• $100(1-\alpha)\%$ CI for θ: confidence interval (CI) $[L, U]$ such that $\underbrace{P[L \le \theta \le U]}_{\text{coverage prob.}} = 1-\alpha$ for all θ. — ⓧ

• A general method for constructing CI:

- Find a pivot — a fn. of $\hat{\theta}$ and θ — whose distribution is completely known.

- Use percentiles of this distribution to get critical points such that ⓧ holds.

- Get the CI by rearranging the terms in the statement so that the prob. in ⓧ is interpreted as a coverage probability.

EX: suppose θ is a parameter, and $\hat{\theta}$ is an unbiased estimator with known variance, i.e., $\hat{\theta} \sim N[\theta, var(\hat{\theta})]$.

Here: Pivot: $Z = \dfrac{\hat{\theta}-\theta}{SE(\hat{\theta})} \sim N(0,1)$. Z

critical points: $z_{\alpha/2}$ and

$P\left[-z_{\alpha/2} \le \dfrac{\hat{\theta}-\theta}{SE(\hat{\theta})} \le z_{\alpha/2}\right] = 1-\alpha$

$= P[L \le \theta \le U]$, $[L, U] = \hat{\theta} \pm z_{\alpha/2}\, SE(\hat{\theta})$

- This interval is approximate if $\hat{\theta}$ is approximately normal.

Special cases: (1) $X \sim N[\mu, \sigma^2]$ ← σ^2 known.

- $\hat{\mu} = \bar{X} \sim N[\mu, \frac{\sigma^2}{n}]$
- CI for μ: $\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

(2) $X \sim$ Any distribution with known variance σ^2, but n is large.

- $\hat{\mu} = \bar{X} \overset{\circ}{\sim} N[\mu, \frac{\sigma^2}{n}]$
- Approx CI for μ: $\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

(3) What if σ^2 is unknown, but n is large?

Estimate σ^2 by $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

↑ sample variance

Since n is large, $s^2 \approx \sigma^2$. The previous interval still works, i.e.,

Approx CI for μ: $\bar{X} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$

Q. How much money I have in my pocket?

<u>Guess 1:</u> $[0, 60]. — less accurate but more precise.

<u>Guess 2:</u> $ [0, 1m] — more accurate but less precise

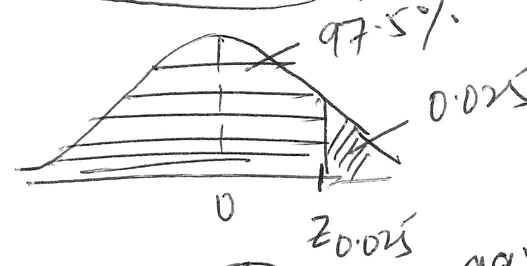
Confidence interval for a normal mean (known variance, cont'd)

$X \sim N[\mu, \sigma^2]$. known.

Q: Given a random sample, which CI for μ would you prefer — a 95% CI or a 99% CI? (Note: `qnorm(0.975) = 1.959964`, `qnorm(0.995) = 2.575829`.)

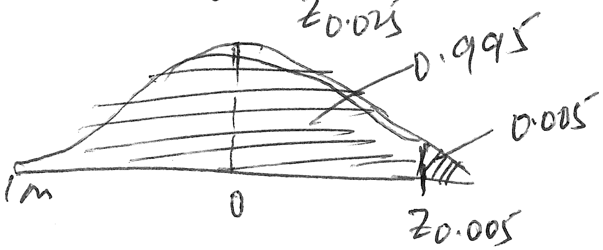
Recall: $\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

95% conf.:



Larger confidence $\Rightarrow$ greater CI $\Rightarrow$ wider CI

99% conf.



Note the tradeoff: b/w accuracy and precision

- The **precision** of a CI is given by its **width**. The **accuracy** of a CI is given its **confidence level**.
- Higher confidence = lower precision (wider).
- The width of a 100% CI is: It is a useless interval — extremely "accurate" but extremely imprecise!

100% CI for μ: $(-\infty, \infty)$ because $z_0 = \infty$.

Q: What can we do to get a narrower CI without lowering the confidence?

- Width = $U - L = 2 z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$
- Increase n to make CI more precise.

Typically we have no control over σ

population SD.

Choosing the sample size n:

$\boxed{\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}}$

margin of error.

- Let w = desired CI width for $1 - \alpha$ confidence.
- Margin of error = $w/2$ → desired.
- Set the CI width to the desired width and solve for n to get

$$2 z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = w$$

$$\Rightarrow \sqrt{n} = \frac{2 z_{\alpha/2} \sigma}{w}.$$

$$\Rightarrow n = \left[\frac{2 z_{\alpha/2} \sigma}{w} \right]^2.$$

→ Round up n if needed.

Ex: Suppose that we wish to estimate the mean CPU service time of a job and we wish to assert with 99% confidence that the estimated value is within less than 0.5 sec of the true value. Suppose that the past experience suggests that CPU service time is normally distributed with standard deviation $\sigma = 1.5$ sec. How many observations should we take?

$1-\alpha$ μ

Here:

$1-\alpha = 0.99 \Rightarrow z_{\alpha/2} = 2.576$

$w = 2(0.5) = 1$ sec.

$\sigma = 1.5$.

μ should be here.

$\bar{x}-0.5 \quad \bar{x} \quad \bar{x}+0.5$

$$n = \left[\frac{2(2.576)(1.5)}{1}\right]^2 = 56.25$$

$$\Rightarrow n = \underline{\underline{57.}}$$

In practice: When σ is unknown, do a pilot experiment to estimate it or get a value from literature or make an intelligent guess.

Note: We ~~know~~ know what to do when n is large.

Confidence interval for a normal mean (unknown variance)

$\rightarrow$ when n is not large.

$\widehat{SE}(\bar{X}) = \frac{\hat{\sigma}}{\sqrt{n}}$

- Unknown variance σ^2 is more realistic.
- Estimate σ^2 by sample variance, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

Take an advanced course to deal with small n for non-normal population.

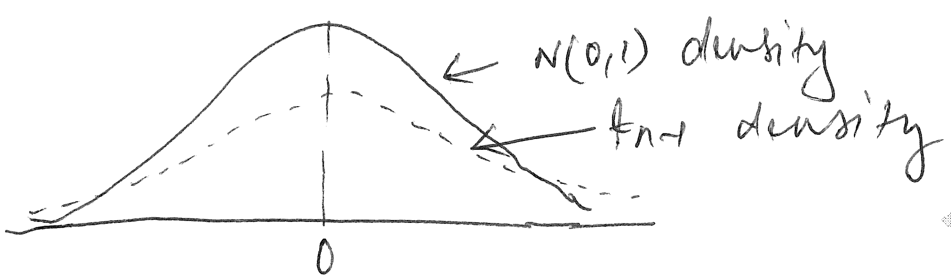
Pivot:

$$T = \frac{\bar{X}-\mu}{\widehat{SE}(\bar{X})} = \frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\sqrt{n}(\bar{X}-\mu)}{S}.$$

Recall: $Z = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$

Result: $T \sim t_{n-1}$, i.e., a t-distribution $(n-1)$ degrees of freedom, instead of the $N(0,1)$ distribution.

- A t_{n-1}-distribution looks like a $N(0,1)$ but it has heavier tails. A heavier tail accounts for the fact that there is there is more uncertainty in T when S is used in place of σ
- When n is large, a t_{n-1}-distribution $\approx N(0,1)$.

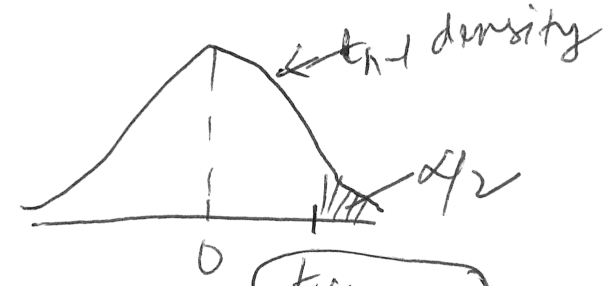


$100(1-\alpha)\%$

Result: CI for μ: $\overline{X} \pm t_{\alpha/2,n-1} S/\sqrt{n}$

$\widehat{se}(\overline{X})$

Proof:



$t_{\alpha/2,n-1}$
$=$
$(1-\alpha/2)$th quantile of t_{n-1} dist.