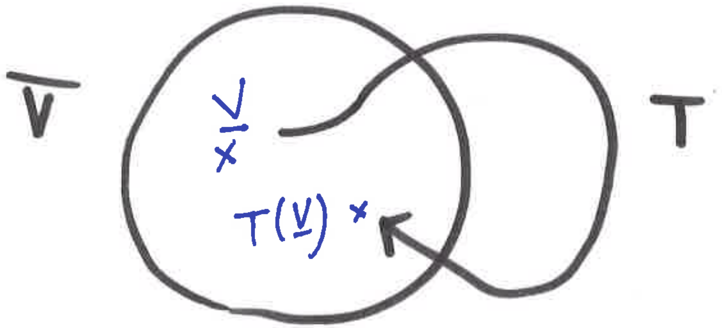
* Choosing a good basis for the matrix of linear operator

· Recall: $[T(\underline{v})]_B = [T]_{B(B)}[\underline{v}]_B$

$$[T(\underline{v})]_D = [T]_{D(D)}[\underline{v}]_D$$



$$[T]_{D(D)} = \overbrace{\underbrace{\underline{\underline{\Phi}}_{D,B}}_{\left[[\underline{b}_1]_D\,[\underline{b}_2]_D \cdots [\underline{b}_n]_D\right]}}^{\underline{\underline{\Phi}}_{B,D}^{-1}} [T]_{B(B)} \underbrace{\underline{\underline{\Phi}}_{B,D}}_{\left[[\underline{d}_1]_B\,[\underline{d}_2]_B \cdots [\underline{d}_n]_B\right]}$$

basis: $B = \{\underline{b}_1, \underline{b}_2, \cdots \underline{b}_n\}$

basis: $D = \{\underline{d}_1, \underline{d}_2, \cdots \underline{d}_n\}$

$\Longleftrightarrow$ diagonalization of matrix

(Recall: $\underline{\underline{D}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$)

· $\underline{\underline{\Phi}}_{D,B} = \underline{\underline{\Phi}}_{B,D}^{-1}$, $\underline{\underline{\Phi}}_{B,D} = \underline{\underline{\Phi}}_{D,B}^{-1}$, $\underline{\underline{\Phi}}_{B,D} \cdot \underline{\underline{\Phi}}_{D,B} = \underline{\underline{\Phi}}_{D,B}\underline{\underline{\Phi}}_{B,D} = \underline{\underline{I}}$

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- At first, we have some basis B, corresponding to which $[T]_{B(B)}$ is probably not in a simple form.
- It would be nice if we can find some basis D

(such that) $\ni$ $[T]_D$ is in some kind of simple form.

↳ diagonal

- We have learned that the trick is to find the eigen-values/vectors of $[T]_B$.
- We need enough l.i. eigenvectors from $[T]_B$ to obtain a diagonal $[T]_D$.

↓

consisting of the e.values of $[T]_B$

• Recheck: $[T]_D = \underbrace{\Phi_{DB}}_{} [T]_B \underbrace{\Phi_{BD}}_{}$ $\quad \Phi_{DB} = \Phi_{BD}^{-1}$

In addition to making $[T]_B$ look simple, we may even further seek to make Φ_{DB} / Φ_{BD} look simple or easy to handle.

• Q: What (more) exactly (speaking) is the so-called simple/easy/useful/helpful property of Φ_{DB}/Φ_{BD} ?

Ans: It would be very nice if the column vectors of Φ_{BD} are o.n. vectors. $\Rightarrow \underbrace{\Phi_{BD}^T \Phi_{BD} = I}$ $\to$ $\Phi_{BD}^{-1} = \Phi_{BD}^T$

o.n. $\hookrightarrow$ orthonormal

• We also know that $\begin{cases} \Phi_{DB}\Phi_{BD} = I \\ \Phi_{DB} = \Phi_{BD}^{-1} \end{cases}$ $\to \Phi_{BD}^{-1} = \Phi_{BD}^T$

- Recall: the def. of "orthogonal matrix":

(in the lecturer's opinion, orthonormal matrix would have been a better terminology)

$\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$ $\quad (\rightsquigarrow \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}})$

column vectors of $\underline{\underline{Q}}^T$ ($\equiv$ row vectors of $\underline{\underline{Q}}$) are o.n.

physical interpretation: column vectors of $\underline{\underline{Q}}$ are o.n.

- Summary: The columns/rows of an orthogonal matrix are o.n. vectors.

- Q: Can we find the afore-mentioned "simple" $\underline{\underline{\Phi}}_{BD}$ for a given $[T]_B$?

Ans: To be answered later.

- If orthogonal matrix $\underline{\underline{\Phi}}_{BD}$ does exist,

then $[T]_D = \underline{\underline{\Phi}}_{BD}^T [T]_B \underline{\underline{\Phi}}_{BD}$

orthogonal diagonalization of matrix

(Recall: $\underline{\underline{D}} = \underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}}$)

- Q: What is the advantage of a diagonal $[T]_D$?

Ans: The linear operation (performed by a Lop) can be performed component-wise (i.e. component by component), by simply scaling-up each component vector, which is an eigenvector, by the corresponding eigen-value.

· **Ex**. $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) \overset{\Delta}{=} \begin{bmatrix}x_1+x_2\\-2x_1+4x_2\end{bmatrix} \overset{?}{=} \underbrace{\begin{bmatrix}1&1\\-2&4\end{bmatrix}}_{\parallel}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$[T]_B$, where $B \overset{\Delta}{=} \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\underline{b}_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\underline{b}_2} \right\}$

$\Phi_{DB} \equiv \underline{\underline{P}}^{-1} = \begin{bmatrix}2&-1\\-1&1\end{bmatrix}$

· e.values e.vectors

$\lambda_1 = 2 \rightsquigarrow \underline{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2 = 3 \rightsquigarrow \underline{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$

$\Phi_{BD} \equiv \underline{\underline{P}} \overset{\Delta}{=} \begin{bmatrix}\underline{v}_1 & \underline{v}_2\end{bmatrix} = \begin{bmatrix}1&1\\1&2\end{bmatrix}$

$= [\underline{d}_2]_B$

$= [\underline{d}]_B$

· $D \overset{\Delta}{=} \left\{ 1\cdot\begin{bmatrix}1\\0\end{bmatrix} + 1\cdot\begin{bmatrix}0\\1\end{bmatrix},\ 1\cdot\begin{bmatrix}1\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\1\end{bmatrix} \right\} = \left\{ \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\underline{d}_1}, \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\underline{d}_2} \right\}$

· (Ex) Let us consider $\underline{v} = \begin{bmatrix}3\\-1\end{bmatrix}$, and

then check its image after the application of T.

$T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) = \begin{bmatrix}1&1\\-2&4\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}2\\-10\end{bmatrix}$ ——— (@1)

· If we want to make $[T]_{@}$ simple, then we adopt $[T]_D$

(at the cost of more effort in finding $[\underline{v}]_D$)

· Let us find $\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)_D$: $\Rightarrow =$

$$\begin{bmatrix}3\\-1\end{bmatrix} \overset{\heartsuit}{=} \alpha\cdot\begin{bmatrix}1\\1\end{bmatrix} + \beta\cdot\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}\alpha\\\beta\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1}}\begin{bmatrix}3\\-1\end{bmatrix} = \underbrace{\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}}\overbrace{\begin{bmatrix}3\\-1\end{bmatrix}}^{[\underline{v}]_B} = \begin{bmatrix}7\\-4\end{bmatrix}$$

This is $\underline{\underline{\Phi}}_{DB}$, which can be calculated in advance

· $$[T(\underline{v})]_D = \underbrace{[T]_D}_{=} \cdot [\underline{v}]_D = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}7\\-4\end{bmatrix} = \begin{bmatrix}14\\-12\end{bmatrix}$$

$$\overbrace{\underline{\underline{P}}^{-1}}^{\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}=}[T]_B\,\underline{\underline{P}} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-2 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$$

λ_2

λ_2

· $$T(\underline{v}) = 14\cdot \underline{d}_1 + (-12)\cdot \underline{d}_2 = 14\cdot\begin{bmatrix}1\\1\end{bmatrix} - 12\cdot\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\-10\end{bmatrix} \quad \text{— (@2)}$$

·Check: (@1) = (@2)

* Orthogonal diagonalizability of matrices

- **Thm** : Consider $\underline{\underline{A}} \in \underbrace{\mathbb{R}}_{\uparrow \text{ N.B.}}{}^{n \times n}$ (i.e. real square matrix).

  $\underline{\underline{A}}$ is orthogonally diagonalizable **iff** $\underline{\underline{A}}$ is symmetric.

  $\underline{\underline{D}} = \underline{\underline{Q}}^T \underline{\underline{A}}\, \underbrace{\underline{\underline{Q}}}_{\text{orthogonal matrix}} \rightarrow$ diagonal

  $\underline{\underline{A}}^T = \underline{\underline{A}}$

- **Prf** : Beyond the scope of this course.

- The consideration/discussions on $\mathbb{R}^{n\times n}$ matrices can be generalized to $\underbrace{\mathbb{C}}_{\text{complex numbers}}{}^{n \times n}$ matrices.

  $a + j \cdot b$

  $(i) = \sqrt{-1}$

$\underline{\underline{D}} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}}$: diagonalization

orthogonal diag.

(Scenario: $\underline{\underline{A}}$: real)

$\underline{\underline{D}} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}}$, where $\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{T}$

※ $\underline{\underline{A}}$ is orthogonally diagonalizable

iff $\underline{\underline{A}}$ is symmetric.

transpose → $\underline{\underline{A}}^{T} = \underline{\underline{A}}$ △

$\underline{\underline{P}}$ is orthogonal

unitary diag.

(Scenario: $\underline{\underline{A}}$: complex)

$\underline{\underline{D}} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}}$, where $\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{H}$

※ $\underline{\underline{A}}$ is unitarily diagonalizable

if ~~f~~ $\underline{\underline{A}}$ is Hermitian.

$\wr\wr$

transpose → conj. symmetric

$\underline{\underline{A}}^{*T}$

conj.

$\wr\wr$ $\underline{\underline{A}}^{H} = \underline{\underline{A}}$

combined $\circ$ → Hermitian (op.) △

$\underline{\underline{P}}$ is unitary

- $\underline{\underline{A}}$: square matrix

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$$
\begin{tikzcd}
\text{real and symmetric} \arrow[r, leftrightarrow] & \text{orthogonally diagonalizable} \\
\text{(complex and) Hermitian} \arrow[r, bend left, "\checkmark"] \arrow[d, shift right, "\checkmark"'] & \text{unitarily diagonalizable} \arrow[l, bend left, "\times"] \arrow[dl, leftrightarrow] \\
\text{(complex and) normal} \arrow[u, shift right, "\times"'] &
\end{tikzcd}
$$

- $\underline{\text{def}}$ normal matrix : $\underline{\underline{A}}^H \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^H$

- cf: unitary matrix : $\underline{\underline{A}}^H \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^H = \underline{\underline{I}}$

- **Thm** The e.values of a Hermitian matrix are real numbers.

Prf

- $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$ — (1)

- $\underline{x}^H \cdot Eq(1) \Rightarrow \underline{x}^H \underline{\underline{A}}\,\underline{x} = \underline{x}^H \cdot \lambda \underline{x} = \lambda\, \underline{x}^H \underline{x} = \lambda \|\underline{x}\|^2$ — (2)

- $(Eq.(2))^H = \underbrace{(\underline{x}^H \underline{\underline{A}}\,\underline{x})^H}_{\underline{x}^H \underline{\underline{A}}^H \underline{x}} = (\lambda \cdot \underbrace{\|\underline{x}\|^2}_{\text{real}})^H = \lambda^* \cdot \|\underline{x}\|^2$

$(\underline{\underline{P}}\,\underline{\underline{Q}})^H = \underline{\underline{Q}}^H \underline{\underline{P}}^H$

$\underline{x}^H \underbrace{\underline{\underline{A}}^H}_{\underline{\underline{A}}} \underline{x}$

$\underline{\underline{A}}$ is Hermitian $\rightarrow$ $\underline{\underline{A}}^H = \underline{\underline{A}}$

$\underline{x}^H \underline{\underline{A}}\,\underline{x} = \lambda^* \cdot \|\underline{x}\|^2$ (3)

- $\left.\begin{matrix}(2)\\(3)\end{matrix}\right\} \Rightarrow \lambda = \lambda^* \Rightarrow \lambda$ is real ※

- **Corollary** The e.values of a symmetric matrix with real entries are real numbers.

Prf real and symmetric $\swarrow$ implies Hermitian

· **Thm** Let $\underline{\underline{A}}$ be a Hemitian matrix. The e.vectors of $\underline{\underline{A}}$ wrt different e.values are orthogonal. (IP: $\langle \underline{a}, \underline{b} \rangle = \left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right\rangle \triangleq \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_n b_n \stackrel{\Delta}{=} [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \underline{a}^H \underline{b}$)

Prf: If $\underline{x}_1$ and $\underline{x}_2$ are eigenvectors wrt distinct eigenvalues λ_1 and λ_2, respectively, then

· $(\underline{\underline{A}}\,\underline{x}_1)^H \underline{x}_2 = \underline{x}_1^H \underbrace{\underline{\underline{A}}^H}_{= \underline{\underline{A}} \ \leftarrow \text{Hermitian}} \underline{x}_2 = \underline{x}_1^H \underbrace{\underline{\underline{A}}\,\underline{x}_2}_{= \lambda_2 \underline{x}_2} = \lambda_2 \underline{x}_1^H \underline{x}_2$ — (#1)

· $(\underline{\underline{A}}\,\underline{x}_1)^H \underline{x}_2 = (\underline{x}_2^H (\underline{\underline{A}}\,\underline{x}_1))^H = (\underline{x}_2^H \overbrace{\underline{\underline{A}}\,\underline{x}_1}^{= \lambda_1 \underline{x}_1})^H = (\lambda_1 \underline{x}_2^H \underline{x}_1)^H = \lambda_1 \underline{x}_1^H \underline{x}_2$ (#2)

· (#1) − (#2): $(\lambda_2 - \lambda_1)\, \underline{x}_1^H \underline{x}_2 = 0 \xRightarrow{\lambda_2 \neq \lambda_1} \underline{x}_1^H \underline{x}_2 = 0 \stackrel{\Delta}{\Rightarrow} \underline{x}_1 \perp \underline{x}_2$ ※

· **Ex** $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 4x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}}_{\shortparallel} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ → real and symmetric

$\underline{\underline{A}} \triangleq [T]_B$, where $B \triangleq \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{b}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\underline{b}_2} \right\}$

· eigen-analysis :

$\lambda_1 = \frac{5-\sqrt{13}}{2}$ $(\simeq .6972)$ $\xrightarrow{\text{e. vector}}$ $\begin{bmatrix} 1 \\ \frac{3-\sqrt{13}}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.3028 \end{bmatrix}$ ($\rightarrow$ norm $\simeq 1.0448$)

$\uparrow$ normalize

$\begin{bmatrix} .9571 \\ -.2898 \end{bmatrix} \triangleq \underline{d}_1$

"$\perp$"

$\lambda_2 = \frac{5+\sqrt{13}}{2}$ $(\simeq 4.3028)$ $\xrightarrow{\text{e. vector}}$ $\begin{bmatrix} 1 \\ \frac{3+\sqrt{13}}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 3.3028 \end{bmatrix}$ ($\rightarrow$ norm $\simeq 3.4508$)

$\downarrow$ normalize

$\begin{bmatrix} .2898 \\ .9571 \end{bmatrix} \triangleq \underline{d}_2$

· We choose

$\underline{\underline{Q}} = [\underline{d}_1 \ \underline{d}_2] = \begin{bmatrix} .9571 & .2898 \\ -.2898 & .9571 \end{bmatrix}$

(Check : $\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}} \rightsquigarrow \underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T$)

• (Ex) Let us consider $\underline{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and try to find $T(\underline{v})$.

• By adopting B as the o.b, $\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} \overset{?}{=} 3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$$\left(T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)\right)_B = [T]_B \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}_B = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{—— } (\star 1)$$

• If we adopt D as the o.b. $\left(D = \{\underline{d}_1, \underline{d}_2\} = \left\{ \begin{bmatrix} .9571 \\ -.2898 \end{bmatrix}, \begin{bmatrix} .2898 \\ .9571 \end{bmatrix} \right\} \right)$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \xrightarrow[\text{wrt. } D]{\text{coord. vector}} \underbrace{\underline{\underline{\Phi}}_{DB}}_{\swarrow} \cdot \overbrace{[\underline{v}]_B}^{= \begin{bmatrix} 3 \\ -1 \end{bmatrix}}$$

$$= \underline{\underline{\Phi}}_{BD}^{-1} = \underbrace{[\underline{d}_1 \ \underline{d}_2]^{-1}}_{} = \underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T = \begin{bmatrix} .9571 & -.2898 \\ .2898 & .9571 \end{bmatrix}$$

$$\overset{\text{ÌÌO}}{\underline{\underline{Q}}} = \begin{bmatrix} .9571 & .2898 \\ -.2898 & .9571 \end{bmatrix}$$

• $$\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)_D = \begin{bmatrix} .9571 & -.2898 \\ .2898 & .9571 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3.1611 \\ -.0877 \end{bmatrix}$$

- $\left[T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right]_D = [T]_D \cdot \left(\begin{bmatrix}3\\-1\end{bmatrix}\right)_D$

$$= \begin{bmatrix}\frac{5-\sqrt{13}}{2} & 0\\ 0 & \frac{5+\sqrt{13}}{2}\end{bmatrix}\begin{bmatrix}3.1611\\-.0877\end{bmatrix} = \begin{bmatrix}2.2040\\-.3775\end{bmatrix}$$

- $T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \overset{\doteqdot x_1}{\overbrace{2.2040}} \cdot \underline{d}_1 + \overset{\doteqdot x_2}{\overbrace{(-.3775)}} \cdot \underline{d}_2 \quad \left(= \begin{bmatrix}\underline{d}_1 & \underline{d}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$

$$= 2.2040 \cdot \begin{bmatrix}.9571\\-.2898\end{bmatrix} - .3775 \cdot \begin{bmatrix}.2898\\.9571\end{bmatrix}$$

$$= \begin{bmatrix}2\\-1\end{bmatrix} \text{ —— } (\bigstar 2)$$

- **Check:** $(\bigstar 1) = (\bigstar 2)$

```
clc;
clear all;
A = [2+i 0 0 ; 0 0 3-2*i ; 0 2+3*i 0]
[P,D] = eig(A)
P_inv = inv(P)
Pinv_A_P = P_inv*A*P
P_herm = P'
Pherm_A_P = P_herm*A*P
```

$$\underline{\underline{A}} = \begin{bmatrix} 2+i & 0 & 0 \\ 0 & 0 & 3-2i \\ 0 & 2+3i & 0 \end{bmatrix} \rightarrow \text{normal}$$

$$(\underline{\underline{A}}^H \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^H)$$

A =

2.0000 + 1.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	3.0000 - 2.0000i
0.0000 + 0.0000i	2.0000 + 3.0000i	0.0000 + 0.0000i

P =

0.0000 + 0.0000i	0.0000 + 0.0000i	1.0000 + 0.0000i
0.7071 + 0.0000i	0.7071 + 0.0000i	0.0000 + 0.0000i
0.5000 + 0.5000i	-0.5000 - 0.5000i	0.0000 + 0.0000i

D =

3.5355 + 0.7071i	0.0000 + 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	-3.5355 - 0.7071i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	2.0000 + 1.0000i

P_inv =

0.0000 + 0.0000i	0.7071 - 0.0000i	0.5000 - 0.5000i
0.0000 + 0.0000i	0.7071 + 0.0000i	-0.5000 + 0.5000i
1.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i

Pinv_A_P =

3.5355 + 0.7071i	0.0000 + 0.0000i	0.0000 + 0.0000i
-0.0000 + 0.0000i	-3.5355 - 0.7071i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	2.0000 + 1.0000i

P_herm =

0.0000 + 0.0000i	0.7071 + 0.0000i	0.5000 - 0.5000i
0.0000 + 0.0000i	0.7071 + 0.0000i	-0.5000 + 0.5000i
1.0000 + 0.0000i	0.0000 + 0.0000i	0.0000 + 0.0000i

Pherm_A_P =

3.5355 + 0.7071i	0.0000 - 0.0000i	0.0000 + 0.0000i
0.0000 + 0.0000i	-3.5355 - 0.7071i	0.0000 + 0.0000i
0.0000 + 0.0000i	0.0000 + 0.0000i	2.0000 + 1.0000i

$$\underline{\underline{A}}\,\underline{P}_1 = \lambda_1 \underline{P}_1$$

$$\underline{\underline{A}}\,\underline{P}_2 = \lambda_2 \underline{P}_2$$

$$\underline{\underline{A}}\,\underline{P}_3 = \lambda_3 \underline{P}_3$$

$$\underline{\underline{P}}^H = \underline{\underline{P}}^{-1}$$

$\Downarrow$

$\underline{\underline{P}}$ is unitary.