* Matrix Arithmetic

- We will talk about the 加.減.乘.除 of matrices.

4則運算 $(+, -, \times, \div)$

Recall:

- $x - y \overset{?}{=} x + (-y)$

  $x \div y \overset{?}{=} x \times \frac{1}{y}$ → also written as y^{-1}

- $y + (-y) = 0$
- $y \times y^{-1} = 1$

⟹ In some sense, matrices have their own $+, -, \times, \div, 0, 1, -y, y^{-1}$, and so on.

- matrix: $\underline{\underline{A}} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} : \underbrace{m \times n}$ ↳ size of $\underline{\underline{A}}$

- row vector:

  $(\underline{r} \overset{a}{=}) \ \vec{r} = [r_1 \ r_2 \ \cdots \ r_n] : 1 \times n$

  → you pick your own choice.

- column vector: $\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} : m \times 1$

· **Equality** of matrices:

def. Consider two matrices $\underline{\underline{A}}: m\times n$, $\underline{\underline{B}}: m\times n$.

$$\underline{\underline{A}} = [a_{ij}]_{m\times n} \doteq [a_{ij}] \doteq \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If you want to stress the size,

$$\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \doteq [b_{ij}] \cong [b_{ij}]_{m\times n}$$

$$\boxed{\underline{\underline{A}} = \underline{\underline{B}}} \text{ iff } \Longleftrightarrow \boxed{a_{ij} = b_{ij} \text{ for } \forall\, i,j}$$

· **scalar multiple** (of a scalar and a matrix)

Consider c: scalar, $\underline{\underline{A}}: m\times n$. $\underline{\underline{B}} = c\cdot\underline{\underline{A}}$ is defined as $b_{ij} = c\cdot a_{ij}$ for $\forall\, i,j$

$\underline{\underline{A}} = [a_{ij}]$

$\underline{\underline{B}} = [b_{ij}] = [b_{ij}]_{m\times n}$

same size as $\underline{\underline{A}}$

· **Ex**:

$$3\cdot\begin{bmatrix} 1 & 0 & 3 & -6 \\ 2 & 6 & -5 & 21 \\ -1 & 7 & 2 & -13 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 9 & -18 \\ 6 & 18 & -15 & 63 \\ -1 & 21 & 6 & -39 \end{bmatrix}$$

- sum of (two) matrices : addition (add)

Consider $\underline{\underline{A}} : m \times n$, $\underline{\underline{B}} : m \times n$. $\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}$ is defined as

$\underline{\underline{A}} = [a_{ij}]$, $\underline{\underline{B}} = [b_{ij}]$, $\underline{\underline{C}} = [c_{ij}] = [c_{ij}]_{m \times n}$

$$c_{ij} = a_{ij} + b_{ij} \text{ for } \forall i, j$$

- Ex:

$$\begin{bmatrix} 1 & 0 & -7 & 3 \\ 2 & 3 & -5 & 0 \\ -4 & 6 & 21 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & -2/3 & -1 & 2 \\ -2 & 1 & -1 & 5 \\ 3 & 7 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 & -8 & 5 \\ 0 & 4 & -6 & 5 \\ -1 & 13 & 21 & 19/2 \end{bmatrix}$$

- difference of matrices: $\underline{\underline{A}} - \underline{\underline{B}} = \underline{\underline{A}} + (-1) \cdot \underline{\underline{B}}$
- Exactly speaking, $\underline{\underline{A}} - \underline{\underline{B}}$ is just an abbreviated expression for $\underline{\underline{A}} + (-1) \cdot \underline{\underline{B}}$.

- product of (two) matrices :

Consider $\underline{\underline{A}} = [a_{ij}]_{m \times n}$ $\underline{\underline{B}} = [b_{ij}]_{n \times r}$. $\boxed{\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}}$

$(\underline{\underline{C}} = [c_{ij}]_{m \times r})$ is defined as $c_{ij} \triangleq \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\begin{bmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \underbrace{a_{i1}\ a_{i2} \cdots a_{in}}_{(i\text{th row})\ \triangleq\ \vec{a}_i} \\ & \vdots & \end{bmatrix} \cdot \begin{array}{c} (j\text{th column}) \\ \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ \cdots & \vdots & \cdots \\ & b_{nj} & \end{bmatrix} \\ \triangleq \underline{b}_j \end{array} = \vec{a}_i \cdot \underline{b}_j$$

(inner product)

- **Ex:**

If

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}_{3\times 2} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}_{2\times 3}$$

then

$$AB = \begin{bmatrix} 3 & -2 \\ \mathbf{2} & \mathbf{4} \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 & \mathbf{3} \\ 4 & 1 & \mathbf{6} \end{bmatrix}$$

$3\times 2 \quad 2\times 3$

$\Downarrow$

3×3

$$= \begin{bmatrix} 3\cdot(-2) - 2\cdot 4 & 3\cdot 1 - 2\cdot 1 & 3\cdot 3 - 2\cdot 6 \\ 2\cdot(-2) + 4\cdot 4 & 2\cdot 1 + 4\cdot 1 & \mathbf{2\cdot 3 + 4\cdot 6} \\ 1\cdot(-2) - 3\cdot 4 & 1\cdot 1 - 3\cdot 1 & 1\cdot 3 - 3\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & \mathbf{30} \\ -14 & -2 & -15 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2\cdot 3 + 1\cdot 2 + 3\cdot 1 & -2\cdot(-2) + 1\cdot 4 + 3\cdot(-3) \\ 4\cdot 3 + 1\cdot 2 + 6\cdot 1 & 4\cdot(-2) + 1\cdot 4 + 6\cdot(-3) \end{bmatrix}$$

$2\times 3 \quad 3\times 2$

$\Downarrow$

2×2

$$= \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix}$$

- **N.B.** $\underline{\underline{A}} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \underline{\underline{A}}$

- size of $\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}$

multiplication

$$\underline{\underline{A}} \cdot \underline{\underline{B}} \rightsquigarrow \underline{\underline{C}}$$

$$m \times n \quad n \times r \quad \Rightarrow \quad m \times r$$

Notice that those two numbers must be the same, for the multiplication to even make sense.

- Next, let us try to talk about the "division" 除法 between two matrices.

  Q: What is it?

- Let us, first, recall the division bet. 2 numbers: $5 \div 4$

  $\overset{\shortparallel}{5 \times \frac{1}{4}} = 5 \times 4^{-1}$

- "1" is something → (such that) $x \times 1 = 1 \times x = x$
- "x^{-1}" is something → $x \times x^{-1} = x^{-1} \times x = 1$.
- Now, back to matrices (from numbers):

number	x	1	x^{-1}
matrix	$\underline{\underline{A}}$	$\underline{\underline{I}}$	$\underline{\underline{A}}^{-1}$

matrix inverse (反矩陣)

identity matrix (單位矩陣)

- $\underline{\underline{A}} \div \underline{\underline{B}} \rightsquigarrow$ some kind of $\underline{\underline{A}} \cdot \underline{\underline{B}}^{-1}$

- **def** An identity matrix of size $n \times n$ (or n, for short) is a square matrix:

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & 1 \end{bmatrix}$$

→ All diagonal elements are 1; all non-diagonal elements are 0.

- **Thm**

$$\underbrace{\underline{\underline{A}}}_{p\times n} \cdot \underbrace{\underline{\underline{I}}}_{n\times n} = \underbrace{\underline{\underline{A}}}_{p\times n}$$

$$\underbrace{\underline{\underline{I}}}_{n\times n} \cdot \underbrace{\underline{\underline{B}}}_{n\times q} = \underbrace{\underline{\underline{B}}}_{n\times q}$$

Prf: Straight forward (Simply apply the def. of identity matrix and matrix multiplication)

- **inverse matrix**:

def: A square matrix $\underset{n\times n}{\underline{\underline{A}}}$ is invertible iff there exists ($\exists$) $\underset{n\times n}{\underline{\underline{B}}} \ni$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{B}} \cdot \underline{\underline{A}} = \underline{\underline{I}}.$$

Then $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are inverse matrices to each other. ($\underline{\underline{B}} = \underline{\underline{A}}^{-1}$, $\underline{\underline{A}} = \underline{\underline{B}}^{-1}$)

- **Thm** Matrix inverse is unique, if it exists.

Prf: Assume that $\underline{\underline{B}}$ is an inverse of $\underline{\underline{A}}$. Suppose that $\underline{\underline{C}}$ is also an inverse of $\underline{\underline{A}}$. Then, we have (by def.).

$$\begin{cases} \underline{\underline{B}}\cdot\underline{\underline{A}} = \underline{\underline{A}}\cdot\underline{\underline{B}} = \underline{\underline{I}} & \text{— (1)} \\ \underline{\underline{C}}\cdot\underline{\underline{A}} = \underline{\underline{A}}\cdot\underline{\underline{C}} = \underline{\underline{I}} & \text{— (2)} \end{cases}$$

associativity of matrix multiplication

- $$\underline{\underline{B}} \underset{\uparrow}{=} \underline{\underline{B}}\cdot\underline{\underline{I}} \overset{(2)}{=} \underline{\underline{B}}\cdot(\underline{\underline{A}}\cdot\underline{\underline{C}}) \overset{(\#)}{=} (\underline{\underline{B}}\cdot\underline{\underline{A}})\cdot\underline{\underline{C}} \overset{(1)}{=} \underline{\underline{I}}\cdot\underline{\underline{C}} \underset{\uparrow}{=} \underline{\underline{C}} \quad \#$$

by def of identity matrix

- (#): **Thm** (associativity of matrix multiplication)

$$\underline{\underline{A}}\cdot(\underline{\underline{B}}\cdot\underline{\underline{C}}) = (\underline{\underline{A}}\cdot\underline{\underline{B}})\cdot\underline{\underline{C}}$$

Prf: Straight forward (although, tedious).

- terminologies :

A matrix is
- invertible $\equiv$ non-singular
- not invertible $\equiv$ singular

- **Thm** $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$ (Assume that $\underline{A}$ and $\underline{B}$ are invertible)

$\underbrace{\underline{\underline{A}}\,\underline{\underline{B}}}_{\to \underline{\underline{P}}}$, $\underbrace{\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}}_{\to \underline{\underline{Q}}}$

Prf:

by (#) (i.e. associativity)

by def. of $\underline{\underline{I}}$ $= \underline{\underline{B}}$

by def of inverse

$$\underline{\underline{Q}}\cdot\underline{\underline{P}} = (\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1})\cdot(\underline{\underline{A}}\,\underline{\underline{B}}) = \underline{\underline{B}}^{-1}(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{\underline{B}} = \underline{\underline{B}}^{-1}\cdot\underline{\underline{I}}\cdot\underline{\underline{B}} = \underline{\underline{B}}^{-1}\cdot\underline{\underline{B}} = \underline{\underline{I}}$$

$$\underline{\underline{P}}\cdot\underline{\underline{Q}} = (\underline{\underline{A}}\cdot\underline{\underline{B}})\cdot(\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}) = \underline{\underline{A}}\cdot(\underline{\underline{B}}\,\underline{\underline{B}}^{-1})\cdot\underline{\underline{A}}^{-1} = \underline{\underline{A}}\cdot\underline{\underline{I}}\cdot\underline{\underline{A}}^{-1} = \underline{\underline{A}}\cdot\underline{\underline{A}}^{-1} = \underline{\underline{I}}$$

$\therefore$ We have $\underline{\underline{Q}}\cdot\underline{\underline{P}} = \underline{\underline{P}}\cdot\underline{\underline{Q}} = \underline{\underline{I}} \Rightarrow \underline{\underline{Q}}$ is the inverse of $\underline{\underline{P}}$ #

- **Thm** $(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$

$\underbrace{\underline{\underline{A}}^{-1}}_{\to \underline{\underline{P}}}$

Prf:

$$\underline{\underline{A}}\cdot\underline{\underline{P}} = \underline{\underline{A}}\cdot\underline{\underline{A}}^{-1} = \underline{\underline{I}}$$
$$\underline{\underline{P}}\cdot\underline{\underline{A}} = \underline{\underline{A}}^{-1}\cdot\underline{\underline{A}} = \underline{\underline{I}}$$

$\Rightarrow$ $\therefore$ We have $\underline{\underline{A}}\cdot\overset{\underline{\underline{A}}^{-1} \,\|\, \leftarrow \text{recall}}{\underline{\underline{P}}} = \underline{\underline{P}}\cdot\underline{\underline{A}} = \underline{\underline{I}}$

$\Rightarrow \underline{\underline{A}}$ is the inverse of $\underline{\underline{P}}$ #