# * Effects of ero's on $\det(\underline{\underline{A}}) \overset{\circ}{=} \delta$

· **Type-I :**

$$\underline{\underline{B}} \overset{\circ}{=} \begin{array}{c} \\ \\ \vec{r}_i \\ \\ \end{array} \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{array}{c} \\ \\ * c \\ \\ \end{array} \quad \underline{\underline{A}} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

$$\leadsto \det(\underline{\underline{B}}) = c \cdot \det(\underline{\underline{A}}) = c \cdot \delta$$

**Prf** Simply perform the cofactor expansion along the $i$th row of $\underline{\underline{B}}$.

$$\boxed{\text{" swapping two rows} \xrightarrow{\text{(effect)}} \text{sign change"}} \quad (\#)$$

· **Type-II :**

mathematical induction

$$\underline{\underline{A}} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}_{n \times n} \overset{\circ}{=} \underline{\underline{B}} \quad \leadsto \det(\underline{\underline{B}}) = -\det(\underline{\underline{A}}) = -\delta$$

**Prf**

① For $2 \times 2$ matrices:

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - bc \overset{\circ}{=} \delta$$

$$\left|\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right| = cb - da = -(ad - bc) = -\delta$$

② Assume that (#) holds for $k \times k$ matrices.

③ For $(k+1) \times (k+1)$ matrix $\underline{\underline{F}}$ when two rows (say, $\vec{r}_{i_1}$ and $\vec{r}_{i_2}$) are swapped to obtain $\underline{\underline{G}}$, let us do the cofactor expansion along any row (say, row-p) 'other' than row-$i_1$ and row-$i_2$.

Then, compare the result to the cofactor expansion of $\underline{\underline{F}}$ along row-p.

Then, apply statement ②.

- **Ex**

$$7\cdot\begin{vmatrix}2&3\\5&6\end{vmatrix}-8\cdot\begin{vmatrix}4&3\\4&6\end{vmatrix}+9\cdot\begin{vmatrix}4&2\\4&5\end{vmatrix}\qquad 7\cdot\begin{vmatrix}5&6\\2&3\end{vmatrix}-8\cdot\begin{vmatrix}4&6\\4&3\end{vmatrix}+9\cdot\begin{vmatrix}4&5\\4&2\end{vmatrix}$$

$$= \begin{vmatrix}4&2&3\\4&5&6\\\boxed{7\ \ 8\ \ 9}\end{vmatrix} \rightsquigarrow \begin{vmatrix}4&5&6\\4&2&3\\\boxed{7\ \ 8\ \ 9}\end{vmatrix} =$$

- **Type-III**:

$$\underline{A}=\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix} \xrightarrow{c} \underline{B} \rightsquigarrow \det(\underline{B})=\det(\underline{A})=\delta$$

**Prf: Omitted.**

- **Ex**

$$\begin{vmatrix}4&2&3\\4&5&6\\7&8&9\end{vmatrix} =: \underline{A} \xrightarrow{-1} \rightsquigarrow \begin{vmatrix}4&2&3\\0&3&3\\7&8&9\end{vmatrix} = -9$$

$$\det(\underline{A}) = -9 =$$

# * Computing determinants by ero's

**Ex**

$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} \cancel{9} & \cancel{-95} & \cancel{51} \\ 99 & -20 & 76 \\ 60 & -25 & -44 \end{bmatrix}}_{\downarrow \det \;\; \delta} * \frac{1}{9} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & -95/9 & 17/3 \\ 99 & -20 & 76 \\ 60 & -25 & -44 \end{bmatrix}}_{\downarrow \det \;\; \frac{1}{9}\delta} \begin{matrix} -99 \\ -60 \end{matrix} \rightarrow \underbrace{\begin{bmatrix} 1 & -95/9 & 17/3 \\ \cancel{0} & \cancel{1025} & \cancel{-485} \\ 0 & 1825/3 & -384 \end{bmatrix}}_{\downarrow \det \;\; \frac{1}{9}\delta} * \frac{1}{1025}$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} 1 & -95/9 & 17/3 \\ 0 & 1 & -97/205 \\ 0 & 1825/3 & -384 \end{bmatrix}}_{\downarrow \det \;\; \frac{1}{9}\delta \cdot \frac{1}{1025} = \frac{1}{9 \cdot 1025}\delta} \frac{-1825}{3} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & -95/9 & 17/3 \\ 0 & 1 & -97/205 \\ \cancel{0} & \cancel{0} & \cancel{-11827/123} \end{bmatrix}}_{\downarrow \det \;\; \frac{1}{9 \cdot 1025}\delta} * \frac{-123}{11827}$$

$$\underline{\underline{R}} = \underbrace{\begin{bmatrix} 1 & -95/9 & 17/3 \\ 0 & 1 & -97/205 \\ 0 & 0 & 1 \end{bmatrix}}_{\downarrow \det}$$

$$\frac{1}{9 \cdot 1025}\delta \cdot \frac{-123}{11827} = \frac{-123}{9 \cdot 1025 \cdot 11827} \cdot \delta$$

It is obvious that $\det(\underline{R}) = 1$. Therefore, we have

$$\frac{-123}{9 \cdot 1025 \cdot 11827} \cdot \delta = 1$$

$$\Rightarrow \delta = \frac{-9 \cdot 1025 \cdot 11827}{123} \text{ ※}$$

• **Thm** ($) $\underline{\underline{E}}$ is e.m. $\rightarrow \det(\underline{\underline{E}}\cdot\underline{\underline{A}}) = \det(\underline{\underline{E}})\cdot\det(\underline{\underline{A}})$

**Prf** · Recall that $\underline{\underline{E}}\,\underline{\underline{A}} = \text{ero}(\underline{\underline{A}})$, where the ero is the one such that $\underline{\underline{I}} \xrightarrow{\text{ero}} \underline{\underline{E}}$.

· Recall the effects of ero's on the determinant of a matrix.

· $\det(\underline{\underline{E}})$ can be evaluated easily (for the three types, respectively), due to the simple structures of e.m's.

• **Thm** $\det(\underline{\underline{A}}\cdot\underline{\underline{B}}) = \det(\underline{\underline{A}})\cdot\det(\underline{\underline{B}})$

**Prf** Omitted. (However, it is based on Thm.($) and Thm.(%))

• **Thm** (%) $\underline{\underline{A}}$ is invertible $\longleftrightarrow \det(\underline{\underline{A}}) \neq 0$

**Prf** Recall that $\underline{\underline{A}}$ is invertible $\longleftrightarrow \text{rref}(\underline{\underline{A}}) = \underline{\underline{I}}$, and then also recall the technique of computing determinants by ero's. And also notice that $\det(\underline{\underline{I}}) = 1 \neq 0$.

# * Finding matrix inverse with determinants

- Recall that $(i,j)^{th}$ cofactor $(C_{ij})$ has been defined for $\underline{A} = [a_{ij}]_{n\times n}$
- **def** adjoint of $\underline{A}$: $adj(\underline{A}) \triangleq \left([C_{ij}]\right)^T$ ← transpose
- **Thm** If $\det(\underline{A}) \neq 0$, then $\underline{A}$ is invertible, and

$$\underline{A}^{-1} = \frac{1}{\det(\underline{A})} \cdot adj(\underline{A})$$

N.B. If $\det(\underline{A})=0$, then $\frac{1}{\det(\underline{A})}$ can NOT be performed!

- **Ex** $\underline{A} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \xrightarrow{\text{cofactors}}$ $C_{11}=12$, $C_{12}=6$, $C_{13}=-16$
$C_{21}=4$, $C_{22}=2$, $C_{23}=16$
$C_{31}=12$, $C_{32}=-10$, $C_{33}=16$

$\xrightarrow{\text{det.}} |\underline{A}| = 64$

$$\underline{A}^{-1} = \frac{1}{64} \cdot \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}^T = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} 12/64 & 4/64 & 12/64 \\ 6/64 & 2/64 & -10/64 \\ -16/64 & 16/64 & 16/64 \end{bmatrix}$$ ※.

# * Solving syst. of l. eqs with determinants (Cramer's rule)

- Consider $\underline{\underline{A}}\,\underline{x} = \underline{b}$

$$\underline{\underline{A}}\,\underline{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\det(\underline{\underline{A}}) \triangleq \Delta} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \triangleq \underline{b}$$

det.

- Let $\underline{\underline{A}}_j$ denote the matrix obtained by replacing the $j^{th}$ column of $\underline{\underline{A}}$ with $\underline{b}$.
- Let $\det(\underline{\underline{A}}_j) \triangleq \Delta_j$
- Then, $x_j = \dfrac{\Delta_j}{\Delta} \sim (\dot{\times})$

- **N.B.** If $\Delta = 0$, then $(\dot{\times})$ can not be performed

$$\Rightarrow \begin{cases} \text{no solution, if at least one } \Delta_j \text{ is not } 0 \\ \text{infinite solutions, if all } \Delta_j\text{'s are } 0 \end{cases}$$

- **Ex**

$$\begin{cases} -2x - 16y - 22z = 25 \\ 50x - 9y + 45z = 94 \\ 10x - 50y - 81z = 12 \end{cases} \sim \overbrace{\begin{bmatrix} -2 & -16 & -22 \\ 50 & -9 & 45 \\ 10 & -50 & -81 \end{bmatrix}}^{\underline{\underline{A}}} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\underline{x}} = \overbrace{\begin{bmatrix} 25 \\ 94 \\ 12 \end{bmatrix}}^{\underline{b}}$$

$$\Delta = \begin{vmatrix} -2 & -16 & -22 \\ 50 & -9 & 45 \\ 10 & -50 & -81 \end{vmatrix}$$

$$= -24938$$

$$\Delta_1 = \begin{vmatrix} 25 & -16 & -22 \\ 94 & -9 & 45 \\ 12 & -50 & -81 \end{vmatrix} = 45035$$

$$\Delta_2 = \begin{vmatrix} -2 & 25 & -22 \\ 50 & 94 & 45 \\ 10 & 12 & -81 \end{vmatrix} = 136288$$

$$\Delta_3 = \begin{vmatrix} -2 & -16 & 25 \\ 50 & -9 & 94 \\ 10 & -50 & 12 \end{vmatrix} = -74874$$

$$\Rightarrow x = \frac{45035}{-24938} = \frac{-45035}{24938}$$

$$y = \frac{136288}{-24938} = \frac{-68144}{12469}$$

$$z = \frac{-74874}{-24938} = \frac{37437}{12469}$$

Compare the results with P.013.

(We obtain the same answer with both methods.)