Chapter 3 Vector Spaces

3.1 Vector Spaces

◎ In general, the components in vectors can be real, complex or any element from a field.

◎ For simplicity, most of our discussions assume that the components in vectors are real.

◎ A vector space is a 4-tuple algebraic structure $(F, V, +, *)$ that satisfies certain axioms.

- F is a field (an algebraic structure that satisfies certain axioms). Its elements are formally called scalars (or informally as numbers).
- In most discussions of the course, F is simply R (set of real numbers).
- Near the end of the course, F is extended to C (set of complex numbers).
- In general, F can be any field.
- $+$: vector addition

◇ $*$: scalar multiplication (more precisely, scalar-vector multiplication)

◇ (F,V,+, $*$) is usually simply denoted as V, for brevity.

◎ The axioms for a vector space:

1. If u and v are objects in V, then u + v is in V.
2. u + v = v + u
3. u + (v + w) = (u + v) + w

4. There is an object 0 in V, called a zero vector for V, such that 0 + u= u + 0= u for all u in V.

5. For each u in V, there is an object –u in V, called a negative of u, such that
u + (-u) = (-u) + u= 0.

6. If k is any scalar and u is any object in V, then ku (abbreviation for k*u) is in V.

7. k (u + v) = ku+ kv

8. $(k + l)\,u = ku + lu$
9. $k(lu) = (kl)\,(u)$
10. $1u = u$

- ◇ N.B. We do not prove axioms. They are simply accepted as the "rules of the game."

◎ Alternatively, we can define a vector space via group and field.

◎ Group $((G\ ,+))$ satisfies

- ◇ closure

- ◇ associativity
- ◇ existence of identity (denoted by 0)
- ◇ existence of inverse
- ◇ commutativity (for abelian/commutative group)

◎ Field ((F,+, $*$)) satisfies

- ◇ (F,+) is a commutative group.
- ◇ ($F\setminus\{0\}$,$*$) is a commutative group.
- ◇ distributiveness: $(a+b)*c = a*c+b*c$

- ◎ Vector space $((F, \mathrm{V}, +, *))$ satisfies
  - ◇ F is a field.
  - ◇ $(\mathrm{V},+)$ is a commutative group
  - ◇ Axioms 6 - 10 on pp. 49 - 50
- ◎ Examples of vector spaces:
  - ◇ R^n or C^n (called "n-Euclidean space")
  - ◇ $R^{m\times n}$ or $C^{m\times n}$ (set of $m\times n$ matrices with real/complex elements)
  - ◇ {real-valued functions}

- ◇ {polynomials of degree $\leqq$ n}
- ◇ A plane through the origin

◎ Some properties of vectors

- ◇ 0**v** =**0** (0**v** is an abbreviation of 0*$\mathbf{v}$)
- ◇ k**0**=**0**
- ◇ (-1) **v** = - **v**
- ◇ k*$\mathbf{v}$=**0** → k=0 or **v**=**0**

3.2 Subspaces

◎ ***Def*** Let (F,V,+, $*$) be a v.s. (vector space). If W is a subset of V and (F,W,+, $*$) is a v.s., then W is a subspace of V.

◎ ***Thm*** (Subspace test): W (a subset of V) is a subspace of V iff for any **u** and **v** in W, c**u**+d**v** is also in W for any scalars c and d..

◎ Examples of subspaces:

◇ Lines through the origin of R^3

◇ {polynomials of degree $\leq$ n} in {polynomials}

- ◇ $\{\mathbf{x}|\mathbf{Ax=0}\}$ in $R^{n\times 1}$ (for $\mathbf{A}$:m×n)
- ◇ $\{\mathbf{Ax}\}$ (i.e. range of $\mathbf{x}$) in $R^{m\times 1}$ (for $\mathbf{A}$:m×n)

◎ Notice that a line or a plane in R^3 that does not pass through the origin is not a subspace.

◎ Linear combinations of vectors

- ◇ $\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_r\mathbf{v}_r$
- ◇ span: $((\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r))=\{k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_r\mathbf{v}_r\}$
- ◇ A span is a subspace.

◇ $((\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r))$ is the smallest subspace that contains $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r\}$.

3.3 Linear Independence

◎ <u>*Def*</u> $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r\}$ is said to be linearly independent (l.i.) iff

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_r\mathbf{v}_r=0 \rightarrow k_1=k_2=\ldots=k_r=0$$

◇ not l.i. ≡ l.d. (linearly dependent)

◎ Some theorems regarding l.i.

◇ S is l.d. if some $\mathbf{v}_i$ = l.c.(other vectors).

◇ S is l.i. if no $\mathbf{v}_i$ = l.c.(other vectors).

◇ $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r,\mathbf{0}\}$ is l.d.

◇ Let S be a subset with r vectors in R^n. If r>n, then S is l.d.

3.4 Basis and Dimension

◎ ***Def*** B=$\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is a basis of V iff it satisfies two conditions:

◇1 B is l.i.

◇2 B spans V (i.e. $((\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n))=V$).

◎ *<u>Thm</u>* Given B: a basis of V. Any vector in V can be written as a unique l.c. of basis vectors (i.e. vectors in B).

◎ *<u>Def</u>* A basis B is said to be an ordered basis (o.b.) when the order of basis vectors is also specified.

◇ coordinate vector of a vector wrt an o.b.

◇ standard basis for $R^{n\times 1}$

◇ standard basis for P_n

◎ A vector space V can be finite-dimensional or infinite-dimensional:

◇ finite-dimensional: V has a basis consisting finite number of vectors

◇ infinite-dimensional: basis consists of infinite number of vectors

◇ We focuses on the finite-dim case.

◎ *Thm* Let V be a finite-dimensional v.s. and $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be a basis of it. Then, the two statements below are true:

◇1 If a set S has more than n vectors, then S is l.d.

◇2 If a set S has fewer than n vectors, then S can not span V.

◇ Consequence of ◇1 and ◇2: All bases of V have the same number of vectors.

◎ ***Def*** dimension: dim(V) = number of basis vectors (in any basis)

◎ Some theorems on basis and dimension:

◇ Given dim(V)=n, then a set of n vectors is a basis if either it is l.i. or it spans V.

◇ Every set that spans V contains a basis for V within it.

◇ Every l.i. set of V can be part of a basis for V.

◇ Let W be a subspace of V. Then, dim(W) $\leqq$ dim(V). Moreover, if dim(W)=dim(V), then W=V.

3.5 Row/Column Space and Nullspace

◎ *Def* Row space of **A**: {l.c.(rows of **A**)}

◇ *Thm* Ero's do not change row-space(**A**).

◎ A procedure for finding a basis of

$S = ((\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n))$:

◇1 Use $\mathbf{v}_i$'s as rows to form a matrix **A**.

◇2 Find **R**=rref(**A**).

◇3 {nonzero rows of **R**} is a basis of S.

◎ ***Def*** Column space of **A**: {l.c.(columns of **A**)}

◇ Define a mapping from $R^{n\times 1}$ to $R^{m\times 1}$ by $\mathbf{y}=T_{\mathbf{A}}(\mathbf{x})=\mathbf{A}\mathbf{x}$. The range of $T_{\mathbf{A}}$ is defined to be $\{\mathbf{A}\mathbf{x} \mid \text{all } \mathbf{x} \text{ in } R^{n\times 1}\}$.

◇ **Ax** is virtually a l.c. of **A**'s col's, with elements of **x** as the coeffs of combination.

◇ Obviously, range($T_{\mathbf{A}}$) = col-space(**A**).

◇ *Thm* **Ax**=**b** is consistent (i.e. has solutions(s)) iff **b** is in col-space(**A**) (i.e. **b** can be expressed as a l.c. of **A**'s col's).

◎ *Def* Null space of **A**: {**x**|**Ax**=**0**}

◇ *Thm* Ero's do not change null-space(**A**).

◎ *Thm* Given **A**:m×n. Row, column, and null spaces are subspaces of $R^{1\times n}$, $R^{m\times 1}$, and $R^{n\times 1}$, respectively.

◎ Solutions to $\mathbf{A}\mathbf{x}=\mathbf{b}$:

◇ particular solution ($\mathbf{x}_p$)

◇ homogeneous solution ($\mathbf{x}_h$)

◇ general solution: $\mathbf{x}_g=\mathbf{x}_p+\mathbf{x}_h$

3.6 Rank and Nullity of a matrix

◎ *Thm* Row-space($\mathbf{A}$) and col-space($\mathbf{A}$) have the same dimension.

◇ This dimension is called **A**'s rank (denoted as rank(**A**)).

◇ rank(**A**)=number of the leading variables in the general solution of **Ax**=**0**.

◎ <u>*Def*</u> Nullity(**A**)=dim(null-space(**A**)).

◇ nullity(**A**)=number of the free parameters in the general solution of **Ax**=**0**.

◎ <u>*Thm*</u> If **A** has n columns, then

rank(**A**)+nullity(**A**)=n.

◎ ***Thm*** If **Ax=b** is a linear system of m equations in n unknowns, then the statements below are equivalent:

◇1 **Ax=b** is consistent for every **b** in $R^{m\times 1}$.

◇2 ((column vectors of **A**))= $R^{m\times 1}$.

◇3 rank(**A**)=m.

◇ If rank(**A**)=r, then the general solution contains n-r parameters.

◎ Some equivalent statements regarding invertibility of a matrix:

If $\boldsymbol{A}$ is an $n{\times}n$ matrix, and if $T_A : R^{n\times 1} \rightarrow R^{n\times 1}$ is multiplication by $\boldsymbol{A}$, then the following are equivalent:

◇ $\boldsymbol{A}$ is invertible.

◇ $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

◇ The reduced row-echelon form of $\boldsymbol{A}$ is I_n.

- ◇ $\boldsymbol{A}$ is expressible as a product of elementary matrices.
- ◇ $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ is consistent for every $n\times 1$ matrix $\mathbf{b}$.
- ◇ $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n\times 1$matrix $\mathbf{b}$.
- ◇ $\det(\boldsymbol{A}) \neq 0$.
- ◇ The range of T_A is $R^{n\times 1}$.
- ◇ T_A is one-to-one.

- The column vectors of $\boldsymbol{A}$ are l.i.
- The row vectors of $\boldsymbol{A}$ are l.i.
- The column vectors of $\boldsymbol{A}$ span $R^{n\times 1}$.
- The row vectors of $\boldsymbol{A}$ span $R^{1\times \mathrm{n}}$.
- The column vectors of $\boldsymbol{A}$ form a basis for $R^{n\times 1}$.
- The row vectors of $\boldsymbol{A}$ form a basis for $R^{1\times \mathrm{n}}$.
- $\boldsymbol{A}$ has rank n.
- $\boldsymbol{A}$ has nullity 0.