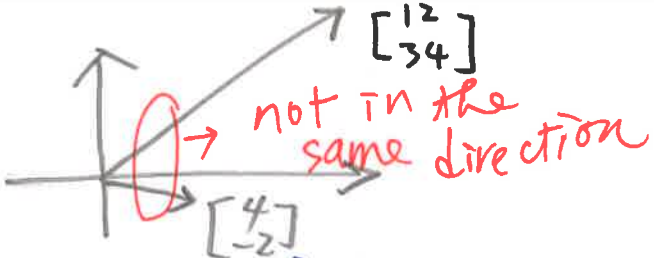
• **Ex** Consider $\underline{\underline{A}} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$



• Let us try $\underline{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. $\Rightarrow$ Test: $\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 34 \end{bmatrix} \neq @ \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ some number

$\Leftarrow$ $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is not an eigenvector.

• Let us try $\underline{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow$

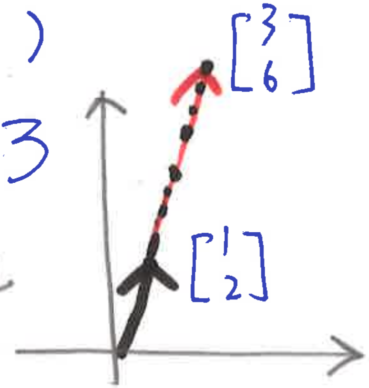
some number

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \end{bmatrix} \neq @ \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ is not an eigenvector.}$$

• Let us try $\underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \underset{\text{some number}}{3} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector (→ e.vector) of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$,

with respect to (corresponding to, associated with) eigenvalue 3
↓ e.value



•**Q**: Are there some systematic methods for finding e.values/e.vectors?

Ex

P.095_1

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

- Find e.values:

$$\det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{vmatrix} = \overbrace{\lambda^3 - 8\lambda^2 + 17\lambda - 4}^{\text{char. poly}} = 0$$

char. equation

$$= (\lambda-4)(\lambda^2-4\lambda+1)$$

$$\lambda^2-4\lambda+1 = (\lambda-(2+\sqrt{3}))(\lambda-(2-\sqrt{3}))$$

$$\overset{\heartsuit}{=} 0$$

$$\Rightarrow \lambda = \underset{\lambda_1}{4},\ \underset{\lambda_2}{2+\sqrt{3}},\ \underset{\lambda_3}{2-\sqrt{3}} \ : \text{e.values}$$

- Find e.vectors:

① For $\lambda = \lambda_1 = 4$, we try to solve $\underline{\underline{A}}\,\underline{x} = \lambda_1 \underline{x} = 4 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Leftrightarrow (\lambda \underline{\underline{I}} - \underline{\underline{A}})\underline{x} = \underline{0}$

$$\cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1/16 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

rref

$$\equiv \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ -4 & 17 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\#1)$$

$\lambda \leftarrow 4$

$\therefore$ The solution to (#1) is $x_1 = \frac{1}{16} x_3$, $x_2 = \frac{1}{4} x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} \xrightarrow{\text{Let } x_3 \overset{\diamond}{=} 16t} = t \cdot \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}$$

② For $\lambda = \lambda_2 = 2+\sqrt{3}$,

$$\left[\begin{array}{ccc} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{3} & -1 & 0 \\ 0 & 2+\sqrt{3} & -1 \\ -4 & 17 & -6+\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda \leftarrow (2+\sqrt{3})$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/(7+4\sqrt{3}) \\ 0 & 1 & -1/(2+\sqrt{3}) \\ 0 & 0 & 0 \end{bmatrix}$$

③ For $\lambda = \lambda_3 = 2-\sqrt{3}$,

$$\left[\begin{array}{ccc} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2-\sqrt{3} & -1 & 0 \\ 0 & 2-\sqrt{3} & -1 \\ -4 & 17 & -6-\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda \leftarrow (2-\sqrt{3})$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/(7-4\sqrt{3}) \\ 0 & 1 & -1/(2-\sqrt{3}) \\ 0 & 0 & 0 \end{bmatrix}$$

- **Ex:**

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 0 & 11 \\ 4 & 3 & -7 \end{bmatrix}$$

$$V = \begin{matrix} \boxed{\begin{matrix} -0.1783 \\ 0.7602 \\ -0.6247 \end{matrix}} \begin{matrix} + 0.0000i \\ + 0.0000i \\ + 0.0000i \end{matrix} & \boxed{\begin{matrix} 0.2805 - 0.2471i \\ -0.9033 + 0.0000i \\ -0.1951 - 0.0792i \end{matrix}} & \boxed{\begin{matrix} 0.2805 + 0.2471i \\ -0.9033 + 0.0000i \\ -0.1951 + 0.0792i \end{matrix}} \end{matrix}$$

$$\begin{matrix} & \lambda_1 & \lambda_2 & \lambda_3 \end{matrix}$$

$$D = \begin{matrix} \boxed{-9.5087} + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & \boxed{1.7544 + 1.5119i} & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & \boxed{1.7544 - 1.5119i} \end{matrix}$$

· **Ex** :

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

(see P.095_1)

$$E_4(\underline{\underline{A}}) = \{\underline{x} \mid \underline{\underline{A}}\,\underline{x} = 4\underline{x}\} = \left\{ t \cdot \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$$E_{2+\sqrt{3}}(\underline{\underline{A}}) = \{\underline{x} \mid \underline{\underline{A}}\,\underline{x} = (2+\sqrt{3})\,\underline{x}\} = \left\{ t \cdot \begin{bmatrix} 1/(7+4\sqrt{3}) \\ 1/(2+\sqrt{3}) \\ 1 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$$E_{2-\sqrt{3}}(\underline{\underline{A}}) = \{\underline{x} \mid \underline{\underline{A}}\,\underline{x} = (2-\sqrt{3})\underline{x}\} = \left\{ t \cdot \begin{bmatrix} 1/(7-4\sqrt{3}) \\ 1/(2-\sqrt{3}) \\ 1 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

· **Thm** $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \longrightarrow \underline{\underline{A}}(c\underline{x}) = \lambda \cdot (c\underline{x})$

Prf $\underline{\underline{A}} \cdot (c\underline{x}) = c \cdot (\underline{\underline{A}}\,\underline{x}) = c \cdot \lambda \underline{x} = \lambda \cdot (c\underline{x})$ ※

Ex.

※ Any nonzero multiple of an e.vector is still an e.vector, with respect to the same e.value.

- **Thm.** $E_\lambda(\underline{\underline{A}})$ is a subspace of $\mathbb{R}^{n\times 1}$.

Prf: (Just apply the subspace test)

- Let $\underline{x}_1, \underline{x}_2 \in E_\lambda(\underline{\underline{A}})$. Then, $\underline{\underline{A}}\,\underline{x}_1 = \lambda \underline{x}_1$, $\underline{\underline{A}}\,\underline{x}_2 = \lambda \underline{x}_2$.
- Next, let us check/test: $a\underline{x}_1 + b\underline{x}_2 \overset{?}{\in} E_\lambda(\underline{\underline{A}})$.

$\circ \leftarrow$ Q.♣

- $\underline{\underline{A}}(\underbrace{a\underline{x}_1 + b\underline{x}_2}_{\underline{z}}) = a\underline{\underline{A}}\,\underline{x}_1 + b\underline{\underline{A}}\,\underline{x}_2 = a\cdot\lambda\underline{x}_1 + b\cdot\lambda\underline{x}_2 = \lambda(\underbrace{a\underline{x}_1 + b\underline{x}_2}_{\underline{z}})$

We have $\underline{\underline{A}}\,\underline{z} = \lambda\underline{z}$

$\therefore$ The answer to Q.♣ is "Yes". $\longrightarrow$ $E_\lambda(\underline{\underline{A}})$ is a subspace.

- **Thm** $\underline{\underline{A}}$ is invertible $\longleftrightarrow$ $\lambda = 0$ is not an e.value

Prf: "$\longleftarrow$" See PP.97–98

"$\longrightarrow$": $\underline{\underline{A}}$ is invertible $\Longrightarrow$ $\underbrace{|\underline{\underline{A}}|}_{\parallel} \neq 0$

$|\underline{\underline{A}} - 0\cdot\underline{\underline{I}}| \neq 0 \Longrightarrow 0$ is not an e.value.

• Ex Let us try to diagonalize $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

$\lambda_1 = 4, \quad \lambda_2 = 2+\sqrt{3}, \quad \lambda_3 = 2-\sqrt{3}$

$\underline{P}_1 \overset{\diamond}{=} \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}, \quad \underline{P}_2 \overset{\diamond}{=} \begin{bmatrix} 1/(7+4\sqrt{3}) \\ 1/(2+\sqrt{3}) \\ 1 \end{bmatrix}, \quad \underline{P}_3 \overset{\diamond}{=} \begin{bmatrix} 1/(7-4\sqrt{3}) \\ 1/(2-\sqrt{3}) \\ 1 \end{bmatrix}$

$$\underline{\underline{P}} = \begin{bmatrix} \underline{P}_1 & \underline{P}_2 & \underline{P}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1/(7+4\sqrt{3}) & 1/(7-4\sqrt{3}) \\ 4 & 1/(2+\sqrt{3}) & 1/(2-\sqrt{3}) \\ 16 & 1 & 1 \end{bmatrix}$$

$$\underline{\underline{P}}^{-1} = \begin{bmatrix} -(-7+4\sqrt{3})(7+4\sqrt{3}) & 4(-2+\sqrt{3})(2+\sqrt{3}) & 1 \\ \frac{2}{3}(-7+4\sqrt{3})(7+4\sqrt{3})\sqrt{3}(2+\sqrt{3})^2 & \frac{-1}{6}(2+\sqrt{3})(-2+\sqrt{3})(7+4\sqrt{3})\sqrt{3}(9+4\sqrt{3}) & \frac{-1}{6}\sqrt{3}(2+\sqrt{3})(7+4\sqrt{3}) \\ -\frac{2}{3}(-7+4\sqrt{3})(7+4\sqrt{3})\sqrt{3}(-2+\sqrt{3})^2 & \frac{1}{6}(2+\sqrt{3})(-7+4\sqrt{3})(-2+\sqrt{3})\sqrt{3}(-9+4\sqrt{3}) & \frac{1}{6}\sqrt{3}(-2+\sqrt{3})(-7+4\sqrt{3}) \end{bmatrix}$$

Then, you can verify that

$$\underline{\underline{P}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{P}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix} \quad ※$$

Ex

A =

$$\begin{bmatrix} 1 & 7 & -3 \\ 2 & 0 & -1 \\ 4 & 13 & -7 \end{bmatrix}$$

[P,D] = eig(A)

P =

$$\begin{bmatrix} \boxed{\begin{matrix} 0.5008 \\ 0.3177 \\ 0.8052 \end{matrix}} & \boxed{\begin{matrix} -0.2033 \\ -0.3205 \\ -0.9252 \end{matrix}} & \boxed{\begin{matrix} 0.4472 \\ -0.0000 \\ 0.8944 \end{matrix}} \end{bmatrix}$$

D =

$$\begin{matrix} 0.6180 & 0 & 0 \\ 0 & -1.6180 & 0 \\ 0 & 0 & -5.0000 \end{matrix}$$

P_inv = inv(A)

$$\begin{bmatrix} 2.8150 & 2.2773 & -1.4075 \\ 2.7911 & -0.8625 & -1.3955 \\ 0.3531 & -2.9422 & 0.9415 \end{bmatrix}$$

B = P_inv * A * P

$$\begin{bmatrix} 0.6180 & 0.0000 & -0.0000 \\ 0.0000 & -1.6180 & 0.0000 \\ -0.0000 & -0.0000 & -5.0000 \end{bmatrix}$$

= D

Ex

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|\lambda \underline{\underline{I}} - \underline{\underline{A}}| = (\lambda-1)(\lambda-2)^2 \overset{\text{set}}{=} 0$$

$\Rightarrow \lambda = 1, 2, 2$

- wrt $\lambda = 1$: $(1 \cdot \underline{\underline{I}} - \underline{\underline{A}}) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is one solution.

- wrt $\lambda = 2$: $(2 \cdot \underline{\underline{I}} - \underline{\underline{A}}) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are l.i. solutions.

- $\underline{\underline{P}} \triangleq \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{P}}^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

- Check: $\underline{\underline{P}}^{-1} \underline{\underline{A}} \, \underline{\underline{P}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Alternative choice: $\underline{\underline{Q}} \triangleq \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$,

then $\underline{\underline{Q}}^{-1} \underline{\underline{A}} \, \underline{\underline{Q}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

· **Ex** $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ $|\lambda \underline{\underline{I}} - \underline{\underline{A}}| = 0 \Rightarrow \lambda = 1, 2, 2$

· Wrt $\lambda = 1 \Rightarrow \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}$ is one solution for $\underline{\underline{A}}\,\underline{x} = 1 \cdot \underline{x}$ ~ ($1)

· Wrt $\lambda = 2 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is one solution for $\underline{\underline{A}}\,\underline{x} = 2 \cdot \underline{x}$ ~ ($2)

And we do NOT have another solution for ($2) that is l.i. with $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore$ We do NOT have enough l.i. e.vectors to construct $\underline{\underline{P}}$ to diagonalize $\underline{\underline{A}}$.

$\Rightarrow \underline{\underline{A}}$ is not diagonalizable.

- **Thm** Eigenvectors corresponding to distinct eigenvalues are l.i.

Prf

- We start with 2 e.vectors $\underline{v}_1$ and $\underline{v}_2$ that correspond to two distinct e.values λ_1 and λ_2.
- $\underline{\underline{A}}\,\underline{x}_1 = \lambda_1 \underline{x}_1$, $\quad \underline{\underline{A}}\,\underline{x}_2 = \lambda_2 \underline{x}_2$ $\quad$ (To show l.i., our goal is to show $k_1 = k_2 = 0$)
- Consider $k_1 \underline{v}_1 + k_2 \underline{v}_2 = \underline{0}$ — (#)
- $$\underline{\underline{A}} \cdot (\#) = \underline{\underline{A}}(k_1 \underline{v}_1 + k_2 \underline{v}_2) = k_1 \underline{\underline{A}}\,\underline{v}_1 + k_2 \underline{\underline{A}}\,\underline{v}_2$$
$$= k_1 \lambda_1 \underline{v}_1 + k_2 \lambda_2 \underline{v}_2 = \underline{\underline{A}}\,\underline{0} = \underline{0} \quad \text{— (\$)}$$
- $\lambda_2 \cdot (\#) - (\$)$: $k_1 \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \underbrace{\underline{v}_1}_{\neq \underline{0}} = \underline{0} \longrightarrow \mathbf{k_1 = 0}$

  $\lambda_1 \neq \lambda_2$

- $\lambda_1 \cdot (\#) - (\$)$: $k_2 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{\underline{v}_2}_{\neq \underline{0}} = \underline{0} \longrightarrow \mathbf{k_2 = 0}$

· Next, let us handle the case of 3 e.vectors $\underline{V}_1, \underline{V}_2, \underline{V}_3$ **P.101_2**

corresponding to 3 distinct e.values $\lambda_1, \lambda_2, \lambda_3$:

(To show l.i, our goal is to show $k_1 = k_2 = k_3 = 0$)

· Consider $k_1 \underline{V}_1 + k_2 \underline{V}_2 + k_3 \underline{V}_3 = \underline{0}$ —— (#)'

· $A \cdot (\#)'$: $\underline{\underline{A}}(k_1 \underline{V}_1 + k_2 \underline{V}_2 + k_3 \underline{V}_3) = k_1 \underline{\underline{A}} \underline{V}_1 + k_2 \underline{\underline{A}} \underline{V}_2 + k_3 \underline{\underline{A}} \underline{V}_3$

$= k_1 \lambda_1 \underline{V}_1 + k_2 \lambda_2 \underline{V}_2 + k_3 \lambda_3 \underline{V}_3 = \underline{\underline{A}} \, \underline{0} = \underline{0}$ —— ($)'

· $\lambda_1 \cdot (\#)' - (\$)'$: $k_2(\lambda_1 - \lambda_2)\underline{V}_2 + k_3(\lambda_1 - \lambda_3)\underline{V}_3 = \underline{0}$ —— (%)

By applying the ~~stat~~ "2 e.vectors wrt 2 distinct e.values",

we have already know that $\underline{V}_2$ and $\underline{V}_3$ are l.i.

· $\left\{\begin{array}{l} \underline{V}_2, \underline{V}_3 : \text{l.i.} \\ (\%) \end{array}\right. \Rightarrow k_2\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} = 0$ and $k_3\underbrace{(\lambda_1 - \lambda_3)}_{\neq 0} = 0$

$\Rightarrow k_2 = 0$ and $k_3 = 0$

· Similarly, it can be shown that $k_1 = 0$.

-※: Then, it can be generalized to "more than 3" cases.