DF (First Midterm—Chapter 1)

Total points: 100 points 2hours to do the work, Oct. 17, 2024

1. Solve the initial value problem.(10 points)

$$y'x\ln x = y\ ; \qquad y(2) = \ln 64$$

$y' = \frac{y}{x\ln x} \quad \Rightarrow \frac{dy}{y} = \frac{dx}{x\ln x}$

兩邊一起積分：$\int \frac{dy}{y} = \int \frac{dx}{x\ln x} \Rightarrow \ln|y| + C_1 = \ln|\ln x| + C_2$

$\Rightarrow \ln|y| = \ln|\ln x| + C$ +3

指數化：$|y| = e^{\ln|\ln x| + C} = K|\ln x|$ ， 其中$K = e^C \Rightarrow y = K\ln x$ +5

帶入初始條件：y(2)=Kln2=ln64 ， $\ln 64 = \ln 4^3 = 3\ln 4$

Kln2=3ln4$\Rightarrow K = \frac{3\ln 4}{\ln 2} = \frac{3(2\ln 2)}{\ln 2} = 6$ +8

$\Rightarrow y = 6\ln x$ +10

2. Determine α so that the equation is exact. Obtain the general solution of the exact equation. (10 points)

$$5x^2 + 2xy^{\alpha} - 3x^2y^{\alpha-1}y' = 0$$

判斷是否為 exact equation：一個方程形式為$M(x,y) + N(x,y)y' = 0$，

則$M(x,y) = 5x^2 + 2xy^{\alpha}$ ， $N(x,y) = -3x^2y^{\alpha-1}$ +2

計算偏導數：$\frac{\partial M}{\partial y} = 2\alpha xy^{\alpha-1}$ ， $\frac{\partial N}{\partial x} = -6xy^{\alpha-1}$

使偏導數相等

$2\alpha xy^{\alpha-1} = -6xy^{\alpha-1}$ ， 如果$xy^{\alpha-1} \neq 0$ ，

則可消去$xy^{\alpha-1}$ $\Rightarrow \alpha = -3$ +5

代入 α 的值：$M(x,y) = 5x^2 + 2xy^{-3}$ ， $N(x,y) = -3x^2y^{-4}$

找到函數$\Psi(x,y)$：$\frac{\partial\Psi}{\partial x} = M$ ， $\frac{\partial\Psi}{\partial y} = N$

對於 M 進行積分：$\Psi(x,y) = \int 5x^2 + 2xy^{-3}dx = \frac{5}{3}x^3 + x^2y^{-3} + h(y)$

$\frac{\partial\Psi}{\partial x} = -3x^2y^{-4} + h'(y)$ ， $-3x^2y^{-4} + h'(y) = -3x^2y^{-4}$ +9

$h'(y) = 0 \Rightarrow h(y) = 0$

Ans：$\frac{5}{3}x^3 + x^2y^{-3} = C$ +10

3. Please show that the first order linear differential equation $(\frac{dy}{dx} + p(x)y = r(x))$ has a general solution, $y = e^{-\int p(x)dx}(\int e^{\int p(x)dx} r(x)dx + c)$ by the method of exact differential equation. (20 points)

$$y' + p(x)y = r(x)$$

$$\frac{dy}{dx} + p(x)y - r(x) = 0$$

$$[p(x)y - r(x)]dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x) \neq \frac{\partial N}{\partial x} = 0$$

$$\mu(x) = e^{\int \frac{1}{1}[p(x)-0]dx} = e^{\int p(x)dx}$$ →+10

$$\varphi(x, y) = \int \mu N dy = \int e^{\int p(x)dx} dy = ye^{\int p(x)dx} + k(x)$$

$$\frac{\partial \varphi}{\partial x} = p(x)ye^{\int p(x)dx} + k'(x) = \mu M = [p(x)y - r(x)]e^{\int p(x)dx}$$

$$k'(x) = -r(x)e^{\int p(x)dx}$$

$$k(x) = \int -r(x)e^{\int p(x)dx} dx + c_1$$

$$\varphi(x,y) = ye^{\int p(x)dx} - \int r(x)e^{\int p(x)dx} dx + c_1 = c_2$$

$$ye^{\int p(x)dx} = \int r(x)e^{\int p(x)dx} dx + c$$

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right)$$ →+20

4. Solve the initial value problem(20 points)

(a)

$\boldsymbol{y' + 5xy = 0}$**, y(0)=**$\boldsymbol{\pi}$

$\Rightarrow y' = -5xy$

$\Rightarrow \frac{1}{y}dy = -5xdx$

$\Rightarrow \log y = -2.5x^2 + k$ +3

$\Rightarrow \mathrm{y} = ce^{-2.5x^2}$ +5

Since y(0) = π, the above equation will give c=π.

Hence y = $\pi e^{-2.5x^2}$. +10

(b)

$y' = y - y^2$ r, y(0)=0.25

$$\Rightarrow \frac{\mathbf{1}}{\boldsymbol{y-y^2}} dy = dx$$

$\Rightarrow(\frac{1}{y}+\frac{1}{1-y})$dy=dx

$\Rightarrow \log y - \log(1-y)$=x+k

$\Rightarrow \frac{y}{1-y}$=$C_1 e^x$+3

$\Rightarrow$y=$\frac{1}{1+ce^{-x}}$+5

Since y(0)=0.25, we get from the above equation, c-3.

Hence y= $\frac{1}{1+3e^{-x}}$. +10

5. Consider $xy' - 5y = 3x^4$. (20 points)

(a) Show that the differential equation is not exact.

(b) Find an integrating factor.

(c) Find the general solution. (perhaps implicitly defined)

(a) $M_y = -5, M_x = 1$ so not exact. +5

(b) $\frac{1}{N}(M_y - M_x) = -\frac{6}{x}$, $u(x) = e^{-\int \frac{6}{x}} = \frac{1}{x^6}$ +5

$$x^{-5}y' - 5x^{-6}y = 3x^{-2}$$

$\because (uv)' = u'v + uv'$

$u = x^{-5},\ v = y$

$-5x^{-6}y + x^{-5}y' = (x^{-5}y)'$

$\therefore (x^{-5}y)' = 3x^{-2}$

$\int (x^{-5}y)'dx = \int 3x^{-2}dx$

$x^{-5}y = -3x^{-1} + C$

(c) $y = -3x^4 + x^5C$ +10

6. Consider $y-xy'=0$. (20 points)

(a) Find an integrating factor $\mu(x)$ that is a function of x alone. (10 points)

(b) Find an integrating factor $\nu(y)$ that is a function of y alone. (10 points)

(a) Since $\frac{1}{N}(M_y - N_x) = -\frac{2}{x}, \mu(x) = \frac{1}{x^2}$ +5,+5

(b) Since $\frac{1}{M}(N_x - M_y) = -\frac{2}{y}, \nu(y) = \frac{1}{y^2}$ +5,+5