Prf (C-S ineq.)

P.110_1

Let u and v be arbitrary vectors in an inner product space over $\mathbb{C}$.

In the special case $v = 0$ the theorem is trivially true. Now assume that $v \neq 0$. Let $\lambda \in \mathbb{C}$ be given by $\lambda = \langle u, v\rangle / \|v\|^2$, then

$$\begin{aligned}
0 &\leq \|u - \lambda \cdot v\|^2 \\
&= \langle u, u\rangle - \langle \lambda \cdot v, u\rangle - \langle u, \lambda \cdot v\rangle + \langle \lambda \cdot v, \lambda \cdot v\rangle \\
&= \langle u, u\rangle - \lambda\langle v, u\rangle - \overline{\lambda}\langle u, v\rangle + \lambda\overline{\lambda}\langle v, v\rangle \\
&= \|u\|^2 - \lambda\overline{\langle u, v\rangle} - \overline{\lambda}\langle u, v\rangle + \lambda\overline{\lambda}\|v\|^2 \\
&= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} - \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \frac{|\langle u, v\rangle|^2}{\|v\|^2} \\
&= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}.
\end{aligned}$$

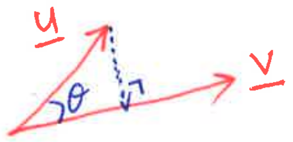
Therefore, $0 \leq \|u\|^2 - \dfrac{|\langle u, v\rangle|^2}{\|v\|^2}$, or $|\langle u, v\rangle| \leq \|u\|\|v\|$.

If the inequality holds as an equality, then $\|u - \lambda \cdot v\| = 0$, and so $u - \lambda \cdot v = 0$, thus u and v are linearly dependent. On the other hand, if u and $v$are linearly dependent, then $|\langle u, v\rangle| = \|u\|\|v\|$, which is immediately established by substuting $u = \lambda v$ into the two sides of the Cauchy-Schwartz inequality.

- For memorization : $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \cdot \|\underline{v}\| \cdot \cos\theta$

$$\Rightarrow |\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \cdot \|\underline{v}\| \cdot |\cos\theta|$$

↳ value: 0~1



※ $|\cos\theta| = 1$ iff $\theta = 0$ or $\theta = \pi$.

*Proof of the triangle ineq.

- (RHS of the ineq.)^2 : $(\|\underline{u}\|+\|\underline{v}\|)^2 = \|\underline{u}\|^2+\|\underline{v}\|^2+2\|\underline{u}\|\cdot\|\underline{v}\|$ — (1)
- (LHS of the ineq.)^2 : $\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v}\rangle$

$$= \|\underline{u}\|^2+\|\underline{v}\|^2+\underbrace{\langle \underline{u},\underline{v}\rangle}_{a+bi}+\underbrace{\langle \underline{v},\underline{u}\rangle}_{a-bi} \quad \text{— (2)}$$

It's a complex number

- $\langle \underline{u},\underline{v}\rangle+\langle \underline{v},\underline{u}\rangle = 2\cdot\underbrace{\mathrm{Re}(\langle \underline{u},\underline{v}\rangle)}_{=a} \leq 2\cdot\underbrace{|\langle \underline{u},\underline{v}\rangle|}_{\sqrt{a^2+b^2}}$ — (3)

C-S ineq.

- $|\langle \underline{u},\underline{v}\rangle| \leq \|\underline{u}\|\cdot\|\underline{v}\|$ — (4)
- (3) & (4) $\Rightarrow$ $\langle \underline{u},\underline{v}\rangle+\langle \underline{v},\underline{u}\rangle \leq 2\|\underline{u}\|\cdot\|\underline{v}\|$ — (5)
- (1) & (2) & (5) $\Rightarrow$ $\|\underline{u}+\underline{v}\|^2 \leq (\|\underline{u}\|+\|\underline{v}\|)^2$ — (6)

$\Rightarrow$ $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\|+\|\underline{v}\|$ — (7)

Ex (Angle bet. vectors)

- Consider $\mathbb{R}^4$, with IP: $\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle \triangleq x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$
- $\underline{u} = (4, 3, 1, -2)$, $\underline{v} = (-2, 1, 2, 3)$
- $\|\underline{u}\| = \sqrt{30}$, $\|\underline{v}\| = \sqrt{18}$, and $\langle \underline{u}, \underline{v} \rangle = -9$.
- $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \cdot \|\underline{v}\| \cdot \cos\theta \Rightarrow \cos\theta = \dfrac{-9}{\sqrt{30}\cdot\sqrt{18}} = \dfrac{-3}{2\sqrt{15}}$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-3}{2\sqrt{15}}\right) \approx 1.968 \text{ (rad.)} \approx 112.8^\circ$$

- **Q:** Continued from the previous example, $\underline{u} \overset{?}{\perp} \underline{v}$

Ans: No. $(\because \langle \underline{u}, \underline{v} \rangle \neq 0)$

* Thm : $\underline{0}$ is orthogonal to any vector.

Prf : $\because \langle \underline{0}, \underline{v} \rangle = 0$ for any $\underline{v}$ (from a theorem on P. 109)

$\therefore$ by def., $\underline{0}$ is orthogonal to $\underline{v}$.

* Thm (Generalized Pythogorean theorem)

Prf
$$\|\underline{u}+\underline{v}\|^2 \overset{\text{by def. of } \|\cdot\|}{=} \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle = \|\underline{u}\|^2 + \|\underline{v}\|^2 + \underbrace{\langle \underline{u}, \underline{v} \rangle}_{=0} + \underbrace{\langle \underline{v}, \underline{u} \rangle}_{=0}$$

$\because \underline{u} \perp \underline{v}$

*Ex Let $\langle f(x), g(x) \rangle \triangleq \int_{-1}^{1} f(x)\, g(x)\, dx$

· Find the angle bet. x and $\cos x$

Sol. $\langle x, \cos x \rangle = \sqrt{\int_{-1}^{1} \underbrace{x}_{\text{odd function}} \cdot \underbrace{\cos x}_{\text{even function}} \, dx} = 0$ $\quad \therefore x \perp \cos x$

odd function → odd function

· Find the angle bet. x and $\sin x$

$\langle x, \sin x \rangle = \sqrt{\int_{-1}^{1} x \cdot \sin x \, dx} = \sqrt{2(\sin 1 - \cos 1)} \simeq .7761$

$\|x\| = \sqrt{\int_{-1}^{1} x^2 \cdot dx} = \sqrt{\frac{2}{3}} \simeq .8165$

$\|\sin x\| = \sqrt{\int_{-1}^{1} (\sin x)^2 dx} = \sqrt{1 - \sin 1 \cdot \cos 1} \simeq \sqrt{.54535} \simeq .7385$

$\cos(\theta) = \dfrac{\|\underline{v}\| \cdot \|\underline{v}\|}{\langle \underline{u}, \underline{v} \rangle} = \dfrac{.8165 \times .7385}{.7761} \simeq .7769$

$\Rightarrow \theta = \cos^{-1}(.7769) \simeq .681 \text{ (rad)} \simeq 39.02^\circ$

· Check the (generalized) Pythogoras theorem:

We know: $x \perp \cos x$

$$\|x+\cos x\|^2 = \overbrace{\int_{-1}^{1}(x+\cos x)(x+\cos x)\,dx}^{甲} = \cos 1 \cdot \sin 1 + \frac{5}{3}$$

$$\|x\|^2 = \overbrace{\int_{-1}^{1} x \cdot x\,dx}^{乙} = \frac{2}{3}$$

$$\|\cos x\|^2 = \overbrace{\int_{-1}^{1}(\cos x)^2 \cdot dx}^{丙} = 1 + \cos 1 \cdot \sin 1$$

$$\frac{2}{3} \oplus (1 + \cos 1 \cdot \sin 1) = \cos 1 \cdot \sin 1 + \frac{5}{3}$$

※ $$甲 = \underbrace{\int_{-1}^{1} x^2\,dx}_{乙} + \underbrace{\int_{-1}^{1}\cos^2 x\,dx}_{丙} + \underbrace{\int_{-1}^{1} x \cdot \cos x\,dx}_{=0} + \underbrace{\int_{-1}^{1}\cos x \cdot x\,dx}_{=0}$$

$\because$ "$\perp$" $\quad \because$ "$\perp$"

(verified, in previous page).

R113-1

Ex:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$\underline{\underline{R}} = \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Let's try to solve for $\underline{\underline{A}}\,\underline{x} = \underline{0}$

$\hookrightarrow \equiv \underline{\underline{R}}\,\underline{x} = \underline{0}$

- null-space $(\underline{\underline{A}}) = \left(\left(\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)\right)$

- $\underline{\underline{A}}\,\underline{x} = \underline{0} \overset{?}{\equiv} (\underbrace{\underline{\underline{A}}^T}_{\overset{\text{IIV}}{\underline{\underline{B}}}})^T \cdot \underline{x} = 0 \equiv \underline{\underline{B}}^T \underline{x} = 0$

$\underline{x}$ is orthogonal to column-space $(\underline{\underline{B}})$, where $\underline{\underline{B}}$ is the transpose of $\underline{\underline{A}}$.

(columns of $\underline{\underline{B}}$ $\overset{\text{"}\equiv\text{"}}{\longleftrightarrow}$ rows of $\underline{\underline{A}}$)

- In this example, the "IP" is the Euclidean IP.

$\hookrightarrow$ i.e. the dot product

P.113_2

- Let $\overline{W} \triangleq \left(\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 13 \\ 5 \end{bmatrix} \right)\right)$

a basis for S

$S \triangleq \left(\left(\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)\right)$ $\quad \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \overset{a}{=} B_S$

- Any vector in S is orthogonal to any vector in $\overline{W}$.
- Any vector in $\overline{W}$ is orthogonal to any vector in S.
- A basis for $\overline{W}$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \overset{a}{=} B_{\overline{W}}$
- $\dim(S) = 2, \quad \dim(\overline{W}) = 3$

$S = W^{\perp}, \quad W = S^{\perp}$

- $B_{\overline{W}} \cup B_S$: a basis for $R^{5\times 1}$
- $\overline{W}$ and S are orthogonal complement to each other in $R^{5\times 1}$.

· **Thm** $W \cap W^{\perp} = \{\underline{0}\}$

(i.e. The only vector common to W and $W^{\perp}$ is $\underline{0}$)

Prf If $\underline{v}$ is common to W and $W^{\perp}$, then

$\langle \underline{v}, \underline{v} \rangle = 0$ ← vectors in W and $W^{\perp}$ are orthogonal (by the definition of "orthogonal complement")

a vector in W ; a vector in $W^{\perp}$

$\Rightarrow \underline{v} = 0$ (by the requirement on "Ip")

(A4 : $\langle \underline{u}, \underline{u} \rangle \geq 0$ with the equality holds iff $\underline{u} = \underline{0}$.)

· We have learned that

$$\underbrace{\text{null-space}(\underline{\underline{A}})}_{n\times 1 \text{ vectors}} \perp \text{column-space}(\underline{\underline{A}}^T) \quad \text{— (\#1)}$$

"≡" (precisely speaking: isomorphic)

$$\underbrace{\text{row-space}(\underline{\underline{A}})}_{1\times n \text{ vectors}}$$

· For simplicity of expression, we say that

$$\underbrace{\text{null-space}(\underline{\underline{A}})}_{\overset{\text{||}}{V}} \perp \underbrace{\text{row-space}(\underline{\underline{A}})}_{\overset{\text{||}}{U}} \quad \text{— (\#2)}$$

· What we actually mean is that: Any vector in V is orthogonal to any vector in U if those two vectors are both written as column vectors (or both as row vectors)

- When we talk about null-space $(\underline{\underline{A}}^T)$: $\underline{\underline{A}}^T \to \underline{\underline{B}}$
  - $\underline{\underline{B}}\,\underline{Y} = \underline{0}$ is being considered.
- $\underline{\underline{B}}\,\underline{Y} = \underline{0} \longleftrightarrow (\underline{\underline{A}}^T)\underline{Y} = \underline{0}$
- From (#2) $\Rightarrow$ $\underline{\underline{A}} \leftarrow \underline{\underline{A}}^T$

null-space$(\underline{\underline{A}}^T) \perp$ row-space$(\underline{\underline{A}}^T)$

$\parallel$

column-space$(\underline{\underline{A}})$

- **Thm** In any IP space V, $V^{\perp} = \{\underline{0}\}$

Prf Suppose that W is the orthogonal complement of V.

Then, for $\underline{w} \in W$, it must be orthogonal to any vector in V, including itself $(\because \underline{w} \in V) \Rightarrow (\underline{w}, \underline{w}) = 0$

$\Rightarrow$ The only possibility is $\underline{w} = \underline{0}$ (by A4 on "IP")

· **Ex** Consider $\mathbb{W} \subseteq \mathbb{R}^{5\times 1}$, where

$$\mathbb{W} = \left(\left(\underbrace{\begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\overset{\circ}{=}\, \underline{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}}_{\overset{\circ}{=}\, \underline{v}_2}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)\right) \overset{\circ}{=} \underline{v}_3, \text{ with the Euclidean IP}$$

i.e. dot product

orthogonal to each other (easily verified)

· Normalization of vectors:

$$\underline{v}_1 \rightsquigarrow \underline{v}_1 / \|\underline{v}_1\| = \underline{v}_1/\sqrt{59} = \begin{bmatrix} -7/\sqrt{59} \\ -3/\sqrt{59} \\ 0 \\ 1/\sqrt{59} \\ 0 \end{bmatrix} \overset{\circ}{=} \tilde{\underline{v}}_1$$

$$\underline{v}_2 \rightsquigarrow \underline{v}_2 / \|\underline{v}_2\| = \underline{v}_2/\sqrt{11} = \begin{bmatrix} 0 \\ 1/\sqrt{11} \\ 1/\sqrt{11} \\ 3/\sqrt{11} \\ 0 \end{bmatrix} \overset{\circ}{=} \tilde{\underline{v}}_2$$

vector's length = 1 ⟷ unit vector

$$\underline{v}_3 \rightsquigarrow \underline{v}_3 / \|\underline{v}_3\| = \underline{v}_3/1 = \underline{v}_3 \overset{\circ}{=} \tilde{\underline{v}}_3$$

· $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is an orthogonal basis for $\mathbb{W}$

· $\{\tilde{\underline{v}}_1, \tilde{\underline{v}}_2, \tilde{\underline{v}}_3\}$ is an orthonormal basis for $\mathbb{W}$.

orthonormal → o.n. (orthogonal and normal)

• **Thm** Let $B=\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ be an o.n. basis.
Then, for $\underline{u} \in V$,

$$\underline{u} = \langle \underline{u}, \underline{v}_1 \rangle \underline{v}_1 + \langle \underline{u}, \underline{v}_2 \rangle \underline{v}_2 + \cdots + \langle \underline{u}, \underline{v}_n \rangle \underline{v}_n$$

Prf • Because B is a basis, we can write $\underline{u}$ as:

$$\underline{u} = k_1 \underline{v}_1 + k_2 \underline{v}_2 + \cdots + k_n \underline{v}_n \sim (1)$$

• $\langle \underline{u}, \underline{v}_1 \rangle = \langle k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_n \underline{v}_n, \underline{v}_1 \rangle$

$$= k_1 \cdot \underbrace{\langle \underline{v}_1, \underline{v}_1 \rangle}_{=1} + k_2 \underbrace{\langle \underline{v}_2, \underline{v}_1 \rangle}_{=0} + \cdots + k_n \underbrace{\langle \underline{v}_n, \underline{v}_1 \rangle}_{=0} = k_1$$

• Similarly, $\langle \underline{u}, \underline{v}_2 \rangle = \cdots = k_2$

• ....

• O.n. basis provides us with great convenience in the computation of coordinates.