Notation[chapter]

[dummy]Theorem Problem[chapter] Exercise[chapter] Example[chapter] Vocabulary[chapter] Definition[section] [dummy]Corollary [dummy]Proposition

[dummy]Lemma

Recall

$N : \mathbb{C}^4 \to \mathbb{C}^4$ is a nilpotent map with the dignatiure $(1, 2, 3, 4)$.

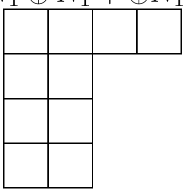
$$[N]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{0} \leftarrow \vec{b}_1 \leftarrow \vec{b}_2 \leftarrow \vec{b}_3 \leftarrow \vec{b}_4$

If $\dim \ker T^2 = 2$, $\dim \ker T^3 = \dim \ker T^4 = 4$, then $[T]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = N_2 \oplus N_2$

Let $T\mathbb{C}^{10} \to \mathbb{C}^{10}$

If $\dim \ker T = 4$, $\dim \ker T^2 = 8$ and $\dim \ker T^3 = 9 and \dim \ker T^4 = 10$, then $[T]_{\mathcal{B}} = N_4 \oplus N_1 \oplus N_1 + \oplus N_1$



■ **Example 0.1 — Single Block.** $T : \mathbb{C}^4 \to \mathbb{C}^4$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

- The eigenvalue is λ with the algebraic multiplicity of 4.
- The geometric multiplicity of λ is 1.

Thus, $T = \lambda \mathrm{id} : \mathbb{C}^4 \to \mathbb{C}^4 : [T - \lambda \mathrm{id}] = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

$T(\vec{b}_1) = \lambda \vec{b}_1$

$T(\vec{b}_i) = \lambda \vec{b}_i + \vec{b}_i$

$[T - \lambda \mathrm{id}]$ has a cycle of $\vec{b}_1 \leftarrow \vec{b}_2 \leftarrow \vec{b}_3 \leftarrow \vec{b}_4$ ■

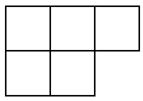
■ **Example 0.2 — Multiple Block.** $T : \mathbb{C}^8 \to \mathbb{C}^8$

$$\exists \text{ a basis } \mathcal{B} \text{ s.t. } [T]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} -i & 1 & \\ & -i & 1 \\ & & -i \end{bmatrix} & & & \\ & \begin{bmatrix} -i & i \\ & -i \end{bmatrix} & & \\ & & \begin{bmatrix} 3 & 1 \\ & 3 \end{bmatrix} & \\ & & & [2] \end{bmatrix}$$ ■

- The eigenvalues are
  - $-i$ with an algebraic multiplicity of 5
  - 3 with an algebraic multiplicity of 2
  - 2 with an algebraic multiplicity of 1
- For $\lambda = -i$

$$[T-(-i)\mathrm{id}] = \begin{bmatrix} \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} & & \\ & & \begin{bmatrix} * & 1 \\ & * \end{bmatrix} & \\ & & & [*] \end{bmatrix}$$

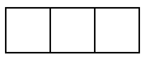
  - the geomrtric mutiplicity of $\lambda = -i$ is 2.
    $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_8)$
    $\vec{0} \leftarrow \vec{b}_1 \leftarrow \vec{b}_2 \leftarrow \vec{b}_3$
    $\vec{0} \leftarrow \vec{b}_4 \leftarrow \vec{b}_5$



  - The generalized eigenspace[1] , $G_i = \mathrm{Sp}(\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5)$
- For $\lambda = 3$

$$[T-3\mathrm{id}]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} * & 1 & \\ & * & 1 \\ & & * \end{bmatrix} & & & \\ & \begin{bmatrix} * & 1 \\ & * \end{bmatrix} & & \\ & & \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} & \\ & & & [*] \end{bmatrix}$$

  - $\vec{0} \leftarrow \vec{b}_6 \leftarrow \vec{b}_7$
  - $G_3 = \mathrm{Sp}(\vec{b}_6, \vec{b}_7)$



- For $\lambda = 2$

$$[T-2\mathrm{id}]_{\mathcal{B}} = [T-3\mathrm{id}]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} * & 1 & \\ & * & 1 \\ & & * \end{bmatrix} & & & \\ & \begin{bmatrix} * & 1 \\ & * \end{bmatrix} & & \\ & & \begin{bmatrix} * & 1 \\ & * \end{bmatrix} & \\ & & & [0] \end{bmatrix}$$

  - $G_2 = \mathrm{Sp}(\vec{b}_8) = E_2$.

■ Example 0.3

Let $\begin{array}{rccl} T: & \mathbb{C}^4 & \to & \mathbb{C}^4 \\ & \vec{x} & \mapsto & A\vec{x} \end{array}$ be a linear transformation.

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[1] $G_\lambda = \ker(T - \lambda \mathrm{id})^l$, $l \geq 1$

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$.

Given the eigenvalue $\lambda_1 = 2$ with an algebraic multiplicity of 3, and the eigenvalue $\lambda_2 = 3$ with an algebraic multiplicity of 1.

Given $\ker(T - 1\mathrm{id}) = mathrmSp(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix})$, $\ker(T - 1\mathrm{id})^2 = \mathrm{Sp}\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$.

Given $\ker T - 2\mathrm{id} = \mathrm{Sp}\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$ [2], and $\ker T - 2\mathrm{id})^2 = \{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$

- $\vec{0} \leftarrow \vec{b}_1 \leftarrow \vec{b}_2$
  Pick a $\vec{b}_2$, compute its image and that will be $\vec{b}_1$.
- $\vec{0} \leftarrow \vec{b}_3$

■

[2] This tells us that we have two Jordan blocks with eigenvalues of 2