Proof of rank(T) + nullity(T) = n (= dim(domain)) **P.079-1**

- Any vector in V can be written as a l.c. of $\underline{v}_1, \underline{v}_2, \cdots \underline{v}_n$:

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_r\underline{v}_r + c_{r+1}\underline{v}_{r+1} + \cdots + c_n\underline{v}_n$$

- Since $\underline{v}_1, \underline{v}_2, \cdots \underline{v}_r$ lie in ker(T), we have $T(\underline{v}_1) = T(\underline{v}_2) = \cdots = T(\underline{v}_r) = \underline{0}$.

So,
$$T(\underline{v}) = c_{r+1}T(\underline{v}_{r+1}) + c_{r+2}T(\underline{v}_{r+2}) + \cdots + c_nT(\underline{v}_n)$$

Thus, we see that $\underbrace{F_1}_{\{T(\underline{v}_{r+1}), T(\underline{v}_{r+2}), \cdots T(\underline{v}_n)\}}$ spans range(T).

- Next, let us show that $T(\underline{v}_{r+1}), T(\underline{v}_{r+2}), \cdots, T(\underline{v}_n)$ are l.i.

Let $\underbrace{k_{r+1}T(\underline{v}_{r+1}) + k_{r+2}T(\underline{v}_{r+2}) + \cdots + k_nT(\underline{v}_n)}_{=\,T(k_{r+1}\underline{v}_{r+1} + k_{r+2}\underline{v}_{r+2} + \cdots + k_n\underline{v}_n)} = \underline{0}_W$

$$T(\underbrace{k_{r+1}\underline{v}_{r+1} + k_{r+2}\underline{v}_{r+2} + \cdots + k_n\underline{v}_n}_{\underline{u}}) \qquad \underline{u} \in \ker(T)$$

$$\Rightarrow \underline{u} = \text{l.c.}(\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_r)$$

$k_{r+1}\underline{v}_{r+1} + k_{r+2}\underline{v}_{r+2} + \cdots + k_n\underline{v}_n = k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_r\underline{v}_r$

$\Rightarrow -k_1\underline{v}_1 - k_2\underline{v}_2 \cdots - k_r\underline{v}_r + k_{r+1}\underline{v}_{r+1} + k_{r+2}\underline{v}_{r+2} + \cdots + k_n\underline{v}_n = \underline{0}_V$

l.i. ($\because$ B is a basis of V)

$\Rightarrow -k_1 = -k_2 = \cdots = -k_r = \underbrace{k_{r+1} = k_{r+2} = \cdots = k_n = 0}$

This is what we need, to show the l.i. among $T(\underline{v}_{r+1}), T(\underline{v}_{r+2}), \cdots, T(\underline{v}_n)$

- B is a basis of V $\longrightarrow$ $\dim(V) = n$ domain
- E is a basis of ker(T) $\longrightarrow$ $\underbrace{\dim(\ker(T))}_{\text{nullity}(T)} = r$
- F is a basis of range(T) $\longrightarrow$ $\underbrace{\dim(\text{range}(T))}_{\text{rank}(T)} = n - r$

- $\therefore$ rank(T) + nullity(T) = dim(domain)