★ Determinants (行列式)

- Determinants are defined for square matrices.
- **def** Given a 1×1 matrix : $\underline{\underline{A}} = [a]$,

$$|\underline{\underline{A}}| \overset{\circ}{=} \det(\underline{\underline{A}}) \triangleq a$$

- Ex: $\det([3]) = 3$, $\det([-2]) = -2$, $\det([37]) = 37$
- Consider $\underline{\underline{A}} = [a_{ij}]_{n\times n}$

$M_{ij} \overset{(\text{def.})}{:}$ the (i,j)th minor of $\underline{\underline{A}}$

$\Vert$

$\boxed{\text{some value}}$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & a_{ij} & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \cdots \end{bmatrix} \xrightarrow{\det.} \boxed{\text{some value}}$$

$(n\times n)$ $\qquad$ $(n-1)\times(n-1)$

Remove row-i and column-j

- $C_{ij} \triangleq (-1)^{i+j}\cdot M_{ij} \overset{(\text{def})}{:}$ the (i,j)th cofactor of $\underline{\underline{A}}$

- **def**: $|\underline{\underline{A}}| = \det(\underline{\underline{A}}) \triangleq$

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

cofactor expansion (along the first row)

·N.B. Computation process:

det. of $n \times n$ matrix

$\uparrow$

n cofactors

$\uparrow$

minor

$\uparrow$

det. of $(n-1)\times(n-1)$ matrix

$\uparrow$

$(n-1)$ cofactors

$\uparrow$

$\vdots$

$\uparrow$

det. of 1×1 matrix #

·Ex (2×2 matrix)

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M_{11} = \det([d]) = d \rightsquigarrow C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = d$$

$$M_{12} = \det([c]) = c \rightsquigarrow C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = -c$$

$$M_{21} = \det([b]) = b \rightsquigarrow C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = -b$$

$$M_{22} = \det([a]) = a \rightsquigarrow C_{22} = (-1)^{2+2} \cdot M_{22} = M_{22} = a$$

$$\det(\underline{\underline{A}}) = a \cdot C_{11} + b \cdot C_{12} = a \cdot d + b \cdot (-c)$$

$$= a \cdot d - b \cdot c$$

·Just for memorization:

(+) (−)

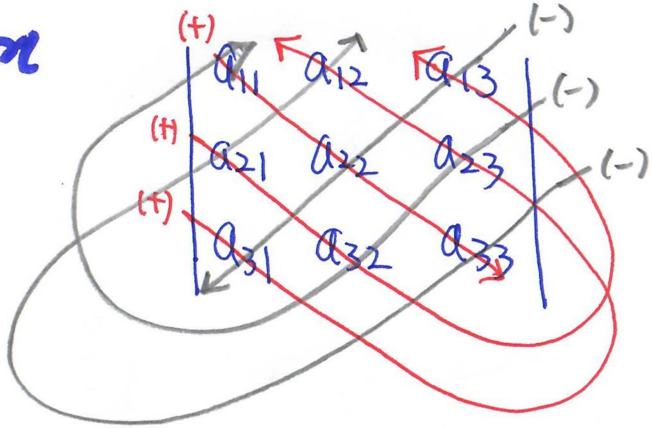
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \doteq \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

· **Ex** (3×3 matrix)

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}\cdot \underbrace{C_{11}}_{=(+1)\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} + a_{12}\cdot \underbrace{C_{12}}_{=(-1)\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} + a_{13}\cdot \underbrace{C_{13}}_{=(+1)\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

$$= a_{11}\cdot a_{22}\cdot a_{33} + a_{21}\cdot a_{32}\cdot a_{13} + a_{31}\cdot a_{23}\cdot a_{12} - a_{13}\cdot a_{22}\cdot a_{31} - a_{23}\cdot a_{32}\cdot a_{11} - a_{33}\cdot a_{21}\cdot a_{12}$$

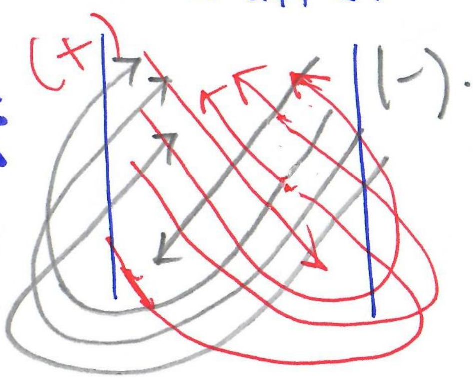
· **Just for memorization**



· **Ex** (4×4 matrix)

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\cdot\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

N.B. $\neq$



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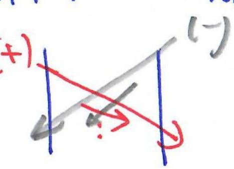

computation for det. is valid only for 2×2 and 3×3 matrices.

Just a coincidence

· **Thm** Determinant of a matrix can be computed by cofactor expansion along any row/column.
(降階展開式)

Prf Omitted (beyond the scope of this course)

· **Thm** Consider $\underline{\underline{A}}$: triangular matrix (i.e. $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ or $\begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ * & * & \cdots & * \end{bmatrix}$).

$\det(\underline{\underline{A}})$ = product of diagonal entries

Prf Obvious.

· **Thm** $\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$ (★) $n \times n$

Prf ①. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ (2x2 matrices)

②. cofactor expansion along the first row of $\underline{\underline{A}}$ = cofactor expansion along the first column of $\underline{\underline{A}}^T$

(★) is valid for $n=2$ (⇝ starting point)
⇓ ← ②
valid for $n=3$
⇓ ← ②
valid for $n=4$
⇓ ← ②
valid for $n=5$
⇓
⋮
(數學歸納法) ← (Proof by mathematical induction)