

# **Linear Algebra II**

**Course Notes**

**Lance Li**

Copyright © 2022 Lance Li

PUBLISHED BY LANCE LI

WWW.MATH.TORONTO.EDU/NHOELL/MAT224

Licensed under the Creative Commons Attribution-NonCommercial 3.0 Unported License (the “License”). You may not use this file except in compliance with the License. You may obtain a copy of the License at <http://creativecommons.org/licenses/by-nc/3.0>. Unless required by applicable law or agreed to in writing, software distributed under the License is distributed on an “AS IS” BASIS, WITHOUT WARRANTIES OR CONDITIONS OF ANY KIND, either express or implied. See the License for the specific language governing permissions and limitations under the License.

*First edition, March 2022*



# Contents

I	Notes
<b>0</b>	<b>Introduction . . . . .</b>
0.1	<b>Introduction</b>
0.1.1	Learning Outcome . . . . .
0.1.2	Course Structure . . . . .
0.1.3	Lecture Rules . . . . .
<b>1</b>	<b>Fields, Complex Numbers and Vector Spaces . . . . .</b>
1.1	<b>Fields and Complex Numbers</b>
1.1.1	Fields . . . . .
1.1.2	Complex Numbers . . . . .
1.3	<b>Vector Spaces</b>
1.3.1	Vector Space . . . . .
1.3.2	Subspace . . . . .
1.4	<b>Linear Combinations</b>
1.5	<b>Linear Dependence and Linear Independence</b>
1.6	<b>Bases and Dimension</b>
<b>2</b>	<b>Linear Transformations . . . . .</b>
2.1	<b>Linear Transformations</b>
2.2	<b>Kernel and Image</b>
2.3	<b>The Inverse of a Linear Transformation</b>
2.4	<b>Change of Basis</b>

<b>3</b>	<b>The Spectral Theorem in <math>\mathbb{R}^n</math></b>	<b>33</b>
<b>3.1</b>	<b>Diagonalizability</b>	<b>33</b>
3.1.1	Eigenvectors, Eigenvalues and Eigenspaces	33
<b>3.2</b>	<b>Diagonalizability</b>	<b>35</b>
<b>3.3</b>	<b>Geometry in <math>\mathbb{R}^n</math></b>	<b>38</b>
<b>3.4</b>	<b>Orthogonal Projection and the Gram-Schmidt Process</b>	<b>43</b>

II

## Part Two

<b>Bibliography</b>	<b>47</b>
<b>Books</b>	<b>47</b>
<b>Index</b>	<b>49</b>

# Notes

<b>0</b>	<b>Introduction .....</b>	<b>7</b>
0.1	Introduction	
<b>1</b>	<b>Fields, Complex Numbers and Vector Spaces .....</b>	<b>11</b>
1.1	Fields and Complex Numbers	
1.3	Vector Spaces	
1.4	Linear Combinations	
1.5	Linear Dependence and Linear Independence	
1.6	Bases and Dimension	
<b>2</b>	<b>Linear Transformations .....</b>	<b>25</b>
2.1	Linear Transformations	
2.2	Kernel and Image	
2.3	The Inverse of a Linear Transformation	
2.4	Change of Basis	
<b>3</b>	<b>The Spectral Theorem in <math>\mathbb{R}^n</math> .....</b>	<b>33</b>
3.1	Diagonalizability	
3.2	Diagonalizability	
3.3	Geometry in $\mathbb{R}^n$	
3.4	Orthogonal Projection and the Gram-Schmidt Process	



# 0. Introduction

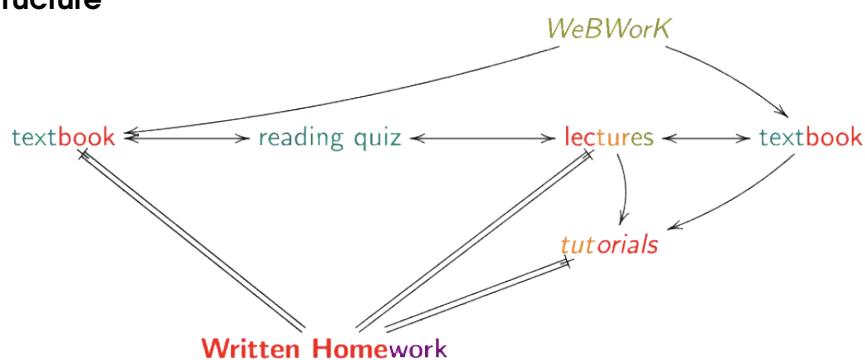
## 0.1 Introduction

Fields, complex numbers, vector spaces over a field, linear transformations, matrix of a linear transformation, kernel, range, dimension theorem, isomorphisms, change of basis, eigenvalues, eigenvectors, diagonalizability, real and complex inner products, spectral theorem, adjoint/self-adjoint/normal linear operators, triangular form, nilpotent mappings, Jordan canonical form.

### 0.1.1 Learning Outcome

- Read and understand new mathematical ideas and concepts on your own
- Communicate mathematical ideas of linear algebra clearly in words and in writing (proofs)
- Connect abstract knowledge to examples
- Approach challenging problems independently
- Use linear algebra as a computational tool

### 0.1.2 Course Structure



Our course has multiple components. Each component is built carefully to support several aspects of the learning outcomes of our course as listed in the syllabus. This page

explains how different components of the course interact with one another and helps you navigate the course

### **Reading Quizzes**

A typical week in our course starts by doing the reading assigned for that week and ends by revisiting the reading assignment for that week. Assigned readings are mainly from our primary textbook [1] followed by a Quercus quiz on your assigned readings for the week and that of the previous week. You should start your reading as soon as they are posted.

You should read all the assigned readings for the week and submit your quiz on Quercus. **The weekly quiz is due at 10 am every Monday.** The best strategy is to spread out the reading throughout the week. You are not expected to understand everything when you read the textbook before the class. Your learning will happen gradually during lectures, tutorials, and mostly when you do the suggested problems from the textbook and the written homework. You will need to review the textbook again after lectures. Reading Quizzes quizzes makeup 5% of your final grade.

### **Lectures**

#### **What to expect from lectures?**

During lectures, your instructor will guide you through important concepts and invite you to participate in lecture activities. Lectures complement your reading, tutorials, and homework. Lectures are not meant to replace textbook reading. There will be concepts and theorems that are only discussed in the textbook or only during your lectures, or only in homework. You are encouraged to attend all lectures. While in your lecture you are expected to actively participate in all activities prompted by your instructor. While the delivery of the course is online, your instructor may record the lectures and make it available for you to watch asynchronously. Note that watching the lectures do not replace attending them as you will miss the in-class activities. You are asked to do a lecture reflection quiz at the end of your lecture week. The reflection due date is a day after your last lecture of the week. Reflections are a tool for your instructor to emphasize important concepts in the lectures and to collect feedback from you, and an opportunity for you to reflect on your understanding of the lecture material. Reflections are part of your participation grade as per our syllabus.

### **Tutorials**

You must be enrolled in a tutorial. You will have weekly tutorial sessions starting on Jan 24th. During tutorials, you will work with your classmates in small groups on tutorial worksheets. Tutorial worksheets are carefully designed to be discussed in groups. Tutorial worksheets are roughly one week behind the lectures.

Ideally, you want to read, understand, and think about all the questions before your tutorial. This involves checking all the definitions and statements of the results you need to know to tackle questions. Set aside 15 minutes before your tutorial to go over questions. Your tutorial sessions are facilitated by your TAs. During your tutorial, your TA will ask you to sit with your tutorial group. You will work with your groupmates on selected problems from the worksheet. Each group works on a shared document that will be checked by the TA during the tutorial. Do not expect your TA to work out the questions for you or to teach the concepts. Your TA will give you feedback on what you already put down in your shared document. That is the more you work among your group the more your TA can help you. Your TA might choose to go over some of the questions for the entire class or answer your questions within your groups. You get the most out of your tutorials if you work on the problems ahead of time and stay active and engaged throughout the tutorial. After all, you can only get an answer to those questions you ask, either from your peers

or the TA. You will get solutions to all tutorial questions at the end of the tutorial week. Tutorials are one of the most important components of the course because they facilitate your communication with your peers. Explain concepts to your group mates and ask them to do the same for you. At the end of each tutorial, you will submit your shared document as a group.

This submission is marked holistically, and not for correctness. You and all your group members will receive the same mark. Your active participation in your tutorial is measured by your group submission marks. Your active participation in tutorials is part of your participation grade.

### Homework

You will have four types of homework. We already talked about two of them. That is reading assignments, due weekly, as part of your Reading Quiz, and Tutorial worksheets that you work on with help of your peers during your tutorial and submit with your group. The other two are

*WeBWork.* WeBWork is an online assignment system. You have weekly WeBWork due every Wednesday 11:59 pm. These questions are straightforward and cover what you learned in the same week and the past week. To access the homework, you should follow the link in the assignment posted under the Homework module. You will be automatically be directed to WeBWork environment. **WARNING:** You should check your grade for WeBWork inside the WebWork environment and not in Quercus. You may occasionally see some grades regarding this homework appear on Quercus. Those numbers are not accurate and will disappear! WeBWork makes 5% of your final grade.

*Written Homework.* There will be five written homework sets. These homework sets are rather long and you have about two weeks to do them. You should submit these homework sets individually. You will receive instructions on how to submit your written homework. Only selected questions from each set are graded. The lowest grade will be dropped. Written homework sets make 10% of your final grade.

### 0.1.3 Lecture Rules

- Come prepared (read the textbook)
- Be fully present
  - No distraction
  - Ready to engage
  - Ready to participate in activities
- Reflect
  - What did you learn?
  - Engage with the textbook
  - Write down your questions
  - Follow up on piazza





# 1. Fields, Complex Numbers and Vector Spaces

## 1.1 Fields and Complex Numbers

### 1.1.1 Fields

**Definition 5.1.4 — Field.** A *field* is a set<sup>a</sup>  $\mathbb{F}$  with two operations, defined on ordered pairs of elements of  $\mathbb{F}$ , called *addition* and *multiplication*. Addition assigns to the pair  $x$  and  $y \in \mathbb{F}$  their *sum*, which is denoted by  $x + y$  and multiplication assigns to the pair  $x$  and  $y \in \mathbb{F}$  their *product*, which is denoted by  $x \cdot y$  or  $xy$ . These two operations must satisfy the following properties for all  $x, y$  and  $z \in \mathbb{F}$ :

- (i) Commutativity of addition  $x + y = y + z$ .
- (ii) Associativity of addition:  $(x + y) + z = x + (y + z)$ .
- (iii) Existence of an additive identity: There is an element  $0 \in \mathbb{F}$ , called zero, such that  $x + 0 = a$ .
- (iv) Existence of additive inverses: For each  $x$  there is an element  $-x \in \mathbb{F}$  such that  $x + (-x) = 0$ .
- (v) Commutativity of multiplication:  $xy = yx$ .
- (vi) Associativity of multiplication:  $(xy)z = x(yz)$ .
- (vii) Distributivity:  $(x + y)z = xz + yz$  and  $x(y + z) = xy + xz$ .
- (viii) Existence of a multiplicative identity: There is an element  $1 \in \mathbb{F}$ , called 1, such that  $x \cdot 1 = x$ .
- (ix) Existence of multiplicative inverses: If  $x \neq 0$ , then there is an element  $x^{-1} \in \mathbb{F}$  such that  $x \cdot x^{-1} = 1$ .

<sup>a</sup>Note that such set must be **non-empty**.

### 1.1.2 Complex Numbers

Goal: build a field containing  $\mathbb{R}$  such that all polynomials (such as  $x^2 + 1$ ) have their roots.

**Definition 5.1.2** The set of *complex numbers*, denoted  $\mathbb{C}$ , is the set of ordered pairs of real numbers  $(a, b)$  with the operations of addition and multiplication defined by

For all  $(a, b)$  and  $(c, d) \in \mathbb{C}$ , the *sum* of  $(a, b)$  and  $(c, d)$  is the complex number defined by  $(a, b) + (c, d) = (a + c, b + d)$

and the *product* of  $(a, b)$  and  $(c, d)$  is the complex number defined by  $(a, b)(c, d) = (ac - bd, ad + cb)$

The subset of  $\mathbb{C}$  consisting of those elements with second coordinate zero,  $\{(a, 0) | a \in \mathbb{R}\}$ , will be identified with the real numbers in the obvious way,  $a \in \mathbb{R}$  is identified with  $(a, 0) \in \mathbb{C}$ . If we apply our rules of addition and multiplication to the subset  $\mathbb{R} \subset \mathbb{C}$ , we obtain

$$(a, 0) + (c, 0) = (a + c, 0)$$

and

$$(a, 0)(c, 0) = (ac - 0 \cdot 0)(a \cdot 0 + c \cdot 0) = (ac, 0)$$

**Proposition 5.1.5** The set of complex numbers is a field with the operations of addition and scalar multiplication as defined previously.

*Proof.* WTS  $\mathbb{C}$  is a field<sup>1</sup>.

(i), (ii), (v), (vi), and (vii) follow immediately.

(iii) The additive identity is  $0 = 0 + 0i$  since

$$(0 + 0i) + (a + bi) = (0 + a) + (0 + b)i = a + bi$$

(iv) The additive inverse of  $a + bi$  is  $(-a) + (-b)i$ .

$$(a + bi) + ((-a) + (-b)i) = (a + (-a)) + (b + (-b))i = 0 + 0i = 0.$$

(viii) The multiplicative identity is  $1 = 1 + 0 \cdot i$  since

$$(1 + 0 \cdot i)(a + bi) = (1 \cdot a - 0 \cdot b) + (1 \cdot b + 0 \cdot a)i = a + bi.$$

(ix) Note first that if  $a + bi \neq 0$ , then either  $a \neq 0$  or  $b \neq 0$  and  $a^2 + b^2 \neq 0$ . Further, note that  $(a + bi)(a + (-b)i) = a^2 + b^2$ .

Therefore  $(a + bi)\frac{a - ib}{a^2 + b^2} = 1$ .

Thus,  $(a + bi)^{-1} = (a - bi)/(a^2 + b^2)$ . ■

**Exercise 1.1** Compute the following.

$$1. (3 - 5i)^{-1}$$

$$\begin{aligned}(3 - 5i)^{-1} &= \frac{(3 + 5i)}{3^2 + 5^2} \\ &= \frac{3}{34} + \frac{5}{34}i\end{aligned}$$

<sup>1</sup>To show  $\mathbb{F}$  is a field, we need to check commutativity, associativity, existence of additive identity and additive inverse, multiplicative identity and multiplicative inverse, and the distributivity between addition and multiplication.

$$2. \frac{4-2i}{3-5i} = (4-2i)(3-5i)^{-1}$$

$$\begin{aligned}\frac{4-2i}{3-5i} &= (4-2i)(3-5i)^{-1} \\ &= (4-2i) \left( \frac{1}{34}(3+5i) \right) \\ &= \frac{1}{34}(4-2i)(3+5i) \\ &= \frac{1}{34}(12+10+20i-6i) \\ &= \frac{1}{34}(22+14i)\end{aligned}$$

■

■ **Example 1.1** Let  $\mathbb{F}_2 = \{0, 1\}$ .

+	0	0
0	0	1
1	1	0

×	0	0
0	0	0
1	0	1

### Discussion

- Is  $+$  commutative? How to see it visually?
  - Yes.
  - The diagonals align up (this tabel is symmetric).
- Is  $\times$  commutative? How to see it visually?
  - Yes.
  - The diagonals align up (this tabel is symmetric).
- Does  $(\mathbb{F}_2, +, \times)$  have additive and multiplicative identities? How to see them visually?
  - Yes, they all have their identities.
  - 0 is the additive identity.  
The corresponding rows and columns of 0 are copies of the index row/column.
  - 1 is the multiplicative identity.  
The corresponding rows and columns of 1 are copies of the index row/column.

■

## 1.3 Vector Spaces

### 1.3.1 Vector Space

Let  $\mathbb{F}$  be a field<sup>2</sup>.

**Definition 5.2.1 — Vector Space over a Field.** A *vector space over  $\mathbb{F}$*  is a set  $V$  (whose elements are called *vectors*) together with two operations:

- A binary operation called **vector addition**, which for each pair of vectors  $\vec{v}, \vec{w} \in V$  produces a vector denoted  $\vec{v} + \vec{w} \in V$ , and
- an operation called **multiplication by a scalar**<sup>a</sup> (a field element), which for each

<sup>2</sup>The differences between a field and a vector space:

- Over *fields*, we have two binary operations;
- Over *vector spaces*, we have one binary operation and one scalar multiplication.

vector  $\vec{v} \in V$ , and each scalar  $c \in \mathbb{F}$  produces a vector denoted  $c\vec{v} \in V$ .

<sup>a</sup>This is also called a **scalar multiplication**

$$\begin{array}{ccc} + : & V \times V & \rightarrow V \\ & (\vec{v}, \vec{w}) & \mapsto \vec{v} + \vec{w} \end{array} \quad \begin{array}{ccc} \times : & \mathbb{F} \times V & \rightarrow V \\ & (c, \vec{v}) & \mapsto c\vec{v} \end{array}$$

Furthermore, the two operations must satisfy the following axioms:

- (1) For all vectors  $\vec{u}, \vec{v}$ , and  $\vec{w} \in V$ ,  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ . (addition is *associative*)
- (2) For all vectors  $\vec{v}$  and  $\vec{w} \in V$ ,  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ . (addition is *commutative*)
- (3) There exists a vector  $\vec{0} \in V$  with the property that  $\vec{v} + \vec{0} = \vec{v}$  for all vectors  $\vec{v} \in V$ . ( $\exists$  an *additive identity*)
- (4) For each vector  $\vec{v} \in V$ , there exists a vector denoted  $-\vec{v}$  with the property that  $\vec{v} + -\vec{v} = \vec{0}$ . ( $\exists$  *additive inverse*)
- (5) For all vectors  $\vec{v}$  and  $\vec{w} \in V$  and all scalars  $c \in \mathbb{F}$ ,  $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ . (*distributive* property 1)
- (6) For all vectors  $\vec{v} \in V$ , and all scalars  $c$  and  $d \in \mathbb{F}$ ,  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ . (*distributive* property 2)
- (7) For all vectors  $\vec{v} \in V$ , and all scalars  $c$  and  $d \in \mathbb{F}$ ,  $(cd)\vec{v} = c(d\vec{v})$ . (multiplication is *associative*)
- (8) For all vectors  $\vec{v} \in V$ ,  $1\vec{v} = \vec{v}$ . ( $\exists$  an *multiplicative identity*)

### ■ Example 1.2

Define  $\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, a_i \in \mathbb{R} \right\}$ ,  $\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, a_i \in \mathbb{C} \right\}$ .

Take  $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ ,  $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ , then  $\vec{z} + \vec{w} = \begin{pmatrix} z_1 + w_1 \\ \vdots \\ z_n + w_n \end{pmatrix}$ <sup>3</sup>.

In general,  $\mathbb{F}^n := \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, a_i \in \mathbb{F} \right\}$  is a vector space over the field  $\mathbb{F}$ <sup>4</sup>.

- Additive identity:  $\vec{0} := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ <sup>5</sup>

- Take  $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{F}^n$ , then  $-\vec{v} = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$ <sup>6</sup>.

### ■ Example 1.3

Define  $P_n(\mathbb{F}) = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mid a_i \in \mathbb{F}\}$ .

- $P_n(\mathbb{F})$  is an  $\mathbb{F}$ -V.S.

Take  $p(x) = a_0 + a_1x + \dots + a_nx^n$ ,  $q(x) = b_0 + b_1x + \dots + b_nx^n \in P_n(\mathbb{F})$ .  
 $p(x) + q(x) := (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$ .

<sup>3</sup>Note that the two +'s ( $\vec{z} + \vec{w}$  and  $z_i + w_i$ ) here are different. The former is vector addition, while the latter is addition over  $\mathbb{C}$ .

<sup>4</sup>"A vector space over the field  $\mathbb{F}$ " can be shortened to " $\mathbb{F}$  vector space", or " $\mathbb{F}$ -V.S."

<sup>5</sup>Note that  $\vec{0}$  is the additive identity of  $\mathbb{F}^n$ , while the 0's are the additive identity over the field  $\mathbb{F}$ .

<sup>6</sup>Note that the two -'s ( $-\vec{v}$  and  $-v_i$ ) here are different. The former is the symbol for the additive inverse inside  $\mathbb{F}^n$ , while the later is additive inverse over the field  $\mathbb{F}$ .

$$cp(x) := ca_0 + \underbrace{ca_1}_{c \times a_1 \text{ on } \mathbb{F}} x + \cdots + ca_n x^n.$$

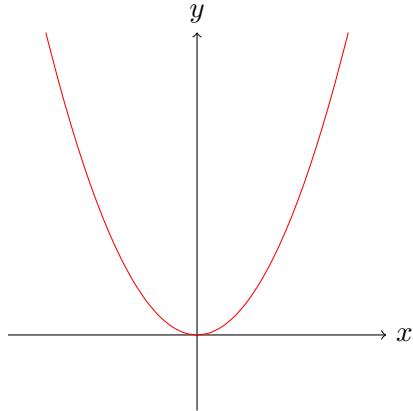
■

**Example 1.4**

$F(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is a function}\}$  is a real vector space.

Consider  $f(x) = x^2 \in F(\mathbb{R})$ .

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$



- Define vector addition and scalar multiplication

Take  $f, g \in F(\mathbb{R})$ .

- $(f + g)(x) := f(x) + g(x)$  for all  $x \in \mathbb{R}$ .
- $(rf)(x) = r(f(x))$  for all  $x \in \mathbb{R}$  and all  $r \in \mathbb{R}$ .

- Additive identity of  $F(\mathbb{R})$ :

$$\begin{aligned} 0_{F(\mathbb{R})} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto 0_{\mathbb{R}} \end{aligned}$$

- Prove that  $0_{F(\mathbb{R})}$  is the additive identity of  $F(\mathbb{R})$ .

*Proof.* WTS  $\forall f \in F(\mathbb{R}), f + 0_{F(\mathbb{R})} = f$ .

Pick arbitrary  $f \in F(\mathbb{R}), f : \mathbb{R} \rightarrow \mathbb{R}$ .

WTS  $\forall x \in \mathbb{R}, (f + 0_{F(\mathbb{R})})(x) = f(x)$ .

$$\begin{aligned} \text{Pick } x \in \mathbb{R}, (f + 0_{F(\mathbb{R})})(x) &= f(x) + 0_{F(\mathbb{R})}(x) && \text{by the definition of } + \text{ in } F(\mathbb{R}) \\ &= f(x) + 0_{\mathbb{R}} && \text{by the definition of } 0_{F(\mathbb{R})} \\ &= f(x) && \text{since } 0_{\mathbb{R}} \text{ is the additive identity in } \mathbb{R} \end{aligned}$$

We have shown that  $\forall x \in \mathbb{R}, (f + 0_{F(\mathbb{R})})(x) = f(x)$ . ■

- Prove that  $\forall f \in F(\mathbb{R}), \exists -f \in F(\mathbb{R})$  s.t.  $f + (-f) = 0_{F(\mathbb{R})}$ .

*Proof.* WTS  $\forall f \in F(\mathbb{R}), \exists -f \in F(\mathbb{R})$  s.t.  $f + (-f) = 0_{F(\mathbb{R})}$ .

Pick arbitrary  $f \in F(\mathbb{R})$ .

Let  $-f$  be the function

$$\begin{aligned} -f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto -f(x) \end{aligned}$$

$$\text{WTS } f + (-f) = \underset{F(\mathbb{R})}{0}.$$

$$\text{WTS } \forall x \in \mathbb{R}, (f + (-f))(x) = \underset{F(\mathbb{R})}{0}(x).$$

Pick arbitrary  $x \in \mathbb{R}$ .

$$\begin{aligned} (f + (-f))(x) &= f(x) + (-f)(x) && \text{by the definition of } + \text{ in } \mathbb{R} \\ &= f(x) + (-f(x)) && \text{by the definition of } -f \\ &= \underset{\mathbb{R}}{0} && \text{since } \underset{\mathbb{R}}{0} \text{ is the additive identity in } \mathbb{R} \\ &= \underset{F(\mathbb{R})}{0} && \text{since } \forall x \in \mathbb{R}, \underset{F(\mathbb{R})}{0}(x) = \underset{\mathbb{R}}{0} \end{aligned}$$

$$\text{Thus, } \forall f \in F(\mathbb{R}), \exists -f \in F(\mathbb{R}) \text{ s.t. } f + (-f) = \underset{F(\mathbb{R})}{0}. \quad \blacksquare$$

■

■

■ **Example 1.5** Prove  $M_{2 \times 3}(\mathbb{R}) \{[a_{ij}] \mid a_{ij} \in \mathbb{R}\}$  is a vector space.

Example:  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$ .

- $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$
- $r [a_{ij}] = [ra_{ij}]$
- Additive identity in  $M_{2 \times 3}(\mathbb{R})$  is  $\underset{2 \times 3}{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .
- Given  $A = [a_{ij}] \in M_{2 \times 3}(\mathbb{R})$ ,  $-A = [-a_{ij}]$  is the additive inverse of  $A$ .

■

**Proposition 1.1.6** Let  $V$  be a vector space. Then

- a) The zero vector  $\vec{0}$  (additive identity in  $V$ ) is unique.

*Proof.* WTS Additive identity in  $V$  is unique<sup>7</sup>.

Suppose  $\vec{0}$  and  $\vec{0}'$  are additive identity in  $V$ .

Since  $\vec{0}$  is additive identity,  $\forall \vec{v} \in V$ ,  $\vec{v} + \vec{0} = \vec{v}$ . In particular,  $\vec{0}' + \vec{0} = \vec{0}'$ .

Since  $\vec{0}'$  is additive identity,  $\forall \vec{v} \in V$ ,  $\vec{v} + \vec{0}' = \vec{v}$ . In particular,  $\vec{0} + \vec{0}' = \vec{0}'$ .

So  $\vec{0}' = \vec{0} + \vec{0}' = \vec{0}$ , so there is only one additive identity in  $V$ . ■

- b) For all  $\vec{v} \in V$ ,  $\underset{\mathbb{F}}{0}\vec{v} = \vec{0}_V$ .

*Proof.* WTS  $\forall \vec{v} \in V$ ,  $\underset{\mathbb{F}}{0}\vec{v} = \vec{0}_V$ .

---

<sup>7</sup>To prove that something is unique, a common technique is to assume we have two examples of the object in question, then show that those two examples must in fact be equal.

Pick  $\vec{v} \in V$ .

$$\begin{aligned} \underset{\mathbb{F}}{0\vec{v}} &= (\underset{\mathbb{F}}{0} + \underset{\mathbb{F}}{0})\vec{v} \\ &= \underset{\mathbb{F}}{0\vec{v}} + \underset{\mathbb{F}}{0\vec{v}} \\ \underset{\mathbb{F}}{(-0\vec{v})} + \underset{\mathbb{F}}{0\vec{v}} &= \underset{\mathbb{F}}{(-0\vec{v})} + (\underset{\mathbb{F}}{0\vec{v}} + \underset{\mathbb{F}}{0\vec{v}}) \\ \underset{\mathbb{F}}{-0\vec{v}} + \underset{\mathbb{F}}{0\vec{v}} &= \underset{\mathbb{F}}{(-0\vec{v})} + \underset{\mathbb{F}}{0\vec{v}} + \underset{\mathbb{F}}{0\vec{v}} \\ \vec{0}_V &= \vec{0}_V + \underset{\mathbb{F}}{0\vec{v}} \\ &= \underset{\mathbb{F}}{0\vec{v}} \end{aligned}$$

Thus,  $\forall \vec{v} \in V$ ,  $\underset{\mathbb{F}}{0\vec{v}} = \vec{0}_V$ . ■

c) For each  $\vec{v} \in V$ , the additive inverse  $-\vec{v}$  is unique.

*Proof.* We use the same idea as in the proof of part a. Given  $\vec{v} \in V$ , if  $-\vec{v}$  and  $(-\vec{v})'$  are two additive inverses of  $\vec{v}$ , then on one hand we have  $\vec{v} + -\vec{v} + (-\vec{v})' = (\vec{v} + -\vec{v}) + (-\vec{v})' = \vec{0} + (-\vec{v})' = (-\vec{v})'$ , by axioms 1, 4, and 3. On the other hand, if we use axiom 2 first before associating, we have  $\vec{v} + -\vec{v} + (-\vec{v})' = \vec{v} + (-\vec{v})' + -\vec{v} = (\vec{v} + (-\vec{v})') + -\vec{v} = \vec{0} + -\vec{v} = -\vec{v}$ . Hence  $-\vec{v} = (-\vec{v})'$ , and the additive inverse of  $\vec{v}$  is unique. ■

d) For all  $\vec{v} \in V$ , and all  $c \in \mathbb{R}$ ,  $(-c)\vec{v} = -(c\vec{v})$ .

*Proof.* We have  $c\vec{v} + (-c)\vec{v} = (c + -c)\vec{v} = 0\vec{v} = \vec{0}$  by axiom 6 and part b. Hence  $(-c)\vec{v}$  also serves as an additive inverse for the vector  $c\vec{v}$ . By part c, therefore, we must have  $(-c)\vec{v} = -(c\vec{v})$ . ■

### 1.3.2 Subspace

■ **Example 1.6** Let  $W \subseteq \mathbb{R}^3$ .

Let  $\vec{w}_1, \vec{w}_2 \in W$ .

Then,  $\vec{w}_1 + \vec{w}_2 \in W$ .

$$\begin{array}{ccc} + : & W \times W & \rightarrow W \\ & (\vec{w}_1, \vec{w}_2) & \mapsto \vec{w}_1 + \vec{w}_2 \end{array} \quad \cdot : \quad \begin{array}{ccc} \mathbb{R} \times W & \rightarrow & W \\ (r, \vec{w}) & \mapsto & r\vec{w} \end{array}$$

We can conclude that  $W$  is a subspace of  $\mathbb{R}^3$ . <sup>8</sup> ■

**Definition 1.2.6 — Subspace.** Let  $V$  be a vector space and let  $W \subseteq V$  be a *non-empty* subset. Then  $W$  is a (vector) *subspace* of  $V$  if  $W$  is a vector space itself under the operations of vector sum and scalar multiplication from  $V$ .

**Theorem 1.2.8 — Subset Test.** Let  $V$  be a vector space, and let  $W$  be a non-empty subset of  $V$ . Then  $W$  is a subspace of  $V$  if and only if for all  $\vec{v}, \vec{w} \in W$ , and all  $c \in \mathbb{R}$ . we have  $c\vec{v} + \vec{w} \in W$ .<sup>a</sup>

<sup>a</sup>This is equivalent to “ $W$  is closed under vector addition and under scalar multiplication of  $V$ ”.

<sup>8</sup>A short hand of “ $W$  is a subspace of  $V$ ” is “ $W \underset{S.S.}{\subseteq} V$ ”.

■ **Example 1.7** Is  $\mathbb{R}^2$  a subspace of  $\mathbb{R}^3$ ?

Note that  $\mathbb{R}^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$  but  $\mathbb{R}^3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in \mathbb{R} \right\}$ . Clearly,  $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$ . ■

■ **Example 1.8** Let  $\mathbb{F}$  be a field.

Define  $\mathbb{F}^3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in \mathbb{F} \right\}$ .

Let  $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{F}^3 \mid a_1 + a_2 + a_3 = 0_{\mathbb{F}} \right\}$ .

*Proof.* WTS  $W$  is a subspace of  $\mathbb{F}^3$ .

- Show that  $W$  is non-empty.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W, \text{ so } W \text{ is non-empty.}$$

- Show that  $W \subseteq \mathbb{F}^3$ .

Since  $\forall \vec{v} \in W, \vec{v} \in \mathbb{F}^3$ , we have  $W \subseteq \mathbb{F}^3$ .

- Show that  $W$  is closed under vector addition.

Let  $\vec{w}_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in W, a_i, b_i \in \mathbb{F}$ .

$$\text{Then, } \vec{w}_1 + \vec{w}_2 = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}.$$

Note that  $(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) = 0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}}$ .

Thus,  $\vec{w}_1 + \vec{w}_2 \in W$ .

That is,  $W$  is closed under vector addition.

- Show that  $W$  is closed under scalar multiplication.

Pick  $r \in \mathbb{F}$ , pick  $\vec{w} \in W$ .

$$r\vec{w} = r \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ra_1 \\ ra_2 \\ ra_3 \end{bmatrix}.$$

$$ra_1 + ra_2 + ra_3 = r(a_1 + a_2 + a_3) = r(0) = 0_{\mathbb{F}}.$$

Thus,  $r\vec{w} \in W$ .

Thus is,  $W$  is closed under scalar multiplication.

Thus,  $W \subseteq \mathbb{F}^3$ . ■

## 1.4 Linear Combinations

**Definition 1.3.1** Let  $S$  be a subset of a vector space  $V$ .

- A **linear combination** of vectors in  $S$  is any sum  $a_x \vec{v}_1 + \dots + a_n \vec{v}_n$ , where the  $a_i \in \mathbb{R}$ , and the  $\vec{v}_i \in S$ .
- If  $5 \neq \emptyset$  (the empty subset of  $V$ ), the set of all linear combinations of vectors in  $S$  is called the (linear) **span** of  $S$ , and denoted  $\text{Span}(S)$ . If  $S = \emptyset$ , we define

$\text{Span}(S) = \{\vec{0}\}$ .

- If  $W = \text{Span}(S)$ , we say  $S$  *spans* (or generates)  $W$ .
- We think of the span of a set  $S$  as the set of all vectors that can be “built up” from the vectors in  $S$  by forming linear combinations.

How do we describe  $\text{Span}(S)$  in set builder notation?

- If  $S$  is finite or infinite

$$\text{Span}(S) = \{a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n \mid a_i \in \mathbb{F}, n \in \mathbb{N}, \vec{x}_i \in S\}$$

- If  $S$  is finite

$$S = \{\vec{w}_1, \dots, \vec{w}_n\}$$

$$\text{Span}(S) = \{a_1\vec{w}_1 + \cdots + a_k\vec{w}_k \mid a_i \in \mathbb{F}, \vec{w}_i \in S\}$$

### ■ Example 1.9

Let  $S = \{\sin^2 x, \cos^2 x\} \subseteq F(\mathbb{R})$ .

- Describe  $\text{Span}(S)$ .

$$\text{Span}(S) = \{a_1 \sin^2 x + a_2 \cos^2 x \mid a_i \in \mathbb{R}\}$$

- True or False:  $\text{Span}(S)$  contains all constant functions

True.

Let  $f(x) = c$ ,  $c \in \mathbb{R}$ .

Then,  $f(x) = c \cdot 1$

$$= c(\sin^2 x + \cos^2 x)$$

$$= c \sin^2 x + c \cos^2 x \in \text{Span}(S)$$

Let  $S' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \subseteq M_{2 \times 2}(\mathbb{R})$ .

- Describe  $\text{Span}(S)$ .

$$\text{Span}(S) = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$$

- True or False:  $\text{Span}(S')$  contains no symmetric matrix.

False.

$$\text{Consider } 0_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

### ■ Example 1.10

Let  $\mathcal{P}(\mathbb{R}) :=$  the set of all polynomials with real coefficients.

- Give a spanning set for  $\mathcal{P}(\mathbb{R})$ .

$$S = \{x^i \mid i \in \mathbb{N} \cup \{0\}\}$$

- Can we find a finite spanning set for ?

No.

■

**Theorem 1.4.4** Let  $V$  be a vector space and let  $S$  be any subset of  $V$ . Then  $\text{Span}(S)$  is a subspace of  $V$ .

*Proof.*  $V$  is an  $\mathbb{F}$ -V.S.  $S \subseteq V$ . WTS  $\text{Span}(S) \subseteq_{S.S.} V$ .

- Show  $\text{Span}(S) \subseteq V$ .

Pick  $\vec{v} \in \text{Span}(S) = \{a_1\vec{v}_1 + \cdots + a_m\vec{v}_m \mid a_i \in \mathbb{F}, \vec{v}_i \in S, m \in \mathbb{N}\}$ .

Then,  $\vec{v} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m$  for some  $\vec{v}_1, \dots, \vec{v}_m \in S$  and  $a_1, \dots, a_m \in \mathbb{F}$ .

Since  $\vec{v}_i \in S \subseteq V$ ,  $\vec{v}$  is a linear combination of vectors in  $V$ , and  $V$  is a vector space. Thus,  $\vec{v} \in V$ .

- Show  $\text{Span}(S) \neq \emptyset$

- Show  $\text{Span}(S)$  is closed under vector addition.

- Show  $\text{Span}(S)$  is closed under scalar multiplication.

■

■ **Example 1.11**

Let  $S_1 = \{1, 2\sin^2 x, 3\cos^2 x\}$ ,  $S_2 = \{\sin^2 x, \cos^2 x\}$ .

$\overset{?}{\text{Span}}(S_1) = \overset{?}{\text{Span}}(S_2)$ ?

- $\text{Span}(S_1) \overset{?}{\subseteq} \text{Span}(S_2)$ ?

Yes.

$S_1 \subseteq \text{Span}(S_2)$ .

$\vec{v} = a \underset{\in \text{Span}(S_2)}{\cdot} \sin^2 x + b \underset{\in \text{Span}(S_2)}{\cdot} \cos^2 x \in \text{Span}(S_2)$  since  $\text{Span}(S_2) \underset{\text{S.S.}}{\subseteq} V$ .

- $\text{Span}(S_2) \overset{?}{\subseteq} \text{Span}(S_1)$ ?

Yes.

$S_2 \subseteq \text{Span}(S_1)$ .

$\vec{v} = a \underset{\in \text{Span}(S_1)}{\cdot} 1 + b \underset{\in \text{Span}(S_1)}{\cdot} \sin^2 x + c \underset{\in \text{Span}(S_1)}{\cdot} \cos^2 x \in \text{Span}(S_1)$  since  $\text{Span}(S_1) \underset{\text{S.S.}}{\subseteq} V$ .

■

## 1.5 Linear Dependence and Linear Independence

**Definition 1.4.2 — Linear Dependence.** Let  $V$  be a vector space, and let  $S$  be a subset of  $V$ .

- A *linear dependence*<sup>a</sup> among the vectors of  $S$  is an equation

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = 0$$

where the  $\vec{v}_i \in S$  and the  $a_i \in \mathbb{R}$  are not all zero (i.e., at least one of the  $a_i \neq 0$ ).

- the set  $S$  is said to be *linearly dependent* if there exists a linear dependence among the vectors in  $S$ .

<sup>a</sup>A linear dependence is often called a *non-trivial relation*.

■ **Example 1.12**

Is  $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq M_{2 \times 2}(\mathbb{R})$  linearly independent?

Yes.

$$0 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}_{2 \times 2}$$

■

■ **Example 1.13** Let  $V$  be a vector space over the field  $\mathbb{F}$ .

Is  $S' = \{\vec{0}, \vec{v}_1, \dots, \vec{v}_n\} \subseteq V$  linearly independent?

Yes.

$$1_{\mathbb{F}} \vec{0} + 0_{\mathbb{F}} \vec{v}_1 + \cdots + 0_{\mathbb{F}} \vec{v}_n = \vec{0}$$

■

**Definition 1.4.4 — Linearly Independent.** A subset  $S$  of a vector space  $V$  is *linearly independent* if whenever we have  $a_i \in \mathbb{R}$  and  $\vec{v}_i \in S$  such that  $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = 0$ , then  $a_i = 0$  for all  $i$ .

■ **Example 1.14** Is  $S = \{e^x, e^{2x}, e^{3x}\} \subseteq \mathcal{C}(\mathbb{R})$  linearly independent?

No.

Assume  $\forall x \in \mathbb{R}, a_1 e^x + a_2 e^{2x} + a_3 e^{3x} = \vec{0}$ .

In particular when  $x = 0$ ,  $a_1 + a_2 + a_3 = 0$ .

Taking the derivative on both sides.  $\forall x \in \mathbb{R}, a_1 e^x + 2a_2 e^{2x} + 3a_3 e^{3x} = \vec{0}$ .

Again, when  $x = 0$ ,  $a_1 + 2a_2 + 3a_3 = 0$ .

Take the derivative again,  $\forall x \in \mathbb{R}, a_1 e^x + 4a_2 e^{2x} + 9a_3 e^{3x} = \vec{0}$ .

When  $x = 0$ ,  $a_1 + 4a_2 + 9a_3 = 0$ .

Thus,  $a_1 = a_2 = a_3 = 0$ .

So  $S$  is linearly independent. ■

### Proposition 1.4.7

- Let  $S$  be a linearly dependent subset of a vector space  $V$ , and let  $S'$  be another subset of  $V$  that contains  $S$ . Then  $S'$  is also linearly dependent.
- Let  $S$  be a linearly independent subset of a vector space  $V$  and let  $S'$  be another subset of  $V$  that is contained in  $S$ . Then  $S'$  is also linearly independent.

■ **Example 1.15** Let  $S_1$  and  $S_2$  be linearly independent subsets of a vector space  $V$ .

- a) Is  $S_1 \cup S_2$  always linearly independent? Why or why not?

No.

$\{1\} \in \mathbb{R}$ ,  $\{2\} \in \mathbb{R}$  and they are linearly independent.

$\{1\} \cup \{2\} = \{1, 2\} \in \mathbb{R}$  and it is linearly dependent.

- b) Is always linearly independent? Why or why not?

Yes.

$S_1 \cap S_2 \subseteq S_1$  and  $S_1 \cap S_2 \subseteq S_2$ . Then by the proposition  $S_1 \cap S_2$  is linearly independent.

- c) Is always linearly independent? Why or why not?

Yes.

$S_1 \setminus S_2 \subseteq S_1$ . Then by the proposition  $S_1 \setminus S_2$  is linearly independent. ■

**Lemma 1.6.8** Let  $S$  be a linearly independent subset of  $V$  and let  $\vec{v} \in V$ , but  $\vec{v} \notin S$ . Then  $S \cup \{\vec{v}\}$  is linearly independent if and only if  $\vec{v} \notin \text{Span}(S)$ .

*Proof.* WTS  $S \cup \{\vec{v}\}$  is linearly independent if and only if  $\vec{v} \notin \text{Span}(S)$ .

- : Assume  $S \cup \{\vec{v}\}$  is linearly independent. WTS  $\vec{v} \notin \text{Span}(S)$ .

Proof by contradiction. Assume  $\vec{v} \in \text{Span}(S)$ .

Then,  $\exists r_1, \dots, r_m \in \mathbb{F}$ ,  $\vec{s}_1, \dots, \vec{s}_m \in S$  s.t.  $\vec{v} + r_1 \vec{s}_1 + \dots + r_m \vec{s}_m$ .

i.e  $\vec{v} - r_1 \vec{s}_1 - \dots - r_m \vec{s}_m = \vec{0}$ . This is a non-trivial relation among vectors in  $S \cup \{\vec{v}\}$ ,  $S \cup \{\vec{v}\}$  is linearly dependent.

Contradiction with the assumption.

Thus,  $\vec{v} \notin \text{Span}(S)$ .

- ←: Assume  $\vec{v} \notin \text{Span}(S)$ . WTS  $S \cup \{\vec{v}\}$  is linearly independent.

(Prove the contrapositive  $\neg(S \cup \{\vec{v}\}$  is linearly independent)  $\implies \neg(\vec{v} \notin \text{Span}(S))$ , that is, we prove that “ $S \cup \{\vec{v}\}$  is linearly dependent  $\implies \vec{v} \in \text{Span}(S)$ ”).

There exists a dependency relation among vectors in  $S \cup \{\vec{v}\}$ .

This relation should contain  $\vec{v}$ .

$$\exists r_1, \dots, r_m, a \in \mathbb{F} \text{ s.t. } r_1\vec{s}_1 + \dots + r_m\vec{s}_m + a\vec{v} = \vec{0}.$$

Thus,  $a\vec{v} = -r_1\vec{s}_1 - \dots - r_m\vec{s}_m$

$$\vec{v} = -\frac{r_1}{a}\vec{s}_1 - \dots - \frac{r_m}{a}\vec{s}_m$$

That is,  $x \in \text{Span}(S)$ .

Thus,  $S \cup \{\vec{v}\}$  is linearly independent if and only if  $\vec{v} \notin \text{Span}(S)$ . ■

## 1.6 Bases and Dimension

**Definition 1.6.1 — Basis.** A subset  $S$  of a vector space  $V$  is called a *basis* of  $V$  if  $V = \text{Span}(S)$  and  $S$  is *linearly independent*.

**Theorem 1.6.3** Let  $V$  be a vector space, and let  $S$  be a nonempty subset of  $V$ . Then  $S$  is a basis of  $V$  if and only if every vector  $\vec{v} \in V$  may be written uniquely as a linear combination of the vectors in  $S$ .

*Proof.* WTS ■

### ■ Example 1.16

$$\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{F} \right\}$$

Find a basis  $\mathcal{B}$  of  $\mathbb{F}^n$ .

$$\text{Note that } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \begin{bmatrix} 1_{\mathbb{F}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1_{\mathbb{F}} \\ \vdots \\ 0 \end{bmatrix} + \dots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1_{\mathbb{F}} \end{bmatrix}.$$

$$\vec{e}_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1_{\mathbb{F}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ with the } i\text{-th element } 1_{\mathbb{F}} \text{ and the rest 0.}$$

$\{\vec{e}_1, \dots, \vec{e}_n\}$  is linearly independent, and  $\text{Span}(\vec{e}_1, \dots, \vec{e}_n) = \mathbb{F}^n$ .

Then  $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$  is a basis for  $\mathbb{F}^n$ . ■

### ■ Example 1.17

Consider  $\mathbb{C}^2 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mid z_i \in \mathbb{C} \right\}$ .

- Suppose  $\mathbb{C}^2$  is a  $\mathbb{C}$ -V.S., what is  $\dim \mathbb{C}^2$ ?

$$- w \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} wz_1 \\ wz_2 \end{pmatrix}, w \in \mathbb{C}.$$

$$- \mathcal{E} = \left\{ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

- Suppose  $\mathbb{C}^2$  is a  $\mathbb{R}$ -V.S., what is  $\dim \mathbb{C}^2$ ?

- $r \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} rz_1 \\ rz_2 \end{pmatrix}, r \in \mathbb{R}.$
- $\mathcal{V} = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ i \end{pmatrix} \right\}$
- $z_1 = a_1 + b_1i, z_2 = a_2 + b_2i$ , then  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 = a_1\vec{v}_1 + a_2\vec{v}_2 + b_1\vec{v}_3 + b_2\vec{v}_4$

■

**Theorem 1.6.10** Let  $V$  be a *finitely generate* vector space and let  $S$  be a spanning set for  $V$ , which has  $m$  elements. Then no linearly independent set in  $V$  can have more than  $m$  elements.

**Corollary 1.6.11** Let  $V$  be a vector space and let  $S$  and  $S'$  be two bases of  $V$ , with  $m$  and  $m'$  elements, respectively. Then  $m = m'$ .

*Proof.* WTS if  $S$  and  $S'$  are two bases of  $V$ , with  $m$  and  $m'$  elements, respectively, then  $m = m'$ .

$S$  is a basis for  $V \xrightarrow{\text{by definition of basis}}$   $S$  is linearly independent.

$S'$  is a basis for  $V \xrightarrow{\text{by definition of basis}}$   $\text{Span}(S') = V$ .

Applying the theorem above,  $m = |S| \leq |S'| = m'$ .

$S$  is a basis for  $V \xrightarrow{\text{by definition of basis}}$   $\text{Span}(S) = V$ .

$S'$  is a basis for  $V \xrightarrow{\text{by definition of basis}}$   $S'$  is linearly independent.

Applying the theorem above,  $m' = |S'| \leq |S| = m$ .

So  $m \leq m'$  and  $m' \leq m$ . That is,  $m = m'$ .

■





## 2. Linear Transformations

### 2.1 Linear Transformations

**Definition 2.1.1 — Linear Transformation.** A function  $T : V \rightarrow W$  is called a *linear mapping* or a *linear transformation* if it satisfies

- i)  $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$  for all  $\vec{v}_1$  and  $\vec{v}_2 \in V$ .
- ii)  $T(r\vec{v}) = rT(\vec{v})$  for all  $r \in \mathbb{R}$  and  $\vec{v} \in V$ .

$V$  is called the *domain* of  $T$  and  $W$  is called the *target* of  $T$ .

■ **Example 2.1** Consider the following examples.

$$\begin{array}{lll} \text{id} : V \rightarrow V & R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ \vec{x} \mapsto \vec{x} & \vec{x} \mapsto \vec{x} \text{ rotated through } \theta \\ P_l : \mathbb{R}^2 \rightarrow \mathbb{R}^2 & T_M : \mathbb{R}^3 \rightarrow \mathbb{R} \\ \vec{x} \mapsto \text{proj}_l \vec{x} & \vec{x} \mapsto [1 \ 2 \ 2] \vec{x} \end{array}$$

These are all commonly seen linear transformations. ■

Given a  $m \times n$  matrix  $A$ ,  $T_A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation.  
$$\vec{x} \mapsto A\vec{x}$$

Conversely for every linear transformation  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $\exists n \times n$  matrix  $M_T$  s.t.  
 $T(\vec{v}) = M_T \vec{v}$ .

Given  $V, W$  be  $\mathbb{F}$ -V.S., and a linear transformation  $T : V \rightarrow W$ .

Can we describe  $T$  using matrices?

■ **Example 2.2** Which function is a linear transformation?

- a) Let  $\mathcal{C}^\infty(\mathbb{R})$  be the space of smooth functions

$$\begin{array}{lll} D : \mathcal{C}^\infty(\mathbb{R}) & \rightarrow & \mathcal{C}^\infty(\mathbb{R}) \\ f & \mapsto & f' \end{array}$$

b) Let  $\mathcal{C}[0, 1]$  be the space of continuous functions on  $[0, 1] \rightarrow \mathbb{R}$ .

$$\begin{array}{rccc} I : & \mathcal{C}[0, 1] & \rightarrow & \mathbb{R} \\ & f & \mapsto & \int_0^1 f(t) dt \end{array}$$

- a) is a linear transformation.

Let  $f, g \in \mathcal{C}^\infty(\mathbb{R})$ ,  $r \in \mathbb{R}$ .

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

$$D(rf) = (rf)' = rf' = rD(f).$$

- b is also a linear transformation.

Let  $f, g \in \mathcal{C}[0, 1]$ ,  $r \in \mathbb{R}$ .

$$I(f + g) = \int_0^1 (f + g)(t) dt = \int_0^1 f(t) + g(t) dt = \int_0^1 f(t) dt + \int_0^1 g(t) dt = I(f) + I(g).$$

$$I(rf) = \int_0^1 (rf)(t) dt = r \int_0^1 f(t) dt = rI(f).$$

■

### Properties of a Linear Transformation

- Linear transformations  $\vec{0}$  to  $\vec{0}$ .
- Linear transformations preserve linear combinations

**Lemma 2.0.1** Let  $T : V \rightarrow W$  be a linear transformation. Then  $T(\vec{0}_V) = \vec{0}_W$ .

*Proof.* WTS  $T(\vec{0}_V) = \vec{0}_W$ .

$$\begin{aligned} T(\vec{0}_V) &= T(\vec{0}_V + \vec{0}_V) && \text{since } \vec{0}_V \text{ is the additive identity in } V \\ &= T(\vec{0}_V) + T(\vec{0}_V) && \text{by linearity of } T \\ T(\vec{0}_V) + (-T(\vec{0}_V)) &= (T(\vec{0}_V) + T(\vec{0}_V)) + (-T(\vec{0}_V)) && \text{since the additive inverse is unique} \\ \vec{0}_W &= T(\vec{0}_V) + (T(\vec{0}_V) + (-T(\vec{0}_V))) && \text{since addition is associative} \\ &= T(\vec{0}_V) + \vec{0}_W \\ &= T(\vec{0}_V) && \text{since } \vec{0}_W \text{ is the additive identity in } W \end{aligned}$$

Thus,  $T(\vec{0}_V) = \vec{0}_W$ . ■

**Proposition 2.1.2** A function  $T : V \rightarrow W$  is a linear transformation if and only if for all  $a$  and  $b \in \mathbb{F}$  and all  $\vec{u}$  and  $\vec{v} \in V$

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

*Proof.* WTS a function  $T : V \rightarrow W$  is a linear transformation if and only if for all  $a$  and  $b \in \mathbb{F}$  and all  $\vec{u}$  and  $\vec{v} \in V$ ,  $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ .

- $\rightarrow$ : WTS  $T : V \rightarrow W$  is a linear transformation  $\implies T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ .

$$\begin{aligned} T(a\vec{u} + b\vec{v}) &= T(a\vec{u}) + T(b\vec{v}) && \text{by (1) of the definition of L.T.} \\ &= aT(\vec{u}) + bT(\vec{v}) && \text{by (2) of the definition of L.T.} \end{aligned}$$

- $\leftarrow$ : WTS  $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v}) \implies T : V \rightarrow W$  is a linear transformation. ■

**Corollary 2.1.3** A function  $T : V \rightarrow W$  is a linear transformation if and only if for all

$a_1, \dots, a_k \in \mathbb{F}$  and for all  $\vec{v}_1, \dots, \vec{v}_k \in V$ :

$$T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i)$$

*Proof.* WTS  $T : V \rightarrow W$  is a linear transformation if and only if for all  $a_1, \dots, a_k \in \mathbb{F}$  and for all  $\vec{v}_1, \dots, \vec{v}_k \in V$ ,  $T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i)$ .

- $\rightarrow$ : WTS  $T : V \rightarrow W$  is a linear transformation  $\implies T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i)$ .

Prove by induction.

- **Base case:** WTS  $T(a_1 \vec{v}_1) = a_1 T(\vec{v}_1)$

This follows immediately from (2) of the definition of linear transformation.

- **Induction hypothesis:**  $T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i)$ .

$$\text{WTS } T\left(\sum_{i=1}^{k+1} a_i \vec{v}_i\right) = \sum_{i=1}^{k+1} a_i T(\vec{v}_i).$$

By (1) of the definition of linear transformation,  $T\left(\sum_{i=1}^{k+1} a_i \vec{v}_i\right) = T\left(\sum_{i=1}^k a_i \vec{v}_i\right) +$

$$T(a_{k+1} \vec{v}_{k+1}).$$

$T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i)$  by the induction hypothesis.

$T(a_{k+1} \vec{v}_{k+1}) = a_{k+1} T(\vec{v}_{k+1})$  by (2) of the definition of linear transformation.

$$\begin{aligned} \text{So } T\left(\sum_{i=1}^{k+1} a_i \vec{v}_i\right) &= T\left(\sum_{i=1}^k a_i \vec{v}_i\right) + T(a_{k+1} \vec{v}_{k+1}) \\ &= \sum_{i=1}^k a_i T(\vec{v}_i) + a_{k+1} T(\vec{v}_{k+1}) \\ &= \sum_{i=1}^{k+1} a_i T(\vec{v}_i) \end{aligned}$$

$$\text{Thus, } T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i).$$

- $\leftarrow$ : WTS  $T\left(\sum_{i=1}^k a_i \vec{v}_i\right) = \sum_{i=1}^k a_i T(\vec{v}_i) \implies T : V \rightarrow W$  is a linear transformation.

■

**Proposition 2.1.14** If  $T : V \rightarrow W$  is a linear transformation and  $V$  is finite-dimensional, then  $T$  is uniquely determined by its value on the members of a basis of  $V$ .

**Definition 2.0.2** Let  $A, B$  be sets.

Let  $f : A \rightarrow B$  be a function.

- $f$  is **injective** (one to one) if  $\forall a_1, a_2 \in A$ ,  $f(a_1) = f(a_2) \implies a_1 = a_2$ .

- $f$  is **surjective** (onto) if  $\forall b \in B, \exists a \in A$  s.t.  $f(a) = b$ .
- $f$  is **bijective** if  $f$  is both injective and surjective.

## 2.2 Kernel and Image

**Definition 2.3.1 — Kernel.** The **kernel**<sup>a</sup> of  $T$ , denoted  $\ker(T)$ , is the subset of  $V$  consisting of all vectors  $\vec{v} \in V$  such that  $T(\vec{v}) = \vec{0}$ .

$$\ker T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$$

<sup>a</sup>Kernel is equivalent to the null space.

**Definition 2.3.10 — Image.** The subset of  $W$  consisting of all vectors  $\vec{w} \in W$  for which there exists a  $\vec{v} \in V$  such that  $T(\vec{v}) = \vec{w}$  is called the **image** of  $T$  and is denoted by  $\text{Im}(T)$ .

$$\text{Im } T = \{\vec{w} \in W \mid \text{there exists } \vec{v} \in V \text{ such that } T(\vec{v}) = \vec{w}\}$$

**Proposition 2.0.1** Let  $T$  be a linear transformation.  $T$  is injective if and only if  $\ker T = \{\vec{0}\}$ .

*Proof.* WTS  $T$  is injective iff  $\ker T = \{\vec{0}\}$ .

- $\rightarrow$ : Assume  $T$  is injective. WTS  $\ker T = \{\vec{0}\}$ .  
Pick  $\vec{v} \in T$ .  
Suppose  $T(\vec{v}) = \vec{0}_W$ .

Then,  $T(\vec{v}) = \vec{0}_W = T(\vec{0})$ .

Since  $T$  is injective,  $\vec{v} = \vec{0}$ .

Thus,  $\forall \vec{v} \in \ker T, \vec{v} = \vec{0}$ .

That is,  $\ker T = \vec{0}$ .

- $\leftarrow$ : Assume  $\ker T = \{\vec{0}\}$ . WTS  $T$  is injective.  
WTS  $T(\vec{v}_1) = T(\vec{v}_2) \implies \vec{v}_1 = \vec{v}_2$ .  
Pick  $\vec{v}_1, \vec{v}_2 \in V$ .  
Suppose  $T(\vec{v}_1) = T(\vec{v}_2)$ .

Then,  $T(\vec{v}_1 - \vec{v}_2) = \vec{0}_W$ .

By linearity of  $T$ ,  $\vec{0}_W = T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$ .

Then,  $\vec{v}_1 - \vec{v}_2 \in \ker T = \{\vec{0}_V\}$ .

Thus,  $\vec{v}_1 - \vec{v}_2 = \vec{0}$ .

That is,  $\vec{v}_1 = \vec{v}_2$ .

Thus,  $T$  is injective iff  $\ker T = \{\vec{0}\}$ . ■

**Proposition 2.0.2** Let  $T$  be a linear transformation.  $T$  is surjective if and only if  $\text{Im } T = W$ .

■ **Example 2.3** Let  $D : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R})$  be a linear transformation.

- Describe:  
–  $\ker D$ .

$$\begin{aligned} \ker D &= \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid D(f) = \vec{0}\} \\ &= \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid f' = \vec{0}\} \\ &= \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid f(x) = c \text{ for some } c \in \mathbb{R}\} \end{aligned}$$

- $\text{Im } D$ .

$$\begin{aligned}\text{Im } D &= \{g \in \mathcal{C}^\infty(\mathbb{R}) \mid g = D(f) \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R})\} \\ &= \{g \in \mathcal{C}^\infty(\mathbb{R}) \mid g = f' \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R})\} \\ &= \mathcal{C}^\infty(\mathbb{R})\end{aligned}$$

- Is  $D$ :

- injective?

No.

The kernel is NOT trivial.

- surjective?

Yes.

$\text{Im } g = \mathcal{C}^\infty(\mathbb{R})$  (the image is equal to the codomain). ■

■ **Example 2.4** Let  $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$  be a linear transformation.

$$\begin{array}{ccc} p(x) & \mapsto & xp(x) \end{array}$$

- Describe:

- $\ker D$ .

$$\begin{aligned}\text{Ker } T &= \{p(x) \in P_2(x) \mid xp(x) = 0\} \\ &= \{\vec{0}\}\end{aligned}$$

- $\text{Im } D$ .

$$\text{Im } T = \{xp(x) \mid p(x) \in P_2(\mathbb{R})\}$$

- Is  $D$ :

- injective?

Yes.

The kernel of  $T$  is trivial.

- surjective?

No.

Take  $p(x) = 2 \in P_2(\mathbb{R})$ .  $2 \notin \text{Im } T$ . ■

**Proposition 2.3.1** Let  $T : V \rightarrow W$  be a linear transformation.  $\ker(T)$  is a subspace of  $V$ .

*Proof.* We will use subspace test.

- Show that  $\ker T \subseteq V$ .

This follows immediately from the definition of kernel.

- Show that  $\ker T \neq \emptyset$ .

Since  $T(\vec{0}) = \vec{0}$ ,  $\vec{0} \in \ker T$ . So  $\ker T \neq \emptyset$ .

- Show that  $\ker T$  is closed under addition.

Pick  $\vec{v}_1, \vec{v}_2 \in \ker T$ .

$$\begin{aligned}T(\vec{v}_1 + \vec{v}_2) &= T(\vec{v}_1) + T(\vec{v}_2) && \text{by linearity of } T \\ &= \vec{0}_W + \vec{0}_W && \text{since } \vec{v}_1, \vec{v}_2 \in \ker T \\ &= \vec{0}_W\end{aligned}$$

- Show that  $\ker T$  is closed under scalar multiplication.

Pick  $\vec{v} \in V$ ,  $r \in \mathbb{F}$ .

$$\begin{aligned} T(r\vec{v}) &= rT(\vec{v}) && \text{by linearity of } T \\ &= r\vec{0}_W && \text{since } \vec{v} \in \ker T \\ &= \vec{0}_W \end{aligned}$$

Thus,  $\ker T \subseteq_{S.S.} V$ . ■

■ **Example 2.5** True or false.

- Let  $T$  be a linear transformation and  $\dim \ker T = 0$ . Then  $T$  is injective.  
True.
- Let  $T$  be a linear transformation and  $\dim \text{Im } T = \dim W$ . Then  $T$  is surjective.  
True
  - $\text{img } T \subseteq_{S.S.} W \xrightarrow{?} \text{img } T = W$
  - Assume  $\dim(\text{img } T) = \dim W$ .
  - Let  $\mathcal{B} = (\vec{b}_1, \dots, \vec{b}_k)$  be a basis for  $\text{img } T \implies \mathcal{B}$  is linearly independent in  $W$ .
  - $\mathcal{B}$  is a basis for  $W$ , and  $\mathcal{B}$  has  $k$  elements.■

■ **Example 2.6**

$$\text{Let } A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

$$A \sim \underbrace{\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & -8 & -3 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{REF} \sim \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{RREF}.$$

$$T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

$$\vec{x} \mapsto A\vec{x}$$

1. Find a basis for  $\underbrace{\ker T_A}_{\text{Nul}(A)}$  and  $\dim(\ker T_A)$ .

$$\ker T_A = \text{Nul } A = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\}.$$

Solution to  $A\vec{x} = \vec{0}$ .

From RREF of :  $x_3$  is a free variable.

- $x_1 = \frac{1}{2}x_3$
- $x_2 = -\frac{3}{8}x_3$
- $x_4 = 0$

$$\vec{x} = x_3 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{8} \\ 1 \\ 0 \end{pmatrix}, x_3 \in \mathbb{R}.$$

$$\ker T_A = \text{Nul } A = \text{Span} \left( \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{8} \\ 1 \\ 0 \end{pmatrix} \right).$$

That is, a basis for  $\ker T_A = \left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{8} \\ 1 \\ 0 \end{pmatrix} \right\}$ ,  $\dim(\ker T_A) = 1$ .

2. Find a basis for  $\underbrace{\text{img } T_A}_{\text{Col}(A)}$  and  $\dim(\text{img } T_A)$ .

$$\begin{aligned} \text{img } T_A &= \text{Cal}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^4\} \\ &= \text{Span} \left( \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right) \end{aligned}$$

Take  $x_3 = 1$ :

$$\begin{aligned} A \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{8} \\ 1 \\ 0 \end{pmatrix} &= \vec{0} \\ \vec{a}_3 &= -\frac{1}{2}\vec{a}_1 + \frac{3}{8}\vec{a}_2 \end{aligned}$$

Thus, a basis for  $\text{Im } T_A = \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$ .  $\dim(\text{img } T_A) = 3$ .

3. Find the relationship between the rank and nullity of  $T_A$ .

$$\dim \text{dom } A = \dim \text{img } T_A + \dim \ker T_A$$

■

**Theorem 2.3.17 — Rank-Nullity Theorem.** If  $V$  is a **finite-dimensional** vector space and  $T : V \rightarrow W$  is a linear transformation, then

$$\dim V = \dim \text{img } T + \dim \ker T$$

*Proof.* Proof: WTS  $\dim V = \dim \text{img } T + \dim \ker T$ .

Consider a basis  $\vec{u}_1, \dots, \vec{u}_m$  for  $\ker T$ .

$\vec{u}_1, \dots, \vec{u}_m$  is linearly independent in  $V$ , thus it can be extended to a basis for  $V$ .

Suppose  $\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_n$  is a basis for  $V$  we get.

Note that  $\dim \ker T = m$ , and  $\dim V = m + n$ .

WTS  $\dim \text{img } T = n$ .

Take  $\vec{w} \in \text{img } T$ .

Then,  $\exists \vec{v} \in V$  s.t.  $T(\vec{v}) = \vec{w}$ .

$$\begin{aligned} \vec{w} &= T(\vec{v}) \\ &= T(a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_n\vec{v}_n) \text{ for some } a_i, b_i \in \mathbb{F} \\ &= a_1T(\vec{u}_1) + \dots + a_mT(\vec{u}_m) + b_1T(\vec{v}_1) + \dots + b_nT(\vec{v}_n) \end{aligned}$$

Since  $\vec{v}_1, \dots, \vec{v}_n$  is a basis for  $\ker T$ ,  $a_i T(\vec{u}_i) = \vec{0}$ .

$$\vec{w} = b_1 T(\vec{v}_1) + \cdots + b_n T(\vec{v}_n)$$

Thus,  $\text{img } T = \text{Span}(T(\vec{v}_1), \dots, T(\vec{v}_n))$ .

Show that  $T(\vec{v}_1), \dots, T(\vec{v}_n)$  is linearly independent.  
 $c_1 T(\vec{v}_1), \dots, c_n T(\vec{v}_n) = \vec{0}$ , WTS  $c_i = 0$ .

$$\begin{aligned} T(\underbrace{c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n}_{\in \ker T}) &= \vec{0} \\ c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n &= r_1 \vec{u}_1 + \cdots + r_m \vec{u}_m \text{ for some } r_i \in \mathbb{F} \\ c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n - r_1 \vec{u}_1 - \cdots - r_m \vec{u}_m &= \vec{0} \end{aligned}$$

$\{\vec{v}_i, \vec{u}_j \mid i \leq u \leq m, 1 \leq j \leq n\}$  is a basis for  $V$ .

Thus,  $c_1 = \cdots = c_n = r_1 = \cdots = r_n = 0$ .

So  $T(\vec{v}_1), \dots, T(\vec{v}_n)$  is linearly independent, and so it is a basis for  $V$ .  
 $\dim \text{img } T = n$ . ■

### 2.3 The Inverse of a Linear Transformation

### 2.4 Change of Basis



## 3. The Spectral Theorem in $\mathbb{R}^n$

### 3.1 Diagonalizability

#### 3.1.1 Eigenvectors, Eigenvalues and Eigenspaces

**Definition 4.1.2 — Eigenvector and Eigenvalue.** Let  $V$  be a vector space over the field  $\mathbb{F}$ .

Let  $T : V \rightarrow V$  be a linear mapping.

a) A vector  $\vec{v} \in V$  is called an *eigenvector of  $T$*  if  $\vec{v} \neq \vec{0}$  and there exists a scalar  $\lambda \in \mathbb{R}$  such that  $T(\vec{v}) = \lambda\vec{v}$ .

b) If  $\vec{v}$  is an eigenvector of  $T$  and  $T(\vec{v}) = \lambda\vec{v}$ , the scalar  $\lambda$  is called the *eigenvalue of  $T$  corresponding to  $\vec{v}$* .

**Definition 4.1.6 — Eigenspace.**  $\forall \lambda \in \mathbb{F}$ , the  *$\lambda$ -eigenspace of  $T$* , denoted  $E_\lambda$ , is the set

$$E_\lambda = \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\} = \{\text{all eigenvectors of } \lambda\} \cup \{\vec{0}\}$$

That is,  $E_k$  is the set containing all the eigenvectors of  $T$  with eigenvalue  $\lambda$ , together with the vector  $\vec{0}$ . (If  $\lambda$  is not an eigenvalue of  $T$ , then we have  $E_\lambda = \vec{0}$ .)

■ **Example 3.1**

Consider the linear transformation  $D : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R})$ .

$$\begin{matrix} f & \mapsto & f' \end{matrix}$$

- $1$  is a eigenvector with a eigenvalue of  $0$ .
- $e^x$  is an eigenvector is with a eigenvalue of  $1$ .
- $e^{Mx}$  is an eigenvector with the eigenvalue of  $M$ .

■ **Example 3.2** Define the plane  $W : x + y + z = 0$ .

Consider the linear transformation  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\vec{x} \mapsto \vec{x} \text{ reflected w.r.t. } W$$

$\forall \vec{x} \in W$ ,  $R(\vec{v}) = 1\vec{v}$ , so  $\lambda = 1$  is an eigenvalue, and  $E_1 = W$ .

$R(\vec{w}) = -\vec{w}$  for all  $\vec{w} \perp W$ , so  $\lambda = -1$  is an eigenvalue, and  $E_{-1} = \text{Sp}(\vec{w})$ .

**Proposition 4.1.7**  $E_\lambda$  is a subspace of  $V$  for all  $\lambda$ .

*Proof.* •  $T(\vec{0}) = \vec{0} = \lambda\vec{0} \implies \vec{x} \in E_\lambda$

- Pick  $\vec{v}, \vec{w} \in E_\lambda$ .

WTS  $\vec{v} + \vec{w} \in E_\lambda$ .

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \lambda\vec{v} + \lambda\vec{w} = \lambda(\vec{v} + \vec{w}) \in E_\lambda$$

- Pick  $\vec{v} \in E_\lambda, r \in \mathbb{F}$ .

WTS  $T(r\vec{v}) = rT(\vec{v})$ .

$$T(r\vec{v}) = \lambda(r\vec{v}) = r(\lambda\vec{v}) = rT(\vec{v}) \in E_\lambda.$$

■

### Eigenbasis

Consider the same linear transformation  $R$  discussed in example 2.

$$R(\vec{x}) = A\vec{x}$$

$$A = [R]_{\mathcal{E}} = \begin{bmatrix} | & | & | \\ R(\vec{e}_1) & R(\vec{e}_2) & R(\vec{e}_3) \\ | & | & | \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Eigenbasis: linearly independent and spans  $\mathbb{R}^3$ .

$$\text{For example: } \mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{w}} \right\}$$

$$\text{Then, } [R]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [R(\vec{v}_1)]_{\mathcal{B}} & [R(\vec{v}_2)]_{\mathcal{B}} & [R(\vec{w})]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[R(\vec{v}_1)]_{\mathcal{B}} = [\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[R(\vec{v}_2)]_{\mathcal{B}} = [\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[R(\vec{w})]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Let  $T : V \rightarrow V$  be a linear transformation.

$\vec{v} \in V$  is an eigenvector if and only if  $T(\vec{v}) = \lambda\vec{v}$ , by the definition.

$$T(\vec{v}) - \lambda\vec{v} = \vec{0}$$

$$T(\vec{v}) - \lambda id(\vec{v}) = \vec{0}$$

$$(T - \lambda id)\vec{v} = 0$$

by sum of linear transformations

$$\vec{v} \in \ker(T - \lambda id)$$

for some  $\lambda \in \mathbb{F}$

**Proposition 4.1.5** A vector  $\vec{v}$  is an eigenvector of  $T$  with eigenvalue  $\lambda$  if and only if  $\vec{v} \neq \vec{0}$  and  $\vec{v} \in \ker(T - \lambda id)$ .

### ■ Example 3.3

Let  $\dim V = n$ , and fix a basis  $\mathcal{B}$  for  $V$ .

$$\text{Then, } \mathcal{L}(V, V) \xrightarrow{\cong} M_{n \times n}(\mathbb{F})$$

$$T \mapsto [T]_{\mathcal{B}} = B$$

Suppose  $\vec{v}$  is an eigenvector of  $T$ .

$T(\vec{v}) - \lambda\vec{v}$  means that  $[T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} = [\lambda\vec{v}]_{\mathcal{B}} = \lambda[\vec{v}]_{\mathcal{B}}$ .

$$\begin{aligned}[T - \lambda id]_{\mathcal{B}} &= [T]_{\mathcal{B}} - \lambda[id]_{\mathcal{B}} \\ &= B - \lambda I_{n \times n}\end{aligned}$$

$\lambda$  is an eigenvalue for  $T$  iff  $B - \lambda I$  is not invertible.

$\lambda$  is an eigenvalue for  $T$  iff  $\det(B - \lambda I) = 0$ .

This is called that **characteristic polynomial** of the linear transformation  $T$ . ■

**Definition 4.1.11 — Characteristic Polynomial.** Let  $A \in M_{n \times n}(\mathbb{R})$ . The polynomial  $\det(A - \lambda I)$  is called the *characteristic polynomial* of  $A$ .

■ **Example 3.4**

$$\begin{array}{rcl} T : \mathcal{P}_3(\mathbb{R}) & \rightarrow & P_3(\mathbb{R}) \\ p & \mapsto & p' + 2p \end{array}$$

1. Find the characteristic polynomial of  $T$ .

$$\text{Pick the basis } \mathcal{B} = (1, x, x^2, x^3). \text{ Then, } B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$\text{char}(T) = \det(B - \lambda I) = (\lambda - 2)^4 \quad ^1$$

**Note:**  $\text{char}(T)$  is well defined (does NOT depend on the choice of basis).

2. Find all eigenvalues and the corresponding eigenspace of  $T$ .

$$E_2 = \text{nul}(B - \lambda I) = \text{sp} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim(E_2) = 1$  is the *geometric multiplicity* of  $\lambda = 2$ . ■

## 3.2 Diagonalizability

**Definition 4.2.1 — Diagonalizable.** Let  $V$  be a finite-dimensional vector space, and let  $T : V \rightarrow V$  be a linear mapping.  $T$  is said to be *diagonalizable* if there exists a basis of  $V$ , all of whose vectors are eigenvectors of  $T$ .

**Proposition 4.2.2**  $T : V \rightarrow V$  is diagonalizable if and only if, for any basis  $\alpha$  of  $V$ , the matrix  $[T]_{\alpha}^{\alpha}$  is similar to a diagonal matrix.

*Proof.*

- $\rightarrow$ : WTS  $T$  is diagonalizable  $\implies \exists$  a basis of  $\mathcal{B}$  s.t. is diagonal.

$T$  has an eigenbasis  $\mathcal{B} = (\vec{b}_1, \dots, \vec{b}_n)$ .

Note that  $T(\vec{b}_i) = \lambda_i \vec{b}_i$  for some  $\lambda_i \in \mathbb{F}$ .

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [T(\vec{b}_1)]_{\mathcal{B}} & \dots & [T(\vec{b}_n)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [\lambda_1 \vec{b}_1]_{\mathcal{B}} & \dots & [\lambda_n \vec{b}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

<sup>1</sup>Note that the power of 4 means the *algebraic multiplicity* of  $\lambda = 2$  is 4.

- $\leftarrow$ : WTS  $\exists$  a basis  $\mathcal{B}$  of  $T$  s.t.  $[T]_{\mathcal{B}}$  is diagonal  $\implies T$  is diagonalizable.

$$\exists \text{ a basis } \mathcal{B} = (\vec{b}_1, \dots, \vec{b}_n) \text{ s.t. } [T]_{\mathcal{B}} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | \\ [T(\vec{b}_1)]_{\mathcal{B}} & \dots & [T(\vec{b}_n)]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$$

$$\forall i, [T(\vec{b}_i)]_{\mathcal{B}} = a_i \vec{e}_i$$

$$[a_i \vec{b}_i]_{\mathcal{B}} = a_i \vec{e}_i$$

So  $T(\vec{b}_i) = a_i \vec{b}_i$ , since  $\gamma_{\mathcal{B}}$  is injective. ■

**Proposition 3.0.1**  $T : V \rightarrow V$  is diagonalizable iff  $\exists$  a basis  $\mathcal{B}$  of  $V$  s.t.  $[T]_{\mathcal{B}}$  is diagonal

**Theorem 4.2.7** (Computational Criteria)

Let  $V$  be a vector space,  $\dim V = n$ .

Let  $T : V \rightarrow V$  be a linear transformation.

Let  $\lambda_1, \dots, \lambda_n$  be **distinct** eigenvalues of  $T$ .

Let  $m_i$  be the algebraic multiplicity of  $\lambda_i$ .

Let  $M_i$  be the geometric multiplicity of  $\lambda_i$ .

$T$  is diagonalizable if and only if

1.  $\sum_{i=1}^k m_i = n$
2.  $\forall i, M_i = m_i$ .

- $\text{char}(T) = (x - \lambda_i)^{m_i} g(x)$
- $\dim(E_{\lambda_i}) = M_i$

■ **Example 3.5** Let  $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$  be a linear transformation.

Let  $\mathcal{A} = (x^2, x, 1)$  be a basis for  $\mathcal{P}_2(\mathbb{R})$ .

$$A = [T]_{\mathcal{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{char}(T) = \det(A - \lambda I) = (2 - \lambda^2)(1 - \lambda)$$

$$E_2 = \text{Span} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$E_1 = \text{Span} \left( \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right)$$

■

$$\text{Nul}(A - 2I) = \text{Span} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

- For  $\lambda = 2$ , the algebraic multiplicity is 2.
- For  $\lambda = 1$ , the algebraic multiplicity is 1.

$$2 + 1 = 3$$

The geometric multiplicity of  $\lambda = 2$  is  $\dim \ker(T - 2\text{id}) = \dim E_2$   
 $\dim(\text{Nul}(A - 2I)) = 2$

- The geometric multiplicity of  $\lambda = 2$  is 2.
- The geometric multiplicity of  $\lambda = 1$  is 1.

$$\text{Nul}(A - 2I) = \text{Span} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) \underset{\text{S.S.}}{\subseteq} \mathbb{R}^3$$

$$\ker(T - 2\text{id}) = E_2$$

$$\begin{array}{ccc} E_2 & \overset{\subseteq}{\underset{\text{S.S.}}{\subseteq}} & \mathcal{P}_2(\mathbb{R}) \\ \cong & & \downarrow \lambda_A \\ \text{Nul}(A - 2I) & \overset{\subseteq}{\underset{\text{S.S.}}{\subseteq}} & \mathbb{R}^3 \end{array}$$

$$\dim \text{Nul}(A - 2I) = 2$$

$$\text{Nul}(A - 2I) \cong E_2$$

$$\dim E_2 = \dim \text{Nul}(A - 2I)$$

Find an eigenbasis  $\mathcal{B}$  for  $T$ .

**Note:** We need the eigenbasis for  $\mathcal{P}_2(\mathbb{R})$ , not  $\mathbb{R}^3$ !

$\{x, -x^2 + 1, -x + 1\}$  is an eigenbasis for  $T$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $E_{\lambda_1} = \{x_1, \dots, x_{l_1}\}$
- $E_{\lambda_2} = \{x_1, \dots, x_{l_2}\}$
- $E_{\lambda_k} = \{x_1, \dots, x_{l_1}\}$

$E_{\lambda_1} \cup \dots \cup E_{\lambda_k}$  is an eigenbasis for  $T$ .

**Proposition 4.2.6** Let  $V$  be finite-dimensional, and let  $T : V \rightarrow V$  be linear.

Let  $A$  be an eigenvalue of  $T$ , and assume that  $A$  is an  $m$ -fold root of the characteristic polynomial of  $T$ . Then we have

$$1 \leq \dim(E_\lambda) \leq m$$

**Proposition 4.2.4** Let  $\vec{v}_1, \dots, \vec{v}_n$  be eigenvectors corresponding to distinct eigenvalues of  $T : V \rightarrow V$ . Then  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is linearly independent.

*Proof.* We know that  $T(\vec{v}_i) = \lambda_i \vec{v}_i$ ,  $1 \leq i \leq k$  and  $\lambda_i \neq \lambda_j$  for  $i \neq j$ .

WTS  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is linearly independent.

### Rough Work

$k = 1$ ,  $0 \neq \vec{v}_1$  is linearly independent.

$k = 2$ ,  $\vec{v}_1, \vec{v}_2$ ,  $T(\vec{v}_1) = T(\alpha \vec{v}_2) = \alpha T(\vec{v}_1) = \alpha \lambda_2 \vec{v}_2 = \lambda_2 \alpha \vec{v}_2 = \lambda_2 \vec{v}_1$ . Contradiction, because the  $\lambda$ 's should be distinct.

Induction on  $k$

- **Base case:**  $k = 1$

$0 \neq \vec{v}_1$  is linearly independent.

- **Induction hypothesis:** suppose  $\vec{v}_1, \dots, \vec{v}_k$  is linearly independent.

Prove that  $\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}$  is linearly independent.

Suppose  $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} = \vec{0}$ . (\*)

WTS  $a_1 = \dots = a_k = a_{k+1} = 0$ .

Apply  $T$  to (\*):

$$\begin{aligned} a_1T(\vec{v}_1) + a_kT(\vec{v}_k) + a_{k+1}T(\vec{v}_{k+1}) &= \vec{0} \\ a_1\lambda_1\vec{v}_1 + a_k\lambda_k\vec{v}_k + a_{k+1}\lambda_{k+1}\vec{v}_{k+1} & \end{aligned} \quad (1)$$

Multiply (\*) by  $\lambda_{k+1}$  (if  $\lambda_{k_i} = 0$ , switch with  $\lambda_k$ ):

$$a_1\lambda_{k+1}\vec{v}_1 + \dots + a_k\lambda_{k+1}\vec{v}_k + a_{k+1}\lambda_{k+1}\vec{v}_{k+1} \quad (2)$$

$$1 - 2: a_1(\lambda_1 - \lambda_{k+1})\vec{v}_1 + \dots + a_k(\lambda_k - \lambda_{k+1})\vec{v}_k = \vec{0}$$

We have  $a_1 = \dots = a_k = 0$ .

Plug back into (\*):

$a_{k+1} = 0$ . So  $\vec{v}_1, \dots, \vec{v}_{k+1}$  are linearly independent. ■

### 3.3 Geometry in $\mathbb{R}^n$

Let  $\mathbb{R}^n = \begin{pmatrix} r_1 \\ r_1 \\ \vdots \\ r_n \end{pmatrix}$ .

We know that  $V \xrightarrow{\gamma_B} \mathbb{F}^n$ ,  $v \mapsto [v]_B$  and  $T \mapsto [T]_B$   
In  $\mathbb{R}^3$  and  $\mathbb{R}^3$ , we have these concepts intuitively

- Norm
- Distance
- Angle

In  $\mathbb{R}^n$ ,

- $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$
- $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$
- $\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$
- $\theta = \cos^{-1} \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2 \|\vec{w}\|^2}$

**Definition 4.3.1** The *standard inner product* (or *dot product*) on  $\mathbb{R}^n$  is the function  $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by the following rule. If  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{y} = (y_1, \dots, y_n)$  in standard coordinates, then  $\langle \vec{x}, \vec{y} \rangle$  is the scalar:  $\langle \vec{x}, \vec{y} \rangle = x_1y_1 + \dots + x_ny_n$ .

**Definition 3.0.1 — Inner Product.**

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\rightarrow \mathbb{F} \\ (\vec{v}, \vec{w}) &\mapsto \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

An *inner product* on  $V$  is a function that takes each ordered pair of  $(u, v)$  of elements of  $V$  to a number  $\langle u, v \rangle \in \mathbb{F}$  and has the following properties:

a) **Positivity** $\langle v, v \rangle \geq 0$  for all  $v \in V$ ;b) **Definiteness** $\langle v, v \rangle = 0$  if and only if  $v = 0$ c) **Additivity in the first slot** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$  for all  $u, v, w \in V$ ;d) **Homogeneity in the first slot** $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$  for all  $u, v \in V$ e) **Conjugate symmetry** $\langle u, v \rangle = \langle v, u \rangle$  for all  $u, v \in V$ .**Proposition 4.3.3**

- a) For all  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$  and all  $c \in \mathbb{R}$ ,  $\langle c\vec{x} + \vec{y}, \vec{z} \rangle = c\langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ . (In other words, as a function of the first variable, the inner product is a *linear* mapping.)
- b) For all  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$  (i.e., the inner product is “*symmetric*”).
- c) For all  $\vec{x} \in \mathbb{R}^n$ ,  $\langle \vec{x}, \vec{x} \rangle \geq 0$ , and  $\langle \vec{x}, \vec{x} \rangle = 0$  if and only if  $\vec{x} = \vec{0}$ . (We say the standard inner product is “*positive-definite*”.)

*Proof.* Many of the simpler facts about the standard inner product may be derived by expanding out the expressions involved, using Definition (4.3.1).

- a) Write  $\vec{x} = (x_1, \dots, x_n)$ ,  $\vec{y} = (y_1, \dots, y_n)$ , and  $\vec{z} = (z_1, \dots, z_n)$ . Then  $\langle c\vec{x} + \vec{y} \rangle = (cx_1 + y_1, \dots, x_n + y_n)$ , so we have

$$\langle \vec{x} + \vec{y}, \vec{z} \rangle = (cx_1 + y_1)z_1 + \dots + (cx_n + y_n)z_n$$

After multiplying this out and rearranging the terms, we obtain

$$\langle \vec{x} + \vec{y}, \vec{z} \rangle = c(x_1z_1 + \dots + x_nz_n) + \dots + (y_1z_1 + \dots + y_nz_n) = c\langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$$

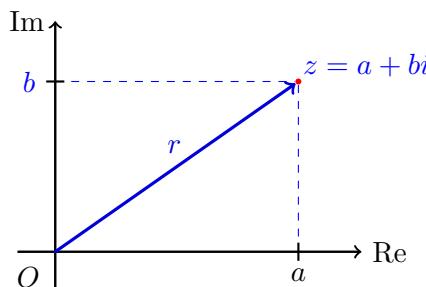
- b) We have:

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= x_1y_1 + \dots + x_ny_n \\ &= y_1x_1 + \dots + y_nz_n \\ &= \langle \vec{y}, \vec{x} \rangle. \end{aligned}$$

- c) By the definition,  $\langle \vec{x}, \vec{x} \rangle = x_1^2 + \dots + x_n^2$ . Since the components of  $\vec{x}$  are all real numbers, each  $x_i^2 \geq 0$ , so  $\langle \vec{x}, \vec{x} \rangle \geq 0$ . Furthermore, if  $\langle \vec{x}, \vec{x} \rangle = 0$ , then it is clear that  $x_i = 0$  for all  $i$ , or in other words, that  $\vec{x} = \vec{0}$ . ■

Note that  $\mathbb{C} \cong \mathbb{R}^2$

Let  $z \in \mathbb{C}$ ,  $z = a + bi$



$$\begin{aligned}\|z\| &= \sqrt{a^2 + b^2} \\ \bar{z} &= a - bi \\ a^2 + b^2 &= z\bar{z} \\ \|z\|^2 &= a^2 + b^2 = z\bar{z} \\ \|\vec{z}\|^2 &= z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n = \langle \vec{z}, \vec{z} \rangle_{\mathbb{C}}\end{aligned}$$

■ **Example 3.6** The following are commonly seen examples of inner products.

- dot product in  $\mathbb{R}^n$   
 $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ny_n$
- Euclidian inner product (or Hermitian inner product)

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\langle \vec{z}, \vec{x} \rangle = z_1\bar{x}_1 + \cdots + z_n\bar{x}_n$$

■

**Definition 5.3.1 — Hermitian Inner Product.** Let  $V$  be a complex vector space. A *Hermitian inner product* on  $V$  is a complex valued function on pairs of vectors in  $V$ , denoted by  $\langle \vec{u}, \vec{v} \rangle \in \mathbb{C}$  for  $\vec{u}, \vec{v} \in V$ , which satisfies the following properties:

- For all  $\vec{u}, \vec{v}$ , and  $\vec{w} \in V$  and  $a, b \in \mathbb{C}$ ,  $\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle$ ,
- For all  $\vec{u}, \vec{v} \in V$ ,  $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ , and
- For all  $\vec{v} \in V$ ,  $\langle \vec{v}, \vec{v} \rangle \geq 0$ , and  $\langle \vec{v}, \vec{v} \rangle = 0$  implies  $\vec{v} = \vec{0}$ .

We call a complex vector space with a Hermitian inner product a *Hermitian inner product space*.

$$\langle z, w \rangle_{\mathbb{C}} \stackrel{?}{=} \overline{\langle w, z \rangle} \quad ^2$$

$$\langle \vec{z}, \vec{w} \rangle_{\mathbb{C}} = \sum_{i=1}^n z_i \bar{w}_i = \sum_{i=1}^n \overline{w_i} z_i = \overline{\sum_{i=1}^n \overline{w_i} z_i} = \overline{\sum_{i=1}^n \overline{w_i} z_i} = \overline{\sum_{i=1}^n w_i \bar{z}_i} = \overline{\langle \vec{w}, \vec{z} \rangle_{\mathbb{C}}}$$

**Definition 3.0.2 — Inner Product Space.** An *inner product space* is a vector space  $V$  along with an inner product on  $V$ .

**Definition 4.3.3 — Norm.**

The *length* (or *norm*) of  $\vec{v} \in \mathbb{R}^n$  is the scalar

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

- b)  $\vec{v}$  is called a *unit vector* if  $\|\vec{v}\| = 1$ .

**Definition 4.3.5** The angle,  $\theta$ , between two nonzero vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$  is defined to be

$$\theta = \cos^{-1} \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \cdot \|\vec{y}\|}.$$

**Definition 4.3.7 — Orthogonal.** Two vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$  are said to be *orthogonal* (or *perpendicular*) if  $\langle \vec{x}, \vec{y} \rangle = 0$ .

■ **Example 3.7** Define  $V = \mathcal{C}^0[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} | f \text{ continuous}\}$

---


$${}^2\overline{w+z} = \overline{w} + \overline{z}.$$

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx$$

Show  $\langle \cdot, \cdot \rangle$  is an inner product on  $V$ .

- Show that  $\langle \cdot, \cdot \rangle$  has positivity

$$\langle f, f \rangle = \int_0^1 f(x) dx \geq 0. \langle \cdot, \cdot \rangle \text{ is positive}$$

- Show that  $\langle \cdot, \cdot \rangle$  has definiteness

$$\int_0^1 f^2(x) dx = 0 \text{ if and only if } f(x) = 0. \langle \cdot, \cdot \rangle \text{ is definite}$$

- Show that  $\langle \cdot, \cdot \rangle$  has additivity and homogeneity in the first slot

$$\begin{aligned} \langle rf + g, h \rangle &= \int_0^1 (rf + g)(x)h(x) dx \\ &= \int_0^1 (rf(x) + g(x))h(x) dx \\ &= \int_0^1 rf(x)h(x) + g(x)h(x) dx \\ &= r \int_0^1 f(x)h(x) dx + \int_0^1 g(x)h(x) dx \\ &= r\langle f, h \rangle + \langle g, h \rangle \end{aligned}$$

- Show that  $\langle \cdot, \cdot \rangle$  has conjugate symmetry

$$\begin{aligned} \langle g, h \rangle &= \int_0^1 g(x)h(x) dx \\ &= \int_0^1 \overline{h(x)g(x)} dx \\ &= \overline{\langle h, g \rangle} \end{aligned}$$

■

Note that inner products have the following properties:

- $\langle \vec{0}, \vec{u} \rangle = 0$  for every  $\vec{u} \in V$ .
- $\langle \vec{u}, \vec{0} \rangle = 0$  for every  $\vec{u} \in V$
- $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$  for all  $\vec{u}, \vec{v}, \vec{w} \in V$
- $\langle \vec{u}, \lambda \vec{v} \rangle = \bar{\lambda} \langle \vec{u}, \vec{v} \rangle$  for all  $\lambda \in \mathbb{F}$  and  $\vec{u}, \vec{v} \in V$ .

Let  $\vec{v} \in V$

- $\|\vec{v}\| = 0$  if and only if  $\vec{v} = \vec{0}$ .
- $\|\lambda \vec{v}\| = |\lambda| \|\vec{v}\|$  for all  $\lambda \in \mathbb{F}$ .

### Definition 3.0.3 — Trace.

Let  $A \in M_{n \times n}(\mathbb{F})$ . The *trace* of  $A$ , denoted  $\text{tr}(A)$ , is defined by

$$\text{tr}(A) = \sum_{i=1}^{\infty} a_{ii}, A_{n \times n} = a_{ii}$$

(the sum of the diagonal entries).

- $\text{tr}(A + B) = \text{tr}A + \text{tr}B$
- $\text{tr}(AB) = \text{tr}(BA)$

**■ Example 3.8**

Let  $V = M_{2 \times 2}(\mathbb{R})$  be an inner product space.

Define  $\langle \cdot, \cdot \rangle_V : V \times V \rightarrow \mathbb{R}$   
 $(A, B) \mapsto \frac{1}{2} \text{tr}(A^T B)$ .

Let  $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ . What is  $\|A\|$ ?

$$\begin{aligned} \|A\|^2 &= \langle A, A \rangle \\ &= \frac{1}{2} \text{tr}(A^T A) \\ &= \frac{1}{2}(2) \\ &= 1 \end{aligned}$$

So  $\|A\| = \sqrt{1} = 1$ . ■

**Definition 4.3.9 — Orthogonal (set) and Orthonormal.**

- a) A set of vectors  $S \subset \mathbb{R}^n$  is said to be *orthogonal* if for every pair of vectors  $\vec{x}, \vec{y} \in S$  with  $\vec{x} \neq \vec{y}$ , we have  $\langle \vec{x}, \vec{y} \rangle = 0$ .
- b) A set of vectors  $S \subset \mathbb{R}^n$  is said to be *orthonormal* if  $S$  is orthogonal and, in addition, every vector in  $S$  is a unit vector.

**Pythagorean Theorem**

Suppose  $u$  and  $v$  are orthogonal vectors in  $V$ . Then

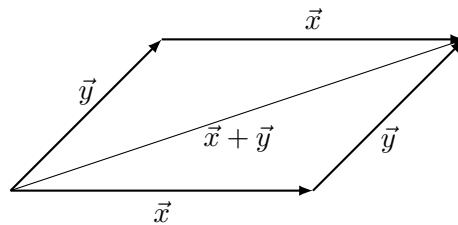
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

*Proof.* WTS  $\vec{u}, \vec{v} \in V$  s.t.  $\langle \vec{u}, \vec{v} \rangle = 0$ . Then  $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ .

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} + \vec{v} \rangle + \langle \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \underbrace{\langle \vec{u}, \vec{u} \rangle}_{0} + \underbrace{\langle \vec{u}, \vec{v} \rangle}_{0} + \underbrace{\langle \vec{v}, \vec{u} \rangle}_{0} + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$
■

**Proposition 4.3.4 — The Triangle Inequality and The Cauchy-Schwarz Inequality.**

- a) (The triangle inequality) For all vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ .  
 (Geometrically, in a triangle, the length of any side is less than or equal to the sum of the lengths of the other two sides. )



b) (The Cauchy-Schwarz inequality<sup>3</sup>) For all vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,

$$|\langle \vec{x}, \vec{x} \rangle| \leq \|\vec{x}\| \cdot \|\vec{y}\|.$$

$$\langle \vec{u}_i, \vec{u}_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

■ **Example 3.9** Let  $\mathbb{R}^2$  with the dot product be an inner product space.

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$  is a basis for  $\mathbb{R}^2$ .

$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$  is an orthogonal basis for  $\mathbb{R}^2$ .

$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$  is an orthonormal basis for  $\mathbb{R}^2$ . ■

■ **Example 3.10** Example of Inner Product Space

$W \subseteq V$ ,  $\dim W < \infty$

$$W^\perp = \{ \vec{v} \in V | \langle \vec{v}, \vec{w} \rangle = 0 \ \forall \vec{w} \in W \}$$

$$V^\perp = ?$$

- $\langle \vec{0}, \vec{v} \rangle = 0$ , so  $\vec{0} \in V^\perp$
- $\vec{v} \in V^\perp$ :  $\langle \vec{v}, \vec{w} \rangle = 0 \ \forall \vec{w} \in V$   
 $\langle \vec{v}, \vec{v} \rangle = 0 \implies \vec{v} = \vec{0}$   
So  $V^\perp = \{\vec{0}\}$

$$\{\vec{0}\}^\perp = ?$$

- $\{\vec{0}\}^\perp = \{ \vec{v} \in V | \langle \vec{v}, \vec{0} \rangle = 0 \} = V$  ■

### Proposition 4.4.3 — Orthogonal Decomposition.

- a) For every subspace  $W$  of  $\mathbb{R}^n$ ,  $W^\perp$  is also a subspace of  $\mathbb{R}^n$ .
- b) We have  $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n) = n$ .
- c) For all subspaces  $W$  of  $\mathbb{R}^n$ ,  $W \cap W^\perp = \{\vec{0}\}$ .
- d) Given a subspace  $W$  of  $\mathbb{R}^n$ , every vector  $\vec{v} \in \mathbb{R}^n$  can be written **uniquely** as  $\vec{v} = \vec{v}_1 + \vec{v}_2$ , where  $\vec{v}_1 \in W$  and  $\vec{v}_2 \in W^\perp$ . In other words,  $\mathbb{R}^n = W \oplus W^\perp$ .

## 3.4 Orthogonal Projection and the Gram-Schmidt Process

### Definition 3.0.4 — Orthogonal Projection.

Let  $V$  be an inner product space.

The orthogonal projection of the vector  $\vec{v} \in V$  onto the subspace  $W \subseteq V$ , denoted  $\text{proj}_W(\vec{v})$ , is the mapping

$$\begin{aligned} \text{proj}_W(\vec{v}) &: V \rightarrow V \\ \vec{v} &\mapsto \vec{v}_1 \end{aligned}$$

where  $\vec{v} = \vec{v}_1 + \vec{v}_2$ , with  $\vec{v}_1 \in W$  and  $\vec{v}_2 \in W^\perp$ .

<sup>3</sup>This inequality is an equality if and only if one of  $u, v$  is a scalar multiple of the other.

**Proposition 4.4.5**

- a)  $\text{proj}_W$  is a linear mapping.
- b)  $\text{im}(\text{proj}_W) = W$ , and if  $\vec{w} \in W$ , then  $\text{proj}_W(\vec{w}) = \vec{w}$ .
- c)  $\ker(\text{proj}_W) = W^\perp$ .

**Proposition 4.4.6** Let  $\{\vec{w}_1, \dots, \vec{w}_n\}$  be an orthonormal basis for the subspace  $W \subseteq V$ .

- a) For each  $w \in W$ , we have

$$\vec{w} = \sum_{i=1}^n \langle \vec{w}, \vec{w}_i \rangle \vec{w}_i$$

- b) For all  $\vec{v} \in V$ , we have

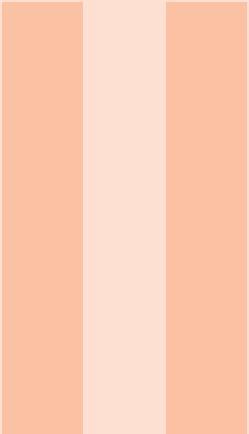
$$\text{proj}_W(\vec{v}) = \sum_{i=1}^n \langle \vec{v}, \vec{w}_i \rangle \vec{w}_i$$

*Proof.* (idea) say  $\{\vec{u}_1, \vec{u}_2\}$  ( $\dim V = 2$ ),  $\vec{v} \in V$

$$\vec{v} = r_1 \vec{u}_1 + r_2 \vec{u}_2, \quad r_i \in \mathbb{F}.$$

$$\begin{aligned} \langle \vec{v}, \vec{u}_1 \rangle &= \langle r_1 \vec{u}_1 + r_2 \vec{u}_2, \vec{u}_1 \rangle \\ &= r_1 \langle \vec{u}_1, \vec{u}_1 \rangle + r_2 \langle \vec{u}_2, \vec{u}_1 \rangle \\ &= r_1 \cdot 1 + r_2 \cdot 0 \\ &= r_1 \end{aligned}$$

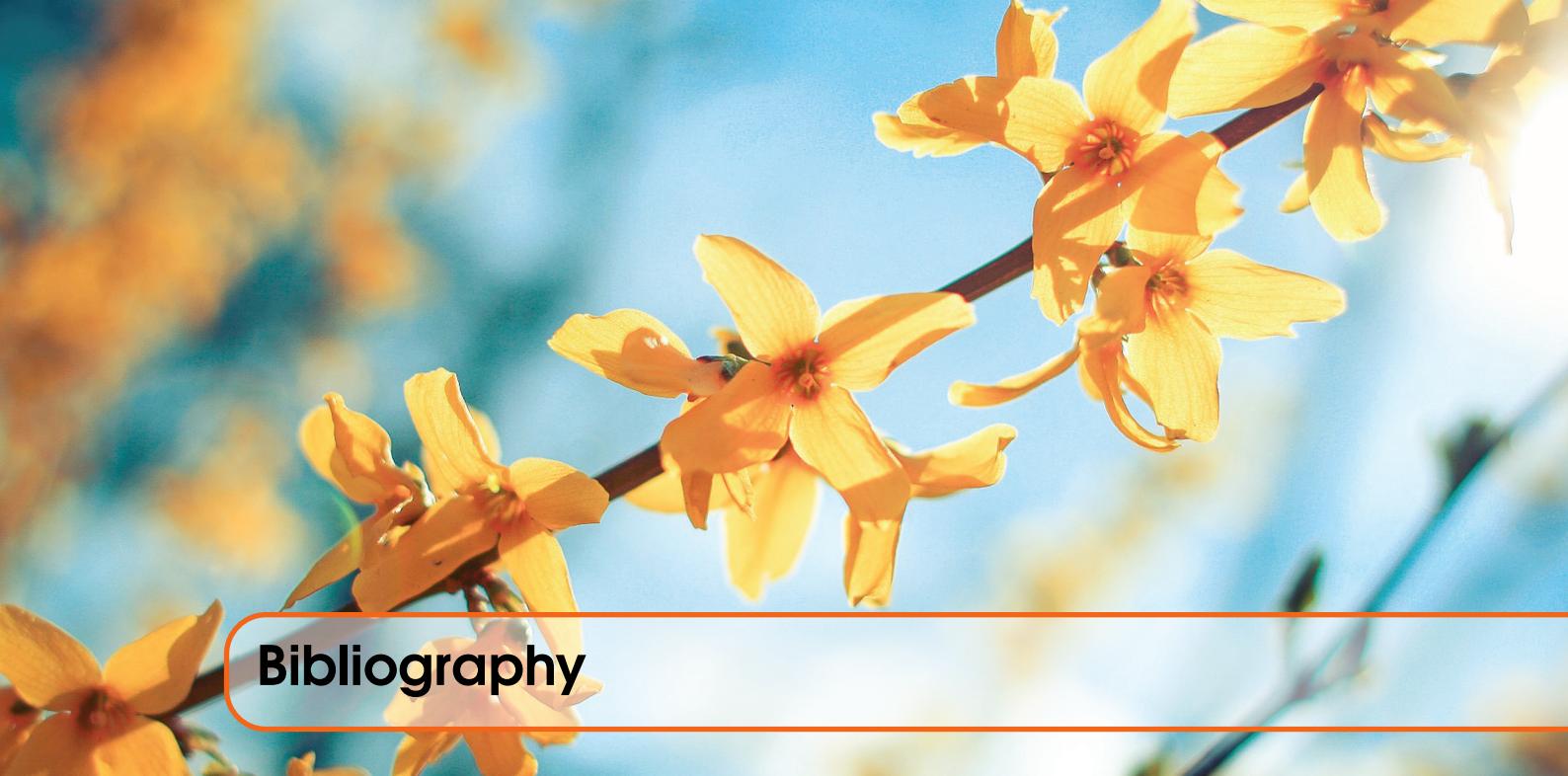
■



## Part Two

Bibliography .....	47
Books	
Index .....	49





## Bibliography

### Books

- [DL11] David B. Diamiano and John B. Little. *A Course in Linear Algebra*. Dover edition. 2011 (cited on page 8).





# Index

## A

Addition ..... 11

## B

Basis ..... 22  
Bijective ..... 28

## C

Complex Numbers ..... 11

## D

Dimension Theorem ..... 31  
Domain ..... 25  
Dot Product ..... 38

## E

Empty Set of a Vector Space ..... 18

## F

Field ..... 11  
Fundamental Theorem of Linear Algebra 31

## H

Hermitian Inner Product ..... 40  
Hermitian Inner Product Space ..... 40

## I

Image ..... 28  
Injective ..... 28  
Inner Product Space ..... 40

## K

Kernel ..... 28

## L

Linear Combination ..... 18  
Linear Dependence ..... 20  
Linear Independence ..... 20  
Linear Mapping ..... 25  
Linear Transformation ..... 25

## M

Multiplication ..... 11

## N

Norm ..... 40

## O

Orthogonal ..... 40  
Orthogonal (set) ..... 42  
Orthogonal Decomposition ..... 43

Orthogonal Projection ..... 43  
Orthonormal ..... 42

**P**

Perpendicular ..... 40  
Product ..... 11  
Pythagorean Theorem ..... 42

**R**

Rank-Nullity Theorem ..... 31

**S**

Span ..... 18  
Standard Inner Product ..... 38  
Subspace ..... 17  
Sum ..... 11  
Surjective ..... 28

**T**

Target ..... 25  
The Cauchy-Schwarz Inequality ..... 42  
The Triangle Inequality ..... 42  
Trace ..... 41

**U**

Unit Vector ..... 40

**V**

Vector Space over a Field ..... 13