· **Thm** Let $B = \{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n\}$ be an o.n. basis.

If $[\underline{u}]_B = [u_1, u_2, \cdots, u_n]^T$

$[\underline{v}]_B = [v_1, v_2, \cdots v_n]^T$,

then $\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Prf

$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + \cdots + u_n \underline{e}_n$$

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots + v_n \underline{e}_n$$

$$\langle \underline{u}, \underline{v} \rangle = \langle u_1 \underline{e}_1 + u_2 \underline{e}_2 + \cdots + u_n \underline{e}_n, v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots + v_n \underline{e}_n \rangle$$

$$= u_1 v_1 \overbrace{\langle \underline{e}_1, \underline{e}_1 \rangle}^{=1} + u_2 v_2 \overbrace{\langle \underline{e}_2, \underline{e}_2 \rangle}^{=1} + \cdots + u_n v_n \overbrace{\langle \underline{e}_n, \underline{e}_n \rangle}^{=1}$$

$$+ \sum_{i \neq j} u_i v_j \underbrace{\langle \underline{e}_i, \underline{e}_j \rangle}_{=0}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad \text{※}$$

• Thm (Projection Theorem): Let $\mathbb{W}$ and $\mathbb{W}^{\perp}$ be orthogonal complement of $\mathbb{V}$ $\xrightarrow{}$ dim $= n$. Then, any vector $\underline{u} \in \mathbb{V}$ can be expressed, uniquely, as

$$\underline{u} = \underline{w}_1 + \underline{w}_2, \quad \text{with } \underline{w}_1 \in \mathbb{W} \text{ and } \underline{w}_2 \in \mathbb{W}^{\perp}.$$

Prf Let $B_1 = \{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_r\}$ and $B_2 = \{\underline{f}_1, \underline{f}_2, \cdots, \underline{f}_{n-r}\}$ be o.n. bases of $\mathbb{W}$ and $\mathbb{W}^{\perp}$, respectively. Then, $B = B_1 \cup B_2$ is an o.n. basis of $\mathbb{V}$.

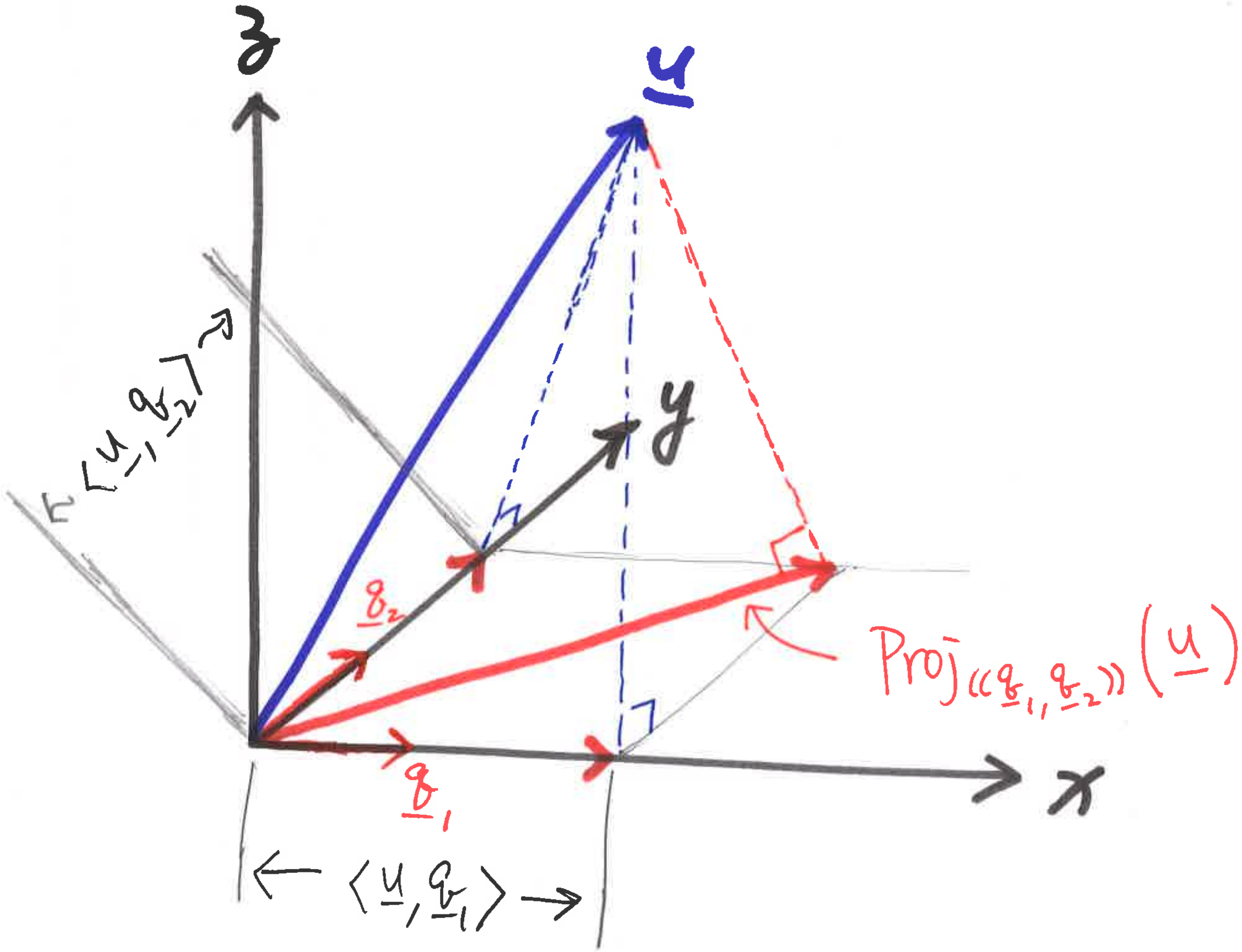
$$\underline{u} = \underbrace{k_1 \underline{e}_1 + k_2 \underline{e}_2 + \cdots + k_r \underline{e}_r}_{\in \mathbb{W},\ \cong\ \underline{x}} + \underbrace{k_{r+1} \underline{f}_1 + k_{r+2} \underline{f}_2 + \cdots + k_n \underline{f}_{n-r}}_{\in \mathbb{W}^{\perp},\ \cong\ \underline{y}}$$

• **To show the uniqueness:** Suppose that $\underline{u} = \underline{x} + \underline{y} = \underbrace{\underline{a}}_{\in \mathbb{W}} + \underbrace{\underline{b}}_{\in \mathbb{W}^{\perp}}$ ← maybe another combination

$$\Rightarrow \underbrace{\underline{x} - \underline{a}}_{\in \mathbb{W}} = \underbrace{\underline{y} - \underline{b}}_{\in \mathbb{W}^{\perp}} \in \mathbb{W} \cap \mathbb{W}^{\perp} = \{\underline{0}\}$$

$$\Rightarrow \underline{x} - \underline{a} = \underline{0} \text{ and } \underline{y} - \underline{b} = \underline{0} \Rightarrow \underline{x} = \underline{a},\ \underline{y} = \underline{b} \quad \#$$

· Vizualization of projection (in $\mathbb{R}^3$)



· Gram-Schmidt process

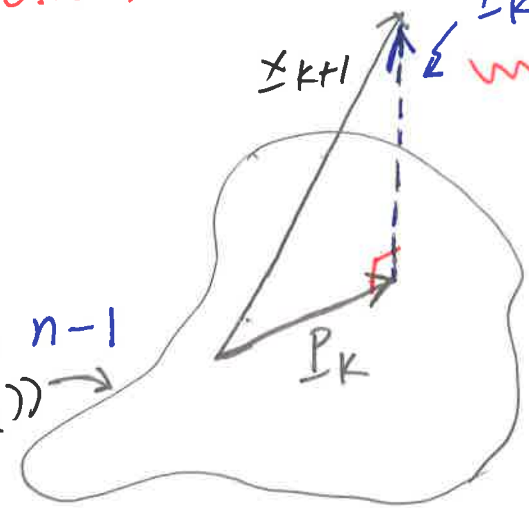
(Scenario : $\underbrace{\{\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n\}}_{\text{some basis}} \xrightarrow{\text{G.-S.}} \underbrace{\{\underline{q}_1, \underline{q}_2, \cdots, \underline{q}_n\}}_{\text{an o.n. basis}}$)

Process : Initialization : $\underline{q}_1 \overset{\Diamond}{=} \dfrac{\underline{x}_1}{\|\underline{x}_1\|}$, $\|\underline{x}_1\| \overset{\circ}{=} r_{11}$

Recursively compute $\underline{q}_2, \underline{q}_3, \cdots \underline{q}_n$ by

$$\underline{q}_{k+1} \overset{\Diamond}{=} \frac{\underline{x}_{k+1} - \underline{P}_k}{\underbrace{\|\underline{x}_{k+1} - \underline{P}_k\|}_{\overset{\circ}{=} r_{k+1,k+1}}}, \text{ for } k = 1, 2, \cdots, n-1$$

$((\underline{q}_1, \underline{q}_2, \cdots, \underline{q}_k))$



$\underline{x}_{k+1} - \underline{P}_k$

normalize

$\underline{q}_{k+1}$

where $$\underline{P}_k = \underbrace{\langle \underline{x}_{k+1}, \underline{q}_1 \rangle}_{\overset{\circ}{=} r_{1,k+1}} \underline{q}_1 + \underbrace{\langle \underline{x}_{k+1}, \underline{q}_2 \rangle}_{\overset{\circ}{=} r_{2,k+1}} \underline{q}_2 + \cdots + \underbrace{\langle \underline{x}_{k+1}, \underline{q}_k \rangle}_{\overset{\circ}{=} r_{k,k+1}} \underline{q}_k$$

· QR decomposition

$$\underline{\underline{A}} \overset{\circ}{=} \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{bmatrix} = \overbrace{\begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_n \end{bmatrix}}^{\overset{\circ}{=} \underline{\underline{Q}}} \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & 0 & r_{33} & & \vdots \\ \vdots & \vdots & 0 & & \\ 0 & 0 & 0 & & r_{nn} \end{bmatrix}}^{\overset{\circ}{=} \underline{\underline{R}}}$$

Ex (G.-S.) We are given a basis: $\left\{ \begin{array}{l} [2\ 1\ 0\ -1] \to \underline{x}_1, \\ [1\ 0\ 2\ -1] \to \underline{x}_2, \\ [0\ -2\ 1\ 0] \to \underline{x}_3 \end{array} \right\}$

- $\underline{q}_1 = \dfrac{\underline{x}_1}{\underbrace{\|\underline{x}_1\|}_{\sqrt{6}}} \triangleq r_{11}$ $= \left[\dfrac{2}{\sqrt{6}}\ \dfrac{1}{\sqrt{6}}\ 0\ \dfrac{-1}{\sqrt{6}}\right]$

- $\underline{P}_2 = \mathrm{Proj}_{\langle\langle \underline{q}_1 \rangle\rangle}(\underline{x}_2) = \overbrace{\langle \underline{x}_2, \underline{q}_1 \rangle}^{\triangleq r_{12}} \underline{q}_1 = \dfrac{\sqrt{6}}{2}\left[\dfrac{2}{\sqrt{6}}\ \dfrac{1}{\sqrt{6}}\ 0\ \dfrac{-1}{\sqrt{6}}\right] = \left[1\ \dfrac{1}{2}\ 0\ \dfrac{-1}{2}\right]$

  $\hookrightarrow = \dfrac{\sqrt{6}}{2}$

$\underline{q}_2 = \dfrac{\underline{x}_2 - \underline{P}_2}{\|\underline{x}_2 - \underline{P}_2\|} \triangleq r_{22}$ $= \dfrac{\left[0\ -\frac{1}{2}\ 2\ -\frac{1}{2}\right]}{3/\sqrt{2}} = \left[0\ \dfrac{-\sqrt{2}}{6}\ \dfrac{2\sqrt{2}}{3}\ \dfrac{-\sqrt{2}}{6}\right]$

$\dfrac{\sqrt{6}}{2} = \dfrac{3}{\sqrt{2}}$ //

- $\underline{P}_3 = \mathrm{Proj}_{\langle\langle \underline{q}_1, \underline{q}_2 \rangle\rangle}(\underline{x}_3) = \overbrace{\langle \underline{x}_3, \underline{q}_1 \rangle}^{\triangleq r_{13}} \underline{q}_1 + \overbrace{\langle \underline{x}_3, \underline{q}_2 \rangle}^{\triangleq r_{23}} \underline{q}_2 = \cdots$

  $\hookrightarrow = \dfrac{-\sqrt{6}}{3}$ $\quad\hookrightarrow = \sqrt{2}$

$\underline{q}_3 = \dfrac{\underline{x}_3 - \underline{P}_3}{\|\underline{x}_3 - \underline{P}_3\|} \triangleq r_{33}$ $= \cdots = \left[\dfrac{2}{\sqrt{21}}\ \dfrac{-4}{\sqrt{21}}\ \dfrac{-1}{\sqrt{21}}\ 0\right]$

$\dfrac{\sqrt{21}}{3}$ //

l.i.

Ans: $\{\underline{q}_1, \underline{q}_2, \underline{q}_3\}$ is an o.n. basis for $\langle\langle \underline{x}_1, \underline{x}_2, \underline{x}_3 \rangle\rangle$ #

Ex QR decomp. (continued from the previous ex.)

$$\left(\begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \right)$$

Check:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & 2/\sqrt{21} \\ 1/\sqrt{6} & -\sqrt{2}/6 & -4/\sqrt{21} \\ 0 & 2\sqrt{2}/3 & -1/\sqrt{21} \\ -1/\sqrt{6} & -\sqrt{2}/6 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 & -\sqrt{6}/3 \\ 0 & \sqrt{6}/2 & \sqrt{2} \\ 0 & 0 & \sqrt{21}/3 \end{bmatrix}$$

Ex (QR)

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

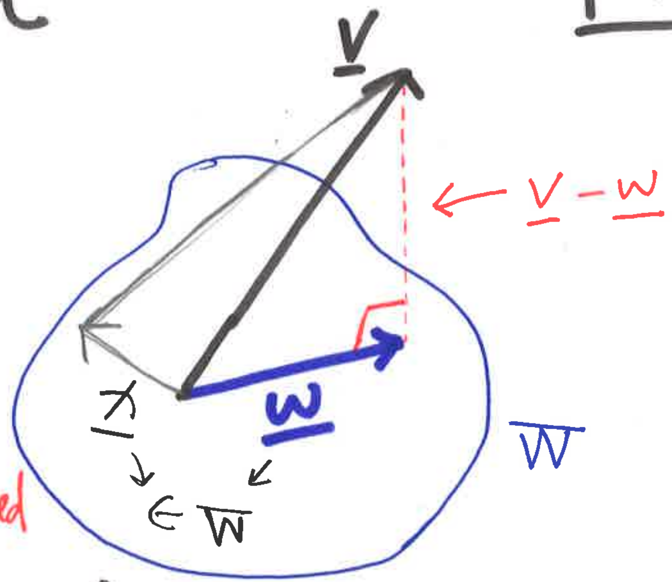
* QR decomp. is helpful in solving $\overbrace{\text{least-square(-error)}}^{\text{LS(E)}}$ solution for overdetermined systems of linear eqs.

· Visualization for "LS" problem

$$d(\underline{v},\underline{x}) \overset{\text{if } \underline{x} \neq \underline{w}}{>} d(\underline{v},\underline{w})$$

for $\forall\, \underline{x} \in \overline{W}$,

where $\underline{w} = \mathrm{Proj}_{\overline{W}}(\underline{v})$.



· Geometric view of an O.D. system of l. eqs:

overdetermined

$$\rightsquigarrow \underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\begin{cases} 3x+2y-z=7 \\ -2x+5y+2z=-1 \\ x-3y+z=5 \\ -x+4y+5z=-2 \\ 0x+2y-7z=1 \end{cases} \rightsquigarrow \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 2 \\ 1 & -3 & 1 \\ -1 & 4 & 5 \\ 0 & 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

We can only hope for them to be as "close" as possible.

$$\hookrightarrow = x \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 5 \\ -3 \\ 4 \\ 2 \end{bmatrix} + z \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 5 \\ -7 \end{bmatrix} = \underline{b}$$

· $\underline{b} \notin$ column-space$(\underline{\underline{A}})$ $\Rightarrow$ No exact solution exists!

· **Ex** (LS solution to an O.D. system of l. eqs)

$$\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 2 \\ 1 & -3 & 1 \\ -1 & 4 & 5 \\ 0 & 2 & 7 \end{bmatrix}}_{\downarrow\circ\ \underline{\underline{A}}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\downarrow\circ\ \underline{x}} = \underbrace{\begin{bmatrix} 7 \\ -1 \\ 5 \\ -2 \\ 1 \end{bmatrix}}_{\downarrow\circ\ \underline{b}} \qquad (\underline{\underline{A}}\,\underline{x} = \underline{b})$$

Sol.

$$\underline{x}_{LS} = (\underline{\underline{A}}^T \underline{\underline{A}})^{-1} \underline{\underline{A}}^T \underline{b}$$

$$= \left(\begin{bmatrix} 3 & -2 & 1 & -1 & 0 \\ 2 & 5 & -3 & 4 & 2 \\ -1 & 2 & 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 2 \\ 1 & -3 & 1 \\ -1 & 4 & 5 \\ 0 & 2 & 7 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & -2 & 1 & -1 & 0 \\ 2 & 5 & -3 & 4 & 2 \\ -1 & 2 & 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1934 \\ 0.0969 \\ 0.1669 \end{bmatrix} \text{※}$$

· $\tilde{\underline{b}} \overset{\circ}{=} \underline{\underline{A}} \cdot \underline{x}_{LS} = \begin{bmatrix} 6.6072 \\ -3.5686 \\ 2.0696 \\ -0.9715 \\ 1.3618 \end{bmatrix}$

· $\| \underline{b} - \tilde{\underline{b}} \| = \underline{4.0655}$

· **Thm** Solving for $\underline{x}_{LS}$ of $\underset{=}{A}\,\underline{x} = \underline{b}$

$$\equiv \text{ Solve } \underset{=}{R}\,\underline{x} = \underset{=}{Q}^T \underline{b}$$

$$\underset{=}{A} = \underset{=}{Q}\,\underset{=}{R}$$

Prf In the derivation of the LS solution, we require

$$\underbrace{\underset{=}{A}^T}_{=\underset{=}{Q}\,\underset{=}{R}} \underbrace{\underset{=}{A}}_{=\underset{=}{Q}\,\underset{=}{R}} \underline{x} = \underbrace{\underset{=}{A}^T}_{=\underset{=}{Q}\,\underset{=}{R}} \underline{b}$$

$$\Rightarrow (\underset{=}{Q}\,\underset{=}{R})^T \underset{=}{Q}\,\underset{=}{R}\,\underline{x} = (\underset{=}{Q}\,\underset{=}{R})^T \underline{b}$$

$$\Rightarrow \underset{=}{R}^T \underbrace{\underset{=}{Q}^T \underset{=}{Q}}_{=\underset{=}{I}} \underset{=}{R}\,\underline{x} = \underset{=}{R}^T \underset{=}{Q}^T \underline{b} \xRightarrow{(\underset{=}{R}^T)^{-1}\cdot(LHS=RHS)} \overbrace{\underset{=}{R}}^{\text{upper triangular}}\,\underline{x} = \overbrace{\underset{=}{Q}^T \underline{b}}^{=\underline{y}}$$

Inverse exists

some column vector

$\underline{x}$ can be found efficiently by back-substitution.