Matrices and Systems of Equations

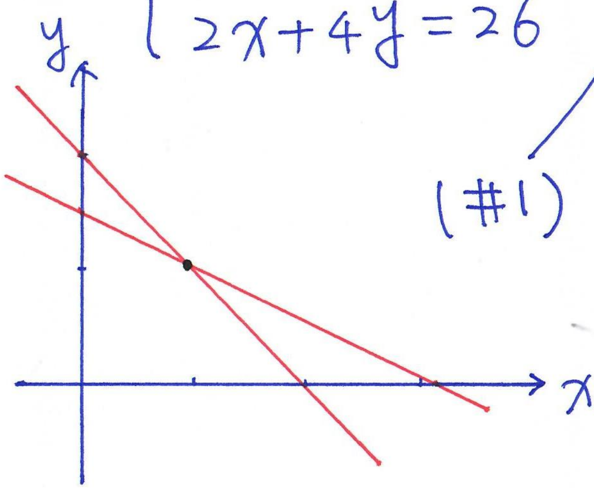
Systems of linear equations (線性联立方程式)

條件式 $\equiv$ 方程式

線性 $\leftrightarrow$ "一次"多項式

· 「雞兔同籠」問題:

$$\begin{cases} x+y=8 \\ 2x+4y=26 \end{cases} \quad (\#1)$$



* equation $\rightsquigarrow$ condition

· General form:

$$(\bigstar)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & (\bigstar 1) \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & (\bigstar 2) \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n & (\bigstar n) \end{cases}$$

Given: a_{ij}, b_i $\quad i = 1, 2, \cdots, m$

unknowns: x_j $\quad j = 1, 2, \cdots, n$

n unknowns, m equations

· A solution to ($\bigstar$) is a n-tuple (ordered seq. of n numbers): $(x_1, x_2, \cdots, x_n)$ that satisfies ($\bigstar 1$), ($\bigstar 2$), $\cdots$, ($\bigstar n$)

- A system of linear equations can be consistent or inconsistent.

consistent → at least one solution exists

inconsistent → no solution exists

solution → some $(x_1, x_2, \cdots, x_n)$ that satisfies (#)

at least one → can be one and only one or many

one and only one $\equiv$ unique

- Let's try to solve (#1):

Recall (from 國中): 代入消去法

(乘除)加減消去法

Here we focus on this

$$\text{A}\ \begin{cases} x+y=8 \xrightarrow{*-2} \\ 2x+4y=26 \xrightarrow{\text{copy}} \end{cases} \quad \text{B}\ \begin{cases} -2x-2y=-16 \xrightarrow{\text{copy}} \\ 2x+4y=26 \xrightarrow{+} \end{cases} \quad \text{C}\ \begin{cases} -2x-2y=-16 \xrightarrow{\text{copy}} \\ 0x+2y=10 \xrightarrow{*\frac{1}{2}} \end{cases}$$

$$\text{D}\ \begin{cases} -2x-2y=-16 \xrightarrow{\text{copy}} \\ 0x+1\cdot y=5 \xrightarrow{*2} \end{cases} \quad \text{E}\ \begin{cases} -2x-2y=-16 \xrightarrow{+} \\ 0x+2y=10 \xrightarrow{\text{copy}} \end{cases} \quad \text{F}\ \begin{cases} -2x+0y=-6 \xrightarrow{*\frac{-1}{2}} \\ 0x+2y=10 \xrightarrow{*\frac{1}{2}} \end{cases}$$

$$\text{G}\ \begin{cases} 1\cdot x+0\cdot y=3 \\ 0\cdot x+1\cdot y=5 \end{cases} \qquad \text{Ans: } \begin{cases} x=3 \\ y=5 \end{cases} \ ※$$

※: A, B, C, D, E, F, G are equivalent systems (of linear equations)

have the same set of solutions

- Obviously, it is much easier to find the solation from G than from A.
- We would like to reduce G to A, if possible, via some operations.

- Q: What kind of operations are allowed?

Ans: Those operations can be classified into 3 classes:

Type 1. Interchange two eqs. in (☆)

Type 2. Multiply an equation by a nonzero number

Type 3. Add a nonzero of an equation to another equation

- "A + B" ∈ Type 3 ; C ∈ Type 2 ;

  "D + E" ∈ Type 3 ; F : two Type-2 ops

- An example of Type-1: $\begin{cases} x+y=8 \\ 2x+4y=26 \end{cases} \equiv \begin{cases} 2x+4y=26 \\ x+y=8 \end{cases}$

- A complex system (of l. eqs) $\xrightarrow[\text{(type 1, 2, 3)}]{\text{elementary ops}}$ A simple system (of l. eqs)

- In the reduction(s) from Ⓐ to Ⓖ, x and y (unknowns) can be seen as nothing but place holders.
- It is the coeffs (to x and y) and the numbers on the RHS (of eqs.) that are the main thing.
- Ⓐ $\rightsquigarrow$ Ⓖ can be seen/regarded as $\left(\xrightarrow{?}\right)$:

Ⓐ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} 1 & 1 & 8 \\ 2 & 4 & 26 \end{bmatrix}$$

Ⓑ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} -2 & -2 & -16 \\ 2 & 4 & 26 \end{bmatrix}$$

Ⓒ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} -2 & -2 & -16 \\ 0 & 2 & 10 \end{bmatrix}$$

Ⓓ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} -2 & -2 & -16 \\ 0 & 1 & 5 \end{bmatrix}$$

Ⓔ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} -2 & -2 & -16 \\ 0 & 2 & 10 \end{bmatrix}$$

Ⓕ
$$\square \cdot x + \square \cdot y = \square$$
$$\begin{bmatrix} -2 & 0 & -6 \\ 0 & 2 & 10 \end{bmatrix}$$

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$\square \cdot x + \square \cdot y = \square$

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} \rightsquigarrow$ rectangular array of numbers

$\downarrow$ Δ ((by) def)

matrix

• Some terminologies:

matrix

$$\underset{=}{A} = \begin{array}{cc} & \begin{matrix} 1 & 2 & \cdots j & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ & & \vdots & & \\ \cdots & \cdots & a_{ij} & \cdots & \cdots \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \end{array}$$

$\rightarrow$ the i^{th} row

the $(i,j)^{th}$ element

the j^{th} column

size of $\underset{=}{A}$: $m \times n$ (m rows, n columns)

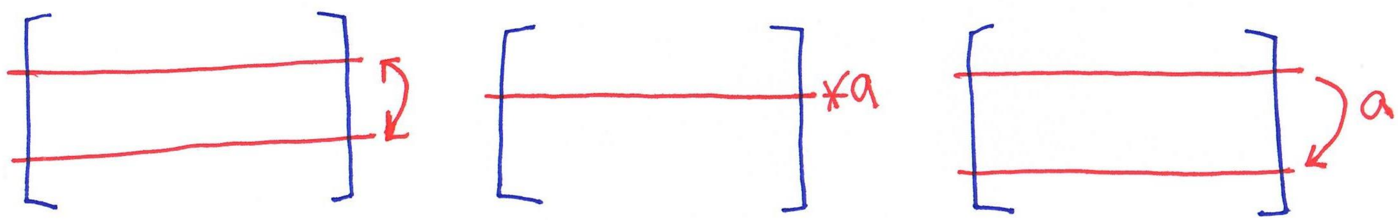
• If $m=n$, $\underset{=}{A}$ is a square matrix.

- elementary ops for system of l. eqs

$\Downarrow$

elementary row operations (ero's) for matrices

Type I Type II Type III



- Q: With ero's, we try to reduce a complex matrix (in some sense) to a simple matrix (, so that the solution to the corresponding system of l. eqs is immediately obtained). But, what is the "simple" matrix ?

Ans: ref or rref