{x\to0^+} \dfrac{\ln x}{x^{-2}}$

$= \lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{-2x^{-3}}$

$= \lim\limits_{x\to0^+} \dfrac{x^2}{-2} = 0$

3. $\lim\limits_{x\to0^+} \dfrac{\sin^{-1}(x^2)}{(\sin^{-1}x)^2}$

$\overset{L'}{=} \lim\limits_{x\to0^+} \dfrac{\frac{1}{\sqrt{1-(x^2)^2}}\cdot\frac{d}{dx}x^2}{2\sin^{-1}x\cdot\frac{d}{dx}\sin^{-1}x}$

$\overset{L'}{=} \lim\limits_{x\to0^+} \dfrac{\frac{1}{\sqrt{1-x^4}}\cdot 2x}{2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}}$

$\overset{L'}{=} \lim\limits_{x\to0^+} \dfrac{x}{\sin^{-1}x\sqrt{1+x^2}}$

$\overset{L'}{=} \lim\limits_{x\to0^+} \dfrac{1}{\frac{1}{\sqrt{1-x^2}}\sqrt{1+x^2}+\sin^{-1}x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x}$

$= 1$ #

4. Show that $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

Let $y = \sinh^{-1}x$

$\sinh y = x = \dfrac{e^y-e^{-y}}{2}$

$e^y - e^{-y} = 2x$

$e^y - 2x - e^{-y} = 0$

$e^{2y} - 2xe^y - 1 = 0$

Let $A = e^y$

$A^2 - 2Ax - 1 = 0$

$\Rightarrow A = \dfrac{2x\pm\sqrt{(-2x)^2-4\times1\times(-1)}}{2\times1}$

$= \dfrac{2x\pm\sqrt{4x^2+4}}{2}$

$= x\pm\sqrt{x^2+1}$

$\because A = e^y > 0 \quad \therefore e^y = A = x+\sqrt{x^2+1}$

i.e. $y = \ln(x+\sqrt{x^2+1})$

5. $\ln x$ grows slower than $x^{\frac{1}{n}}$ $\forall n\in\mathbb{Z}^+$ as $x\to\infty$.

$\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{\frac{1}{n}}} = \lim\limits_{x\to\infty} n\dfrac{1}{x^{\frac{1}{n}}}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{n}x^{\frac{1}{n}-1}} = 0$

$\Rightarrow \ln x = o(x^{\frac{1}{n}})$

6. $O(n^3)$, $O(n\log n)$ $O(n\log^2 n)$

$O(n^3) > O(n\log^2 n) > O(n\log n)$

Three time complexity $= f(n), g(n), h(n)$

$f(n) = O(n^3) \Rightarrow f(n) \le M_1n^3 \ \forall n>n_0, \exists M_1>0$

$g(n) = O(n\log^2 n) \Rightarrow g(n) \le M_2 n\log^2 n, \ \forall n>n_1, \exists M_2>0$

$h(n) = O(n\log n) \Rightarrow h(n) \le M_3 n\log n, \ \forall n>n_2, \exists M_3>0$

When $\forall n>\max(n_1,n_2,n_3)$, n is extremely large here

$M_1n^3 \ge f(n) \ge M_2n\log^2 n \ge g(n) \ge M_3 n\log n \ge h(n)$

i.e. $O(n^3) \ge O(n\log^2 n) \ge O(n\log n)$

7. $\int \log_2 x\,dx$

Let $u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$= \dfrac{1}{\ln 2}\int \ln x\,dx$

$= \dfrac{1}{\ln 2}\left(x\ln x - \int x\cdot\frac{1}{x}dx\right) + C$

$= \dfrac{1}{\ln 2}(x\ln x - x) + C$ #

可轉頁再寫。

8、sol1:

$$\int \cos^2(2x)\sin x\,dx$$

$$= \int \frac{1+\cos(4x)}{2} \sin x\,dx$$

$\cos(4x)\sin x \qquad \sin(4x \pm x) = \sin 4x \cos x \pm \cos 4x \sin x$

$= \frac{1}{2}\left[\sin 5x - \sin 3x\right]$

$$= \frac{1}{2}\int \left(\sin x + \cos(4x)\sin x\right) dx$$

$$= \frac{1}{2}\left[-\cos x + \int \frac{1}{2}(\sin 5x - \sin 3x)\,dx\right]$$

$$= \frac{1}{2}\left[-\cos x + \frac{1}{2}\left(\frac{-1}{5}\cos(5x) + \frac{1}{3}\cos(3x)\right)\right] + C$$

$$= \frac{-1}{2}\cos x - \frac{1}{20}\cos(5x) + \frac{1}{12}\cos(3x) + C \;\#$$

8、sol2:

Let $u = \cos x \quad du = -\sin x\,dx$

$$\int \cos^2(2x)\sin x\,dx$$

$$= \int (2\cos^2 x - 1)^2 \sin x\,dx$$

$$= \int (4\cos^4 x - 4\cos^2 x + 1)\sin x\,dx$$

$$= -\int (4u^4 - 4u^2 + 1)\,du$$

$$= -\frac{4}{5}u^5 + \frac{4}{3}u^3 - u + C$$

$$= -\frac{4}{5}\cos^5 x + \frac{4}{3}\cos^3 x - \cos x + C \;\#$$

9. $\int \cos(3x)\cos(4x)\,dx$

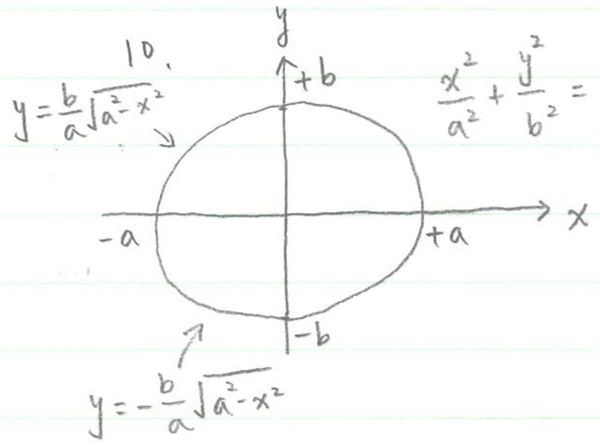
$$= \frac{1}{2}\int \left[\cos(3+4)x + \cos(3-4)x\right] dx$$

$$= \frac{1}{2}\int (\cos 7x + \cos x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{7}\sin 7x + \sin x\right) + C$$

$$= \frac{1}{14}\sin 7x + \frac{1}{2}\sin x + C \;\#$$

10.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$$= \pm\frac{b}{a}\sqrt{a^2 - x^2}, \quad a^2 \geq x^2$$

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

Let $x = a\sin\theta \qquad x = 0 \Rightarrow \theta = 0$

$dx = a\cos\theta\,d\theta \qquad x = a \Rightarrow \theta = \frac{\pi}{2}$

$$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\,(a\cos\theta\,d\theta)$$

$\Rightarrow a^2 - x^2 = a^2\cos^2\theta$

$\Rightarrow \sqrt{a^2 - x^2} = a|\cos\theta|$

$= a\cos\theta \quad (\because 0 < \theta < \frac{\pi}{2})$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 4ab \cdot \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 2ab\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \pi ab \;\#$$