Total: 100 points

1. A function is given as

$$f(x) = 1 + \sqrt{2}\csc x + \cot x$$

   (a) (**20** points) Find an equation for the tangent to the curve at $P(\frac{\pi}{4}, 4)$.

   (b) (**20** points) Find the horizontal tangent to the curve at $Q$.

**Solution:**

(a) $f'(x) = -\sqrt{2}\csc x \cot x - \csc^2 x = -\dfrac{1}{\sin x} \cdot \dfrac{\sqrt{2}\cos x + 1}{\sin x}$

$f'(\dfrac{\pi}{4}) = -4 \Rightarrow$ tangent line: $y - 4 = -4(x - \dfrac{\pi}{4}) \Rightarrow y = -4x + (\pi + 4)$

(b) Find $x$ when $f'(x) = 0 \Rightarrow \sqrt{2}\cos x + 1 = 0 \Rightarrow x = \dfrac{3\pi}{4} \Rightarrow f(\dfrac{3\pi}{4}) = 2.$

The horizontal tangent line is $y = 2$.

2. Find $f'(x)$ for the following functions

   (a) (**10** points) $f(x) = 2\left(\dfrac{1}{\sqrt{x}} + \sqrt{x}\right)$.

   (b) (**10** points) $f(x) = \dfrac{\sin x + \cos x}{\cos x}$.

**Solution:**

(a) $f'(x) = -x^{-3/2} + x^{-1/2}$

(b) $f'(x) = \sec^2 x$

3. A function is given as

$$f(x) = \begin{cases} 0 & x \le 0 \\ 5 - x & 0 < x < 4 \\ \frac{1}{5-x} & 4 \le x \end{cases}$$

(a) (**10** points) Find left-hand derivative $f'_-(4)$

(b) (**10** points) Find right-hand derivative $f'_+(4)$.

(c) (**10** points) Where is $f$ discontinuous?

(d) (**10** points) Where is $f$ not differentiable?

Hint: Sketch $f(x)$ may be helpful.

**Solution:**

(a) $f'_-(4) = \lim_{h\to 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0^-} \frac{[5-(4+h)] - (5-4)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

(b) $f'_+(4) = \lim_{h\to 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{5-(4+h)} - \frac{1}{5-4}}{h} = \lim_{h\to 0^+} \frac{\frac{1}{1-h} - 1}{h} = \lim_{h\to 0^+} \frac{h}{h(1-h)} = 1$

(c) Discontinuity: $x = 0, x = 5$.

(d) Not differentiable: $x = 0, x = 4, x = 5$.