1.

a.

$$\lim_{x\to\infty}\frac{(1+x^2)^{-1}}{x^{-2}} = \lim_{x\to\infty}\frac{x^2}{1+x^2} = 1$$

b.

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x} = \lim_{x\to\infty} e^{\ln\left(1+\frac{2}{x}\right)^{3x}} = e^{\lim_{x\to\infty}3x\ln\left(1+\frac{2}{x}\right)} \quad \left(取y=\frac{1}{x}\right)$$

$$= e^{3\lim_{y\to0}\frac{\frac{2}{1+2y}}{1}} = e^{3\cdot2} = e^{6}$$

2.

a.

$$原式 = \int_0^1 \tan^{-1}x\ d\left(\frac{x^2}{2}\right) = \frac{x^2}{2}\tan^{-1}x\,|_0^1 - \int_0^1 \frac{x^2}{2}d\ (\tan^{-1}x)$$

$$= \left(\frac{1}{2}.\frac{\pi}{4}-0\right) - \frac{1}{2}\int_0^1 x^2.\frac{1}{1+x^2}\ dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1\left(\frac{x^2+1-1}{1+x^2}\right)dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 (1-\left(\frac{1}{1+x^2}\right))dx = \frac{\pi}{8} - \frac{1}{2}(x-\tan^{-1}x)|_0^1 = \pi - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$$

b.

$$\int \sin^2 x\cos^5 x\,dx = \int \sin^2 x\cos^4 x\cos x\ \ dx$$

$$= \int \sin^2 x(1-\sin^2 x)^2 d(\sin x) = \int (\sin^2 x - 2\ \ sin^4 x + \sin^6\ \ x\ \ )d\sin x$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$$

c. 令 $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

解方程式 得 C=1 A= -1 B= -1 原式 $= \int(-\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{1}{x-2})dx = \ -\ln|x-1| + \frac{1}{x-1} +$

$\ln|x-2| + c = \ln\left|\frac{x-2}{x-1}\right| + \frac{1}{x-1} + c$

d. 取 $x = a\,sin\,y$

原式 $= \int \frac{a\cos y}{(a^2-a^2\sin^2 y)^{\frac{3}{2}}}dy = \frac{1}{a^2}\int \sec^2 y\,dy = \frac{1}{a^2}\tan y + c = \frac{1}{a^2}\frac{\sin y}{\cos y} + c$

$$= \frac{1}{a^2}\frac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}} + c \quad = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + c$$

3.	a. 原式 $= \int_0^{1^-} \frac{dx}{\sqrt[3]{(x-1)^2}} + \int_{1^+}^{3} \frac{dx}{\sqrt[3]{(x-1)^2}} = \lim_{t\to 1^-} \int_0^t (x-1)^{-\frac{2}{3}} dx + \lim_{s\to 1^+} \int_s^3 (x-1)^{-\frac{2}{3}} dx$ $= \lim_{t\to 1^-} \{3(x-1)^{\frac{1}{3}}\|_0^t\} + \lim_{x\to 1^+} \{3(x-1)^{\frac{1}{3}}\|_s^3\} = 3 + 3\sqrt[3]{2}$ b. 原式 $= \int_1^{\infty} \frac{1}{x^2+x^4} dx = \int_1^{\infty} \frac{dx}{x^2(1+x^2)} = \int_1^{\infty} \left(\frac{1}{x^2} - \left(\frac{1}{1+x^2}\right)\right) dx = \lim_{t\to\infty} \left(-\frac{1}{x} - \tan^{-1} x\right) \|_1^t$ $= -\frac{\pi}{2} + 1 + \frac{\pi}{4} = 1 - \frac{\pi}{4}$
4.	a. 由 $\int_1^{\infty} \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^{\infty} = 1$，故 $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ 收斂, by Integral Test b. 發散， $\lim_{k\to\infty} \left(\frac{3k+5}{2k-5}\right)^k = \infty \neq 0$，故發散, by nth − Term Test c. $\frac{1}{k^2} \sin\left(\frac{\pi}{k}\right) \leq \frac{1}{k^2}$ ， $\sum_{k=1}^{\infty} \frac{1}{k^2}$ 收斂 ， $\therefore \sum_{k=1}^{\infty} \frac{1}{k^2} \ sin\left(\frac{\pi}{k}\right)$ 收斂 d. $\lim_{k\to\infty} \sqrt[k]{k^3 \left(\frac{k}{2k-1}\right)^k} = \lim_{k\to\infty} (k^{\frac{1}{k}})^3 \cdot \left(\frac{k}{2k-1}\right) = \frac{1}{2} < 1 \quad \therefore \sum_{k=0}^{\infty} a_k$ 收斂 by Root Test e. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{\lim_{k\to\infty}((k+1)^3 e^{-(k+1)})}{k^3 e^{-k}} = \frac{1}{e} < 1$，故收斂 by Ratio Test f. $\because \frac{2}{\ln(n+1)} < \frac{2}{\ln(n)}$ ，且 $\lim_{n\to\infty} \frac{2}{\ln(n+1)} = 0$ 故由交錯級數審練法知原級數為收斂, by Leibniz’s Test
5.	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = \lim_{n\to\infty} \left\|\frac{\frac{(x-3)^{n+1}}{(n+2)2^{n+1}}}{\frac{(x-3)^n}{(n+1)2^n}}\right\| = \frac{\|x-3\|}{2} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{\|x-3\|}{2} < 1$ $\therefore \|x-3\| < 2 \longrightarrow 1 < x < 5$，現在討論端點如下: $(1) x = 5$ 時， $\sum_{n=2}^{\infty} \frac{1}{n+1}$ 為發散 　 $(2) x = 1$ 時， $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$ 為收斂 故得收斂區間為 $1 \leq x < 5$