ENGINEERING MATHEMATICS (II) – MIDTERM Solutions

Winter 2023

PROBLEM 1

(a) *If you type in the following MATLAB commands*

A=[1 3 2; 4 6 -7; 9 8 7];

$B = A(:, 3)$

then what is shown on your screen?

Sol: The first command produces $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -7 \\ 9 & 8 & 7 \end{bmatrix}$. After the second command, which shows the third column, we can get $\begin{bmatrix} 2 \\ -7 \\ 7 \end{bmatrix}$.

(b) *a) continued. If you continue to type in the command* $\mathbf{C} = A([2,3],[1,2])'$, *then what is shown on your screen?*

Sol: $\mathbf{C}$ is the transpose of a submatrix of $\mathbf{A}$ (2-3 rows and 1-2 columns), and thus we can see $\begin{bmatrix} 4 & 6 \\ 9 & 8 \end{bmatrix}' = \begin{bmatrix} 4 & 9 \\ 6 & 8 \end{bmatrix}$

(c) *Suppose that* $\mathbf{EA} = \mathbf{B}$, *where* $\mathbf{E}$ *is a* 3×3 *matrix,*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 4 \\ 2 & 3 & 1 & 5 \end{bmatrix}, \quad \textit{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 4 \\ 4 & 6 & 2 & 10 \end{bmatrix}$$

then what is $\mathbf{E}^{-1}$*?*

Sol: We can see that the difference between $\mathbf{A}$ and $\mathbf{B}$ is that the 3rd row of $\mathbf{B}$ is two times that of $\mathbf{A}$, so $\mathbf{E}$ is the 2nd type elementary matrix given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore,

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) *Suppose that* $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \;\forall\; \mathbf{x} \neq \mathbf{0}$, *where* $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{x}$ *is an* $n \times 1$ *column vector, then is it true that all diagonal elements of* $\mathbf{A}$ *are larger than zero?* **Hint**: *What is* $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ *if* $\mathbf{x} = \mathbf{e}_i$, *where* $\mathbf{e}_i$ *is an* $n \times 1$ *column vector whose* i^{th} *element is zero and the remaining elements are zeros?*

Sol: Note that $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$, *i.e.* the i^{th} column of $\mathbf{A}$. Thereby, $\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i = [\mathbf{A}]_{i,i}$. Since $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, for all $\mathbf{x}$, thus if we choose $\mathbf{x} = \mathbf{e}_i,\ i = 1,\ldots,n$, we have $[\mathbf{A}]_{i,i} > 0,\ i = 1,\ldots,n$, which implies that all diagonal elements of $\mathbf{A}$ are larger than zero.

(e) *Plot* $Span\left(\begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix}\right)$.

Sol: $Span\left(\begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix}\right) = \alpha\begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.5\alpha \end{bmatrix} =$ z-axis in $\mathcal{R}^3$.

(f) *Suppose that* $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *is the only solution to* $\mathbf{C}\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 6 \\ 2 \end{bmatrix}$. *Determine the number of linearly independent columns of* $\mathbf{C}$.

Sol: We can first note that $\mathbf{C}$ is a 5×3 matrix. Since $\mathbf{C}\mathbf{x}$, which is equal to the weighted summation of the columns of $\mathbf{C}$, has exactly one solution, all columns of $\mathbf{C}$ are linearly independent, *i.e.* the number of linearly independents of $\mathbf{C}$ is equal to 3. Or you can find that as now $\mathcal{N}(\mathbf{C}) = \mathbf{0}$, so $dim(\mathcal{R}(\mathbf{C}) = n = 3$, which implies that all columns are linearly independent.

(g) *(f) continued. Determine the reduced row echelon form of* $\mathbf{C}$.

Sol: From (f), $rank(\mathbf{C}) = 3 = dim(col(\mathbf{C}))$ (three nonzero pivots in the (reduced) row echelon form), so the reduced row echelon form of $\mathbf{C}$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(h) *Assume that* $rank(\mathbf{A}) = r$, *then determine the rank of the following matrices:*

$$2\mathbf{A}^T,\ \begin{bmatrix} \mathbf{A} & \mathbf{A} \end{bmatrix},\ \begin{bmatrix} \mathbf{A} \\ \mathbf{A} \end{bmatrix},\ \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}$$

Sol: Note that $rank(\mathbf{A})$ is equal to the number of linearly independent columns (rows) of $\mathbf{A}$. Also, with extra zeros, the linearly independent rows/columns will double in the last matrix. Therefore,

$$rank(2\mathbf{A}^T) = r,\ rank(\begin{bmatrix} \mathbf{A} & \mathbf{A} \end{bmatrix}) = r,\ rank(\begin{bmatrix} \mathbf{A} \\ \mathbf{A} \end{bmatrix}) = r,\ rank(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}) = 2r$$

PROBLEM 2

Consider the following system of linear equations

$$\begin{aligned} x + 4y - z &= 1 \\ 2x + 6y + \gamma z &= \beta \\ x + 6y - 5z &= 5 \end{aligned}$$

Determine the values of γ and β such that the above system of linear equations has no solution, one solution, and infinitely many solutions. Also, **determine the corresponding solution set***.*

Sol: To solve it, form the augmented matrix first and then reduce it to the row echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 2 & 6 & \gamma & \beta \\ 1 & 6 & -5 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & -2 & \gamma+2 & \beta-2 \\ 0 & 2 & -4 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 2 & -4 & 4 \\ 0 & -2 & \gamma+2 & \beta-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & \gamma-2 & \beta+2 \end{array}\right].$$

Therefore, the solution sets can be divided into the following three cases:

- If $\gamma = 2$ and $\beta \neq -2$, then there is no solution.
- If $\gamma = 2$ and $\beta = -2$, then there is infinitely many solutions. Now the problem becomes

$$\begin{aligned} x + 4y - z &= 1 \\ 2y - 4z &= 4 \end{aligned}$$

  Setting the free variable $z = \alpha$, we can obtain the solution

$$\left\{(-7 - 7\alpha, 2 + 2\alpha, \alpha) \middle| \ \alpha \text{is any arbitrary real numbers}\right\}$$

- If $\gamma \neq 2$, then there is exactly one solution. The problem now becomes

$$\begin{aligned} x + 4y - z &= 1 \\ 2y - 4z &= 4 \\ (\gamma - 2)z &= \beta + 2 \end{aligned}$$

  Solving the above, we can obtain the solution

$$\left\{(-7 - 7\frac{\beta+2}{\gamma-2}, 2 + 2\frac{\beta+2}{\gamma-2}, \frac{\beta+2}{\gamma-2}\right\}$$

PROBLEM 3

Suppose that

$$adj(\mathbf{A}^{-1}) = \begin{bmatrix} -1 & -3 & -5 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$$

Also, $det(\mathbf{A}) > 0$.

(a) *Determine* $det(3\mathbf{A}^2\mathbf{A}^T)$. **Hint**: *The fact* $\mathbf{A}(adj\ \mathbf{A}) = det(\mathbf{A})\mathbf{I}$ *is of help.*

Sol: Conducting the cofactor expansion along the first row of $adj(\mathbf{A}^{-1})$ yields

$$det(adj(\mathbf{A})^{-1}) = (-1) \times \Big(6 \times 3 - 1 \times 8\Big) - (-3) \times \Big(2 \times 3 - 1 \times 3\Big) + (-5) \times \Big(2 \times 3 - 1 \times 3\Big) = 9$$

Also, taking the determinant on both sides of $\mathbf{A}(adj\ \mathbf{A}) = det(\mathbf{A})\mathbf{I}$ yields $det(adj(\mathbf{A})) = det(\mathbf{A})^{n-1}$ (already shown this in HW 2). It follows that $det(adj(\mathbf{A})^{-1}) = \frac{1}{det(\mathbf{A})^{n-1}}$. Therefore, we have $det(\mathbf{A}) = \frac{1}{3}$ as now $n = 2$. Consequently, we have

$$det(3\mathbf{A}^2\mathbf{A}^T) = 3^3(det(\mathbf{A}))^2 det(\mathbf{A}^T) = 27 \times (\frac{1}{3})^3 \times \frac{1}{3} = 1$$

where we have used the fact that $det(\mathbf{A}^T) = det(\mathbf{A}) = \frac{1}{3}$.

(b) *What is* $\mathbf{A}$*?*

Sol: Replacing $\mathbf{A}$ with $\mathbf{A}^{-1}$ in $\mathbf{A}(adj\ \mathbf{A}) = det(\mathbf{A})\mathbf{I}$ yields $\mathbf{A}^{-1}(adj\ \mathbf{A}^{-1}) = det(\mathbf{A}^{-1})\mathbf{I}$. It follows that

$$\mathbf{A} = \frac{1}{det(\mathbf{A}^{-1})}(adj\ \mathbf{A}^{-1}) = det(\mathbf{A})(adj\ \mathbf{A}^{-1}) = \frac{1}{3}\begin{bmatrix} -1 & -3 & -5 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$$

PROBLEM 4

$$\mathsf{V} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix},\ a_{i+2} = a_i + a_{i+1},\ i = 1, 2, 3 \right\} \quad and \quad \mathsf{W} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix},\ a_1 + a_2 = 0, a_3 + a_4 = 0 \right\}$$

.

(a) *Find* $dim(\mathsf{V})$.

Sol: Based on the rule $a_{i+2} = a_i + a_{i+1}$, we have $a_3 = a_1 + a_2$, $a_4 = a_2 + a_3 = a_2 + a_1 + a_2 = a_1 + 2a_2$, and $a_5 = a_3 + a_4 = (a_1 + a_2) + (a_1 + 2a_2) = 2a_1 + 3a_2$. Thereby,a typical element of V is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_1 + a_2 \\ a_1 + 2a_2 \\ 2a_1 + 3a_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

It is easy to verify that these two spanning vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ are linearly independent and thus is a basis, so $dim(\mathsf{V}) = 2$.

(b) *Find dim(V∩W).*

Sol: Note that based on its definition, $a_1 + a_2 = 0$, $(a_1 + 2a_2) + (2a_1 + 3a_2) = 3a_1 + 5a_2 = 0$. It follows that $a_1 = 0$ and $a_2 = 0$. Therefore, a typical element of W is given by $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. So dim(V∩W)=0.

PROBLEM 5 (10 pts)

Assume that $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ *and* $\mathbf{I}$ *is a* 5×5 *identity matrix, then determine* $(\mathbf{I} + \mathbf{v}\mathbf{v}^T)^{-1}\mathbf{v}$.

Hint: *You can assume that* $(\mathbf{I} + \mathbf{v}\mathbf{v}^T)^{-1} = \mathbf{I} + k\mathbf{v}\mathbf{v}^T$ *and find* k *first.*

Sol: It can be readily shown that $\mathbf{v}^T\mathbf{v} = 5$. Assume that $(\mathbf{I} + \mathbf{v}\mathbf{v}^T)^{-1} = \mathbf{I} + k\mathbf{v}\mathbf{v}^T$, then it follows that

$$\mathbf{I} = (\mathbf{I} + \mathbf{v}\mathbf{v}^T)(\mathbf{I} + k\mathbf{v}\mathbf{v}^T) = \mathbf{I} + \mathbf{v}\mathbf{v}^T + k\mathbf{v}\mathbf{v}^T + k\mathbf{v}\overbrace{\mathbf{v}^T\mathbf{v}}^{=5}\mathbf{v}^T = \mathbf{I} + (1 + k + 5k)\mathbf{v}\mathbf{v}^T$$

which leads to $k = -\frac{1}{6}$. Consequently,

$$(\mathbf{I} + \mathbf{v}\mathbf{v}^T)^{-1}\mathbf{v} = (\mathbf{I} - \frac{1}{6}\mathbf{v}\mathbf{v}^T)\mathbf{v} = \mathbf{v} - \frac{1}{6}\mathbf{v}\overbrace{\mathbf{v}^T\mathbf{v}}^{=5} = \frac{1}{6}\mathbf{v} = \frac{1}{6}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$