112-2 Midterm (I) Solution

Chapter: 7-1~7-3 & 7-5~7-8

Total: 50 pts

1. $f(x) = \frac{e^{2x}+2}{e^{2x}-2}$ and $e^{2x} \neq 2$, find $f^{-1}(x) = ?$ (10 pts)

$y = f(x) = \frac{e^{2x}+2}{e^{2x}-2} \rightarrow$ exchange x *and* y

$x = \frac{e^{2y}+2}{e^{2y}-2} \rightarrow x(e^{2y}-2) = e^{2y}+2 \rightarrow xe^{2y} - 2x - e^{2y} = 2 \rightarrow e^{2y} = \frac{2x+2}{x-1} \rightarrow \rightarrow$

take ln both side $\rightarrow 2y = \ln\left(\frac{2x+2}{x-1}\right) \rightarrow y = \frac{1}{2}\ln\left(\frac{2x+2}{x-1}\right)$

$$\therefore f^{-1}(x) = \frac{1}{2}\ln\left|\frac{2x+2}{x-1}\right|$$

2. $g(x) = (\sqrt{x+12})^{\sqrt{4x}}$, find $g'(4) = ?$ (10 pts)

$g(x) = \left(\sqrt{x+12}\right)^{\sqrt{4x}} \rightarrow e^{\ln(\sqrt{x+12})^{\sqrt{4x}}} \rightarrow e^{\sqrt{4x}\cdot\frac{1}{2}\ln(x+12)} \rightarrow e^{\sqrt{x}\ln(x+12)}$

$\rightarrow g'(x) = e^{\sqrt{x}\ln(x+12)}[(\frac{\ln(x+12)}{2\sqrt{x}} + \frac{\sqrt{x}}{x+12})]$

$\rightarrow g'(4) = e^{2\ln 16}\left(\frac{\ln(16)}{4} + \frac{2}{16}\right) = e^{\ln 16^2}\left(\ln 2 + \frac{1}{8}\right) = 256\ln 2 + 32$

3. Find the following limits. (10 pts)

a. $\lim_{x\to 0^+}(1 + sin4x)^{\cot x}$ (5 pts) *b.* $\lim_{x\to 0}\frac{x - \tan^{-1}x}{\sin^{-1}x - x}$ (5 pts)

$a.\ \lim_{x\to 0^+}(1+sin4x)^{\cot x} = \lim_{x\to 0^+} e^{cotx\ln(1+sin4x)}\ \left(\frac{0}{0}\right) \rightarrow \lim_{x\to 0^+} e^{\frac{\ln(1+sin4x)}{tanx}} \rightarrow$

$$\rightarrow \lim_{x\to 0^+} e^{\frac{\frac{4cos4x}{1+sin4x}}{sec^2x}} \rightarrow \lim_{x\to 0^+} e^{\frac{\frac{4\cdot 1}{1+0}}{1}} = e^4$$

$$b.\ \lim_{x\to 0}\frac{x - \tan^{-1}x}{\sin^{-1}x - x} = \lim_{x\to 0}\frac{1 - \frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}} - 1}$$

$$\rightarrow \lim_{x\to 0}\frac{\frac{2x}{(1+x^2)^2}}{-\frac{1}{2}\cdot\frac{-2x}{(\sqrt{1-x^2})^3}} \rightarrow \lim_{x\to 0}\frac{\frac{2}{(1+x^2)^2}}{\frac{1}{(\sqrt{1-x^2})^3}} \rightarrow \lim_{x\to 0}\frac{2(\sqrt{1-x^2})^3}{(1+x^2)^2} = 2$$

4. Verify the integration formulas: $\int \frac{\tan^{-1}x}{x^2}dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1}x}{x} + C$ (10 pts)

$$\int \frac{\tan^{-1}x}{x^2}dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1}x}{x} + C \quad \text{(Differential both side)}$$

$$\rightarrow \frac{\tan^{-1}x}{x^2} = \frac{1}{x} - \frac{1}{2}\cdot\frac{2x}{1+x^2} - \frac{\frac{x}{1+x^2} - \tan^{-1}x}{x^2}$$

$$= \frac{1}{x} - \frac{x}{1+x^2} - \frac{x-(1+x^2)\cdot\tan^{-1}x}{(1+x^2)x^2}$$

$$= \frac{1}{(1+x^2)x} - \frac{x-(1+x^2)\cdot\tan^{-1}x}{(1+x^2)x^2}$$

$$= \frac{x}{(1+x^2)x^2} - \frac{x-(1+x^2)\cdot\tan^{-1}x}{(1+x^2)x^2} = \frac{(1+x^2)\cdot\tan^{-1}x}{(1+x^2)x^2}$$

$$= \frac{\tan^{-1}x}{x^2} \rightarrow \textit{Verified}$$

5. Evaluate the integral $\int \frac{e^{sinx}\cos(x)}{\sqrt{e^{2\sin(x)}-1}}dx.$ (10 pts)

$$\int \frac{e^{sinx}\cos(x)}{\sqrt{e^{2\sin(x)}-1}}dx \rightarrow u = e^{sinx}$$

$$\rightarrow du = e^{sinx}\cdot\cos x\, dx \rightarrow dx = \frac{du}{\cos x\cdot e^{\sin x}}$$

$$\int \frac{e^{sinx}\cos(x)}{\sqrt{e^{2\sin(x)}-1}}dx$$

$$= \int \frac{e^{sinx}\cos(x)}{\sqrt{u^2-1}}\cdot\frac{du}{\cos x\cdot e^{\sin x}} = \int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(u) + C$$

$$= \cosh^{-1}\left(e^{sinx}\right) + C$$