Engineering mathematics II **Final exam., 6/22/2021**

This is an open-book test. The total score is 105 points. Please show your computations.

1. Consider some transformations that map vectors in $\mathcal{R}^2$ to vectors in $\mathcal{R}^2$, as listed below:

$$
\begin{aligned}
T_1(x,y) &= (x^2+y, 2x-y) \\
T_2(x,y) &= (3x+2y, 1) \\
T_3(x,y) &= (2x-3y, x+y) \\
T_4(x,y) &= (\cos(x+y), \sin(x-y)) \\
T_5(x,y) &= (2x+2y, 3x+3y) \\
T_6(x,y) &= (x+y, 0)
\end{aligned}
$$

In the mean time, please notice that $\mathcal{B} = \{(1,0),(0,1)\}$ and $\mathcal{D} = \{(1,1),(1,-1)\}$ are bases of $\mathcal{R}^2$.

(a). (5%) Some of the transformations are linear transformations (LT's). Please identify them.

(b). (5%) Among the LT's that had been found in the preceeding subproblem, one of them is invertible. Please identify it. Then, for the covenience of some description which is to appear later, let us refer to this LT as T.

(c). (5%) Find the matrix of T with respect to $\mathcal{B}$.

(d). (5%) Find the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{D}$.

(e). (5%) Find the matrix of T with respect to $\mathcal{D}$.

(f). (5%) Let the inverse transformation of T be denoted as S. Then, $S(1,5)=$?

2. Consider the matrix below:

$$
\underline{\underline{A}} = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}
$$

It is known that -1 is one of the eigenvalues of $\underline{\underline{A}}$.

(a). (6%) Find all of the eigenvalues of $\underline{\underline{A}}$ (including -1).

(b). (6%) With respect to all of the eigenvalue(s) that is not -1, find the corresponding eigenvector(s).

(c). (8%) Let us denote the the eigenspace of $\underline{\underline{A}}$ with respect to the eigenvalue -1 as $\mathcal{F}$. Show that the dimension of $\mathcal{F}$ is 2 .

(d). (5%) Is $\underline{\underline{A}}$ diagonalizable ?

3. Consider an inner-product space $\mathcal{P}_2$, where $\mathcal{P}_2$ is the vector space of polynomials of degree 2 or less, with the inner product being defined as

$$< f(x), g(x) >= \int_0^1 f(x)g(x)\,dx$$

Let us consider two functions, which are also regarded vectors: $\underline{u} = x^2 + 1$, and $\underline{v} = x - 1$.

(a). (5%) Show that $\underline{u}$ and $\underline{v}$ are not orthogonal to each other.

(b). (7%) Let the angle between $\underline{u}$ and $\underline{v}$ be denoted as θ. Then, $\cos(\theta) = ?$

(c). (8%) Find the projection of $\underline{u}$ onto $\underline{v}$.

4.(10%) In this problem, let us try to maximize the value of $J = (2x + 3y - 5z)^2$ under the constraint of $x^2 + y^2 + z^2 = 1$, by applying the Cauchy-Schwarz inequality theorem. The trick is to view J as the inner product of (x, y, z) and $(2, 3, 5)$. Then, the maximization of J ia achieved only if (x, y, z) and $(2, 3, 5)$ are aligned in the same direction (i.e. they are scalar multiples of each other). Please find the maximum value of J.

5.(10%) Consider the system of over-determined system of linear equations, wherein there are more equations than unknowns:

$$\begin{cases} 2x + y = 11 \\ 3x - 2y = 7 \\ x + y = 5 \\ 4x - 7y = -3 \end{cases}$$

Find the LSE (least square error) solution to this system of linear equations.

6. Consider the complex-valued matrices below:

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 2-i \\ 2-i & -1 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 3 & 2-i \\ 2+i & -1 \end{bmatrix}, \quad \underline{\underline{C}} = \begin{bmatrix} i & 7 \\ 7 & i \end{bmatrix}$$

wherein $i = \sqrt{-1}$.

(a). (5%) One of the three matrices is a Hermitian matrix. Please identify it. For the convenience of some description that is to appear later, let us also denote it as $\underline{\underline{Q}}$.

(b). (5%) Among the four values: $4 + 2i$, $2 - 3i$, $3i$, and 4, one of them is an eigenvalue of $\underline{\underline{Q}}$. Please identify it.