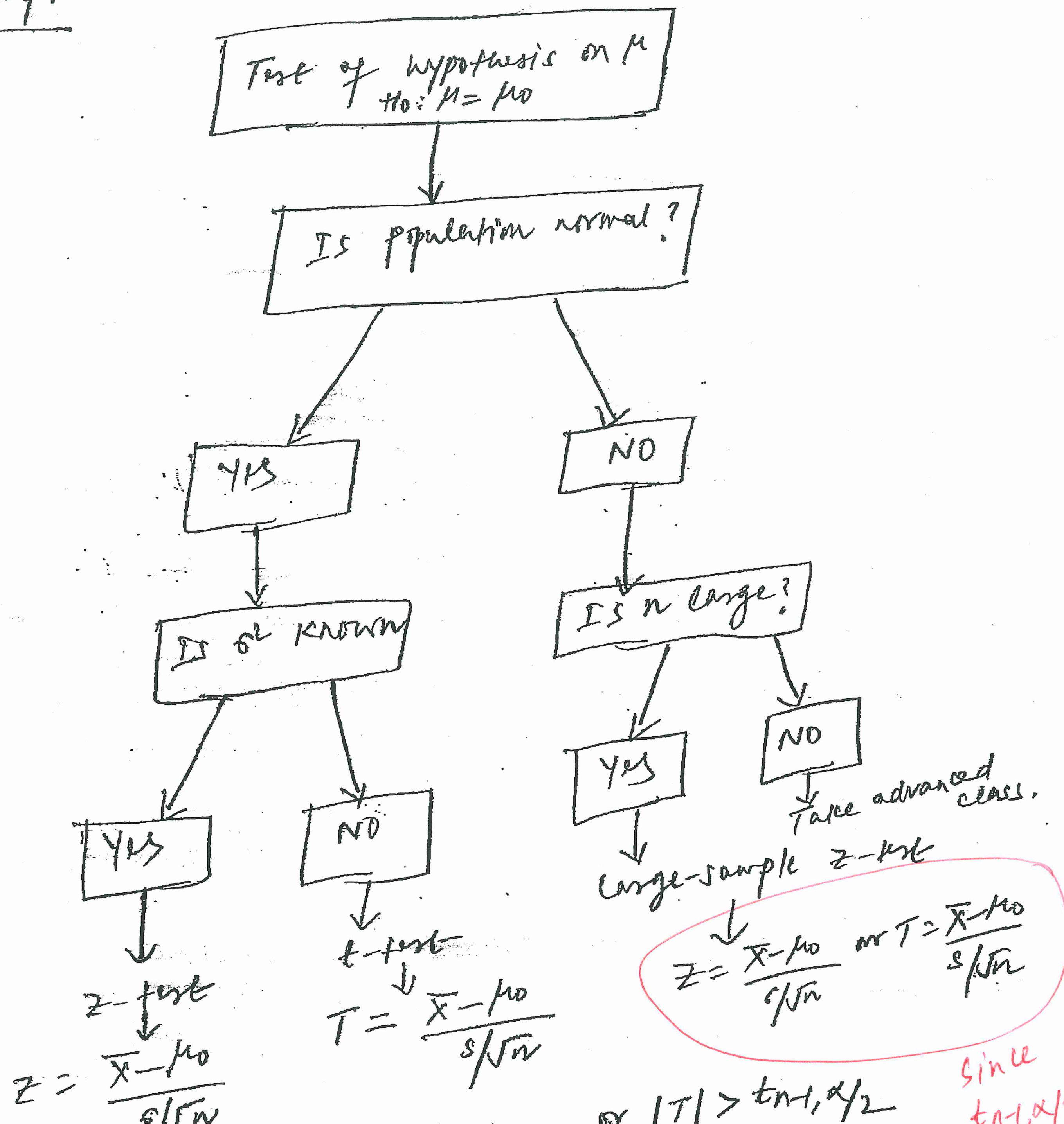
Recap:



$H_1: \mu \neq \mu_0$: reject H_0 if $|Z| > z_{\alpha/2}$ or $|T| > t_{n-1,\alpha/2}$

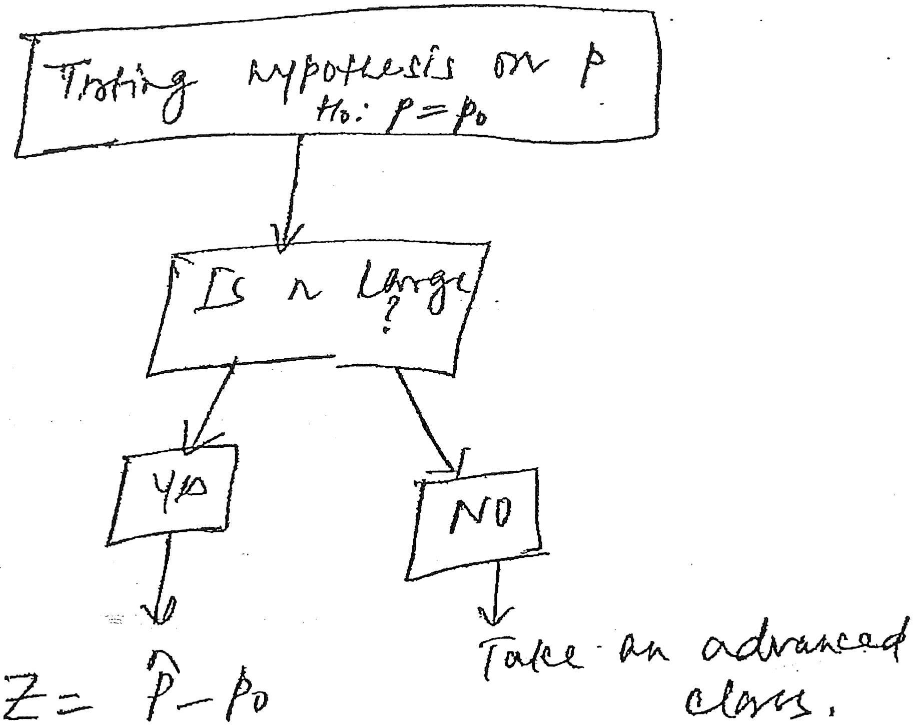
$H_1: \mu > \mu_0$: Reject H_0 if $Z > z_\alpha$ or $T > t_{n-1,\alpha}$

$H_1: \mu < \mu_0$: Reject H_0 if $Z < -z_\alpha$ or $T < -t_{n-1,\alpha}$

Since $t_{n-1,\alpha/2} \approx z_{\alpha/2}$

not $z_{\alpha/2}$

as $t_{n-1,\alpha/2}$ may not be in the table.



$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$H_1: p \neq p_0$: reject H_0 if $|Z| > Z_{\alpha/2}$

$H_1: p > p_0$: reject H_0 if $Z > Z_\alpha$

$H_1: p < p_0$: reject H_0 if $Z < -Z_\alpha$.

A general large-sample z-test.

Recall: Often, $\hat{\theta} \overset{\cdot}{\sim} N[\theta, \hat{V}]$ if n is large.

EX1: θ = mean of a population.

EX2: θ = parameter, $\hat{\theta}$ is MLE.; $\hat{V} = I^{-1}$; I = (observed) information = Hessian matrix.

$\uparrow$ any model parameter.

$H_0: \theta = \theta_0$

Then: $$z = \frac{\hat{\theta} - \theta_0}{\sqrt{\hat{V}}} \overset{\cdot}{\sim} N(0,1) \text{ if } n \text{ is large.}$$

— Can do a z-test.

Ex 2: The number of concurrent users for an ISP has historically averaged 5000. After a marketing campaign, the management would like to know if it has resulted in an increase in the number of concurrent users. To test this, data were collected by observing the number of concurrent users at 100 (n) randomly selected moments of time. Suppose that the average and the standard deviation of the sample data are 5200 ($\bar{x}$) and $s =$ 800, respectively. Is there evidence that the mean number of concurrent users has increased? Assume 5% level of significance.

Recall: $X = \#$ concurrent users. $\theta = E[X]$

$H_0: \underbrace{\theta = 5000}_{\text{campaign not successful}}$ vs $H_1: \underbrace{\theta > 5000}_{\text{campaign successful}}$

Large-sample z-test. $z_{obs} = \dfrac{5200 - 5000}{800/\sqrt{100}} = 2.5$

p-value $= P[\underbrace{Z}_{N(0,1)} > 2.5] = 0.006$. Since p-value $< \alpha = 0.05$, reject H_0.

$\Rightarrow$ Campaign is successful.

Note: $z_{0.05} = 1.645$. Same conclusion with the critical pt.

Ex 3: A recent poll of 1,000 American people estimated that the approval rating of the current congress is 31% $\hat{\theta}$. Do these data give evidence that less than 30% of the American people approve the performance of the congress? Assume 5% level of significance.

Recall: θ = proportion of Americans who approve the performance of congress.

$H_0: \theta = 0.3$ vs $H_1: \theta < 0.3$

Large-sample z-test

$$z = \frac{0.31 - 0.3}{\sqrt{\frac{(0.3)(1-0.3)}{1000}}} = 0.69.$$

$$\text{p-value} = P[\underset{N(0,1)}{Z} < 0.69] = 0.75 \text{ (verify)}.$$

Since p-value is large [it is $> \alpha = 0.05$], accept H_0. There is no significant evidence that less than 30% people approved the performance of the congress.

Note: $-z_{0.05} = -1.645 = \text{qnorm}(0.05)$

Since $z_{obs} > -z_{0.05} \Rightarrow$ accept H_0.

Two-sample tests for $\mu_X - \mu_Y$ for normal populations

Set up: $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$

- X sample: $X_1, \ldots, X_n$ — i.i.d. as X
- Y sample: $Y_1, \ldots, Y_m$ — i.i.d. as Y
- $H_0 : \mu_X - \mu_Y = \Delta$, where Δ is given and may be zero

X and Y are dependent.

Case 1: Paired samples, i.e., (X_i, Y_i) comes from subject $i = 1, \ldots, n$.

- $D = X - Y \sim N(\mu_D, \sigma_D^2)$ where $\mu_D = \mu_X - \mu_Y$, $\sigma_D^2 = \sigma_X^2 + \sigma_Y^2 - 2\,\mathrm{Cov}(X,Y)$
- Define the differences $D_i = X_i - Y_i$ — i.i.d. as D.
- Apply one-sample procedures to the differences — **paired z-test** or **paired t-test**

$H_0: \mu_X - \mu_Y = \Delta$

Case 2: Independent samples with known variances σ_X^2 & σ_Y^2

Test statistic:

$$z = \frac{(\bar{X} - \bar{Y}) - \Delta}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1) \text{ when } H_0 \text{ is true.}$$

- Know how to get critical points and p-values for z-test
- **Two-sample z-test**

<u>Case 3:</u> Independent samples with unknown variances σ_X^2 & σ_Y^2

Test statistic:

$$t = \frac{\bar{X} - \bar{Y} - \Delta}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \;\dot\sim\; t_\nu$$ when H_0 is true, where

ν is approximated using Satterthwaite's formula (seen before).

- Know how to get get critical points and p-values for a t-test
- Approximate **Two-sample t-test**
- No assumption regarding equality of variances

Case 4: Independent samples with unknown but equal variances $\sigma_X^2 = \sigma_Y^2$

Estimation of common variance σ^2:

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$$

Test statistic:

$$t = \frac{(\bar{X}-\bar{Y}) - \Delta}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim t_{n+m-2} \quad \text{when } H_0 \text{ is true.}$$

- Know how to get get critical points and p-values for a t-test
- **Two-sample t-test**

Two-sample tests for $\mu_X - \mu_Y$ for non-normal populations

Set up: Same as before but the populations are non-normal

Test statistic:

Here ~~we~~ can use z-test provided both n and m are large.

$$z = \frac{(\bar{X}-\bar{Y}) - \Delta}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \approx \frac{(\bar{X}-\bar{Y}) - \Delta}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \;\dot{\sim}\; N(0,1) \text{ when } H_0 \text{ is true.}$$

- Large-sample z-test. Its level is approximately α

Two-sample test for difference in proportions, $p_X - p_Y$

[Two samples are indep.]

As before, apply large-sample z-test we the populations here follow Bernoulli distributions. Use *pooled sample proportion* in case of $H_0 : p_X = p_Y$.

Test statistic:

$$z = \frac{(\hat{p}_X - \hat{p}_Y) - 0}{se(\hat{p}_X - \hat{p}_Y)} \sim N(0,1) \text{ when } H_0 \text{ is true.}$$

$$\mathrm{var}(\hat{p}_X - \hat{p}_Y) = \frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{m}$$

$$\overset{p}{=} \frac{p(1-p)}{n} + \frac{p(1-p)}{m}$$

H_0 is true

$$= p(1-p)\left(\frac{1}{n} + \frac{1}{m}\right).$$

Estimate it by: $\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)$,

where $\hat{p}$ = pooled estimator of p

= prop. of 1's in the combined sample of size $n+m$.

- The level of the test is approximately α