3.7 Change of Basis

◎ Recall: coordinate vector

◇ Given an ordered basis (o.b.) $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space V, any vector $\mathbf{v}$ in V can be uniquely written as a l.c. of the basis vectors:

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

◇ $[k_1, k_2, \ldots, k_n]^T$ is called the coordinate vector of $\mathbf{v}$ wrt B. It is denoted as $[\mathbf{v}]_B$.

◎ Change of basis:

◇ Problem: Given $[\mathbf{v}]_B$, how do we convert it to $[\mathbf{v}]_D$, where $D = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is another o.b. ?

- Answer: $[\mathbf{v}]_D = Q\,[\mathbf{v}]_B$, where
  $Q = [[\mathbf{v}_1]_D, [\mathbf{v}_2]_D, \ldots, [\mathbf{v}_n]_D]$
- Q is called the transition matrix (aka. Change-of-basis matrix) from B to D.
- $P=Q^{-1}$ is equal to the transition matrix from D to B. In other words,
  $P = [[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_n]_B]$, and
  $[\mathbf{v}]_B = P\,[\mathbf{v}]_D$

• Effect of change of basis on coordinate vectors:

• $B = \{\underline{v}_1, \underline{v}_2, \cdots \underline{v}_n\}$, $D = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n\}$

• Let us write the ~~original~~ old basis vectors as l.c.'s of the new basis vectors:

$$\underline{v}_1 = c_{11}\underline{u}_1 + c_{21}\underline{u}_2 + c_{31}\underline{u}_3 + \cdots + c_{n1}\underline{u}_n \rightsquigarrow [\underline{v}_1]_D = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}$$

$$\underline{v}_2 = c_{12}\underline{u}_1 + c_{22}\underline{u}_2 + c_{32}\underline{u}_3 + \cdots + c_{n2}\underline{u}_n \rightarrow [\underline{v}_2]_D = \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}$$

$$\vdots$$

$$\underline{v}_n = c_{1n}\underline{u}_1 + c_{2n}\underline{u}_2 + c_{3n}\underline{u}_3 + \cdots + c_{nn}\underline{u}_n \rightarrow [\underline{v}_n]_D = \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix}$$

• $\underline{v} = k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_n\underline{v}_n$

$$= k_1(c_{11}\underline{u}_1 + c_{21}\underline{u}_2 + \cdots + c_{n1}\underline{u}_n)$$
$$+ k_2(c_{12}\underline{u}_1 + c_{22}\underline{u}_2 + \cdots + c_{n2}\underline{u}_n)$$
$$+ \cdots$$
$$+ k_n(c_{1n}\underline{u}_1 + c_{2n}\underline{u}_2 + \cdots + c_{nn}\underline{u}_n)$$

$$= (c_{11}k_1 + c_{12}k_2 + \cdots + c_{1n}k_n)\underline{u}_1$$
$$+ (c_{21}k_1 + c_{22}k_2 + \cdots + c_{2n}k_n)\underline{u}_2$$
$$+ \cdots$$
$$+ (c_{n1}k_1 + c_{n2}k_2 + \cdots + c_{nn}k_n)\underline{u}_n$$

$$\therefore [\underline{v}]_D = \begin{bmatrix} c_{11}k_1 + c_{12}k_2 + \cdots + c_{1n}k_n \\ c_{21}k_1 + c_{22}k_2 + \cdots + c_{2n}k_n \\ \cdots \\ c_{n1}k_1 + c_{n2}k_2 + \cdots + c_{nn}k_n \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ & \cdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$\begin{matrix} \| & \| & & \| & \| \\ [\underline{v}_1]_D & [\underline{v}_2]_D & & [\underline{v}_n]_D & [\underline{v}]_B \end{matrix}$$