MAT235

Multivariable Calculus

Sinan Li

PUBLISHED BY SINAN LI

ARTSCI.CALENDAR.UTORONTO.CA/COURSE/MAT235Y1

First digital, May 2022

Contents

I

Lecture Notes

Introduction

Parametric equations and polar coordinates. Vectors, vector functions and space curves. Differential and integral calculus of functions of several variables. Line integrals and surface integrals and classic vector calculus theorems. Examples from life sciences and physical science applications.

This is a second-year **Multivariable Calculus** course. The depth of mathematics will be taught at the standard level accessibly to all second-year undergraduate students who has fully finished any first year Introduction to Calculus course. The course will have significant emphasis on computation. In general, theorems will be stated without proofs, but with an indication of the mathematical ideas involved. No emphasis will be put on rigorous mathematic proofs.

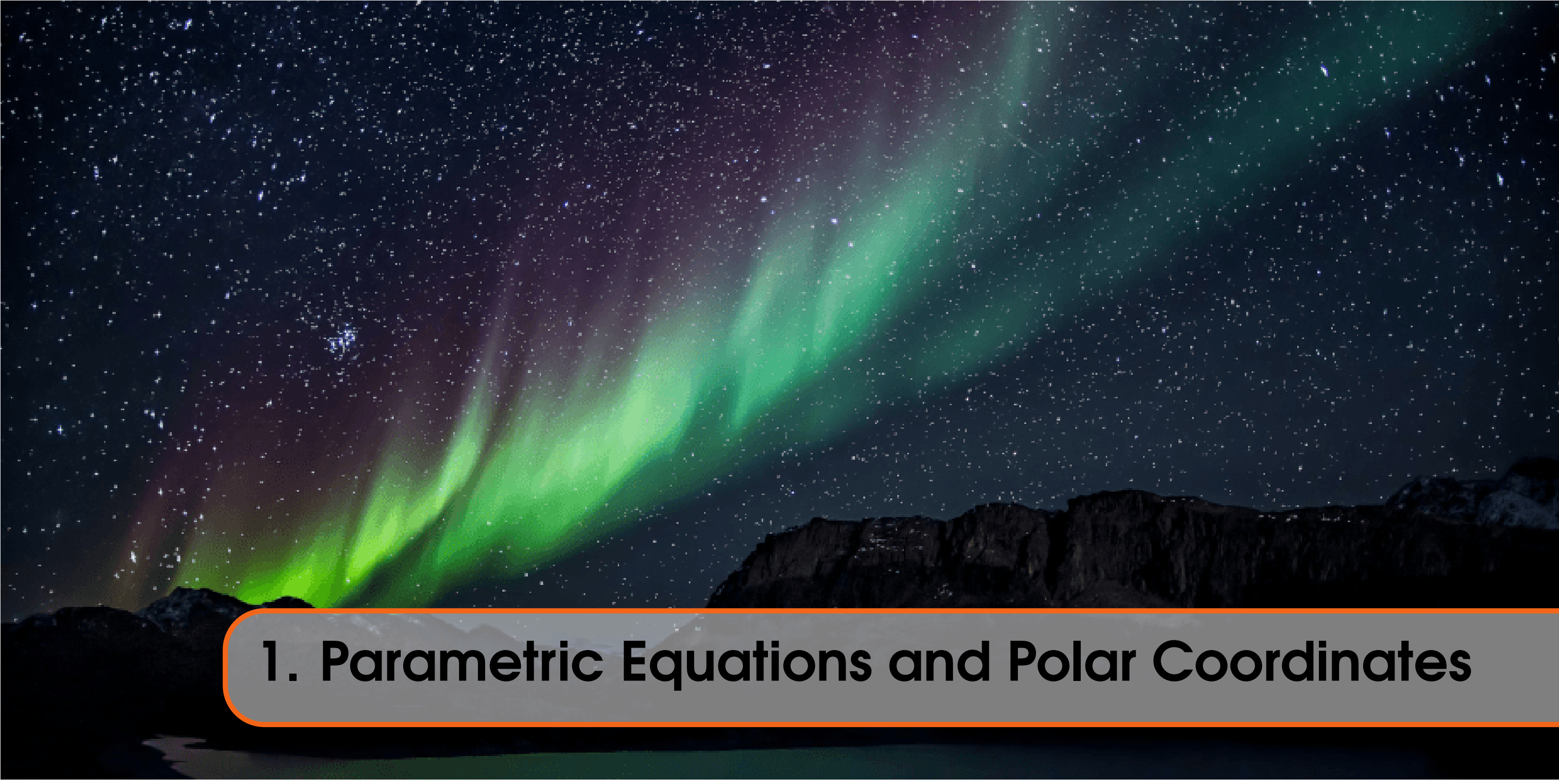
Requirements

Any full first-year Introduction to Calculus course is acceptable, preferably with depth taught at the standard level for first-year undergraduate students. Some examples of the courses include, but are not limited to: MAT135/136; MAT137; MAT157; MAT133; There will be a review of key concepts that we need from Linear Algebra at the beginning of the course, so it is not necessary to have any background in linear algebra.

Textbook

- Single Variable Calculus: Early Transcendentals, Stewart (8th/9th Edition) [1]
- Advanced Calculus, Folland

The textbooks are optional, and are not necessary to the course.



1. Parametric Equations and Polar Coordinates

1.1 Curves Defined by Parametric Equations

1.1.1 Parametric Equations

Definition 1.1.1 — Parametric Equations. Suppose that x and y are both given as functions of a third variable t, called a *parameter*, by the equations

$$x = f(t) \qquad y = g(t)$$

which are called *parametric equations*.

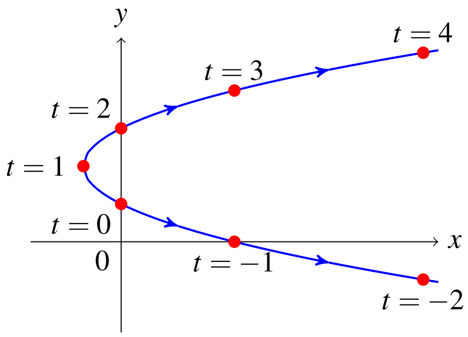
Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$, varies and traces out a curve called a *parametric curve*.

■ **Example 1.1** Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 1$, then $x = -1$, $y = 2$ and so the corresponding point is $(-1, 2)$. We plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



■

In the above example, we found a Cartesian equation in x and y whose graph coincided with the curve represented by parametric equations. This process is called *eliminating the parameter*.

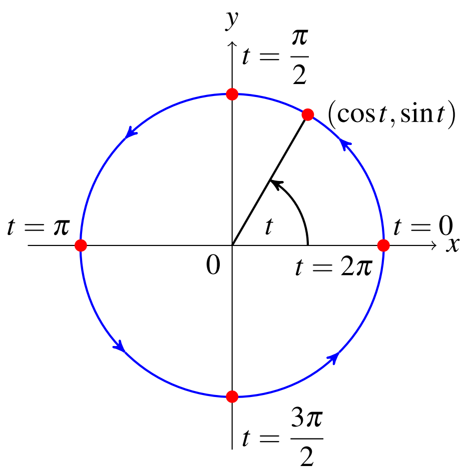
■ **Example 1.2** Sketch the curve represented by

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

If we plot points, it appears that the curve is a circle. We can confirm this by eliminating the parameter t. Observe that

$$x^2 + y^2 + \cos^2 t + \sin^2 t = 1$$

Because $x^2 + y^2 = 1$ is satisfied for all pairs of x and y values generated by the parametric equations, the point (x, y) moves along the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians). As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

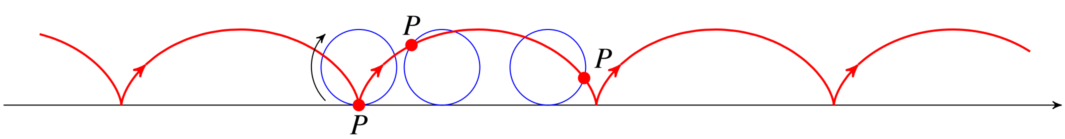


■

1.1.2 Cycloid

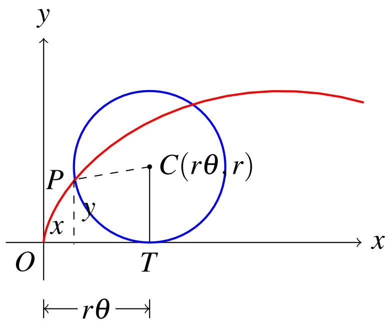
Definition 1.1.2 — Cycloid. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a *cycloid*. If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid are

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$



We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated throug θ radians. Because the circle has been in contact with the line, we see that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$



Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \le \theta \le 2\pi$.

Although it is possible to eliminate the parameter θ from the equations above, the resulting Cartesian equation in x and y is very complicated [1], and not as convenient to work with as the parametric equations.

1.2 Calculus with Parametric Curves

1.2.1 Tangents

Suppose f and t are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we can solve for $\frac{dy}{dt}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dx}{dt} \neq 0$$

The curve has a horizontal tangent when $\frac{dy}{dt}$, provided that $\frac{dx}{dt} \neq 0$, and it has a vertical tangent when $\frac{dx}{dt} = 0$, provided that $\frac{dy}{dt} \neq 0$ (if both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, then we would need to use other methods to determine the slope of the tangent).

$\frac{d^2y}{dx^2}$ can also be found by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

[1] $x = r\arccos\left(1 - \frac{y}{r}\right)$, which only gives half of one arch.

1.2.2 Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, them we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

1.3 Polar Coordinates

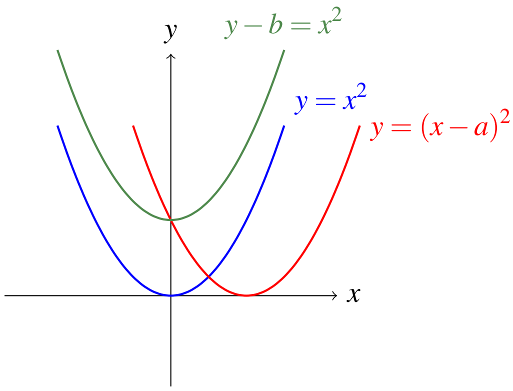
1.4 Calculus in Polar Coordinates

1.5 Conic Sections

1.5.1 Curved Lines

An equation involving x and y gives a *curve* in the plane.

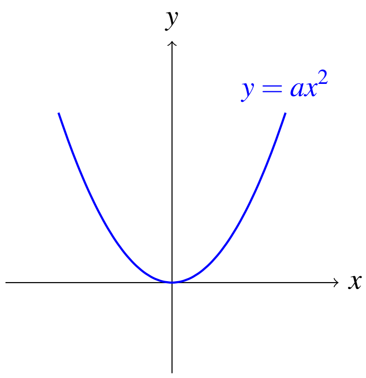
By replacing x with $x-a$, the graph is shifted to the right by a. By replacing y with $y-b$, the graph is shifted upward by b.

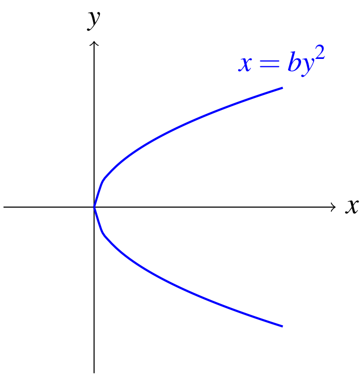


Definition 1.5.1 — Parabola. The *parabola* (with vertex) at the origin is defined by

$$y = ax^2 \qquad x = by^2$$

- $y = ax^2$ opens up for $a > 0$, and opens down for $a < 0$.
- $x = by^2$ opens to the right for $b > 0$, and opens to the left for $b < 0$.





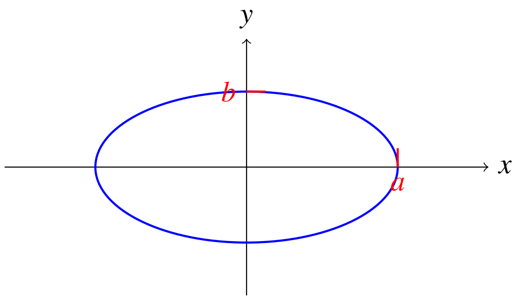
Definition 1.5.2 — Ellipse. The *ellipse* (with center) at the origin is defined by

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

with $(a,0)$ being the right-most point on the x-axis, and $(0,b)$ as the top-most point on the y-axis.

When $a=b$, an ellipse becomes a *circle*, with $a=b=R$ as the radius of the circle [a].

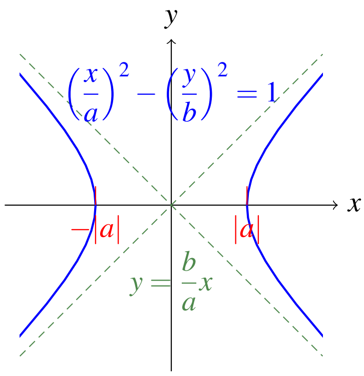
[a] $\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2=1 \implies x^2+y^2=R^2$



Definition 1.5.3 — Hyperbola. The *hyperbola* (with center) at the origin is defined by

$$\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=1 \qquad \left(\frac{y}{a}\right)^2-\left(\frac{x}{b}\right)^2=1$$

A hyperbola has 2 curves, and a center line that is not crossed. The 2 curves goes in the opposite direction of the center.



Since the right side is positive, and a square is positive, the position of the negative sign determines the direction of the curves.

To determine the direction, we can either set $x=0$ or $y=0$. For example, in the case of $x^2-y^2=1$, by setting $x=0$, we attain $-y^2=1$ is impossible, so the vertical line $x=0$ is the center is never crossed. Thus the 2 curves open towards left and right.

To find the **slant asymptotes** when x and y are both large, change the number 1 on the right side of the equation to 0.

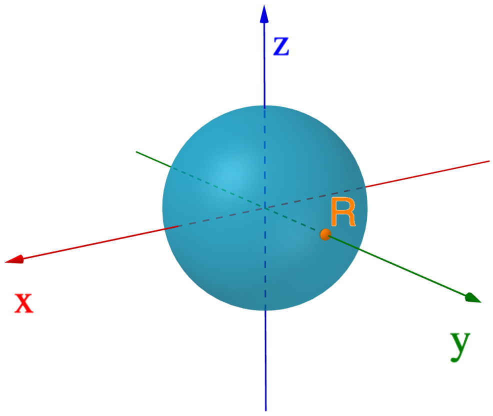
The above 3 curves together are sometimes called: *Conic Sections*.

1.5.2 Curved Surfaces

An equation involving x, y, and z gives a *curved surface* in 3D space. In general, this surface is difficult to imagine and sketch in 3D.

Definition 1.5.4 — 3D Sphere. The *3D Sphere* (with center) at the origin with radius R is given by

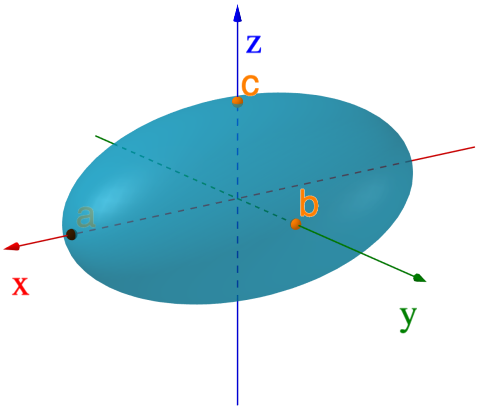
$$x^2+y^2+z^2=R^2$$



Definition 1.5.5 — Ellipsoid. The 3D *ellipsoid* (with center) at the origin is given by

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$$

$(a,0,0)$, $(0,b,0)$, $(0,0,c)$ are the 3 points on the ellipsoid, similar to the ellipse.
When $a=b=c$, an ellipsoid becomes a *sphere*, with $a=b=c=R$ as the radius.



Surfaces Attained Through Rotation: $r^2=x^2+y^2$
When the variables x and y appear together as x^2+y^2, this signals that the surface is attained through rotation. Set a new variable r, with $r^2=x^2+y^2$. We graph the equation involving z and r in the rz-plane, with $r>0$ only. We then revolve the curve around the z-axis to attain the surface in 3D: the r-axis stands for both x-axis and y-axis.

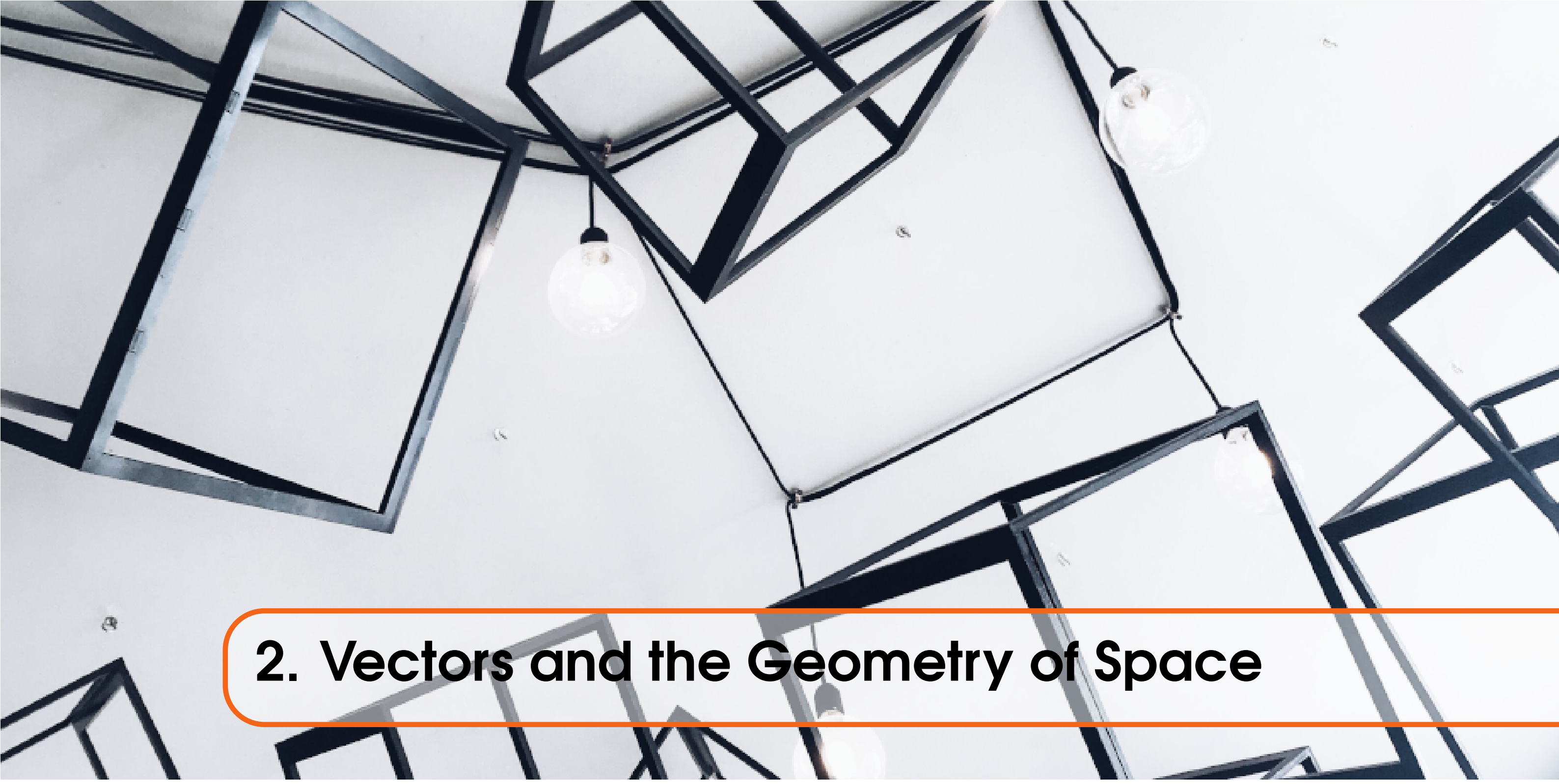
Exercise 1.1 Sketch the following curves.

1. $x=2y^2$
2. $\dfrac{x^2}{2}+\dfrac{y^2}{9}=1$
3. $x^2+2y^2=4$
4. $x^2+3y^2+2x-12y+10=0$
5. $y^2-x^2=1$
6. $\dfrac{(x-2)^2}{4}-\dfrac{(y+2)^2}{9}=1$

Exercise 1.2 Sketch the following surfaces.

1. $x^2+y^2=1$
2. $z=x^2+y^2$
3. $z^2=x^2+y^2$
4. $x^2+y^2+\frac{z^2}{4}=1$
5. $z=\left(\sqrt{x^2+y^2}-1\right)^2$
6. $x^2+y^2+z^2=2z$

■



2. Vectors and the Geometry of Space

2.1 Vector and Matrix

A point in 2D needs 2 coordinates. It is typically written as, for example:

$$\vec{a} = 1 \quad \text{or} \quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This is the point $x = 1$ and $y = 2$ on the 2D xy-plane.

A point in 3D needs 3 coordinates. It is typically written as, for example:

$$\vec{b} = (0,2,1) \quad \text{or} \quad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

This is the point $x = 0$, $y = 2$, and $z = 1$ in 3D space.

The point can also be interpreted as a *vector*, written in the same way. We can also think of a vector as an *arrow* from the origin to the point.

Typically, it doesn't matter if we write the vector horizontally or vertically.

Definition 2.1.1 — Matrix. A *matrix* is a block of numbers written in a rectangle (or square) in a specific order.

For example, A is a 2×3 (2 by 3) matrix, with 2 rows and 3 columns:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

We may think of vectors as $1 \times n$ or $n \times 1$ matrix (for $n = 2$ or 3). We can add matrices, and multiply matrices by a scalar (number).

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

- $r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb & rc & rd \end{bmatrix}$

Standard arithmetic rules apply. For matrices A, B, and scalar c:

- $A+B=B+A$
- $cA=Ac$
- $c(A+B)=cA+cB$

2.2 Determinant

The determinant of a **square** matrix is a number.

Starting at the first row, first position, write down this value, and remove this row and this column from the original matrix. Multiply this value by the determinant of the remaining matrix.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} \dots$$

Now, move right, write down this value, and remove this row and this column from the original matrix. Multiply this value to the determinant of the remaining matrix with a **minus sign**.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

Now, move right again and repeat until we have reached the last column. The positive and negative signs need to **alternate**.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Each time we apply the algorithm, we end up with several new determinants to calculate, but with matrices of *smaller sizes*.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad = bc$$

Determinant does not behave well with addition and scalar multiplication:

$$\det(A+B) \neq \det(A)+\det(B) \qquad \det(cA) \neq c \cdot \det(A)$$

Exercise 2.1 Find the determinant.

1. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

2.3 Dot product

2 vectors can form a *dot product*, and the result is a scalar (number).

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots$$

Note that this dot product is different from scalar multiplication, as we are multiplying 2 vectors together, not a scalar with a vector.

The *length* (absolute value, or *norm*) of a vector using the Pythagoras Theorem:

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{(x_1)^2 + (x_2)^2 + \cdots}$$

Similarly, define the distance between two points using pythagoras Theorem:

$$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots}$$

Going from a point a to point c, is always shorter than going to some other point b first and then back to c. This is called the *Triangle Inequality*:

$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$$

$$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$$

Let θ be the angle between $\vec{x}$ and $\vec{y}$, the dot product is also given by

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta$$

Thus, 2 vectors are *orthogonal* (perpendicular) if the dot product is zero.

The dot product have the usual properties of multiplication:

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $c(\vec{a} \cdot \vec{b}) = (c\vec{a}) \cdot \vec{b} = a \cdot (c\vec{b})$

Note that since the dot product needs two vectors and produce a number, a quantity such as $\vec{a} \cdot \vec{b} \cdot \vec{c}$ is not well defined.

Exercise 2.2 Find the length of the vectors. Find $\vec{a} \cdot \vec{b}$ and the angle between them.

1. $\vec{a} = (4, 3)$, $\vec{b} = (2, -1)$
2. $\vec{a} = (4, 0, 2)$, $\vec{b} = (2, -1, 0)$

■

2.4 Cross product

2 vectors (in 3D) can form a *cross product*, and the result is a *vector* (in 3D).

Let $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$ in 3D.

We typically write the cross product using determinant:

$$\vec{x} \times \vec{y} = \det \begin{bmatrix} i & i & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\vec{x} \times \vec{y} = i(x_2 y_3 - x_3 y_2) - j(x_1 y_3 - x_3 y_1) + k(x_1 y_2 - x_2 y_1)$$

$$\vec{x} \times \vec{y} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$$

The notations $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ are very easy to use, where the quantity attached to i is the first component of the vector, and the quantity attached to j would be the second, and k would be the third.

The vector produced by the cross product, $\vec{x} \times \vec{y}$, would be orthogonal to both vectors $\vec{x}$ and $\vec{y}$. Notice that in most cases, there would be 2 such vectors with this property. They are exactly

$$\vec{x} \times \vec{y} \qquad \text{and} \qquad -\vec{x} \times \vec{y}$$

In fact, we have

$$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$$

Notice that the order of a dot product does not matter, but the order of cross product matters up to a negative sign.

The other usual properties of multiplication hold:

- $(a\vec{a} \times \vec{b}) = c(\vec{a} \times \vec{b}) = a \times (c\vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Similar to the dot product, we can also relate the angle between the 2 vectors,

$$|\vec{x} \times \vec{y}| = |\vec{x}||\vec{y}| \sin\theta$$

Note that since $\vec{x} \times \vec{y}$ is a vector, we need to take its absolute value. So 2 vectors are *parallel* if the cross product is zero.

Exercise 2.3 Let $\vec{x} = (3,-2,1)$, $\vec{y} = (1,-1,1)$. Find $\vec{x} \times \vec{y}$. ■

2.5 Lines and Planes

For a *line* to be defined, we need a direction $\vec{v}$ and a point on the line $\vec{r}_0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} := \vec{r} = t\vec{v} + \vec{r}_0$$

The vector $\vec{r}$ and the value t do not need to be determined [1].

For a *plane* to be defined, we need a **normal vector** $\vec{n} = (a,b,c)$ and a point on the plane $\vec{r}_0$.

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$ax + by + cz = \vec{n} \cdot \vec{r}_0$$

where $\vec{r}$ is the same as above and does not need to be determined. The *normal vector* $\vec{n}$ is orthogonal to the plane [2].

Plane Defined by 3 Points

Consider 3 points $\vec{u}$, $\vec{v}$, and $\vec{w}$. We can connect any 2 pairs of points together to create 2 direction vectors which are inside the plane. For example, we may take $\vec{x} = \vec{u} - \vec{v}$, and$\vec{y} = \vec{w} - \vec{v}$. From there we can form the cross product to attain a vector that is orthogonal to both vectors inside the plane, so the cross product would be the normal vector, which is orthogonal to the plane.

[1] $\vec{v}$ and $\vec{r}_0$ are not unique, and any correct pair would give the right equation.

[2] Similar to lines, $\vec{v}$ and $\vec{r}_0$ are not unique.

Exercise 2.4 Find the equation of the line or plane.

1. The line through $(-8,0,4)$ and $(3,-2,4)$.
2. The plane through the origin and perpendicular to the vector $(1,5,2)$.
3. The plane through $(2,4,6)$ parallel to the plane $x+y-z=5$.
4. The plane through $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.
5. The plane equidistant from point $(3,1,5)$ and $(-2,0,0)$.

■

3. Partial Derivatives

3.1 Multivariable Function

In first year, we study 1D functions: *functions of one variable*, as $f(x)$. This can be plotted in 2D plane, with $y = f(x)$. Now, we can study functions of multiple variables.

Definition 3.1.1 — 2D Function. A *2D function*, $f(x,y)$, has 2 inputs being x, y, and 1 output.

Sometimes, we can label the output as $z = f(x,y)$.

We can plot this 2D function in 3D space, where the height z above the point (x,y) on the xy-plane is equal to $f(x,y)$.

This would give a graph, which is a surface.

Alternatively, $z = f(x,y)$ can be thought of as an equation which involves x, y, and z, which would also give a surface in 3D space.

However, similar to 1D functions, given an equation involving x, y, and z, it is not always possible to isolate z into $z = f(x,y)$.

Examples of 2D functions include:

$$f(x,y) = x^2 + 2y \qquad f(x,y) = xe^{xy} \qquad f(x,y) = \sin(xy^2)$$

Definition 3.1.2 — 3D Function. A *3D function*, $f(x,y,z)$, has 3 inputs being x, y, and z, and 1 output.

It is possible to label the output as $w = f(x,y,z)$, but this is typically not useful as this introduces a 4th variable. The "plot" of this function would be in 4D space, which does not exist.

Examples of 3D functions include:

$$f(x,y,z = x^2 + 2z - 1) \qquad f(x,y,z) = ye^{xz}$$

Definition 3.1.3 — Level Set. The *level set* of f at k is given by setting f to be equal to the constant k. This will reduce the dimension of f by 1.

Exercise 3.1
- Draw the level set of $f(x,y) = 4x^2 + y^2 + 1$ for $k = 2$ and $k = 5$.
- Draw the level set of $f(x,y,z) = x^2 + y^2 - z$ for $k = 0$ and $k = 2$.

3.2 Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if

$$\lim_{(x,y)\to\vec{a}} f(x,y) = f(\vec{a})$$

Every standard function we know are continuous in their domains. Any function transformations (such as $+$, $-$, $\times$, $/$, composition) of continuous functions, are also continuous (except division by 0).

As most functions are continuous, most limits can be obtained by putting $(x,y) = \vec{a}$ in the formula of $f(x,y)$.

Different from 1D functions, computing limits in 2D is much more difficult. Most limit evaluation techniques from 1D functions does not work for 2D functions. In particular, there is no L'Hôpital's rule Rule for 2D functions.

Fortunately, since most functions are continuous except when dividing by 0, we typically only need to focus on the point where the function is dividing by 0 (and state the function is continuous everywhere else).

It turns out that it is much easier to show a limit does not exist.

Test for limit that does not exist in $\mathbb{R}^2$

Given $f(x,y)$, compute the limit as (x,y) approaches $\vec{a} = (0,0)$.

1. Replace y with a easy curve $y = g(x)$, passing through $\vec{a} = (0,0)$, (such as $y = 0$, $y = x$, $y = x^2$), then let x approach 0 to attain a (1D) limit.
2. Replace x with a easy curve $x = h(y)$, passing through $\vec{a} = (0,0)$, (such as $x = 0$, $x = y^2$), then let y approach 0 to attain a (1D) limit.
3. Try with several different $g(x)$ and $h(y)$ to attain many (1D) limits.
4. If you find 2 **different** (1D) limits generated by 2 different curves, then the (2D) limit of (x,y) approaches $\vec{a} = (0,0)$ of $f(x,y)$ does not exist.

Note that if all the 1D limits are the same, that is **not** enough for you to conclude the 2D limit exists. To show the limit exists, we typically must use *Squeeze Theorem*.

Since most limit evaluation techniques does not work for 2D functions, it is a very fortunate fact that Squeeze Theorem does work for 2D functions. The formulation is almost the same as the 1D version.

Theorem 3.2.1 — Squeeze Theorem.
To attain $\lim_{(x,y)\to\vec{a}} f(x,y)$, we can try to find $g(x,y)$ and $h(x,y)$ such that

1. $g(x,y) \le f(x,y) \le h(x,y)$ near the point $\vec{a}$
2. $\lim_{(x,y)\to\vec{a}} g(x,y) = L = \lim_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y) = L$.

Use Squeeze Theorem to show limit exist

In practice, we typically want to show $\lim_{(x,y)\to\vec{a}} f(x,y) = 0$.

Start at $|f(x,y)|$, create a *chain of inequalities*, and simplify $|f(x,y)|$ to attain $|g(x,y)|$, which has limit 0 as (x,y) approaches $\vec{a}$:

$$|f(x,y)| \leq \cdots \leq |g(x,y)| \to 0 \quad \text{i.e.} \quad \lim_{(x,y)\to\vec{a}} |g(x,y)| = 0$$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y) = 0$.

The typical strategy in constructing the above inequality is to remove positive quantities from the denominator of $f(x,y)$.

Exercise 3.2

$$f(x,y) = \frac{x^2 y}{x^4+y^2} \quad \text{and} \quad f(0,0) = 0$$

Show the limit at $(x,y) = (0,0)$ does not exist. ■

Exercise 3.3

$$f(x,y) = \frac{xy}{\sqrt{x^4+y^2}} \quad \text{and} \quad f(0,0) = 0$$

Show the function **continuous** at $(0,0)$. ■

Exercise 3.4 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{2x^4+y^2}$ ■

3.3 Derivative

Consider multivariable function $f(x,y)$ or $f(x,y,z)$.

Define the *derivative* to be:

$$\text{For } f(x,y) \quad \Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\text{For } f(x,y,z) \quad \Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

where Δf is called *'del' f*, or *gradient* of f.

We can interpret Δf as a vector, even though $f(x,y)$ or $f(x,y,z)$ is a scalar.

The quantities in Δf such as $\frac{\partial f}{\partial x}$ are called *partial derivatives*. When taking partial derivatives with respect to a variable, we regard **all other variables** as *constants* and take the derivative as usual. Sometimes, we use the notations

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y \qquad \frac{\partial f}{\partial z} = f_z$$

■ **Example 3.1** Let $f(x,y) = x^2y^3 + x$.

To take the partial derivative with respect to x, $\frac{\partial f}{\partial x}$, we view y as constant.

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

To take the partial derivative with respect to y, $\frac{\partial f}{\partial y}$, we view x as constant.

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

■

■ **Example 3.2** Let $f(x,y,z) = x^2z + yz^2$.

To take the partial derivative with respect to x, $\frac{\partial f}{\partial x}$, we view **both** y and z as constants. Similarly, for other partial derivatives,

$$\frac{\partial f}{\partial x} = 2xz \qquad \frac{\partial f}{\partial y} = z^2 \qquad \frac{\partial f}{\partial z} = x^2 + 2yz$$

■

Exercise 3.5 Computer Δf.

1. $f(x,y) = (x^2-1)(y+1)$
2. $f(x,y) = (xy-2)^2$
3. $f(x,y) = \frac{2}{x+3y}$
4. $f(x,y) = x^{x+4y}$
5. $f(x,y,z) = x^2 + 2z - 1$
6. $f(x,y,z) = ye^{xz}$

■

3.4 Directional Derivative

Remember one dimensional derivatives in first year?

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

where f is defined on the real line.

We start at the point $x = a$, and we ask, if we move just a tiny bit away from $x = a$, how much does the function change?

We generalize this idea of moving a tiny bit away in higher dimensions. In the real line, we can move only left or right. In 2D, we can move in any direction we like (on the plane).

We can describe the direction of movement by a *unit vector*, similar to the direction vector of the equation of a line. The *directional derivative* toward the direction $\vec{u}$ at the point $\vec{a}$ [1]:

$$D_{\vec{u}}f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

This is also the *rate of change* of the function in the direction $\vec{u}$ at the point $\vec{a}$.

Given a vector $\vec{v}$, we can turn it into a *unit vector* $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$.

Definition 3.4.1 — Unit Vector. A *unit vector* is a vector whose length (norm) is exactly 1.

Some useful facts:

1. Partial derivatives are directional derivatives with $\vec{u}$ being in a *coordinate direction*. For example, for $f(x,y)$,
   - $\frac{\partial f}{\partial x} = D_{\vec{u}}f$ with $\vec{u} = (1,0)$;
   - $\frac{\partial f}{\partial y} = D_{\vec{u}}f$ with $\vec{u} = (0,1)$;

[1] This is a 1D limit, as h is a number (the length of the movement).

2. If f is **differentiable**, then all directional derivative exist and

$$D_{\vec{u}} f = \Delta f \dot{u}$$

where $\vec{u}$ is a **unit vector**.
This formula gives an easy way to calculate directional derivative.

3. We can interpret $\Delta f(\vec{a})$ as a vector.
If we evaluate Δf at a point $\vec{a}$, and if we interpret the vector $\Delta f(\vec{a})$ as a direction, then the directional derivative in the direction of $\Delta f(\vec{a})$ at the point $\vec{a}$ is the largest, and the value of this maximum directional derivative is equal to $|\Delta f(\vec{a})|$.

4. Recall equation of plane: $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
For $f(x,y,z) = k$ for some constant k, gives a *level set* surface. Define the *tangent plane* at some fixed point $\vec{r}_0 = (x_0, y_0, z_0)$ by the **normal vector** $\vec{n} = \Delta f(x_0, y_0, z_0)$.

Exercise 3.6 $f(x,y,z) = e^{2x}y + y^2 + 4$. At the point $(0,0)$, find the directional derivative along the direction given by $\vec{u} = (1,3)$ (first turn $\vec{v}$ into a unit vector). ■

Exercise 3.7 $f(x,y,z) = x^2y + z$. At the point $(2,2,1)$, find the maximum rate of change, and the direction with the maximum (the direction with maximum rate of change is Δf, and the value of this maximum rate of change is $|\Delta f|$). ■

Exercise 3.8 Define $f(x,y,z) = xy^2e^z$. Let $f(x,y,z) = e$.
At the point $(1,1,1)$, find the equation of the tangent plane. ■

Exercise 3.9 Find all directional derivatives at $(0,0)$ (including the partials), if they exist. Is the function **continuous** at $(0,0)$?

a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

b) $f(x,y) = \sqrt{|xy|}$

Strategy:
When the function has division by 0, or some other problems at the origin (such as absolute value or square root), we must use the definition of directional derivative, because f **may not be differentiable** at $(0,0)$ so the formula $\Delta f \cdot \vec{u}$ does not work. Set $\vec{a} = (0,0)$ and $\vec{u} = (a,b)$ where $a^2 + b^2 = 1$. ■

3.5 Differentiation

Definition 3.5.1 — Differentiable. f is *differentiable* if the gradient vector $\Delta f(\vec{a})$ satisfies

$$\lim_{\vec{h} \to \vec{0}} = \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - \Delta f(\vec{a}) \cdot \vec{h}}{|\vec{h}|} = 0$$

Note that this is a *2D limit*. This limit is typically used when $\vec{a} = (0,0)$, so we may set $\vec{h} = (x,y)$ and show the limit is zero using Squeeze Theorem.

Some useful facts:

1. If f is C^1 at $\vec{a}$ [2], then f is differentiable at $\vec{a}$ (Δf exists).
2. Being differentiable is a stronger condition than having directional derivative. If f is differentiable, then all directional derivative is given by $D_{\vec{u}} f = \Delta f \cdot \vec{u}$.
3. If a function f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.

Exercise 3.10 Define

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show the directional derivatives exist at $(0,0)$, but f is **not differentiable** at $(0,0)$.

Strategy:
There are 2 ways to approach the problem.

1. If the function is **not continuous** at $(0,0)$, then it is **not differentiable** at $(0,0)$. So we may try to show it is not continuous at $(0,0)$. However, some functions **are continuous** at $(0,0)$, so we can attain no conclusion in such cases.
2. Compute all the directional derivatives. If f is differentiable, then every directional derivative should satisfy $D_{\vec{u}} = \Delta f \cdot \vec{u}$.

   Check this equation for $\vec{u} = (1,0)$, $\vec{u} = (0,1)$, and $\vec{u} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ (easy unit vectors).

Exercise 3.11 Show whether the function is differentiable at $(0,0)$.

a) $f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

3.6 Higher Order Derivative

We may take (partial) derivatives on top of partial derivatives.

In 2 dimensions, for $f(x,y)$, there are 4 ways of taking second derivatives. We put them into the *Hessian Matrix*:

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial xy} \\ \dfrac{\partial^2 f}{\partial yx} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) & \dfrac{\partial f}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) \\ \dfrac{\partial f}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) & \dfrac{\partial f}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) \end{bmatrix} = \begin{bmatrix} f_{xx} & fxy \\ fyx & fyy \end{bmatrix}$$

If f is C^2 at $\vec{a}$ [3], then **mixed partial derivatives commute**:

The order of taking derivatives does not change the final answer: $f_{xy} = f_{yx} \rightarrow$ Taking derivative in x then y is the same as taking in y then x.

[2] This means all partial derivatives exist near $\vec{a}$ and at $\vec{a}$, and the partial derivatives are **continuous** (as functions) at $\vec{a}$.

[3] This means all the 2nd order partial derivatives exist near $\vec{a}$ and at $\vec{a}$, and the 2nd order partial derivatives are continuous (as functions) at $\vec{a}$ (In practice, a function can fail to be continuous when there is division by zero).

In other words, if f is C^2, then **the Hessian Matrix is symmetric**.

We can also construct the Hessian for $f(x,y,z)$, which will be a 3×3 matrix. We can also construct higher order derivatives, such as 3rd order f_{xyz} or f_{xyy}. In general, if f is C^k, then mixed partial derivatives commute (up to order k).

Exercise 3.12 Find the Hessian Matrix.

1. $f(x,y) = 3x^2 + 4xy + 5y^2$
2. $f(x,y) = \cos(x+2y)$
3. $f(x,y) = x^{2x+y}$
4. $f(x,y) = e^{x^2+y}$
5. $f(x,y) = e^x \sin(y)$
6. $f(x,y,z) = x^2y + xz + z^2$

■

Exercise 3.13

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find f_x and f_y at $(x,y) = (0,0)$ **and** $(x,y) \neq (0,0)$. Then show $f_{xy} \neq f_{yx}$ at $(0,0)$. i.e. Taking 2nd derivatives in different orders give different answers.

Is f C^2 at $(0,0)$ (and have we reacher a contradiction)? ■

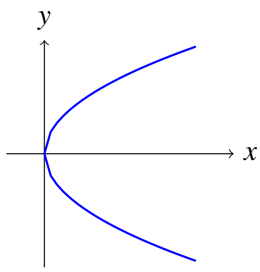
II Appendices

Solutions to Exercise Questions

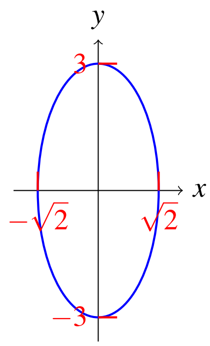
Chapter 1

Exercise 1.1

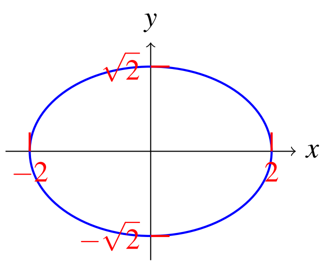
1. $x = 2y^2$



2. $$\frac{x^2}{2} + \frac{y^2}{9} = 1$$
$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$



3. $$x^2 + 2y^2 = 4$$
$$\frac{x^2}{4} + \frac{2y^2}{4} = 1$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$

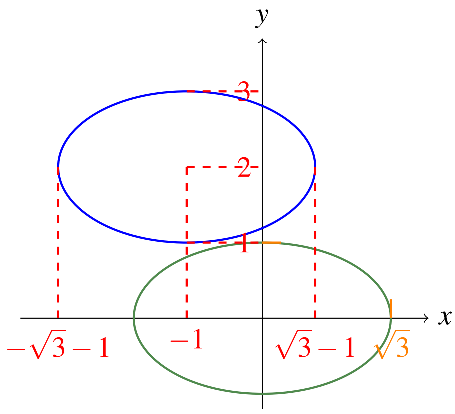


4. $$x^2+3y^2+2x-12y+10=0$$
$$(x^2+2x+1)+3(y^2-4y+4)-3=0$$
$$(x+1)^2+3(y-2)^2=3$$
$$\left(\frac{x+1}{\sqrt{3}}\right)^2+\left(\frac{y-2}{1}\right)^2=1$$

Ellipse before the shift (in green):

$$\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y}{1}\right)^2=1$$

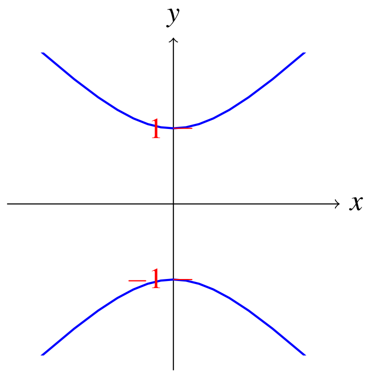
Then, shift 1 unit left and 2 units up.



5. $y^2-x^2=1$

If $y=0$, then $-x^2=1$, not possible. Thus, the graph must not cross the horizontal axis, and the hyperbola opens **up and down**.

If $x=0$, then $y^2=1$, $y=\pm 1$.



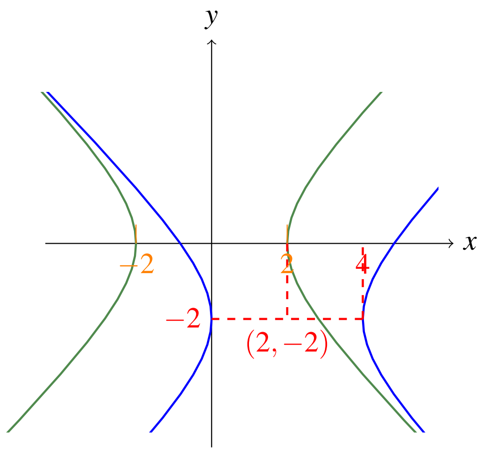
6. $$\frac{(x-2)^2}{4}-\frac{(y+2)^2}{9}=1$$
$$\left(\frac{x-2}{2}\right)^2-\left(\frac{y+2}{3}\right)^2=1$$

Hyperbola before the shift (in green):

$$\left(\frac{x}{2}\right)^2-\left(\frac{y}{3}\right)^2=1$$

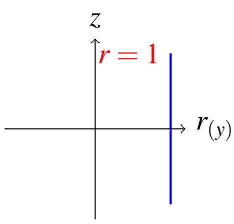
If $x=0$, then $-y^2=1$, not possible. Thus, the graph must not cross the vertical axis, and the hyperbola opens **left and right**.

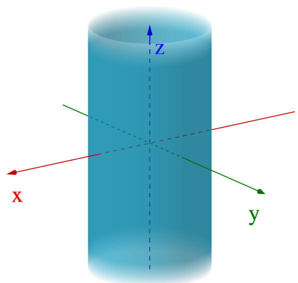
If $y=0$, then $\left(\frac{x}{2}\right)^2=1$, $x=\pm 2$.



Exercise 1.2

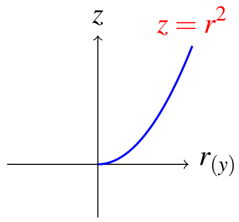
1. $$x^2+y^2=1$$
$$r^2=1$$
$$r=1$$

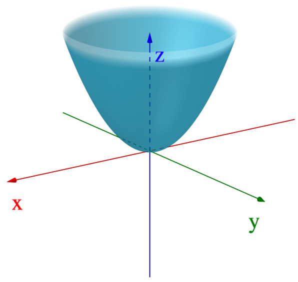




2. $z = x^2 + y^2$

$z = r^2$

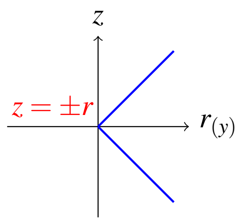


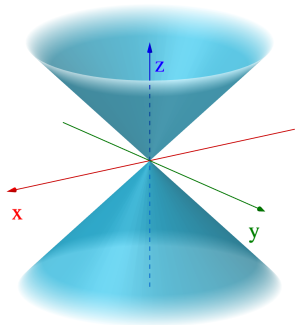


3. $z^2 = x^2 + y^2$

$z^2 = r^2$

$z = \pm r$

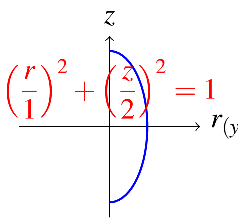


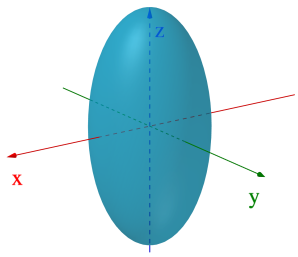


4. $$x^2 + y^2 + \frac{z^2}{4} = 1$$

$$r^2 + \frac{z^2}{4} = 1$$

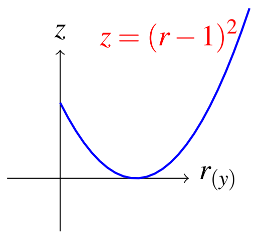
$$\left(\frac{r}{1}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

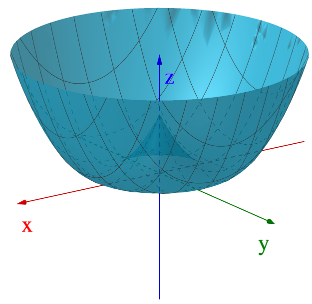




5. $$z = \left(\sqrt{x^2 + y^2} - 1\right)^2$$

$$z = (r-1)^2$$

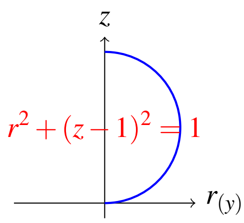


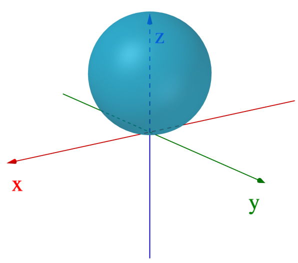


6. $$x^2 + y^2 + z^2 = 2z$$

$$r^2 + z^2 - 2z + 1 = 1$$

$$r^2 + (z-1)^2 = 1$$





Chapter 2

Exercise 2.1

1. $$\det\begin{pmatrix}3 & -1\\ 2 & 5\end{pmatrix} = 3\times 5 - 2\times(-1)$$
$$= 15+2$$
$$= 17$$

2. $$\det\begin{pmatrix}1 & 3 & -3\\ -3 & -5 & 2\\ -4 & 4 & -6\end{pmatrix} = 1\cdot\det\begin{pmatrix}-5 & 2\\ 4 & -6\end{pmatrix} - 3\cdot\det\begin{pmatrix}-3 & 2\\ -4 & -6\end{pmatrix} + (-3)\cdot\det\begin{pmatrix}-3 & -5\\ -4 & 4\end{pmatrix}$$
$$= 1\cdot 22 - 3\cdot 26 + (-3)\cdot(-32)$$
$$= 22 - 78 + 96$$
$$= 40$$

3. $$\det\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 4\\ 0 & 0 & 4\end{pmatrix} = 0\cdot\det\begin{pmatrix}1 & 0\\ 3 & 4\end{pmatrix} - 0\cdot\det\begin{pmatrix}3 & 0\\ 1 & 4\end{pmatrix} + 4\cdot\det\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} \quad \text{since} \begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$
$$= 0 - 0 + 4\cdot 8$$
$$= 32$$

Exercise 2.2

1. $$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{4^2+3^2} = 5$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{2^2+(-1)^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = 4\cdot 2 + 3\cdot(-1) = 5$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{5\cdot\sqrt{5}}$$
$$\theta = \arccos\left(\frac{1}{\sqrt{5}}\right)$$

2. $$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{4^2+0^2+2^2} = 2\sqrt{5}$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{2^2+(-1)^2+0^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = 4\cdot 2 + 0\cdot(-1) + 2\cdot 0 = 8$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{8}{2\sqrt{5}\cdot\sqrt{5}}$$
$$\theta = \arccos\left(\frac{4}{5}\right)$$

Exercise 2.3

$$
\begin{aligned}
\vec{x}\times\vec{y} &= \det\begin{pmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \\
&= i\cdot\det\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} - j\cdot\det\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} + k\cdot\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix} \\
&= (-1,-2,-1)
\end{aligned}
$$

Exercise 2.4

1. $$
\begin{aligned}
\vec{v} &= (3,-2,4)-(-8,0,4) \\
&= (11,-2,0)
\end{aligned}
$$
$$
\vec{r} = t\begin{bmatrix} 11 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}
$$
2. $\begin{cases} \vec{r}_0 = (0,0,0) \\ \vec{n} = (1,5,2) \end{cases}$
$$
\begin{aligned}
\vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}_0 \\
\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}\cdot\vec{r} &= \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}\cdot\vec{r} &= 0
\end{aligned}
$$
3. $\begin{cases} \vec{r}_0 = (2,4,6) \\ \vec{n} \ = (1,1,-1) \end{cases}$
$$
\begin{aligned}
\vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}_0 \\
\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\cdot\vec{r} &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} \\
\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\cdot\vec{r} &= 0
\end{aligned}
$$
4. $\vec{x} = (1,0,1)-(0,1,1) = (1,-1,0)$, and $\vec{y} = (1,1,0)-(0,1,1) = (1,0,-1)$.
$$
\begin{aligned}
\vec{n} &= \vec{x}\times\vec{y} \\
&= \det\begin{pmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
&= i\cdot\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - j\cdot\det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + k\cdot\det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$\begin{cases}\vec{r}_0 = (0,1,1)\\ \vec{n} \ = (1,1,1)\end{cases}$$

$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{r} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{r} = 0$$

5. $$\begin{aligned}\vec{r}_0 &= \frac{(3,1,5)+(-2,0,0)}{2}\\ &= \frac{1}{2}(1,1,5)\end{aligned}$$

$$\begin{aligned}\vec{n} &= (-2,0,0)-(3,1,5)\\ &= (-5,-1,-5)\end{aligned}$$

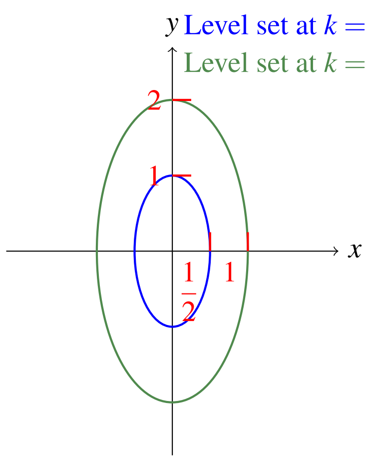
$$\begin{aligned}\vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}_0\\ (-5,1,-5)\cdot\vec{r} &= (-5,1,-5)\cdot\frac{1}{2}(1,1,5)\\ -5x-y-5z &= \frac{1}{2}(-5+1-25)\\ &= \frac{-31}{2}\end{aligned}$$

Chapter 3

Exercise 3.1

1. At $k=2$, $$\begin{aligned}4x^2+y^2+1 &= 2\\ 4x^2+y^2 &= 1\\ \left(\frac{x}{\frac{1}{2}}\right)^2+\left(\frac{y}{1}\right)^2 &= 1\end{aligned}$$

At $k=5$, $$\begin{aligned}4x^2+y^2+1 &= 5\\ 4x^2+y^2 &= 4\\ x^2+\frac{y^2}{4} &= 1\\ \left(\frac{x}{1}\right)^2+\left(\frac{y}{2}\right)^2 &= 1\end{aligned}$$

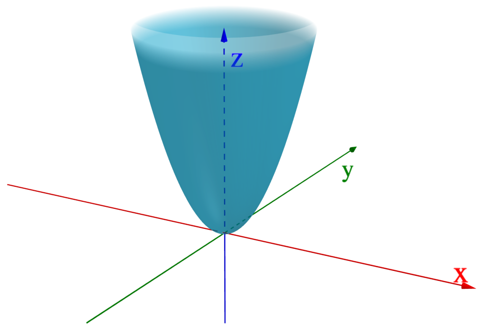


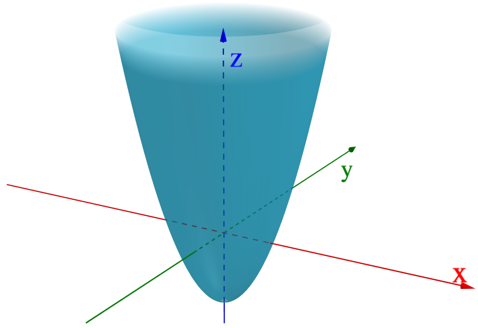
2. At $k=0$, $x^2+y^2-z=0$

$$x^2+y^2=z$$
$$r^2=z$$

At $k=2$, $x^2+y^2-z=2$

$$x^2+y^2-2=z$$
$$r^2-2=z$$





Exercise 3.2

- If $y=0: f = \dfrac{0}{x^4+0}$
  $= 0$
  Since $x^4 \neq 0$
- If $x=0: f = \dfrac{0}{0+y^2}$
  $= 0$
  Since $y^2 \neq 0$
- If $y=x: f = \dfrac{x^2x}{x^4+x^2}$
  $= \dfrac{x}{x+1} \to 0$
- If $y=x^2: f = \dfrac{x^2x^2}{x^4+(x^2)^2}$
  $= \dfrac{x^4}{2x^4}$
  $= \dfrac{1}{2}$ Since $x^4 \neq 0$

Exercise 3.3

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^4+y^2}}\right| \leq \left|\frac{xy}{\sqrt{y^2}}\right| = \left|\frac{xy}{y}\right|$$
$$= |x| \to 0$$

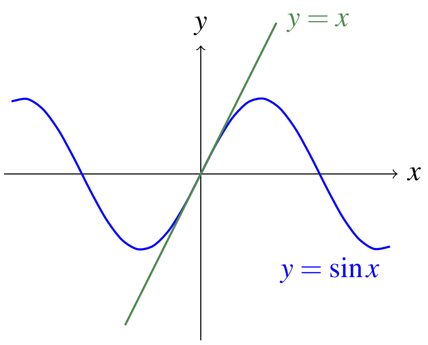
Thus, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^4+y^2}} = 0 = f(0,0)$.

Thus, the function is continuous at $(0,0)$.

Exercise 3.4

By removing $2x^4$ from the denominator, we obtain $\left|\dfrac{x^2\sin^2 y}{2x^4+y^2}\right| \leq \left|\dfrac{x^2\sin^2 y}{y^2}\right|$.

Note that $|\sin\theta| \leq |\theta|$, and that as $\theta \to 0$, $\sin\theta \approx \theta$. This is the *small angle approximation*.



So $\left|\frac{x^2\sin^2 y}{y^2}\right| \approx \left|\frac{x^2\cdot y^2}{y^2}\right| = |x^2| \to 0.$

Then, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{2x^4+y^2} = 0.$

Exercise 3.5

1. $\frac{\partial f}{\partial x} = 2x(y+1)$ and $\frac{\partial f}{\partial y} = x^2-1.$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x(y+1), x^2-1).$
2. $\frac{\partial f}{\partial x} = 2y(xy-2)$ and $\frac{\partial f}{\partial y} = 2x(xy-2).$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2y(xy-2), 2x(xy-2)).$
3. $\frac{\partial f}{\partial x} = \frac{-2}{(x+3y)^2}$ and $\frac{\partial f}{\partial y} = \frac{-6}{(x+3y)^2}.$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{-2}{(x+3y)^2}, \frac{-6}{(x+3y)^2}\right).$
4. $\frac{\partial f}{\partial x} = e^{x+4y}$ and $\frac{\partial f}{\partial y} = 4e^{x+4y}.$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^{x+4y}, 4e^{x+4y}).$
5. $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 2.$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 0, 2).$
6. $\frac{\partial f}{\partial x} = yze^{xz}$, $\frac{\partial f}{\partial y} = e^{xy}$, and $\frac{\partial f}{\partial z} = xye^{xz}.$
   So $\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (yze^{xz}, e^{xz}, xye^{xz}).$

Exercise 3.6

$$\begin{aligned}\vec{u} &= \frac{\vec{v}}{|\vec{v}|}\\ &= \frac{(1,3)}{\sqrt{1^2+3^2}}\\ &= \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\Delta f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\ &= \begin{bmatrix}2ye^{2x}\\ e^{2x}+2y\end{bmatrix}\\ \Delta f(0,0) &= \begin{bmatrix}2\cdot 0\cdot e^{2\cdot 0}\\ e^{2\cdot 0}+2\cdot 0\end{bmatrix}\\ &= \begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{So } D_{\vec{u}}f &= \Delta f\cdot\vec{u}\\ &= \begin{bmatrix}0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}\\ &= \frac{3}{\sqrt{10}}\end{aligned}$$

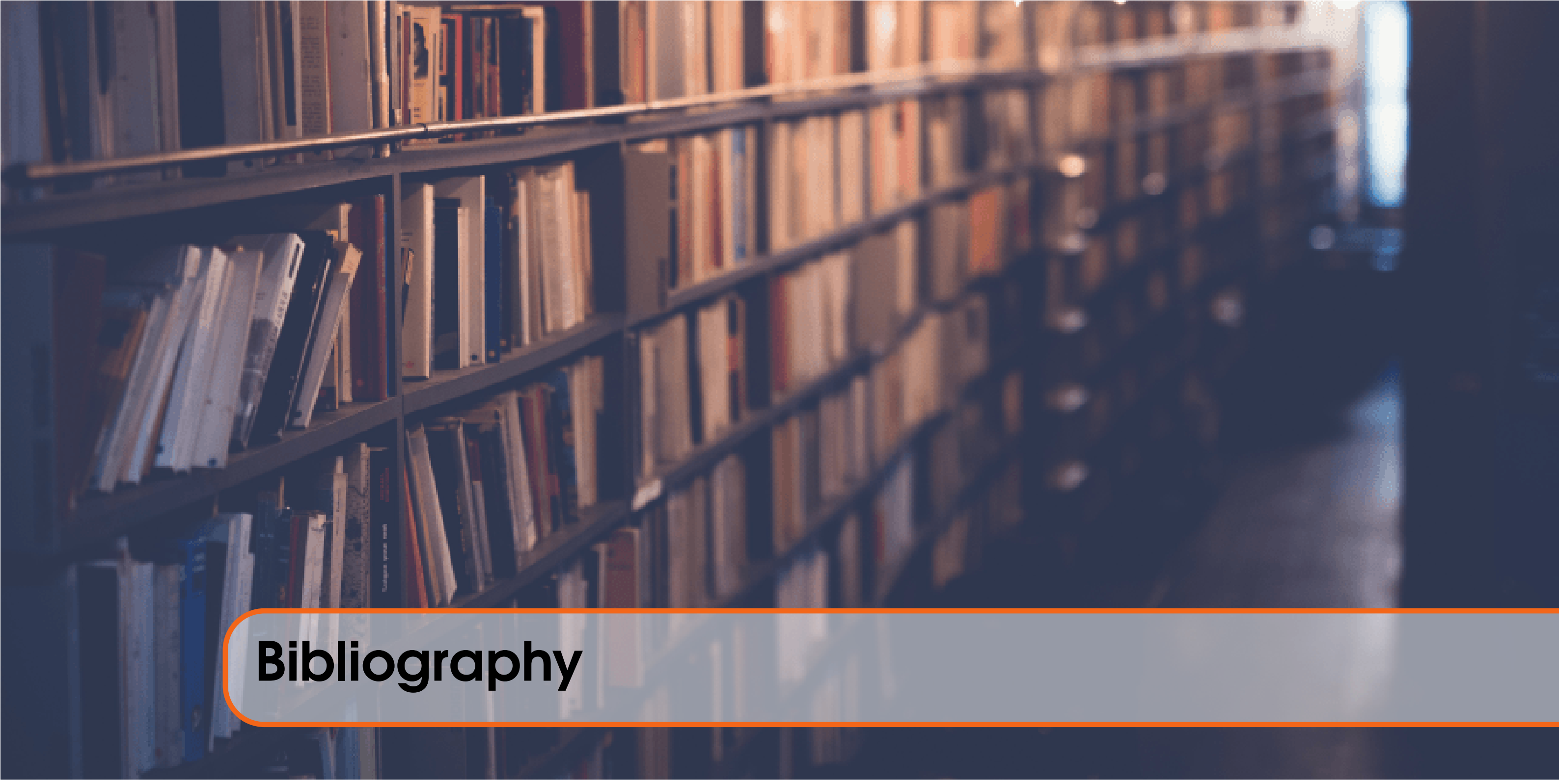
Exercise 3.7

$$\Delta f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \begin{bmatrix}2xy\\ x^2\\ 1\end{bmatrix}.$$

The directional vector, $\Delta f(2,2,1) = \begin{bmatrix} 2\cdot 2\cdot 2 \\ 2^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$.

The maximum rate of change is $|\Delta f(2,2,1)| = \sqrt{8^2+4^2+1} = 9$.

Thus, the maximum rate of change at $(2,2,1)$ is 9 in the $\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$ direction.



Bibliography

[SCW19] James Stewart, Daniel Clegg, and Saleem Watson. *Calculus: Early Transcendentals*. 9th edition. Boston: Cengage Learning, Inc., 2019 (cited on page 7).

Index

P

R

S

T

U

V