107 暨質②

1. 繪 $f(x)=\dfrac{2x}{\sqrt{x^2+1}}$

$f(x)=2x(x^2+1)^{-\frac{1}{2}}$

① Domain: $\{x \mid x\in R\}$

② $f(x)$ is continuous on R

③ $\lim\limits_{x\to+\infty}\dfrac{2x}{\sqrt{x^2+1}}=\lim\limits_{x\to+\infty}\dfrac{2}{\sqrt{1+\frac{1}{x^2}}}=2$

$\lim\limits_{x\to-\infty}\dfrac{2x}{\sqrt{x^2+1}} \underset{y\to\infty}{\overset{令 y=-x}{=\!=}} \lim\limits_{y\to\infty}\dfrac{-2y}{\sqrt{y^2+1}}=\lim\limits_{y\to\infty}\dfrac{-2}{\sqrt{1+\frac{1}{y^2}}}=-2$

$y=2$、$y=-2$ are horizontal asymptotes of $f(x)$

④ $f(-x)=\dfrac{-2x}{\sqrt{x^2+1}}=-f(x) \Rightarrow$ 原點為對稱點(奇函數)

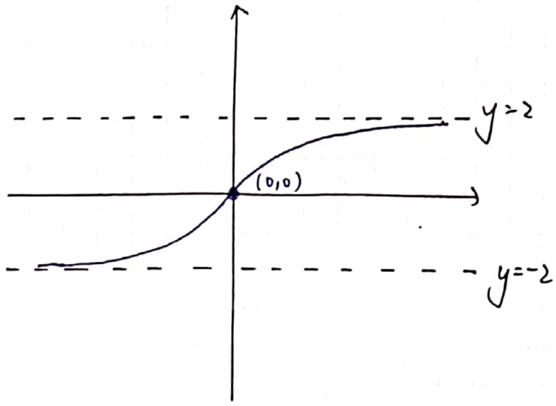
$f'(x)=2(x^2+1)^{-\frac{1}{2}}+2x\cdot\frac{-1}{2}(x^2+1)^{-\frac{3}{2}}\cdot(2x)$

$=\dfrac{2}{\sqrt{x^2+1}}+\dfrac{-2x^2}{\sqrt{(x^2+1)^3}}=\dfrac{\cancel{2x^2}+2-\cancel{2x^2}}{\sqrt{(x^2+1)^3}}=\dfrac{2}{\sqrt{(x^2+1)^3}}=2(x^2+1)^{-\frac{3}{2}}$

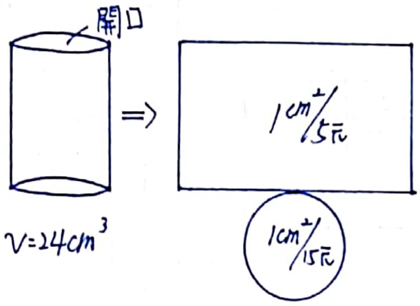
$f''(x)=\cancel{2}\cdot\frac{-3}{\cancel{2}}(x^2+1)^{-\frac{5}{2}}\cdot 2x$

$=-6x(x^2+1)^{-\frac{5}{2}}=\dfrac{-6x}{\sqrt{(x^2+1)^5}}$ Let $f''(x)=0$, $x=0$

x	$(-\infty,0)$	0	$(0,+\infty)$
$f(x)$		0	
$f'(x)$	$+$	$+$	$+$
$f''(x)$	$+$	0	$-$
	↗	反曲點	↗



2.



$\pi r^2 h=24\pi$

$h=\dfrac{24}{r^2}$

surface area of the can: $A=\pi r^2+2\pi rh$

cost $=15\pi r^2+5\cdot 2\pi rh$

$=15\pi r^2+10\pi rh=15\pi r^2+10\pi r\cdot\dfrac{24}{r^2}$

$=15\pi r^2+\dfrac{240\pi}{r}$

$\dfrac{d}{dr}(\text{cost})=30\pi r+\dfrac{-240\pi}{r^2}=0$

$30\cancel{\pi}r=\dfrac{240\cancel{\pi}}{r^2}$

$r^3=8$, $r=2$

$\to h=\dfrac{24}{r^2}=\dfrac{24}{4}=6$

$r=2$, $h=6$ 時

A: 成本最低

Ïhý uú

3. (a) $\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

Sol1:

$$\int_1^4 \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx$$

$$\left(\frac{d}{dx}2\sqrt{x} = \frac{1}{\sqrt{x}} \qquad d\,2\sqrt{x} = \frac{1}{\sqrt{x}}\,dx\right)$$

$$= \int_1^4 \frac{1}{\sqrt{1+\sqrt{x}}}\,d\,2\sqrt{x}$$

$$= 2\int_1^4 (1+\sqrt{x})^{-\frac{1}{2}}\,d\sqrt{x}$$

$$= 2\cdot 2\cdot (1+\sqrt{x})^{\frac{1}{2}}\Big|_1^4$$

$$= 2\cdot 2\cdot\left(3^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)$$

$$= \underline{4(\sqrt{3}-\sqrt{2})} \#$$

Sol2:

$$\int_1^4 \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx$$

令 $\sqrt{1+\sqrt{x}} = u$, $\quad x=4 \to u=\sqrt{3}$, $\quad x=1 \to u=\sqrt{2}$

$$u = \left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}\left(1+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}} = \frac{1}{4\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}$$

$$\therefore\ 4\cdot du = \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx$$

$$\Rightarrow \int_{\sqrt{2}}^{\sqrt{3}} 4\,du = 4\int_{\sqrt{2}}^{\sqrt{3}} 1\,du$$

$$= 4\cdot u\Big|_{\sqrt{2}}^{\sqrt{3}}$$

$$= \underline{4(\sqrt{3}-\sqrt{2})} \#$$

(b)

$$\int_0^1 \frac{(e^x+1)^2}{e^x}\,dx$$

$$= \int_0^1 \frac{e^{2x}+2e^x+1}{e^x}\,dx$$

$$= \int_0^1 e^x + 2 + e^{-x}\,dx$$

$$= e^x + 2x - e^{-x}\Big|_0^1$$

$$= \left(e+2-\frac{1}{e}\right) - (1+0-1)$$

$$= \underline{e+2-\frac{1}{e}} \#$$

(c)

$$\int_1^8 \frac{\sqrt{1+\log_2 x}}{x}\,dx$$

$$\frac{1}{x}\,dx = d(\ln x)$$

$$\left(\because \frac{d}{dx}(\ln x) = \frac{1}{x} \quad \therefore \frac{1}{x}\,dx = d(\ln x)\right)$$

$$\log_2 x = \frac{\ln x}{\ln 2}$$

$$= \int_1^8 \left(1+\frac{\ln x}{\ln 2}\right)^{\frac{1}{2}} d(\ln x)$$

$$= \ln 2\cdot\int_1^8 \left(1+\frac{\ln x}{\ln 2}\right)^{\frac{1}{2}}\cdot d\left(\frac{\ln x}{\ln 2}\right)$$

$$= \ln 2\cdot\frac{2}{3}\left(1+\frac{\ln x}{\ln 2}\right)^{\frac{3}{2}}\Big|_1^8$$

$$= \ln 2\cdot\frac{2}{3}\cdot\left[\left(1+\frac{3\ln 2}{\ln 2}\right)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\ln 2\cdot[8-1]$$

$$= \underline{\frac{14}{3}\ln 2} \#$$

Ïhý uú

4. 若 $e^x\cot y = xe^{2y}$，求 $\frac{dy}{dx}=?$

$$\frac{d}{dx}(e^x\cot y)=\frac{d}{dx}(xe^{2y})$$

$$e^x\cdot\cot y+\underline{e^x\cdot(-\csc^2 y)\cdot\frac{dy}{dx}}=e^{2y}+\underline{x\cdot e^{2y}\cdot 2\cdot\frac{dy}{dx}}$$

$$\therefore \frac{dy}{dx}=\frac{e^x\cot y-e^{2y}}{2xe^{2y}+e^x\csc^2 y}$$ ※

5. 若 $f(x)=\int_x^{e^x}\sqrt{1+t^2+t^4}\,dt$ 求 $f'(0)$ （微積分基本定理一：$\frac{d}{dx}\int_a^x f(t)dt=f(x)$）

$$f'(x)=\frac{d}{dx}\int_x^{e^x}\sqrt{1+t^2+t^4}\,dt=\frac{d}{dx}\left[\int_0^{e^x}\sqrt{1+t^2+t^4}\,dt+\int_x^0\sqrt{1+t^2+t^4}\,dt\right]=\frac{d}{dx}\left[\int_0^{e^x}\sqrt{1+t^2+t^4}\,dt-\int_0^x\sqrt{1+t^2+t^4}\,dt\right]$$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$$=\underbrace{\frac{d}{dx}\int_0^{e^x}\sqrt{1+t^2+t^4}\,dt}_{令\ u=e^x}-\frac{d}{dx}\int_0^x\sqrt{1+t^2+t^4}\,dt=\left(\frac{d}{du}\int_0^u\sqrt{1+t^2+t^4}\,dt\right)\cdot\frac{du}{dx}-\sqrt{1+x^2+x^4}$$

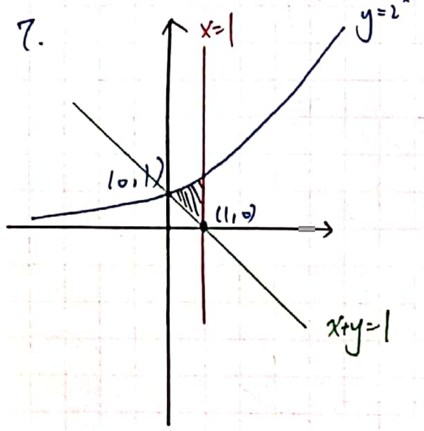
$$=\sqrt{1+u^2+u^4}\cdot\frac{de^x}{dx}-\sqrt{1+x^2+x^4}=\sqrt{1+e^{2x}+e^{4x}}\cdot e^x-\sqrt{1+x^2+x^4}$$

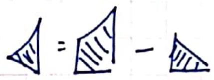
$$\therefore f'(0)=\sqrt{1+e^0+e^0}\cdot e^0-\sqrt{1+0+0}=\sqrt{3}\cdot 1-1=\underline{\sqrt{3}-1}$$ ※

6. $y=x^{\ln x}$，求 $\frac{dy}{dx}=?$

$$y=x^{\ln x}=e^{\ln(x^{\ln x})}=e^{\ln x\cdot\ln x}\quad\therefore\frac{dy}{dx}=x^{\ln x}\cdot\left(\frac{1}{x}\cdot\ln x+\ln x\cdot\frac{1}{x}\right)=\underline{x^{\ln x}\cdot\frac{2\ln x}{x}}$$ ※

7.





$$=\int_0^1 2^x\,dx-1\cdot1\cdot\frac{1}{2}$$

$$=\frac{2^x}{\ln 2}\Big|_0^1-\frac{1}{2}$$

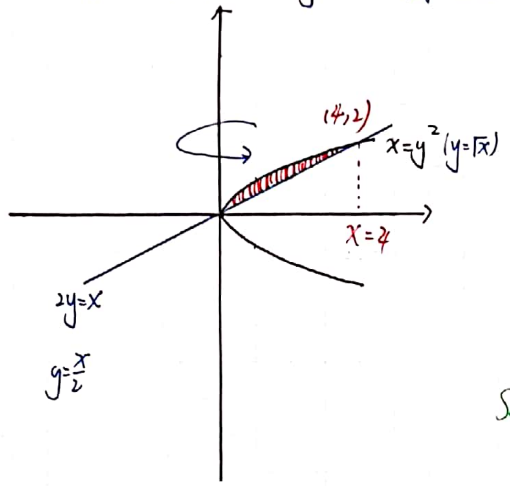
$$=\frac{2}{\ln 2}-\frac{1}{\ln 2}-\frac{1}{2}$$

$$=\frac{1}{\ln 2}-\frac{1}{2}$$

$$=\underline{\frac{2-\ln 2}{2\ln 2}}$$ ※

Ïhý uú

8. 求曲線 $y^2=x$, $2y=x$ 所圍成之區域繞 y 軸旋轉所旋轉體的體積



$y^2=2y$ $\quad y(y-2)=0$

$y=0$ or 2

Sol1 (shell)

$$V=\int_0^4 2\pi\cdot(x-0)\left(\sqrt{x}-\frac{x}{2}\right)dx$$

$$=2\pi\int_0^4 x^{\frac{3}{2}}-\frac{1}{2}x^2\,dx=2\pi\cdot\left[\left(\frac{2}{5}x^{\frac{5}{2}}-\frac{1}{6}x^3\right)\Big|_0^4\right]$$

$$=2\pi\left(\frac{2}{5}\times32-\frac{64}{6}\right)=128\pi\cdot\frac{1}{30}=\frac{64\pi}{15}$$

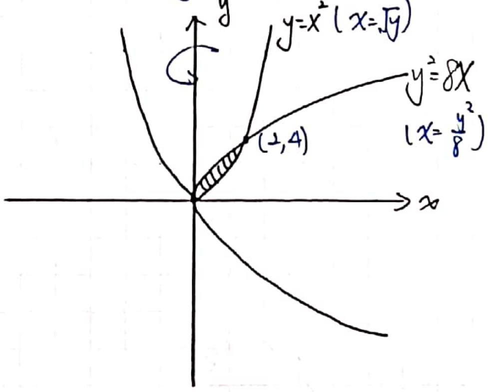
Sol2 (washer)

$$V=\int_0^2\pi\left[R(y)^2-r(y)^2\right]dy=\int_0^2\pi\left(4y^2-y^4\right)dy$$

$$=\pi\left[\left(\frac{4}{3}y^3-\frac{1}{5}y^5\right)\Big|_0^2\right]=\pi\left[\frac{4}{3}\times8-\frac{32}{5}\right]=\pi\left(\frac{32}{3}-\frac{32}{5}\right)$$

$$=\pi\cdot32\cdot\frac{2}{15}=\frac{64\pi}{15}$$

9. 求曲線 $y=x^2$ 和 $y^2=8x$ 所圍之區域繞 y 軸旋轉所旋轉體之體積



$\sqrt{y}=\frac{y^2}{8}$ $\quad 64y=y^4$

$8\sqrt{y}=y^2$ $\quad y(y^3-64)=0$

$y=0$ or 4

Sol1 (shell)

$$V=\int_0^2 2\pi(x-0)\left(\sqrt{8x}-x^2\right)dx=2\pi\int_0^2 2\sqrt{2}x^{\frac{3}{2}}-x^3\,dx$$

$$=2\pi\left[\left(2\sqrt{2}\times\frac{2}{5}x^{\frac{5}{2}}-\frac{1}{4}x^4\right)\Big|_0^2\right]=2\pi\left[2\sqrt{2}\times\frac{2}{5}\times4\sqrt{2}-4\right]$$

$$=2\pi\cdot\left[\frac{32}{5}-4\right]=2\pi\cdot\frac{12}{5}=\frac{24}{5}\pi$$

Sol2 (washer)

$$V=\int_0^4\pi\left(R(y)^2-r(y)^2\right)dy=\pi\int_0^4\left(y-\frac{1}{64}y^4\right)dy$$

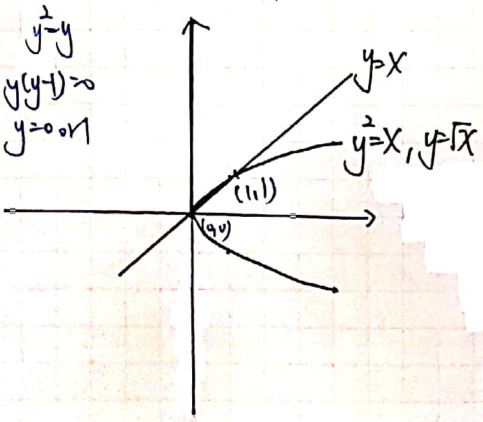
$$=\pi\left[\left(\frac{1}{2}y^2-\frac{1}{64\cdot5}y^5\right)\Big|_0^4\right]=\pi\left[8-\frac{64\cdot16}{64\cdot5}\right]=\pi\cdot\frac{40-16}{5}=\frac{24}{5}\pi$$

10. 一個立體底部在 $y=x$ 和 $y^2=x$ 所圍成之區域，垂直 x 軸的截面為半圓，直徑落在此區域，求體積？

$y^2=y$

$y(y-1)=0$

$y=0$ or 1



$A=\pi r^2=\frac{\pi}{4}(D^2)$, $\frac{A}{2}=\frac{\pi}{8}(D^2)$

$$A(x)=\frac{\pi}{8}\left(\sqrt{x}-x\right)^2=\frac{\pi}{8}\left(x-2x\sqrt{x}+x^2\right)$$

$$V=\int_0^1 A(x)\,dx=\int_0^1\frac{\pi}{8}\left(x-2x^{\frac{3}{2}}+x^2\right)dx$$

$$=\frac{\pi}{8}\left[\left(\frac{1}{2}x^2-\frac{4}{5}x^{\frac{5}{2}}+\frac{1}{3}x^3\right)\Big|_0^1\right]=\frac{\pi}{8}\cdot\left(\frac{1}{2}-\frac{4}{5}+\frac{1}{3}\right)$$

$$=\frac{\pi}{8}\times\frac{15-24+10}{30}=\frac{\pi}{240}$$

Ïhý uú

11. 求曲 $y+\frac{1}{4x}+\frac{x^3}{3}=0$ 從 $(2,-\frac{67}{24})$ 到 $(3,-\frac{109}{12})$ 的弧長

$$y=\frac{-1}{4}x^{-1}-\frac{1}{3}x^3$$

$$f'(x)=\frac{1}{4}x^{-2}-x^2$$

$$1+[f'(x)]^2=\frac{1}{16}x^{-4}-\frac{1}{2}+x^4+1$$

$$=\frac{1}{16}x^{-4}+\frac{1}{2}+x^4$$

$$=\left(\frac{1}{4}x^{-2}+x^2\right)^2$$

$$L=\int_2^3\sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_2^3\frac{1}{4}x^{-2}+x^2\,dx$$

$$=\left(\frac{-1}{4}x^{-1}+\frac{1}{3}x^3\right)\Big|_2^3$$

$$=\frac{-1}{4}\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{1}{3}(27-8)=\frac{-1}{4}\times\frac{-1}{6}+9-\frac{8}{3}$$

$$=\frac{1+216-64}{24}=\frac{153}{24}=\underline{\frac{51}{8}}\text{ #}$$

12. 求曲線 $4x=y^2$ 從 $(0,0)$ 到 $(1,2)$ 的弧繞 x 軸所成的旋轉曲面的面積

$$y=2\sqrt{x}=2x^{\frac{1}{2}}$$

$$\frac{dy}{dx}=x^{\frac{-1}{2}}=\frac{1}{\sqrt{x}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{x}=\frac{x+1}{x}$$

$$S=\int_0^1 2\pi y\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\int_0^1 2\cancel{\sqrt{x}}\cdot\sqrt{\frac{x+1}{\cancel{x}}}\,dx$$

$$=2\pi\int_0^1 2\cdot(x+1)^{\frac{1}{2}}\,dx$$

$$=4\pi\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^1$$

$$=4\pi\cdot\frac{2}{3}\cdot(2\sqrt{2}-1)$$

$$=\underline{\frac{8\pi}{3}(2\sqrt{2}-1)}\text{ #}$$

Ïhý uú