Total: 100 points

1. (**30** points) Find the shortest distance from the point $(8,1)$ to the curve $y = 1 + x^{3/2}$

Solution:
Let S be the distance from the point $(8,1)$ to the curve $y = 1 + x^{3/2}$.
$f = S^2 = (x-8)^2 + (y-1)^2 = (x-8)^2 + (1 + x^{3/2} - 1)^2 = x^3 + x^2 - 16x + 64$
The domain of $y = 1 + x^{3/2}$ is $[0, \infty)$. Therefore, we should notice that $x \geq 0$.
$\dfrac{df}{dx} = 3x^2 + 2x - 16 = 0 \Rightarrow x = -\dfrac{8}{3}$ or $x = 2$. Note that x cannot be negative. Only $x = 2$ is feasible.
If $x = 2 \Rightarrow y = 1 + x^{3/2} = 1 + \sqrt{8} \Rightarrow f = S^2 = (-6)^2 + (\sqrt{8})^2 = 44$.
If $x = 0 \Rightarrow y = 1 \Rightarrow f = S^2 = 64 > 44 \Rightarrow$ The shortest distance is $\sqrt{44}$.

2. (**20** points) Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$$

(Hint): $\displaystyle\int_a^b \sin(kx)\,dx = \frac{\cos(ka) - \cos(kb)}{k}$

Solution:
$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right) = \lim_{n\to\infty}\sum_{j=1}^{n}\left[\sin\left(\frac{j}{n}\pi\right)\right]\left(\frac{1}{n}\right) = \int_0^1 \sin(\pi x)\,dx = \frac{2}{\pi}$$

3. (**20** points) Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the area of this triangle?

(Hint): Area $= A = \dfrac{1}{2}ab\sin\theta$

Solution:
$A = \dfrac{1}{2}ab\sin\theta$. Note that $0 < \theta < \pi$. $\dfrac{dA}{d\theta} = \dfrac{1}{2}ab\cos\theta \Rightarrow$ When $\dfrac{dA}{d\theta} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$.

4. (**30** points) A particle moves along the x-axis. Its acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -t^2.$$

At $t = 0$, the particle is at the origin ($s(0) = 0$). In the course of its motion, it reaches the point $x = 81/4$ (m), but no point beyond $x = 81/4$ (m). When it reaches $x = 81/4$ (m), the particle is stopped. Determine its velocity at $t = 0$.

Solution:
$a(t) = s''(t) = -t^2 \Rightarrow v(t) = s'(t) = -\dfrac{t^3}{3} + C_1 \Rightarrow s(t) = -\dfrac{t^4}{12} + C_1t + C_2$

However, $s(0) = 0 \Rightarrow C_2 = 0 \Rightarrow s(t) = -\dfrac{t^4}{12} + C_1t$

Assume that when $t = t_1$, the location is b.

Because no point beyond $x = 81/4$, the maximum value of $s(t)$ is $s(t_1) = 81/4$.

At this time, $v(t_1) = s'(t_1) = 0 \Rightarrow -\dfrac{t_1^3}{3} + C_1 = 0 \Rightarrow t_1 = (3C_1)^{1/3}$

$s(t_1) = 81/4 \Rightarrow -\dfrac{t_1^4}{12} + C_1t_1 = 81/4 \Rightarrow -\dfrac{3C_1 \cdot (3C_1)^{1/3}}{12} + C_1 \cdot (3C_1)^{1/3} = 81/4$

Therefore, $\dfrac{(3C_1)^{4/3}}{4} = 81/4 \Rightarrow C_1 = 9 \Rightarrow v(0) = -\dfrac{0^3}{3} + C_1 = 9 \Rightarrow v(0) = 9(\mathrm{m/s})$.