# CSC320

*Introduction to Visual Computing*

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# CONTENTS

# Part I

# Foundations

CHAPTER

# IMAGE GEOMETRY 1

## 1.1 Linear 2D Transforms

### 1.1.1 Homogeneous 2D Point Coordinate

**Basic Notational Conventions**

- Column-Vector Representation

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- Row-Vector Representation

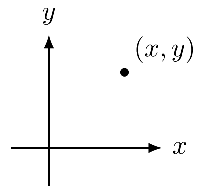
$$\begin{bmatrix} x & y \end{bmatrix}$$

- Matrix Transpose Operation

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top = \begin{bmatrix} x & y \end{bmatrix}$$

**Homogeneous Coordinates**

In the standard Euclidean coordinate system, a point is represented by a pair of coordinates $(x, y)$, where $x$ and $y$ are the coordinates of the point along the $x$ and $y$ axes, respectively.




In homogeneous coordinates, a point is represented by a triple of coordinates $(x, y, w)$, where $x$ and $y$ are the coordinates of the point along the $x$ and $y$ axes, respectively, and $w$ is a scaling factor that is normally set to 1. The homogeneous coordinates of a point are not unique, since any multiple of the coordinates $(x, y, w)$ represents the same point. For example, $(2, 3, 1)$ and $(4, 6, 2)$ represent the same point. The homogeneous coordinates of a point are unique only up to a scaling factor.

### Remark

Two vector of homogeneous coordinates $(x, y, w)$ and $(x', y', w')$ represent the same point if and only if there exists a non-zero scalar $\lambda$ such that

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

**Example.** The homogeneous coordinates of the point $(2, 3)$ are $(2, 3, 1)$, $(4, 6, 2)$, $(6, 9, 3)$, etc. ♢

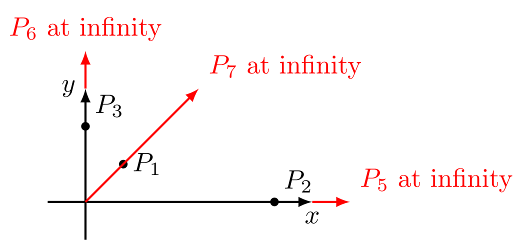
#### Homogeneous to Euclidean Coordinat

To convert a homogeneous coordinate $(x, y, w)$ to a Euclidean coordinate $(x', y')$, we divide the first two coordinates of the homogeneous coordinate by the third coordinate, i.e.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

**Example.** Plot the following points in the Euclidean plane:

$$\begin{array}{lll} P_1 = (2, 2, 2) & P_2 = (10, 0, 2) & P_3 = (0, 8, 4) \\ P_4 = (1, 0, 0.01) & P_5 = (1, 0, 0) & P_6 = (0, 1, 0) \\ P_7 = (1, 1, 0) & & \end{array}$$




Note that $P_5$, $P_6$, and $P_7$ are at infinity. This is a very important property of homogeneous coordinates. ♢

**Remark**

Points infinitely far away from the origin in the Euclidean plane have a finite representation in homogeneous coordinates (i.e. $(x, y, 0)$). These points are sometimes called **ideal points**.

### 1.1.2 Homogeneous 2D Line Coordinates

**Homogeneous 2D Line Coordinates**

**Remark**

The general equation of a line in the Euclidean plane is

$$ax + by + c = 0$$

where $a$, $b$, and $c$ are real numbers and $a$ and $b$ are not both zero.
In homogeneous coordinates, using the matrix notation, the general equation of a line is

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

Or, equivalently,

$$\ell^\top p = 0$$

where $\ell$ is the vector holding line coefficients and $p$ is the vector holding point coordinates.

**Definition 1.1.1**

The **homogeneous coordinates of a line** with the equation

$$ax + by + c = 0$$

is the vector

$$\ell = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

**Example.** The homogeneous coordinates of the line $y = x$ is

$$\ell = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\diamondsuit$

**Line Passing Through Two Points**

In this case, $\ell$ must satisfy the following equations:

$$\ell^\top P_1 = 0 \qquad \ell^\top P_2 = 0$$

where $p_1$ and $p_2$ are the homogeneous coordinates of the two points.

We can take the cross product of $p_1$ and $p_2$ to obtain $\ell$:

$$\ell = P_1 \times P_2$$

> **Remark**
>
> To compute the cross product of two vectors, we can use the following two methods.
>
> $$\ell = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$
>
> As matrix multiplication:
>
> $$P_1 \times P_2 = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} P_2$$
>
> As determinant:
>
> $$P_1 \times P_2 = \begin{vmatrix} \color{red}{i} & \color{red}{j} & \color{red}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

**Intersection of Two Lines**

In this case, $p$ must satisfy the following equations:

$$\ell_1^\top p = 0 \qquad \ell_2^\top p = 0$$

where $\ell_1$ and $\ell_2$ are the homogeneous coordinates of the two lines.

We can take the cross product of $\ell_1$ and $\ell_2$ to obtain $p$:

$$p = \ell_1 \times \ell_2$$

> **Remark**
>
> For parallel lines, we cannot compute its standard Euclidean intersection point. Instead, we can compute the homogeneous intersection point, which will be at infinity.

### 1.1.3 Affine 2D Transformations

- **Identity**

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Stretching** (along $x$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} sx \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

**Stretching** (along $y$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} x \\ sy \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Shearing** (along $y$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} x \\ hx + y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Rotation** (about the origin bt angle $\phi$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} (\cos\phi)x - (\sin\phi)y \\ (\sin\phi)x + (\cos\phi)y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Translation** (along $x$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} x + d \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

**Translation** (along $y$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \cong \begin{bmatrix} x \\ y + d \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can combine these transformations to form more complex transformations. For example, we can combine a rotation and a translation to form a **rotation about a point** transformation.

### Definition 1.1.2 Affine Transformation (Intuitive)

An **affine transformation** is any combination of scaling, shearing, rotation, and translation.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix}$$

where $g \neq 0$ is a scaling factor that does not affect the transformation.

### Remark

An affine transformation is a transformation that preserves parallelism and ratios of distances along parallel lines.

In the original image, two parallel lines would intersect at some infinity $(x, y, 0)$. After an affine transformation, the two lines would still intersect at some infinity $(x', y', 0)$.

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

**Definition 1.1.3 Affine Transformation**

An affine transformation is any invertible $3 \times 3$ matrix that preserves the points at infinity.

But what if the last row of the matrix is not $(0, 0, 1)$?

### 1.1.4 Homographies

**Homographies** are also called **projective transformations** or **perspective transformations**. They still preserve linearity, but they do not preserve parallelism.

**Definition 1.1.4**

A **homography** is any 2D transformation of homogeneous coordinates that is represented by an invertible $3 \times 3$ matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ l & m & g \end{bmatrix}$$

All homographies are invertible. This is because Homographies map unique points to unique points.

**Remark**

Lines remain straight after a homography.
Let $\ell^T p = 0$ be a line. It suffices to show that all points on this line are mapped to points on a line.

**Example.** A naive image warping algorithm is forward mapping.

```
input: src_image, H
output: dest_image

for c=1 to num_columns              // cycle over all source pixels
   for r=1 to num_rows
        x, y = pixel_xy(r,c)         // get the source pixel's (x,y) coordinates
        p = homogeneous_coords(x, y) // convert to vector homogeneous coords
        p_prime = H*p                // apply homography
        x_prime, y_prime = euclidean_coords(p_prime)  // convert to Euclidean coords
        r_prime, c_prime = pixel_rc(x_prime, y_prime) // calculate the dest pixel
```

```
      dest_image(r_prime, c_prime) = src_image(r,c) // copy pixel color
```

This algorithm has some problems:

- Magnification causes "holes" in destination image
- Minification causes overwriting of pixel contents

◊

**Backward-Mapping Algorithm**

This is also known as the **inverse mapping algorithm**. This algorithm does not create "holes" in the destination image.

```
input: src_image, H
output: dest_image

for c_prime=1 to num_columns                            // cycle over all dest px
   for r_prime=1 to num_rows
       x_prime, y_prime = pixel_xy(r_prime,c_prime)    // get dest px's (x,y)
       p_prime = homogeneous_coords(x_prime, y_prime) // convert to homo. coords
       p = inverse(H)*p_prime                          // apply inverse homography
       x, y = euclidean_coords(p)                      // convert to Euclidean
       r, c = pixel_rc(x, y)                           // calculate source px
       dest_image(r_prime, c_prime) = src_image(r,c)  // copy px color
```

## 1.2 Perspective Viewing

### 1.2.1 Aperture

### 1.2.2 Geometry of Perspective Projection

## 1.3 Alignment and Stitching

CHAPTER

# IMAGE FILTERING 2

## 2.1 Linear Shift-Invariant Filters

### 2.1.1 Definition

A filter transforms one signal into another. Filters are used to describe image formation (lens burring, sensor noise, etc.), as well as to implement operations on images (edge detection, noise reduction, etc.).


**Definition 2.1.1 Linear Transformation**
A transformation $T$ is **linear** if and only if it satisfies

$$T[a_1 f_1(x) + a_2 f_2(x)] = a_1 T[f_1(x)] + a_2 T[f_2(x)]$$

for any $a_1, a_2 \in \mathbb{R}$ and continuous functions $f_1, f_2$.

Every linear transformation between functions $f, g$ can be expressed as an integral of the form

$$g(x) = \int_{-\infty}^{\infty} h(x, \tau) f(\tau)\, d\tau,$$

where the $h(x, \tau)$ is the contribution of position $\tau$ of the input to position $x$ of the output.

**Definition 2.1.2 Shift-Invariant Filter**

A filter $h(x, \tau)$ is **shift-invariant** if and only if shifted inputs produce identical but shifted outputs.

Imagine we have the shifted input

$$f'(x') = f(x - x_0)$$

and the shifted output

$$g'(x') = g(x - x_0).$$

The **shift invariance property** states that

$$T[f(x - x_0)] = g(x - x_0)$$

### 2.1.2 The Impulse Function

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# MODEL FITTING

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# COLOR IMAGING AND DISPLAYING 4

# Part II

# Image Representation for CV

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# VECTOR-BASED

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# MULTI-SCALE

# Part III

# Appendices

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