104-02 Calculus Midterm Exam (Date: 2016/04/26 110 minutes, total: 120 points)

1. Evaluate

   a. $\lim_{x\to 0}\frac{x-\tan^{-1}x}{\sin^{-1}x-x}$ (8 points) **b.** $\lim_{n\to\infty}(\ln n)^{\frac{1}{n}}$ (8 points)

2. Evaluate

   a. $\int x\ln x\,dx$ (8 points) **b.** $\int \sin^3 x\cos^3 x\,dx$ (8 points)

   c. $\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}}dx$ (8 points) **d.** $\int \frac{dx}{x^4+x^2-2}$ (8 points)

3. The gamma function is defined as $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\,,\quad x>0$

   a. Show that $\Gamma(1)=1$ (6%) **b.** Show that $\Gamma(x+1)=x\Gamma(x)$ when $x>0$ (6%)

4. Determine convergence or divergence

   a. $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ (5 points) **b.** $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$ (5 points)

   c. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ (5 points) **d.** $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}$ (5 points)

   e. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ (5 points) **f.** $\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{\ln(n+1)}$ (5 points)

5. Determine the convergence interval of $\sum_{n=2}^{\infty}\frac{(x-3)^n}{(n+1)2^n}$ (10%)

6. Find the Taylor series for $f(x)=\frac{1-e^{-x}}{x}$ at $x=0$, $f(0)=1$.(Hint: find the Taylor series for e^x first) (10 points)

7. If $f(x)=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\cdots+\frac{2^n x^n}{n}+\cdots$, and $x\in\left(-\frac{1}{2},\frac{1}{2}\right)$, what is $f(x)$? (10 points)