Calculus, Final Exam. (105-1)   2017/1/10

1. $f(x) = e^{-\frac{x^2}{2}}$

$f(-x) = f(x) \Rightarrow$ The graph of f is symmetric w.r.t. the y-axis.

$\lim\limits_{x\to\pm\infty} f(x) = 0 \Rightarrow y=0$ (x-axis) is a horizontal asymptote of the graph of f.

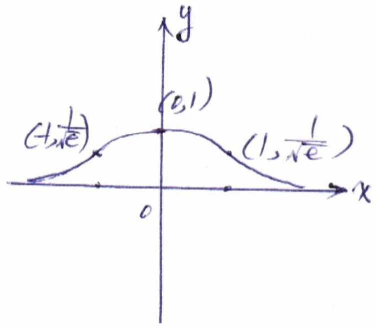
$$f'(x) = e^{-\frac{x^2}{2}} \frac{d}{dx}\left(-\frac{x^2}{2}\right) = -x e^{-\frac{x^2}{2}}$$

$$f''(x) = -\frac{d}{dx}\left(x e^{-\frac{x^2}{2}}\right) = -\left(e^{-\frac{x^2}{2}} + x \cdot e^{-\frac{x^2}{2}}(-x)\right) = -e^{-\frac{x^2}{2}}(1-x^2)$$

$$= e^{-\frac{x^2}{2}}(x^2-1)$$

Let $f'(x) = 0 \Rightarrow x = 0$.   Let $f''(x) = 0 \Rightarrow x = \pm 1$

x	0	(0,1)	1	(1,∞)
$f(x)$	1		$\frac{1}{\sqrt{e}}$	
$f'(x)$	0	−	−	−
$f''(x)$	−	−	0	+
conclusion	rel. max.	↘	Pt. of inflection	↘



2. $F = \dfrac{kx}{(x^2+r^2)^{5/2}}$

$$\frac{dF}{dx} = k\left[\frac{(x^2+r^2)^{\frac{5}{2}} \cdot 1 - x \cdot \frac{5}{2}(x^2+r^2)^{\frac{3}{2}}(2x)}{(x^2+r^2)^5}\right]$$

$$= k \cdot \frac{(x^2+r^2) - 5x^2}{(x^2+r^2)^{7/2}}$$

Let $\frac{dF}{dx} = 0 \Rightarrow 4x^2 = r^2 \Rightarrow x^2 = \frac{r^2}{4} \Rightarrow x = \frac{r}{2}$

x	0	$(0, \frac{r}{2})$	$\frac{r}{2}$	$(\frac{r}{2}, \infty)$
F	0		$\frac{16k}{25\sqrt{5}\,r^4}$	
$\frac{dF}{dx}$	+	+	0	−
conclusion	0	↗	max	↘

F will attain it's maximum when $x = \frac{r}{2}$

3

(a)
$$f'(x) = \frac{[e^{2x} + \ln(x+1)]\frac{d}{dx}10^x - 10^x \frac{d}{dx}[e^{2x} + \ln(x+1)]}{[e^{2x} + \ln(x+1)]^2}$$

$$= \frac{[e^{2x} + \ln(x+1)]10^x \ln 10 - 10^x(2e^{2x} + \frac{1}{x+1})}{[e^{2x} + \ln(x+1)]^2}$$

$$f'(x_0) = f'(0) = \frac{(e^0 + \ln 1) \cdot 10^0 \ln 10 - 10^0 (2 \cdot e^0 + 1)}{(e^0 + \ln 1)^2}$$

$$= \frac{1 \cdot \ln 10 - 1 \cdot (2+1)}{1^2} = -3 + \ln 10$$

(b) Let $y = g(x) = (\pi + \sin x)^{e^x} \Rightarrow \ln y = \ln(\pi + \sin x)^{e^x} = e^x \ln(\pi + \sin x)$

$$\frac{d}{dx}\ln y = \frac{d}{dx}[e^x \ln(\pi + \sin x)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = e^x \ln(\pi + \sin x) + e^x \frac{d}{dx}\ln(\pi + \sin x)$$

$$= e^x \ln(\pi + \sin x) + e^x \cdot \frac{1}{\pi + \sin x} \frac{d}{dx}(\pi + \sin x)$$

$$= e^x \ln(\pi + \sin x) + \frac{e^x \cos x}{\pi + \sin x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[e^x \ln(\pi + \sin x) + \frac{e^x \cos x}{\pi + \sin x}\right]$$

$$= (\pi + \sin x)^{e^x}\left[e^x \ln(\pi + \sin x) + \frac{e^x \cos x}{\pi + \sin x}\right]$$

$$g'(x_0) = g'(0) = \left.\frac{dy}{dx}\right|_{x=0} = (\pi + \sin 0)^{e^0}\left[e^0 \ln(\pi + \sin 0) + \frac{e^0 \cos 0}{\pi + \sin 0}\right]$$

$$= \pi\left(\ln \pi + \frac{1}{\pi}\right) = \pi \ln \pi + 1$$

4 (a) Let $u=\cot\theta \Rightarrow du=-\csc^2\theta\, d\theta$

$\theta=\frac{\pi}{6} \Rightarrow u=\sqrt{3}$; $\theta=\frac{\pi}{2} \Rightarrow u=0$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{e^{\cot\theta}}{\sin^2\theta}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} e^{\cot\theta}\csc^2\theta\, d\theta = -\int_{\sqrt{3}}^{0} e^u\, du = \int_0^{\sqrt{3}} e^u\, du$$

$$= e^u \Big|_0^{\sqrt{3}} = e^{\sqrt{3}} - e^0 = e^{\sqrt{3}} - 1$$

(b) Let $u=\ln(1+\sin\theta) \Rightarrow du = \frac{\cos\theta}{1+\sin\theta}\, d\theta$

$$\int \frac{\cos\theta\, d\theta}{\sqrt[3]{\ln(1+\sin\theta)}\,(1+\sin\theta)} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}}\, du = \frac{3}{2}u^{\frac{2}{3}} + C$$

$$= \frac{3}{2}\left[\ln(1+\sin\theta)\right]^{\frac{2}{3}} + C$$

(c) Let $u=x^3+1 \Rightarrow du=3x^2dx \Rightarrow x^2dx=\frac{1}{3}du$

$x^3=u-1$

$$\int \sqrt[3]{x^3+1}\; x^5 dx = \int \sqrt[3]{x^3+1}\cdot x^3\cdot x^2 dx = \int u^{\frac{1}{3}}(u-1)\cdot\frac{1}{3}du$$

$$= \frac{1}{3}\int \left(u^{\frac{4}{3}} - u^{\frac{1}{3}}\right)du = \frac{1}{3}\left(\frac{3}{7}u^{\frac{7}{3}} - \frac{3}{4}u^{\frac{4}{3}}\right) + C$$

$$= \frac{1}{7}u^{\frac{7}{3}} - \frac{1}{4}u^{\frac{4}{3}} + C = \frac{1}{7}(x^3+1)^{\frac{7}{3}} - \frac{1}{4}(x^3+1)^{\frac{4}{3}} + C$$

5 (a) Let $u=\ln x$

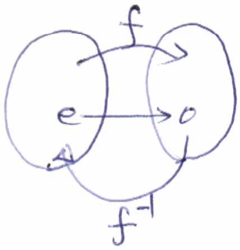
$$f'(x) = \frac{d}{dx}\int_1^{\ln x} \sqrt{\cos^2 t + e^t}\, dt = \left(\frac{d}{du}\int_1^u \sqrt{\cos^2 t + e^t}\, dt\right)\left(\frac{du}{dx}\right)$$

$$= \sqrt{\cos^2 u + e^u}\cdot\frac{d}{dx}\ln x$$

$$= \frac{1}{x}\sqrt{\cos^2(\ln x) + e^{\ln x}} = \frac{1}{x}\sqrt{\cos^2(\ln x) + x}$$

$\because x>0 \Rightarrow f'(x)>0 \Rightarrow f$ is an increasing function $\Rightarrow f$ is a one-to-one function

(b) Let $f(x)=0 \Rightarrow \ln x = 1 \Rightarrow x = e$

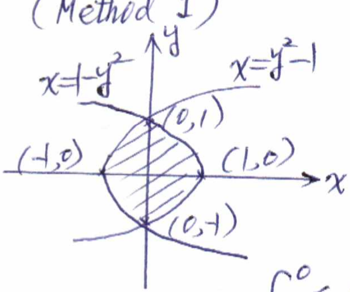


$$(f^{-1})'(0) = \frac{1}{f'(e)} = \frac{1}{\frac{1}{e}\sqrt{\cos^2(\ln e)+e}}$$

$$= \frac{e}{\sqrt{\cos^2(\ln e)+e}} = \frac{e}{\sqrt{\cos^2 1 + e}}$$

6

(Method 1)



Let $y^2-1=1-y^2 \Rightarrow 2y^2=2 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$x=y^2-1 \Rightarrow y^2=x+1 \Rightarrow y=\pm\sqrt{x+1}$

$x=1-y^2 \Rightarrow y^2=1-x \Rightarrow y=\pm\sqrt{1-x}$

$$A=\int_{-1}^{0}[\sqrt{x+1}-(-\sqrt{x+1})]\,dx+\int_0^1[\sqrt{1-x}-(-\sqrt{1-x})]\,dx$$

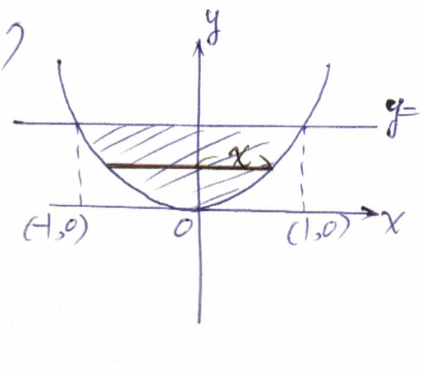
$$=2\int_{-1}^{0}(x+1)^{\frac{1}{2}}dx+2\int_0^1(1-x)^{\frac{1}{2}}dx$$

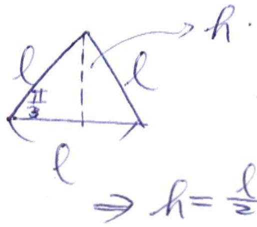
$$=2\left[\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_{-1}^{0}-\frac{2}{3}(1-x)^{\frac{3}{2}}\Big|_0^1\right]=2\left(\frac{2}{3}-\left(-\frac{2}{3}\right)\right)=2\cdot\frac{4}{3}=\frac{8}{3}$$

(Method 2) $$A=\int_{-1}^{1}[(1-y^2)-(y^2-1)]\,dy=\int_{-1}^{1}(2-2y^2)\,dy=2\int_{-1}^{1}(1-y^2)\,dy$$

$$=4\int_0^1(1-y^2)\,dy=4\left(y-\frac{1}{3}y^3\right)\Big|_0^1=4\left(1-\frac{1}{3}\right)=4\cdot\frac{2}{3}=\frac{8}{3}$$

7





$$\tan\frac{\pi}{3}=\frac{h}{\frac{l}{2}}$$

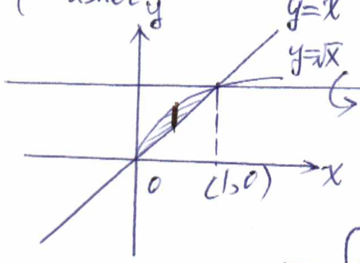
$$\Rightarrow h=\frac{l}{2}\tan\frac{\pi}{3}=\frac{l}{2}\cdot\sqrt{3}$$

$$A = \frac{1}{2} l h = \frac{1}{2} l \cdot \frac{l}{2}\sqrt{3} = \frac{\sqrt{3}}{4} l^2$$

$$V \left(= \int A\, dy\right) = \int_0^1 \frac{\sqrt{3}}{4}\left[x-(-x)\right]^2 dy = \frac{\sqrt{3}}{4}\int_0^1 (2x)^2 dy$$

$$= \sqrt{3}\int_0^1 x^2 dy = \sqrt{3}\int_0^1 y\, dy = \sqrt{3}\cdot \frac{1}{2} y^2 \Big|_0^1 = \frac{\sqrt{3}}{2}$$

8 (Washer Method)



$$x = \sqrt{x} \Rightarrow x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \text{ or } 1$$

$$\Rightarrow y = 0 \text{ or } y = 1$$

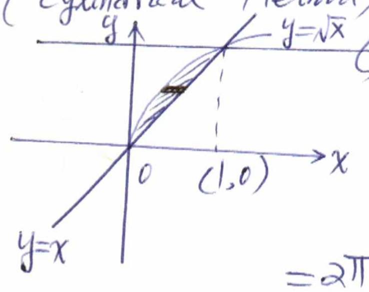
$$A = \pi \int_0^1 \left[(1-y_{small})^2 - (1-y_{large})^2\right] dx$$

$$= \pi \int_0^1 \left[(1-x)^2 - (1-\sqrt{x})^2\right] dx = \pi \int_0^1 \left[(1-2x+x^2) - (1-2x^{\frac{1}{2}}+x)\right] dx$$

$$= \pi \int_0^1 \left(x^2 - 3x + 2x^{\frac{1}{2}}\right) dx = \pi \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}}\right)\Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3}\right) = \pi\left(\frac{5}{3} - \frac{3}{2}\right) = \frac{\pi}{6}$$

(Cylindrical Method)



$$A = \int_0^1 2\pi (1-y)\left(y_{large} - y_{small}\right) dy$$

$$= 2\pi \int_0^1 (1-y)(y-y^2)\, dy$$

$$= 2\pi \int_0^1 (y - 2y^2 + y^3)\, dy = 2\pi\left(\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}$$

9

$$L = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2} - \frac{\ln x}{4}\right) = x - \frac{1}{4x}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(x-\frac{1}{4x}\right)^2 = x^2 - 2x\cdot\frac{1}{4x} + \frac{1}{16x^2} = x^2 - \frac{1}{2} + \frac{1}{16x^2}$$

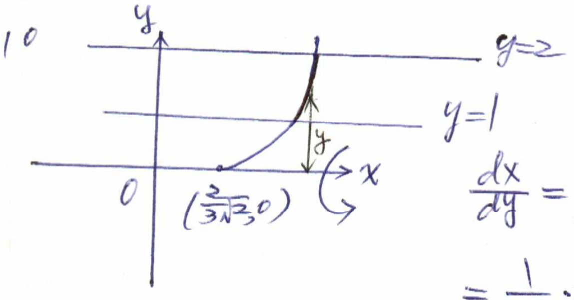
$$1+\left(\frac{dy}{dx}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2} = x^2 + 2x\cdot\frac{1}{4x} + \left(\frac{1}{4x}\right)^2$$

$$= \left(x+\frac{1}{4x}\right)^2$$

$$L = \int_2^4 \sqrt{\left(x+\frac{1}{4x}\right)^2}\, dx = \int_2^4 \left(x+\frac{1}{4x}\right) dx = \frac{1}{2}x^2 + \frac{1}{4}\ln|x| \Big|_2^4$$

$$= \left(\frac{1}{2}\cdot 4^2 + \frac{1}{4}\ln 4\right) - \left(\frac{1}{2}\cdot 2^2 + \frac{1}{4}\ln 2\right) = \left(8 + \frac{1}{4}\ln 4\right) - \left(2 + \frac{1}{4}\ln 2\right)$$

$$= 6 + \frac{1}{4}(\ln 4 - \ln 2) = 6 + \frac{1}{4}\ln\frac{4}{2} = 6 + \frac{1}{4}\ln 2$$



$$A = \int_1^2 2\pi y\, dl = 2\pi\int_1^2 y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{d}{dy}\left[\frac{1}{3}(y^2+2)^{\frac{3}{2}}\right] = \frac{1}{2}(y^2+2)^{\frac{1}{2}}\frac{d}{dy}(y^2+2)$$

$$= \frac{1}{2}\cdot 2y\cdot(y^2+2)^{\frac{1}{2}} = y\,(y^2+2)^{\frac{1}{2}}$$

$$\left(\frac{dx}{dy}\right)^2 = y^2(y^2+2) = y^4+2y^2$$

$$1+\left(\frac{dx}{dy}\right)^2 = y^4+2y^2+1 = (y^2+1)^2$$

$$A = 2\pi\int_1^2 y\sqrt{(y^2+1)^2}\, dy = 2\pi\int_1^2 y(y^2+1)\, dy = 2\pi\int_1^2 (y^3+y)\, dy$$

$$= 2\pi\left(\frac{1}{4}y^4 + \frac{1}{2}y^2\right)\Big|_1^2 = 2\pi\left[\left(\frac{1}{4}\cdot 16 + \frac{1}{2}\cdot 4\right) - \left(\frac{1}{4} + \frac{1}{2}\right)\right]$$

$$= 2\pi\left(4+2-\frac{3}{4}\right) = 2\pi\cdot\frac{21}{4} = \frac{21}{2}\pi$$