Calculus Midterm Solution

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1.

a. $\lim_{x\to 0}\frac{x}{\sqrt[3]{8+x}-\sqrt[3]{8+x^3}}=$

$$\lim_{x\to 0}\frac{x\left[\sqrt[3]{(8+x)^2}+\sqrt[3]{(8+x)(8+x^3)}+\sqrt[3]{(8+x^3)^2}\right]}{\left[\sqrt[3]{8+x}-\sqrt[3]{8+x^3}\right]\left[\sqrt[3]{(8+x)^2}+\sqrt[3]{(8+x)(8+x^3)}+\sqrt[3]{(8+x^3)^2}\right]}=$$

$$\lim_{x\to 0}\frac{x\left[\sqrt[3]{(8+x)^2}+\sqrt[3]{(8+x)(8+x^3)}+\sqrt[3]{(8+x^3)^2}\right]}{(8+x)-(8+x^3)}=$$

$$\lim_{x\to 0}\frac{x\left[\sqrt[3]{(8+x)^2}+\sqrt[3]{(8+x)(8+x^3)}+\sqrt[3]{(8+x^3)^2}\right]}{x(1-x^2)}=12$$

b. $\lim_{x\to 0}\frac{|x|-x}{|x|-x^3}$

有變號的情形，需考慮左右極限

$$\lim_{x\to 0^+}\frac{|x|-x}{|x|-x^3}=\lim_{x\to 0^+}\frac{x-x}{x-x^3}=0$$

$$\lim_{x\to 0^-}\frac{|x|-x}{|x|-x^3}=\lim_{x\to 0^-}\frac{-x-x}{-x-x^3}=2$$

左右極限不相等，故極限不存在

c. $\lim_{x\to 0}\frac{\lfloor x+1\rfloor+|x|}{x}$

有變號的情形，需考慮左右極限

$$\lim_{x\to 0^+}\frac{\lfloor x+1\rfloor+|x|}{x}=\lim_{x\to 0^+}\frac{1+x}{x}=\infty$$

$$\lim_{x\to 0^-}\frac{\lfloor x+1\rfloor+|x|}{x}=\lim_{x\to 0^-}\frac{0-x}{x}=-1$$

左右極限不相等，故極限不存在

d. $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

$$=\lim_{x\to 0}\frac{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{1+\sin x-1+\sin x}{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\left(\frac{2\sin x}{x}\frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)=1$$

2.

$$f(x)=\frac{1+\sin x}{x\cos x}$$

$$f'(x)=\frac{\cos x\cdot x\cos x-(\cos x-x\sin x)(1+\sin x)}{(x\cos x)^2}$$

$$=\frac{(x-\cos x)(\sin x+1)}{x^2\cos^2 x}$$

3.

$$\frac{d}{dx}(2x^3-x^2y^2+4y^3=16)$$

$$\Rightarrow\quad 6x^2-2xy^2-2x^2y\frac{dy}{dx}+12y^2\frac{dy}{dx}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{2xy^2-6x^2}{12y^2-2x^2y}$$

$$\Rightarrow\quad \left.\frac{dy}{dx}\right|_{(2,1)}=\left.\frac{2xy^2-6x^2}{12y^2-2x^2y}\right|_{(2,1)}=-5$$

4.

水深 h，水面半徑 r

$$\frac{h}{20}=\frac{r}{10}\quad\Rightarrow\quad h=2r$$

體積 V

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{1}{12}\pi h^3$$

$$\frac{dV}{dh}=\frac{1}{4}\pi h^2\quad\Rightarrow\quad\frac{\frac{dV}{dt}}{\frac{dh}{dt}}=\frac{1}{4}\pi h^2\quad\Rightarrow\frac{dV}{dt}=3$$

$$\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{1}{4}\pi h^2}=\frac{3}{\frac{1}{4}\pi\cdot 2^2}=\frac{3}{\pi}\ \text{ m/min}$$

5.

$f(x)=2x^7+x-1$, $f(0)=-1<0$, $f(1)=2>0$

在 (0, 1) 內存在 ξ 使得 $f(\xi)=0$

設 $f(x)$ 有兩相異實根 ξ_1 ,ξ_2 ，且 $\xi_1<\xi_2$ ，則有 $f(\xi_1)=f(\xi_2)=0$

由 Rolle 定理知存在 $x\in(\xi_1,\xi_2)$ 使得 $f'(x)=0$

$f'(x)=14x^6+1\neq 0$，矛盾，故 $f(x)$ 恰有一實根。

若 $\xi_1 = \xi_2 = \xi$ 為重根，則 $f(\xi_1) = f(\xi_2) = 0 \quad \Rightarrow \quad f'(\xi) = 0$

但 $f'(x) \neq 0$ ，故 $\xi_1 \neq \xi_2$ ，故得証，恰有一根

6.

$$\lim_{x \to 0} \frac{\sin bx}{\sin ax} = \lim_{x \to 0} \left(\frac{\sin bx}{bx} \cdot \frac{ax}{\sin ax} \cdot \frac{b}{a} \right) = \frac{b}{a}$$

7.

If $\varepsilon > 0$, we find $\delta > 0$, so that when $0 < |x - 3| < \delta$, $|(2x + 2) - 8| < \varepsilon$.

$$|(2x + 2) - 8| < \varepsilon \Rightarrow |2(x - 3)| < \varepsilon \Rightarrow |x - 3| < \frac{\varepsilon}{2}$$

we choose $\delta = \frac{\varepsilon}{2}$, so that when $|x - 3| < \delta$, $|(2x + 2) - 8| < \varepsilon$.

8.

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x^2 - 1) - 0}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{-(x^2 - 1) - 0}{x - 1} = \lim_{x \to 1^-} (-x - 1) = -2$$

so that $f(x)$ is not differentiable at $x = 1$.

9.

$$f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow f(x + h) \approx f(x) + f'(x)h$$

$$x = 4, \quad h = 0.02, \quad \sqrt{4.02} = 2 + \frac{1}{2} \cdot (4)^{-\frac{1}{2}} \cdot 0.02 = 2.005$$

10.

$$A(t) = 4\pi r^2, \quad V(t) = \frac{4\pi r^3}{3}, \quad \begin{cases} \dfrac{dA}{dt} = 8\pi r \dfrac{dr}{dt} \\ \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt} \end{cases}$$

$$\frac{dV}{dt} = \frac{dA}{dt} \cdot \frac{r}{2} = 4 \cdot \frac{10}{2} = 20 \quad \frac{cm^3}{sec}$$

11.

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} (x + 3)^{\frac{2}{3}} + \frac{2}{3} x^{\frac{1}{3}} (x + 3)^{-\frac{1}{3}} = \frac{x+1}{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$. Hence, $x = -1, 0, -3$ are critical points.

$f''(x) = \frac{-2}{x^{\frac{5}{3}}(x+3)^{\frac{4}{3}}}$, $x = 0$ or $x = -3$ are possible inflection points.

x	-3	-1	0
$f(x)$	0	$-\sqrt[3]{4}$	0
$f'(x)$	+ $-$	$-$ +	+ +

$f(x)$is increasing on $(-\infty, -3)$ and $(-1, \infty)$; $f(x)$is decreasing on $(-3, -1)$