Total: 100 points

1. (**20** points) Answer the following question:

   (a) Evaluate $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)\,d\theta$ (b) If $f(x) = \int_{1/x}^{x} \frac{1}{t}\,dt$, find $f'(x)$.

Solution:

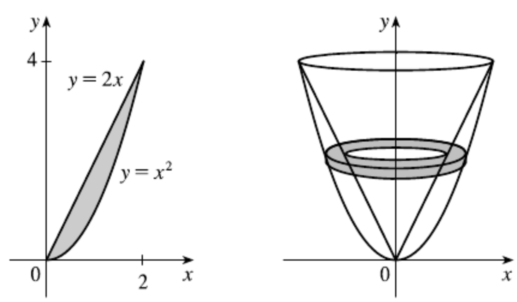
(a) Let $u = \tan\left(\frac{\theta}{6}\right) \Rightarrow du = \frac{1}{6}\sec^2\left(\frac{\theta}{6}\right)d\theta,\ \cot^5\left(\frac{\theta}{6}\right) = \frac{1}{\tan^5\left(\frac{\theta}{6}\right)} = \frac{1}{u^5},\ u : \frac{1}{\sqrt{3}} \to 1$

Therefore, $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta = \int_{\frac{1}{\sqrt{3}}}^{1} 6u^{-5}\,du = -\frac{3}{2} + \frac{27}{2} = 12$

(b) $\frac{d}{dx}\left(\int_{1/x}^{x} \frac{1}{t}\,dt\right) = \frac{1}{x}\cdot 1 - \frac{1}{1/x}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$

2. (**30** points) Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 2x$ about the y-axis.

Solution:

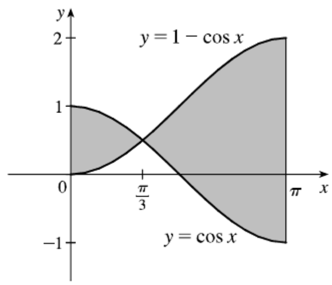


- The cross-section is a washer. $V = \int_0^4 \left[\pi\left(\sqrt{y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right]dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]\Big|_0^4 = \frac{8}{3}\pi$

3. (**20** points) Find the areas of the regions **enclosed** by the curves:

$$y = \cos x, \quad y = 1 - \cos x, \quad 0 \le x \le \pi$$

Solution:



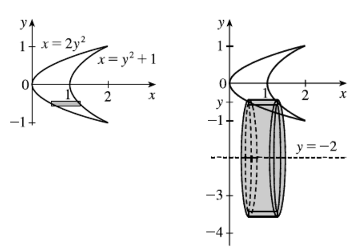
- The intersection of these two curves on $[0, \pi]$ can be found by solving $\cos x = 1 - \cos x$. Thus, the curves intersects when $2\cos x = 1 \Rightarrow \cos x = 1/2 \Rightarrow x = \pi/3$.

  The area is $A = \int_0^{\pi/3} [\cos x - (1 - \cos x)]\, dx + \int_{\pi/3}^{\pi} [(1 - \cos x) - \cos x]\, dx$

  $\Rightarrow A = [2\sin x - x]\Big|_0^{\pi/3} + [x - 2\sin x]\Big|_{\pi/3}^{\pi} = 2\sqrt{3} + \dfrac{\pi}{3}$

4. (**30** points) Use <u>**the method of cylindrical shells**</u> to find the volume generated by rotating the region bounded by $x = 2y^2$ and $x = y^2 + 1$ about the axis $y = -2$.

Solution:



- The shell has radius $y - (-2) = y + 2$, circumference $2\pi(y+2)$, and height $(y^2+1) - 2y^2 = 1 - y^2$.

  Therefore, $V = \int_{-1}^{1} \left[2\pi(y+2)\cdot(1-y^2)\right] dy = 2\pi \int_{-1}^{1} \left(-y^3 - 2y^2 + y + 2\right) dy = \dfrac{16\pi}{3}$

  In this case, one can also utilize the symmetry (even/odd function) to simplify the integral.

  $\int_{-1}^{1}(-y^3 + y)\, dy = 0$ and $\int_{-1}^{1}(-2y^2 + 2)\, dy = 2\int_0^1 (-2y^2 + 2)\, dy = 2 \cdot \dfrac{4}{3} = \dfrac{8}{3}$

  Therefore, $V = 2\pi \int_{-1}^{1} \left(-y^3 - 2y^2 + y + 2\right) dy = 2\pi \cdot 2 \int_0^1 (-2y^2 + 2)\, dy = 2\pi \cdot \dfrac{8}{3} = \dfrac{16\pi}{3}$