106 final exam

頡

1. Let $f(x)=xe^{-x}$ (a) Find all absolute extreme values for f
(b) Find all inflection points for f

$f'(x)=1\cdot e^{-x}+x\cdot e^{-x}\cdot(-1)$

$=e^{-x}(1-x) \Rightarrow$ Let $f'(x)=0,\ x=1$

$f''(x)=e^{-x}\cdot(-1)+(-1)\cdot e^{-x}+(-x)\cdot e^{-x}\cdot(-1)$

$=-e^{-x}-e^{-x}+xe^{-x}$

$=-2e^{-x}+xe^{-x}$

$=(x-2)e^{-x} \Rightarrow$ Let $f''(x)=0,\ x=2$

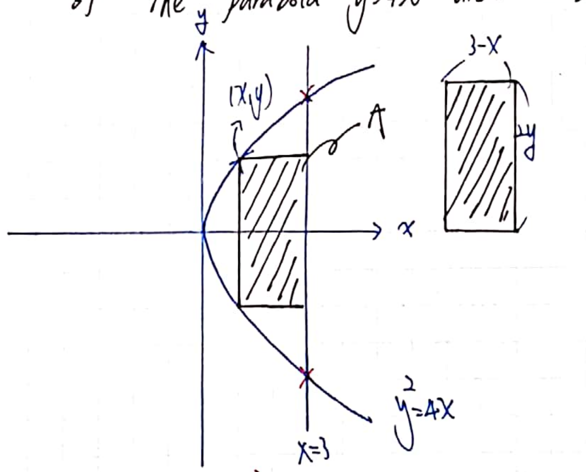
x	$(-\infty,1)$	1	$(1,2)$	2	$(2,\infty)$
$f(x)$		$\frac{1}{e}$		$\frac{2}{e^2}$	
$f'(x)$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	0	$+$
graph		絕對極大值		反曲點	

$f(x)$ has an absolute maximum $\frac{1}{e}$ at $x=1$

$f(x)$ has an inflection point at $x=2$

The inflection point is $(2,\frac{2}{e^2})$

2. Find the maximum area A of the rectangle that can be inscribed in the portion of the parabola $y^2=4x$ and $x=3$



$y^2=12$

$y=\pm2\sqrt{3}$

$A=(3-x)\cdot(2y)$

$=(3-\frac{y^2}{4})\cdot(2y)$

$=6y-\frac{1}{2}y^3=-\frac{1}{2}(y^3-12y)=\frac{-y}{2}(y^2-12)=-\frac{1}{2}y(y+2\sqrt{3})(y-2\sqrt{3})$

$\frac{dA}{dy}=6-\frac{3}{2}y^2 \Rightarrow$ Let $\frac{dA}{dy}=0,\ 6=\frac{3y^2}{2},\ y^2=4,\ y=2$

y	$(0,2)$	2	$(2,2\sqrt{3})$
A		8	
A'	$+$	0	$-$
graph		絕對極大值	

The maximum area is 8 #

A has an absolute maximum 8

at $y=2$

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3. Find the derivative of the following functions. (a) $f(x)=(\sin x)^{\ln x}$

(b) $g(t)=e^{-\sin t}\cdot\ln(2^t+e)$

(c) $h(y)=\log_{10}(\ln(y^2+y+1))$

(a)

Sol1.

Let $y=(\sin x)^{\ln x}$

$\xrightarrow{\text{取}\ln}$ $\ln y=\ln(\sin x)^{\ln x}$

$=\ln x\cdot\ln(\sin x)$

$\xrightarrow{\frac{d}{dx}}$ $\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}\cdot\ln(\sin x)+\ln x\cdot\frac{\cos x}{\sin x}$

$\Rightarrow \frac{dy}{dx}=(\sin x)^{\ln x}\cdot\left[\frac{\ln(\sin x)}{x}+\ln x\cdot\cot x\right]$ ※

Sol2.

$y=(\sin x)^{\ln x}$

$=e^{\ln(\sin x)^{\ln x}}=e^{\ln x\cdot\ln(\sin x)}$

$y'=e^{\ln x\cdot\ln(\sin x)}\cdot\left(\frac{1}{x}\cdot\ln(\sin x)+\ln x\cdot\frac{\cos x}{\sin x}\right)$

$=(\sin x)^{\ln x}\left(\frac{\ln x}{x}+\ln x\cdot\cot x\right)$ ※

(b)

$g'(t)=e^{-\sin t}\cdot(-\cos t)\cdot\ln(2^t+e)+e^{-\sin t}\cdot\frac{2^t\cdot\ln 2+0}{2^t+e}$

$=-e^{-\sin t}\cdot\cos t\cdot\ln(2^t+e)+\frac{e^{-\sin t}\cdot 2^t\ln 2}{2^t+e}$ ※

Note: $\frac{d}{dt}(2^t)=\frac{d}{dt}\left(e^{t\ln 2}\right)$

$=2^t\cdot(1\cdot\ln 2)$

$=2^t\cdot\ln 2$

(c) $h(y)=\log_{10}(\ln(y^2+y+1))=\frac{\ln(\ln(y^2+y+1))}{\ln 10}$

$h'(y)=\frac{1}{\ln 10}\cdot\left(\frac{\frac{2y+1}{y^2+y+1}}{\ln(y^2+y+1)}\right)=\frac{2y+1}{(\ln 10)[\ln(y^2+y+1)](y^2+y+1)}$ ※

4. $f(x)=\int_1^x\frac{1}{\sqrt{t^4+2t^2+3}}\,dt$ (a) show that f is 1-1 function (b) Find $(f^{-1})'(0)$

(a) $f'(x)=\frac{d}{dx}\int_1^x\frac{1}{\sqrt{t^4+2t^2+3}}\,dt=\frac{1}{\sqrt{x^4+2x^2+3}}$

$\Rightarrow f'(x)>0$

$\Rightarrow f(x)$ is an increasing function

$\Rightarrow$ "Horizontal Test Line"

$\Rightarrow f(x)$ is a one-to-one function ※

(b) $f(f^{-1}(x))=x$

$f'(f^{-1}(x))\cdot(f^{-1})'(x)=1 \quad \therefore (f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(0)=\frac{1}{f'(f^{-1}(0))}$ $\quad\because f(1)=0 \quad \therefore f^{-1}(0)=1$

$=\frac{1}{f'(1)}$

$=\frac{1}{\sqrt{6}}$ ※

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5. Evaluate the following integrals: (a) $\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{x}\cdot\cos\sqrt{x}}{\sqrt{x}}\,dx$ (b) $\int_2^4 \frac{\log_3(x-1)}{x-1}\,dx$

(c) $\int_0^1 \frac{x\cdot\exp(x^2)}{e+\exp(x^2)}\,dx$

(a) 令 $u=\cos\sqrt{x}$

$du=-\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$

$\to -2\,du=\frac{\sin\sqrt{x}}{\sqrt{x}}$

$\begin{cases} x=\pi^2 \Rightarrow u=\cos\pi=-1 \\ x=\frac{\pi^2}{4} \Rightarrow u=\cos\frac{\pi}{2}=0 \end{cases}$

$\Rightarrow -2\int_0^{-1} u\,du = -\cancel{2}\cdot\frac{1}{\cancel{2}}u^2\Big|_0^{-1} = -(1-0) = \underline{-1}$ #

(b)

$\int_2^4 \frac{\log_3(x-1)}{x-1} = \int_2^4 \frac{\ln(x-1)}{\ln 3\cdot(x-1)}\,dx$ 令 $u=\ln(x-1)$, $du=\frac{1}{x-1}dx$ $\begin{cases} x=4,\ u=\ln 3 \\ x=2,\ u=\ln 1=0 \end{cases}$

$=\frac{1}{\ln 3}\int_0^{\ln 3} u\cdot du = \frac{1}{\ln 3}\cdot\frac{1}{2}u^2\Big|_0^{\ln 3}$

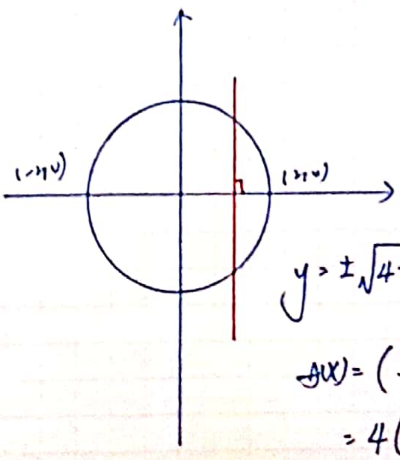
$=\frac{\ln 3\cdot\cancel{\ln 3}}{2\cancel{\ln 3}} = \underline{\frac{\ln 3}{2}}$ #

(c) $\int_0^1 \frac{x\cdot e^{x^2}}{e+e^{x^2}}\,dx$ 令 $u=e+e^{x^2}$, $du=e^{x^2}\cdot 2x\,dx$, $\frac{1}{2}du=x\cdot e^{x^2}dx$ $\begin{cases} x=1,\ u=2e \\ x=0,\ u=e+1 \end{cases}$

$=\frac{1}{2}\int_{1+e}^{2e}\frac{1}{u}\,du$

$=\left[\frac{1}{2}\cdot\ln|u|\right]\Big|_{1+e}^{2e} = \frac{1}{2}\cdot\left[\ln 2e-\ln(1+e)\right] = \underline{\frac{1}{2}\cdot\ln\left(\frac{2e}{1+e}\right)}$ #

6. The base consists the region inside the circle $x^2+y^2=4$, Cross-section perpendicular to the x-axis are squares. Find the volume of the solid



$V=\int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} 4(4-x^2)\,dx = 4\int_{-2}^{2} 4-x^2\,dx$

$=4\cdot\left[4x-\frac{1}{3}x^3\Big|_{-2}^{2}\right] = 4\cdot\left(16-\frac{16}{3}\right) = 4\cdot\frac{32}{3} = \underline{\frac{128}{3}}$ #

$y=\pm\sqrt{4-x^2}$

$A(x)=\left(2\sqrt{4-x^2}\right)^2$

$=4(4-x^2)$

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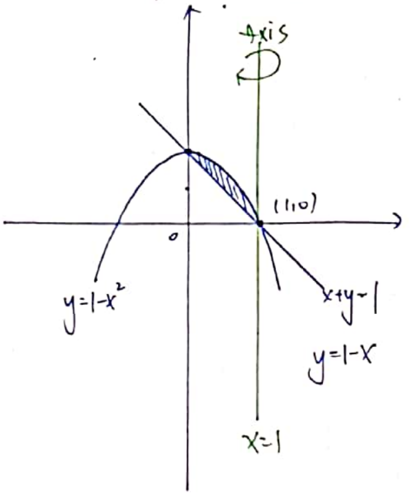
7. Use the cylindrical shell method to find the volume of the solid obtained by revolving the region bounded by $y=1-x^2$, $x+y=1$ about $x=1$



$$V=\int_0^1 2\pi(1-x^2-1+x)(1-x)\,dx$$

$$=2\pi\int_0^1 (1-x)(-x^2+x)\,dx$$

$$=2\pi\int_0^1 -x^2+x+x^3-x^2\,dx$$

$$=2\pi\int_0^1 -2x^2+x^3+x\,dx$$

$$=2\pi\cdot\left[\tfrac{-2}{3}x^3+\tfrac{1}{4}x^4+\tfrac{1}{2}x^2\Big|_0^1\right]$$

$$=2\pi\cdot\left(\tfrac{-2}{3}+\tfrac{1}{4}+\tfrac{1}{2}\right)$$

$$=2\pi\cdot\frac{-8+3+6}{12}$$

$$=\frac{\pi}{6}\ \text{※}$$

8. (a) Find the arc length of the curve $y=\frac{1}{24}(x^2+16)^{\frac{3}{2}}$, $0\le x\le 3$

(b) Find the area of the surface of revolution by revolving the same curve in (a) about y-axis

(a)
$$y'=\frac{dy}{dx}=\frac{1}{\cancel{24}_{8}}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot(x^2+16)^{\frac{1}{2}}\cdot(\cancel{2}x)=\frac{1}{8}x\cdot(x^2+16)^{\frac{1}{2}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2\cdot(x^2+16)}{64}=\frac{64+x^4+16x^2}{64}=\frac{(x^2+8)^2}{64}$$

$$\Rightarrow L=\int_0^3\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^3\frac{(x^2+8)}{8}\,dx$$

$$=\frac{1}{8}\cdot\left(\frac{1}{3}x^3+8x\Big|_0^3\right)$$

$$=\frac{1}{8}\cdot(9+24)$$

$$=\frac{33}{8}\ \text{※}$$

(b)
$$A=\int_0^3 2\pi\cdot x\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\cdot\int_0^3\frac{1}{8}x\cdot(x^2+8)\,dx$$

$$=\frac{\pi}{4}\cdot\int_0^3 x^3+8x\,dx$$

$$=\frac{\pi}{4}\cdot\left[\frac{1}{4}x^4+4x^2\Big|_0^3\right]$$

$$=\frac{\pi}{4}\cdot\left(\frac{81}{4}+36\right)$$

$$=\frac{\pi}{4}\cdot\frac{225}{4}$$

$$=\frac{225\pi}{16}\ \text{※}$$

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