Second Midterm—Chapter 2

Total points: 100 points 2 hours to do the work, Nov. 29, 2018

1. Consider the 2nd order differential equation $xy'' + 2y' + xy = 0$

(A) Reduce the 2nd order differential equation to a 1st order differential equation if one solution of the above 2nd order differential equation is $y_1 = \frac{cosx}{x}$.

(5%)

$y_1 = x^{-1}cosx \rightarrow y_1' = -x^{-2}cosx - x^{-1}sinx$

$\rightarrow y_1'' = 2x^{-3}cosx + x^{-2}sinx + x^{-2}sinx - x^{-1}cosx$

Let $y_2 = uy_1 \rightarrow y_2' = u'y_1 + uy_1'$

$\rightarrow y_2'' = u''y_1 + 2u'y_1' + uy_1''$ → 代入化簡錯誤+3

原式 $\rightarrow x(u''y_1 + 2u'y_1' + uy_1'') + 2(u'y_1 + uy_1') + x(uy_1) = 0$

$\rightarrow x(u''x^{-1}cosx + 2u'(-x^{-2}cosx - x^{-1}sinx) + u(2x^{-3}cosx + 2x^{-2}sinx - x^{-1}cosx)) + 2(u'x^{-1}cosx + u-x^{-2}cosx - x^{-1}sinx) + x(ux^{-1}cosx) = 0$

$\rightarrow u''cosx - 2u'sinx = 0$ → +4

Let $U = u'$, 原式 $\rightarrow U'cosx - 2Usinx = 0$ → +5

(B) Solve the 1st order differential equation in (A). (4%)

$U'cosx - 2Usinx = 0 \rightarrow U' - 2Utanx = 0 \rightarrow \frac{dU}{U} = 2tanxdx$ → +2

$ln|U| = -2\,ln|cosx| + c$ → +3

$U = cos^{-2}x \cdot c = c \cdot sec^2x$ → +4

(C) Find the second solution for the 2nd order differential equation. (3%)

$U = u' \rightarrow \mathrm{u} = \int U\,dx = \int c \cdot sec^2x = ctanx + \tilde{c}$

$y_2 = uy_1 \rightarrow (ctanx + \tilde{c}) \cdot \frac{cosx}{x} = \frac{csinx}{x} + \frac{\tilde{c}cosx}{x}$ → +2

$\because y_1$與y_2為線性獨立

$\therefore y_2 = \frac{sinx}{x}$ → +3

(D) Solve the 2nd order differential equation with $y(\frac{\pi}{2}) = \frac{2}{\pi}$ and $y'(\frac{\pi}{2}) = 0$.

(3%)

解出c_1、c_2 → +2

$$y(x) = c_1\frac{cosx}{x} + c_2\frac{sinx}{x}$$

$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi} = c_2 \cdot \frac{2}{\pi} \rightarrow c_2 = 1$, $y'\left(\frac{\pi}{2}\right) = 0 = c_1\left(-\frac{2}{\pi}\right) - (\frac{4}{\pi^2}) \rightarrow c_1 = -\frac{2}{\pi}$

$\therefore y(x) = \frac{-2cosx}{\pi x} + \frac{sinx}{x}$ → +3

2. Consider the 2nd order differential equation $y'' - 2y' + y = e^x + x$

(A) Find the homogeneous solution $y_h(x)$. (3%)

Let $y = e^{\lambda x}$ → $y' = \lambda e^{\lambda x}$ → $y'' = \lambda^2 e^{\lambda x}$

原式 → $\lambda^2 e^{\lambda x} - 2(\lambda e^{\lambda x}) + (e^{\lambda x}) = 0$

→$e^{\lambda x}(\lambda^2 - 2\lambda + 1) = 0$ → λ=1, 1（重根） → +2

$\therefore\ y_h(x) = c_1 e^x + c_2 x e^x$ → +3

(B) Find the particular solution $y_p(x)$ using **the Method of Undetermined Coefficients**. (10%)

$y_p(x) = y_{p1} + y_{p2}$

$\begin{cases} y_{p1} = Ae^x \cdot x^2 \\ y_{p2} = Bx + C \end{cases}$ → +4

$y_{p1} = Ae^x \cdot x^2$ → $y'_{p1} = Ae^x \cdot x^2 + Ae^x \cdot 2x$

→ $y''_{p1} = Ae^x \cdot x^2 + Ae^x \cdot 2x + Ae^x \cdot 2x + Ae^x \cdot 2$

原式 → $(Ae^x \cdot x^2 + Ae^x \cdot 4x + Ae^x \cdot 2) - 2(Ae^x \cdot x^2 + Ae^x \cdot 2x) +$
$(Ae^x \cdot x^2) = e^x$

→ $2Ae^x = e^x$ → $A = \frac{1}{2}$ →$y_{p1} = \frac{1}{2}e^x x^2$ → +7

$y_{p2} = Bx + C$ → $y'_{p2} = B$

→ $y''_{p2} = 0$

原式 → $-2B + Bx + C = x$ → B=1 , C=2 → $y_{p2} = x + 2$ → +9

$\therefore\ y_p(x) = y_{p1} + y_{p2} = \frac{1}{2}e^x x^2 + x + 2$ → +10

(C) Find the general solution. (2%)

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 x e^x + \frac{1}{2}e^x x^2 + x + 2$$

3. Consider the 2nd order differential equation $x^2y'' - 3xy' + 3y = xlnx$

(A) Find the homogeneous solution $y_h(x)$. (3%)

Let $y = x^m \rightarrow y' = mx^{m-1} \rightarrow y'' = m(m-1)x^{m-2}$

原式 $\rightarrow m(m-1) - 3m + 3 = 0$

$\rightarrow m^2 - 4m + 3 = 0 \rightarrow m = 1\,,3 \rightarrow +2$

$\therefore y_h(x) = c_1x + c_2x^3 \rightarrow +3$

(B) Find the particular solution $y_p(x)$ using **the Method of Variation of Parameters**. (10%)

$x^2y'' - 3xy' + 3y = xlnx \rightarrow y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \frac{lnx}{x}$

$y_1 = x\,, \quad y_2 = x^3$

$\mathrm{w} = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3 \rightarrow +2$

$w_1 = \begin{vmatrix} 0 & x^3 \\ 1 & 3x^2 \end{vmatrix} = -x^3 \rightarrow +3$

$w_2 = \begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix} = x \rightarrow +4$

$\therefore y_p(x) = y_1 \int \frac{w_1}{\mathrm{w}}\left(\frac{lnx}{x}\right)dx + y_2 \int \frac{w_2}{\mathrm{w}}\left(\frac{lnx}{x}\right)dx$ → 代錯+5

$= x\int \frac{-x^3}{2x^3}\left(\frac{lnx}{x}\right)dx + x^3\int \frac{x}{2x^3}\left(\frac{lnx}{x}\right)dx \rightarrow +6$

$= \frac{-1}{2}x\left(\frac{1}{2}(lnx)^2\right) + \frac{x^3}{2}\left(-\frac{1}{2}x^{-2}lnx - \frac{1}{4}x^{-2}\right) \rightarrow +9$

$= -\frac{1}{4}x(lnx)^2 - \frac{1}{4}xlnx - \frac{1}{8}x \rightarrow +10$

(C) Find the general solution. (2%)

$$y(x) = y_h(x) + y_p(x) = c_1x + c_2x^3 - \frac{1}{4}x(lnx)^2 - \frac{1}{4}xlnx - \frac{1}{8}x$$

4. Solve the initial value problem to find the general solution. (15%)

$x^2y'' + xy' + 9y = 0$, $y(1) = 0$, $y'(1) = 2.5$

Let $y = x^m \rightarrow y' = mx^{m-1} \rightarrow y'' = m(m-1)x^{m-2}$

原式 $\rightarrow m(m-1) + m + 9 = 0 \rightarrow m = \pm 3i$ → +3

$y_1 = x^{3i} = e^{3i(lnx)} = cos3lnx + isin3lnx$

$y_2 = x^{-3i} = e^{-3i(lnx)} = cos3lnx - isin3lnx$

解出$\boldsymbol{y_1}$、$\boldsymbol{y_2}$ → +5

$y_3 = \frac{1}{2}(y_1 + y_2) = cos3lnx$

解出$\boldsymbol{y_3}$、$\boldsymbol{y_4}$ → +9

$y_4 = \frac{1}{2i}(y_1 - y_2) = sin3lnx$

$\therefore$ $general\ solution: y(x) = c_1(cos3lnx) + c_2(sin3lnx)$ → +10

$y'(x) = -c_1(sin3lnx) \cdot \frac{3}{x} + c_2(cos3lnx) \cdot \frac{3}{x}$

$y(1) = 0 \rightarrow c_1 = 0$ → +13

$y'(1) = 2.5 \rightarrow 3c_2 = 2.5 \rightarrow c_2 = \frac{5}{6}$ → +14

$\therefore$ $y(x) = \frac{5}{6}(sin3lnx)$ → +15

5. Consider the two functions described in (A) and (B), respectively. (i) Find a second-order homogeneous linear differential equation for which the given functions are solutions. (ii) Show linear independence by their **Wronskian**.

(A) $y_1 = cosh1.8x$, $y_2 = sinh1.8x$ (15%)

(i) $y_1 = \frac{e^{1.8x}+e^{-1.8x}}{2}$, $y_2 = \frac{e^{1.8x}-e^{-1.8x}}{2}$

$y_3 = y_1 + y_2 = e^{1.8x}$ 解出y_3、y_4 → +5

$y_4 = y_1 - y_2 = e^{-1.8x}$

$let\ y = e^{\lambda x}$, $\lambda = \pm 1.8$ → +6

$y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$let\ ODE\ is\ y'' + by' + cy = 0$

$\lambda = 1.8$ 代入得 → $3.24 + 1.8b + c = 0$

$\lambda = -1.8$ 代入得 → $3.24 - 1.8b + c = 0$

$\therefore b = 0, c = -3.24$ 解出 b → +8 解出 c → +9

$\therefore y'' - 3.24y = 0$ → +10

(ii) $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ → 計算出結果+3, 表達出≠ 0 +2

$w = 1.8cosh1.8x \cdot cosh1.8x - 1.8sinh1.8x \cdot sinh1.8x = 1.8 \neq 0$

$\therefore y_1$, y_2 *are linear indepent*

(B) $y_1 = e^{-2.5x}cos0.5x$, $y_2 = e^{-2.5x}sin0.5x$ (15%)

(i) $y_3 = \frac{1}{2}(y_1 + y_2) = e^{-2.5x}e^{0.5xi}$

解出y_3、y_4→ +5

$y_4 = \frac{1}{2i}(y_1 + y_2) = e^{-2.5x}e^{-0.5xi}$

$\lambda = -2.5 \pm 0.5i$ → +6

$let\ ODE\ is\ y'' + by' + cy = 0$ → $\frac{-b\pm\sqrt{b^2-4c}}{2} = -2.5 \pm 0.5i$

$\therefore b = 5, c = \frac{13}{2}$ 解出 b → +8 解出 c → +9

$\therefore y'' + 5y' + \frac{13}{2}y = 0$ → +10

(ii) $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ → 計算出結果+3, 表達出≠ 0 +2

$$w = e^{-2.5x}cos0.5x(-2.5e^{-2.5x}sin0.5x + 0.5e^{-2.5x}cos0.5x)$$
$$-e^{-2.5x}sin0.5x(-2.5e^{-2.5x}cos0.5x - 0.5e^{-2.5x}sin0.5x)$$

$= 0.5e^{-5x} \neq 0$

$\therefore y_1 \ , \ y_2 \ are\ linear\ indepent$

6. Given a 2nd order differential equation : $xy'' + 2y' + xy = 2sinx$.

Let $y = u(x)x^{-1}$ and transfer the given differential equation to be a differential equation with constant coefficients with respect to u. (10%)

$y = u(x)x^{-1}$ → +1

$y' = u'x^{-1} + (-1)ux^{-2}$ → +1

$y'' = (u''x^{-1} + (-1)u'x^{-2}) - (u'x^{-2} + (-2)ux^{-3})$ → +1

原式 → $(u'' - u'x^{-1}) - (u'x^{-1} - 2ux^{-2}) + 2(u'x^{-1} - ux^{-2}) + u = 2sinx$

→$u'' + u = 2sinx$ → +10