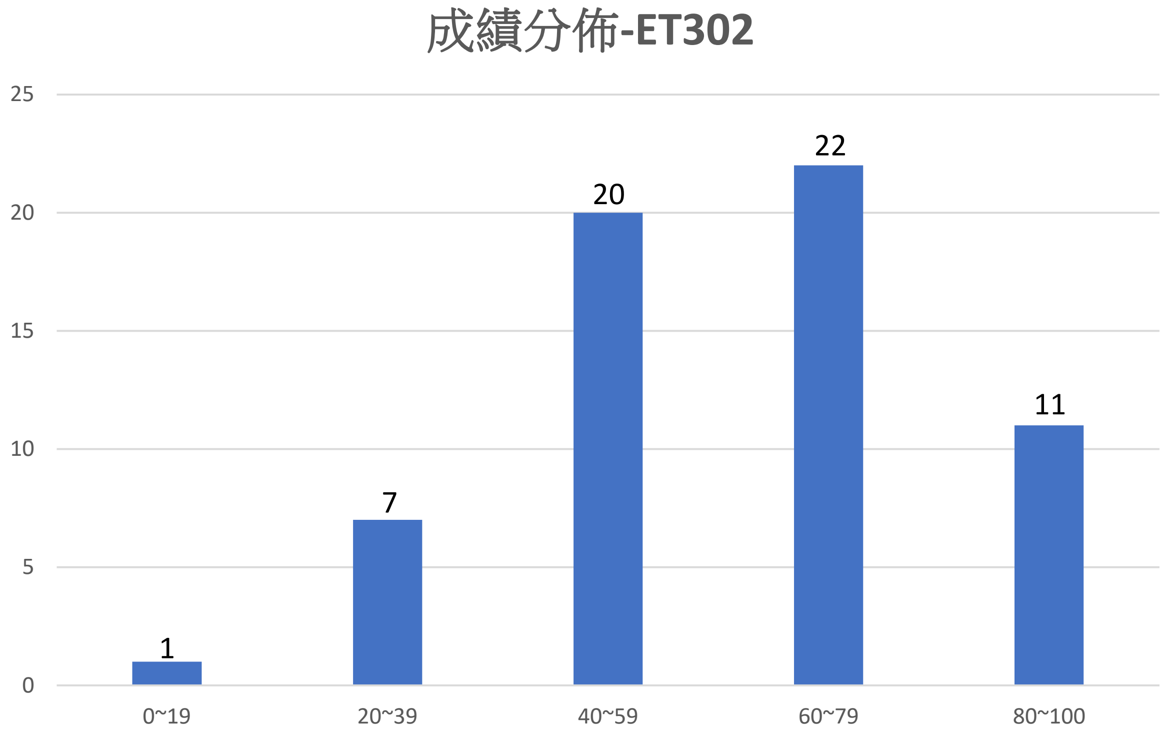
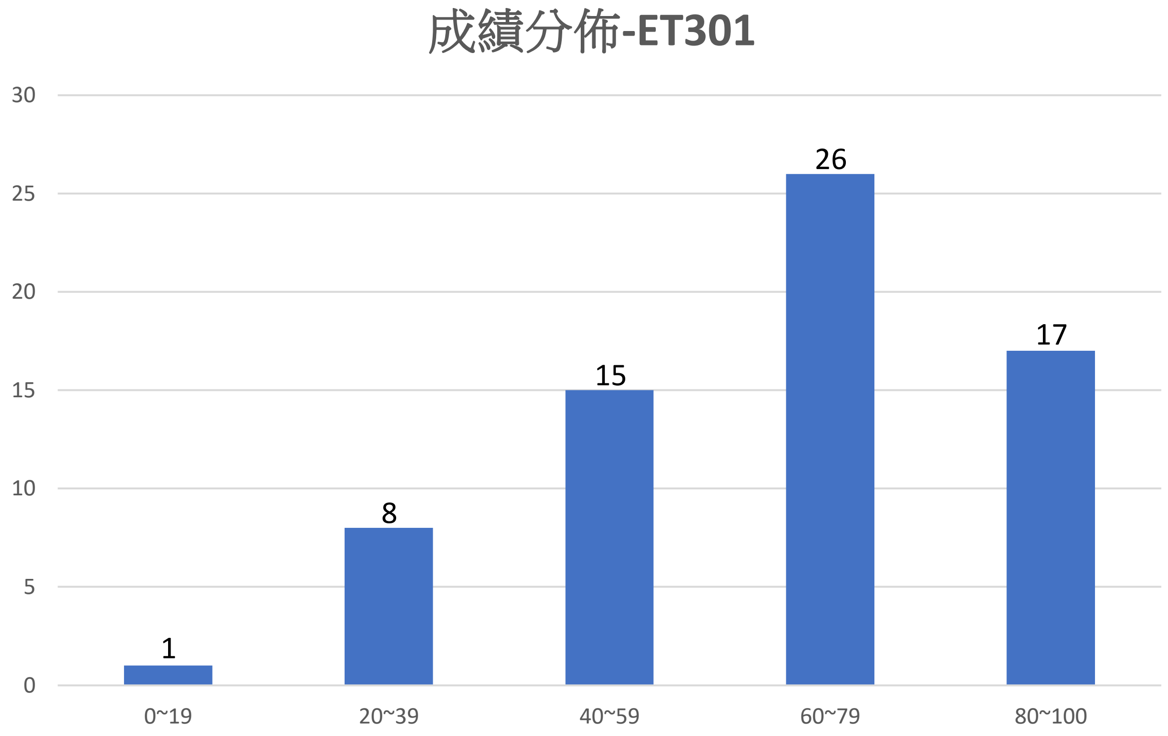
考卷皆有拍照做紀錄，拿到考卷後請勿更改，有問題需加分者在簡略說明題目後至前方找助教加分

題目與解答在Moodle有公佈





ET301 : Avg=69.22
ET302 : Avg=65.76

(10%) 1. Express the base vector $\hat{a}_R$, $\hat{a}_\Theta$ and $\hat{a}_\emptyset$ of a spherical coordinate system in terms of the cylindrical base vector $\hat{a}_r$, $\hat{a}_\emptyset$, $\hat{a}_z$ and coordinate r , Ø and z .

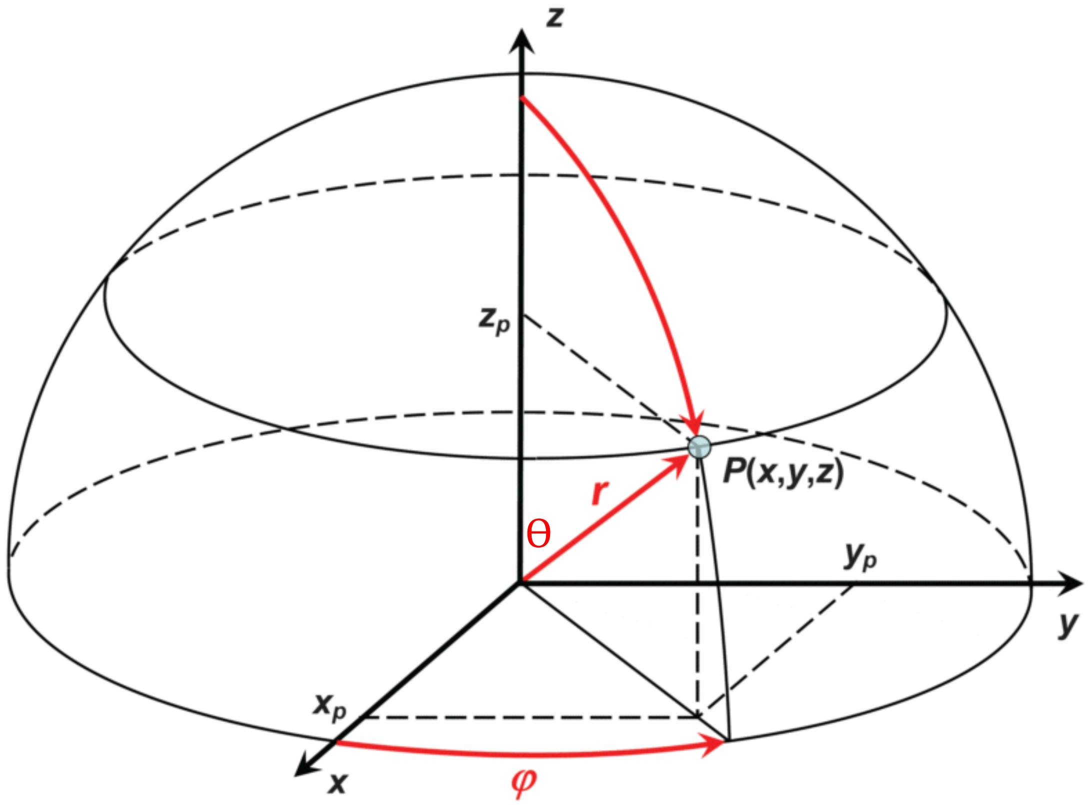
使用圓柱坐標$\hat{a}_r$, $\hat{a}_\emptyset$, $\hat{a}_z$與座標r , Ø , z 表達球座標$\hat{a}_R$, $\hat{a}_\Theta$, $\hat{a}_\emptyset$

使用圓柱轉球座標矩陣:

$$\begin{bmatrix} \hat{a}_R \\ \hat{a}_\Theta \\ \hat{a}_\emptyset \end{bmatrix} = \begin{bmatrix} sin\Theta & 0 & cos\Theta \\ cos\Theta & 0 & -sin\Theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_r \\ \hat{a}_\emptyset \\ \hat{a}_z \end{bmatrix}$$

故:

$$\hat{a}_R = \hat{a}_r\ sin\Theta + \hat{a}_z cos\Theta$$
$$\hat{a}_\Theta = \hat{a}_r\ cos\Theta - \hat{a}_z\ sin\Theta$$
$$\hat{a}_\emptyset = \hat{a}_\emptyset$$



(10%) 2. In Fig.1 , verify the divergence theorem by the vector field $\overrightarrow{F} = \hat{a}_R cos^2\emptyset/R^3$ existing in the region between two spherical shells defined by R=2 and R=3.

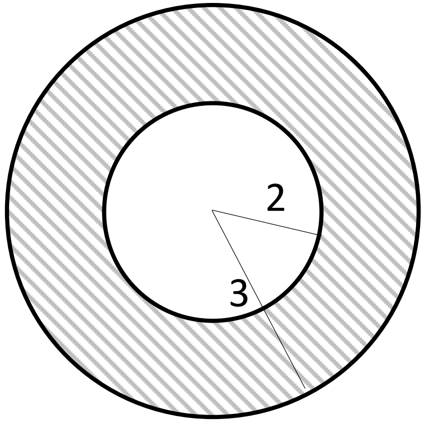


Fig.1

$$\nabla\cdot\overrightarrow{F} = \frac{\partial}{R^2\partial R}(R^2F_R) = \frac{-cos^2\emptyset}{R^4}$$

$$\int_v \nabla\cdot\overrightarrow{F}\,dv = \int\frac{-cos^2\emptyset}{R^4}\cdot R^2 sin\theta\, dR\, d\theta\, d\emptyset = -\int_2^3\frac{1}{R^2}dR\cdot\int_0^{\pi}sin\theta\, d\theta\cdot\int_0^{2\pi}cos^2\emptyset\, d\emptyset = -\frac{\pi}{3}$$

$$\oint_s\overrightarrow{F}\cdot d\overrightarrow{s} = \int 外 + \int 內 = \int\frac{cos^2\emptyset}{3^2}\hat{a}_R\cdot 3^2\, sin\theta\, d\theta\, d\emptyset\,\hat{a}_R + \int\frac{cos^2\emptyset}{2^3}\hat{a}_R\cdot 2^2\, sin\theta\, d\theta\, d\emptyset(-\hat{a}_R)$$
$$= \frac{1}{3}\int_0^{\pi}sin\theta\, d\theta\int_0^{2\pi}cos^2\emptyset\, d\emptyset - \frac{1}{2}\int_0^{\pi}sin\theta\, d\theta\int_0^{2\pi}cos^2\emptyset\, d\emptyset = \left(\frac{1}{3}-\frac{1}{2}\right)2\pi = -\frac{\pi}{3}$$

$\therefore \int_v \nabla\cdot\overrightarrow{F}\,dv = \oint_s\overrightarrow{F}\cdot d\overrightarrow{s}$, 得證

$$\nabla\cdot\mathbf{A} = \operatorname{div}\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

(15%) 3. Given a vector function E=$a_x y$+$a_y x$, evaluate the scalar line integral $\int E \cdot d\ell$ from P_1(2,1,-1) to P_2(8,2,-1)

a) alone the parabola x=2y^2,
b) alone the straight line joining the two points.
c) Evaluate $\int E \cdot d\ell$ from P_3(3,4,-1) to P_4(4,3,-1) by converting both E and the positions of P_3 and P_4 into cylindrical coordinates.

a) x=2$y^2, dx = 4ydy, \quad \int E \cdot d\ell = \int_{P_1}^{P_2} ydx + xdy = \int_{(2,1,-1)}^{(8,2,-1)} y(4y)dy + 2y^2dy = 14$

b) m $= \frac{2-1}{8-2} = \frac{1}{6}$ (斜率) , x=6y-4 , y=(x+4)/6 , $\int E \cdot d\ell = \int_2^8 \frac{x+4}{6} dx + \int_1^2 (6y-4)dy = 14$

c) $\begin{bmatrix} \hat{a}_r \\ \hat{a}_\emptyset \\ \hat{a}_z \end{bmatrix} = \begin{bmatrix} cos\emptyset & sin\emptyset & 0 \\ -sin\emptyset & cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} rsin\emptyset \\ rcos\emptyset \\ 0 \end{bmatrix}$, E= $a_r rsin2\emptyset + a_\emptyset rcos2\emptyset$, $E \cdot d\ell = \int rsin2\emptyset dr + r^2 cos2\emptyset d\emptyset$

P_3(3,4,-1) ⇒ P_3(5,$tan^{-1}\frac{4}{3}$, -1) , P_4(4,3,-1) ⇒ P_4(5, $tan^{-1}\frac{3}{4}$,-1) , r=5

$$\therefore \int_{tan^{-1}\frac{4}{3}}^{tan^{-1}\frac{3}{4}} r^2 cos2\emptyset d\emptyset = 25 \int_{tan^{-1}\frac{4}{3}}^{tan^{-1}\frac{3}{4}} cos2\emptyset d\emptyset = 25\left(\frac{1}{2} sin2\emptyset\right)\Bigg|_{tan^{-1}\frac{4}{3}}^{tan^{-1}\frac{3}{4}} = 25(sin\emptyset cos\emptyset)\Bigg|_{tan^{-1}\frac{4}{3}}^{tan^{-1}\frac{3}{4}}$$

$$= 25\left[\left(\frac{\frac{3}{4}}{\sqrt{(\frac{3}{4})^2+1}} \times \frac{1}{\sqrt{1+(\frac{3}{4})^2}}\right)-\left(\frac{\frac{4}{3}}{\sqrt{(\frac{4}{3})^2+1}} \times \frac{1}{\sqrt{1+(\frac{4}{3})^2}}\right)\right] = 0$$

$\sin 2x = 2\sin x cos x$

$\sin(tan^{-1}x) = \frac{x}{\sqrt{x^2+1}}$

$cos(tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

(16%) 4. Given three vectors A , B and C as follows,

A=a_x+$a_y 2$-$a_z 3$, B=$-a_y 4$+a_z , C=$a_x 5$-$a_z 2$

Find

a) a_A

b) |A-B|

c) A · B

d) Θ_{AB}

e) the component of A in the direction of C

f) A×C

g) A · (B×C) and (A×B) ·C

h) (A×B)×C and A× (B×C)

Sol:

a) $\bar{a}_A = \frac{\bar{A}}{A} = \frac{\bar{a}_x+\bar{a}_y 2-\bar{a}_z 3}{\sqrt{1^2+2^2+(-3)^2}} = \frac{1}{\sqrt{14}}\left(\bar{a}_x+\bar{a}_y 2-\bar{a}_z 3\right)$

b) $|\bar{A}-\bar{B}|$= $\left|\bar{a}_x+\bar{a}_y 6-\bar{a}_z 4\right|$=$\sqrt{1^2+6^2+(-4)^2}$= $\sqrt{53}$

c) $\bar{A}\cdot\bar{B} = 0+2(-4)+(-3) = -11$

d) Θ_{AB}= $cos^{-1}(\bar{A}\cdot\bar{B}/\mathrm{AB})$= $cos^{-1}(\frac{-11}{\sqrt{14}\sqrt{17}})$= 135.5°

e) $\bar{A}\cdot\bar{a}_c = \bar{A}\cdot\frac{\bar{C}}{C} = \bar{A}\cdot\frac{1}{\sqrt{29}}(\bar{a}_x 5-\bar{a}_z 2) = \frac{11}{\sqrt{29}}$

f) A×C= $-\bar{a}_x 4-\bar{a}_y 13-\bar{a}_z 10$

g) $\bar{A}\cdot(\bar{B}\times\bar{C})$= $(\bar{A}\times\bar{B})\cdot\bar{C}$= -42

h) $(\bar{A}\times\bar{B})\times\bar{C}$= $\bar{B}(\bar{A}\cdot\bar{C})$- $\bar{A}(\bar{C}\cdot\bar{B})$= $\bar{a}_x 2-\bar{a}_y 40+\bar{a}_z 5$

$\bar{A}\times(\bar{B}\times\bar{C})$= $\bar{B}(\bar{A}\cdot\bar{C})$- $\bar{C}(\bar{A}\cdot\bar{B})$= $\bar{a}_x 55-\bar{a}_y 44-\bar{a}_z 11$

(10%) 5. In Fig.3 , calculate the electric field *E* at the center of an equilateral triangle .

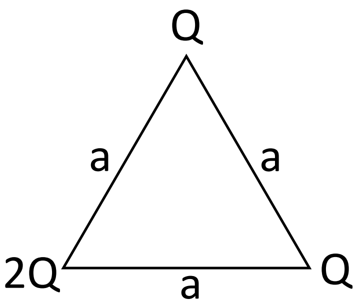
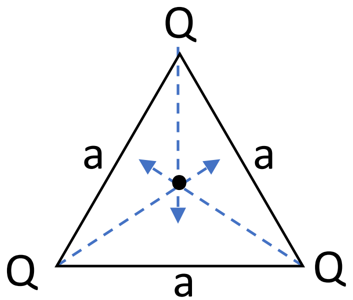


Fig.3

可簡化成:



合力為0

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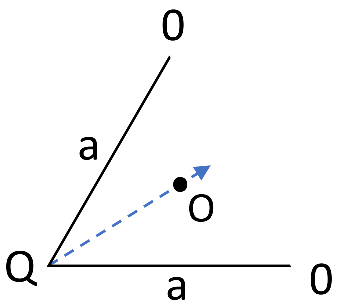
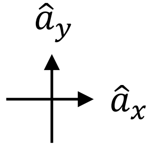
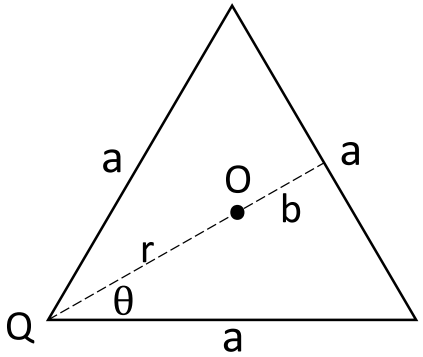




可自設座標軸

$$\therefore \vec{E} = 0 + \frac{1}{4\pi\varepsilon_0} \times \frac{Q}{\left(\frac{\sqrt{3}a}{3}\right)^2}\hat{a}_O = \frac{1}{4\pi\varepsilon_0} \times \frac{Q}{\left(\frac{\sqrt{3}a}{3}\right)^2} \times \left(\cos 30^\circ \hat{a}_x + sin\, 30^\circ\, \hat{a}_y\right) = \frac{Q}{4\pi\varepsilon_0\left(\frac{\sqrt{3}a}{3}\right)^2}\left(\frac{\sqrt{3}}{2}\hat{a}_x + \frac{1}{2}\hat{a}_y\right)$$



正三角形 : 每個內角60度 , θ=30度

$\overline{Qa'} = \frac{\sqrt{3}}{2}$ a , O為重心 , 故 r = $\frac{2}{3}\,\frac{\sqrt{3}}{2}$a $= \frac{\sqrt{3}a}{3}$

因題目關係，答案向量純量皆可

(15%) 6. In Fig.4 , verify Stokes's Theorem with vector function $\overrightarrow{F} = \hat{a}_{\emptyset} 3\sin(\frac{\emptyset}{2})$ for a hemispherical and the boundary of hemispherical with radius r=4 .

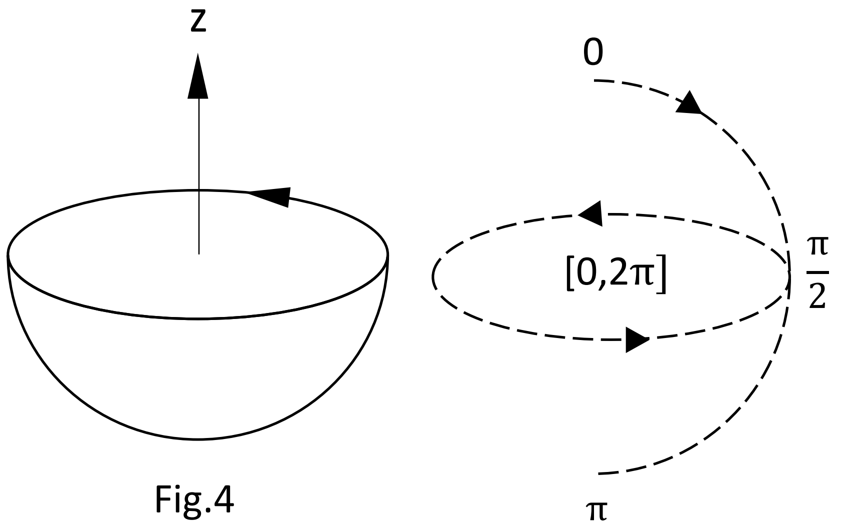


Fig.4

hint: $\nabla \times \overrightarrow{F} = \hat{a}_R \frac{3cos\Theta sin\frac{\emptyset}{2}}{Rsin\Theta}$ - $\hat{a}_\Theta \frac{3sin\frac{\emptyset}{2}}{R}$

$$\int_s \nabla \times \overrightarrow{F} \cdot d\overrightarrow{s} = \frac{3cos\Theta sin\frac{\emptyset}{2}}{4sin\Theta} \hat{a}_R$$

$$= \frac{3}{4} \int \frac{cos\Theta sin\frac{\emptyset}{2}}{sin\Theta} \hat{a}_R \cdot R^2 sin\theta d\Theta d\emptyset(-\hat{a}_R) = -12 \int_{\frac{\pi}{2}}^{\pi} cos\theta d\Theta \cdot \int_0^{2\pi} \sin\left(\frac{\pi}{2}\right) d\emptyset = -12[sin\theta]\Bigg|_{\frac{\pi}{2}}^{\pi} \cdot \left[-2\cos\left(\frac{\pi}{2}\right)\right]\Bigg|_0^{2\pi} = 48$$

$$\oint_c \overrightarrow{F} \cdot d\overrightarrow{l} = \int_0^{2\pi} \hat{a}_\emptyset 3\sin\left(\frac{\emptyset}{2}\right) \cdot 4d\emptyset \hat{a}_\emptyset = 12 \cdot \left[-2\cos\left(\frac{\emptyset}{2}\right)\right]\Bigg|_0^{2\pi} = 48$$

$\int_s \nabla \times \overrightarrow{F} \cdot d\overrightarrow{s} = \oint_c \overrightarrow{F} \cdot d\overrightarrow{l}$, 得證

(12%) 7. A uniform electron cloud (Fig.5) with density $\rho_{(r)} = \rho_0(1-\frac{r^2}{a^2})$, find the electric field E at :

a) r<a
b) r>a
c) Write the integral expression of charge Q.

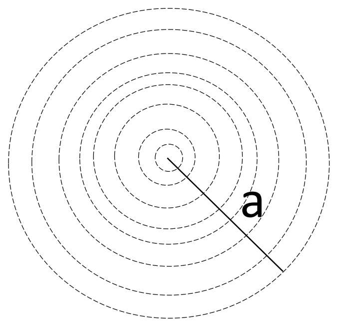


Fig.5

從題目可知他是一個均勻的電子雲，且電荷密度為$\rho_{(r)}$=...(常數且應為定值)，但$1-\frac{r^2}{a^2}$不是定值，故本題**送分**。題目應為不均勻電子雲。

另外r為一代數，表在某處的意思，與球體座標R無關聯與影響。

校正後題目: A non-uniform electron cloud (Fig.5) with density $\rho_{(r)} = \rho_0(1-\frac{r^2}{a^2})$, find the electric field E at :

a) r<a(球內)
b) r>a(球外)
c) Write the integral expression of charge Q.

a) r<a , $E_{in} \times 4\pi r^2 = \frac{\int_0^r \rho_0(1-\frac{r^2}{a^2})4\pi r^2}{\varepsilon_0} dr$, $E_{in} = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)$

b) r>a , $E_{out} \times 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^a \rho_0(1-\frac{r^2}{a^2})4\pi r^2 dr$, $E_{out} = \frac{2\rho_0 a^3}{15\varepsilon_0 r^2}$

c) $Q = \int \rho_{(r)} dv = \int_0^a \rho_0(1-\frac{r^2}{a^2})4\pi r^2 dr$

(12%) 8. Proof :

a) $\nabla \cdot (\nabla \times \overrightarrow{A}) = 0$

b) $\nabla \times (\nabla V) = 0$

$$a)\ \int_{v} \nabla \cdot \left(\nabla \times \overrightarrow{A}\right) dv = \oint_{s} \nabla \times \overrightarrow{A}\, d\overrightarrow{s} = \oint_{c} \overrightarrow{A} \cdot dl = 0$$

$$b)\ \int_{s} \nabla \times (\nabla V) \cdot d\overrightarrow{s} = \oint_{c} \nabla V \cdot dl = \oint_{v} dv = 0$$

直接用矩陣展開計算、垂直與平行方式證明，亦給分