Total: 100 points

1. (**30** points) For what value of a and b is

$$f(x) = \begin{cases} ax + 2b & x \leq 0 \\ x^2 + 3a - b & 0 < x \leq 2 \\ 3x - 5 & 2 < x \end{cases}$$

continuous at every x.

Solution:
$\lim_{x \to 0^-} f(x) = a \cdot 0 + 2b = 2b$, and $\lim_{x \to 0^+} f(x) = 0^2 + 3a - b = 3a - b$.

For $f(x)$ to be continuous at $x = 0$, we must have $2b = 3a - b \Rightarrow 3a = 3b \Rightarrow a = b$.

$\lim_{x \to 2^-} f(x) = 2^2 + 3a - b = 3a - b + 4$, and $\lim_{x \to 2^+} f(x) = 3 \cdot 2 - 5 = 1$.

For $f(x)$ to be continuous at $x = 2$, we must have $3a - b + 4 = 1 \Rightarrow 3a - b = -3$.

After solving the equation: $a = b$ and $3a - b = -3$, one can find that $a = b = -\frac{3}{2}$.

2. (**30** points) Find the limit or show that it does not exist.

$$\lim_{x \to -\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$$

Solution: Note: When $x < 0$, $x^3 = -\sqrt{x^6}$.

$$\lim_{x \to -\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3} = \lim_{x \to -\infty} \frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2-x^3}{x^3}} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{1}{x^6} + 4}}{(\frac{2}{x^3} - 1)} = \frac{-\sqrt{\lim_{x \to -\infty} \frac{1}{x^6} + \lim_{x \to -\infty} 4}}{2 \lim_{x \to -\infty} \frac{1}{x^3} - \lim_{x \to -\infty} 1} = \frac{-\sqrt{0 + 4}}{2 \cdot 0 - 1} = 2$$

3. (**20** points) A function is expressed as $f(x) = \dfrac{\sqrt{x^2+4}}{x}$. Use various limits to find all of its asymptotes.

Solution:

$\lim_{x\to\infty} \dfrac{\sqrt{x^2+4}}{x} = 1$, $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+4}}{x} = -1 \Rightarrow$ Horizontal asymptote: $y = 1$ and $y = -1$.

$\lim_{x\to 0^+} \dfrac{\sqrt{x^2+4}}{x} = \infty$, $\lim_{x\to 0^-} \dfrac{\sqrt{x^2+4}}{x} = -\infty \Rightarrow$ Vertical asymptote: $x = 0$.

4. (**20** points) Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

$$\text{Equation : } \sin x = x^2 - x \quad \text{Interval : } (1, 2)$$

Note: $\sin 1 \approx 0.84$, $\sin 2 \approx 0.91$

Solution:

Let $f(x) = x^2 - x - \sin x$.

$f(1) = 1^2 - 1 - 0.84 = -0.84 < 0, f(2) = 2^2 - 2 - 0.91 = 1.09 > 0 \Rightarrow f(1) \cdot f(2) < 0.$

Therefore, there is a solution between 1 and 2.