ENGINEERING MATHEMATICS (II) – MIDTERM Solutions

Winter 2021

PROBLEM 1

(a) *If you type in the following MATLAB codes*

$\mathbf{C} = [\mathbf{1\ 3\ 5; 2\ 4\ 6;\ 2\ 3\ 1}]; \mathbf{C}(:,\mathbf{2}) = \mathbf{C}(:,\mathbf{1}) - \mathbf{C}(:,\mathbf{3})$

then what is that shows on your screen?

Sol: The first command produces $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 2 & 3 & 1 \end{bmatrix}$. After the second command, which subtracts the third column from the first column to obtain the second one, we can get

$$\begin{bmatrix} 1 & -4 & 5 \\ 2 & -4 & 6 \\ 2 & 1 & 1 \end{bmatrix}$$

.

(b) *(a) continued. If you type in the following MATLAB codes*

$\mathbf{rref}(\mathbf{C})$

then what does that show on your screen?

Sol: There are two possible solutions for this problem.

- If you use the original matrix $\mathbf{C} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 2 & 3 & 1 \end{bmatrix}$, then you will see the **reduced row echelon form** of $\mathbf{C}$ on the screen as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  This is due to the fact that all three columns (rows) are linearly independent.

- On the other hand, if you use the new $\mathbf{C} = \begin{bmatrix} 1 & -4 & 5 \\ 2 & -4 & 6 \\ 2 & 1 & 1 \end{bmatrix}$, then you will see the **reduced row echelon form** of $\mathbf{C}$ on the screen as

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

  because there are now two are linearly independent columns (rows).

(c) *Suppose that both of* $\mathbf{A}$ *and* $\mathbf{B}$ *are symmetric matrices, then which of the following matrices are symmetric?*

$$\mathbf{A}^2 + \mathbf{B}^2,\ (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}),\ \mathbf{ABA}$$

Sol: Since $\mathbf{A}$ and $\mathbf{B}$ are symmetric, *i.e.* $\mathbf{A}^T = \mathbf{A}$ and $\mathbf{B}^T = \mathbf{B}$, we have

$$\left(\mathbf{A}^2 + \mathbf{B}^2\right)^T = \mathbf{A}^T\mathbf{A}^T + \mathbf{B}^T\mathbf{B}^T = \mathbf{A}\mathbf{A} + \mathbf{B}\mathbf{B} = \mathbf{A}^2 + \mathbf{B}^2$$

$$\begin{aligned}\left((\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}\right)^T &= \left(\mathbf{A}-\mathbf{B}\right)^T\left(\mathbf{A}+\mathbf{B}\right)^T = \left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}+\mathbf{B}\right) = \mathbf{A}^2 + \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} + \mathbf{B}\mathbf{B}^2 \\ &\neq \mathbf{A}^2 - \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} + \mathbf{B}\mathbf{B}^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})\end{aligned}$$

$$(\mathbf{A}\mathbf{B}\mathbf{A})^T = \mathbf{A}^T\mathbf{B}^T\mathbf{A}^T = \mathbf{A}\mathbf{B}\mathbf{A}$$

In summary, only $\mathbf{A}^2 + \mathbf{B}^2$ and $\mathbf{A}\mathbf{B}\mathbf{A}$ are symmetric.

(d) *Suppose an $m \times n$ matrix $\mathbf{A}$ has rank r, then which of the following matrices also have rank r?*

$$\mathbf{A}^T, \quad -2\mathbf{A}, \ \ \mathbf{A}^2, \ \ \begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is an $m \times n$ zero matrix.

Sol: Note that $rank(\mathbf{A})$ is equal to the number of linearly independent columns (rows) of $\mathbf{A}$. As such, $\mathbf{A}^T$, $-2\mathbf{A}$ and $\begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix}$ all have rank r.

(e) Determine a 4×4 *elementary matrix* $\mathbf{E}$ such that

$$\begin{bmatrix} 2 & -1 & 0 & 5 \\ -1 & 2 & 1 & 6 \\ 0 & -1 & 2 & -1 \end{bmatrix} \mathbf{E} = \begin{bmatrix} 2 & 0 & -1 & 5 \\ -1 & 1 & 2 & 6 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

Sol: Note that after postmultiplying by $\mathbf{E}$, the second column and the third one are swaped while the others remain fixed, so $\mathbf{E}$ is equal to swap the second column and the third one of $\mathbf{I}$ given by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f) *Suppose that* $\mathbf{G}\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ *has exactly one solution* $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$*. Then determine the rank of* $\mathbf{G}$*.*

Sol: First, note that $\mathbf{G}$ is a 5×3 matrix. Since $\mathbf{G}\mathbf{w}$ has exactly one solution (no free variables), $rank(\mathbf{G}) = n = 3$.

(g) *((f) continued) If* $\mathbf{Gz} = \mathbf{Gy}$*, is it true that* $\mathbf{z} = \mathbf{y}$*?*

Sol: From (f), $rank(\mathbf{G}) = 3 = dim(col(\mathbf{G}))$ which implies that all columns of G are linearly independent. Therefore,

$$\mathbf{Gz} = \mathbf{Gy} \Rightarrow z_1\mathbf{g}_1 + z_2\mathbf{g}_2 + z_3\mathbf{g}_3 = y_1\mathbf{g}_1 + y_2\mathbf{g}_2 + y_3\mathbf{g}_3 \Rightarrow z_i = y_i, \; i = 1, 2, 3$$

where $\mathbf{g}_i$ denotes the i^{th} column of $\mathbf{G}$ and z_i (y_i) denotes the i^{th} element of $\mathbf{z}$ ($\mathbf{y}$), and we have used the fact the linear combination coefficients are unique for linearly independent vectors. Please note that you can NOT (pre-)multiplying $\mathbf{G}^{-1}$ on both sides to arrive at this conclusion, as $\mathbf{G}$ is not a square matrix and does not have an inverse.

(h) *((f) continued) How many solution(s) does* $\mathbf{Gx} = \mathbf{b}$ *have for any* $\mathbf{b} \in \mathcal{R}^5$*?*

Sol: From (f), $dim(col(\mathbf{G})) = rank(\mathbf{G}) = 3 < 5 = dim(\mathcal{R}^5)$, so there are two cases for the solutions of $\mathbf{Gx} = \mathbf{b}$:

- If $\mathbf{b} \in col(\mathbf{G})$, there is exactly one solution.
- If $\mathbf{b} \notin col(\mathbf{G})$, there is no solution.

PROBLEM 2

Consider the following system of linear equations

$$\begin{aligned} x + y + \alpha z &= 1 \\ x + \alpha y + z &= 3 \\ \alpha x + y + z &= 2\alpha \end{aligned}$$

Determine the values of α *such that the above system of linear equations has no solution, one solution, and infinitely many solutions. Also,* **determine the corresponding solution set** *only when this system of linear equations has* **infinite number of solutions**.

Sol: To solve this problem, form the augmented matrix first and then reduce it to the row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 3 \\ \alpha & 1 & 1 & 2\alpha \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha-1 & 1-\alpha & 2 \\ 0 & 1-\alpha & 1-\alpha^2 & \alpha \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha-1 & 1-\alpha & 2 \\ 0 & 0 & 2-\alpha^2-\alpha & \alpha+2 \end{array}\right].$$

Note that $2-\alpha^2-\alpha = -(\alpha^2+\alpha-2) = -(\alpha+2)(\alpha-1)$. Therefore, the solution sets can be divided into the following three cases:

- If $\alpha = 1$, then there is no solution.
- If $\alpha = -2$, then there is ∞ number of solutions. Now the problem becomes

$$\begin{aligned} x + y + -2z &= 1 \\ -3y + 3z &= 2 \end{aligned}$$

Setting the free variable $z=\beta$, we can obtain the solution

$$(x,y,z)=(\frac{5}{3}+\beta,-\frac{2}{3}+\beta,\beta)$$

where β is any arbitrary real numbers.

- If $\alpha \neq 1$ and $\alpha \neq -2$, then there is exactly one solution.

PROBLEM 3

Consider two matrices

$$\mathbf{A}=\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad \textit{and} \quad \mathbf{B}=\begin{bmatrix} 0 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

(a) *Determine* $det(2\mathbf{A}^{-1}\mathbf{B}^2)+det(\mathbf{A}^T\mathbf{B})$.

Sol: Conducting the cofactor expansion along the first row of $\mathbf{A}$ yields $det(\mathbf{A})=4\Big(1\times(-2)-1\times(-1)\Big)-(-1)\Big((-1)\times(-2)-1\times 0\Big)=-2$. Likewise, expanding along the first row of $\mathbf{B}$ yields $det(\mathbf{B})=3\times(0\times 5-(-1)\times 1)=3$. So

$$det(2\mathbf{A}^{-1}\mathbf{B}^2)+det(\mathbf{A}^T\mathbf{B})=2^3\frac{1}{det(\mathbf{A})}\Big[det(\mathbf{B})\Big]^2+det(\mathbf{A})det(\mathbf{B})=-36+(-6)=-42$$

(b) *Determine* $\Big[adj(\mathbf{A})\Big]^{-1}$.

Sol: Since $\mathbf{A}\cdot(adj\ \mathbf{A})=det(\mathbf{A})\cdot\mathbf{I}$, it follows that $\Big[adj(\mathbf{A})\Big]^{-1}=\frac{1}{det(\mathbf{A})}\mathbf{A}=-\frac{1}{2}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$.

PROBLEM 4

Consider two subsets V *and* W *of* $\mathcal{R}^5$ *which are, respectively, given by*

$$\mathsf{V}=\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix},\ a_i=-a_{6-i},\ i=1,\cdots,5\right\} \quad \textit{and} \quad \mathsf{W}=\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix},\ a_{i+1}=(-1)^i a_i,\ i=1,\cdots,4\right\}$$

.

(a) *Determine* $dim(\mathsf{V})$.

Sol: A typical element of V (the elements are asymmetric with respect to the central element) is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ -a_2 \\ -a_1 \end{bmatrix}=a_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}+a_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

It is easy to verify that these two spanning vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ are linearly independent and thus is a basis, so $dim(\mathsf{V}) = 2$.

(b) *Determine dim(*$\mathsf{V}\cap\mathsf{W}$*).*

Sol: Note that based on its definition, $a_2 = -a_1$, $a_3 = a_2 = -a_1$, $a_4 = -a_3 = a_1$, and $a_5 = a_4 = a_1$. Therefore, a typical element of W is given by $\begin{bmatrix} a_1 \\ -a_1 \\ -a_1 \\ a_1 \\ a_1 \end{bmatrix}$. Note that the 3^{rd} element in every vector of $\mathsf{V}\cap\mathsf{W}$ must be equal to zero, so $-a_1 = 0$ which amounts to $a_1 = 0$. Therefore, the only element in $(\mathsf{V}\cap\mathsf{W})$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so dim($\mathsf{V}\cap\mathsf{W}$)=0.

PROBLEM 5

Consider two $n \times 1$ column vectors $\mathbf{x}$ and $\mathbf{y}$ which satisfy $\mathbf{x}^T\mathbf{y} = 1$. Suppose the inverse of $(\mathbf{I} + \mathbf{x}\mathbf{y}^T)$ is $\mathbf{I} + \gamma\mathbf{x}\mathbf{y}^T$, where $\mathbf{I}$ is an $n \times n$ identity matrix, determine γ.

Sol: Since the inverse of $(\mathbf{I} + \mathbf{x}\mathbf{y}^T)$ is $\mathbf{I} + \gamma\mathbf{x}\mathbf{y}^T$, by definition we have

$$(\mathbf{I} + \mathbf{x}\mathbf{y}^T)(\mathbf{I} + \gamma\mathbf{x}\mathbf{y}^T) = \mathbf{I} \Rightarrow \mathbf{I} + \mathbf{x}\mathbf{y}^T + \gamma\mathbf{x}\mathbf{y}^T + \gamma\mathbf{x}\overbrace{\mathbf{y}^T\mathbf{x}}^{=\mathbf{x}^T\mathbf{y}=1}\mathbf{y}^T = \mathbf{I} \Rightarrow \mathbf{I} + (1 + \gamma + \gamma)\mathbf{x}\mathbf{y}^T = \mathbf{I}$$

Consequently, $\gamma = -\frac{1}{2}$.