The following answers except 9(a) and 9(b) are checked using fx-991EX

1. $\lim_{x\to 0} \dfrac{\tan x - x}{x^3} \Rightarrow ? \quad \dfrac{1}{3}$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{2\cdot \sec x \cdot \sec x \tan x}{6x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{2(\sec^2 x \tan^2 x + \sec^2 x \cdot \sec^2 x)}{6} = \boxed{\frac{1}{3}}$$

#1

2. $f(x) = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\cdot \sin^{-1}x - 2x, \quad f'(x) = ? \quad (\sin^{-1}x)^2$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}} \Bigg|_{|x|<1}$$

$$\Rightarrow f'(x) = \left[(\sin^{-1}x)^2 + x\cdot 2\cdot \sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}\right] + 2\cdot\left[\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1-x^2}}\cdot(-\cancel{2}x)\cdot \sin^{-1}x + \cancel{\sqrt{1-x^2}}\cdot \frac{1}{\cancel{\sqrt{1-x^2}}}\right] - 2$$

$$= (\sin^{-1}x)^2 + \cancel{\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}} + \cancel{\frac{2(-x)\cdot \sin^{-1}x}{\sqrt{1-x^2}}} + \cancel{2} - \cancel{2} = \boxed{(\sin^{-1}x)^2}$$

#2

3. $\displaystyle\int \underbrace{\underbrace{2x}_{}\,\underbrace{\tanh^{-1}x}_{u_1}\,dx}_{dv_1} \Rightarrow ?$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$du_1 = \frac{d}{dx}\tanh^{-1}x = \frac{1}{\operatorname{sech}^2(\tanh^{-1}x)} = \frac{1}{1-x^2}\,dx$$

$v_1 = x^2$

$$x^2\tanh^{-1}x - \int \frac{x^2}{1-x^2}\,dx$$

$$x^2\tanh^{-1}x - \int \frac{1}{1-x^2} - \cancel{\frac{1-x^2}{1-x^2}}^{\,1}\,dx$$

$$= \boxed{x^2\tanh^{-1}x - \tanh^{-1}x + x + C}$$

#3

$$x^2\tanh^{-1}x - \int \frac{x^2}{1-x^2}\,dx$$

$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$= x^2\tanh^{-1}x - \int \frac{\sin^2\theta \cdot \cancel{\cos\theta}}{\cos^2\theta}\,d\theta$$

$$= x^2\tanh^{-1}x - \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= x^2\tanh^{-1}x - \int \sec\theta\,d\theta + \sin\theta$$

$$= x^2\tanh^{-1}x - \ln|\sec\theta + \tan\theta| + \sin\theta + C$$

$$= x^2\tanh^{-1}x - \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + x + C$$

$$= \boxed{x^2\tanh^{-1}x - \ln|1+x| + \frac{1}{2}\ln|1-x^2| + x + C}$$

#3

4. (a)

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} 2x^2 = \infty \;\; \boxed{\Rightarrow x^2 \text{ grows faster than } \ln x}$$

#4(a)

4(b)

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{(2\sqrt{x}-1)^2} = \lim_{x\to\infty} \frac{x}{4x} = \frac{1}{4} \quad \because \frac{1}{4} \text{ is a finite positive integer}$$

$$\therefore \boxed{\sqrt{x^2+5} \text{ grows as fast as } (2\sqrt{x}-1)^2}$$

#4(b)

5 $\displaystyle\int \frac{\ln(\tan^{-1}x)}{1+x^2}\,dx = ?$

$u = \tan^{-1}x \Rightarrow du = \frac{1}{1+x^2}\,dx$

$$= \int \underbrace{\ln u}_{u_1}\,\underbrace{du}_{dv_1}$$

$du_1 = \frac{1}{u}\,du$

$v_1 = u$

$$= u\ln u - \int du$$

$$= u\ln u - u + C$$

$$= \boxed{\tan^{-1}x \ln|\tan^{-1}x| - \tan^{-1}x + C}$$

#5

6. (a) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx \; \frac{u = \cos x}{du = -\sin x\,dx} \; -\int \frac{1-u^2}{u^{\frac{1}{2}}}\,du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du = \boxed{\frac{2}{5}(\cos x)^{\frac{5}{2}} - 2(\cos x)^{\frac{1}{2}} + C}$

#6(a)

6(b)

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$

$$\int \frac{\sin x + \csc x}{\sec x}\,dx = \int \sin x\cos x + \cot x\,dx \; \frac{u = \sin x}{du = \cos x\,dx} \; \int u\,du + \ln|\sin x|$$

$$= \boxed{\frac{1}{2}\sin^2 x + \ln|\sin x| + C}$$

#6(b)

7. $\int \frac{1}{x^2\sqrt{2-x}}\,dx = ?$

$u = 2-x \Rightarrow x = 2-u$

$\Rightarrow du = -dx$

$= -\int \frac{1}{(2-u)^2\cdot\sqrt{u}}\,du$

$v = \sqrt{u} \Rightarrow dv = \frac{1}{2\sqrt{u}}\,du$

$= -2\int \frac{1}{(2-v^2)^2}\,dv$

$v = \sqrt{2}\sin\theta \Rightarrow dv = \sqrt{2}\cos\theta\, d\theta$

$= -2\int \frac{\sqrt{2}\,\cancel{\cos\theta}}{4\cos^{\cancel{4}3}\theta}\,d\theta$

$= -\frac{\sqrt{2}}{2}\int \sec^3\theta\, d\theta$

$= -\frac{\sqrt{2}}{2}\int \underbrace{\sec\theta}_{u_1}\cdot\underbrace{\sec^2\theta\, d\theta}_{dv_1}$

$du_1 = \sec\theta\tan\theta\, d\theta$

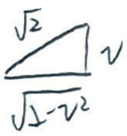
$v_1 = \tan\theta$

$= -\frac{\sqrt{2}}{2}\left(\sec\theta\tan\theta - \int \cancel{\tan^2\theta}^{(\sec^2\theta - 1)}\sec\theta\, d\theta\right)$

$= -\frac{\sqrt{2}}{2}\sec\theta\tan\theta + \frac{\sqrt{2}}{2}\int \sec^3\theta\, d\theta - \frac{\sqrt{2}}{2}\ln|\sec\theta+\tan\theta|$

$\Rightarrow +\cancel{\sqrt{2}}\int \sec^3\theta\, d\theta = +\frac{\cancel{\sqrt{2}}}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)$

$= -\frac{\sqrt{2}}{4}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) + C$



$= -\frac{\sqrt{2}}{4}\left(\frac{\sqrt{2}\cdot v}{2-v^2} + \ln\left|\frac{\sqrt{2}+v}{\sqrt{2-v^2}}\right|\right) + C$

$v = \sqrt{u} = \sqrt{2-x}$

$= \boxed{-\frac{\sqrt{2}}{4}\left(\frac{\sqrt{4-2x}}{x} + \ln|\sqrt{2}+\sqrt{2-x}| - \frac{1}{2}\ln|x|\right) + C}$

#7

8(a) $\displaystyle\int \frac{x^2+5}{(x+1)(x^2-2x+3)}\,dx = ?$

$$= \int \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}\,dx$$

$$x^2+5 = A(x^2-2x+3) + (Bx+C)(x+1)$$

$x=-1$:

$$6 = 6A \Rightarrow A=1$$

$$\Rightarrow 2x+2 = (Bx+C)(x+1)$$

$x=0$:

$$2 = C$$

$$\Rightarrow 2x+2 = Bx^2+Bx+2x+2 \Rightarrow B=0$$

$$= \int \frac{1}{x+1} + \frac{2}{x^2-2x+3}\,dx$$

$$= \ln|x+1| + 2\cdot\underbrace{\int \frac{1}{(x-1)^2+2}\,dx}_{u=x-1 \Rightarrow du=dx}$$

$$= \int \frac{1}{\underbrace{u^2+(\sqrt{2})^2}_{\substack{u=\sqrt{2}\tan\theta \\ \Rightarrow du=\sqrt{2}\sec^2\theta\, d\theta}}}\,du = \int \frac{\sqrt{2}\cancel{\sec^2\theta}}{2\cancel{\sec^2\theta}}\,d\theta = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$= \boxed{\ln|x+1| + \sqrt{2}\cdot\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C}$$

#8(a)

8(b)

$$\int \frac{2x+2}{(x-1)(x^2+1)^2}\,dx$$

$$= 2\int \frac{x+1}{(x-1)(x^2+1)^2}\,dx$$

$$= 2\int \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\,dx$$

$x+1 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)$

$x=1$:

$2 = 4A \Rightarrow \boxed{A = \frac{1}{2}}$

$x=0$:

$1 = A + C(-1)(1) + E(-1) \Rightarrow \frac{1}{2} = -C-E \quad —(1)$

$x=-1$:

$0 = 4A + (C-B)(-2)(2) + (E-D)(-2) \Rightarrow -2 = -4C+4B-2E+2D$

$x+1 = [(x^2+1)(Bx+C) + (Dx+E)](x-1) + A(x^2+1)^2$

$= [Bx^3 + Cx^2 + Bx + C + Dx + E](x-1) + \frac{1}{2}(x^4 + 2x^2 + 1)$

$\Rightarrow \boxed{B = -\frac{1}{2}}$

$= \dots -Bx^3 + Cx^3 \dots$

$\Rightarrow \boxed{C = -\frac{1}{2}}$

$C = -\frac{1}{2}$ 代 (1): $\boxed{E = 0}$

$= \dots (-\overset{\frac{1}{2}}{\cancel{B}} + \overset{-\frac{1}{2}}{\cancel{C}} - D - \overset{0}{\cancel{E}})x \dots$

$\Rightarrow -D = 1 \Rightarrow \boxed{D = -1}$

check.

$-2 = -\overset{2}{\cancel{4C}} + \overset{-2}{\cancel{4B}} - \overset{0}{\cancel{2E}} + \overset{-2}{\cancel{2D}}$ ok

$$= 2\int \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}(x+1)}{x^2+1} + \frac{-x}{(x^2+1)^2}\,dx$$

$$= \ln|x-1| - \int \frac{x+1}{x^2+1}\,dx - 2\int \frac{x}{(x^2+1)^2}\,dx$$

$$= \boxed{\ln|x-1| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + \frac{1}{x^2+1} + C}$$

#8(b)

The following answers are checked using MatLab.

9(a) $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx \Rightarrow ?$

$$= \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx \underset{du=dx}{\overset{u=x-1}{=}} \int_{-1}^0 u^{-\frac{2}{3}}\,du + \int_0^2 u^{-\frac{2}{3}}\,du = 3\lim_{b\to 0^-}\left[u^{\frac{1}{3}}\right]_{-1}^{b} + 3\lim_{c\to 0^+}\left[u^{\frac{1}{3}}\right]_c^2$$

$$= 3(0-(-1)) + 3(\sqrt[3]{2}-0)$$

$$= \boxed{3(1+\sqrt[3]{2})}$$

#9(a)

9(b) $\int_1^\infty \frac{1}{x^4+x^2}\,dx \Rightarrow ?$

$$= \int_1^\infty \frac{1}{x^2(x^2+1)}\,dx = \int_1^\infty \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+1}\,dx$$

$$1 = (Ax+B)(x^2+1) + (Cx+D)(x^2)$$

$$= (A+C)x^3 + (B+D)x^2 + \underbrace{Ax}_{\Rightarrow \boxed{A=0}} + B$$

subs. 0 for A:

$$(\overset{0}{A}+C)x^3 = 0 \Rightarrow \boxed{C=0}$$

Let $x=0$:

$$\boxed{1=B}$$

subs 1 for B

$$0 = (B+D)x^2 \Rightarrow \boxed{D=-1}$$

$$= \int_1^\infty \frac{1}{x^2} - \frac{1}{x^2+1}\,dx$$

$$= -\lim_{b\to\infty}\left[\frac{1}{x}\right]_1^b - \lim_{b\to\infty}\left[\tan^{-1}x\right]_1^b$$

$$= -(0-1) - \left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$= \boxed{1-\frac{\pi}{4}}$$

#9(b)