DE (First Midterm—Chapter 1)

Total points: 100 points 2hours to do the work,

1. Solve the initial value problem.(10 points)

$y'xlnx = y$ ; $y(7) = ln343$

$$\frac{dy}{dx}x\ln x = y$$

$$\int \frac{dy}{y} = \int \frac{1}{x\ln x}dx$$ ←3

Let $u = \ln x$

$$du = \frac{1}{x}dx$$

$$\int \frac{dy}{y} = \int \frac{du}{u}$$

$$\ln|y| = \ln|u| + c_1$$ ←5

$$y = ce^{\ln u} = c\ln x$$

$$y(7) = ln343 = cln7$$

$$c = \frac{ln343}{ln7} = \frac{3ln7}{ln7} = 3$$

$$\therefore y = 3\ln x$$ ←10

2. Solve the initial value problem. (15 points)

$1 + e^{\frac{y}{x}} - \left(\frac{y}{x}\right)e^{\frac{y}{x}} + e^{\frac{y}{x}}y' = 0$ ; $y(1)$=-5

Let $\frac{y}{x} = u$ y=ux $y' = \frac{du}{dx}x + u$ ←2

$1 + e^u - ue^u + e^u(\frac{du}{dx}x + u) = 0$ ←3

$$1 + e^u + e^u\frac{du}{dx}x = 0$$

$$e^u\frac{du}{dx}x = -(1 + e^u)$$

$\frac{e^u}{1+e^u}du = -\frac{dx}{x}$ ←8

ln$(1 + e^u)$=-lnx+c

ln$(1 + e^u)$=ln$x^{-1}$+ln$e^c$ ←10

$1 + e^u = \frac{e^c}{x}$ let $e^c = A$

$$e^{\frac{y}{x}} = \frac{A}{x} - 1$$

$\frac{y}{x}$ $=ln\,ln\left(\frac{A}{x} - 1\right)$ $y = xln(\frac{A}{x} - 1)$ ←12

y(1)=-5=ln(A-1) A=$e^{-5} + 1$

$y = xln(\frac{e^{-5}+1}{x} - 1)$ ←15

3. Solve the initial value problem. (10 points)

$$xy' = y + x^2\,sec\,sec\left(\frac{y}{x}\right) ; \qquad y(1) = \pi$$

$y' = \frac{y}{x} + x\sec\frac{y}{x}$

Let $u = \frac{y}{x}$ ←2

$y = ux$

$y' = u'x + u$

$u'x + u = u + x\sec u$

$\frac{du}{dx} = \sec u$ ←5

$\int \cos u du = \int dx$

$\sin u = x + c$ ←7

$u = \sin^{-1}(x+c)$

$y = x\sin^{-1}(x+c)$

$\pi = \sin^{-1}(1+c)$

$1 + c = \sin\pi = 0$ ←9

$c = -1$

$\therefore y = x\sin^{-1}(x-1)$ ←10

4. Find an integrating factor, use it to find the general solute on of the differential equation, and then obtain the solution of the initial value problem. (20 points)

$(x-y) \ +cos\ cos\ (x-y) \ -cos\ cos\ (x-y)\ y' = 0$ ; $y(0) = \frac{7\pi}{6}$

$[\sin(x-y)+\cos(x-y)]dx + [-\cos(x-y)]dy = 0$ 若有明確嘗試解 u□2 分

$\frac{\partial M}{\partial y} = -\cos(x-y)+\sin(x-y) \neq \frac{\partial N}{\partial x} = \sin(x-y)$ 證明未正和 5 分

$\mu = e^{\int \frac{1}{-\cos(x-y)}[-\cos(x-y)+\sin(x-y)-\sin(x-y)]dx} = e^{\int dx} = e^x$ ←9

$\left\{e^x[\sin(x-y)+\cos(x-y)]dx\right\} + \left\{-e^x\left[\cos(x-y)\right]\right\}dy = 0$

$\frac{\partial M}{\partial N} = \frac{\partial N}{\partial x} = -e^x\cos(x-y) + e^x\sin(x-y)$

$\varphi(x,y)=\int -e^{x}[\cos(x-y)]dy+k(x)=e^{x}\sin(x-y)+k(x)$

$M=\frac{\partial u}{\partial x}=e^{x}\sin(x-y)+e^{x}\cos(x-y)+k^{'}(x)=e^{x}\sin(x-y)+e^{x}\cos(x-y)$

$k^{'}(x)=0\ \ k(x)=c_1$

$\therefore u=e^{x}\sin(x-y)=c_1$

G.S. : $e^{x}\sin(x-y)+c_1=c_2$ ←18

$\therefore e^{x}\sin(x-y)=c_3$

代入 $(0,\frac{7\pi}{6})$

$\sin\frac{-7\pi}{6}=c_3=\sin\frac{5\pi}{6}=\frac{1}{2}$

$\therefore e^{x}\sin(x-y)=\frac{1}{2}$

←20

5. Solve $x^2y'+xy=-y^{\frac{-3}{2}}$ (15 points)

原式同除$x^2$另成式 1 ←2

Let u=$y^{\frac{5}{2}}$ $u'=\frac{5}{2}y^{\frac{3}{2}}y'$ $y'=\frac{2}{5}y^{\frac{-3}{2}}u'$ 帶入式 1 ←3

$\frac{2}{5}y^{\frac{-3}{2}}u'+\frac{y}{x}=-y^{\frac{-3}{2}}x^{-2}$ 同除$y^{\frac{-3}{2}}$

$u'+\frac{5u}{2x}=-\frac{5}{2x^2}$第二式

$u=e^{\int\frac{5}{2x}dx}=x^{\frac{5}{2}}$乘回二式

$x^{\frac{5}{2}}u'+\frac{5}{2}x^{\frac{3}{2}}u=-\frac{5}{2}x^{\frac{1}{2}}$ ←10

$$(x^{\frac{5}{2}}u)'=-\frac{5}{2}x^{\frac{1}{2}}$$

$$x^{\frac{5}{2}}u=-\frac{5}{3}x^{\frac{3}{2}}+C$$

$$y^{\frac{5}{2}}=-\frac{5}{3}x^{-1}+Cx^{-\frac{5}{2}}$$

$y = (-\frac{5}{3}x^{-1} + Cx^{-\frac{5}{2}})^{\frac{2}{5}}$ ←15

6. Please solve the differential equation. (15 points)

$$y' = 6(y - 2.5)tanh(1.5x)$$

$\frac{dy}{dx} = 6(y - 2.5)\tanh(1.5x)$ ←2

$\int \frac{dy}{y - 2.5} = \int 6\tanh(1.5x)dx$ ←4

Let $u = \cosh(1.5x)$

$du = 1.5\sinh(1.5x)dx$

$\int \frac{dy}{y - 2.5} = 6\int \frac{1}{1.5}\frac{1}{u}du$ ←8

$\ln|(y - 2.5)| = 4\ln|u| = 4\ln|\cosh(1.5x)| + c_1$

$e^{\ln|(y-2.5)|} = e^{4\ln|\cosh(1.5x)|+c_1}$ ←12

$y - 2.5 = c\cosh^4(1.5x)$ ←15

7. Solve $\frac{dy}{dx} = (y+x)^3 - 1$ (15 points)

Let z=y+x   y=z-x  dy=dz-dx   ←3 嘗試用其他方法未解出也 3 分

代入原式 可得$\frac{dz}{dx} = z^3$

同乘$\frac{1}{z^3}dx$ 設 z≠0 $\frac{1}{z^3}dz = dx$   ←8 若 z≠0 沒寫-1 分

$$-\frac{1}{2}\frac{1}{z^2} = x + c$$

$$z^2 = -\frac{1}{2} * \frac{1}{x+c}$$

$$z = \pm\sqrt{\frac{-1}{2} * \frac{1}{x+c}}$$

y=z-x=-x$\pm\sqrt{\frac{-1}{2} * \frac{1}{x+c}}$ 為通解   ←13

當 z=0 y=-x 為奇異解   ←15