1. (**15** points) Evaluate the following integrals. (5 points for each)

   (a) $\int x\sqrt{2-5x}\,dx$ (b) $\int \frac{\cos(\pi/x)}{x^2}\,dx$ (c) $\int \frac{1}{\sqrt{x\sqrt{x}+x}}\,dx$

Solution:

(a) Let $u = 2-5x \Rightarrow du = -5dx,\ x = \frac{2-u}{5}$

$$\int x\sqrt{2-5x}\,dx = \int \left[\frac{1}{5}(2-u)\cdot u^{\frac{1}{2}}\cdot\left(-\frac{1}{5}\right)\right]du = -\frac{1}{25}\int(2-u)u^{\frac{1}{2}}\,du = -\frac{1}{25}\int\left(2u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du$$
$$= -\frac{1}{25}\left[\frac{4}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]+C = -\frac{4}{75}(2-5x)^{\frac{3}{2}}+\frac{2}{125}(2-5x)^{\frac{5}{2}}+C$$

(b) Let $u = \frac{\pi}{x} \Rightarrow du = -\frac{\pi}{x^2}dx \Rightarrow \frac{1}{x^2}dx = -\frac{1}{\pi}$

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\right)du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right)+C$$

(c) Let $u = \sqrt{x}+1 \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du$

$$\int \frac{1}{\sqrt{x\sqrt{x}+x}}\,dx = \int \frac{1}{\sqrt{\sqrt{x}+1}}\frac{dx}{\sqrt{x}} = \int \frac{2}{\sqrt{u}}\,du = 4u^{\frac{1}{2}}+C = 4\sqrt{\sqrt{x}+1}+C$$

2. (**10** points) The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate maximum error and the relative error in the calculated area of the disk.

Solution:

- $A = \pi r^2 \Rightarrow dA = 2\pi r dr$.

  Maximum error is $dA = 2\pi r dr = 2\pi\cdot 24\cdot 0.2 = 9.6\pi$

  Relative error is $\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = 2\frac{dr}{r} = 2\cdot\frac{0.2}{12} = \frac{1}{60}$

3. Let $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} \left[1+\sin(t^2)\right]dt$.

   (a) (**2** points) Find $g(\frac{\pi}{2})$

   (b) (**4** points) Find $g'(x)$

   (c) (**4** points) Find $f'(\frac{\pi}{2})$

Solution:

(a) $g(\frac{\pi}{2}) = \int_0^{\cos\frac{\pi}{2}} \left[1+\sin(t^2)\right]dt = \int_0^0 \left[1+\sin(t^2)\right]dt = 0$

(b) $g'(x) = \frac{d}{dx}\int_0^{\cos x} \left[1+\sin(t^2)\right]dt = (-\sin x)\left[1+\sin(\cos^2 x)\right]$

(c) $f'(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt = g'(x)\frac{1}{\sqrt{1+\left[g(x)\right]^3}} \Rightarrow f'(\frac{\pi}{2}) = g'(\frac{\pi}{2})\frac{1}{\sqrt{1+\left[g(\frac{\pi}{2})\right]^3}}$

Therefore, $f'(\frac{\pi}{2}) = g'(\frac{\pi}{2})\frac{1}{\sqrt{1+0^3}} = g'(\frac{\pi}{2}) = \left(-\sin\frac{\pi}{2}\right)\left[1+\sin(\cos^2\frac{\pi}{2})\right] = -1$

4. (**10** points) Find the volume of the solid generated by revolving the plane region enclosed by $y = 2x - x^2$ and $y = 0$ about the line $x = 4$.

Solution:

- Use the shell method.

$$V = \int_0^2 2\pi(4-x)(2x-x^2)\,dx = 2\pi\int_0^2 (8x-6x^2+x^3)\,dx = 2\pi\left[4x^2-2x^3+\frac{1}{4}x^4\right]\Bigg|_0^2 = 8\pi$$

5. (**10** points) Prove that the equation $3x + 1 - \sin x = 0$ has exactly one real solution.

Solution:

- $f(x) = 3x + 1 - \sin x$. $f(x)$ is differentiable for all x.
  $f(-\pi) = -3\pi + 1 < 0$ and $f(0) = 1 > 0$. Therefore, $f(x) = 0$ has a solution in $(-\pi, 0)$.
  Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$.
  By Rolle's Theorem, there exist some point $x = a$ such that $f'(a) = 0$
  But $f'(x) = 3 - \cos x > 0$ for all x. So $f(x)$ has exactly one real solution.

6. Let $f(x) = (x^3 + x^2)^{1/3}$
   (a) (**6** points) Find the intervals of increase or decrease.
   (b) (**6** points) Find the intervals of concavity.
   (c) (**2** points) Find the local maximum and minimum values.
   (d) (**1** points) Find the inflection points.

Solution:

(a) $f'(x) = \dfrac{3x+2}{3x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}} \Rightarrow f'(x) > 0$ for $x < -\dfrac{2}{3}$ or $x > 0$, and $f'(x) < 0$ when $-\dfrac{2}{3} < x < 0$.

Therefore, $f(x)$ is increasing on $(-\infty, -\dfrac{2}{3})$ and $(0, \infty)$, decreasing on $(-\dfrac{2}{3}, 0)$

(b) $f''(x) = -\dfrac{2}{9x^{\frac{4}{3}}(x+1)^{\frac{5}{3}}} \Rightarrow f''(x) > 0$ for $x < -1$, and $f''(x) < 0$ for $-1 < x < 0$ or $x > 0$.

Therefore, $f(x)$ is concave upward on $(-\infty, -1)$, concave downward on $(-1, 0)$ and $(0, \infty)$.

(c) Based on the result of (a), one can find that $f'(x)$ changes its sign at $x = -\dfrac{2}{3}$ and $x = 0$.

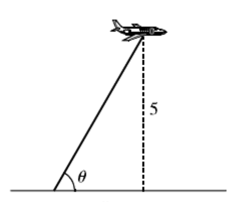
Therefore, $f(-\dfrac{2}{3}) = \dfrac{\sqrt[3]{4}}{3}$ is its local maximum. $f(0) = 0$ is its local minimum.

(d) Based on the result of (b), one can find that $f''(x)$ changes its sign only at $x = -1$.

$f(x)$ is continuous at $x = -1$, too. Therefore, $f(-1) = 0$ is its inflection point.

7. (**10** points) A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\dfrac{\pi}{3}$, this angle is decreasing at a rate of $\dfrac{\pi}{6}$ rad/min. How fast is the plane traveling at that time?

Solution:



- $\cot\theta = \dfrac{x}{5} \Rightarrow -\csc^2\theta \dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dx}{dt}$. When $\theta = \dfrac{\pi}{3}, \dfrac{d\theta}{dt} = -\dfrac{\pi}{6} \Rightarrow \dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10\pi}{9}$ (km/min)

8. A curve $C: \; x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ on xy-plane. There is a point $P(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$ on the curve C.
   (a) (**5** points) Find the lines that are tangent to the curve C at the point P.
   (b) (**5** points) Find the lines that are normal to the curve C at the point P.
   (c) (**10** points) Find the arc length of the curve C from the point $P(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$ to the point $Q(1, 0)$ on the curve C.

Solution:

(a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot y' = 0 \Rightarrow y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

When $x = \frac{\sqrt{2}}{4}, y = \frac{\sqrt{2}}{4}$, $\left.\frac{dy}{dx}\right|_{x=\sqrt{2}/4} = -1.$

The tangent line is $y - \frac{\sqrt{2}}{4} = (-1)\,(x - \frac{\sqrt{2}}{4}) \Rightarrow x + y = \frac{\sqrt{2}}{2}$

(b) The slope of the normal line at P is $\frac{-1}{\left.\frac{dy}{dx}\right|_{x=\sqrt{2}/4}} = \frac{-1}{-1} = 1$

The normal line is $y - \frac{\sqrt{2}}{4} = 1 \cdot (x - \frac{\sqrt{2}}{4}) \Rightarrow y = x$

(c) $y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \Rightarrow y' = -\frac{\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}}}{x^{\frac{1}{3}}} \Rightarrow L = \int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{1 + \left[\frac{(1 - x^{2/3})^{1/2}}{x^{1/3}}\right]^2}\, dx = \int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{1 + \frac{1 - x^{2/3}}{x^{2/3}}}\, dx$

$\Rightarrow L = \int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{x^{-2/3}}\, dx = \int_{\frac{\sqrt{2}}{4}}^{1} x^{-1/3}\, dx = \left[\frac{3}{2}x^{\frac{2}{3}}\right]\Big|_{\frac{\sqrt{2}}{4}}^{1} = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}.$

9. (**10** points) If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R+r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

Solution:

- $P(R) = \dfrac{E^2 R}{(R+r)^2} \Rightarrow P'(R) = \dfrac{dP}{dR} = \dfrac{E^2(R+r)^2 - E^2 \cdot 2R(R+r)}{(R+r)^4} = \dfrac{E^2(r-R)}{(R+r)^3}$

  $P'(R) = 0 \Rightarrow R = r$ is the extrema of $P(R)$.

  When $R < r$, $P'(R) > 0$ (increasing)

  When $R > r$, $P'(R) < 0$ (decreasing)

  Therefore, $P(R)$ have maximum value at $R = r$. The maximum value of the power is $P(r) = \dfrac{E^2}{4r}$.

10. (**10** points) A curve $x = \sqrt{r^2 - y^2}$, $0 \le y \le r/2$ is rotated about y-axis. Please find the area of the resulting surface.

Solution:

- $x = \sqrt{r^2 - y^2} \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{2}(r^2 - y^2)^{-\frac{1}{2}}(-2y) = -\dfrac{y}{\sqrt{r^2 - y^2}} \Rightarrow 1 + (\dfrac{dx}{dy})^2 = \dfrac{r^2}{r^2 - y^2}$

  Thus, the surface area of this sphere is

$$S = \int_0^{\frac{r}{2}} \left(2\pi x\sqrt{1 + (\frac{dx}{dy})^2}\right) dy = \int_0^{\frac{r}{2}} \left(2\pi\sqrt{r^2 - y^2}\sqrt{\frac{r^2}{r^2 - y^2}}\right) dy = 2\pi r \int_0^{\frac{r}{2}} dy = 2\pi r \cdot \frac{r}{2} = \pi r^2.$$