ENGINEERING MATHEMATICS (II): LINEAR ALGEBRA MIDTERM SOLUTIONS

Winter 2022

PROBLEM 1

(a) *If you type in the following MATLAB commands*

A=[1 2 3 5; ones(1,4); 9 -3 2 6; 1 3 8 5]; D $= A([1,3],[2,4])$

then what does that show on your screen?

Sol: These MATLAB commands produce

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 1 & 1 & 1 \\ 9 & -3 & 2 & 6 \\ 1 & 3 & 8 & 5 \end{bmatrix}.$$

Since $\mathbf{D}$ is equal to the intersection of the first row and the third row with the second column and the fourth column. Therefore, on your screen you will see

$$\mathbf{D} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

(b) *Write down the MATLAB code to generate a matrix* $\mathbf{C}$ *which has 2 rows. The first row of* $\mathbf{C}$ *is equal to the summation of the first row and the second row of* $\mathbf{A}$*, and the second row of* $\mathbf{C}$ *is equal to 2 times the third row of* $\mathbf{A}$.

Sol: The MATLAB command is C=[A(1,:)+A(2,:);2*A(3,:)].

(c) *Suppose* $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{Ax} = \mathbf{0}$ *for* <u>*ALL*</u> $n \times 1$ *column vector* $\mathbf{x}$. *Is it true that* $\mathbf{A} = \mathbf{0}$*?*

Sol: The answer is yes. If we choose $\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ is a vector with the i^{th} element being equal to 1 and zeros elsewhere, then

$$\mathbf{Ae}_i = \mathbf{a}_i = \mathbf{0}$$

where $\mathbf{a}_i$ denotes the i^{th} column of $\mathbf{A}$. The above holds for $i = 1, \cdots, n$, so $\mathbf{A} = \mathbf{0}$. This completes the proof.

(d) *Suppose*

$$\mathbf{C} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix} [2, 4, 6, 8, 10]$$

Then is $\mathbf{C}$ invertible?

Sol: As we have discussed in the classes (or HWs), any matrix of the form $\mathbf{A} = \mathbf{x}\mathbf{y}^T$ (outer product) has rank=1. This is because $rank(\mathbf{A})$ is equal to the number of linearly independent columns (or rows) of $\mathbf{A}$. Thereby, $\mathbf{C}$ is not invertible, as $rank(\mathbf{C}) = 1 \neq 3$.

(e) *Let S be the set of ordered pair of real number with addition and scalar multiplication defined, respectively, as*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + y_2, y_1 + x_2) \\ c(x, y) &= (cx, cy)\end{aligned}$$

Is S a vector space with these two operations?

Sol: No. with this new definition of addition, Axiom A2, the commutative rule, does not hold. For example, $(1, 5) + (2, -1) = (0, 7) \neq (7, 0) = (2, -1) + (1, 5)$.

(f) *Suppose that $\mathbf{G}$ is a 5×4 matrix, where $\mathbf{g}_1 - 2\mathbf{g}_2 + \mathbf{g}_4 = \mathbf{0}$, in which $\mathbf{g}_i$ denotes the i^{th} column of $\mathbf{G}$. Then how many solution(s) does the linear system $\mathbf{G}\mathbf{x} = \mathbf{0}$ have?*

Sol: There are two ways to solve this problem. The first way is to note that since the columns of $\mathbf{G}$ are linear dependent, $dim(col(\mathbf{G}) < 4$. This implies that

$$dim(\mathcal{N}(\mathbf{G})) = n - dim(col(\mathbf{G})) > 0$$

Therefore, $\mathbf{G}\mathbf{x} = \mathbf{0}$ have infinitely many solutions.

For the second way, we can note that $\mathbf{x} = [1, -2, 0, 1]^T$ is a solution to $\mathbf{G}\mathbf{x} = \mathbf{0}$. It is obvious that $\mathbf{x} = \alpha[1, -2, 0, 1]^T$ $\forall\alpha$ is also a solution to $\mathbf{G}\mathbf{x} = \mathbf{0}$, so it has infinitely many solutions.

(g) *Let $\mathbf{A}$ be a 4×4 matrix with reduced row echelon form given by*

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$ *and* $\mathbf{a}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$, *then determine $\mathbf{a}_3$, where $\mathbf{a}_i$ denotes the i^{th} column of $\mathbf{A}$.*

Sol: The key words are: the linear dependency between columns hold after invertible transformation. It can readily found that $\mathbf{u}_3 = 2\mathbf{u}_1 - \mathbf{u}_2$, where $\mathbf{u}_i$ denotes the i^{th} column of $\mathbf{U}$.

Thereby, $\mathbf{a}_3 = 2\mathbf{a}_1 - \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 3 \\ -5 \end{bmatrix}$

PROBLEM 2

Consider the following system of linear equations

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 5z &= 5 \\ 2x + 5y + \lambda z &= \gamma \end{aligned}$$

Determine the values of λ and γ such that the above system of linear equations has no solution, one solution, and infinitely many solutions. Also **determine the corresponding solution set** *when this system of linear equations is consistent..*

Sol: To solve it, form the augmented matrix first and then reduce it to the row echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -5 & 5 \\ 2 & 5 & \lambda & \gamma \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -3 & 4 \\ 0 & -3 & \lambda+4 & \gamma-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & \lambda+1 & \gamma+2 \end{array}\right].$$

Therefore, the solution sets can be divided into the following three cases:

- If $\lambda = -1$ and $\gamma \neq -2$, then there is no solution.
- If $\lambda = -1$ and $\gamma = -2$, then there is infinitely many solutions. Now the problem becomes

$$\begin{aligned} x + 4y - 2z &= 1 \\ 3y - 3z &= 4 \end{aligned}$$

  Setting the free variable $z = \alpha$, we can obtain the solution

$$\left\{ \left(\frac{-13}{3} - 2\alpha, \frac{4}{3} + \alpha, \alpha\right) \middle| \ \alpha \text{is any arbitrary real numbers} \right\}$$

- If $\lambda \neq 1$, then there is exactly one solution. The problem now becomes

$$\begin{aligned} x + 4y - 2z &= 1 \\ 3y - 3z &= 4 \\ (\lambda+1)z &= \gamma + 2 \end{aligned}$$

  Solving the above, we can obtain the solution

$$\left\{ \left(-\frac{13}{3} - 2\frac{\gamma+2}{\lambda+1}, \frac{4}{3} + \frac{\gamma+2}{\lambda+1}, \frac{\gamma+2}{\lambda+1}\right) \right\}$$

PROBLEM 3

Consider two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 5 \\ 3 & -1 & 2 \end{bmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) *Determine the inverse of* $\mathbf{B}$.

Sol: Form the augmented matrix as

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

so the inverse is $\begin{bmatrix} 1 & -1 & 3 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) *Determine* $det(2\mathbf{A}^2\mathbf{B}) + det(\mathbf{A}^{-1}\mathbf{B}^T)$.

Sol: Conducting the cofactor expansion along the first column of $\mathbf{A}$ yields $det(\mathbf{A}) = 1\Big(-3\times 2 - 5\times(-1)\Big) + 3\Big(2\times 5 - (-3)\times(-3)\Big) = 2$. $\mathbf{B}$ is a diagonal matrix, so $det(\mathbf{B}) = 1\times 2\times(-1) = -2$. So

$$det(2\mathbf{A}^2\mathbf{B}) + det(\mathbf{A}^{-1}\cdot\mathbf{B}^T) = 2^3 det(\mathbf{A})^2 det(\mathbf{B}) + \frac{1}{det(\mathbf{A})} det(\mathbf{B}) = 8\times 4\times -2 - 1 = -64 + (-1) = -65$$

(c) *Determine the nullspace of* $\mathbf{B}^3$.

Sol: $det(\mathbf{B}^3) = det(\mathbf{B})^3 = (-2)^3 = -8 \neq 0$, which implies that $\mathbf{B}^3$ is nonsingular and rank($\mathbf{B}^3$)=3. Invoking the rank-nullity theorem, we have dim($\mathcal{N}(\mathbf{B}^3)$)=3-3=0 and thus

$$\mathcal{N}(\mathbf{B}^3) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(d) *Determine* $adj(\mathbf{A}^{-1})$.

Sol: Since $\mathbf{A} \cdot (adj\ \mathbf{A}) = det(\mathbf{A}) \cdot \mathbf{I} \Rightarrow \mathbf{A}^{-1} \cdot (adj(\mathbf{A}^{-1}) = det(\mathbf{A}^{-1}) \cdot \mathbf{I}$, it follows that

$$adj(\mathbf{A}^{-1}) = \frac{1}{det(\mathbf{A})}\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

PROBLEM 4

Consider two subspaces V *and* W *of* $\mathbf{P}_5$*, where* $\mathbf{P}_5$ *denotes the set of all polynomials of degrees less than* 5. V *and* W *are defined, respectively, as*

$$\mathsf{V} = \{p(x) : p(x) = p(-x)\}$$

and

$$\mathsf{W} = \{q(x) : q(1) = 0\}$$

(a) *Determine dim(V).*

Sol: Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ be an element in V. Since $p(x) = p(-x)$, we have

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = a_0 + a_1(-x) + a_2(-x)^2 + a_3(-x)^3 + a_4(-x)^4$$

$$\Rightarrow a_0 = a_0, a_1 = -a_1, a_2 = a_2, a_3 = -a_3, a_4 = a_4 \Rightarrow a_1 = 0, a_3 = 0$$

Therefore, A typical element (an even function) of V is

$$p(x) = a_0 + a_2x^2 + a_4x^4 = a_0 \cdot 1 + a_2 \cdot x^2 + a_4 \cdot x^4$$

It can readily shown that (say by Wronskian) $\{1, x^2, x^4\}$ are linearly independent and thus is a basis of V). Thereby, $dim(\mathsf{V}) = 3$.

(b) *Determine dim(V∩W).*

Sol: Note that every element of W has 1 as its root. Therefore, together the results in (a) above, for every element in $\mathsf{V} \cap \mathsf{W}$, we have

$$p(x)|_{x=1} = 0 \Rightarrow a_0 + a_2 + a_4 = 0 \Rightarrow a_4 = -a_0 - a_2$$

Consequently, A typical element (an even function) of V is

$$p(x) = a_0 + a_2x^2 + (-a_0 - a_2)x^4 = a_0 \cdot (1 - x^4) + a_2 \cdot (x^2 - x^4)$$

Again, it can readily shown that $\{1 - x^4, x^2 - x^4\}$ are linearly independent and thus is a basis of $\mathsf{V} \cap \mathsf{W}$. Thereby, $dim(\mathsf{V} \cap \mathsf{W}) = 2$.