Total: 100 points

1. (**20** points) If $f(x) = x\sqrt{3+x^2}$, find $(f^{-1})'(-2) =?$

Solution:

- If $y = f^{-1}(x)$, then $x = y\sqrt{3+y^2}$. Do differentiation for both side

$$\Rightarrow 1 = y'\sqrt{3+y^2} + y\frac{2yy'}{2\sqrt{3+y^2}} \quad \Rightarrow \quad y' = \frac{\sqrt{3+y^2}}{3+2y^2}$$

When $x = -2$ for $f^{-1}(x)$:

$$-2 = y\sqrt{3+y^2} \Rightarrow y^2(3+y^2) = 4 \Rightarrow (y^2+4)(y^2-1) = 0$$

After solving the equation, one can find that when $y = 1$ or $y = -1$, $(y^2+4)(y^2-1) = 0$.

However, when $y = 1$, $x = 2$. This solution cannot make x be equal to -2. Therefore, y must be -1.

Thus,

$$(f^{-1})'(-2) = \left.\frac{\sqrt{3+y^2}}{3+2y^2}\right|_{y=-1} = \frac{2}{5}$$

2. (**30** points) Find the derivative of the following functions. (10 points for each)
   (a) $f(x) = \ln[\ln(\ln x)]$
   (b) $f(x) = e^x + e^{e^x} + e^{e^{e^x}}$
   (c) $f(x) = x^{\sin x}$

Solution:

(a) $f(x) = \ln[\ln(\ln x)]$

$$f'(x) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x(\ln x)(\ln \ln x)}$$

(b) $f(x) = e^x + e^{e^x} + e^{e^{e^x}}$

$$f'(x) = e^x + e^{e^x} \cdot e^x + e^{e^{e^x}} \cdot e^{e^x} \cdot e^x$$

(c) $f(x) = y = x^{\sin x} \Rightarrow \ln y = \sin x \ln x$

$$\frac{y'}{y} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \Rightarrow y' = \left(x^{\sin x}\right) \cdot \left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right)$$

3. (**20** points) When a simple electrical circuit containing inductance and resistance (**RL circuit**) but no capacitance has the electromotive force removed, the rate of decrease of the current is proportional to the current. If the current is $I(t)$ amperes t second after cutoff, and if $I = 50$ when $t = 0$, and $I = 10$ when $t = 0.01$, find a formula for $I(t)$.

Solution:

- $\frac{dI}{dt} = kI \Rightarrow \frac{1}{I}\,dI = k\,dt \Rightarrow \int \frac{1}{I}\,dI = \int k\,dt \Rightarrow \ln|I| = kt + C_1 \Rightarrow I(t) = I_0 e^{kt}$ where $I_0 = e^{C_1}$

  Because we have known that $I(0) = 50 \Rightarrow I_0 = 50, \quad I(0.01) = 10 = I_0 e^{0.01k} = 50e^{0.01k}$

  $\Rightarrow k = \frac{\ln(10/50)}{0.01} = 100 \ln\left(\frac{1}{5}\right) \quad \Rightarrow I(t) = 50e^{100\ln(1/5)t} = 50\left(e^{\ln(1/5)}\right)^{100t} = 50\left(\frac{1}{5}\right)^{100t}$

4. (**30** points) Evaluate the integrals (10 points for each)

(a) $\int e^x \csc(e^x+1)\cot(e^x+1)\,dx$

(b) $\int \cot\left(\frac{\theta}{4}\right)\,d\theta$

(c) $\int \frac{\ln x}{x+4x(\ln x)^2}\,dx$

Solution:

(a) Let $u = e^x + 1 \Rightarrow du = e^x\,dx$

$$\int e^x \csc(e^x+1)\cot(e^x+1)\,dx = \int \csc u \cot u\,du = -\csc u + C = -\csc(e^x+1) + C$$

(b) $\int \cot\frac{\theta}{4}\,d\theta = \int \frac{\cos(\theta/4)}{\sin(\theta/4)}\,d\theta.$ Let $u = \sin\left(\frac{\theta}{4}\right) \Rightarrow du = \frac{1}{4}\cos\left(\frac{\theta}{4}\right)d\theta$

$$\int \cot\frac{\theta}{4}\,d\theta = 4\int \frac{1}{u}\,du = 4\ln\left|\sin\left(\frac{\theta}{4}\right)\right| + C$$

(c) $\frac{\ln x}{x+4x(\ln x)^2} = \frac{\ln x}{x}\cdot\frac{1}{1+4(\ln x)^2}.$ Let $u = 1+4(\ln x)^2 \Rightarrow du = 8\cdot\frac{\ln x}{x}\,dx$

$$\int \frac{\ln x}{x+4x(\ln x)^2}\,dx = \int \frac{\ln x}{x}\cdot\frac{1}{1+4(\ln x)^2}\,dx = \frac{1}{8}\int\frac{1}{u}\,du = \frac{1}{8}\ln\left[1+4(\ln x)^2\right] + C$$