Probability and Statistics

Final exam., 12/19/2024

This is an open-book test, which means: (1). lecture notes and your annotations on them can be in electronic form, (2). your own notes can be in electronic form, (3). all other materials must be in print-out form, (4). access to the internet is absolutely prohibited. Please show your computations. The total score is 110 points.

1. Suppose that we want to generate random numbers that follow the probability density function (pdf):

$$f_X(x) = \begin{cases} A \cdot \sqrt{x} \,, & \text{if } 0 \le x \le 1 \,, \\ 0 \,, & \text{otherwise.} \end{cases}$$

In the meantime, let us assume that available to us is a random number generator GenUniRand, and it generates random numbers according to the Uniform$(0, 1)$ distribution.

(a). (5%) $A = ?$

(b). (10%) How do we generate random numbers that meets the $f_X(\cdot)$ distribution?
<Hint> Apply the inverse-transformation method.

2. In this problem, we consider some random experiment, which is denoted as $\mathcal{A}$, wherein some traffic events are to be observed. Let us try to analyze it with Poisson and exponential distributions. Let us denote the number of occurrences per unit-time, when $\mathcal{A}$ is observed, as X. We assume that $X \sim$ Poisson(1.73) (i.e. X is distributed according to the Poisson(1.73) distribution). In the meantime, let Y denote the inter-arrival time of adjacent events, when $\mathcal{A}$ is observed.

(a). (5%) Find the moment generating function of X.

(b). (5%) Prob$(1 \le Y \le 2)$ =?

(c). (5%) Prob$(Y \le 1 | Y \le 2)$ =?

3. Recall that the gamma function is defined by

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt \ .$$

(a). (5%) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
<Hint> Apply the change of variables: $t = \frac{x^2}{2}$.

(b). (5%) $\Gamma(\frac{13}{2})$ =?

4. The joint probability mass function (pmf) of random variables X and Y, denoted as $P_{XY}(x, y) = \text{Prob}(X = x, Y = y)$, are shown below:

$t^{-\frac{1}{2}}$
$\sqrt{t}^{-1}$
$\frac{1}{\sqrt{t}}$

$P_{XY}(x,y)$	$x=1$	$x=2$	$x=3$
$y=1$	3/27	5/27	1/27
$y=2$	4/27	2/27	3/27
$y=3$	1/27	4/27	4/27

(a). (5%) Find the marginal pmf's of X and Y.

(b). (5%) Find the conditional pmf: Prob$(X = x|Y = 3)$.

(c). (5%) Suppose that we have observed: $Y = 3$, and we want to make an estimate about X. Then, what is the best estimate for X, if the criterion for measuring the goodness of estimation is mean squared error.

(d). (5%) Let Z be defined as $Z = X + Y$. Find the pmf of Z.

(e). (5%) Continued from the preceeding subproblem, find the moment generating function (mgf) of Z. Let us denote it as $M_Z(t)$.

(f). (5%) Let the mgf's of X and Y be denoted as $M_X(t)$ and $M_Y(t)$, respectively. Is it true that the equation holds: $M_Z(t) = M_X(t) \cdot M_Y(t)$?

5. Two continuous random variables X and Y are specified by their joint probability density function (pdf):

$$f_{XY}(x,y) = \begin{cases} 1, & \text{if } 0 \le x, y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a). (5%) Show that X and Y are independent.

(b). (5%) Let S be defined as $X + Y$. Find the pdf of S.

(c). (5%) Let Z be defined as $\max(X, Y)$. Find the pdf of Z.
<Hint> Start by finding the cummulative distribution function (cdf) of Z: $F_Z(t) = \text{Prob}(Z \le t)$. In the meantime, you need to convert the event "$Z \le t$" into some constraints on X and Y.

(d). (5%) Continued from the preceeding subproblem, find the variance of Z.

(e). (5%) Let Q be defined as $\min(X, Y)$. Find the pdf of Q.

(f). (5%) Continued from the preceeding subproblem, find the mean of Q.

6.(10%) A fair die is rolled 100 times. Let us apply the central limit theorem to find the approximate probability that the sum of the outcomes lies between 325 and 375. What value is it? Please express your answer in terms of $\Phi(\cdot)$: the cummulative distribution function of the standard Gaussian random variable.