Total: 100 points

1. Let $f(x) = \dfrac{\sqrt{\pi x}(x-7)}{|x-7|}$

   (a) (**20** points) Find $\displaystyle\lim_{x\to 7^+} \frac{\sqrt{\pi x}(x-7)}{|x-7|}$

   (b) (**20** points) Find $\displaystyle\lim_{x\to 7^-} \frac{\sqrt{\pi x}(x-7)}{|x-7|}$

Solution:

(a) When $x > 7, |x-7| = x-7 \Rightarrow \displaystyle\lim_{x\to 7^+} \frac{\sqrt{\pi x}(x-7)}{|x-7|} = \lim_{x\to 7^+} \frac{\sqrt{\pi x}(x-7)}{(x-7)} = \lim_{x\to 7^+} \sqrt{\pi x} = \sqrt{7\pi}$.

(b) When $x < 7, |x-7| = -(x-7) \Rightarrow \displaystyle\lim_{x\to 7^-} \frac{\sqrt{\pi x}(x-7)}{|x-7|} = \lim_{x\to 7^-} \frac{\sqrt{\pi x}(x-7)}{-(x-7)} = \lim_{x\to 7^-} -\sqrt{\pi x} = -\sqrt{7\pi}$.

2. Assume $f(x) = x^4 \cos\left(\dfrac{2}{x}\right)$.

   (a) (**20** points) Show that $-x^4 \le x^4 \cos\left(\dfrac{2}{x}\right) \le x^4$

   (b) (**20** points) Find $\displaystyle\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$

Solution:

(a) Because $-1 \le \cos\left(\dfrac{2}{x}\right) \le 1$ and $x^4 \ge 0$ for all x, thus $-x^4 \le x^4 \cos\left(\dfrac{2}{x}\right) \le x^4$.

(b) From (a), $-x^4 \le x^4 \cos\left(\dfrac{2}{x}\right) \le x^4$. And we know that $\displaystyle\lim_{x\to 0}\left(-x^4\right) = 0$ and $\displaystyle\lim_{x\to 0}\left(x^4\right) = 0$.
By using Sandwich theorem, $\displaystyle\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$

3. (**20** points) Use $\epsilon - \delta$ definition to prove the following limit.

$$\lim_{x \to 2} (2 - 3x) = -4$$

Solution:

$\forall \epsilon > 0$, one can choose $\delta = \dfrac{\epsilon}{3}$. If $0 < |x-2| < \delta = \dfrac{\epsilon}{3} \Longrightarrow 0 < |(-3) \cdot (x-2)| < 3\delta = \epsilon$.

$\Longrightarrow 0 < |-3x+6| < \epsilon \Longrightarrow 0 < |(2-3x)+4| < \epsilon \Longrightarrow 0 < |(2-3x)-(-4)| < \epsilon$.

Thus, by the definition of limits, $\lim_{x \to 2} (2 - 3x) = -4$