國立臺灣科技大學答案卷
National Taiwan University of Science and Technology Answer Sheet

評分 Score	教師簽章 Signature of Lecturer
97	

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科目/Course title 工程數學 日期/Date 112.6.13

記分欄 從此邊開始寫起。試卷用紙務須節用，非經主試認可不得續用其他紙張作答。/Please write from here.

1 double the length:

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$A = [a_1 ; a_2]$ $\qquad A\begin{bmatrix} x \\ y \end{bmatrix} = a_1 x + a_2 y$

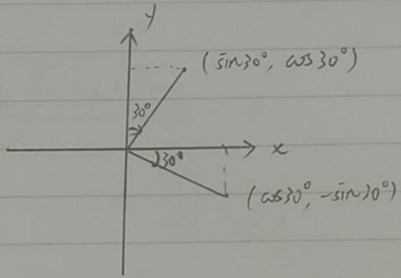
$a_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

rotate 30° clockwise

$B = [b_1 ; b_2]$

$b_1 = \begin{bmatrix} \cos 30^\circ \\ -\sin 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

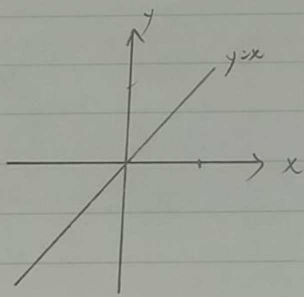
$b_2 = \begin{bmatrix} \sin 30^\circ \\ \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ $\Rightarrow B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$



reflect about $x = y$

$C = [c_1 ; c_2]$

$c_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



The standard matrix expression is:

$E = CBA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}}$ #1(a)

$T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = E\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix} = x\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} + y\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

solve it!

$\ker\left(T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix}, E\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ $\Rightarrow$ Only $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ can transformed to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ $\boxed{T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \text{ is one-to-one}}$ #1

$E\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x+\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \end{array}\right] \xrightarrow{+\sqrt{3}} \left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ 0 & 4 & 0 \end{array}\right] \Rightarrow x = 0,\ y = 0$

$\begin{array}{cc} e_1 & e_2 \end{array}$

$\sqrt{3}\left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ 0 & 4 & 0 \end{array}\right] \Rightarrow \operatorname{rank}(E) = 2 = n \Rightarrow$ single solution

$\Rightarrow c_1 e_1 + c_2 e_2 = 0$ iff $c_1 = c_2 = 0 \Rightarrow \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ are l.i.

$\Rightarrow \dim\left(T_E\begin{bmatrix} x \\ y \end{bmatrix}\right) =$ $\qquad \Rightarrow \dim\left(T_E\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2 = \dim(\mathbb{R}^2) \Rightarrow \boxed{T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \text{ is onto}}$ #1(c)

記分欄 轉頁從此開始寫起。

2. $\langle f(t), g(t)\rangle = \int_{-3}^{3} f(t)\, g(t)\, dt$

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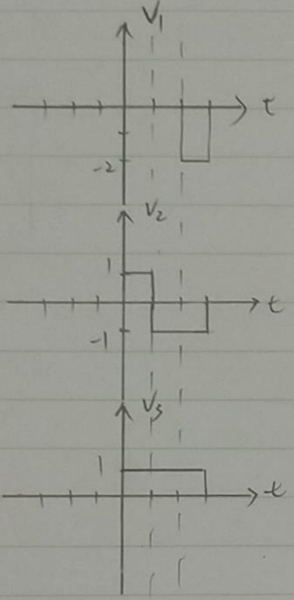
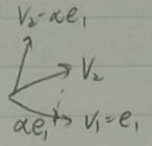
$\|f(t)\| = \sqrt{\langle f(t), f(t)\rangle}$



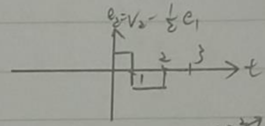


Let $e_1 = v_1$

$$\langle e_1, e_2\rangle = \langle e_1, v_2-\alpha e_1\rangle = \langle e_1, v_2\rangle - \alpha\|e_1\|^2 = 0 \Rightarrow \alpha = \frac{\langle e_1, v_2\rangle}{\|e_1\|^2}$$

$$e_2 = v_2 - \frac{\langle e_1, v_2\rangle}{\|e_1\|^2} e_1$$

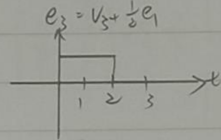
$$= v_2 - \frac{\underbrace{\int_{-3}^{0} 0\, dt}_{0} + \underbrace{\int_{0}^{1} 0\cdot 1\, dt}_{0} + \underbrace{\int_{1}^{2} 0\cdot(-1)\, dt}_{0} + \underbrace{\int_{2}^{3} (-2)(-1)\, dt}_{2[t]_2^3 = 2}}{\underbrace{\int_{-3}^{2} 0^2\, dt}_{0} + \underbrace{\int_{2}^{3} (-2)^2\, dt}_{4[t]_2^3 = 4}} \cdot e_1 = v_2 - \frac{1}{2} e_1$$



$$e_3 = v_3 - \frac{\langle e_1, v_3\rangle}{\|e_1\|^2} e_1 - \frac{\langle e_2, v_3\rangle}{\|e_2\|^2} e_2$$

$$= v_3 - \frac{\underbrace{\int_{-3}^{0} 0\, dt}_{0} + \underbrace{\int_{0}^{2} 0\cdot 1\, dt}_{0} + \underbrace{\int_{2}^{3} (-2)(1)\, dt}_{(-2)[t]_2^3 = (-2)}}{4} e_1 - \frac{\underbrace{\int_{-3}^{0} 0\, dt}_{0} + \underbrace{\int_{0}^{1} 1\cdot 1\, dt}_{1} + \underbrace{\int_{1}^{2} (-1)\cdot 1\, dt}_{-[t]_1^2 = -1} + \underbrace{\int_{2}^{3} 0\cdot 1\, dt}_{0}}{\underbrace{\int_{-3}^{0} 0^2\, dt}_{0} + \underbrace{\int_{0}^{1} 1^2\, dt}_{1} + \underbrace{\int_{1}^{2} (-1)^2\, dt}_{1} + \underbrace{\int_{2}^{3} 0^2\, dt}_{0}} e_2$$

$$= v_3 + \frac{1}{2} e_1$$

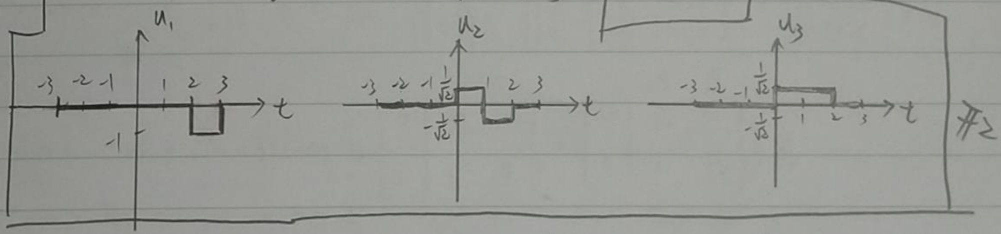


$$\Rightarrow u_1 = \frac{e_1}{\|e_1\|} = \frac{v_1}{\sqrt{\underbrace{\int_{-3}^{2} 0^2\, dt}_{0} + \underbrace{\int_{2}^{3} (-2)^2\, dt}_{4}}} = \frac{v_1}{2}$$

$$\Rightarrow u_2 = \frac{e_2}{\|e_2\|} = \frac{e_2}{\sqrt{\underbrace{\int_{-3}^{0} 0^2\, dt}_{0} + \underbrace{\int_{0}^{1} 1^2\, dt}_{1} + \underbrace{\int_{1}^{2} (-1)^2\, dt}_{1} + \underbrace{\int_{2}^{3} 0^2\, dt}_{0}}} = \frac{v_2 - \frac{1}{2}v_1}{\sqrt{2}} = \frac{v_2}{\sqrt{2}} - \frac{1}{2\sqrt{2}} v_1$$

$$\Rightarrow u_3 = \frac{e_3}{\|e_3\|} = \frac{v_3 + \frac{1}{2}v_1}{\sqrt{\underbrace{\int_{-3}^{0} 0^2\, dt}_{0} + \underbrace{\int_{0}^{2} 1^2\, dt}_{2} + \underbrace{\int_{2}^{3} 0^2\, dt}_{0}}} = \frac{v_3}{\sqrt{2}} + \frac{v_1}{2\sqrt{2}}$$

$$\Rightarrow \{u_1, u_2, u_3\} = \left\{\frac{v_1}{2},\ \frac{v_2}{\sqrt{2}} - \frac{v_1}{2\sqrt{2}},\ \frac{v_3}{\sqrt{2}} + \frac{v_1}{2\sqrt{2}}\right\}$$



#2

$0\sim1: \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$

$1\sim2: -\frac{1}{\sqrt{2}} - 0 = \frac{-1}{\sqrt{2}}$

$2\sim3: -\frac{1}{\sqrt{2}} - \frac{-2}{2\sqrt{2}} = 0$

$0\sim2: \frac{1}{\sqrt{2}} + \frac{0}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

$2\sim3: \frac{1}{\sqrt{2}} + \frac{-2}{2\sqrt{2}} = 0$

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3.(a) $A = \begin{bmatrix} 2 & 6 \\ \eta & \alpha \end{bmatrix}$, $C_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -6 \\ -\eta & \lambda-\alpha \end{bmatrix}\right) = \lambda^2 - (2+\alpha)\lambda + \underbrace{2\alpha-6\eta}_{\lambda_1\lambda_2}$

$= (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$

$\Rightarrow \lambda_1+\lambda_2 = 2+\alpha$ —(1)

$\lambda_1\lambda_2 = 2\alpha - 6\eta$ —(2)

Subs. $\lambda_1 = 2$, $\lambda_2 = 6$ into (1) (2)

$8 = 2+\alpha$
$12 = 2\alpha - 6\eta$ $\Rightarrow \boxed{\alpha = 6} \Rightarrow 12 = 12 - 6\eta \Rightarrow \boxed{\eta = 0}$ #3(a)

$\boxed{\text{The eigenvalues of } (2A-I)^{-1} \text{ are:}}$

$\dfrac{1}{2\cdot 2-1} = \dfrac{1}{3}$

$\dfrac{1}{6\cdot 2-1} = \dfrac{1}{11}$ #3(a)

3(b) $Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 6 \\ \eta & \alpha \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 \\ 2\eta+\alpha \end{bmatrix} = \begin{bmatrix} 2\lambda_1 \\ \lambda_1 \end{bmatrix} \Rightarrow \lambda_1 = 5 \Rightarrow 2\eta + \alpha = 5$

$\Rightarrow \begin{bmatrix} 2 & 6 \\ \eta & \alpha \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 \\ 3\eta+\alpha \end{bmatrix} = \begin{bmatrix} 3\lambda_2 \\ \lambda_2 \end{bmatrix} \Rightarrow \lambda_2 = 4 \Rightarrow 3\eta + \alpha = 4$

$\Rightarrow \boxed{\eta = -1}$ #3(b)

$\boxed{\alpha} = 5 - 2(-1) = \boxed{7}$ #3(b)

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3(c) $A = \begin{bmatrix} 2 & 6 \\ \eta & -2 \end{bmatrix}$

$AP = PD$

$C_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -6 \\ -\eta & \lambda+2 \end{bmatrix}\right) = \lambda^2 - 4 - 6\eta \Rightarrow \lambda = \pm\sqrt{4+6\eta}$ (1)

A matrix is not diagonalizable when the multiplicity > the dimension of the eigenspace, in this case

$\lambda = 4 + 6\eta = 0 \Rightarrow \eta = -\frac{2}{3}$ (the multiplicity of $\lambda = 0$ is 2)

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Subs. $\lambda = 0$ into $\lambda I - A = 0$:

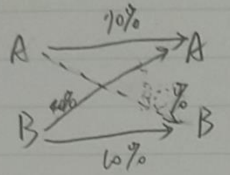
$\eta = -\frac{2}{3}$

$\left[\begin{array}{cc|c} -2 & -6 & 0 \\ \frac{2}{3} & 2 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1/3)}} \begin{array}{cc} a & b \\ \left[\begin{array}{cc|c} -2 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \text{let } b = \alpha \Rightarrow -2a = 6\alpha \Rightarrow a = -3\alpha \Rightarrow X = \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\Rightarrow \dim(N(\lambda I - A)) = 1 < \#(\lambda = 0) = 2$

$\Rightarrow \boxed{\text{When } \eta = -\frac{2}{3}, \text{ A is not diagonalizable}}$ #3(C)

4



$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}}_{C}^{n} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$$

diagonalize C:

$$C_C(\lambda) = \det\left(\begin{bmatrix} \lambda-0.7 & -0.4 \\ -0.3 & \lambda-0.6 \end{bmatrix}\right) = \lambda^2 - 1.3\lambda + \cancel{0.42 - 0.12}^{\,0.3}$$

$$\Rightarrow \lambda = \frac{1.3 \pm \sqrt{1.3^2 - 1.2}}{2}$$

Subs. $\lambda = \dfrac{1.3 + \cancel{\sqrt{1.69-1.2}}^{\,0.7}}{2}$ into $\lambda I - A = 0$:

$$+1\begin{bmatrix} 0.3 & -0.4 & 0 \\ -0.3 & 0.4 & 0 \end{bmatrix} = \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 0.3 & -0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow \text{let } b = \alpha: \quad \Rightarrow x = \alpha\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

$0.3a = 0.4\alpha$

$a = \frac{4}{3}\alpha$

Subs. $\lambda = \dfrac{1.3 - \cancel{\sqrt{1.69-1.2}}^{\,0.7}}{2}$ into $\lambda I - A = 0$

0.6

$$-\frac{3}{4}\begin{bmatrix} -0.4 & -0.4 & 0 \\ -0.3 & -0.3 & 0 \end{bmatrix} = \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} -0.4 & -0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow \text{let } b = \alpha \quad \Rightarrow x = \alpha\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$-0.4a = 0.4\alpha$

$a = -\alpha$

$$\Rightarrow AP = PD, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix} \quad P = \begin{bmatrix} \frac{4}{3} & -1 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \cancel{\frac{1}{\frac{4}{3}+1}}^{\,\frac{3}{7}}\begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{bmatrix}$$

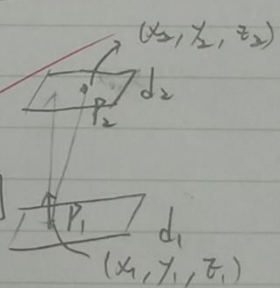
$$\Rightarrow A^n = PD^nP^{-1}$$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{3} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 0.3^n \end{bmatrix}}\underbrace{\begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{4}{3} & -0.3^n \\ 1 & 0.3^n \end{bmatrix} \qquad = \begin{bmatrix} \frac{3}{7}\cdot\frac{3}{10} + \frac{3}{7}\cdot\frac{7}{10} \\ -\frac{3}{7}\cdot\frac{3}{10} + \frac{4}{7}\cdot\frac{7}{10} \end{bmatrix} \begin{bmatrix} \frac{3}{7} \\ \frac{-9}{70} + \frac{28}{70} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{19}{70} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3}\cdot\frac{3}{7} - 0.3^n \cdot \frac{19}{70} \\ \frac{3}{7} + 0.3^n \cdot \frac{19}{70} \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{\text{number of particle } A}{\text{number of particle } B} = \frac{\frac{4}{7}}{\frac{3}{7}} = \boxed{\frac{4}{3}} \quad \#4$$



$ax + by + cz = a + b + c$

5. $ax + by + cz = d_1$ $\qquad a x_1 + b y_1 + c z_1 = d_1$

$ax + by + cz = d_2$ $\qquad a x_2 + b y_2 + c z_2 = d_2$

$$x = \left|\overrightarrow{P_2P_1} \cdot \frac{\vec{V}}{\|V\|}\right| = \left|\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}^T \cdot \frac{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{a(x_2-x_1) + b(y_2-y_1) + c(z_2-z_1)}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{d_2 - d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$\Rightarrow$ $\boxed{\text{distance between them is } \dfrac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}}$ #5