11/07

Probability and Statistics

Midterm exam., 10/31/2024

This is an open-book test, which means: (1). lecture notes and your annotations on them can be in electronic form, (2). your own notes can be in electronic form, (3). all other materials must be in print-out form, (4). access to the internet is absolutely prohibited. Please show your computations. The total score is 110 points.

1. Three women and seven men are to be seated around a round table randomly. Let A denote the event that all three women are sitting together. Let B denote the event that no women are sitting next to each other. Let C denote the event that exactly two of the three women are sitting together.

(a). (5%) Prob(A) = ? $\frac{1}{12}$
< Hint: > Use one of the three women as a reference position. This will simplify your reasoning.

(b). (5%) Prob(B) = ? $\frac{5}{12}$

(c). (5%) Prob(C) = ? $\frac{1}{2}$

(d). (5%) Is it true that A and B are independent events? No

(e). (5%) Is it true that A, B, and C constitute a partition of the sample space? Yes

2. Suppose that three cards are drawn from a standard deck of poker cards (52 cards, no Joker). Let X denote the number of different suits , out of the four possible suits (i.e. spade, heart, diamond, and club), among those three cards. Let Y denote an odd-sum indicator – if the sum (of the three cards) is odd, then Y assumes the values of 1; otherwise, Y assumes the values of 0.

(a). (5%) Find the probability mass function (pmf) of X. $P_X(x) = \begin{cases} 0.0518, & x=1 \\ 0.5506, & x=2 \\ 0.3976, & x=3 \\ 0, & \text{otherwise} \end{cases}$
< Hint: > Only at x = 1,2,3 can X assume some non-zero probability. So, just try to focus on those situations, respectively.

(b). (5%) Find the cummulative distribution function (cdf) of X. $F_X(x) = \begin{cases} 0, & x<1 \\ 0.0518, & 1 \le x < 2 \\ 0.0518+0.5506, & \\ 1, & x \ge 3 \end{cases}$

(c). (5%) Find the pmf of Y. $P_Y(y) = \begin{cases} 0.5021, & y=0 \\ 0.497, & y=1 \\ 0, & \text{otherwise} \end{cases}$
< Hint: > Out of the 52 cards, there are 24 even-numbered cards and 28 odd-numbered cards.

(d). (5%) Find the cdf of Y. $P(A|B) = \frac{P(AB)}{P(B)}$

(e). (5%) E(X) = ? Var(X) = ? $E(X) = 2.3458$ $Var(X) = 0.3298$

(f). (5%) Find the value of λ that minimizes $J(\lambda)$, where $J(\lambda) = \mathbf{E}((X-\lambda)^2)$.
< Hint: > Take the derivative of $J(\lambda)$ with respect to λ, and then set the derivative to zero. By the way, this particular value of λ is called the minimum mean squared estimate (MMSE) of X. $\lambda = 2.3458$

1 2 3 4 5 6 7 8 9

7 6

(g). (5%) Prob($Y = 1 | X = 1$) = ?

(h). (5%) Prob($Y = 1$ and $X = 1$) = ?

(i). (5%) Are X and Y independent ? No

3. In a certain factory, three machines, referred to as M_1, M_2, and M_3, respectively, make 18%, 35%, and 47%, respectively, of the products. It is known from the past experience that 0.023%, 0.036%, and 0.017% of the products made by each machine, respectively, are defective.

(a). (5%) Assume that a product is chosen randomly from the factory. What is the probabilty that it is defective? 0.02473%

(b). (5%) Assume that a product was chosen randomly and found to be defective. What is the probability that this defect product was made by machine M_1? 16.741%

4. (10%) Let X denote a geometric random variable, with a parameter of p. It is already known that the mean of X is $\frac{1}{p}$. Show that the variance of X is $\frac{1-p}{p^2}$.

5. Let X be a continuous random variable, whose probability density function (pdf) is:

$$f_X(x) = A \cdot e^{-\lambda|x|} \ , \ -\infty < x < \infty$$

where λ is a positive parameter, and A is some constant that makes $f_X(x)$ a pdf.

(a). (5%) $A = ?$ Please express your answer in terms of λ. $A = \frac{\lambda}{2}$

(b). (5%) Let $F_X(x)$ denote the cummulative distribution function (cdf) of X. Find $F_X(x)$. $F_X(x) = \begin{cases} \frac{1}{2}e^{\lambda x}, & x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x}, & x > 0 \end{cases}$

(c). (10%) A random variable Y is related to X via the function: $Y = \ln(|X|)$. Find the pdf of Y. $\lambda e^{y} \cdot e^{-\lambda e^{y}}$

$\le x < 3$

$\frac{e^{\lambda x}}{x} - \frac{1}{\lambda}$

$e^{y} = x$

c^{1+}

$w(y) = x = e^{y}$

$w(x)$