Total: **120** points
Note: To get full points, you should write down the procedure **in detail**.

1. Find the Taylor series centered at $x = a$ for the following function:
   (a) (**5** points) $f(x) = \dfrac{1}{1-x}, \quad a = 2$
   (b) (**5** points) $f(x) = 3^x, \quad a = 1$

Solution:

(a) $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}, \quad a = 2 \Rightarrow f(2) = -1.$

$$\begin{aligned}
f'(x) &= (1-x)^{-2} \Rightarrow f'(2) = (-1)^2 \\
f''(x) &= 2(1-x)^{-3} \Rightarrow f''(2) = 2! \cdot (-1)^3 \\
f^{(3)}(x) &= 3 \cdot 2 \cdot (1-x)^{-4} \Rightarrow f^{(3)}(2) = 3! \cdot (-1)^4 \\
f^{(n)}(x) &= n! \cdot (1-x)^{-(n+1)} \Rightarrow f^{(n)}(2) = n! \cdot (-1)^{n+1}
\end{aligned}$$

Therefore, the Taylor series centered at $x = 2$ for $f(x) = \dfrac{1}{1-x}$ is

$$-1 + (x-2) - (x-2)^2 + (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n + \cdots = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n$$

(b) $f(x) = 3^x, \quad a = 1 \Rightarrow f(1) = 3$

$$\begin{aligned}
f'(x) &= (\ln 3) \cdot 3^x \Rightarrow f'(1) = (\ln 3) \cdot 3 \\
f''(x) &= (\ln 3)^2 \cdot 3^x \Rightarrow f''(1) = (\ln 3)^2 \cdot 3 \\
f^{(3)}(x) &= (\ln 3)^3 \cdot 3^x \Rightarrow f^{(3)}(1) = (\ln 3)^3 \cdot 3 \\
f^{(n)}(x) &= (\ln 3)^n \cdot 3^x \Rightarrow f^{(n)}(1) = (\ln 3)^n \cdot 3
\end{aligned}$$

Therefore, the Taylor series centered at $x = 1$ for $f(x) = 3^x$ is

$$3 + 3\ln 3(x-1) + \frac{3(\ln 3)^2}{2!}(x-1)^2 + \cdots + \frac{3(\ln 3)^n}{n!}(x-1)^n + \cdots = \sum_{n=0}^{\infty} \frac{3(\ln 3)^n}{n!}(x-1)^n$$

2. A infinite geometric series

$$\sum_{n=2}^{\infty}(1+c)^{-n} = \frac{1}{(1+c)^2} + \frac{1}{(1+c)^3} + \cdots + \frac{1}{(1+c)^n} + \cdots$$

(a) (**5** points) If this geometric series converges, what is the range of c?

(b) (**5** points) If $\sum_{n=2}^{\infty}(1+c)^{-n} = 2$, what is the value of c?

Solution:

(a) If it is convergent,

$$\left|\frac{1}{1+c}\right| < 1 \Rightarrow |1+c| > 1 \Rightarrow 1+c > 1 \quad \text{or} \quad 1+c < -1 \Rightarrow c > 0 \quad \text{or} \quad c < -2.$$

(b) The common ratio is $\frac{1}{1+c}$, and the first term is $\frac{1}{(1+c)^2}$. Therefore

$$\frac{\frac{1}{(1+c)^2}}{1-\frac{1}{1+c}} = 2 \Rightarrow 1 = 2(1+c)^2 - 2(1+c) \Rightarrow 2c^2+2c-1 = 0 \Rightarrow c = \frac{-2\pm 2\sqrt{3}}{4} = \frac{-1\pm\sqrt{3}}{2}.$$

However, $-2 < \frac{-1+\sqrt{3}}{2} < 0$ which is **NOT** satisfied the condition in (a). Thus $c = \frac{-1-\sqrt{3}}{2}$.

3. (**10** points) Find the length of the parametric curve $x = 1+3t^2,\ y = 4+2t^3,\quad 0 \le t \le 1.$

Solution:

- $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$. The length of the curve is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{36t^2+36t^4}\,dt = 6\int_0^1 t\sqrt{1+t^2}\,dt \quad (u = 1+t^2,\ du = 2tdt)$$

$$= 3\int_1^2 \sqrt{u}\,du = 3\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Big|_0^2 = 4\sqrt{2}-2.$$

4. (**10** points) Find the area of the region common to the two regions bounded by the curves

$$r = -6\cos\theta \quad \text{and} \quad r = 2 - 2\cos\theta.$$

Solution:

- The polar graph $r = -6\cos\theta$ is a circle. The polar graph $r = 2(1-\cos\theta)$ is a cardioid. First step is to find the intersection of these two polar graphs.

$$-6\cos\theta = 2 - 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \quad \theta = \frac{4\pi}{3}$$

Because both of them are symmetric about the x axis, the area is $A = 2A_1$ where

$$\begin{aligned} A_1 &= \frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (2-2\cos\theta)^2 \, d\theta \\ &= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta \, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} \left(4 - 8\cos\theta + 4\cos^2\theta\right) d\theta \\ &= 9\int_{\pi/2}^{2\pi/3} (1+\cos 2\theta) \, d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta) \, d\theta \\ &= 9\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \left(\pi + \frac{9\sqrt{3}}{4}\right) = \frac{5\pi}{2} \end{aligned}$$

Therefore, the area is $A = 2A_1 = 5\pi$.

5. (**10** points) Let $f(x, y) = \sqrt{x^2 + y^2}$. Find

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Solution:

- $f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$.

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x = x\left(x^2 + y^2\right)^{-\frac{1}{2}}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y = y\left(x^2 + y^2\right)^{-\frac{1}{2}}$$

and,

$$\frac{\partial^2 f}{\partial x^2} = \left(x^2 + y^2\right)^{-\frac{1}{2}} + x\left(-\frac{1}{2}\right)\left(x^2 + y^2\right)^{-\frac{3}{2}} \cdot 2x = \left(x^2 + y^2\right)^{-\frac{1}{2}} - x^2\left(x^2 + y^2\right)^{-\frac{3}{2}}$$
$$\frac{\partial^2 f}{\partial y^2} = \left(x^2 + y^2\right)^{-\frac{1}{2}} + y\left(-\frac{1}{2}\right)\left(x^2 + y^2\right)^{-\frac{3}{2}} \cdot 2y = \left(x^2 + y^2\right)^{-\frac{1}{2}} - y^2\left(x^2 + y^2\right)^{-\frac{3}{2}}.$$

Therefore,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 2\left(x^2 + y^2\right)^{-\frac{1}{2}} - \left(x^2 + y^2\right)\left(x^2 + y^2\right)^{-\frac{3}{2}} \\ &= 2\left(x^2 + y^2\right)^{-\frac{1}{2}} - \left(x^2 + y^2\right)^{-\frac{1}{2}} \\ &= \left(x^2 + y^2\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2 + y^2}}.\end{aligned}$$

6. (**10** points) Evaluate the following integrals. (5 points for each)
Hint: You can change the order of integration if necessary.

(a) $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10}\frac{1}{\ln y}\,dy\,dx$ (b) $\displaystyle\int_0^{1}\int_{3y}^{3}e^{x^2}\,dx\,dy$

Solution:

(a) Change the order of integration.

$$\int_0^{\ln 10}\int_{e^x}^{10}\frac{1}{\ln y}\,dy\,dx=\int_1^{10}\int_0^{\ln y}\frac{1}{\ln y}\,dx\,dy=\int_1^{10}\left[\frac{x}{\ln y}\right]\Bigg|_0^{\ln y}dy=\int_1^{10}1\,dy=9.$$

(b) Change the order of integration.

$$\int_0^{1}\int_{3y}^{3}e^{x^2}\,dx\,dy=\int_0^{3}\int_0^{x/3}e^{x^2}\,dy\,dx=\int_0^{3}\left[ye^{x^2}\right]\Bigg|_0^{x/3}dx=\int_0^{3}\frac{x}{3}e^{x^2}\,dx=\frac{1}{3}\int_0^{3}xe^{x^2}\,dx$$
$$=\left[\frac{1}{6}e^{x^2}\right]\Bigg|_0^{3}=\frac{1}{6}\left(e^9-1\right).$$

7. (**15** points) Find all the local maxima, local minima, and saddle point(s) of the function

$$f(x,y)=x^3-3x^2+6y^2+5$$

Solution:

- $f_x=3x^2-6x,\ f_y=12y\Rightarrow f_{xx}=6x-6,\ f_{yy}=12,\ f_{xy}=0.$

$$f_x=3x^2-6x=3x(x-2)=0\Rightarrow x=0,\ x=2$$
$$f_y=12y=0\Rightarrow y=0.$$

Thus, the critical points are $(0,0)$, $(2,0)$.

Point $(0,0)$: $f_{xx}f_{yy}-(f_{xy})^2=-72<0\Rightarrow$ saddle point.

Point $(2,0)$: $f_{xx}f_{yy}-(f_{xy})^2=72<0,\ f_{xx}=6>0\Rightarrow$ local minima.

8. In order to find the absolute extreme value of $f(x,y) = x^2 + 3y^2 + 2y$ on the disk $x^2 + y^2 \leq 1$, one can solve this problem by answering the following questions:
   (a) (**5** points) Find the extreme value located at the interior of the disk by **finding the critical points** of $f(x,y)$ inside the disk.
   (b) (**10** points) Find the extreme value of $f(x,y)$ on the circle $g(x,y) = x^2 + y^2 - 1 = 0$.
   (c) (**5** points) Based on the results of (a) and (b), find the absolute maximum and minimum of $f(x,y)$ on the disk $x^2 + y^2 \leq 1$.

Solution:

(a) Find critical points located on $x^2 + y^2 < 1$.

$$f_x(x,y) = 2x = 0 \Rightarrow x = 0$$
$$f_y(x,y) = 6y + 2 = 0 \Rightarrow y = -\frac{1}{3}$$

and $(x,y) = (0, -\frac{1}{3})$ is located inside the disk. The critical point is $(0, -\dfrac{1}{3})$, and $f\left(0, -\frac{1}{3}\right) = -\frac{1}{3}$.

(b) Use Lagrange multiplier method to find the extreme value on the boundary.

$$\nabla f = (2x)\hat{\mathbf{i}} + (6y+2)\hat{\mathbf{j}}, \qquad \nabla g = (2x)\hat{\mathbf{i}} + (2y)\hat{\mathbf{j}}.$$

Because $\nabla f = \lambda \nabla g \;\Rightarrow\; 2x = 2x\lambda,\ 6y + 2 = 2y\lambda \;\Rightarrow\; 2x(1-\lambda) = 0 \;\Rightarrow\; \lambda = 1$ or $x = 0$.

Case 1 For $\lambda = 1, 6y + 2 = 2y \;\Rightarrow\; y = -\dfrac{1}{2}$.

$$g(x,y) = x^2 + y^2 - 1 = 0 \Rightarrow x^2 + \frac{1}{4} - 1 = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{2}.$$

For the point $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ and $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$,

$$f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{1}{2}.$$

Case 2 For $x = 0$,

$$g(x,y) = x^2 + y^2 - 1 = 0 \Rightarrow 0 + y^2 - 1 = 0 \Rightarrow y = \pm 1.$$

When $(x,y) = (0,1) \Rightarrow f(0,1) = 5$, and when $(0,-1) \Rightarrow f(0,-1) = 1$.

Therefore, the extreme value of $f(x,y)$ on the circle is 5 and $\dfrac{1}{2}$.

(c) Absolute maximum value is 5 at $(0,1)$. Absolute minimum value is $-\dfrac{1}{3}$ at $\left(0, -\dfrac{1}{3}\right)$.

9. (**25** points) Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$. (5 points for each)

(a) Find the gradient of f.

(b) Find the directional derivative of f at the point $A(1,1)$ in the direction toward the point $B(3,3)$.

(c) Find the maximum increasing rate of change of f at the point $A(1,1)$. Which is the direction of the maximum increasing rate of change?

(d) Find the tangent plane of $z = f(x,y)$ at the point $(1,1,\frac{1}{\sqrt{2}})$.

(e) Use linear approximation of $f(x,y)$ at $(1,1)$ to estimate the value of $f(1.01, 0.99)$.

Solution:

(a) $\nabla f = -\dfrac{x}{(x^2+y^2)^{3/2}}\hat{\mathbf{i}} - \dfrac{y}{(x^2+y^2)^{3/2}}\hat{\mathbf{j}}$

(b) At $A(1,1)$, $\nabla f|_A = -\dfrac{\sqrt{2}}{4}\hat{\mathbf{i}} - \dfrac{\sqrt{2}}{4}\hat{\mathbf{j}}$. Direction: $\hat{\mathbf{u}} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} + \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$. Therefore, the directional derivative is $\nabla f|_A \cdot \hat{\mathbf{u}} = -\dfrac{1}{2}$

(c) The maximum **increasing** rate of change of f at the point $A(1,1)$ is $|\nabla f|_A| = \frac{1}{2}$. The direction is $\hat{\mathbf{u}} = -\dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$

(d) Tangent plane is

$$z - \frac{1}{\sqrt{2}} = f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$\Rightarrow z - \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{4}(x-1) - \frac{\sqrt{2}}{4}(y-1) \Rightarrow \frac{\sqrt{2}}{4}x + \frac{\sqrt{2}}{4}y + z = \sqrt{2}$$

(e) Linear approximation at $(1,1)$ is

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}(x-1) - \frac{\sqrt{2}}{4}(y-1).$$

Therefore,

$$f(1.01, 0.99) \approx L(1.01, 0.99) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}(1.01-1) - \frac{\sqrt{2}}{4}(0.99-1) = \frac{1}{\sqrt{2}}.$$