Total: 100 points

1. Evaluate the following problems

   (a) (15 points) $\dfrac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy$

   (b) (15 points) $\dfrac{d}{dx}\int_{1/x}^{4} \sqrt{1+\dfrac{1}{t}}\, dt$

Solution:

(a) $\dfrac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy = \sec^2(\tan\theta)\cdot \dfrac{d}{d\theta}(\tan\theta) = \left[\sec^2(\tan\theta)\right]\sec^2\theta.$

(b) $\dfrac{d}{dx}\int_{1/x}^{4} \sqrt{1+\dfrac{1}{t}}\, dt = -\sqrt{1+\dfrac{1}{1/x}}\cdot\left(-x^{-2}\right) = \dfrac{\sqrt{1+x}}{x^2}$

2. (20 points) Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution:

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} \Rightarrow L = \int_0^4 \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1+\dfrac{9}{4}x}\, dx$

Let $u = 1+\dfrac{9}{4}x \Rightarrow du = \dfrac{9}{4}\, dx$. Because $x : 0 \to 4$, thus $u : 1 \to 10$

Therefore, $L = \int_1^{10} \dfrac{4}{9}u^{\frac{1}{2}}\, du = \dfrac{4}{9}\cdot\left(\dfrac{2}{3}u^{\frac{3}{2}}\Big|_1^{10}\right) = \dfrac{8}{27}\left(10\sqrt{10}-1\right)$

3. (20 points) Find the volume of the solid generated by revolving the region R about the axis $\underline{x=-2}$. The region R is bounded by $y=\sqrt{x}$ and the lines $y = 2$ and $x = 0$.

Solution:

$r(y) = 0-(-2) = 2, R(y) = y^2-(-2) = y^2+2$

Thus, $V = \int_0^2 \pi\left[(R(y))^2-(r(y))^2\right]dy = \pi\int_0^2\left(y^4+4y^2\right)dy = \pi\cdot\left[\dfrac{1}{5}y^5+\dfrac{4}{3}y^3\right]\Bigg|_0^2 = \dfrac{256}{15}\pi$

4. Evaluate the integrals

(a) (10 points) $\displaystyle\int_0^{\pi/6} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

(b) (10 points) $\displaystyle\int x^{\frac{1}{2}} \sin\left(x^{\frac{3}{2}}+1\right)\,dx$

(c) (10 points) $\displaystyle\int_0^1 \frac{1}{(1+\sqrt{x})^3}\,dx$

Solution:

(a) Let $u=\cos\theta \Rightarrow du=-\sin\theta\ d\theta$. When $\theta: 0\to\frac{\pi}{6}$, then $u: 1\to\frac{\sqrt{3}}{2}$.

Therefore, $\displaystyle\int_0^{\pi/6} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_1^{\frac{\sqrt{3}}{2}} \frac{-1}{u^2}\,du = \left[\frac{1}{u}\right]\Bigg|_1^{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}-1.$

(b) Let $u=x^{\frac{3}{2}}+1 \Rightarrow du=\frac{3}{2}x^{\frac{1}{2}}\ dx$.

Therefore, $\displaystyle\int x^{\frac{1}{2}}\sin\left(x^{\frac{3}{2}}+1\right)\,dx = \int \frac{2}{3}\sin u\ du = -\frac{2}{3}\cos u + C = -\frac{2}{3}\cos\left(x^{\frac{3}{2}}+1\right)+C$

(c) Let $u=1+\sqrt{x} \Rightarrow du=\frac{1}{2}x^{-1/2}\,dx \Rightarrow x^{1/2}=u-1,\ dx=2x^{1/2}\,du=2(u-1)\,du$.

When $x: 0\to 1$, then $u: 1\to 2$.

Therefore, $\displaystyle\int_0^1 \frac{1}{(1+\sqrt{x})^3}\,dx = \int_1^2 \frac{2(u-1)}{u^3}\,du = \int_1^2 2(u^{-2}-u^{-3})\,du = 2\left(-\frac{1}{u}+\frac{1}{2}\frac{1}{u^2}\right)\Bigg|_1^2 = \frac{1}{4}$