國立臺灣科技大學答案卷

National Taiwan University of Science and Technology Answer Sheet

評分 Score	教師簽章 Signature of Lecturer
70	

姓名/Name ______ 學號/Student ID ______ 班級/Class ______

科目/Course title 工程數學 日期/Date 111. 10. 13

記分欄 從此處開始寫起。試卷用紙務須節用，非經主試認可不得續用其他紙張作答。/Please write from here.

1. $xy' = y + x^2\sin^2\left(\frac{y}{x}\right)$; $y(1) = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{y}{x} + x\sin^2\left(\frac{y}{x}\right) \Rightarrow \cancel{x\cos^2\left(\frac{y}{x}\right) + \frac{dy}{dx} = x}$

$+\sin^2 x$

$\cancel{\cos^2 x - \sin^2 x = \cos(2x) \Rightarrow \frac{1-\cos(2x)}{2} = \sin^2 x}$

$u = \frac{y}{x} \Rightarrow \frac{dy}{dx} = u + xu'$

$\cancel{u} + xu' = \cancel{u} + \cancel{x}\sin^2(u) \Rightarrow u' = \sin^2(u) \Rightarrow \cancel{u = \int \frac{1-\cos(2x)}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

$\Rightarrow \frac{du}{dx} = \sin^2(u) \Rightarrow \int \frac{1}{\sin^2(u)}du = x + C \Rightarrow -\cot(u) = x + C \Rightarrow -\cot\left(\frac{y}{x}\right) = x + C$

$= \int \csc^2(u)\,du = -\cot(u)$

$\xrightarrow{y = \frac{\pi}{4},\ x=1} -\cot\left(\frac{\pi}{4}\right) = 1 + C \Rightarrow C = -2 \Rightarrow -\cot\left(\frac{y}{x}\right) = x - 2 \Rightarrow \frac{y}{x} = \cot^{-1}(2-x)$

$\Rightarrow y = x\cdot\tan^{-1}\left(\frac{1}{2-x}\right)$

The solution is:

$\boxed{y = x\cdot\tan^{-1}\left(\frac{1}{2-x}\right)}$ #1

$\Rightarrow u = \cot^{-1}(-x-C) \Rightarrow \frac{y}{x} = \cot^{-1}(-x-C)$

$\xrightarrow{y=\frac{\pi}{4},\ x=1} \frac{\pi}{4} = \cot^{-1}(-1-C) \Rightarrow C = -2 \Rightarrow y = x\cot^{-1}(-x-(-2))$

$\Rightarrow y = x\cdot\tan^{-1}\left(\frac{1}{2-x}\right)$

1. $xy' = y + x^2\sin^2\left(\frac{y}{x}\right)$, $y(1) = \frac{\pi}{4}$

$y' = \frac{y}{x} + x\sin^2\left(\frac{y}{x}\right)$ ... ①

令 $u = \frac{y}{x}$, $y = ux$, $dy = u\,dx + x\,du$ 代①

$u\,dx + x\,du = [u + x\sin^2 u]\,dx$

+5 $\frac{1}{\sin^2 u}du = dx$

+8 $-\cot u = x + C$

$-\cot\frac{y}{x} = 1 + C$ → $y(1) = \frac{\pi}{4}$ 代入

$C = -2$

$\tan\left(\frac{y}{x}\right) = \frac{1}{-x+2}$

+10 $y = x\tan^{-1}\left(\frac{1}{-x+2}\right)$ # or $y = -x\cot^{-1}(x-2)$

or $y = x\cot^{-1}(2-x)$

2. $y'x\ln x = y$, $y(2) = \ln 8$

+10

2. $y'x\ln x = y$; $y(2) = \ln 8$

$y' = \frac{1}{x\ln x}\cdot y \Rightarrow \underbrace{y' - \frac{1}{x\ln x}}_{p(x)}\cdot y = \underbrace{0}_{q(x)} \Rightarrow e^{\int p(x)dx} = e^{-\int\frac{1}{x\ln x}dx} \xrightarrow[du = \frac{1}{x}dx]{u = \ln x} e^{-\int\frac{1}{u}du} = e^{-\ln|\ln x|} = |\ln x|^{-1}$

$\Rightarrow (\ln x)^{-1}y' - \frac{1}{x(\ln x)^2}\cdot y = 0 \Rightarrow (\ln x)^{-1}\cdot y = \int 0\,dx = C \Rightarrow y = C\cdot\ln x \xrightarrow{y=\ln 8,\ x=2} \ln 8 = C\cdot\ln 2 \Rightarrow C = 3$

The solution is:

$\boxed{y = 3\ln x}$ #2

+10

3. $y' = 9(y-7.5)\tanh(4.5x)$ $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \tanh = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ ✓

$\int \frac{1}{9}\cdot\frac{1}{(y-7.5)}\cdot dy = \int \tanh(4.5x)\,dx = \int \frac{e^{4.5x} - e^{-4.5x}}{e^{4.5x} + e^{-4.5x}}dx \xrightarrow[du = (4.5e^{4.5x} - 4.5e^{-4.5x})dx]{u = e^{4.5x} + e^{-4.5x}} \frac{1}{4.5}\int\frac{1}{u}du = \frac{1}{4.5}\ln|e^{4.5x} + e^{-4.5x}| + C$

$= 4.5(e^{4.5x} - e^{-4.5x})dx$

$\Rightarrow \frac{1}{\cancel{9}}\cdot\ln|y-7.5| = \frac{\cancel{2}}{\cancel{4.5}}\ln|e^{4.5x} + e^{-4.5x}| + C \Rightarrow |y-7.5| = e^{\ln(|e^{4.5x} + e^{-4.5x}|)^2 + C} = (e^{4.5x} + e^{-4.5x})^2\cdot C_2 \xrightarrow{C_3 = \pm C_2} y - 7.5 = (e^{4.5x} + e^{-4.5x})^2\cdot C_3$

$C_2 = e^C$

The solution is

$\boxed{y = (e^{4.5x} + e^{-4.5x})^2\cdot C_3 + 7.5}$ #3

+15

4. $y' = \frac{2x^2 - y}{x\ln(x)}$

$y' = \frac{2x^{\cancel{2}}}{\cancel{x}\ln(x)} - \frac{1}{x\ln(x)}\cdot y \Rightarrow y' + \underbrace{\frac{1}{x\ln|x|}}_{p(x)}\cdot y = \underbrace{\frac{2x}{\ln(x)}}_{q(x)} \Rightarrow e^{\int p(x)dx} = e^{\int\frac{1}{x\ln(x)}dx} \xrightarrow[du = \frac{1}{x}dx]{u = \ln(x)} e^{\int\frac{1}{u}du} = e^{\ln|\ln(x)|} = |\ln(x)|$

$\Rightarrow (\ln(x))\cdot y' + \frac{1}{x}\cdot y = 2x \Rightarrow \ln(x)\cdot y = x^2 + C \Rightarrow y = \frac{x^2}{\ln(x)} + \frac{C}{\ln(x)}$

The solution is:

$\boxed{y = \frac{x^2}{\ln(x)} + \frac{C}{\ln(x)}}$ #4

+15

5. $\frac{dy}{dx} + xy = xy^4$

let $v = y^{1-n} \Rightarrow v = y^{-3} \Rightarrow y = v^{-\frac{1}{3}}$, $y' = -\frac{1}{3}v^{-\frac{4}{3}}\cdot v' \Rightarrow -\frac{1}{3}v^{-\frac{4}{3}}v' + xv^{-\frac{1}{3}} = x\,v^{-\frac{4}{3}} \xrightarrow{x(-3)\,v^{\frac{4}{3}}} v' \underbrace{-3x}_{p(x)}v = \underbrace{-3x}_{q(x)} \Rightarrow e^{\int p(x)dx} = e^{-\frac{3}{2}x^2}$

$\Rightarrow e^{-\frac{3}{2}x^2}v' - 3e^{-\frac{3}{2}x^2}xv = -3x\cdot e^{-\frac{3}{2}x^2} \Rightarrow e^{-\frac{3}{2}x^2}\cdot v = -3\int x\cdot e^{-\frac{3}{2}x^2}dx \xrightarrow[du = -3x]{u = -\frac{3}{2}x^2} \int e^u du = e^{-\frac{3}{2}x^2} + C$

$\Rightarrow v = 1 + C\cdot e^{\frac{3}{2}x^2} \Rightarrow y = v^{-\frac{1}{3}} = (1 + C\cdot e^{\frac{3}{2}x^2})^{-\frac{1}{3}}$

The solution is:

$\boxed{y = (1 + Ce^{\frac{3}{2}x^2})^{-\frac{1}{3}}}$ #5

可轉頁書寫

記分欄 轉頁從此開始寫起。

6. $\sin(x-y)+\cos(x-y)-\cos(x-y)y'=0$; $y(0)=\frac{7}{6}\pi$

X

訂正過程請參見P.3

$\sin x\cos y-\cos x\sin y+\cos x\cos y+\sin x\sin y-(\cos x\cos y+\sin x\sin y)y'=0$

$\cos x\cdot(\cos y-\sin y)+\sin x\cdot(\cos y+\sin y)$

$\cancel{\tan(x-y)+1-y'=0}\Rightarrow\underbrace{\mu\cdot(\tan(x-y)+1)}_{M}\underbrace{-\mu y'}_{N}=0$, $\frac{\partial M}{\partial y}=\mu\cdot(\sec^2(x-y)\cdot(-1))+\frac{\partial\mu}{\partial y}(\tan(x-y)+1)$

$\frac{\partial N}{\partial x}=-\frac{\partial\mu}{\partial x}$

If $\mu(x,y)=\mu(x)$:

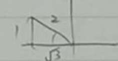
$+\mu\cdot\sec^2(x-y)=+\frac{\partial\mu}{\partial x}\Rightarrow\int\sec^2(x-y)\,\partial x=\int\frac{1}{\mu}\,\partial\mu\Rightarrow\tan(x-y)=\ln|\mu|\Rightarrow\boxed{\mu=e^{\tan(x-y)}}$

$\Rightarrow\phi=\begin{cases}\int M\,\partial x=\int e^{\tan(x-y)}\cdot[\tan(x-y)+1]\,\partial x=\int e^{\tan(x-y)}\partial x+\int\tan(x-y)\cdot e^{\tan(x-y)}=\int e^{\tan(x-y)}\partial x+g(y)\\ \int N\,\partial y=-\int e^{\tan(x-y)}\partial y=-\int e^{\tan(x-y)}\partial y+h(x)\end{cases}$

$\Rightarrow\phi=\int e^{\tan(x-y)}\partial x-\int e^{\tan(x-y)}\partial y=C\xrightarrow{y=\frac{7}{6}\pi,\ x=0}\int e^{\tan(-\frac{7}{6}\pi)}\partial x-\int e^{\tan(-\frac{7}{6}\pi)}\partial y=C\rightarrow C=e^{-\frac{1}{\sqrt3}}x-e^{-\frac{1}{\sqrt3}}y$

$-\frac{1}{\sqrt3}$

$=-\frac{7}{6}\pi\cdot e^{-\frac{1}{\sqrt3}}$



The solution is

$$\boxed{\int e^{\tan(x-y)}\partial x-\int e^{\tan(x-y)}\partial y=-\frac{7}{6}\pi\cdot e^{-\frac{1}{\sqrt3}}}\ \#6$$

7. $\frac{dy}{dx}+p(x)y=r(x)$

$-r(x)+p(x)y+\frac{dy}{dx}=0$

+20

Multiply by the integrating factor $\mu=e^{\int p(x)dx}$:

$\underbrace{[-r(x)+p(x)y]\cdot e^{\int p(x)dx}}_{M}+\underbrace{e^{\int p(x)dx}}_{N}\frac{dy}{dx}=0$ ✓

$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}-r(x)\cdot e^{\int p(x)dx}+p(x)\cdot e^{\int p(x)dx}\cdot y=p(x)\cdot e^{\int p(x)dx}$

$\Rightarrow$ Exact D.E ✓

$\frac{\partial N}{\partial x}=p(x)\cdot e^{\int p(x)dx}$

So. $\phi=\begin{cases}\int M\,\partial x=\int e^{\int p(x)dx}(-r(x)+p(x)y)\,dx=\left[y\int p(x)\cdot e^{\int p(x)dx}dx\right]-\int r(x)\cdot e^{\int p(x)dx}dx\ \overset{u=\int p(x)dx}{\underset{du=p(x)dx}{=}}\ ye^{\int p(x)dx}-\int e^{\int p(x)dx}\cdot r(x)dx+g(y)\\ \int N\,\partial y=e^{\int p(x)dx}\cdot y+h(x)\end{cases}$

$\Rightarrow\phi=ye^{\int p(x)dx}-\int e^{\int p(x)dx}\cdot r(x)dx$

The implicit solution is :

$c=ye^{\int p(x)dx}-\int e^{\int p(x)dx}\cdot r(x)dx$

We can transform it to the explicit form :

$$\boxed{y=e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}\cdot r(x)dx+c\right)}\ \#7$$

6. Find an integrating factor, use it to find the general solution of the differential equation, and then obtain the solution of the initial value problem. (20 points)

(1) (2) (3)

$$\sin(x-y)+\cos(x-y)-\cos(x-y)\ y'=0\ ;\ y(0)=\frac{7\pi}{6}$$

6

$$\frac{\partial M}{\partial y}=-\cos(x-y)+\sin(x-y)\neq\frac{\partial N}{\partial x}=\sin(x-y)\quad\rightarrow +8$$

$$\mu=e^{\int\frac{1}{-\cos(x-y)}[-\cos(x-y)+\sin(x-y)-\sin(x-y)]dx}=e^{\int dx}=e^{x}\quad\rightarrow +10\ ✓$$

$$\cancel{\{e^{x}[\sin(x-y)+\cos(x-y)]dx\}+\{-e^{x}[\cos(x-y)]\}dy=0}$$

✓

要先 否則

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-e^{x}\cos(x-y)+e^{x}\sin(x-y)$$

$$\varphi(x,y)=\int -e^{x}[\cos(x-y)]dy+k(x)=e^{x}\sin(x-y)+k(x)$$

$$M=\frac{\partial\varphi}{\partial x}=e^{x}\sin(x-y)+e^{x}\cos(x-y)+k'(x)=e^{x}\sin(x-y)+e^{x}\cos(x-y)$$

$$k'(x)=0\quad k(x)=c_1\quad\rightarrow +15$$

$$\therefore u=e^{x}\sin(x-y)=c_1$$

G.S.: $e^{x}\sin(x-y)+c_1=c_2$

$$\therefore e^{x}\sin(x-y)=c_3\quad\rightarrow +18$$

代入 $(0,\frac{7\pi}{6})$

$$\sin\frac{-7\pi}{6}=c_3=\sin\frac{5\pi}{6}=\frac{1}{2}$$

$$\therefore e^{x}\sin(x-y)=\frac{1}{2}\quad\rightarrow +20$$

$$\underbrace{\sin(x-y)+\cos(x-y)}_{M}\ \underbrace{-\cos(x-y)}_{N}y'=0$$

$$\frac{\partial M}{\partial y}=\cos(x-y)\cdot(-1)+(-\sin(x-y))\cdot(-1)=\sin(x-y)-\cos(x-y)$$

$$\frac{\partial N}{\partial x}=\sin(x-y)$$

$\because \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ $\therefore$ The D.E is not exact

Find an I.F. s.t. $\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x}$:

$$\frac{\partial}{\partial y}\mu(x,y)(\sin(x-y)+\cos(x-y))=\frac{\partial}{\partial x}\mu(x,y)(-\cos(x-y))$$

$$\Rightarrow\left(\frac{\partial\mu}{\partial y}\right)\cdot(\sin(x-y)+\cos(x-y))+\mu(x,y)\cdot(-\cos(x-y)+\sin(x-y))$$

$$=\left(\frac{\partial\mu}{\partial x}\right)(-\cos(x-y))+\mu(x,y)(\sin(x-y))$$

If $\mu(x,y)=\mu(x)$:

$$\left(\cancel{\frac{\partial\mu}{\partial y}}^{0}\right)(\sin(x-y)+\cos(x-y))+\mu(x)\cdot(\sin(x-y)-\cos(x-y))$$

$$=\left(\frac{\partial\mu}{\partial x}\right)(-\cos(x-y))+\mu(x)\cdot(\sin(x-y))$$

$$\Rightarrow\mu(x)(\cancel{\sin(x-y)}-\cos(x-y))=\left(\frac{\partial\mu}{\partial x}\right)(-\cos(x-y))+\cancel{\mu(x)\cdot(\sin(x-y))}\Rightarrow+\cancel{\cos(x-y)}\cdot\mu(x)=\frac{\partial\mu}{\partial x}(+\cancel{\cos(x-y)})$$

$$\Rightarrow\int dx=\int\frac{1}{\mu(x)}d(\mu(x))\Rightarrow x+C_1=\ln|\mu(x)|+C_2\Rightarrow C_3e^{x}=\mu(x)$$

The I.F. is:

$\boxed{\mu(x)=e^{x}}$ #6(1)

$$\Rightarrow\frac{\partial}{\partial y}\underbrace{e^{x}(\sin(x-y)+\cos(x-y))}_{M}=\frac{\partial}{\partial x}\underbrace{e^{x}\cdot(-\cos(x-y))}_{N}\Rightarrow\phi=\int N\,dy=e^{x}\cdot(-1)(+\sin(x-y))+h(x)\quad\text{—(1)}$$

$$\frac{\partial(1)}{\partial x}\Rightarrow e^{x}\cdot\sin(x-y)+e^{x}\cos(x-y)+h'(x)=e^{x}(\sin(x-y)+\cos(x-y))+h'(x)\ \text{v.s.}\ M\Rightarrow h'(x)=0$$

$$\Rightarrow h(x)=C_0$$

The general solution $\phi=C$ is

$\boxed{e^{x}\sin(x-y)=C}$ #6(2)

Subs. $\begin{cases}x=0\\ y=\frac{7}{6}\pi\end{cases}$ into the general solution:

$$\sin(-\tfrac{7}{6}\pi)=-\sin(\tfrac{7}{6}\pi)=\tfrac{1}{2}=C$$

The solution of the initial value problem is:

$\boxed{e^{x}\sin(x-y)=\frac{1}{2}}$ #6(3)

If $\mu(x,y) = \mu(y)$:

$$\left(\frac{\partial \mu}{\partial y}\right) \cdot \left(\sin(x-y)+\cos(x-y)\right) + \mu(y) \cdot \left(-\cos(x-y) + \cancel{\sin(x-y)}\right)$$

$$= \cancel{\left(\frac{\partial \mu}{\partial x}\right)}^{0} \left(-\cos(x-y)\right) + \cancel{\mu(y)\left(\sin(x-y)\right)}$$

$$\Rightarrow \left(\frac{\partial \mu}{\partial y}\right)\left(\sin(x-y)+\cos(x-y)\right) = \mu(y) \cos(x-y)$$

$$\Rightarrow \left(\frac{\partial \mu}{\partial y}\right) \cdot \frac{1}{\mu(y)} = \frac{\cos(x-y)}{\sin(x-y)+\cos(x-y)} \Rightarrow \int \frac{1}{\mu(y)} \partial \mu = \int \frac{\cos(x-y)}{\sin(x-y)+\cos(x-y)} \partial y$$

$$= \int \frac{1}{\tan(x-y)+1} \partial y$$

(hard to determine)