# Point estimation (Chapter 9)

**Problem:** $X \sim f_\theta(x)$, where $\theta$ is an unknown parameter. This $\theta$ may be a vector.

Population: $X \sim f_\theta(x)$

**Data:** $X_1, \ldots, X_n$ — a random sample of $X$.

$x_1, x_2, \ldots, x_n$ pdf/pmf

We have seen a number of descriptive statistics and what they estimate. But the choice of an $\hat{\theta}$ of $\theta$ may not be obvious.

- Form of $f$ is known; but the parameter is unknown.

**Two general methods of parameter estimation:**

- Method of moments
- Method of maximum likelihood

Often (but not always) the two agree. The former is generally easier, but the latter tends to be better if $n$ is large.

# Method of moments

**$r$-th population moment:** $(r = 1, 2, 3, \ldots)$. $\mu_r = E[X^r]$

- First population moment: $\mu_1 = E[X] (= \mu)$ mean of population.
- Second population moment: $\boxed{\mu_2 = E[X^2]}$

Recall: $\sigma^2 = E[(X-\mu)^2] = E[X^2] - \mu^2$ $\Rightarrow$ $\boxed{\mu_2 = E[X^2] = \sigma^2 + \mu^2}$

**$r$-th sample moment:** $\boxed{M_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r}$

- First sample moment: $M_1 = \bar{X}$
- Second sample moment: $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Recall: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left[X_i^2 + \bar{X}^2 - 2X_i\bar{X}\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right] = \frac{nM_2}{n-1} - \frac{n}{n-1}\bar{X}^2$

$\Rightarrow \boxed{M_2 = \frac{\{(n-1)s^2 + n\bar{X}^2\}}{n}}$

<u>**Method of Moments:**</u> Suppose there are $d$ unknown parameters. Set up a system of $d$ equations by equating the first $d$ population moments to their sample counterparts, i.e.:

involves $\theta$ $\rightarrow \mu_r = M_r$ (involve sample data), $r = 1, 2, \ldots, d$. — (*)

Solve this system for the unknown parameters — the solution is $\hat{\theta}$, the method of moment estimators (MOME) of $\theta$.

Try to use numerical techniques to solve (*) if analytical soln. is not available.

**Ex:** Suppose $X_1, X_2, \ldots, X_n$ denote a random sample from a Bernoulli $(p)$ distribution, where $p$ is unknown. Find its MOME.

$p$ = proportion of successes (or 1's) in the population.

0's and 1's

Population: $X \sim \text{Bernoulli}(p)$.

unknown → $p = E[X]$

$X_1, X_2, \ldots, X_n$ (random sample).

0's and 1's.

<u>MOME:</u> $d = 1$.

$\mu_1 = E[X] = M_1 = \bar{X} = \hat{p}$

$\bar{X} = \frac{\sum_i X_i}{n}$ = prop. of 1's in the sample $= \hat{p}$

$\Rightarrow$ $\boxed{\hat{p}_{MOME} = \hat{p} = \text{prop. of 1's in the sample}}$

## Some properties of $\hat{p}$:

- $E[\hat{p}] = p$ for all $p$, $\Rightarrow$ $\hat{p}$ is unbiased
- $\text{var}[\hat{p}] = \frac{p(1-p)}{n}$
- $\hat{p}$ is consistent
- saw before that: If $n$ is large, $\hat{p} \stackrel{\cdot}{\sim} N\left[p, \frac{p(1-p)}{n}\right]$.

(CLT).

EX: $p$ = prop. of Texas residents who like snow.

1 = like snow
0 = don't like.

$n=10$. Data: 1, 1, 1, 0, 1, 0, 0, 1, 1, 1

$X_1$, $X_2$, $X_3$, …, $X_{10}$

$\hat{p} = \frac{7}{10} = 0.70 \Rightarrow$ Estimate that 70% of Tx residents like snow.

$$\widehat{SE}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{(0.7)(0.3)}{10}} = ?$$

$E(X)$ $\quad Var(X)$

**Ex:** Suppose $X_1, X_2, \ldots, X_n$ denote a random sample from a Normal $(\mu, \sigma^2)$ distribution, where both parameters are unknown. Find their MOME.

$d = 2$

$$\mu = \mu_1 = M_1 = \bar{X} \quad \text{— (1)}$$

$$\sigma^2 + \mu^2 = \mu_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \quad \text{— (2)}$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

from (1)

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

As before.

$$\left.\begin{aligned}\hat{\mu}_{MOME} &= \bar{X}\\ \hat{\sigma}^2_{MOME} &= \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{(n-1)S^2}{n}\end{aligned}\right\}$$

Properties:

(1) $\bar{X}$ is unbiased, consistent, $Var(\bar{X}) = \frac{\sigma^2}{n}$

(2) If $n$ is large, $\bar{X} \sim N\left[\mu, \frac{\sigma^2}{n}\right]$

(3) $E[\hat{\sigma}^2_{MOME}] = \frac{(n-1)}{n} E[S^2] = \frac{(n-1)}{n}\sigma^2 \Rightarrow \hat{\sigma}^2_{MOME}$ is biased. $\Rightarrow$ underestimate the variance.

(4) Can see that $\hat{\sigma}^2_{MOME}$ is consistent and asymptotically normal. ~~Will~~ Can derive SE of $\hat{\sigma}^2_{MOME}$ explicitly, but will see an alternative simulation-based method later.