MAT235

Multivariable Calculus

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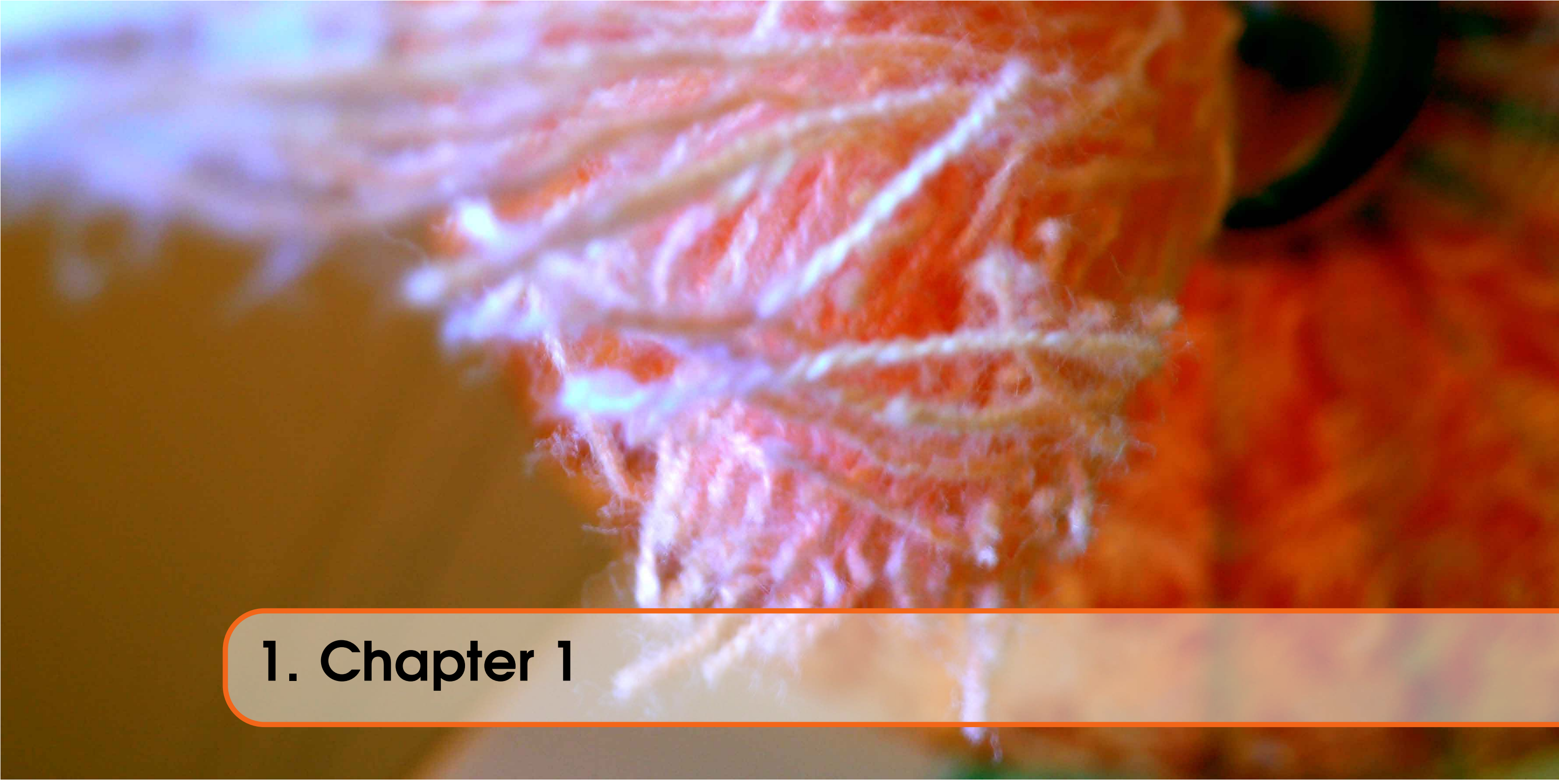


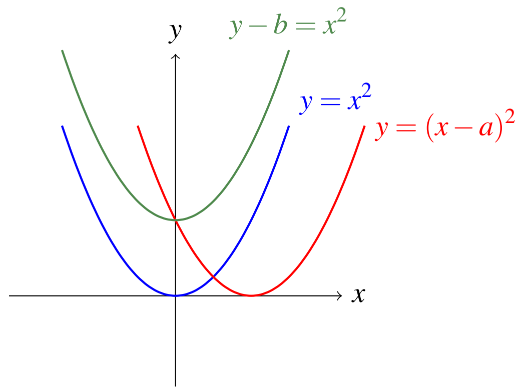
1. Chapter 1

1.1 Curved Lines and Surfaces

1.1.1 Curved Lines

An equation involving x and y gives a *curve* in the plane.

By replacing x with $x-a$, the graph is shifted to the right by a. By replacing y with $y-b$, the graph is shifted upward by b.



Definition 1.1.1 — Parabola. The *parabola* (with vertex) at the origin is defined by

$$y = ax^2 \qquad x = by^2$$

- $y = ax^2$ opens up for $a > 0$,and opens down for $a < 0$.
- $x = by^2$ opens to the right for $b > 0$, and opens to the left for $b < 0$.

Definition 1.1.2 — Ellipse. The *ellipse* (with center) at the origin is defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

with $(a,0)$ being the right-most point on the x-axis, and $(0,b)$ as the top-most point on the y-axis.

When $a=b$, an ellipse becomes a *circle*, with $a=b=R$ as the radius of the circle.

Definition 1.1.3 — Hyperbola. The *hyperbola* (with center) at the origin is defined by

$$\left(\frac{x}{2}\right)^2-\left(\frac{y}{b}\right)^2=1 \qquad \left(\frac{y}{a}\right)^2-\left(\frac{x}{b}\right)^2=1$$

A hyperbola has 2 curves, and a center line that is not crossed. The 2 curves goes in the opposite direction of the center.

Since the right side is positive, and a square is positive, the position of the negative sign determines the direction of the curves.

To determine the direction, we can either set $x=0$ or $y=0$. For example, in the case of $x^2-y^2=1$, by setting $x=0$, we attain $-y^2=1$ is impossible, so the vertical line $x=0$ is the center is never crossed. Thus the 2 curves open towards left and right.

To find the **slant asymptotes** when x and y are both large, change the number 1 on the right side of the equation to 0.

The above 3 curves together are sometimes called: *Conic Sections*.

1.1.2 Curved Surfaces

An equation involving x, y, and z gives a *curved surface* in 3D space. In general, this surface is difficult to imagine and sketch in 3D.

Definition 1.1.4 — 3D Sphere. The *3D Sphere* (with center) at the origin with radius R is given by

$$x^2+y^2+z^2=R^2$$

Definition 1.1.5 — Ellipsoid. The 3D *ellipsoid* (with center) at the origin is given by

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$$

$(a,0,0)$, $(0,b,0)$, $(0,0,c)$ are the 3 points on the ellipsoid, similar to the ellipse.

When $a=b=c$, an ellipsoid becomes a *sphere*, with $a=b=c=R$ as the radius.

Surfaces Attained Through Rotation: $r^2=x^2+y^2$

When the variables x and y appear together as x^2+y^2, this signals that the surface is attained through rotation. Set a new variable r, with $r^2=x^2+y^2$. We graph the equation involving z and r in the rz-plane, with $r>0$ only. We then revolve the curve around the z-axis to attain the surface in 3D: the r-axis stands for both x-axis and y-axis.

Exercise 1.1 Sketch the following curves.

1. $x=2y^2$
2. $\dfrac{x^2}{2}+\dfrac{y^2}{9}=1$
3. $x^2+2y^2=4$
4. $x^2+3y^2+2x-12y+10=0$
5. $y^2-x^2=1$
6. $\dfrac{(x-2)^2}{4}-\dfrac{(y+2)^2}{9}=1$

■

Exercise 1.2 Sketch the following surfaces.

1. $x^2+y^2=1$
2. $z=x^2+y^2$
3. $z^2=x^2+y^2$
4. $x^2+y^2+\frac{z^2}{4}=1$
5. $z=\left(\sqrt{x^2+y^2}-1\right)^2$
6. $x^2+y^2+z^2=2z$

1.2 Vector and Matrix

A point in 2D needs 2 coordinates. It is typically written as, for example:

$$\vec{a}=1 \quad \text{or} \quad \vec{a}=\begin{bmatrix}1\\2\end{bmatrix}$$

This is the point $x=1$ and $y=2$ on the 2D xy-plane.

A point in 3D needs 3 coordinates. It is typically written as, for example:

$$\vec{b}=(0,2,1) \quad \text{or} \quad \vec{b}=\begin{bmatrix}0\\2\\1\end{bmatrix}$$

This is the point $x=0$, $y=2$, and $z=1$ in 3D space.

The point can also be interpreted as a *vector*, written in the same way. We can also think of a vector as an *arrow* from the origin to the point.

Typically, it doesn't matter if we write the vector horizontally or vertically.

Definition 1.2.1 — Matrix. A *matrix* is a block of numbers written in a rectangle (or square) in a specific order.

For example, A is a 2×3 (2 by 3) matrix, with 2 rows and 3 columns:

$$A=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$$

We may think of vectors as $1\times n$ or $n\times 1$ matrix (for $n=2$ or 3). We can add matrices, and multiply matrices by a scalar (number).

- $\begin{bmatrix}a & b\\ c & d\end{bmatrix}+\begin{bmatrix}e & f\\ g & h\end{bmatrix}=\begin{bmatrix}a+e & b+f\\ c+g & d+h\end{bmatrix}$
- $r\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}ra & rb & rc & rd\end{bmatrix}$

Standard arithmetic rules apply. For matrices A, B, and scalar c:

- $A+B=B+A$
- $cA=Ac$
- $c(A+B)=cA+cB$

1.2.1 Determinant

The determinant of a **square** matrix is a number.

Start at the first row, first position, write down this value, and remove this row and this column from the original matrix. Multiply this value to the determinant of the remaining matrix.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} \cdots$$

Now move right, write down this value, and remove this row and this column from the original matrix. Multiply this value to the determinant of the remaining matrix with a **minus sign**.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

Now move right again and repeat until we reached the last column. The positive and negative sign need to **alternate**.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Each time we apply the algorithm, we end up with several new determinants to calculate, but with matrices of *smaller sizes*.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad = bc$$

Determinant does not behave well with addition and scalar multiplication:

$$\det(A+B) \neq \det(A) + \det(B) \qquad \det(cA) \neq c \cdot \det(A)$$

Exercise 1.3 Find the determinant.

1. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

■

1.3 Dot product

2 vectors can form a *dot product*, and the result is a scalar (number).

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots$$

Note that this dot product is different from scalar multiplication, as we are multiplying 2 vectors together, not a scalar with a vector.

The *length* (absolute value) of a vector using the Pythagoras Theorem:

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{(x_1)^2 + (x_2)^2 + \cdots}$$

Similarly, define the distance between two points using pythagoras Theorem:

$$d(\vec{x},\vec{y}) = |\vec{x}-\vec{y}| = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots}$$

Going from a point a to point c, is always shorter than going to some other point b first and then back to c. This is called the *Triangle Inequality*:

$$d(\vec{a},\vec{c}) \leq d(\vec{a},\vec{b})+d(\vec{b},\vec{c})$$

$$|\vec{x}+\vec{y}| \leq |\vec{x}|+|\vec{y}|$$

Let θ be the angle between $\vec{x}$ and $\vec{y}$, the dot product is also given by

$$\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos\theta$$

Thus, 2 vectors are *orthogonal* (perpendicular) if the dot product is zero.

The dot product have the usual properties of multiplication:

- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- $c(\vec{a}\cdot\vec{b}) = (c\vec{a})\cdot\vec{b} = a\cdot(c\vec{b})$

Note that since the dot product needs two vectors and produce a number, a quantity such as $\vec{a}\cdot\vec{b}\cdot\vec{c}$ is not well defined.

Exercise 1.4 Find the length of the vectors. Find $\vec{a}\cdot\vec{b}$ and the angle between them.

1. $\vec{a} = (4,3), \vec{b} = (2,-1)$
2. $\vec{a} = (4,0,2), \vec{b} = (2,-1,0)$

■

1.4 Cross product

2 vectors (in 3D) can form a *cross product*, and the result is a *vector* (in 3D).

Let $\vec{x} = (x_1,x_2,x_3)$, $\vec{y} = (y_1,y_2,y_3)$ in 3D.

We typically write the cross product using determinant:

$$\vec{x}\times\vec{y} = \det\begin{bmatrix} i & i & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\vec{x}\times\vec{y} = i(x_2y_3-x_3y_2)-j(x_1y_3-x_3y_1)+k(x_1y_2-x_2y_1)$$

$$\vec{x}\times\vec{y} = (x_2y_3-x_3y_2, x_1y_3-x_3y_1, x_1y_2-x_2y_1)$$

The notations $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ are very easy to use, where the quantity attached to i is the first component of the vector, and the quantity attached to j would be the second, and k would be the third.

The vector produced by the cross product, $\vec{x}\times\vec{y}$, would be orthogonal to both vectors $\vec{x}$ and $\vec{y}$. Notice that in most cases, there would be 2 such vectors with this property. They are exactly

$$\vec{x}\times\vec{y} \qquad \text{and} \qquad -\vec{x}\times\vec{y}$$

In fact, we have

$$\vec{x}\times\vec{y} = -\vec{y}\times\vec{x}$$

Notice that the order of a dot product does not matter, but the order of cross product matters up to a negative sign.

The other usual properties of multiplication hold:

- $(a\vec{a}\times\vec{b}) = c(\vec{a}\times\vec{b}) = a\times(c\vec{b})$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Similar to the dot product, we can also relate the angle between the 2 vectors,

$$|\vec{x}\times\vec{y}| = |\vec{x}||\vec{y}|\sin\theta$$

Note that since $\vec{x}\times\vec{y}$ is a vector, we need to take its absolute value. So 2 vectors are *parallel* if the cross product is zero.

Exercise 1.5 Let $\vec{x} = (3,-2,1)$, $\vec{y} = (1,-1,1)$. Find $\vec{x}\times\vec{y}$. ■

1.5 Lines and Planes

For a *line* to be defined, we need a direction $\vec{v}$ and a point on the line $\vec{r}_0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} := \vec{r} = t\vec{v}+\vec{r}_0$$

The vector $\vec{r}$ and the value t do not need to be determined [1].

For a *plane* to be defined, we need a **normal vector** $\vec{n} = (a,b,c)$ and a point on the plane $\vec{r}_0$.

$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$

$$ax+by+cz = \vec{n}\cdot\vec{r}_0$$

where $\vec{r}$ is the same as above and does not need to be determined. The *normal vector* $\vec{n}$ is orthogonal to the plane [2].

Plane Defined by 3 Points

Consider 3 points $\vec{u}$, $\vec{v}$, and $\vec{w}$. We can connect any 2 pairs of points together to create 2 direction vectors which are inside the plane. For example, we may take $\vec{x} = \vec{u}-\vec{v}$, and$\vec{y} = \vec{w}-\vec{v}$. From there we can form the cross product to attain a vector that is orthogonal to both vectors inside the plane, so the cross product would be the normal vector, which is orthogonal to the plane.

Exercise 1.6 Find the equation of the line or plane.

1. The line through $(-8,0,4)$ and $(3,-2,4)$.
2. The plane through the origin and perpendicular to the vector $(1,5,2)$.
3. The plane through $(2,4,6)$ parallel to the plane $x+y-z=5$.
4. The plane through $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.
5. The plane equidistant from point $(3,1,5)$ and $(-2,0,0)$.

■

1.6 Multivariable Function

In first year, we study 1D functions: *functions of one variable*, as $f(x)$. This can be plotted in 2D plane, with $y=f(x)$. Now, we can study functions of multiple variables.

[1] $\vec{v}$ and $\vec{r}_0$ are not unique, and any correct pair would give the right equation.
[2] Similar to lines, $\vec{v}$ and $\vec{r}_0$ are not unique.

Definition 1.6.1 — 2D Function. A *2D function*, $f(x,y)$, has 2 inputs being x, y, and 1 output.

Sometimes, we can label the output as $z = f(x,y)$.

We can plot this 2D function in 3D space, where the height z above the point (x,y) on the xy-plane is equal to $f(x,y)$.

This would give a graph, which is a surface.

Alternatively, $z = f(x,y)$ can be thought of as an equation which involves x, y, and z, which would also give a surface in 3D space.

However, similar to 1D functions, given an equation involving x, y, and z, it is not always possible to isolate z into $z = f(x,y)$.

Examples of 2D functions include:

$$f(x,y) = x^2 + 2y \qquad f(x,y) = xe^{xy} \qquad f(x,y) = \sin(xy^2)$$

Definition 1.6.2 — 3D Function. A *3D function*, $f(x,y,z)$, has 3 inputs being x, y, and z, and 1 output.

It is possible to label the output as $w = f(x,y,z)$, but this is typically not useful as this introduces a 4th variable. The "plot" of this function would be in 4D space, which does not exist.

Examples of 3D functions include:

$$f(x,y,z = x^2 + 2z - 1) \qquad f(x,y,z) = ye^{xz}$$

Definition 1.6.3 — Level Set. The *level set* of f at k is given by setting f to be equal to the constant k. This will reduce the dimension of f by 1.

Exercise 1.7

- Draw the level set of $f(x,y) = 4x^2 + y^2 + 1$ for $k = 2$ and $k = 5$.
- Draw the level set of $f(x,y,z) = x^2 + y^2 - z$ for $k = 0$ and $k = 2$.

1.6.1 Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if

$$\lim_{(x,y)\to\vec{a}} f(x,y) = f(\vec{a})$$

Every standard function we know are continuous in their domains. Any function transformations (such as $+$, $-$, $\times$, $/$, composition) of continuous functions, are also continuous (except division by 0).

As most functions are continuous, most limits can be obtained by putting $(x,y) = \vec{a}$ in the formula of $f(x,y)$.

Different from 1D functions, computing limits in 2D is much more difficult. Most limit evaluation techniques from 1D functions does not work for 2D functions. In particular, there is no L'Hôpital's rule Rule for 2D functions.

Fortunately, since most functions are continuous except when dividing by 0, we typically only need to focus on the point where the function is dividing by 0 (and state the function is continuous everywhere else).

It turns out that it is much easier to show a limit does not exist.

Test for limit that does not exist in $\mathbb{R}^2$

Given $f(x,y)$, compute the limit as (x,y) approaches $\vec{a}=(0,0)$.

1. Replace y with a easy curve $y=g(x)$, passing through $\vec{a}=(0,0)$, (such as $y=0$, $y=x$, $y=x^2$), then let x approach 0 to attain a (1D) limit.
2. Replace x with a easy curve $x=h(y)$, passing through $\vec{a}=(0,0)$, (such as $x=0$, $x=y^2$), then let y approach 0 to attain a (1D) limit.
3. Try with several different $g(x)$ and $h(y)$ to attain many (1D) limits.
4. If you find 2 **different** (1D) limits generated by 2 different curves, then the (2D) limit of (x,y) approaches $\vec{a}=(0,0)$ of $f(x,y)$ does not exist.

Note that if all the 1D limits are the same, that is **not** enough for you to conclude the 2D limit exists. To show the limit exists, we typically must use *Squeeze Theorem*.

Since most limit evaluation techniques does not work for 2D functions, it is a very fortunate fact that Squeeze Theorem does work for 2D functions. The formulation is almost the same as the 1D version.

Theorem 1.6.1 — Squeeze Theorem. To attain $\lim_{(x,y)\to\vec{a}} f(x,y)$, we can try to find $g(x,y)$ and $h(x,y)$ such that

1. $g(x,y)\le f(x,y)\le h(x,y)$ near the point $\vec{a}$
2. $\lim_{(x,y)\to\vec{a}} g(x,y)=L=\lim_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y)=L$.

Use Squeeze Theorem to show limit exist

In practice, we typically want to show $\lim_{(x,y)\to\vec{a}} f(x,y)=0$.

Start at $|f(x,y)|$, create a *chain of inequalities*, and simplify $|f(x,y)|$ to attain $|g(x,y)|$, which has limit 0 as (x,y) approaches $\vec{a}$:

$$|f(x,y)|\le\cdots\le|g(x,y)|\to 0 \quad \text{i.e.} \quad \lim_{(x,y)\to\vec{a}}|g(x,y)|=0$$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y)=0$.

The typical strategy in constructing the above inequality is to remove positive quantities from the denominator of $f(x,y)$.

Exercise 1.8

$$f(x,y)=\frac{x^2y}{x^4+y^2} \quad \text{and} \quad f(0,0)=0$$

Show the limit at $(x,y)=(0,0)$ does not exist. ■

Exercise 1.9

$$f(x,y)=\frac{xy}{\sqrt{x^4+y^2}} \quad \text{and} \quad f(0,0)=0$$

Show the function **continuous** at $(0,0)$. ■

Exercise 1.10 Find $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{2x^4+y^2}$ ■