105 final exam

頃

1. Sketch the graph of $f(x) = \exp(-\frac{x^2}{2})$ $f(x) = e^{-\frac{x^2}{2}}$

1. Domain, $\{x \mid x \in R\}$

$f(-x) = -\frac{(-x)^2}{2} = -\frac{x^2}{2} = f(x) \Rightarrow f(x)$ is symmetical with respect to the y-axis (even function)

2. $f'(x) = e^{-\frac{x^2}{2}} \cdot -x = -xe^{-\frac{x^2}{2}}$

$f''(x) = -e^{-\frac{x^2}{2}} + (-x) \cdot e^{-\frac{x^2}{2}} \cdot (-x) = (x^2-1)e^{-\frac{x^2}{2}} = (x+1)(x-1)e^{-\frac{x^2}{2}}$

3. Let $f'(x) = 0$, $x = 0$ $\because f'(0) = 0$ and $f''(0) < 0$

critical point, $x = 0$ $\Rightarrow f(x)$ has a local maximum at $x = 0$

4. by the sign graph $\Rightarrow f' = (+++|----)$ $-\infty \quad 0 \quad \infty$

$f'(x) > 0$ when x is on $(-\infty, 0) \Rightarrow$ increasing

$f'(x) < 0$ when x is on $(0, \infty) \Rightarrow$ decreasing

5. Let $f''(x) = 0$, $x = \pm 1$

inflection points: $x = 1$, $x = -1$

by the sign graph $\Rightarrow f'' = (+++|----|+++)$ $-\infty \quad -1 \quad 1 \quad \infty$

$f''(x) > 0$ when x is on $(-\infty, -1) \Rightarrow$ concave up

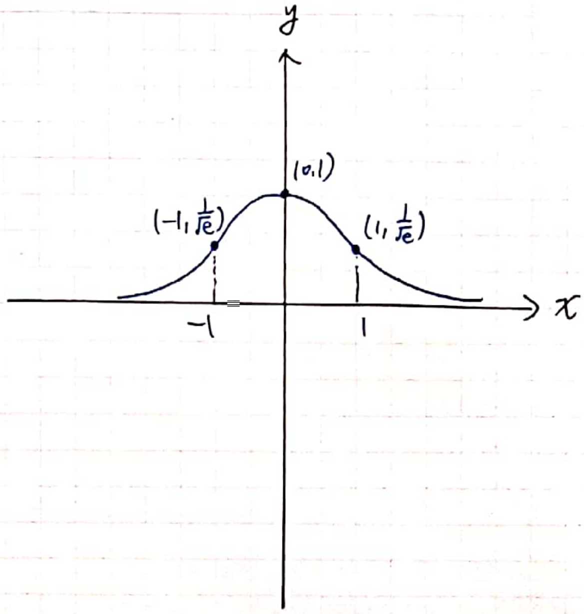
$f''(x) < 0$ when x is on $(-1, 1) \Rightarrow$ concave down

$f''(x) > 0$ when x is on $(1, +\infty) \Rightarrow$ concave up

6. $\lim\limits_{x\to+\infty} e^{-\frac{x^2}{2}} = \lim\limits_{x\to+\infty} \frac{1}{e^{\frac{x^2}{2}}} = 0$; $\lim\limits_{x\to-\infty} e^{-\frac{x^2}{2}} = 0$

$\Rightarrow y = 0$ is a horizontal asymptote of $f(x)$

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,+\infty)$
$f(x)$		$\frac{1}{\sqrt{e}}$		1		$\frac{1}{\sqrt{e}}$	
$f'(x)$	+	+	+	0	−	−	−
$f''(x)$	+	0	−	−	−	0	+
graph	↗	反曲點	↗	相對極大值	↘	反曲點	↘



2. An electric current, when flowing in a circular coil of radius r, exerts a force $F = \frac{kx}{(x^2+r^2)^{\frac{5}{2}}}$ on a small magnet located at a distance x above the center of the coil. Find the value of x that will maximize F, where k is a constant.

$$\frac{dF}{dx} = \frac{k\cdot(x^2+r^2)^{\frac{5}{2}} - \frac{5}{2}(x^2+r^2)^{\frac{3}{2}}(2x)\cdot kx}{(x^2+r^2)^5} = \frac{k(x^2+r^2)^{\cancel{\frac{5}{2}}} - 5x^2k\,(x^2+r^2)^{\cancel{\frac{3}{2}}}}{(x^2+r^2)^{\cancel{5}\,\frac{7}{2}}} = \frac{k(x^2+r^2) - 5x^2k}{(x^2+r^2)^{\frac{7}{2}}}$$

$$= k\,\frac{r^2-4x^2}{(x^2+r^2)^{\frac{7}{2}}} = k\,\frac{-4(x+\frac{r}{2})(x-\frac{r}{2})}{(x^2+r^2)^{\frac{7}{2}}}$$

r^2-4x^2
$= -4(x^2-\frac{r^2}{4})$
$= -4(x+\frac{r}{2})(x-\frac{r}{2})$

Let $\frac{dF}{dx}=0$, $x=\frac{r}{2}$, by the sign graph $\Rightarrow f' = (+++|---)$
$0 \quad \frac{r}{2} \quad +\infty$

x	0	$(0,\frac{r}{2})$	$\frac{r}{2}$	$(\frac{r}{2},\infty)$
F	0		$\frac{16k}{25\sqrt{5}r^4}$	
$\frac{dF}{dx}$	$+$	$+$	0	$-$
graph		↗	abs max	↘

$$F(\tfrac{r}{2}) = \frac{\frac{kr}{2}}{(\frac{r^2}{4}+r^2)^{\frac{5}{2}}} = \frac{kr}{2(\frac{5r^2}{4})^{\frac{5}{2}}}$$

$$= \frac{k\cancel{r}}{\cancel{2}\cdot(\frac{5r^2}{4})(\frac{5r^2}{4})(\frac{\sqrt{5}r}{\cancel{2}})} = \frac{16k}{25\sqrt{5}r^4}$$

F has an absolute maximum $\frac{16k}{25\sqrt{5}r^4}$ at $x=\frac{r}{2}$ ※

3. Find the derivative of the following functions at the given point x_0:

(a) $f(x) = \frac{10^x}{e^{2x}+\ln(x+1)}$, $x_0=0$

$$f'(x) = \frac{10^x\cdot\ln 10\,(e^{2x}+\ln(x+1)) - 10^x\cdot(e^{2x}\cdot 2+\frac{1}{x+1})}{(e^{2x}+\ln(x+1))^2}$$

$$f'(0) = \frac{1\cdot\ln 10\cdot(1+0) - 1\cdot(2+\frac{1}{1})}{1^2} = \frac{\ln 10 - 3}{1} = \underline{\ln 10 - 3}$$ ※

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(b) $g(x) = (\pi + \sin x)^{e^x}$, $x_0 = 0$

$$g(x) = e^{\ln(\pi+\sin x)^{e^x}} = e^{e^x \cdot \ln(\pi+\sin x)}$$

$$g'(x) = (\pi+\sin x)^{e^x} \cdot \frac{d}{dx}\left(e^x \cdot \ln(\pi+\sin x)\right)$$

$$= (\pi+\sin x)^{e^x} \cdot \left(e^x \cdot \ln(\pi+\sin x) + e^x \cdot \frac{\cos x}{\pi+\sin x}\right)$$

$$g'(0) = (\pi+0)^{e^0} \cdot \left(e^0 \cdot \ln\pi + e^0 \cdot \frac{1}{\pi}\right)$$

$$= \pi \cdot \left(\ln\pi + \frac{1}{\pi}\right)$$

$$= \underline{\pi\ln\pi + 1} \ \#$$

4. Evaluate the following integrals : (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{e^{\cot\theta}}{\sin^2\theta}\, d\theta$ (b) $\int \frac{\cos\theta}{\sqrt[3]{\ln(1+\sin\theta)}\,(1+\sin\theta)}\, d\theta$

(c) $\int \left(\sqrt[3]{x^3+1}\right) x^5\, dx$

(a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} e^{\cot\theta}(\csc^2\theta)\, d\theta$ 令 $u = \cot\theta$

$du = -\csc^2\theta\, d\theta$

$$\begin{cases} \theta = \frac{\pi}{2} \Rightarrow u = 0 \\ \theta = \frac{\pi}{6} \Rightarrow u = \sqrt{3} \end{cases}$$

$$= -\int_{\sqrt{3}}^{0} e^u\, du$$

$$= -e^u \Big|_{\sqrt{3}}^{0}$$

$$= \underline{e^{\sqrt{3}} - 1} \ \#$$

(b) 令 $u = \ln(1+\sin\theta)$

$$du = \frac{\cos\theta}{1+\sin\theta}\, d\theta$$

$$\int \frac{\cos\theta}{\sqrt[3]{\ln(1+\sin\theta)}\,(1+\sin\theta)}\, d\theta = \int u^{-\frac{1}{3}}\, du$$

$$= \frac{3}{2}u^{\frac{2}{3}} + C = \frac{3}{2}\left[\ln(1+\sin\theta)\right]^{\frac{2}{3}} + c$$

(c) 令 $u = x^3+1$, $du = 3x^2\, dx$, $\frac{1}{3}du = x^3\, dx$

$$\int \sqrt[3]{x^3+1} \cdot x^3 \cdot x^2\, dx$$

$$= \frac{1}{3}\int u^{\frac{1}{3}} \cdot (u-1)\, du$$

$$= \frac{1}{3} \cdot \int u^{\frac{4}{3}} - u^{\frac{1}{3}}\, du$$

$$= \frac{1}{3} \cdot \left[\frac{3}{7}u^{\frac{7}{3}} - \frac{3}{4}u^{\frac{4}{3}}\right] + C$$

$$= \underline{\frac{1}{7}(x^3+1)^{\frac{7}{3}} - \frac{3}{4}(x^3+1)^{\frac{4}{3}} + C} \ \#$$

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5. $f(x)=\int_{1}^{\ln x}\sqrt{\cos^2 t+e^t}\,dt,\ x>0$, (a) show that f is a one-to-one function

(b) Find $(f^{-1})'(0)$

(a)

$$f'(x)=\frac{d}{dx}\int_{1}^{\ln x}\sqrt{\cos^2 t+e^t}\,dt \quad , 令\ u=\ln x$$

$$=\left(\frac{d}{du}\int_{1}^{u}\sqrt{\cos^2 t+e^t}\,dt\right)\cdot\frac{du}{dx}$$

$$=\sqrt{\cos^2 u+e^u}\cdot\frac{1}{x}$$

$$=\sqrt{\cos^2(\ln x)+e^{\ln x}}\cdot\frac{1}{x}$$

$$=\frac{\sqrt{\cos^2(\ln x)+x}}{x}$$

$\to$ $\because x>0 \Rightarrow f'(x)>0$

$\Rightarrow f$ is an increasing function

$\Rightarrow$ "Horizontal Line Test"

$\Rightarrow f$ is a one-to-one function

(b) $f(f^{-1}(x))=x$

$\Rightarrow f'(f^{-1}(x))\cdot(f^{-1})'(x)=1$

$\therefore (f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$

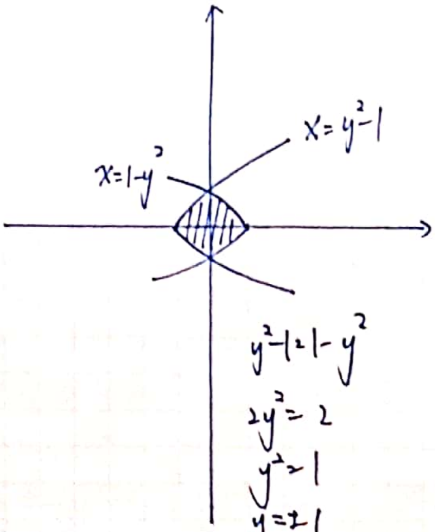
$f(x)=0$, $\ln x=1$, $x=e$

$f(e)=\int_{1}^{1}\sqrt{\cos^2 t+e^t}\,dt=0$

$\therefore f^{-1}(0)=e$

$$(f^{-1})'(0)=\frac{1}{f'(f^{-1}(0))}=\frac{1}{f'(e)}=\frac{1}{\frac{\sqrt{\cos^2(\ln e)+e}}{e}}=\frac{e}{\sqrt{\cos^2(1)+e}}$$ ※

6. Find the area (面積) of the region (區域) bounded by $x=1-y^2$ and, $x=y^2-1$



$y^2-1=1-y^2$

$2y^2=2$

$y^2=1$

$y=\pm1$

$$A=\int_{-1}^{1}(1-y^2)-(y^2-1)\,dy$$

$$=\int_{-1}^{1}1-y^2-y^2+1\,dy$$

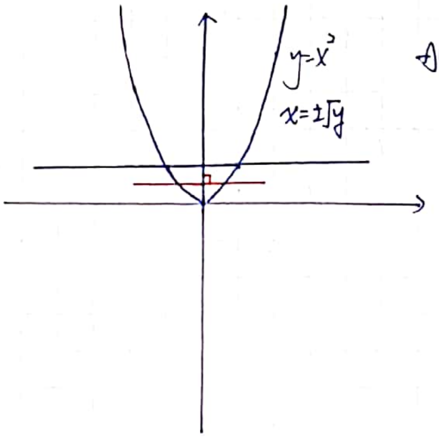
$$=\int_{-1}^{1}2-2y^2\,dy$$

$$=2y-\frac{2}{3}y^3\Big|_{-1}^{1}$$

$$=4-\frac{2}{3}\cdot2$$

$$=\frac{8}{3}$$ ※

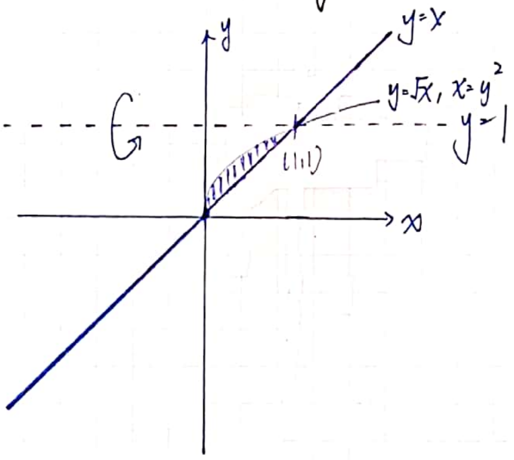
7. Find the Volume of the solid S which has base bounded by $x^2=y$ and $y=1$ and cross-section perpendicular to the y-axis are equilateral triangles (正三角形)



$A(x)=\frac{\sqrt{3}}{4}\times(邊長)^2=\frac{\sqrt{3}}{4}(2\sqrt{y})^2=\frac{\sqrt{3}}{\cancel{4}}\cancel{4}y=\sqrt{3}y$

Volume of $S=\int_0^1\sqrt{3}y\,dy=\sqrt{3}\cdot\frac{1}{2}y^2\Big|_0^1=\underline{\frac{\sqrt{3}}{2}}$ ※

8. Find the Volume of solid obtained by rotating the region bounded by $y=x$ and $y=\sqrt{x}$ about the line $y=1$



Sol 1: shell

$V=\int_0^1 2\pi(1-y)(y-y^2)\,dy$

$=2\pi\int_0^1 y-y^2-y^2+y^3\,dy$

$=2\pi\cdot\int_0^1 y^3-2y^2+y\,dy$

$=2\pi\cdot\left[\frac{1}{4}y^4-\frac{2}{3}y^3+\frac{1}{2}y^2\Big|_0^1\right]$

$=2\pi\cdot\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)$

$=2\pi\cdot\frac{3-8+6}{12}$

$=\underline{\frac{\pi}{6}}$ ※

Sol 2: washer

$V=\int_0^1\pi[R(x)^2-r(x)^2]\,dx$

$=\pi\int_0^1\left[(1-x)^2-(1-\sqrt{x})^2\right]dx$

$=\pi\int_0^1 x^2-2x+1-(x-2\sqrt{x}+1)\,dx$

$=\pi\int_0^1 x^2-3x+2\sqrt{x}\,dx$

$=\pi\cdot\left[\frac{1}{3}x^3-\frac{3}{2}x^2+\frac{4}{3}x^{\frac{3}{2}}\Big|_0^1\right]$

$=\pi\cdot\frac{2-9+8}{6}$

$=\underline{\frac{\pi}{6}}$ ※

9. Find the length of the curve $y=\frac{x^2}{2}-\frac{\ln x}{4},\ 2\le x\le 4$

$\frac{dy}{dx}=x-\frac{1}{4x}$

$L=\int_2^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$1+\left(\frac{dy}{dx}\right)^2=1+x^2-\frac{1}{2}+\frac{1}{16x^2}$

$=x^2+\frac{1}{2}+\frac{1}{16x^2}$

$=\left(x+\frac{1}{4x}\right)^2$

$=\int_2^4 x+\frac{1}{4x}\,dx$

$=\frac{1}{2}x^2+\frac{1}{4}\ln x\Big|_2^4$

$=\frac{1}{2}\times 12+\frac{1}{4}(\ln 4-\ln 2)$

$=6+\frac{1}{4}(2\ln 2-\ln 2)$

$=\underline{6+\frac{1}{4}\ln 2}$ ※

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10. Find the area of the surface obtained by rotating the curve $x=\frac{1}{3}(y^2+2)^{\frac{3}{2}}$, $1\leq y\leq 2$ about the x-axis

$$A=\int_1^2 2\pi\cdot y\cdot\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_1^2 y\cdot(y^2+1)\,dy$$

$$=2\pi\int_1^2 y^3+y\,dy$$

$$=2\pi\cdot\left[\frac{1}{4}y^4+\frac{1}{2}y^2\Big|_1^2\right]$$

$$=2\pi\cdot\left(\frac{15}{4}+\frac{3}{2}\right)$$

$$=2\pi\cdot\frac{21}{4}$$

$$=\underline{\frac{21}{2}\pi}\ \#$$

$$\frac{dx}{dy}=\frac{1}{2}(y^2+2)^{\frac{1}{2}}\cdot(2y)=y(y^2+2)^{\frac{1}{2}}$$

$$1+\left(\frac{dx}{dy}\right)^2=1+y^2(y^2+2)$$

$$=y^4+2y^2+1$$

$$=(y^2+1)^2$$

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