* ref and rref

• leading nonzero (of a row in a matrix)

$$\underline{\underline{A}} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

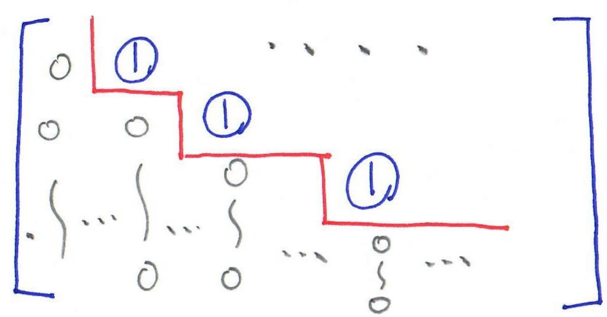
$$\left[\ \underbrace{0\ 0 \cdots 0,}_{\text{all zeros}}\ \overset{\neq 0}{a},\ \underbrace{b_1,\ b_2,\ \cdots}_{\text{can be of any values}}\ \right] \to \text{(a row)}$$

• leading-1 : If the leading nonzero is "1", it is called a leading-1.

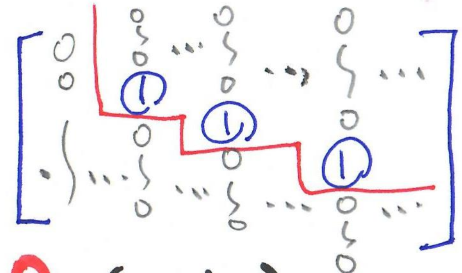
≈ staircase

• def: A matrix is in row echelon form (**ref**) iff

1. All-zero rows, if there is any, must lie at the bottom.
2. All leading nonzeros much be 1 (,and therefor are leading-1's).
3. Leading-1's can be placed on an echelon.



- Notice that all elements/entries **beneath a leading-1** are **0** (zero).
- <u>def</u>: A matrix is in reduced row echelon form (**rref**) iff



1\.
2\.
3\.
} the 3 conditions for ref

4\. All elements **above a leading-1** are **0** (zero).

- Exs:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
	$\begin{bmatrix}1&2&3&4\\0&0&1&2\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&3&0\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&2\\0&0&3\end{bmatrix}$	$\begin{bmatrix}1&4&6\\0&0&1\\0&1&3\end{bmatrix}$	$\begin{bmatrix}1&0&0&1&2\\0&1&0&2&4\\0&0&1&3&6\end{bmatrix}$
ref?:	✓	×	✓	✓	×	×	✓
rref?:	×	×	✓	✓	×	×	✓

- $\underline{\underline{A}} \xrightarrow{\text{ero's}} \underline{\underline{R}}$: ref — **Gaussian elimination/reduction**

$\underline{\underline{A}} \xrightarrow{\text{ero's}} \underline{\underline{R}}$: rref — **Gauss-Jordan elimination/reduction**

· **Ex** (reduction to ref)

$$\left[\begin{array}{ccc|c} 9 & -95 & 51 & 22 \\ 99 & -20 & 76 & 14 \\ 60 & -25 & -44 & 16 \end{array}\right] \xrightarrow{*\frac{1}{9}}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 99 & -20 & 76 & 14 \\ 60 & -25 & -44 & 16 \end{array}\right] \xrightarrow{-99}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 0 & 1025 & -485 & -228 \\ 60 & -25 & -44 & 16 \end{array}\right] \xrightarrow{-60}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 0 & 1025 & -485 & -228 \\ 0 & \frac{1825}{3} & -384 & \frac{-392}{3} \end{array}\right] \xrightarrow{*\frac{1}{1025}}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 0 & 1 & \frac{-97}{205} & \frac{-228}{1025} \\ 0 & \frac{1825}{3} & -384 & \frac{-392}{3} \end{array}\right] \xrightarrow{-\frac{1825}{3}}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 0 & 1 & \frac{-97}{205} & \frac{-228}{1025} \\ 0 & 0 & \frac{-11827}{123} & \frac{572}{123} \end{array}\right] \xrightarrow{*\frac{-123}{11827}}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-95}{9} & \frac{17}{3} & \frac{22}{9} \\ 0 & 1 & \frac{-97}{205} & \frac{-228}{1025} \\ 0 & 0 & 1 & \frac{-572}{11827} \end{array}\right] \quad ※$$

· If the matrix is an augmented matrix corresponding to a syst. l. eqs., then reduction to ref + back substitution is more computationally efficient than reduction to rref (and then immediately get the ~~answer~~ solution) (when the system is large)

· The ref of a matrix is **not unique**.

· **Ex** (reduction to rref)

$$\left[\begin{array}{ccc|c} -2 & -16 & -22 & 25 \\ 50 & -9 & 45 & 94 \\ 10 & -50 & -81 & 12 \end{array}\right] \xrightarrow{*\frac{-1}{2}} \left[\begin{array}{ccc|c} 1 & 8 & 11 & \frac{-25}{2} \\ 50 & -9 & 45 & 94 \\ 10 & -50 & -81 & 12 \end{array}\right] \xrightarrow{-50} \left[\begin{array}{ccc|c} 1 & 8 & 11 & \frac{-25}{2} \\ 0 & -409 & -505 & 719 \\ 10 & -50 & -81 & 12 \end{array}\right] \xrightarrow{-10}$$

$$\left[\begin{array}{ccc|c} 1 & 8 & 11 & \frac{-25}{2} \\ 0 & -409 & -505 & 719 \\ 0 & -130 & -191 & 137 \end{array}\right] \xrightarrow{*\frac{-1}{409}} \left[\begin{array}{ccc|c} 1 & 8 & 11 & \frac{-25}{2} \\ 0 & 1 & \frac{505}{409} & \frac{-719}{409} \\ 0 & -130 & -191 & 137 \end{array}\right] \xrightarrow{-8} \left[\begin{array}{ccc|c} 1 & 0 & \frac{459}{409} & \frac{1279}{818} \\ 0 & 1 & \frac{505}{409} & \frac{-719}{409} \\ 0 & -130 & -191 & 137 \end{array}\right] \xrightarrow{130}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{459}{409} & \frac{1279}{818} \\ 0 & 1 & \frac{505}{409} & \frac{-719}{409} \\ 0 & 0 & \frac{-12469}{409} & \frac{-37437}{409} \end{array}\right] \xrightarrow{*\frac{-409}{37437}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{459}{409} & \frac{1279}{818} \\ 0 & 1 & \frac{505}{409} & \frac{-719}{409} \\ 0 & 0 & 1 & \frac{37437}{12469} \end{array}\right] \xrightarrow{-\frac{459}{409}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-45035}{24938} \\ 0 & 1 & \frac{505}{409} & \frac{-719}{409} \\ 0 & 0 & 1 & \frac{37437}{12469} \end{array}\right] \xrightarrow{-\frac{505}{409}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-45035}{24938} \\ 0 & 1 & 0 & \frac{-68144}{12469} \\ 0 & 0 & 1 & \frac{37437}{12469} \end{array}\right]$$

· The rref of a matrix is **unique**.

· If $\underline{\underline{A}}$ corresponds to a syst. l. eqs (i.e. $\underline{\underline{A}}$ is an aug. matrix), then rref($\underline{\underline{A}}$) gives us the solution immediately !

- **Ex** (solving a syst. l. eqs. by rref)

$$\begin{cases} -x_1+x_2-x_3+3x_4=0 \\ 3x_1+x_2-x_3-x_4=0 \\ 2x_1-x_2-2x_3-x_4=0 \end{cases} \Rightarrow \text{aug. matrix}: \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

homogeneous eqs

rref

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

syst. l. eqs

$$\begin{cases} x_1+0x_2+0x_3-x_4=0 \\ 0x_1+x_2+0x_3+x_4=0 \\ 0x_1+0x_2+x_3-x_4=0 \end{cases}$$

$$\begin{cases} x_1=x_4 \\ x_2=-x_4 \\ x_3=x_4 \end{cases}$$

- x_1, x_2, x_3 are unknowns corresponding to (the positions) of leading-1's.
- x_4 is NOT in the position of a leading-1. It is a free parameter.
- solution: $(x_1, x_2, x_3, x_4) = (x_4, -x_4, x_4, x_4) = (t, -t, t, t)$ ※

  Let $x_4 = t$

- N.B. If a column is all-zero, then ero's keep that column all-zero. One typical example is the rightmost column of an augmented matrix associated with a system of linear homogenous eqs.

- **Exs (solving a syst. l. eqs)**

- $\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{cases} x_1 = 1 - x_2 - x_3 \\ x_4 = 2 \\ x_5 = -1 \end{cases}$

- $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -\frac{3}{5} x_3 + 1 \\ x_2 = -\frac{1}{5} x_3 \end{cases}$

- $\left[\begin{array}{ccc|c} 72 & -59 & -33 & 52 \\ 42 & 12 & -68 & -13 \\ 18 & -62 & -67 & 82 \\ 12 & 7 & -6 & -5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -125207/362190 \\ 0 & 1 & 0 & -14183/12073 \\ 0 & 0 & 1 & -13863/60365 \\ 0 & 0 & 0 & 361816/60365 \end{array}\right]$

  $\Rightarrow$ This is an inconsistent system, because the equation

  $$0x_1 + 0x_2 + 0x_3 = \frac{361816}{60365}$$

  can never be satisfied.

- $\begin{cases} x + y = 8 \\ 2x + 4y = 26 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 2 & 4 & 26 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{cases} x = 3 \\ y = 5 \end{cases}$ ※