1. (a) $\lim_{x\to 9} \frac{\sin(\sqrt{x}-3)}{x-9} = \lim_{x\to 9} \frac{\sin(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = 1 \cdot \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$ #

(b) $\lim_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor}$

$\underbrace{\lim_{x\to\infty} \frac{-1}{\lfloor x\rfloor}}_{=0} \le \lim_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor} \le \underbrace{\lim_{x\to\infty} \frac{1}{\lfloor x\rfloor}}_{=0}$ $\because -1 \le \sin x \le 1$

By sandwich theorem, $\lim_{x\to\infty} \frac{\sin x}{\lfloor x\rfloor} = 0$ #

(c) $\lim_{x\to 0} \frac{\tan 3x}{2x^2+5x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{\cos 3x}}{2x^2+5x} = \lim_{x\to 0} \frac{3\sin 3x}{3x(2x+5)\cos 3x} = 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{1}{(2x+5)\cos 3x}$

$= 3 \cdot 1 \cdot \frac{1}{5} = \frac{3}{5}$ #

(d) $\lim_{x\to\infty} \sqrt{2x^2+1} \cdot \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \sqrt{2x^2+1} \cdot \frac{1}{x} \cdot \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \sqrt{2+\frac{1}{x^2}} \cdot 1 = \sqrt{2}$ #

2. $\lim_{x\to c} \sqrt{x} = \sqrt{c}$, $c>0$

$\forall \varepsilon>0, \exists \delta>0$, s.t. $\forall x$, $0<|x-c|<\delta \Rightarrow |\sqrt{x}-\sqrt{c}|<\varepsilon$

$|\sqrt{x}-\sqrt{c}| = \frac{|\sqrt{x}-\sqrt{c}|\,|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|} = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \le \frac{|x-c|}{\sqrt{c}} < \varepsilon$

$\therefore$ We take $\delta = \sqrt{c}\,\varepsilon$

$\forall \varepsilon>0$, $0<\varepsilon<\sqrt{c}$, $\exists \delta = \sqrt{c}\,\varepsilon$ s.t. $0<|x-c|<\delta=\sqrt{c}\,\varepsilon$

$\Rightarrow 0 < |\sqrt{x}+\sqrt{c}|\,|\sqrt{x}-\sqrt{c}| < \sqrt{c}\,\varepsilon$

$\Rightarrow 0 < |\sqrt{x}-\sqrt{c}| < \frac{\sqrt{c}}{\sqrt{x}+\sqrt{c}}\,\varepsilon \le 1 \cdot \varepsilon$

i.e. $|\sqrt{x}-\sqrt{c}| < \varepsilon$ #

3. $\lim_{x\to -5} \frac{1}{(x+5)^2} = \infty$

$\forall M>0, \exists \delta>0$, s.t. $\forall x$, $\boxed{0<|x-(-5)|<\delta} \Rightarrow \frac{1}{(x+5)^2}>M$

$\frac{1}{(x+5)^2}>M$

$\Rightarrow \frac{1}{x+5} > \sqrt{M} \Rightarrow x+5 < \frac{1}{\sqrt{M}}$ $\therefore$ we choose $\delta = \frac{1}{\sqrt{M}}$

$\forall M>0, \exists \delta = \frac{1}{\sqrt{M}}$

$\boxed{0<|x-(-5)|<\frac{1}{\sqrt{M}}} \Rightarrow 0<|x+5|<\frac{1}{\sqrt{M}} \Rightarrow 0<(x+5)^2<\frac{1}{M} \Rightarrow \frac{1}{(x+5)^2}>M$ #

4. $f(x)=\frac{\tan x}{\tan(\tan x)}$

$$\lim_{x\to 0}\frac{\tan x}{\tan(\tan x)} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{\frac{\sin(\tan x)}{\cos(\tan x)}}$$

$$= \lim_{x\to 0}\frac{\tan x}{\sin(\tan x)}\cdot \lim_{x\to 0}\cos(\tan x) = \lim_{x\to 0}\frac{1}{\frac{\sin(\tan x)}{\tan x}}\cdot \lim_{x\to 0}\cos(\tan x) = 1\cdot 1 = 1$$

Define $F(x) = \begin{cases} \frac{\tan x}{\tan(\tan x)}, & x\neq 0 \\ 1, & x=0 \end{cases}$ #

5.

$f(c)\neq 0$, $f(x)$ is continuous. $\Rightarrow \lim_{x\to c} f(x) = f(c) \neq 0$

$\forall \varepsilon>0, \exists \delta>0$, s.t. $\forall x$, $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

If $f(c)>0$

Let $\varepsilon = \frac{f(c)}{2}$

$|f(x)-f(c)| < \frac{f(c)}{2}$

$\Rightarrow -\frac{f(c)}{2} < f(x)-f(c) < \frac{f(c)}{2}$

$\Rightarrow 0 < f(x) < \frac{3}{2}f(c)$

i.e. $f(x)>0$

If $f(c)<0$

Let $\varepsilon = -\frac{f(c)}{2}$

$|f(x)-f(c)| < -\frac{f(c)}{2}$

$\Rightarrow \frac{f(c)}{2} < f(x)-f(c) < -\frac{f(c)}{2}$

$\Rightarrow \frac{3}{2}f(c) < f(x) < \frac{f(c)}{2} < 0$

i.e. $f(x)<0$

6. $y=\dfrac{x^3-3x^2+3x-1}{x^2+x-2}=\dfrac{(x-1)(x-1)^2}{(x+2)(x-1)}=(x-4)+\dfrac{9}{x+2}$

$$\begin{array}{r|l} 1\;-1 & 1-3+3-1 \\ & 1-1 \\ & -2+3 \\ & -2+2 \\ \hline & 1-1 \end{array} \qquad 1-2+1$$

分子分母次方差為1才能用

$y=x-4$ is an oblique asymptote.

$$\lim_{x\to -2^+}\frac{(x-1)^3}{(x-1)(x+2)}=\infty$$

$x=-2$ is a vertical asymptote.

$$\lim_{x\to -2^-}\frac{(x-1)^3}{(x-1)(x+2)}=-\infty$$ (当x趨近漸近線時，函數值會趨近 ∞ or $-\infty$)

7. $f(x)=\sqrt{5x f(x)}$, $f^2(x)=5x f(x)$

$2f(x)\cdot f'(x)=5\cdot f(x)+5x\cdot f'(x)$, $f'(x)[2f(x)-5x]=5f(x)$

$f'(x)=\dfrac{5f(x)}{[2f(x)-5x]}$ #

8. $\dfrac{dR}{dt}=3$, $\dfrac{dx}{dt}=-2$, find $\dfrac{dz}{dt}$ at $R=10$, $x=20$

$\dfrac{dz}{dt}=\dfrac{1}{2}[R^2+x^2]^{-\frac{1}{2}}\cdot\left[\left(2R\cdot\dfrac{dR}{dt}\right)+\left(2x\cdot\dfrac{dx}{dt}\right)\right]$

$=\dfrac{1}{2}[100+400]^{-\frac{1}{2}}\cdot[(20\cdot3)+(40\cdot -2)]$

$=\dfrac{1}{2}[500]^{-\frac{1}{2}}\cdot[-20]=\dfrac{-10}{\sqrt{500}}=\dfrac{-1}{\sqrt{5}}$ Ω/sec #

9. Let $f(x)=\sqrt[3]{x}$, $a=27=3^3$

$L(x)=f(a)+f'(a)(x-a)$

$f'(x)=\dfrac{1}{3}x^{-\frac{2}{3}}$, $f'(27)=\dfrac{1}{3}(3^3)^{-\frac{2}{3}}=\dfrac{1}{3}\cdot\dfrac{1}{9}=\dfrac{1}{27}$

$L(x)=3+\dfrac{1}{27}(x-27)$, $L(27.12)=3+\underbrace{\dfrac{1}{27}(0.12)}_{=0.004}=3.004=3+\dfrac{1}{225}$

$$\frac{1}{\cancel{27}_{9}}\times\frac{\cancel{12}^{4\,1}}{\cancel{100}_{25}}=\frac{1}{225}$$

10. $f(x) = \dfrac{x}{x^2+1}$

$f'(x) = \dfrac{1(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = \dfrac{(1-x)(1+x)}{(x^2+1)^2}$

critical points : $x=1$ or $x=-1$ $\Rightarrow f'(x)=0$

$f(1) = \dfrac{1}{2}$　　∵ $f(x)$的水平漸近線是 $y=0$

$f(-1) = -\dfrac{1}{2}$　　∴ 是 abs max/min 也是 local max/min #

11.

$f(x) = 6x^5 + 13x + 1$

$f(0) = 1$
$f(-1) = -1$ $\Rightarrow$ By Indemediate value theorem

在 $[-1,0]$ 之間有一根使 $f(x)=0$，確定有一根使 $f(x)=0$

假設 $f(x)$ 有兩個实數解 x_1, x_2 , $x_1 < x_2$, $f(x_1) = f(x_2) = 0$

By Rolle's theorem , $\exists\, c \in (x_1, x_2) \Rightarrow f'(c) = 0$

But $f'(x) = 30x^4 + 13 > 0 \Rightarrow f'(x) \neq 0$

∴ $f(x)$ has exactly one real root #