Note: Don't use the calculator. To get full points, you should write down the procedure **in detail**.

1. (30 points) Let $f(x,y) = x^2e^{-y}$. (6 points for each)
   (a) Find the gradient of f.
   (b) Find the directional derivative of f at the point $P(-2,0)$ in the direction toward the point $Q(2,-3)$.
   (c) Find the maximum increasing rate of change of f at the point $P(-2,0)$. Which is the direction of the maximum increasing rate of change?
   (d) Find the tangent plane of $z = f(x,y)$ at the point $(-2,0,4)$.
   (e) Let $z = f(x,y)$ and $x = u^2 - v^2$, $y = 2uv$. Find $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(0,\sqrt{2})}$.

Solution:

(a) $\nabla f = 2xe^{-y}\hat{\mathbf{i}} + (-x^2e^{-y})\hat{\mathbf{j}}$

(b) At $P(-2,0)$, $\nabla f|_P = -4\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$. Direction: $\hat{\mathbf{u}} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{4}{5}\hat{\mathbf{i}} - \dfrac{3}{5}\hat{\mathbf{j}}$. Therefore, the directional derivative is $\nabla f|_P \cdot \hat{\mathbf{u}} = -\dfrac{4}{5}$

(c) The maximum **increasing** rate of change of f at the point $P(-2,0)$ is $|\nabla f|_P| = 4\sqrt{2}$. The direction is $\hat{\mathbf{u}} = -\dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$

(d) Tangent plane: $z - 4 = f_x(-2,0)(x+2) + f_y(-2,0)(y-0) \Rightarrow z - 4 = -4(x+2) - 4y \Rightarrow 4x + 4y + z = -4$

(e) $u = 0, v = \sqrt{2} \Rightarrow x = -2, y = 0$. Chain rule: $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} = \dfrac{\partial z}{\partial x}\cdot(-2v) + \dfrac{\partial z}{\partial y}\cdot(2u)$.

$\left.\dfrac{\partial z}{\partial x}\right|_{(x,y)=(-2,0)} = -4$, $\left.\dfrac{\partial z}{\partial y}\right|_{(x,y)=(-2,0)} = -4$. Thus, $\left.\dfrac{\partial z}{\partial v}\right|_{(u,v)=(0,\sqrt{2})} = (-4)\cdot(-2\sqrt{2}) + (-4)\cdot 0 = 8\sqrt{2}$

2. (10 points) Find the radius of convergence and interval of convergence of the series.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\cdot 4^n}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

Solution:

(a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x+2)^n}\right| = \frac{|x+2|}{4}\cdot\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x+2|}{4} < 1 \Leftrightarrow |x+2| < 4$. So, $R = 4$.

If $x = -6$, then the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-4)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is converged by alternating series test.

If $x = 2$, then the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(4)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ which is diverged.

Thus the interval of convergence is $[-6,2)$.

(b) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right| = |x-2|\cdot\lim_{n\to\infty}\frac{2}{n+3} = 0 < 1.$

Thus the interval of convergence is $(-\infty,\infty)$, and its radius of convergence is $R = \infty$.

3. Please answer the following questions.
   (a) (3 points) Find the Maclaurin series expansion for $\ln(1+x)$ for $|x|<1$.
   (b) (5 points) Please utilize the result of (a) to find the Maclaurin series expansion of $f(x)=\ln(1+4x+3x^2)$. Write out the general terms.
   (c) (2 points) What is the radius of convergence for the result of (b).

Solution:

(a) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ for $|x|<1$.

(b) $\ln(1+4x+3x^2) = \ln\big((1+x)(1+3x)\big) = \ln(1+x)+\ln(1+3x)$, and $\ln(1+3x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x)^n}{n}$ for $|x|<\dfrac{1}{3}$

Thus the Maclaurin series of $f(x)$ is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n+1}{n}x^n$

(c) Because the radius of convergence for $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ is 1, and the radius of convergence for $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x)^n}{n}$ is $\dfrac{1}{3}$, the radius of convergence for $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n+1}{n}x^n$ is $\dfrac{1}{3}$.

4. (20 points) Evaluate the integrals: (10 points for each)
   (a) $\displaystyle\int_0^2\int_{x^2}^4 \frac{x^5}{\sqrt{x^6+y^3}}\,dy\,dx.$
   (b) $\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (2x+y)\,dx\,dy.$

Solution:

(a) Change the order of the iterated integral first! $\Rightarrow \displaystyle\int_0^2\int_{x^2}^4 \frac{x^5}{\sqrt{x^6+y^3}}\,dy\,dx = \int_0^4\int_0^{\sqrt{y}} \frac{x^5}{\sqrt{x^6+y^3}}\,dx\,dy.$

$\displaystyle\int \frac{x^5}{\sqrt{x^6+y^3}}\,dx = \int \frac{1}{6}u^{-\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{1}{2}} + C = \frac{1}{3}\sqrt{x^6+y^3}+C \quad (u=x^6+y^3,\quad du = 6x^5dx).$

Thus, $\displaystyle\int_0^4\int_0^{\sqrt{y}} \frac{x^5}{\sqrt{x^6+y^3}}\,dx\,dy = \int_0^4 \left(\frac{1}{3}\sqrt{x^6+y^3}\right)\Bigg|_{x=0}^{x=\sqrt{y}} = \int_0^4 \frac{1}{3}\left(\sqrt{2}-1\right)y^{\frac{3}{2}}\,dy = \frac{\sqrt{2}-1}{3}\cdot\left(\frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_0^4 = \frac{64}{15}\left(\sqrt{2}-1\right).$

(b) The region of the double integral is a semi-disk. One can change the integral to polar coordinate system.

$\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (2x+y)\,dx\,dy = \int_0^{\pi}\int_0^3 (2r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi}\left[(2\cos\theta+\sin\theta)\cdot\left[\frac{1}{3}r^3\right]_0^3\right]d\theta$

$\displaystyle\Rightarrow \int_0^{\pi}\left[(2\cos\theta+\sin\theta)\cdot\left[\frac{1}{3}r^3\right]_0^3\right]d\theta = \int_0^{\pi} 9\,(2\cos\theta+\sin\theta)\,d\theta = 9\,[2\sin\theta-\cos\theta]_0^{\pi} = 9\cdot 2 = 18.$

5. A parametric curve $x = 3t - t^3, y = 3t^2$.
   (a) (4 points) Show that the curve intersects itself at the point $(0, 9)$
   (b) (6 points) Find the length of the **loop** of the curve.

Solution:

(a) Assume the point is at $(x, y) = (3t - t^3, 3t^2) = (3s - s^3, 3s^2)$ where $s \neq t$. Because $3t^2 = 3s^2 \Longrightarrow s = -t$.
Thus, $3t - t^3 = 3s - s^3 = 3(-t) - (-t)^3 \Rightarrow 2t(t^2 - 3) = 0$. Because $t \neq s$, one can find that $t \neq 0$. Thus, $t = \pm\sqrt{3}$
When $t = \pm\sqrt{3}$, $(x, y) = (3t - t^3, 3t^2) = (0, 9)$. This curve intersects itself at $(0, 9)$.

(b) From (a), the curve intersects itself at $(0, 9)$ when $t = \pm\sqrt{3}$. From $t = -\sqrt{3}$ to $t = \sqrt{3}$, the curve is a loop.
The length of the loop is $L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\, dt = 12\sqrt{3}$.

6. (10 points) Find all the local maxima, local minima, and saddle point(s) of the function $f(x, y) = (x^2 + y^2)e^{-x}$.

Solution:

- $f_x(x, y) = (2x - x^2 - y^2)e^{-x}, f_y(x, y) = 2ye^{-x} \Rightarrow f_{xx}(x, y) = (x^2 + y^2 - 4x + 2)e^{-x}, f_{yy}(x, y) = 2e^{-x}, f_{xy}(x, y) = -2ye^{-x}$.
  $f_y = 0$ implies $y = 0$. Therefore, $f_x = 0$ gives $2x - x^2 = 0 \Rightarrow x = 0$ or $x = 2$. Critical point: $(0, 0)$ and $(2, 0)$.
  At $(0, 0)$, Hessian is $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} = 2 > 0$. Thus, $f(0, 0) = 0$ is a local minimum.
  At $(2, 0)$, Hessian is $f_{xx}f_{yy} - f_{xy}^2 = -4e^{-4} < 0$. Thus, $(2, 0)$ is a saddle point.

7. (10 points) Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

- Let $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Then $\nabla f = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\nabla g = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$.
  Use Lagrange multiplier method: to solve $\nabla f = \lambda \nabla g \Rightarrow 1 = 2x\lambda, 1 = 2y\lambda, -1 = 2z\lambda$ and $x^2 + y^2 + z^2 - 1 = 0$
  $\lambda = \frac{1}{2x} = \frac{1}{2y} = -\frac{1}{2z} \Rightarrow x = y = -z$. Substitute $x = y = -z$ into $x^2 + y^2 + z^2 - 1 = 0 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$
  When $x = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}, z = -\frac{1}{\sqrt{3}} \Rightarrow f(x, y, z) = f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \sqrt{3}$
  When $x = -\frac{1}{\sqrt{3}} \Rightarrow y = -\frac{1}{\sqrt{3}}, z = \frac{1}{\sqrt{3}} \Rightarrow f(x, y, z) = f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\sqrt{3}$
  Maximum value is $\sqrt{3}$ which is located at the point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$
  Minimum value is $-\sqrt{3}$ which is located at the point $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

8. Let C_1 be the curve $(x^2+y^2)^2 = 2a^2xy$ and C_2 be the curve $x^2+y^2 = \dfrac{a^2}{2}$ where $a > 0$
   (a) (6 points) Find polar equations for the curves C_1 and C_2.
   (b) (4 points) Find all points of intersection of C_1 and C_2.
   (c) (10 points) Find the area of the region that lies inside C_1 and C_2.

Solution:

(a) $x^2+y^2 = r^2, x = r\cos\theta, y = r\sin\theta \Rightarrow r^4 = 2a^2r^2\cos\theta\sin\theta \Rightarrow r^2 = a^2\sin 2\theta \Rightarrow C_1 : r^2 = a^2\sin 2\theta, C_2 : r = \dfrac{a}{\sqrt{2}}$.

(b) Use polar equations: $C_1 : r^2 = a^2\sin 2\theta, C_2 : r = \dfrac{a}{\sqrt{2}}$ to solve $\left(\dfrac{a}{\sqrt{2}}\right)^2 = a^2\sin 2\theta \Rightarrow \sin 2\theta = \dfrac{1}{2}$.

Therefore, $\theta = \dfrac{1}{12}\pi, \dfrac{5}{12}\pi, \dfrac{13}{12}\pi$, and $\dfrac{17}{12}\pi$.

The intersections are at the points: $(r,\theta) = \left(\dfrac{a}{\sqrt{2}}, \dfrac{1}{12}\pi\right), \left(\dfrac{a}{\sqrt{2}}, \dfrac{5}{12}\pi\right), \left(\dfrac{a}{\sqrt{2}}, \dfrac{13}{12}\pi\right), \left(\dfrac{a}{\sqrt{2}}, \dfrac{17}{12}\pi\right)$

(c) By symmetry, the area is $A = 2(A_1 + A_2 + A_3)$.

$$A_1 = \int_0^{\pi/12} \left(\frac{1}{2}a^2\sin 2\theta\right) d\theta = \frac{a^2}{2}\cdot\left(\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/12}\right) = \frac{a^2}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$$

$$A_2 = \int_{\pi/12}^{5\pi/12} \left(\frac{1}{2}\cdot\frac{a^2}{2}\right) d\theta = \frac{a^2}{4}\cdot\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) = \frac{a^2}{12}\pi$$

$$A_3 = \int_{5\pi/12}^{\pi/2} \left(\frac{1}{2}a^2\sin 2\theta\right) d\theta = \frac{a^2}{2}\cdot\left(\left[-\frac{1}{2}\cos 2\theta\right]_{5\pi/12}^{\pi/2}\right) = \frac{a^2}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$$

Therefore, the area is $A = 2(A_1 + A_2 + A_3) = a^2\left(1 - \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{6}\right)$.