Total: 100 points

1. (40 points) Find the first derivatives for the following functions.

(a) $f(x) = \dfrac{\cos x}{1+\sin x}$ (b) $f(x) = 2\left(\dfrac{1}{\sqrt{x}} + \sqrt{x}\right)$ (c) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$ (d) $f(x) = \tan^2(nx)$

Solution:

(a) $f'(x) = \dfrac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} = \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = -\dfrac{1+\sin x}{(1+\sin x)^2}$.

$\Longrightarrow f'(x) = -\dfrac{1}{1+\sin x}$.

(b) $f(x) = 2\left(\dfrac{1}{\sqrt{x}} + \sqrt{x}\right) = 2\left(x^{-1/2} + x^{1/2}\right) \Longrightarrow f'(x) = 2\left(-\dfrac{1}{2}x^{-3/2} + \dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{x^{3/2}} + \dfrac{1}{x^{1/2}}$.

(c) **The Chain Rule.** $\Longrightarrow f'(x) = \dfrac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}}\left[1 + \dfrac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\left(1+\dfrac{1}{2}x^{-\frac{1}{2}}\right)\right]$.

(d) **The Chain Rule.** $\Longrightarrow f'(x) = 2\left[\tan(nx)\right] \cdot \sec^2(nx) \cdot n = 2n\tan(nx)\sec^2(nx)$.

2. (10 points) Evaluate the limit

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1}$$

(Hint): Use the definition of $f'(1)$ if $f(x) = x^{1000}$

Solution:

Let $f(x) = x^{1000}$. By the definition of a derivative, $f'(1) = \lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1}\dfrac{x^{1000}-1}{x-1}$.

By the power rule, $f'(x) = 1000x^{999} \Longrightarrow f'(1) = 1000$. Therefore, $\lim\limits_{x\to 1}\dfrac{x^{1000}-1}{x-1} = 1000$

3. (20 points) Is there a value of b that will make

$$g(x) = \begin{cases} x+b & x<0 \\ \cos x & x \geq 0 \end{cases}$$

continuous at $x=0$? Is there a value of b that will make $g(x)$ differentiable at $x=0$?

Solution:

To check the continuity:

$\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} x+b = b, \quad \lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \cos x = 1.$

If g is continuous at $x=0$, $g(0) = \lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^+} g(x) \Longrightarrow b = 1.$

To check the differentiability:

The left-derivative: $f'_-(0) = \dfrac{d}{dx}(x+b)\Big|_{x=0} = 1.$ The right-derivative: $f'_+(0) = \dfrac{d}{dx}\cos x\Big|_{x=0} = 0.$

Therefore, g is not differentiable at $x=0$ for any value b.

4. (10 points) For the graph

$$x^2\cos^2 y - \sin y = 0$$

find the tangent line and normal line at the point $(0,\pi)$.

Solution:

Do implicit differentiation to find y' first.

$2x\cos^2 y + x^2 \cdot 2\cos y \cdot (-\sin y)\cdot y' - \cos y \cdot y' = 0 \Longrightarrow y' = \dfrac{dy}{dx} = \dfrac{2x\cos^2 y}{2x^2\cos y \sin y + \cos y}$

At $(0,\pi)$, the slope of the tangent line is $\dfrac{dy}{dx}\Big|_{(0,\pi)} = \dfrac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}\Big|_{(0,\pi)} = 0.$

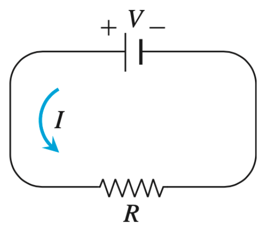
The tangent line is $y=\pi$. Therefore, the normal line is $x=0$.

5. (20 points) The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 (volt/sec) while I is decreasing at the rate of 1/3 (amp/sec). Let t denote time in seconds.

(a) $\dfrac{dV}{dt} = ?$ (b) $\dfrac{dI}{dt} = ?$ (c) What equation relates $\dfrac{dR}{dt}$ to $\dfrac{dV}{dt}$ and $\dfrac{dI}{dt}$.

(d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps.

Is R increasing or decreasing?



Solution:

(a) $\dfrac{dV}{dt} = 1$ (volt/sec)

(b) $\dfrac{dI}{dt} = -\dfrac{1}{3}$ (amp/sec)

(c) $\dfrac{dV}{dt} = R\dfrac{dI}{dt} + I\dfrac{dR}{dt} \implies \dfrac{dR}{dt} = \dfrac{1}{I}\left(\dfrac{dV}{dt} - R\dfrac{dI}{dt}\right) = \dfrac{1}{I}\left(\dfrac{dV}{dt} - \dfrac{V}{I}\cdot\dfrac{dI}{dt}\right)$

(d) $\dfrac{dR}{dt} = \dfrac{1}{2}\left[1 - \dfrac{12}{2}\cdot\left(-\dfrac{1}{3}\right)\right] = \dfrac{3}{2}$ (ohms/sec). R is increasing.