Engineering mathematics II

Final exam., 6/16/2020

This is an open-book test. The total score is 110 points. Please show your computations.

1. Let us consider a linear operator T acting on $\mathcal{P}_1 = \{ax + b | a, b \in \mathcal{R}\}$. It is defined by

$$T(x) = 3x - 7\,, \text{ and } T(1) = -2x + 1\,.$$

(a). (5%) $T(ax+b) = ?$
$<$ Hint: $>$ $T(ax+b) = a \cdot T(x) + b \cdot T(1)$.

(b). (5%) Let the matrix of T with respect to the ordered basis $B = \{x, 1\}$ be denoted as $[T]_B$. Find $[T]_B$.

(c). (5%) Find the eigenvalues of $[T]_B$.

(d). (5%) Is $[T]_B$ orthogonally diagonalizable?

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(e). (5%) Let S denote the inverse transformation of T. For a given pair of real numbers α and β, then, $S(\alpha x + \beta) = ?$

(f). (5%) Find the matrix of S with respect to B.

2.(10%) Assume that $\underline{\underline{A}}$ is an invertible matrix. Furthermore, assume that λ is an eigenvalue of $\underline{\underline{A}}$. Let $\underline{x}$ be an eigenvector of $\underline{\underline{A}}$ corresponding to λ. Show that $\underline{x}$ is also an eigenvector of $\underline{\underline{A}}^{-1}$, and the corresponding eigenvalue is $1/\lambda$.
$<$ Hint: $>$ Start with $\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$.

3. Consider the inner-product space of functions defind over the interval between 0 and 1, with the inner product $< p(x), q(x) > = \int_0^1 p(x)\, q(x)\, dx$. Let us consider two vectors: $f(x) = x$ and $g(x) = e^{-x}$.

(a). (5%) Find the norms of $f(x)$ and $g(x)$, respectively.

(b). (5%) Find the angle between $f(x)$ and $g(x)$.

(c). (5%) Find the distance between $f(x)$ and $g(x)$.

(d). (10%) Let $\mathcal{W}$ denote the span of $f(x)$ and $g(x)$. Find an orthonormal basis for $\mathcal{W}$.

4. Consider the matrix below:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & \sqrt{10} \\ \sqrt{10} & 4 \end{bmatrix}.$$

(a). (10%) Find a matrix $\underline{\underline{P}}$ such that $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{D}}$ is a diagonal matrix.

(b). (5%) Continued from the preceeding subproblem, $\underline{\underline{D}} = ?$

(c). (5%) Is it possible to find a matrix $\underline{\underline{Q}}$ such that $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}}$ is a diagonal matrix?

5.(10%) Consider the system of over-determined system of linear equations, wherein there are more equations than unknowns:

$$\begin{cases} 2x + y = 2 \\ x - 2y = 5 \\ 3x + y = -1 \\ x + 3y = -4 \\ 7x - y = 6 \end{cases}$$

Find the LSE (least square error) solution to this system of linear equations.

6. Consider the complex matrices below:

$$\underline{\underline{A}} = \begin{bmatrix} 1+2i & 3 \\ 1 & 1+2i \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 1 \end{bmatrix}, \quad \underline{\underline{C}} = \begin{bmatrix} 3 & 1+2i \\ 1+2i & 1 \end{bmatrix}.$$

(a). (5%) One of those matrices is an Hermitian matrix. Please identify it.

(b). (5%) Continued from the preceeding subproblem, find the eigenvalues of this Hermitian matrix.

(c). (5%) Continued from the preceeding subproblem, is this Hermitian matrix also a unitary matrix?