CSC263

Data Structures and Analysis

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CONTENTS

Part I

Data Structure

CHAPTER

PRIORITY QUEUES AND HEAPS 1

Data

- Collection of elements
- Each element x has a priority $x.priority$

Operations

- INSERT(Q, x)

  Add x to Q

  Note: $x.priority$ can be non-unique
- MAX(Q)

  Return the element with max priority

  Note: Q is unchanged
- EXTRACT-MAX(Q)

  Remove and return the element with the max priority

1.1 Implementation

1.1.1 Attempts

Implementation 1: Unsorted Array / Linked List

- INSERT takes $\Theta(1)$ time in the worst case
- MAX takes $\Theta(n)$ time in the worst case
- EXTRACTMAX takes $\Theta(n)$ time in the worst case

Implementation 2: Sorted Array / Linked List

- INSERT takes $\Theta(n)$ time in the worst case
- MAX takes $\Theta(1)$ time in the worst case
- EXTRACTMAX takes $\Theta(1)$ time in the worst case

1.1.2 Implementation

We want to combine the advantages of both data structures by having a "partially sorted" ADT – a (binary) *heap*.

There are two kinds of binary heaps: max-heaps and min-heaps. In both kinds, the values in the nodes satisfy a **heap property**, the specifics of which depend on the kind of heap.

- Max heap property: the key of every node x is *larger* than or equal to the keys of its children.

  The largest element in a max-heap is stored at the root.

- Min heap property: the key of every node x is *smaller* than or equal to the keys of its children

  The smallest element in a min-heap is stored at the root,

These are called *heap orders*. There is no ordering between the siblings. A max/min heap is valid if it is a nearly complete binary tree and it satisfies the max/min heap property.

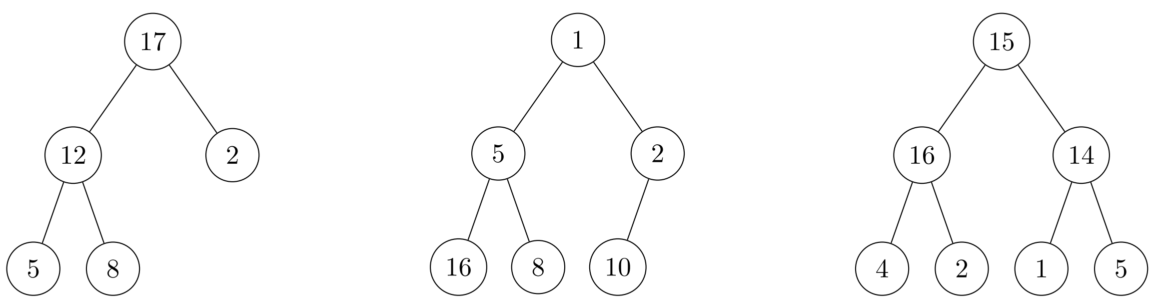


Figure 1.1: A valid max-heap (left), a valid min-heap (middle), and an invalid heap (right)

Although a heap is an **almost complete binary tree** [1], in practice, we usually use an array to store the data in memory. An array H that represents a heap is an object with two attributes: $H.length$, which (as usual) gives the number of elements in the array, and $H.heap\text{-}size$, which represents how many elements in the heap are stored within array H. The root of the tree is $H[1]$, and given the index i of a node, we can compute the indices of its parent, left child, and right child:

- PARENT

  return $\lfloor i/2 \rfloor$

- LEFT

  return $2i$

- RIGHT

  return $2i + 1$

[1] That is, the tree is completely filled on all levels except possibly the lowest, which is filled from the left up to a point.

1.2 Operations

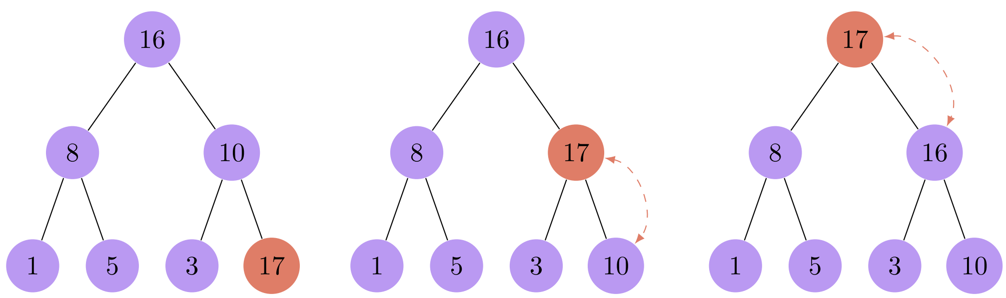
1.2.1 Insert

To insert element with key p into the heap H,

- Increment $H.\textit{heap-size}$ and add a new node with key p to the next available position
- Repeatedly swap the new item with its parent until the heap property is satisfied

  This swapping process is called **bubbling up**
- Worst-case runtime: $\Theta(\lg n)$

For example, consider $\textsc{Insert}(H, 17)$ where $H = [16, 8, 10, 1, 5, 3]$

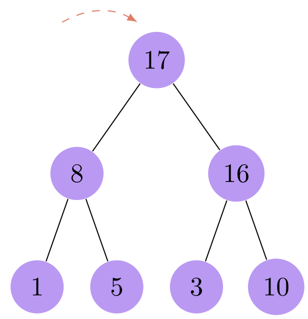



```
procedure MAX-HEAP-INSERT(H, p)
    i ← H.heap-size ← H.heap-size + 1
    H[i] = p
    while PARENT(i) > 0 and H[i] > H[PARENT(i)] do
        swap H[i] with H[PARENT[i]]
        i ← PARENT(i)
    end while
end procedure
```

1.2.2 Find-Max

To find the maximum key in the heap H,

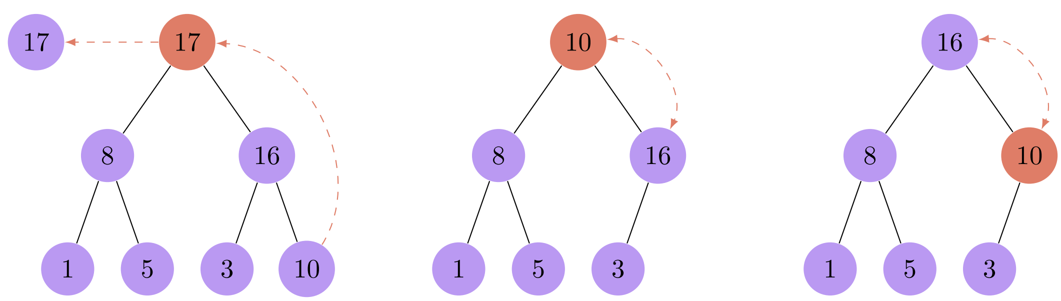
- Simply return the item in the root
- Worst-case runtime: $\Theta(1)$



procedure FIND-MAX(H)
    return $H[1]$
end procedure

1.2.3 EXTRACT-MAX

- Save the item from the root in a temporary variable
- Replace the root with the rightmost item in the lowest level of the tree and decrement $H.heap\text{-}size$
- Repeatedly swap the item we moved with its largest child until the heap property is restored. This swapping process is called **bubble down**.
- Worst-case runtime: $\Theta(\lg n)$



procedure EXTRACT-MAX(H)
    $max \leftarrow H[1]$
    $H[1] \leftarrow H[H.heap\text{-}size]$
    $H.heap\text{-}size \leftarrow H.heap\text{-}size - 1$
    MAX-HEAPIFY($H, 1$)
    return max
end procedure

```
procedure MAX-HEAPIFY(H, i)
    l ← LEFT(i)
    r ← RIGHT(i)
    if l ≤ H.heap-size and H[l] > H[i] then
        largest ← l
    else
        largest ← i
    end if
    if r ≤ H.heap-size and H[r] > H[largest] then
        largest ← r
    end if
    if largest ≠ i then
        swap H[i] with H[largest]
        MAX-HEAPIFY(H, largest)
    end if
end procedure
```

1.2.4 Build-Max-Heap

- Takes an array A of length n and builds a max-heap H from it.
- Worst-case runtime: $\Theta(n)$

```
procedure BUILD-MAX-HEAP(A)
    H.heap-size ← A.length
    for i = ⌊A.length / 2⌋ downto 1 do
        MAX-HEAPIFY(H, i)
    end for
end procedure
```

1.3 Applications

1.3.1 Heap Sort

Implementation 1:

- Run Build-Max-Heap(A)
- Run Extract-Max(A) for $n-1$ times

Implementation 2:

- We modify the Extract-Max algorithm to swap the root with the last item

CHAPTER

2 DICTIONARIES

Data

- A set S
- Each element x has a **unique** key $x.key$

Operations

- SEARCH(S, k)

  Return x in S with $x.key = k$ (or NIL).

- INSERT(S, x)

  Add x to S – if S contains y with $y.key = x.key$, then *replace* y with x.

- DELETE(S, x) [a]

  Remove element x from S.

[a] If we are given the key k instead of the element x, we could do DELETE(S, SEARCH(S, k))

	SEARCH	INSERT	DELETE#
unsorted array	n	n	1
sorted* array	$\lg n$	n	n
unsorted linked list	n	n	1
sorted* linked list	n	n	1
direct access table	1	1	1
hash table	n	n	n
binary search tree	height of the BST	height of the BST	height of the BST
balanced search tree	$\lg n$	$\lg n$	$\lg n$

*: these require the keys to be ordered
#: here we are only doing deletion (without having to search for x)

2.1 Binary Search Trees

A binary search tree is a binary tree with the *binary-search-tree property*:

Let x be a node in a binary search tree. If y is a node in the left subtree of x, then $y.key \leq x.key$. If y is a node in the right subtree of x, then $y.key \geq x.key$.

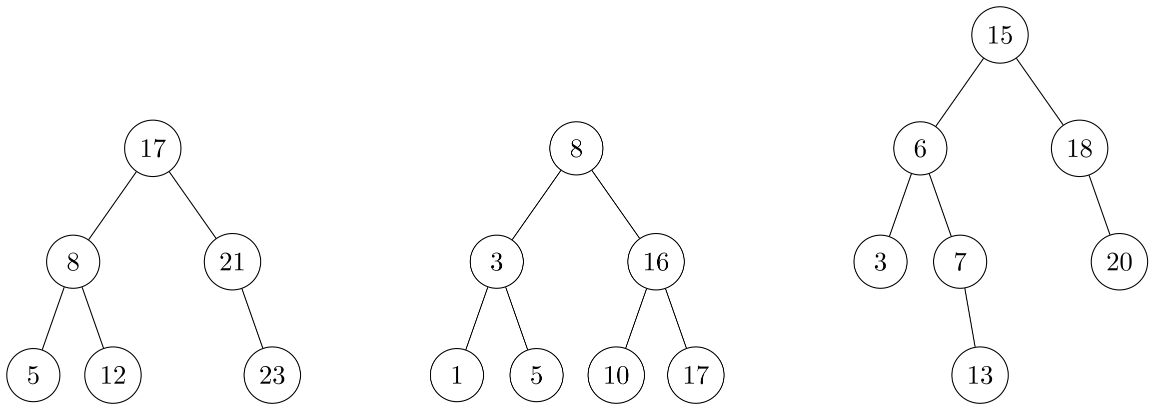


Figure 2.1: Examples of binary search trees

```
class Item:
    key: Any
    value: Any
```

```
class BST_Node:
    item: Item
    left: BST_Node
    right: BST_Node
```

```
class Dictionary:
    root: BST_Node
```

2.1.1 Insert

procedure INSERT(S, x)
    $S.root \leftarrow$ BST-INSERT$(S.root, x)$
end procedure

procedure BST-INSERT$(root, x)$
    # Insert x into stubree at root; return new root
    if $root =$ NIL **then**
        $root \leftarrow$ BST_NODE(x) # Add x
    else if $x.key < root.item.key$ **then**
        $root.left \leftarrow$ BST-INSERT$(root.left, x)$
    else if $x.key > root.item.key$ **then**
        $root.right \leftarrow$ BST-INSERT$(root.right, x)$
    else # $x.key = root.item.key$
        $root.item \leftarrow x$ # replace with x
    end if
    return $root$
end procedure

2.1.2 Search

```
procedure Search(S, k)
    node ← BST-Search(S.root, k)
    if node = NIL then
        return NIL
    end if
    return node.item
end procedure
```

```
procedure BST-Search(root, k)
    # Return node under root with key k (or NIL)
    if root = NIL then
        pass # k not in tree
    else if k < root.item.key then
        root ← BST-Search(root.left, k)
    else if k > root.item.key then
        root ← BST-Search(root.right, k)
    else  # k = root.item.key
        pass
    end if
    return root
end procedure
```

2.1.3 Delete

```
procedure Delete(S, k)
    S.root ← BST-Delete(S.root, k)
end procedure
```

```
procedure BST-Delete(root, x)
    # Delete x from stubree at root; return new root
    if root = NIL then pass # x not in tree
    else if x < root.item.key then
        root.left ← BST-Delete(root.left, x)
    else if x > root.item.key then
        root.right ← BST-Delete(root.right, x)
    else  # x.key = root.item.key
        if root.left = NIL then
            root ← root.right # could be NIL
        else if root.right = NIL then
            root ← root.left
        else  # Replace root.item with its successor
            root.item, root.right ← BST-Del-Min(root.right)
        end if
    end if
    return root
end procedure
```

```
procedure BST-Del-Min(root)
    # Remove element with smallest key under root; return item and root of resulting subtree
Require: root ≠ NIL
    if root.left = NIL then
        return root.item, root.right
    else
        item, root.left ← BST-Del-Min(root.left)
        return item, root
    end if
end procedure
```

2.2 Balanced Search trees

Despite the simplicity of the BST, it is not a very efficient data structure. The worst-case running time of the BST operations is proportional to the height of the tree, which is $\Theta(n)$, where n is the number of elements in the tree. The shape of a BST is determined by the order in which keys are inserted. If the keys are inserted in sorted order, the BST degenerates into a linked list.

We can improve the performance of the BST by making it more balanced. A *balanced BST* (also known as an *AVL tree* – Adelson-Velsky, Landis Tree) is one in which the heights of the two subtrees of any node differ by at most one. The height of a balanced BST is $\Theta(\log n)$, where n is the number of elements in the tree.

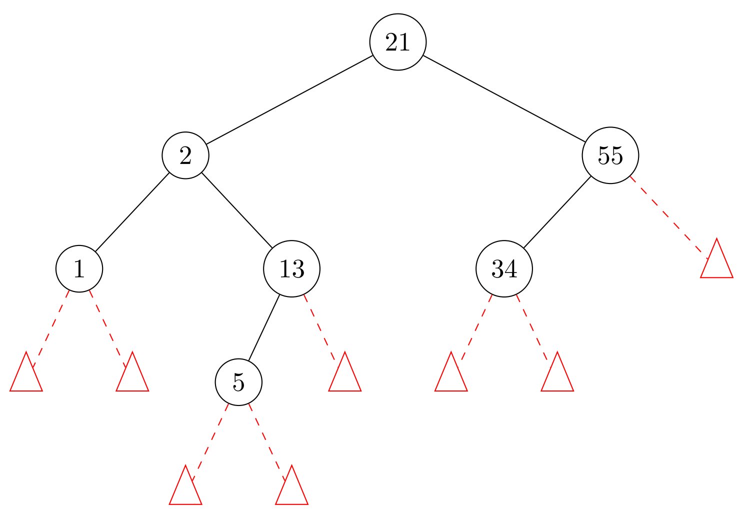


Figure 2.2: AVL balanced binary search tree

To implement an AVL tree, we need a mechanism to detect imbalance in the tree, and a way to restore balance. We will use the following definition of *balance factor* of a node x in a BST:

Definition 2.2.1 Balance Factor

An AVL balanced node x has a balance factor of -1, 0, or 1. If the height of its left subtree is h_L, and the height of its right subtree is h_R, then x has a balance factor of $h_L - h_R$.

- If $h_R - h_L = 0$, then x is balanced.
- If $h_R - h_L = 1$, then x is right-heavy.
- If $h_R - h_L = -1$, then x is left-heavy.

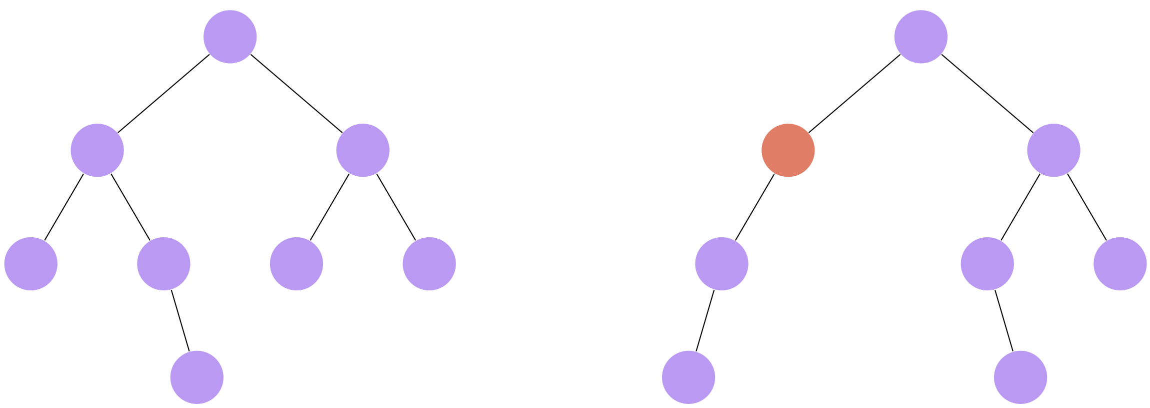


Figure 2.3: AVL balanced tree (left) and unbalanced tree (right)

Rotations

To restore balance, we need to perform a *rotation* on the tree. There are four types of rotations, depending on the balance factor of the node and its children. The following figure shows the four types of rotations.

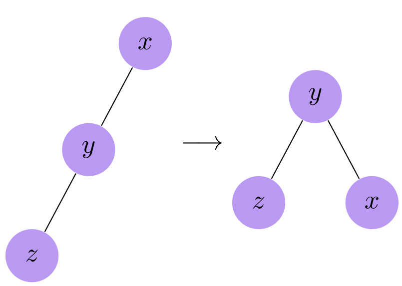


Figure 2.4: Single Left Rotation

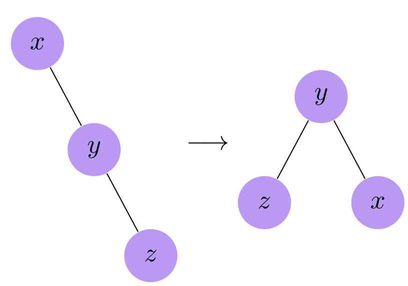


Figure 2.5: Single Left Rotation

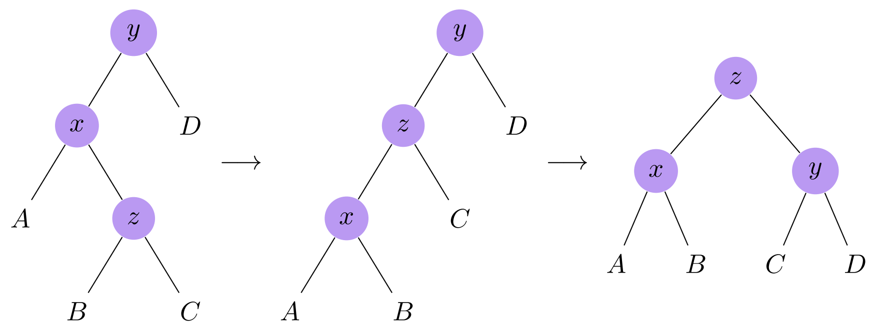


Figure 2.6: Double Left-Right Rotation

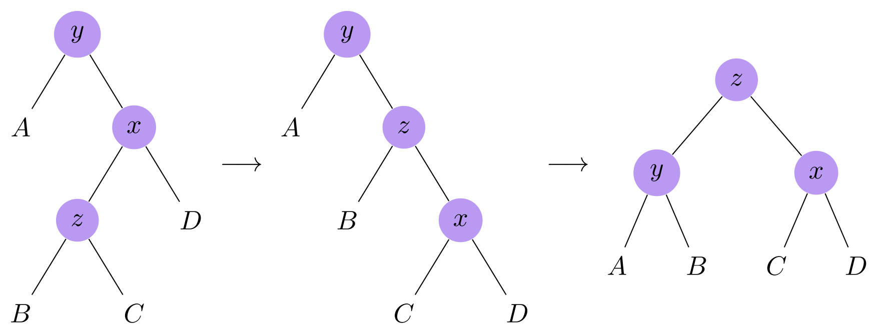


Figure 2.7: Double Right-Left Rotation

2.2.1 Insert

```
procedure AVL-Insert(root, x)
    # Insert x into the tree at root, return new root
    if root = NIL then
        root ← AVL_Node(x) # add x
    else if x.key < root.item.key then
        root.left ← AVL-Insert(root.left, x)
        root ← AVL-Balance-Right(root)
    else if x.key > root.item.key then
        root.right ← AVL-Insert(root.right, x)
        root ← AVL-Balance-Left(root)
    else# x.key = root.item.key
        root.item ← x # replace with x
    end if
    return root
end procedure
```

2.2.2 Delete

---

procedure AVL-DELETE($root$, x)
    # Delete x from the tree at $root$, return new root
    if $root = \text{NIL}$ **then**
        pass # x not in tree
    else if $x.key < root.item.key$ **then**
        $root.left \leftarrow$ AVL-DELETE($root.left, x$)
        $root \leftarrow$ AVL-BALANCE-LEFT($root$)
    else if $x.key > root.item.key$ **then**
        $root.right \leftarrow$ AVL-DELETE($root.right, x$)
        $root \leftarrow$ AVL-BALANCE-RIGHT($root$)
    else # $x.key = root.item.key$
        if $root.left = \text{NIL}$ **then**
            $root \leftarrow root.right$ # could be NIL
        else if $root.right = \text{NIL}$ **then**
            $root \leftarrow root.left$
        else
            if $root.left.height > root.right.height$ **then**
                $root.item, root.left \leftarrow$ AVL-DELETE-MAX($root.left$)
            else
                $root.item, root.right \leftarrow$ AVL-DELETE-MIN($root.right$)
            end if
        end if
        $root.height \leftarrow 1 +$ MAX($root.left.height, root.right.height$)
    end if
    return $root$
end procedure

---

procedure AVL-DEL-MAX($root$)
    # Delete the maximum item from the tree at $root$, return new root and deleted item
Require: $root \neq \text{NIL}$
    if $root.right = \text{NIL}$ **then**
        return $root.item, root.left$
    else
        $item, root.right \leftarrow$ AVL-DELETE-MAX($root.right$)
        $root \leftarrow$ AVL-BALANCE-RIGHT($root$)
        return $item, root$
    end if
end procedure

2.2.3 Rebalancing

```
procedure AVL-Balance-Left(root)
Require: root ≠ NIL
    # First, recalculate height
    root.height ← 1 + Max(root.left.height, root.right.height)
    # Then, rebalance the left, if necessary
    if root.right.height > root.left.height + 1 then
        # Check for double rotation
        if root.right.left.height > root.right.right.height then
            root.right ← AVL-Rotate-Right(root.right)
        end if
        root ← AVL-Rotate-left(root)
    end if
    return root
end procedure
```

```
procedure AVL-Rotate-Left(parent)
Require: parent ≠ NIL, parent.right ≠ NIL
    # Rearrange references
    child ← parent.right
    parent.right ← child.left
    child.left ← parent
    # Update heights; parent first because it is now deeper
    parent.height ← 1 + Max(parent.left.height, parent.right.height)
    child.height ← 1 + Max(child.left.height, child.right.height)
    # Return new parent
    return child
end procedure
```

2.3 Hashing

- Universe U

  The set of all keys. We assume that $|U|$ is very large.

- Hash Table T

  An array of fixed size m. Each location $T[i]$ is called a *bucket*.

- Hash Function h

  The hash function $h : U \to \{0, 1, \dots, m-1\}$ maps each key in U to an index in $\{0, 1, \dots, m-1\}$. For each key $k \in U$, $h(k)$ is called the *home bucket* of k.

  To access item with key k, examine $T[h(k)]$.

A hash table is an effective data structure for implementing dictionaries. Although SEARCH for an element in a hes table can take as long as searching for an element in a linked list – $\Theta(n)$ time in the worst case – in practice, hashing preforms extremely well. Under reasonable assumptions, the average time to search for an element in a hash table is $\mathcal{O}(1)$.

2.3.1 Direct Access Tables

Direct addressing is a simple technique that works well when the universe U of keys is reasonably small. If U is small, then we can use an array T of size $|U|$ to implement a dictionary, called a *direct access table*. The key k is used as an index into T to access the item with key k.

2.3.2 Hash Tables

The downside of direct addressing is apparent: if the universe U is large or infinite. storing a table T of size $|U|$ is impractical, and the set K of keys *actually stored* may be so small relative to Y that most of the space allocated for T would be wasted. Instead, we use a hash table.

However, when $m << |U|$, collisions are unavoidable. A *collision* occurs when two keys k_1 and k_2 (with $k_1 \neq k_2$) are mapped to the same bucket $h(k_1) = h(k_2)$. There are two ways to handle collisions: *open addressing* and *closed addressing / chaining*.

Open Addressing

In open addressing, if $T[h(k)]$ is occupied, then we search for the next available location in T to store the item with key k. We call the original hash function h_1 the *primary hash function*, such that $h_1(k)$ is the home bucket of k. We use the *probe sequence* $h(k, i)$ to determine the bucket to try after i collisions.

- *Linear Probing*

  $h(k, i) = (h_1(k) + i) \mod m$

  Note that long clusters of occupied buckets can occur.

- *Quadratic Probing*

  $h(k, i) = (h_1(k) + c_1 i + c_2 i^2) \mod m$

  c_1 and c_2 are constants dependent on m.

- *Double Hashing*

  $h(k, i) = (h_i(k) + i \cdot (h_2(k))) \mod m$, where $h_2(k)$ is a secondary hash function.

Close Addressing / Chaining

In close addressing, we use a linked list to store the items in each bucket. Each nonempty slot points to a linked list, and all the elements that hash to the same slot go into that slot's linked list.

The average-case performance of the hash table depends on how evenly the hash function h distributes the keys across the buckets in the table. The *simple uniform hashing assumption*(SUHA) states that any given key is equally likely to hash into any of the m slots of the table, independently of where any other elements has hashed to. Under this assumption, the expected number of keys in each bucket is the same.

The expected number of keys in a bucket is $\frac{n}{m}$, where n is the number of items in the table and m is the size of the table. This ratio is called the *load factor* of the hash table, and we denote it by α.

CHAPTER

DYNAMIC ARRAY

3

C styled arrays are static, meaning that they have a fixed size. In this chapter, we will learn how to implement a dynamic array, which is a data structure that can grow and shrink in size.

```
class DynamicArray {
    capacity: integer  # room for elements
    size:     integer  # actual number of elements
};
```

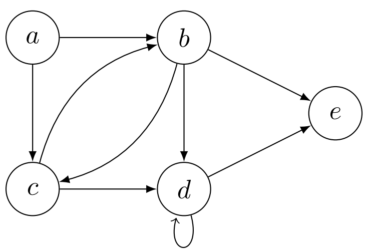
```
procedure INSERT(A, x)
    if A.size = A.capacity then
        A ← RESIZE(A)
    end if
    A.size ← A.size + 1
    A[A.size] ← x
end procedure
```

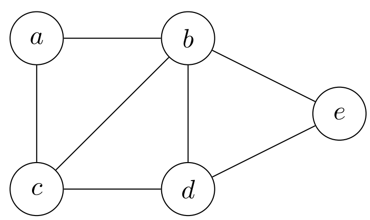
```
procedure RESIZE(A)
    B ← DYNAMICARRAY(2 × A.capacity)
    for i = 1 to A.size do
        INSERT(B, A[i])
    end for
    return B
end procedure
```

To analyze the running time of the above algorithm, see this example using accounting method for amortized analysis. The amortized running time of the above algorithm is $\Theta(1)$.

CHAPTER

4 GRAPHS





4.1 Graphs

4.1.1 Graphs

Define a graph $G = \{V, E\}$

Representations

- Adjacency matrix

	a	b	c	d	e
a	0	1	1	0	0
b	1	0	1	1	1
c	1	1	0	1	0
d	0	1	1	0	1
e	0	1	0	1	0

Complexity: let $n = |V|$ and $m = |E|$

- Space: $O(n^2)$
- Edge query: $\Theta(1)$ time

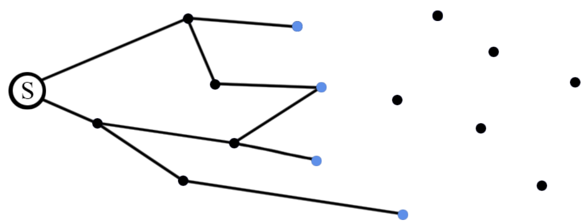
- Adjacency list

  Complexity: let $n = |V|$ and $m = |E|$

  - Space: $\Theta(n + m)$
  - Edge query: $\Theta(n)$ in worst-case time

4.1.2 Breadth-First Search

In breadth first search, we start at a source $s \in V$, and explore every vertex reachable from a, using only edges.



We assign each vertex $v \in V$ a color, which can be one of the following:

- White: v has not been discovered
- Gray: v has been discovered, but not explored
- Black: v has been discovered and explored

Define $\pi[v]$ to be the predecessor of v in the breadth-first search tree.
Define $d[v]$ to be the distance from s to v in the breadth-first search tree.
We use a queue to keep track of the vertices that we have discovered, but not yet explored.

```
procedure BFS(G, s)
    # Initialize tracking info for all vertices
    for v ∈ G.V do
        colour[v] = white
        π[v] ← NIL
        d[v] ← ∞
    end for
    # Initialize empty queue and source vertex tracking info
    Q ← MAKE-QUEUE()
    colour[s] ← gray
    π[s] ← NIL
    d[s] ← 0
    ENQUEUE(Q, s)
    # Main loop. Loop Invariant: Q contains all (and only) gery vertices
    while Q ≠ EMPTY-QUEUE do
        u ← DEQUEUE()
        for v ∈ G.Adj[u] do
            if colour[v] ← white then
                colour[v] ← gray
                π[v] ← u
                d[v] ← d[u] + 1
                ENQUEUE(Q, v)
            end if
        end for
        colour[u] = black
    end while
end procedure
```

In breath first search,

- Each vertex is enqueued at most once
- Each vertex is dequeued at most once
- Each adjacent list is examined at most once
- The time complexity is $\Theta(n+m)$

BFS Finds Shortest Paths

Define $\delta(s,v)$ to be the length of the shortest path from vertex s to vertex v (i.e. the smallest number of edges in any path from s to v). If there is no path from s to v, then $\delta(s,v)=\infty$. Note that this definition will change when we consider **weighted** graphs.

> **Theorem 4.1.1** **Let $G=\{V,E\}$ be a graph, and let $s \in V$. Then, after BFS(G, s), $\forall v \in V$, $\delta(s,v)=d[v]$**

To prove this theorem, we will need to prove the following lemmas first.

Lemma 4.1.1 $\forall (u,v) \in E$**,** $\delta(s,v) \leq \delta(s,u) + 1$

Proof. (idea)

If $\delta(s,u) = \infty$, then the claim holds trivially.

If $\delta(s,u) \neq \infty$, then u is reachable from s.

Thus, v is also reachable from s.

Thus, the shortest path from s to v is no longer than the shortest path from s to u, plus the edge (u,v).

Hence, $\delta(s,v) \leq \delta(s,u) + 1$. ∎

Lemma 4.1.2 **At ant point during BFS,** $\forall v \in V$**,** $d[v] \geq \delta(s,v)$

Proof. (idea)

Use induction on the number of Enqueue operations.

Immediately after we do the first Enqueue operation, $d[s] = 0$, and $\delta(s,s) = 0$.

We also have $d[v] = \infty$, and $\delta(s,v) = \infty$ for all $v \in V - \{s\}$.

Now, consider some vertex v that is first discovered while visiting a vertex u.

By the IH, we have $d[u] \geq \delta(s,u)$.

Hence, $d[v] = d[u] + 1 \geq \delta(s,u) + 1$ by Lemma 4.2.1.

Then v is painted grey and $d[v]$ is not changed for the rest of the algorithm. ∎

Lemma 4.1.3 **If** $Q = \langle v_1, \ldots, v_r \rangle$**, then** $d[v_i] \leq d[v_{i+1}]$ **for all** $i \in \{1, \ldots, r-1\}$ **and** $d[v_r] \leq d[v_1] + 1$

Proof. (sketch)

Use induction on the number of Dequeue / Enqueue operations.

When $Q = \langle s \rangle$, the claim holds trivially.

To prove the inductive step, we need to show that the lemma hold after applying Dequeue / Enqueue to $Q = \langle v_1, \ldots, v_3 \rangle$.

- Case 1

  If we perform a Dequeue operation, then $Q = \langle v_2, \ldots, v_3 \rangle$ afterwards.

  By the IH, $d[v_r] \leq d[v_1] + 1$ and $d[v_1] \leq d[v_2]$.

  Hence, $d[v_r] \leq d[v_2] + 1$.

  All other inequalities are unaffected.

- Case 2

  If we perform a Enqueue operation, then $Q = \langle v_1, \ldots, v_{r+1} \rangle$ afterwards.

  We discover v_{r+1} while visiting some vertex u, so $d[v_{r+1}] = d[u] + 1$.

  Vertex u must have been the previous vertex dequeued from the queue.

Hence, either v_1 was discovered while visiting u, in which case $[v_1] = d[u] + 1$, or Q was equal to $\langle u_2, v_1, \dots \rangle$ at some prior point, in which case $d[u] \le d[v_1]$ by IH.

Hence, $d[v_{r+1}] = d[u] + 1 \le d[v_1] + 1$.

Otherwise, $d[v_r] \le d[u] + 1 = d[v_{r+1}]$ by the IH.

■

Now, we can prove the theorem.

Proof. To derive a contradiction, suppose $d[v] \ne \delta(s, v)$ for some vertex $v \in V$.
Suppose v is a vertex with minimal $\delta(s, v)$ for which this is satisfied.
By Lemma 4.1.2, we have $d[v] > \delta(s, v)$.
Because we chose v with minimal $\delta(s, v)$, we have $d[u] = \delta(s, u)$.
Hence, $d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$.
Consider the colour of v when we first dequeue u from Q.

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4.2 Minimum Spanning Trees

4.3 Disjoint Sets

Part II

Algorithms

CHAPTER

5 SORTING

5.1 Heap Sort

5.2 Quick Sort

```
procedure QuickSort(A)
    if thenLen(A) ≤ 1
        return A
    end if
    L, p, G ← Partition(A)
    return QuickSort(L) + [p] + QuickSort(G)
end procedure
```

5.2.1 Deterministic Quick Sort

In deterministic quick sort, we choose the pivot to be a certain index in the array. We can choose the pivot to be the first element, the last element, or the middle element.

- Run-time depends on input ordering
- Bad ordering would yield bad run-time, while random ordering would generally yield better run-time

5.2.2 Randomized Quick Sort

In randomized quick sort, we choose the pivot to be a random index in the array. Intuitively, the expected run-time would be the same as deterministic quick sort – $\Theta(n \lg n)$ – but the worst case run-time would be much better.

Part III

Analysis

CHAPTER

6 AVERAGE CASE ANALYSIS

In *average case analysis*, we are interested in the average performance of an algorithm. we take the average, or the expected value, over the distribution of the possible inputs.

For each n, define $S_n = \{\text{all inputs of size } n\}$, and if we consider the inputs to be random, then S_n is the sample space. For each $x \in S_n$, define $P(x)$ to be the probability that x will be chosen as the input. Define $t(x)$ as the number of steps preformed on input x. t is the random variable.

Then, the average case running time is defined as

$$\begin{aligned} T(n) &= E[t] \\ &= \sum_{x \in S_n} P(x) \cdot t(x) \end{aligned}$$

Example. Consider the linear search algorithm on a linked list L.

```
procedure LinSearch(L, x)
    z ← L.head
    while z ≠ NIL and z.data! = x do
        z ← z.next
    end while
    return z
end procedure
```

Let S_n be the sample space of all linked lists of size n. Let $P(x)$ be the probability that x is chosen as the input. Let $t(x)$ be the number of steps performed on input x.

- We need to know S_n with probability

Consider the inputs $\texttt{input}_1 = ([1,2,3],2)$ and $\texttt{input}_2 = ([\text{“}a\text{”}, \text{“}b\text{”}, \text{“}c\text{”}], \text{“}b\text{”})$, note that they will take the same steps. We only need one input for each possible value of t.

Define $S_n = \{([1,2,\ldots,n],1), ([1,2,\ldots,n],2), \ldots, ([1,2,\ldots,n],n), ([1,2,\ldots,n],0)\}$.

- We assume all the inputs happen equally likely, then $P(x) = \frac{1}{n+1}$.
- We need an exact formula for $t(x)$.

  In practice, we choose some “key operations” s.t. counting **only** these operations is within a constant factor of total time – then set $t(x) =$ number of key operations.

  Here, we choose line 3, $z.data \neq x$, as the key operation.

Then, we have
$$\begin{aligned}
T(n) &= \sum_{(L,i)\in S_n} t(L,i)\cdot P(L,i) \\
&= \frac{1}{n+1}\sum_{i=0}^{n} t([1,2,\ldots,n],i) \\
&= \frac{1}{n+1}\left(t([1,2,\ldots,n],0) + \sum_{i=1}^{n} i\right) \\
&= \frac{1}{n+1}\left(n + \frac{n(n+1)}{2}\right) \\
&= \frac{n}{n+1} + \frac{n}{2}
\end{aligned}$$

Example. Consider $\textsc{Search}(T,k)$ on a hash table T for a key k.

Assume t has m slots, and uses chaining to resolve collisions. Assume that prior to applying the textscSearch algorithm, the hash table contains n keys.

Assume the key k is samples uniformly at random from U.

Let $N(k)$ be the number of keys examined during search for k. $N(k)$ is the key operation.

$$\begin{aligned}
E[N(k)] &= \sum_{k\in U} P[k]\cdot N(k) \\
&= \sum_{i=0}^{m-1} \sum_{\substack{k\in U\\ h(k)=i}} P[k]\cdot N(k) && \text{regroup terms} \\
&\leq \sum_{i=0}^{m-1} \sum_{\substack{k\in U\\ h(k)=i}} P[k]\cdot L_i && \text{since } N(k)\leq L_i \text{ when } h(k)=i \\
&= \sum_{i=0}^{m-1} L_i \cdot \sum_{\substack{k\in U\\ h(k)=i}} P[k] \\
&= \sum_{i=0}^{m-1} L_i \cdot P[h(k)=i]
\end{aligned}$$

$$= \sum_{i=0}^{m-1} L_i \cdot \frac{1}{m}$$
$$= \frac{1}{m} \sum_{i=0}^{m-1} L_i$$
$$= \frac{n}{m}$$

Example. Consider QUICKSORT on an array A of size n.

Let $S_n = \{\text{all permutations of } [1, 2, dots, n]\}$

We assume an uniform distribution of the inputs, then $P(x) = \frac{1}{n!}$.

Let the random variable $T(A)$ be the total number of comparisons between elements of A.

Define $X_{i,j} = \begin{cases} 1 & \text{if } i \text{ is compared to } j \\ 0 & \text{otherwise} \end{cases}$ for $1 \le i < j \le n$.

Then, $T(A) = \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}$.

The probability $P(X_{i,j} = 1) = P(i \text{ or } j \text{ appear in } A \text{ before all other values in range } [i...j])$

$$= \frac{1}{j-i+1} + \frac{1}{j-i+1}$$
$$= \frac{2}{j-i+1}$$

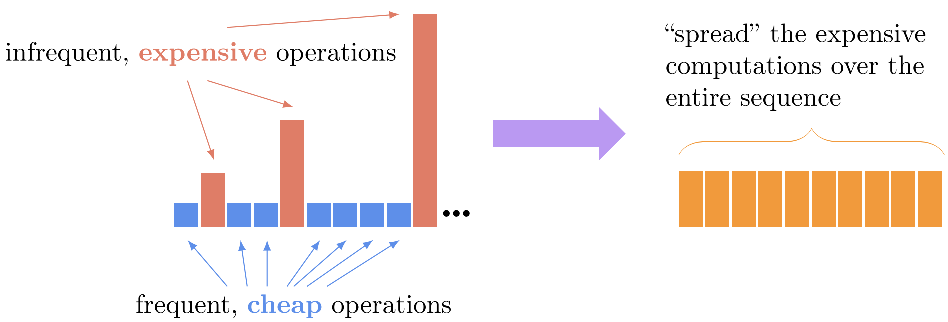
Then, $E[T(A)] = \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}]$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(X_{i,j} = 1)$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{\alpha+1} \qquad \text{substitute } j-i \text{ with } \alpha$$
$$< \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{k}$$
$$= \sum_{i=1}^{n-1} \Theta(\lg n)$$
$$= \Theta(n \lg n)$$

CHAPTER 7

AMORTIZED ANALYSIS

In amortized analysis, we analyze the cost of a sequence of operations, not just a single operation. The amortized cost of a sequence of operations is the average cost per operation.



- The *worst-case sequence complexicy* of a sequence S of k operations is the maximum possible total steps performed by S (taken over all possible inputs to the operations in S).

- The worst-case sequence complexity is **at most** $k \times$ the worst-case complexity of any individual operation in S.

- Suppose that the worst-case sequence complexity of a sequence of k operations is $T(k)$. Then the (worst-case) *amortized complexity* per operation of this sequence is $\frac{T(k)}{k}$.

- With amortized analysis, we take the average of the costs of multiple opertions (as opposed to average-case analysis, where we calculate the cost of a single operation by averaging over the input distribution)

7.1 Aggregated Method

In the *aggregated method*, we determine the upper bound $T(n)$ on the total cost of a sequence of N operations, then calculate the average cost per operation as $\frac{T(n)}{n}$.

Example. Consider when we insert into an array. We increase the size of the array by 4 when it is $\frac{3}{4}$ full.

For simplicity, suppose that each insert with no resizing requires c steps, for some constant $c \in \mathbb{N}^+$ (so each insert with resizing requires $c \cdot n + c$ steps, where n is the number of items in the array prior to resizing).

Operation Number	Cost
1	c
2	c
3	c
4	$4c$
5	c
6	c
7	c
8	$7c$
9	c
10	$10c$
$\vdots$	$\vdots$

Within a sequence of k insertions, we need to resize $\left\lfloor \frac{k-1}{3} \right\rfloor$ times. For all k insert operations, the total cost is

$$T(k) = c \cdot \sum_{i=1}^{\left\lfloor \frac{k-1}{3} \right\rfloor} (3i+1) + c \cdot \left(k - \left\lfloor \frac{k-1}{3} \right\rfloor\right)$$

Note that
$$\begin{aligned} c \cdot \sum_{i=1}^{\left\lfloor \frac{k-1}{3} \right\rfloor} (3i+1) &= c \cdot \sum_{i=1}^{\left\lfloor \frac{k-1}{3} \right\rfloor} 3i + \sum_{i=1}^{\left\lfloor \frac{k-1}{3} \right\rfloor} 1 \\ &= 3c \cdot \sum_{i=1}^{\left\lfloor \frac{k-1}{3} \right\rfloor} i + c \cdot \left\lfloor \frac{k-1}{3} \right\rfloor \\ &= \frac{3}{2} c \cdot \left\lfloor \frac{k-1}{3} \right\rfloor \cdot \left(\left\lfloor \frac{k-1}{3} \right\rfloor + 1 \right) + c \cdot \left\lfloor \frac{k-1}{3} \right\rfloor \in \Theta(k^2) \end{aligned}$$

Thus, $T(k)$ is $\Theta(k^2)$. The amortized cost per operation is $\frac{T(k)}{k} = \Theta(k)$.

Example. Consider a k digit binary counter.

In this problem, we count the total number of bits changed in the counter. We can do this by counting the number of times each bit changes.

Bit Number	Number of Changes
0	m
1	$\approx \frac{m}{2}$
2	$\approx \frac{m}{4}$
$\vdots$	$\vdots$
i	$\approx \frac{m}{2^i}$
$\vdots$	$\vdots$

$$\begin{aligned}\text{Then, } T &= \sum_{i=0}^{(\lg m)-1} \frac{m}{2^i} \\ &= m \cdot \sum_{i=0}^{(\lg m)} \frac{1}{2^i} \\ &< m \sum_{i=0}^{\infty} \frac{1}{2^i} \\ &= 2m\end{aligned}$$

7.2 Accounting Method

The *accounting method* is a form of aggregate analysis which assigns to each operation an amortized cost which may differ from its actual cost. Early operations have an amortized cost higher than their actual cost, which accumulates a saved "credit" that pays for later operations having an amortized cost lower than their actual cost. Because the credit begins at zero, the actual cost of a sequence of operations equals the amortized cost minus the accumulated credit. Because the credit is required to be non-negative, the amortized cost is an upper bound on the actual cost. Usually, many short-running operations accumulate such credit in small increments, while rare long-running operations decrease it drastically.

Example. Consider a sequence of m INSERT for dynamic array.

Recall that the "cost" is the actual run-time, while the "charge" is the estimated amortized time.

Note that for $k = 2^n + 1$, we need $2^n = k - 1$ for reading and $2^n + 1 = k$ for writing. Thus, the cost is $2k + 1$.

Then, we know that $\text{cost}(\textsc{Insert}(k)) = \begin{cases} 2k+1 & \text{if } k = 2^n + 1 \\ 1 & \text{otherwise} \end{cases}$

Define $\text{charge}(\textsc{Insert}(k)) = \$5$. We need \$1 for writing the new element, and save \$4 as credit.

We need to prove our credit invariant: every element in the second half of the array has \$4 credit.

Proof. Proof by induction on the number of operations done.

- init: 0 elements, 0 credits
- Consider one INSERT Assuming credit invariant holds

– If the array does not grow, then the new element gets \$4 credit. The credit invariant holds.

– If the array does grow, it must be full, the total credits is $\$4 \cdot \frac{n}{2} = \$2n$, enough toi cover copying n elements. The new element gets \$4 credit. The credit invariant holds.

■

$$
\begin{aligned}
\text{Thus, Amortized} \leq \frac{\text{WCSC}}{m} &= \frac{\text{total cost}}{m} \\
&\leq \frac{\text{total charge}}{m} \quad \text{because of credit invariant} \\
&= \frac{5m}{m} \\
&= 5
\end{aligned}
$$

Part IV

Appendices

BIBLIOGRAPHY

Books

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