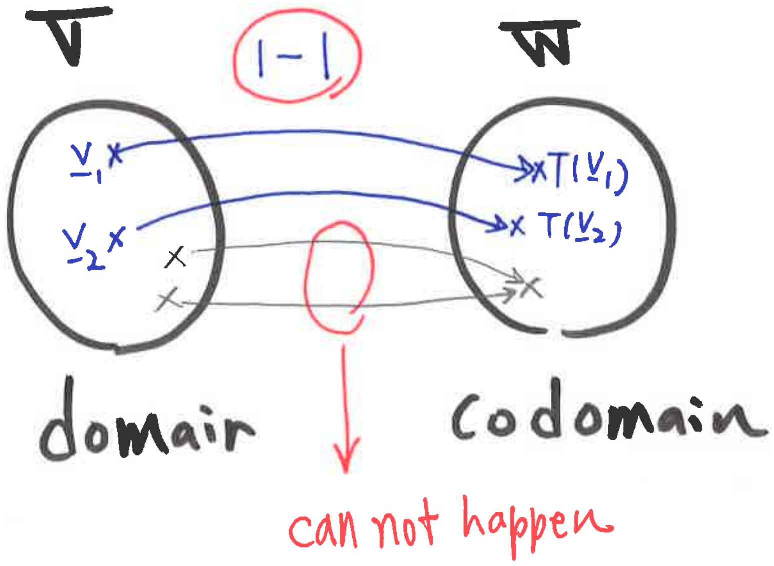


$$\boxed{\underline{v}_1 \neq \underline{v}_2 \rightarrow T(\underline{v}_1) \neq T(\underline{v}_2)}$$

$$\equiv$$

$$\boxed{T(\underline{v}_1) = T(\underline{v}_2) \rightarrow \underline{v}_1 = \underline{v}_2}$$

• **Thm** T is 1-1 $\leftrightarrow$ $\ker(T) = \{\underline{0}_V\}$

$\ker(T)$ || aka null-space(T)

(Recall: $p \rightarrow q \equiv \sim q \rightarrow \sim p$)

Prf "$\rightarrow$"

- We know $T(\underline{0}_V) = \underline{0}_W$
- If $\underline{v} \neq \underline{0}$, then $T(\underline{v}) \neq T(\underline{0}) = \underline{0}_W$ (due to "1-1")
- $\therefore$ No nonzero vectors in V can be transformed into $\underline{0}_W$
- $\therefore$ The only vector that can to transformed into $\underline{0}_W$ is $\underline{0}_V$.

In other words, $\ker(T) = \{\underline{0}\}$

"$\Leftarrow$"

- Assume that we have $T(\underline{v}_1)=T(\underline{v}_2) \Rightarrow T(\underline{v}_1)-T(\underline{v}_2)=\underline{0}$ —— (#1)
- $T(\underline{v}_1)-T(\underline{v}_2)=T(\underline{v}_1-\underline{v}_2)$ —— (#2)
- (#2) & (#1) & $\ker(T)=\{\underline{0}\} \Rightarrow \underline{v}_1-\underline{v}_2=\underline{0} \Rightarrow \underline{v}_1=\underline{v}_2$ ※

- **Thm** $T: V \to V$ is 1-1 iff $\text{range}(T)=V$. (☆)

Prf We know that

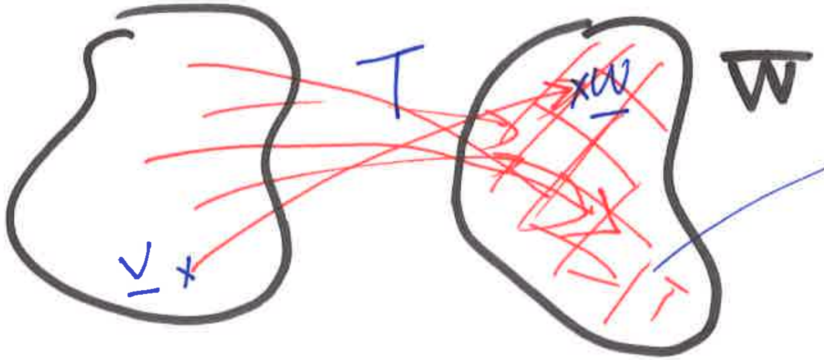
$$\underbrace{\text{rank}(T)}_{\substack{\| \\ \dim(\cancel{\text{null-space}}\ \text{range}(T)) \\ \| \leftarrow (☆) \\ \dim(V)}} + \underbrace{\text{nullity}(T)}_{\substack{\| \\ \dim(\text{null-space}(T))}} = \underbrace{\dim(\text{domain})}_{\substack{\| \\ V}}$$

$\dim(V) \rightarrow \text{nullity}(T)=0$

$\Downarrow$

$\text{null-space}(T)=\{\underline{0}\}$ ※

· onto:

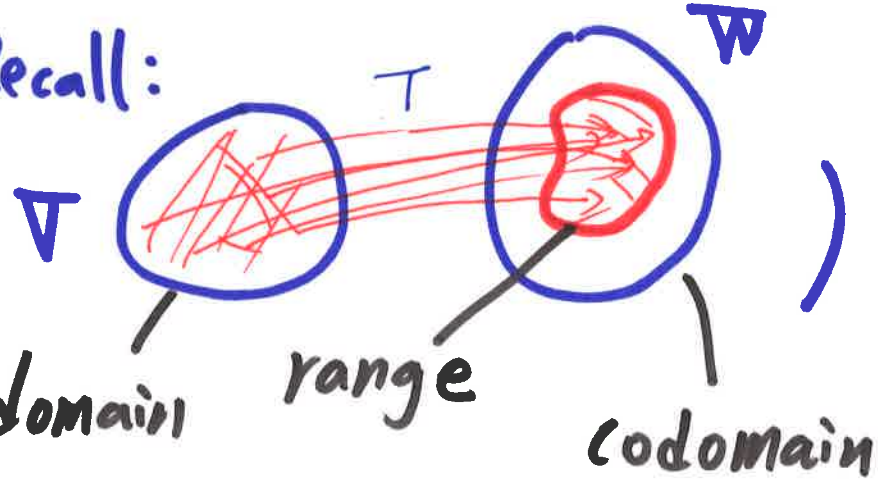


All vectors in W can be seen as images of some vectors in V.

· Thm $T: V \to W$ is onto iff range$(T) = W$

Prf: It is obvious from the definition of "range".

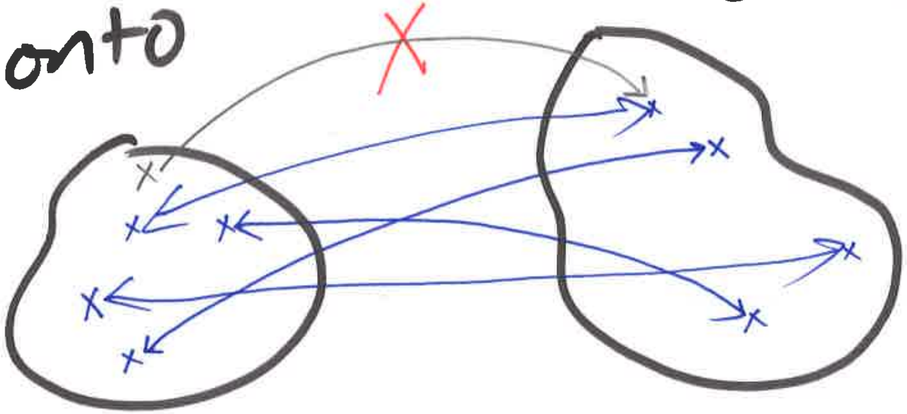
(Recall:

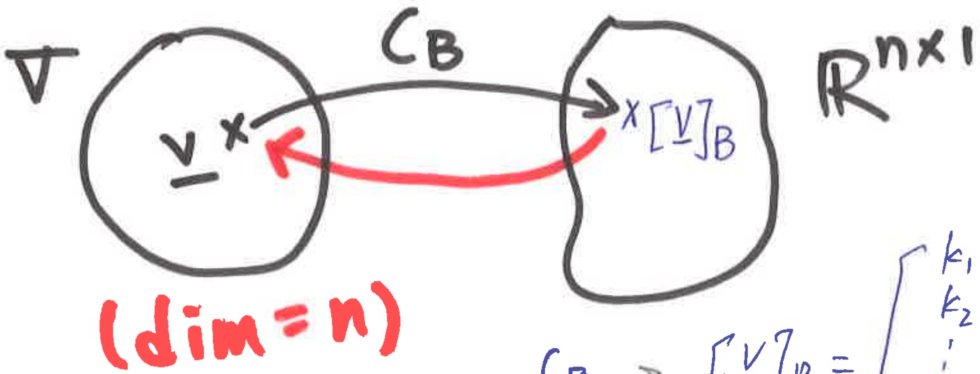


)

· 1-1 and onto

can not happen, because of "1-1"

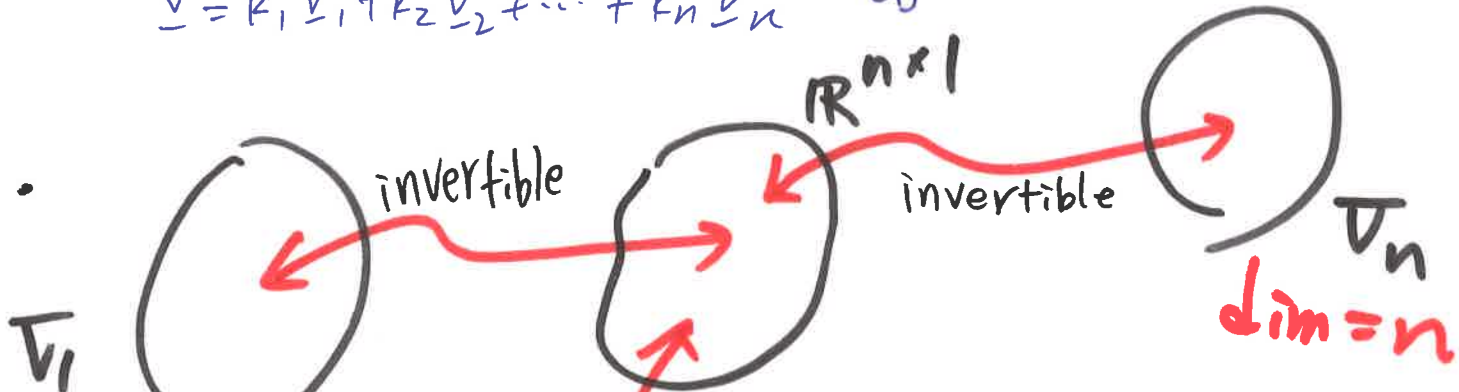
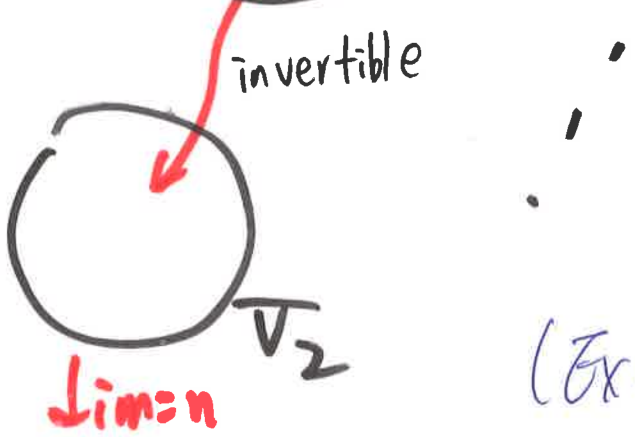




$B = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$

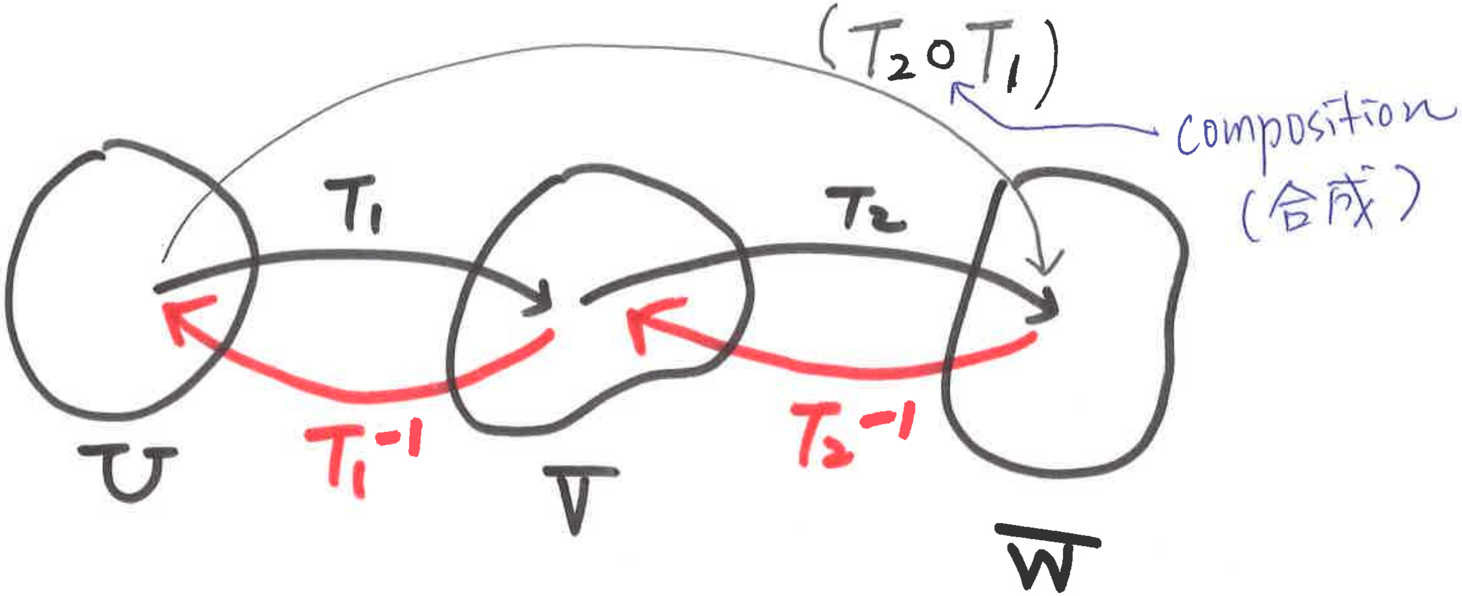
C_B: $[\underline{v}]_B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$

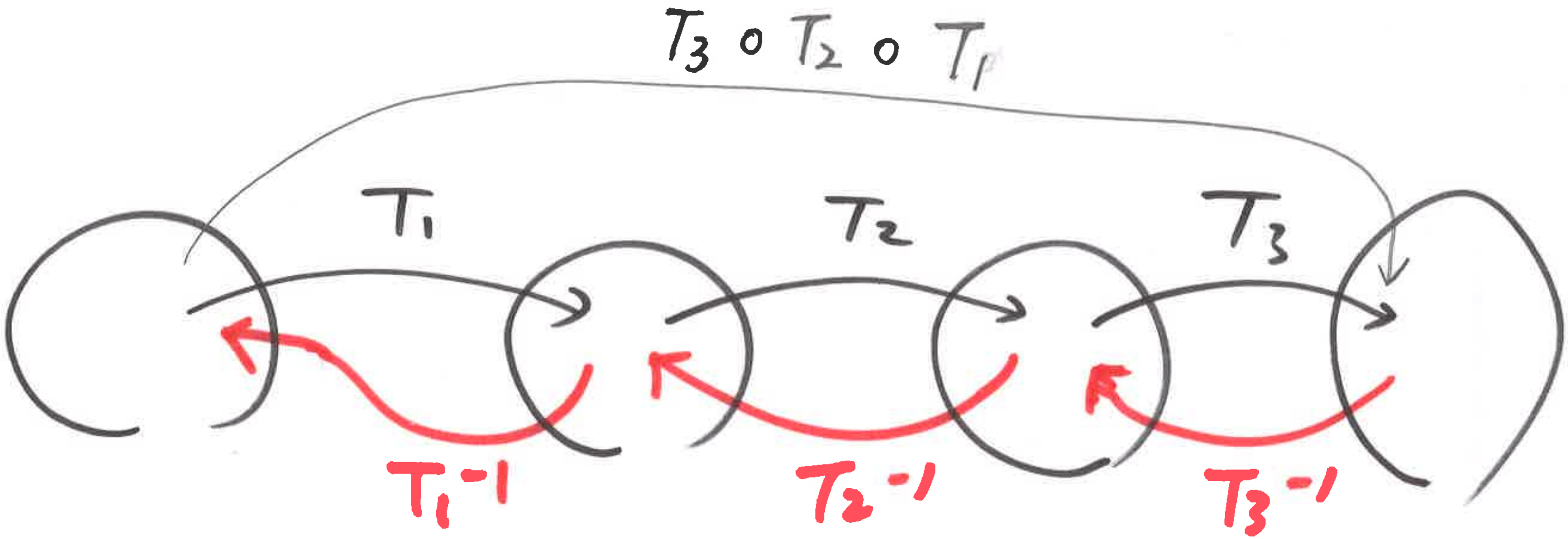
C_B^{-1}: $\underline{v} = k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_n\underline{v}_n$

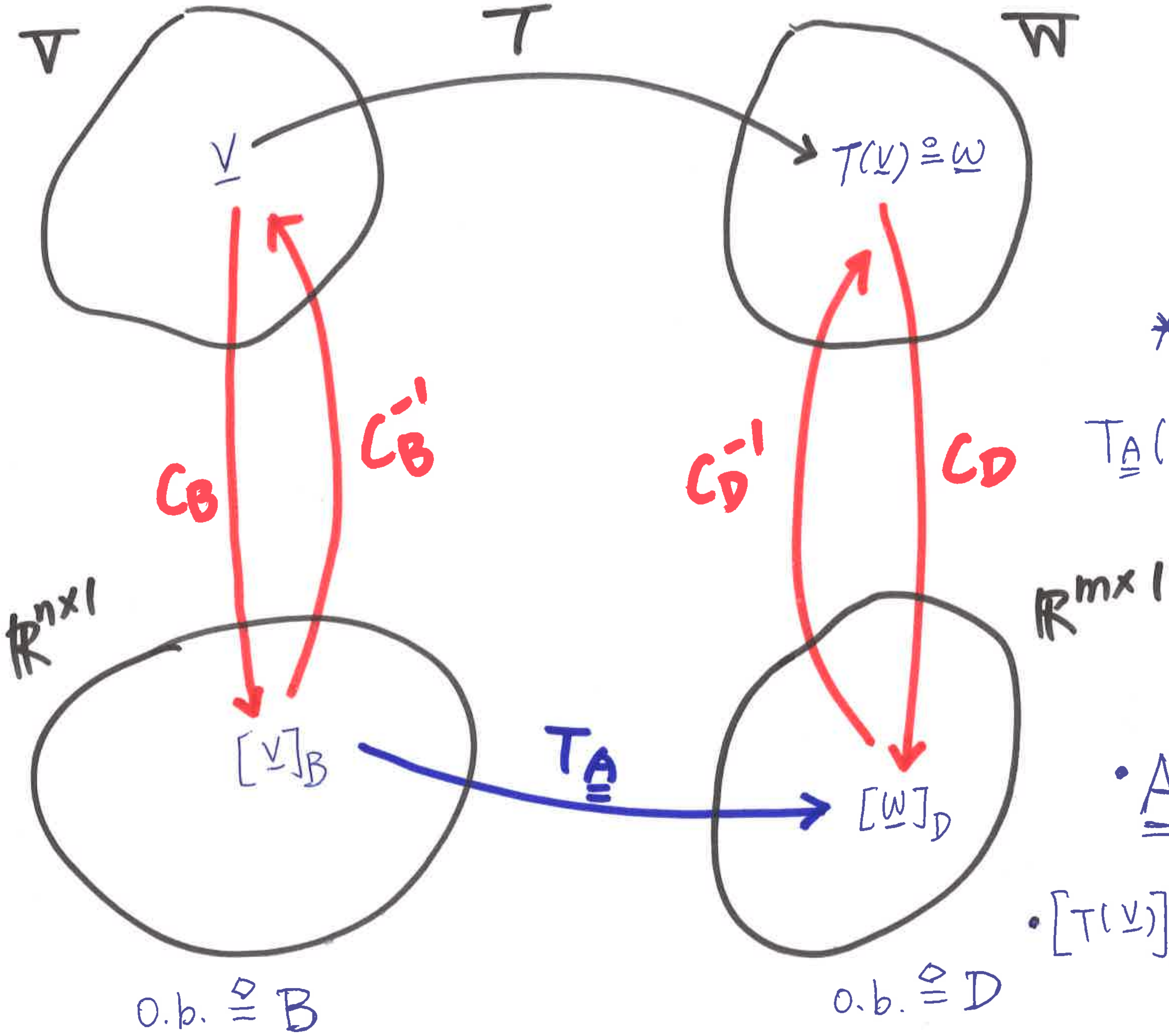
※ If, somehow, one V.S. is easier to investigate than the other isomorphic V.S., then we may want to make investigation in that particular V.S.

(Ex: In many cases, $\mathbb{R}^{n\times 1}$ is such a choice.)

- 

- 

- Generalization is straightforward.



$*\ T_{\underline{\underline{A}}} : \mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$

$T_{\underline{\underline{A}}}(\underline{x}) \triangleq \underline{\underline{A}}\,\underline{x}\ : m\times 1$

$m\times n \qquad n\times 1$

- $\underline{\underline{A}} \triangleq [T]_{D,B}$
- $[T(\underline{v})]_D = [T]_{D,B}[\underline{v}]_B$

$\ast$ Q: How do we obtain $[T]_{D,B} \stackrel{\circ}{=} \underline{\underline{A}}$

$B = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$, $D = \{\underline{w}_1, \underline{w}_2, \cdots \underline{w}_m\}$

- $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- $[T(\underline{v})]_D = \underline{\underline{A}} \cdot [\underline{v}]_B$ — (@)

- We require (@) to hold for any vectors in V.

In particular, (@) must hold for all basis vectors $\underline{v}_1, \underline{v}_2, \cdots \underline{v}_n$. That is,

$$\underline{\underline{A}}[\underline{v}_1]_B = [T(\underline{v}_1)]_D, \quad \underline{\underline{A}}[\underline{v}_2]_B = [T(\underline{v}_2)]_D, \cdots, \underline{\underline{A}}[\underline{v}_n]_B = [T(\underline{v}_n)]_D \quad \text{— (@2)}$$

- But $[\underline{v}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $[\underline{v}_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$ $[\underline{v}_n]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, so

$$\underline{\underline{A}}[\underline{v}_1]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \underline{\underline{A}}[\underline{v}_2]_B = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots \underline{\underline{A}}[\underline{v}_k]_B = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix}, \cdots \quad \text{(@3)}.$$

- Substituting (@3) into (@2), we have

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(\underline{v}_1)]_D, \cdots \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = [T(\underline{v}_k)]_D, \cdots$$

- Summary:

$$[T]_{D,B} = \left[[T(\underline{v}_1)]_D, \; [T(\underline{v}_2)]_D, \cdots [T(\underline{v}_n)]_D \right]$$

• **Ex**: LT: $T: \mathbb{R}^{2\times 1} \to \mathbb{R}^{3\times 1}$ (i.e. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$)

* $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$

• If $B \triangleq \{ \underset{\underline{v}_1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}, \underset{\underline{v}_2}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}} \}$, $D \triangleq \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$,

then $T(\underline{v}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow [T(\underline{v}_1)]_D = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$T(\underline{v}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow [T(\underline{v}_2)]_D = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\therefore [T]_{D,B} = \left[[T(\underline{v}_1)]_D \;\middle|\; [T(\underline{v}_2)]_D \right] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$

• Alternatively, let us adopt $C \triangleq \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $E \triangleq \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Due to the simplicity of the bases (for $\mathbb{R}^{2\times 1}$ and $\mathbb{R}^{3\times 1}$, respectively):

$$\begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This is $[T]_{E,C}$ #

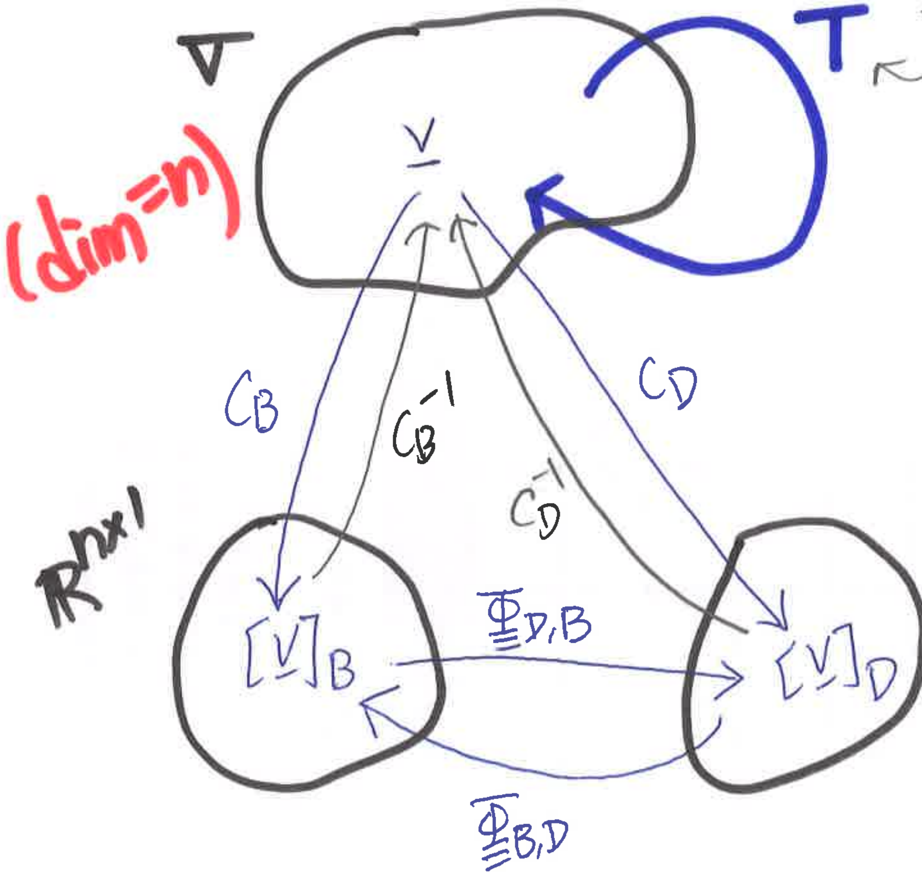
· LT $T: V \to W$

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
V \ni \underline{v} \arrow[rr, "T"] \arrow[d, shift right, "C_B"'] \arrow[dd, bend right=60, "C_C"'] & & \underline{w} \in W \arrow[d, shift left, "C_D"] \arrow[dd, bend left=60, "C_E"] \\
\mathbb{R}^{n\times 1} \ni [\underline{v}]_B \arrow[rr, "{[T]_{D,B}}"] \arrow[u, shift right, "C_B^{-1}"'] \arrow[d, shift right, "\Phi_{C,B}"'] & & {[\underline{w}]_D} \in \mathbb{R}^{m\times 1} \arrow[u, shift left, "C_D^{-1}"] \arrow[d, shift left, "\Phi_{E,D}"] \\
\mathbb{R}^{n\times 1} \ni [\underline{v}]_C \arrow[rr, "{[T]_{E,C}}"] \arrow[u, shift right, "\Phi_{B,C}"'] \arrow[uu, bend left=40, "C_C^{-1}"] & & {[\underline{w}]_E} \in \mathbb{R}^{m\times 1} \arrow[u, shift left, "\Phi_{D,E}"] \arrow[uu, bend right=40, "C_E^{-1}"']
\end{tikzcd}
$$

?

$$[T]_{E,C} = \Phi_{E,D}\,[T]_{D,B}\,\Phi_{B,C}$$

- Lop $T : V \to V$



wrt B: $[T]_{B,B} \doteq [T]_B$

wrt D: $[T]_{D,D} \doteq [T]_D$

$\to n \times n$

- Q: $[T]_B \overset{?}{\leftrightsquigarrow} [T]_D$

- Ans:

$$[T]_{D,D} = \underbrace{\underline{\underline{\Phi}}_{D,B}}_{=\,\underline{\underline{\Phi}}_{B,D}^{-1}} [T]_{B,B} \, \underline{\underline{\Phi}}_{B,D}$$

$$\Rightarrow \quad [T]_{D,D} = \underline{\underline{\Phi}}_{B,D}^{-1} [T]_{B,B} \, \underline{\underline{\Phi}}_{B,D} \quad \text{--- } (\$)$$

- ($) is in the form of $\underline{\underline{B}} = \underline{\underline{P}}^{-1} \underline{\underline{A}} \, \underline{\underline{P}}$,

where $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{P}}$ are $n \times n$ matrices.

· **Thm** $\underline{\underline{A}} \sim \underline{\underline{B}} \Longrightarrow |\underline{\underline{A}}| = |\underline{\underline{B}}|$

$\underbrace{\underline{\underline{A}} \sim \underline{\underline{B}}}_{\text{iff}}$

$\underline{\underline{A}}$ is similar to $\underline{\underline{B}}$

Prf $|\underline{\underline{B}}| = |\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}| = |\underline{\underline{P}}^{-1}| \cdot |\underline{\underline{A}}| \cdot |\underline{\underline{P}}|$

$= |\underline{\underline{P}}^{-1}| \cdot |\underline{\underline{P}}| \cdot |\underline{\underline{A}}| = |\underline{\underline{P}}^{-1} \cdot \underline{\underline{P}}| \cdot |\underline{\underline{A}}|$

$= |\underline{\underline{I}}| \cdot |\underline{\underline{A}}| = 1 \cdot |\underline{\underline{A}}| = |\underline{\underline{A}}|$.

(By the way, $|\underline{\underline{P}}^{-1}| = \dfrac{1}{|\underline{\underline{P}}|}$)

for 4/28/2020 AM 11:20 ~ 12:10

↑

4/29/2020

· **Thm** Invertibility is similarity-invariant.

Prf : $\begin{cases} \text{• Determinant is similarity-invariant.} \\ \text{• Determinant determines invertibility.} \end{cases}$

· **Thm** Rank is similarity-invariant.

Pf $\underline{\underline{B}} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}} \Rightarrow \underline{\underline{P}}\, \underline{\underline{B}} = \underline{\underline{A}}\, \underline{\underline{P}}$ — $\mathrm{rank}(\underline{\underline{P}}\,\underline{\underline{B}}) = \mathrm{rank}(\underline{\underline{A}}\,\underline{\underline{P}})$ (%1)

(%4)

· $\underline{\underline{B}}$ is row equivalent to $\underline{\underline{P}}\,\underline{\underline{B}}$ ($\because \underline{\underline{P}}$ is invertible)

$\Rightarrow$ row-rank$(\underline{\underline{B}})$ = row-rank$(\underline{\underline{P}}\,\underline{\underline{B}})$

$\Rightarrow$ rank$(\underline{\underline{B}})$ = rank$(\underline{\underline{P}}\,\underline{\underline{B}})$ ~ (%2)

$(\underline{\underline{P}}^T)^{-1} = (\underline{\underline{P}}^{-1})^T$

· Let us consider $(\underline{\underline{A}}\,\underline{\underline{P}})^T = \underline{\underline{P}}^T \underline{\underline{A}}^T$

· $\underline{\underline{A}}^T$ is row equivalent to $\underline{\underline{P}}^T \underline{\underline{A}}^T$ ($\because \underline{\underline{P}}^T$ is invertible)

$\Rightarrow$ row-rank$(\underline{\underline{A}}^T)$ = row-rank$(\underline{\underline{P}}^T \underline{\underline{A}}^T)$

$\quad\quad$ ‖ $\quad\quad\quad\quad\quad\quad\quad\quad$ ‖

column-rank$(\underline{\underline{A}})$ $\quad\quad$ column-rank$(\underline{\underline{A}}\,\underline{\underline{P}})$

$\quad\quad$ ‖ $\quad\quad\quad\quad\quad\quad\quad\quad$ ‖

rank$(\underline{\underline{A}})$ $=$ rank$(\underline{\underline{A}}\,\underline{\underline{P}})$ ~ (%3)

· (%1) & (%2) & (%3) $\Rightarrow$ rank$(\underline{\underline{B}})$ = rank$(\underline{\underline{A}})$ ※

· **Thm** Nullity is similarity-invariant.

Prf Consider $\underline{\underline{B}} = \underline{\underline{P}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{P}}$: $n \times n$

- $\text{rank}(\underline{\underline{B}}) = \text{rank}(\underline{\underline{A}})$ (due to the theorem on P.092-2)
- $\text{rank}(\underline{\underline{A}}) + \text{nullity}(\underline{\underline{A}}) = \#(\text{columns of } \underline{\underline{A}}) = n$

  $\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

  $\text{rank}(\underline{\underline{B}}) + \text{nullity}(\underline{\underline{B}}) = \#(\text{columns of } \underline{\underline{B}}) = n$

$\Longrightarrow$ **$\text{nullity}(\underline{\underline{B}}) = \text{nullity}(\underline{\underline{A}})$**

· trace of a square matrix $\underline{\underline{A}} = [a_{ij}]$: $n \times n$

$$\text{tr}(\underline{\underline{A}}) \triangleq \sum_{k=1}^{n} a_{kk} = \text{sum of elements on the diagonal}$$

· Ex: $\text{tr}\left(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 9 & -1 \\ 7 & 6 & 4 \end{bmatrix}\right) = 1 + 9 + 4 = 14$

· **Thm** $T: V \to V$. $\dim(V) = n$

T is invertible iff $[T]_B$ is invertible.

any basis of V

· **Ex** $T(x, y, z) = (x+2y,\ y-z,\ x+3y-z)$

$$\begin{bmatrix} x+2y \\ y-z \\ x+3y-z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow[\text{(suppose)}]{\heartsuit} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have more than ~~two~~ one vectors that can be mapped into $\underline{0}$ → violates "1-1" !

a vector