Linear Algebra II

Course Notes

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0. Introduction

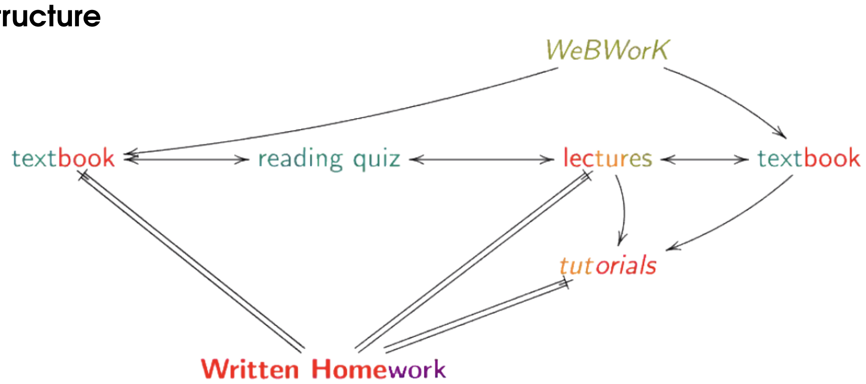
0.1 Introduction

Fields, complex numbers, vector spaces over a field, linear transformations, matrix of a linear transformation, kernel, range, dimension theorem, isomorphisms, change of basis, eigenvalues, eigenvectors, diagonalizability, real and complex inner products, spectral theorem, adjoint/self-adjoint/normal linear operators, triangular form, nilpotent mappings, Jordan canonical form.

0.1.1 Learning Outcome

- Read and understand new mathematical ideas and concepts on your own
- Communicate mathematical ideas of linear algebra clearly in words and in writing (proofs)
- Connect abstract knowledge to examples
- Approach challenging problems independently
- Use linear algebra as a computational tool

0.1.2 Course Structure



Our course has multiple components. Each component is built carefully to support several aspects of the learning outcomes of our course as listed in the syllabus. This page explains how different components of the course interact with one another and helps you navigate the course

Reading Quizzes

A typical week in our course starts by doing the reading assigned for that week and ends by revisiting the reading assignment for that week. Assigned readings are mainly from our primary textbook followed by a Quercus quiz on your assigned readings for the week and that of the previous week. You should start your reading as soon as they are posted.

You should read all the assigned readings for the week and submit your quiz on Quercus. The weekly quiz is due at 10 am every Monday. The best strategy is to spread out the reading throughout the week. You are not expected to understand everything when you read the textbook before the class. Your learning will happen gradually during lectures, tutorials, and mostly when you do the suggested problems from the textbook and the written homework. You will need to review the textbook again after lectures. Reading Quizzes quizzes makeup 5% of your final grade.

Lectures

What to expect from lectures?

During lectures, your instructor will guide you through important concepts and invite you to participate in lecture activities. Lectures complement your reading, tutorials, and homework. Lectures are not meant to replace textbook reading. There will be concepts and theorems that are only discussed in the textbook or only during your lectures, or only in homework. You are encouraged to attend all lectures. While in your lecture you are expected to actively participate in all activities prompted by your instructor. While the delivery of the course is online, your instructor may record the lectures and make it available for you to watch asynchronously. Note that watching the lectures do not replace attending them as you will miss the in-class activities. You are asked to do a lecture reflection quiz at the end of your lecture week. The reflection due date is a day after your last lecture of the week. Reflections are a tool for your instructor to emphasize important concepts in the lectures and to collect feedback from you, and an opportunity for you to reflect on your understanding of the lecture material. Reflections are part of your participation grade as per our syllabus.

Tutorials

You must be enrolled in a tutorial. You will have weekly tutorial sessions starting on Jan 24th. During tutorials, you will work with your classmates in small groups on tutorial worksheets. Tutorial worksheets are carefully designed to be discussed in groups. Tutorial worksheets are roughly one week behind the lectures.

Ideally, you want to read, understand, and think about all the questions before your tutorial. This involves checking all the definitions and statements of the results you need to know to tackle questions. Set aside 15 minutes before your tutorial to go over questions. Your tutorial sessions are facilitated by your TAs. During your tutorial, your TA will ask you to sit with your tutorial group. You will work with your groupmates on selected problems from the worksheet. Each group works on a shared document that will be checked by the TA during the tutorial. Do not expect your TA to work out the questions for you or to teach the concepts. Your TA will give you feedback on what you already put down in your shared document. That is the more you work among your group the more your TA can help you. Your TA might choose to go over some of the questions for the entire class or answer your questions within your groups. You get the most out of your tutorials if you work on the problems ahead of time and stay active and engaged throughout the tutorial. After all, you can only get an answer to those questions you ask, either from your peers or the TA. You will get solutions to all tutorial questions at the end of the tutorial week. Tutorials are one of the most important components of the course because they facilitate your communication with your peers. Explain concepts to your group mates and ask them to do the same for you. At the end of each tutorial, you will submit your shared document as a group.

This submission is marked holistically, and not for correctness. You and all your group members will receive the same mark. Your active participation in your tutorial is measured by your group submission marks. Your active participation in tutorials is part of your participation grade.

Homework

You will have four types of homework. We already talked about two of them. That is reading assignments, due weekly, as part of your Reading Quiz, and Tutorial worksheets that you work on with help of your peers during your tutorial and submit with your group. The other two are

WeBWork. WeBWork is an online assignment system. You have weekly WeBWork due every Wednesday 11:59 pm. These questions are straightforward and cover what you learned in the same week and the past week. To access the homework, you should follow the link in the assignment posted under the Homework module. You will be automatically be directed to WeBWork environment. WARNING: You should check your grade for WeBWork inside the WebWork environment and not in Quercus. You may occasionally see some grades regarding this homework appear on Quercus. Those numbers are not accurate and will disappear! WeBWork makes 5% of your final grade.

Written Homework. There will be five written homework sets. These homework sets are rather long and you have about two weeks to do them. You should submit these homework sets individually. You will receive instructions on how to submit your written homework. Only selected questions from each set are graded. The lowest grade will be dropped. Written homework sets make 10% of your final grade.

0.1.3 Lecture Rules

- Come prepared (read the textbook)
- Be fully present
  - No distraction
  - Ready to engage
  - Ready to participate in activities
- Reflect
  - What did you learn?
  - Engage with the textbook
  - Write down your questions
  - Follow up on piazza

1. Fields, Complex Numbers and Vector Spaces

1.1 Fields and Complex Numbers

1.1.1 Fields

Definition 5.1.4 — Field. A *field* is a set[a] $\mathbb{F}$ with two operations, defined on ordered pairs of elements of $\mathbb{F}$, called *addition* and *multiplication*. Addition assigns to the pair x and $y \in \mathbb{F}$ their *sum*, which is denoted by $x+y$ and multiplication assigns to the pair x and y their *product*, which is denoted by $x \cdot y$ or xy. These two operations must satisfy the following properties for all x, y and $z \in \mathbb{F}$:

(i) Commutativity of addition $x+y=y+z$.
(ii) Associativity of addition: $(x+y)+z=x+(y+z)$.
(iii) Existence of an additive identity: There is an element $0 \in \mathbb{F}$, called zero, such that $x+0=a$.
(iv) Existence of additive inverses: For each x there is an element $-x \in \mathbb{F}$ such that $x+(-x)=0$.
(v) Commutativity of multiplication: $xy=yx$.
(vi) Associativity of multiplication: $(xy)z=x(yz)$.
(vii) Distributivity: $(x+y)z=xz+yz$ and $x(y+z)=xy+xz$.
(viii) Existence of a multiplicative identity: There is an element $1 \in \mathbb{F}$, called 1, such that $x \cdot 1 = x$.
(ix) Existence of multiplicative inverses: If $x \neq 0$, then there is an element $x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$.

[a] Note that such set must be **non-empty**.

1.1.2 Complex Numbers

Goal: build a field containing $\mathbb{R}$ such that all polynomials (such as x^2+1) have their roots.

Definition 5.1.2 The set of *complex numbers*, denoted $\mathbb{C}$, is the set of ordered pairs of real numbers(a,b) with the operations of addition and multiplication defined by

For all (a,b) and $(c,d) \in \mathbb{C}$, the *sum* of (a,b) and (c,d) is the complex number defined by $(a,b)+(c,d) = (a+c, b+d)$

and the *product* of (a,b) and (c,d) is the complex number defined by $(a,b)(c,d) = (ac - bd, ad+cb)$

The subset of $\mathbb{C}$ consisting of those elements with second coordinate zero, $\{(a,0) | a \in \mathbb{R}\}$, will be identified with the real numbers in the obvious way, $a \in \mathbb{R}$ is identified with $(a,0) \in \mathbb{C}$. If we apply our rules of addition and multiplication to the subset $\mathbb{R} \subset \mathbb{C}$, we obtain

$$(a,0)+(c,0) = (a+c,0)$$

and

$$(a,0)(c,0) = (ac - 0\cdot 0)(a\cdot 0 + c\cdot 0) = (ac,0)$$

Proposition 5.1.5 The set of complex numbers is a field with the operations of addition and scalar multiplication as defined previously.

Proof. WTS $\mathbb{C}$ is a field[1].

(i), (ii), (v), (vi), and (vii) follow immediately.

(iii) The additive identity is $0 = 0 + 0i$ since
$(0+0i)+(a+bi) = (0+a)+(0+b)i = a+bi$

(iv) The additive inverse of $a+bi$ is $(-a)+(-b)i$.
$(a+bi)+((-a)+(-b)i) = (a+(-a))+(b+(-b))i = 0+0i = 0.$

(viii) The multiplicative identity is $1 = 1+0\cdot i$ since
$(1+0\cdot i)(a+bi) = (1\cdot a - 0\cdot b)+(1\cdot b + 0\cdot a)i = a+bi.$

(ix) Note first that if $a+bi \neq 0$, then either $a \neq 0$ or $b \neq 0$ and $a^2+b^2 \neq 0$. Further, note that $(a+bi)(a+(-b)i) = a^2+b^2$.

Therefore $(a+bi)\dfrac{a-ib}{a^2+b^2} = 1$.

Thus, $(a+ib)^{-1} = (a-ib)/(a^2+b^2)$. ■

Exercise 1.1 Compute the following.

1. $(3-5i)^{-1}$

$$\begin{aligned}(3-5i)^{-1} &= \frac{(3+5i)}{3^2+5^2}\\ &= \frac{3}{34}+\frac{5}{34}i\end{aligned}$$

[1] To show $\mathbb{F}$ is a field, we need to check commutativity, associativity, existence of additive identity and additive inverse, multiplicative identity and multiplicative inverse, and the distributivity between addition and multiplication.

2. $\frac{4-2i}{3-5i} = (4-2i)(3-5i)^{-1}$

$$\begin{aligned}
\frac{4-2i}{3-5i} &= (4-2i)(3-5i)^{-1} \\
&= (4-2i)\left(\frac{1}{34}(3+5i)\right) \\
&= \frac{1}{34}(4-2i)(3+5i) \\
&= \frac{1}{34}(12+10+20i-6i) \\
&= \frac{1}{34}(22+14i)
\end{aligned}$$

■

■ **Example 1.1** Let $\mathbb{F}_2 = \{0,1\}$.

$+$	0	0
0	0	1
1	1	0

$\times$	0	0
0	0	0
1	0	1

Discussion

- Is $+$ commutative? How to see it visually?
  - Yes.
  - The diagonals align up (this tabel is symmetric).
- Is $\times$ commutatiove? How to see it visually?
  - Yes.
  - The diagonals align up (this tabel is symmetric).
- Does $(\mathbb{F}_2, +, \times)$ have additive and multiplicative identities? How to see them visually?
  - Yes, they all have their identities.
  - 0 is the additive identity.
    The corresponding rows and columns of 0 are copies of the index row/column.
  - 1 is the multiplicative identity.
    The corresponding rows and columns of 1 are copies of the index row/column.

■

1.3 Vector Spaces

Let $\mathbb{F}$ be a field.

Definition 5.2.1 — Vector Space over a Field. A *vector space over* $\mathbb{F}$ is a set V (whose elements are called *vectors*) together with two operations:

- A binary operation called **vector addition**, which for each pair of vectors $\vec{v}, \vec{w} \in V$ produces a vector denoted $\vec{v} + \vec{w} \in V$, and
- an operation called **multiplication by a scalar**[a] (a field element), which for each vector $\vec{v} \in V$, and each scalar $c \in \mathbb{F}$ produces a vector denoted $c\vec{v} \in V$.

[a]This is also called a **scalar multiplication**

$$+: \begin{array}{ccc} V \times V & \to & V \\ (\vec{v}, \vec{w}) & \mapsto & \vec{v} + \vec{w} \end{array} \qquad \times: \begin{array}{ccc} \mathbb{F} \times V & \to & V \\ (c, \vec{v}) & \mapsto & c\vec{v} \end{array}$$

Furthermore, the two operations must satisfy the following axioms:

(1) For all vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w} \in V$, $(\overrightarrow{u}+\overrightarrow{v})+\overrightarrow{w} = \overrightarrow{u}+(\overrightarrow{v}+\overrightarrow{w})$. (addition is *associative*)
(2) For all vectors $\overrightarrow{v}$ and $\overrightarrow{w} \in V$, $\overrightarrow{v}+\overrightarrow{w} = \overrightarrow{w}+\overrightarrow{v}$. (addition is *commutative*)
(3) There exists a vector $\overrightarrow{0} \in V$ with the property that $\overrightarrow{x}+\overrightarrow{0} = \overrightarrow{v}$ for all vectors $\overrightarrow{v} \in V$. ($\exists$ an *additive identity*)
(4) For each vector $\overrightarrow{v} \in V$, there exists a vector denoted $-\overrightarrow{v}$ with the property that $\overrightarrow{v}+-\overrightarrow{v} = \overrightarrow{0}$. ($\exists$ *additive inverse*)
(5) For all vectors $\overrightarrow{v}$ and $\overrightarrow{w} \in V$ and all scalars $c \in \mathbb{F}$, $c(\overrightarrow{v}+\overrightarrow{w}) = c\overrightarrow{v}+c\overrightarrow{w}$. (*distributive* property 1)
(6) For all vectors $\overrightarrow{v} \in V$, and all scalars c and $d \in \mathbb{F}$, $(c+d)\overrightarrow{v} = c\overrightarrow{v}+d\overrightarrow{v}$. (*distributive* property 2)
(7) For all vectors $\overrightarrow{v} \in V$, and all scalars c and $d \in \mathbb{F}$, $(cd)\overrightarrow{v} = c(d\overrightarrow{v})$. (multiplication is *associative*)
(8) For all vectors $\overrightarrow{v} \in V$, $1\overrightarrow{v} = \overrightarrow{v}$. ($\exists$ an *multiplicative identity*)

TODO: FINISH THIS PART

2. Linear Transformations

3. The Spectral Theorem in $\mathbb{R}^n$

3.1 Diagonalizability

3.1.1 Eigenvectors, Eigenvalues and Eigenspaces

Definition 4.1.2 — Eigenvector and Eigenvalue. Let V be a vector space over the field $\mathbb{F}$.

Let $T : V \to V$ be a linear mapping.

a) A vector $\overrightarrow{v} \in V$ is called an *eigenvector of T* if $\overrightarrow{v} \neq \overrightarrow{0}$ and there exists a scalar $\lambda \in \mathbb{R}$ such that $T(\overrightarrow{v}) = \lambda \overrightarrow{v}$.

b) If $\overrightarrow{v}$ is an eigenvector of T and $T(\overrightarrow{v}) = \lambda \overrightarrow{v}$, the scalar λ is called the *eigenvalue of T corresponding to $\overrightarrow{v}$*.

Definition 4.1.6 — Eigenspace. $\forall \lambda \in \mathbb{F}$, the *$\lambda$-eigenspace of T*, denoted E_λ, is the set

$$E_\lambda = \{\overrightarrow{v} \in V \mid T(\overrightarrow{v}) = \lambda \overrightarrow{v}\} = \{\text{all eigenvectors of } \lambda\} \cup \{\overrightarrow{0}\}$$

That is, E_k is the set containing all the eigenvectors of T with eigenvalue λ, together with the vector $\overrightarrow{0}$. (If λ is not an eigenvalue of T, then we have $E_\lambda = \overrightarrow{0}$.)

■ **Example 3.1**

Consider the linear transformation $D : \begin{array}{ccc} \mathscr{C}^\infty(\mathbb{R}) & \to & \mathscr{C}^\infty(\mathbb{R}) \\ f & \mapsto & f' \end{array}$.

- 1 is a eigenvector with a eigenvalue of 0.
- e^x is an eigenvector is with a eigenvalue of 1.
- e^{Mx} is an eigenvector with the eigenvalue of M.

■

■ **Example 3.2** Define the plane $W : x + y + z = 0$.

Consider the linear transformation $R : \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \overrightarrow{x} & \mapsto & \overrightarrow{x} \text{ reflected w.r.t. } W \end{array}$

$\forall \overrightarrow{x} \in W$, $R(\overrightarrow{v}) = 1\overrightarrow{v}$, so $\lambda = 1$ is an eigenvalue, and $E_1 = W$.

$R(\overrightarrow{w}) = -\overrightarrow{w}$ for all $\overrightarrow{w} \perp W$, so $\lambda = -1$ is an eigenvalue, and $E_{-1} = \mathrm{Sp}(\overrightarrow{w})$. ■

Proposition 4.1.7 E_λ is a subspace of V for all λ.

Proof.
- $T(\vec{0}) = \vec{0} = \lambda\vec{0} \implies \vec{x} \in E_\lambda$
- Pick $\vec{v}, \vec{w} \in E_\lambda$.
  WTS $\vec{v} + \vec{w} \in E_\lambda$.
  $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \lambda\vec{v} + \lambda\vec{w} = \lambda(\vec{v}+\vec{w}) \in E_\lambda$
- Pick $\vec{v} \in E_\lambda, r \in \mathbb{F}$.
  WTS $T(r\vec{v}) = rT(\vec{v})$.
  $T(r\vec{v}) = \lambda(r\vec{v}) = r(\lambda\vec{x}) = rT(\vec{v}) \in E_\lambda$.

■

Eigenbasis

Consider the same linear transformation R discussed in example 2.

$R(\vec{x}) = A\vec{x}$

$$A = [R]_{\mathscr{E}} = \begin{bmatrix} | & | & | \\ R(\vec{e_1}) & R(\vec{e_2}) & R(\vec{e_3}) \\ | & | & | \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Eigenbasis: linearly independent and spans $\mathbb{R}^3$.

For example: $\mathscr{B} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{\vec{v_1}}, \underbrace{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}_{\vec{v_2}}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{w}} \right\}$

Then, $[R]_{\mathscr{B}} = \begin{bmatrix} | & | & | \\ [R(\vec{v_1})]_{\mathscr{B}} & [R(\vec{v_2})]_{\mathscr{B}} & [R(\vec{w})]_{\mathscr{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$[R(\vec{v_1})]_{\mathscr{B}} = [\vec{v_1}]_{\mathscr{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[R(\vec{v_2})]_{\mathscr{B}} = [\vec{v_2}]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[R(\vec{w})]_{\mathscr{B}} = [\vec{w}]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Let $T : V \to V$ be a linear transformation.

$\vec{v} \in V$ is an eigenvector if and only if $T(\vec{v}) = \lambda\vec{v}$, by the definition.

Proposition 4.1.5 A vector $\vec{v}$ is an eigenvector of T with eigenvalue λ if and only if $\vec{v} \neq \vec{0}$ and $\vec{v} \in \ker(T - \lambda \mathrm{id})$.

$$T(\vec{v}) - \lambda\vec{v} = \vec{0}$$
$$T(\vec{v}) - \lambda\, id(\vec{v}) = \vec{0}$$
$$(T - \lambda\, id)\vec{v} = 0 \qquad \text{by sum of linear transformations}$$
$$\vec{v} \in \ker(T - \lambda \text{ id}) \qquad \text{for some } \lambda \in \mathbb{F}$$

Let $\dim V = n$, and fix a basis $\mathscr{B}$ for V,

$$\begin{array}{rcl} \text{Then, } \mathscr{L}(V,V) & \overset{\cong}{\to} & \underset{n\times n}{M}(\mathbb{F}) \\ T & \mapsto & [T]_{\mathscr{B}} = B \end{array}$$

Suppose $\vec{v}$ is an eigenvector of T.
$T(\vec{v}) - \lambda\vec{v}$ means that $[T]_{\mathscr{B}}[\vec{v}]_{\mathscr{B}} = [\lambda\vec{v}]_{\mathscr{B}} = \lambda[\vec{v}]_{\mathscr{B}}$.

$$\begin{aligned} [T - \lambda\ id]_{\mathscr{B}} &= [T]_{\mathscr{B}} - \lambda[id]_{\mathscr{B}} \\ &= B - \lambda I_{n\times n} \end{aligned}$$

λ is an eigenvalue for T iff $B - \lambda I$ is not invertible. λ is an eigenvalue for T iff $\det(B - \lambda I) = 0$. This is called that **characteristic polynomial** of the linear transformation T.

Definition 4.1.11 — Characteristic Polynomial. Let $A \in M_{n\times n}(\mathbb{R})$. The polynomial $\det(A = \lambda I)$ is called the *characteristic polynomial* of A.

Example 3.3

$$\begin{array}{rcl} T: \ \mathscr{P}_3(\mathbb{R}) & \to & P_3(\mathbb{R}) \\ p & \mapsto & p' + 2p \end{array}$$

1. Find the characteristic polynomial of T.

   Pick the basis $\mathscr{B} = (1, x, x^2, x^3)$. Then, $B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

   $$\text{char}(T) = \det(B - \lambda I) = (\lambda - 2)^4$$ [1]

   Note: char(T) is well defined (does NOT depend on the choice of basis).
2. Find all eigenvalues and the corresponding eigenspace of T.

   $$E_2 = \text{nul}(B - \lambda I) = \text{sp}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   $\dim(E_2) = 1$ is the *geometric multiplicity* of $\lambda = 2$.

■

3.2 Diagonalizability

Definition 4.2.1 — Diagonalizable. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear mapping. T is said to be *diagonalizable* if there exists a basis of V, all of whose vectors are eigenvectors of T.

[1] Note that the power of 4 means the *algebraic multiplicity* of $\lambda = 2$ is 4.

Part Two

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