1102 Calculus Midterm Exam Date:2022/04/14 Total:120

1. $\lim_{x\to 0}\frac{(e^x-1)^3}{(x-2)e^x+x+2} = ?$ 6 (5%)

$\frac{0}{0}$

$= \lim_{x\to 0}\frac{3(e^x-1)^2\cdot e^x}{e^x+(x-2)e^x+1} = \lim_{x\to 0}\frac{6(e^x-1)e^{2x}+3(e^x-1)^2 e^x}{\cancel{e^x}+\cancel{e^x}+(x-2)e^x} = \lim_{x\to 0}\frac{6(e^{2x}-e^x)\ \frac{0}{0}}{x+x-2} = \lim_{x\to 0} 6(2e^{2x}-e^x)+6(e^x-1)e^x = \boxed{6}$

#1

2. Let $f(x) = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x$, find $f'(x)$. (5%)

$= (\sin^{-1}x)^2 + x\left(2\cdot\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}\right) + 2\left(\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot(-2x)\cdot\sin^{-1}x + \cancel{\sqrt{1-x^2}}\cdot\frac{1}{\cancel{\sqrt{1-x^2}}}\right) - 2$

$= (\sin^{-1}x)^2 + \cancel{\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}} + \cancel{\frac{-2x\sin^{-1}x}{\sqrt{1-x^2}}} = \boxed{(\sin^{-1}x)^2}$

#2

3. $\int_0^1 \frac{2}{\sqrt{3+4x^2}}dx = ?$ (10%)

$2x = \sqrt{3}\tan\theta$

$\frac{d}{d\theta}(2x) = \frac{d}{d\theta}(\sqrt{3}\tan\theta)$

$dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$\int \frac{2}{\sqrt{3+3\tan^2\theta}}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta = \int\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\sec\theta}d\theta = \ln|\sec\theta+\tan\theta|$

$\left[\ln\left|\sec\left(\tan^{-1}\frac{2x}{\sqrt{3}}\right)+\frac{2x}{\sqrt{3}}\right|\right]_0^1$ $\frac{\sec x>0,\ -\frac{\pi}{2}<x<\frac{\pi}{2}}{-\frac{\pi}{2}<\tan^{-1}\frac{2x}{\sqrt{3}}<\frac{\pi}{2}}$ $\left[\ln\left|\sqrt{1+\frac{4x^2}{3}}+\frac{2x}{\sqrt{3}}\right|\right]_0^1$

$= \ln\left|\frac{\sqrt{21}}{3}+\frac{2}{\sqrt{3}}\right| - \cancel{\ln 1}^{0} = \boxed{\ln\left(\frac{\sqrt{21}+2\sqrt{3}}{3}\right)}$ #3

4. Compare the growth rates of functions as $x\to\infty$: $x^{\frac{1}{n}}$, $\ln x$ (5%)

$\lim_{x\to\infty}\frac{\ln x}{x^{\frac{1}{n}}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{n}\cdot x^{\frac{1}{n}-1}} = \lim_{x\to\infty}\frac{n}{x^{\frac{1}{n}}} = 0 \Rightarrow$ $\boxed{x^{\frac{1}{n}} \text{ grows faster than } \ln x}$

#4

5. $\int\frac{xe^x}{x^2+2x+1}dx = ?$ (10%)

$u = x+1 \Rightarrow du = dx$

$= \int\frac{xe^x}{(x+1)^2}dx = \int\frac{(u-1)\cdot e^{u-1}}{u^2}du = \int\underbrace{\frac{1}{u}}_{u_2}\cdot\underbrace{e^{u-1}du}_{dv_2} - \int\frac{1}{u^2}\cdot e^{u-1}du = \frac{e^{u-1}}{u} + \cancel{\int e^{u-1}\cdot\frac{1}{u^2}du - \int\frac{1}{u^2}\cdot e^{u-1}du} = \boxed{\frac{e^x}{x+1}+C}$

#5.

6. $\int\tan^4 x\,dx = ?$ (5%)

$\int\tan^2x\cdot(\sec^2x-1)dx = \int\tan^2x\sec^2x\,dx - \int\tan^2x\,dx = \int u^2du - \int\sec^2x-1\,dx = \boxed{\frac{1}{3}\tan^3x - \tan x + x + C}$

$u = \tan x$

$du = \sec^2x\,dx$

#6

7. $\int\frac{1}{x\sqrt{x^4-1}}dx = ?$ (10%)

$x^2 = \sec\theta$

$2x\,dx = \sec\theta\tan\theta\, d\theta$

$= \int\frac{1}{x\cdot\cancel{\tan\theta}}\cdot\frac{1}{2x}\cdot\sec\theta\cancel{\tan\theta}\,d\theta = \int\frac{\cancel{\sec\theta}}{2\cancel{\sec\theta}}d\theta = \boxed{\frac{1}{2}\sec^{-1}(x^2)+C}$

#7

8. $\int\frac{4x^2+5x+6}{(x+2)x^2}dx = ?$ (10%)

$= \int\frac{A}{(x+2)}+\frac{Bx+C}{x^2}dx$

$A(x^2)+(Bx+C)(x+2) = 4x^2+5x+6$

$x=0:\ 2C=6 \Rightarrow \boxed{C=3}$

$x=1:\ A+3B=6 \Rightarrow \boxed{B=1}$

$x=-1:\ A-B=2 \Rightarrow \boxed{A=3}$

$\Rightarrow \int\frac{3}{x+2}+\frac{x+3}{x^2}dx = \boxed{3\ln|x+2|+\ln|x|-\frac{3}{x}+C}$

#8

9. $\int_0^\infty \underset{u}{x}\,\underset{dv}{e^{-2x}dx} = ?$ (10%)

$v = \int e^{-2x}dx$ $\frac{u=-2x}{du=-2dx}$ $-\frac{1}{2}\int e^u du = -\frac{1}{2}e^{-2x}$

$= \left[-\frac{1}{2}xe^{-2x}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}dx = -\frac{1}{2}\left[xe^{-2x}\right]_0^\infty - \frac{1}{4}\left[e^{-2x}\right]_0^\infty = -\frac{1}{2}\lim_{b\to\infty}be^{-2b}+\frac{1}{4} = -\frac{1}{2}\lim_{b\to\infty}\frac{b}{e^{2b}}-\frac{1}{4} = \cancel{-\frac{1}{2}\lim_{b\to\infty}\frac{1}{2\cdot e^{2b}}}^{0}+\frac{1}{4} = \boxed{\frac{1}{4}}$

#9

10. $\frac{1}{1*3*5}+\frac{2}{3*5*7}+\frac{3}{5*7*9}+\ldots = ?$ (10%)

$\sum_{n=1}^\infty\frac{n}{(2n-1)(2n+1)(2n+3)} = \sum_{n=1}^\infty\frac{A}{(2n-1)(2n+1)}+\frac{B}{(2n+1)(2n+3)}$

$A(2n+3)+B(2n-1) = n$

$n=-\frac{3}{2}:\ -4B=-\frac{3}{2} \Rightarrow B=\frac{3}{8}$

$n=\frac{1}{2}:\ 4A=\frac{1}{2} \Rightarrow A=\frac{1}{8}$

$= \frac{1}{8}\left(\sum_{n=1}^\infty\frac{1}{(2n-1)(2n+1)}\right)+\frac{3}{8}\left(\sum_{n=1}^\infty\frac{1}{(2n+1)(2n+3)}\right)$

（背面還有題目）

$= \frac{1}{8}\left(\frac{1}{3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots\right)+\frac{3}{8}\left(\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots\right) = \frac{1}{8}\cdot\frac{1}{2}\left(1-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\ldots\right)+\frac{3}{8}\cdot\frac{1}{2}\left(\frac{1}{3}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\ldots\right)$

$\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)$

$= \frac{1}{16}+\frac{1}{16} = \boxed{\frac{1}{8}}$ #10

11. Check the convergence or divergence of the series (40%)

(a) $\sum_{n=1}^{\infty} \frac{2}{1+e^n} < \sum_{n=1}^{\infty} \frac{2}{n^2} \Rightarrow$ convergence

(b) $\sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx$. A: Convergence

$\sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx$ V.S. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ → P-series $p>1$, convergence

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n(n+1)}$

ratio: $\lim_{n\to\infty} \frac{\frac{\cancel{(n+1)!}}{(n+1)^{n+1}}}{\frac{\cancel{n!}}{n^n}} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n < 1 \Rightarrow$ convergence

(d) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(\ln n)}$

convergence

(e) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

root:

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 < 1 \Rightarrow$ convergence

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11.(b)

by limit comparison test:

$\lim_{n\to\infty} \frac{\int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx}{\frac{1}{n^{\frac{3}{2}}}}$ $\frac{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}{\text{\& L'hopital}}$ $\lim_{n\to\infty} \frac{\frac{d}{dn}\int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx}{\frac{d}{dn}\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\left[\frac{d}{d(\frac{1}{n})}\int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx\right]\cdot\left[\frac{d}{dn}\frac{1}{n}\right]}{-\frac{3}{2}\frac{1}{n^{\frac{5}{2}}}}$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\lim_{n\to\infty} \frac{\frac{\sqrt{\frac{1}{n}}}{1+(\frac{1}{n})^2}\cdot\frac{-1}{\cancel{n^2}}}{-\frac{3}{2}\cdot\frac{1}{\cancel{n^{\frac{5}{2}}}}} = \lim_{n\to\infty} \frac{\frac{\cancel{\sqrt{\frac{1}{n}}}}{1+(\frac{1}{n})^2}}{\cancel{\sqrt{n}}} = 0 \Rightarrow \sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx < \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \Rightarrow$ convergence

$\Rightarrow \sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx$ is convergence, too

11(d)

by power series test:

$\frac{1}{n\ln(\ln n)} > 0$ for all n

$\lim_{n\to\infty} \frac{1}{n\ln(\ln n)} = 0$

$\frac{1}{n\ln(\ln n)} > \frac{1}{(n+1)\ln(\ln(n+1))} \Rightarrow a_n > a_{n+1} \Rightarrow \frac{1}{n\ln(\ln n)}$ is decreasing

$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(\ln n)}$ is convergence