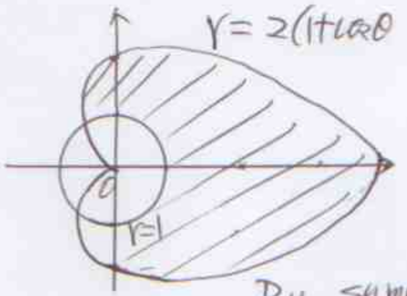
Final Exam 2011/6/16

1.

Sol:



Let $2(1+\cos\theta)=1 \Rightarrow 1+\cos\theta=\frac{1}{2} \Rightarrow \cos\theta=\frac{-1}{2}$

$\Rightarrow \theta=-\frac{2\pi}{3},\ \frac{2\pi}{3}$

By symmetry, $A=2\left[\frac{1}{2}\int_0^{\frac{2\pi}{3}}[2(1+\cos\theta)]^2 d\theta-\frac{1}{2}\int_0^{\frac{2\pi}{3}}1^2 d\theta\right]$

$= \int_0^{\frac{2\pi}{3}}\left[4(1+2\cos\theta+\cos^2\theta)-1\right]d\theta=\int_0^{\frac{2\pi}{3}}(3+8\cos\theta+4\cos^2\theta)\,d\theta$

$= \int_0^{\frac{2\pi}{3}}\left[3+8\cos\theta+2(1+\cos 2\theta)\right]d\theta=\int_0^{\frac{2\pi}{3}}(5+8\cos\theta+2\cos 2\theta)\,d\theta$

$= (5\theta+8\sin\theta+\sin 2\theta)\Big|_0^{\frac{2\pi}{3}}=\frac{10\pi}{3}+8\cdot\frac{\sqrt{3}}{2}+\left(-\frac{\sqrt{3}}{2}\right)=\frac{10\pi}{3}+\frac{7\sqrt{3}}{2}$

2

pf: Let $u=x+ct$, $v=x-ct$. Then $V=f(u)+g(v)$

$\frac{\partial V}{\partial t}=f'(u)\frac{\partial u}{\partial t}+g'(v)\frac{\partial v}{\partial t}=f'(u)\cdot c+g'(v)(-c)=c\left[f'(u)-g'(v)\right]$

$\frac{\partial^2 V}{\partial t^2}=\frac{\partial}{\partial t}\left[c\left(f'(u)-g'(v)\right)\right]=c\left[\frac{\partial}{\partial t}f'(u)-\frac{\partial}{\partial t}g'(v)\right]=c\left[f''(u)\frac{\partial u}{\partial t}-g''(v)\frac{\partial v}{\partial t}\right]$

$=c\left[f''(u)\cdot c-g''(v)(-c)\right]=c^2\left[f''(u)+g''(v)\right]$

$\frac{\partial V}{\partial x}=f'(u)\frac{\partial u}{\partial x}+g'(v)\frac{\partial v}{\partial x}=f'(u)+g'(v)$

$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\frac{\partial}{\partial x}\left[f'(u)+g'(v)\right]=f''(u)\frac{\partial u}{\partial x}+g''(v)\frac{\partial v}{\partial x}=f''(u)+g''(v)$

$\Rightarrow \frac{\partial^2 V}{\partial t^2}=c^2\left[f''(u)+g''(v)\right]=c^2\frac{\partial^2 V}{\partial x^2}$

3
sol: $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k} = -2x\vec{i} - 2y\vec{j} - 4z\vec{k}$

$\nabla T\Big|_{(2,1,1)} = -4\vec{i} - 2\vec{j} - 4\vec{k}$

In order to cool off as rapidly as possible, we should move in the direction $-\nabla T\Big|_{(2,1,1)} = 4\vec{i} + 2\vec{j} + 4\vec{k}$

4
sol: Let $F(x,y,z) = \tan^{-1}\left(\frac{y}{x}\right) - z = 0$

$\nabla F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\vec{i} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\vec{j} - \vec{k}$

$= \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} - \vec{k}$

$\nabla F\Big|_{(1,1,\frac{\pi}{4})} = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} - \vec{k}$

The tangent plane at the point $(1,1,\frac{\pi}{4})$ is $-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - (z - \frac{\pi}{4}) = 0$

or $x - y + 2z = \frac{\pi}{2}$

The normal line at the point $(1,1,\frac{\pi}{4})$ is $\frac{x-1}{-\frac{1}{2}} = \frac{y-1}{\frac{1}{2}} = \frac{z-\frac{\pi}{4}}{-1}$

or $x - 1 = \frac{y-1}{-1} = \frac{z-\frac{\pi}{4}}{2}$ or $\begin{aligned} x &= 1+t \\ y &= 1-t \\ z &= \frac{\pi}{4} + 2t \end{aligned}$

5
pf: $\nabla\left(\frac{-k}{\|\vec{r}\|}\right) = (-k)\nabla\frac{1}{\sqrt{x^2+y^2+z^2}} = (-k)\Big[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}\vec{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}\vec{j}$

$+ \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}\vec{k}\Big]$

$= (-k)\Big[(-\frac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)\vec{i} + (-\frac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}(2y)\vec{j} + (-\frac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}(2z)\vec{k}\Big]$

$= k\Big[\frac{x\,\vec{i}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{y\,\vec{j}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{z\,\vec{k}}{(x^2+y^2+z^2)}\Big] = \frac{k\vec{r}}{\|\vec{r}\|^3}$

6

Sol: $f(x,y) = \sin(\pi x y + \ln y)$

$f(0,1) = \sin(0 + \ln 1) = \sin 0 = 0$

The linear approximation of f at $(0,1)$ is

$$L(x,y) = f(0,1) + \frac{\partial f}{\partial x}\bigg|_{(0,1)}(x-0) + \frac{\partial f}{\partial y}\bigg|_{(0,1)}(y-1)$$

$$\frac{\partial f}{\partial x} = [\cos(\pi x y + \ln y)]\frac{\partial}{\partial x}(\pi x y + \ln y) = [\cos(\pi x y + \ln y)](\pi y)$$

$$\frac{\partial f}{\partial y} = [\cos(\pi x y + \ln y)]\frac{\partial}{\partial y}(\pi x y + \ln y) = [\cos(\pi x y + \ln y)]\left(\pi x + \frac{1}{y}\right)$$

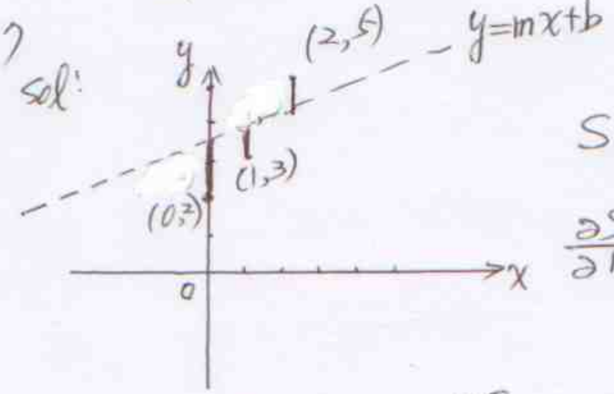
$$\frac{\partial f}{\partial x}\bigg|_{(0,1)} = [\cos(\ln 1)] \cdot \pi \cdot 1 = \pi \quad , \quad \frac{\partial f}{\partial y}\bigg|_{(0,1)} = [\cos(\ln 1)](\pi \cdot 0 + 1) = 1$$

$\therefore \quad L(x,y) = \pi x + y - 1$

$L(0.01, 1.05) = \pi(0.01) + 1.05 - 1 = 0.01\pi + 0.05 \simeq 0.0814$

7

Sol:



$$S(m,b) = (2-b)^2 + [3-(m+b)]^2 + [5-(2m+b)]^2$$

$$\frac{\partial S}{\partial m} = 2[3-(m+b)](-1) + 2[5-(2m+b)](-2)$$
$$= -26 + 10m + 6b$$

$$\frac{\partial S}{\partial b} = 2(2-b)(-1) + 2[3-(m+b)](-1) + 2[5-(2m+b)](-1)$$
$$= -20 + 6m + 6b$$

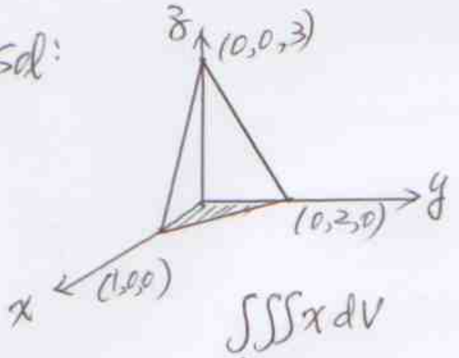
Let $\begin{cases} \frac{\partial S}{\partial m} = 0 \\ \frac{\partial S}{\partial b} = 0 \end{cases} \Rightarrow \begin{cases} 5m + 3b = 13 \\ 3m + 3b = 10 \end{cases} \Rightarrow m = \frac{3}{2},\ b = \frac{11}{6}$

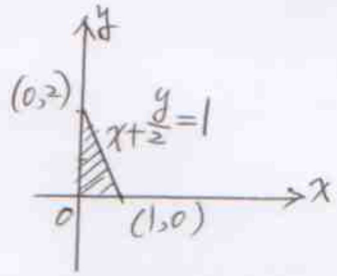
$$\Delta = \begin{vmatrix} \frac{\partial^2 S}{\partial m^2} & \frac{\partial^2 S}{\partial b \partial m} \\ \frac{\partial^2 S}{\partial m \partial b} & \frac{\partial^2 S}{\partial b^2} \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 6 & 6 \end{vmatrix} = 24 > 0, \quad 10 > 0$$

$\Rightarrow$ $m=\frac{3}{2}$, $b=\frac{11}{6}$ will give $S(m,b)$ the minimum

8

Sol:





$$\iiint_D x\,dV$$

$$= \int_0^1 \int_{y=0}^{y=2(1-x)} \int_{z=0}^{z=3(1-x-\frac{y}{2})} x\,dz\,dy\,dx$$

$$= \int_0^1 \int_{y=0}^{y=2(1-x)} xz \Big|_{z=0}^{z=3(1-x-\frac{y}{2})} dy\,dx = \int_0^1 \int_{y=0}^{y=2(1-x)} 3x\left(1-x-\frac{y}{2}\right) dy\,dx$$

$$= 3\int_0^1 x\left[(1-x)y - \frac{y^2}{4}\right]_{y=0}^{y=2(1-x)} dx = 3\int_0^1 x\left[(1-x)\cdot 2(1-x) - \frac{4(1-x)^2}{4}\right] dx$$

$$= 3\int_0^1 x(1-x)^2 dx$$

Let $u=1-x \Rightarrow du=-dx$

$x=0 \Rightarrow u=1$; $x=1 \Rightarrow u=0$

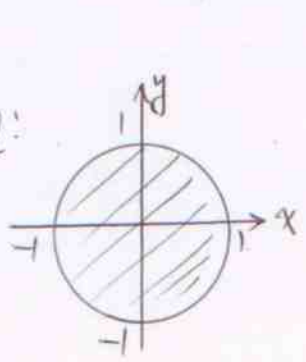
$$\therefore 3\int_0^1 x(1-x)^2 dx = 3\int_1^0 (1-u)u^2(-du) = 3\int_0^1 u^2(1-u)\,du$$

$$= 3\int_0^1 (u^2-u^3)\,du = 3\left(\frac{1}{3}u^3 - \frac{1}{4}u^4\right)\Big|_0^1 = 3\cdot\frac{1}{12} = \frac{1}{4}$$

$$\therefore \iiint_D x\,dV = \frac{1}{4}$$

9

Sol:



Let $x=r\cos\theta$, $y=r\sin\theta$

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \ln(r^2+1)\, r\, dr\, d\theta$$

Let $y = r^2+1 \Rightarrow dy = 2r\,dr$

$r=0 \Rightarrow y=1$; $r=1 \Rightarrow y=2$

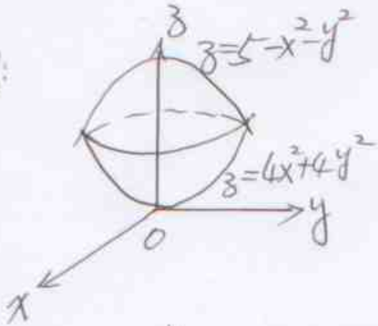
$$\int_0^1 \ln(r^2+1)\, r\, dr = \frac{1}{2}\int_1^2 \ln y\, dy = \frac{1}{2}\left[y\ln y\Big|_1^2 - \int_1^2 y\cdot\frac{1}{y}\, dy \right]$$

$$= \frac{1}{2}\left[2\ln 2 - y\Big|_1^2 \right] = \frac{1}{2}(2\ln 2 - 1)$$

$$\therefore \int_0^{2\pi}\int_0^1 \ln(r^2+1)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}(2\ln 2 - 1)\, d\theta = \frac{1}{2}(2\ln 2-1)\theta\Big|_0^{2\pi}$$

$$= \pi(2\ln 2 - 1)$$

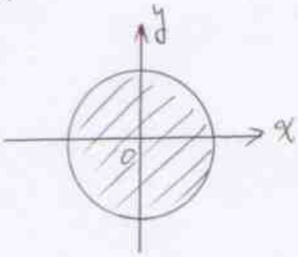
10
sol:



Let $5 - x^2 - y^2 = 4x^2 + 4y^2$

$\Rightarrow 5(x^2+y^2) = 5 \Rightarrow x^2+y^2 = 1$

$$V = \iint_{x^2+y^2\le 1} (5 - x^2 - y^2 - 4x^2 - 4y^2)\, dy\, dx$$



$$= 5\iint_{x^2+y^2\le 1} [1 - (x^2+y^2)]\, dy\, dx$$

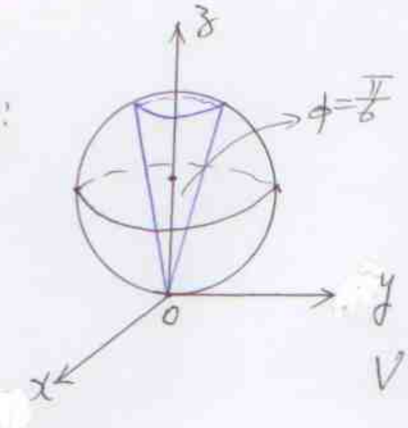
Let $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow V = 5\int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta = 5\int_0^{2\pi}\int_0^1 (r - r^3)\, dr\, d\theta$$

$$= 5\int_0^{2\pi} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1 d\theta = \frac{5}{4}\theta\Big|_0^{2\pi} = \frac{5\pi}{2}$$

11

sol:



$$\rho = 2\cos\phi$$

$$\Rightarrow \rho^2 = 2\rho\cos\phi$$

$$\Rightarrow x^2+y^2+z^2 = 2z \Rightarrow x^2+y^2+(z-1)^2 = 1^2$$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{2\cos\phi} d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{6}} 8\cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \left(-\frac{1}{4}\cos^4\phi\right)\Big|_0^{\frac{\pi}{6}} d\theta$$

$$= \left(-\frac{2}{3}\right)\int_0^{2\pi}\left(\cos^4\frac{\pi}{6} - 1\right)d\theta = \left(-\frac{2}{3}\right)\int_0^{2\pi}\left[\left(\frac{\sqrt{3}}{2}\right)^4 - 1\right]d\theta$$

$$= \frac{2}{3}\int_0^{2\pi}\left(1-\frac{9}{16}\right)d\theta = \frac{2}{3}\cdot\frac{7}{16}\theta\Big|_0^{2\pi} = \frac{7\pi}{12}.$$