Total: 100 points

1. (**20** points) Evaluate $\displaystyle\int \sin x \cos x \ln(\cos x)\ dx$

Solution:

- Do substitution first. $t = \cos x \to dt = -\sin x\, dx$

$$\int \sin x \cos x \ln(\cos x)\ dx = \int -t \ln(t)\ dt$$

Use **integration by part** to solve this integral. $u = \ln(t) \to du = \frac{1}{t}\, dt, \quad dv = -t\, dt \to v = -\frac{1}{2}t^2.$

$$\begin{aligned}\int -t\ln(t)\ dt &= -\frac{1}{2}t^2\ln(t) + \int \frac{1}{2}t^2 \cdot \frac{1}{t}\, dt = -\frac{1}{2}t^2\ln(t) + \frac{1}{2}\int t\, dt \\ &= -\frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2 + C \\ &= -\frac{1}{2}\cos^2 x \ln(\cos x) + \frac{1}{4}\cos^2 x + C\end{aligned}$$

2. (**20** points) Evaluate $\displaystyle\int \cos^3 2x \sin^5 2x\, dx$

Solution:

$$\begin{aligned}\int \cos^3 2x \sin^5 2x\, dx &= \frac{1}{2}\int \cos^2 2x \sin^5 2x \cdot \cos 2x\, d(2x) = \frac{1}{2}\int \left(1 - \sin^2 2x\right)\sin^5 2x\, d(\sin 2x) \\ &= \frac{1}{2}\int \sin^5 2x\, d(\sin 2x) - \frac{1}{2}\int \sin^7 2x\, d(\sin 2x) = \frac{1}{2}\left[\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right] + C\end{aligned}$$

3. (**20** points) Evaluate $\displaystyle\int \left(x^2 + x + 1\right) e^x\, dx$

Solution:

$$\begin{aligned}\int \left(x^2+x+1\right)e^x\, dx &= \left(x^2+x+1\right)e^x - \int (2x+1)\, e^x\, dx \\ &= \left(x^2+x+1\right)e^x - \left[(2x+1)\, e^x - \int 2e^x\, dx\right] \\ &= \left[\left(x^2+x+1\right) - (2x+1) + 2\right]e^x + C = \left(x^2 - x + 2\right)e^x + C\end{aligned}$$

4. (**20** points) Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution: Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow dx = \sec^2\theta\, d\theta$, $\sqrt{x^2+1} = \sec\theta$, $\csc\theta = \dfrac{\sqrt{x^2+1}}{x}$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Let $u = \sin\theta \to du = \cos\theta\, d\theta$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\csc\theta + C = -\frac{\sqrt{x^2+1}}{x} + C$$

5. (**20** points) Evaluate $\displaystyle\int \frac{4x^2+5x+3}{(x-1)(x+2)(x+1)}\,dx$

Solution:

- Do **partial fraction decomposition** first.

$$\frac{4x^2+5x+3}{(x-1)(x+2)(x+1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+1}.$$

Use Heaviside cover-up method to find the coefficients.

$$A = \left.\frac{4x^2+5x+3}{(x+2)(x+1)}\right|_{x=1} = \frac{12}{6} = 2,$$
$$B = \left.\frac{4x^2+5x+3}{(x-1)(x+1)}\right|_{x=-2} = \frac{9}{3} = 3,$$
$$C = \left.\frac{4x^2+5x+3}{(x+2)(x-1)}\right|_{x=-1} = \frac{2}{-2} = -1.$$

Therefore,

$$\begin{aligned}\int \frac{4x^2+5x+3}{(x-1)(x+2)(x+1)}\,dx &= \int \frac{2}{x-1}\,dx + \int \frac{3}{x+2}\,dx - \int \frac{1}{x+1}\,dx \\ &= 2\ln\left|x-1\right| + 3\ln\left|x+2\right| - \ln\left|x+1\right| + K\end{aligned}$$

where K is the integration constant.