MAT235

Multivariable Calculus

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Introduction

Parametric equations and polar coordinates. Vectors, vector functions and space curves. Differential and integral calculus of functions of several variables. Line integrals and surface integrals and classic vector calculus theorems. Examples from life sciences and physical science applications.

This is a second-year **Multivariable Calculus** course. The depth of mathematics will be taught at the standard level accessibly to all second-year undergraduate students who has fully finished any first year Introduction to Calculus course. The course will have significant emphasis on computation. In general, theorems will be stated without proofs, but with an indication of the mathematical ideas involved. No emphasis will be put on rigorous mathematic proofs.

Requirements

Any full first-year Introduction to Calculus course is acceptable, preferably with depth taught at the standard level for first-year undergraduate students. Some examples of the courses include, but are not limited to: MAT135/136; MAT137; MAT157; MAT133; There will be a review of key concepts that we need from Linear Algebra at the beginning of the course, so it is not necessary to have any background in linear algebra.

Textbook

- Single Variable Calculus: Early Transcendentals, Stewart (8th/9th Edition) [1]
- Advanced Calculus, Folland

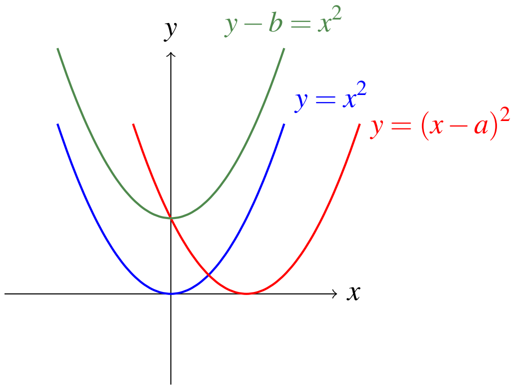
The textbooks are optional, and are not necessary to the course.

1. Conic Sections

1.1 Curved Lines

An equation involving x and y gives a *curve* in the plane.

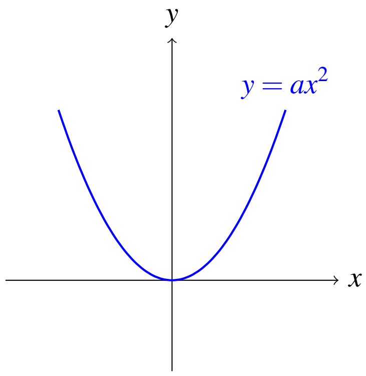
By replacing x with $x-a$, the graph is shifted to the right by a. By replacing y with $y-b$, the graph is shifted upward by b.

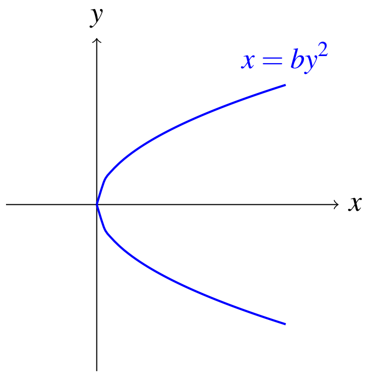


Definition 1.1.1 — Parabola. The *parabola* (with vertex) at the origin is defined by

$$y = ax^2 \qquad x = by^2$$

- $y = ax^2$ opens up for $a > 0$, and opens down for $a < 0$.
- $x = by^2$ opens to the right for $b > 0$, and opens to the left for $b < 0$.





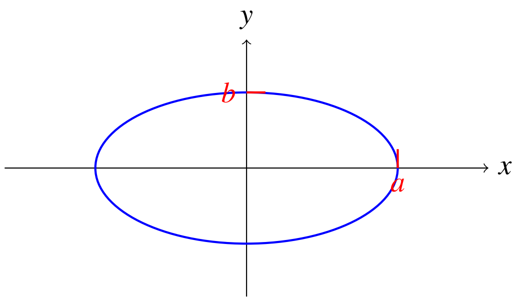
Definition 1.1.2 — Ellipse. The *ellipse* (with center) at the origin is defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

with $(a,0)$ being the right-most point on the x-axis, and $(0,b)$ as the top-most point on the y-axis.

When $a = b$, an ellipse becomes a *circle*, with $a = b = R$ as the radius of the circle [a].

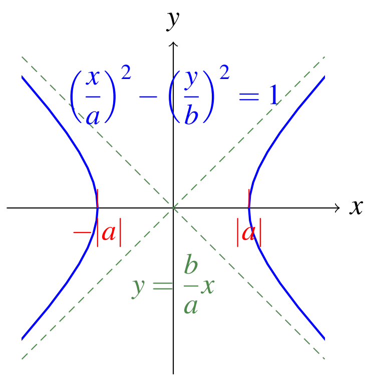
[a] $\left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2 = 1 \implies x^2 + y^2 = R^2$



Definition 1.1.3 — Hyperbola. The *hyperbola* (with center) at the origin is defined by

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \qquad \left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$$

A hyperbola has 2 curves, and a center line that is not crossed. The 2 curves goes in the opposite direction of the center.



Since the right side is positive, and a square is positive, the position of the negative sign determines the direction of the curves.

To determine the direction, we can either set $x = 0$ or $y = 0$. For example, in the case of $x^2 - y^2 = 1$, by setting $x = 0$, we attain $-y^2 = 1$ is impossible, so the vertical line $x = 0$ is the center is never crossed. Thus the 2 curves open towards left and right.

To find the **slant asymptotes** when x and y are both large, change the number 1 on the right side of the equation to 0.

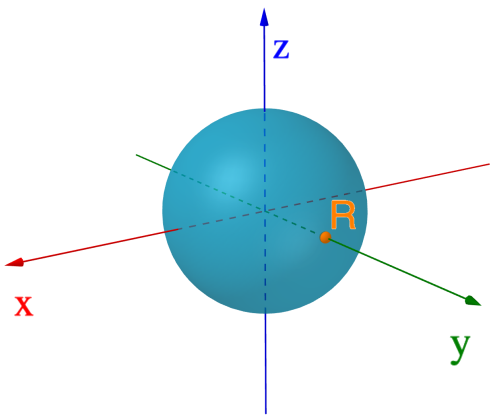
The above 3 curves together are sometimes called: *Conic Sections*.

1.2 Curved Surfaces

An equation involving x, y, and z gives a *curved surface* in 3D space. In general, this surface is difficult to imagine and sketch in 3D.

Definition 1.2.1 — 3D Sphere. The *3D Sphere* (with center) at the origin with radius R is given by

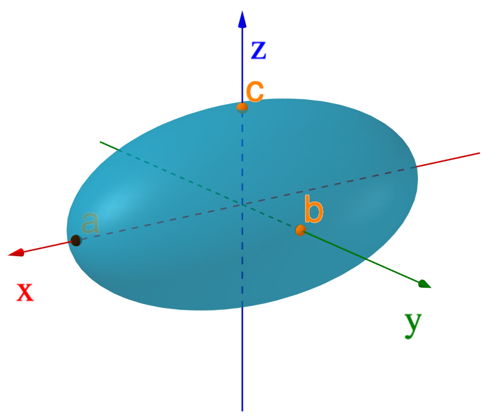
$$x^2 + y^2 + z^2 = R^2$$



Definition 1.2.2 — Ellipsoid. The 3D *ellipsoid* (with center) at the origin is given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

$(a,0,0)$, $(0,b,0)$, $(0,0,c)$ are the 3 points on the ellipsoid, similar to the ellipse.
When $a = b = c$, an ellipsoid becomes a *sphere*, with $a = b = c = R$ as the radius.



Surfaces Attained Through Rotation: $r^2 = x^2 + y^2$

When the variables x and y appear together as $x^2 + y^2$, this signals that the surface is attained through rotation. Set a new variable r, with $r^2 = x^2 + y^2$. We graph the equation involving z and r in the rz-plane, with $r > 0$ only. We then revolve the curve around the z-axis to attain the surface in 3D: the r-axis stands for both x-axis and y-axis.

Exercise 1.1 Sketch the following curves.

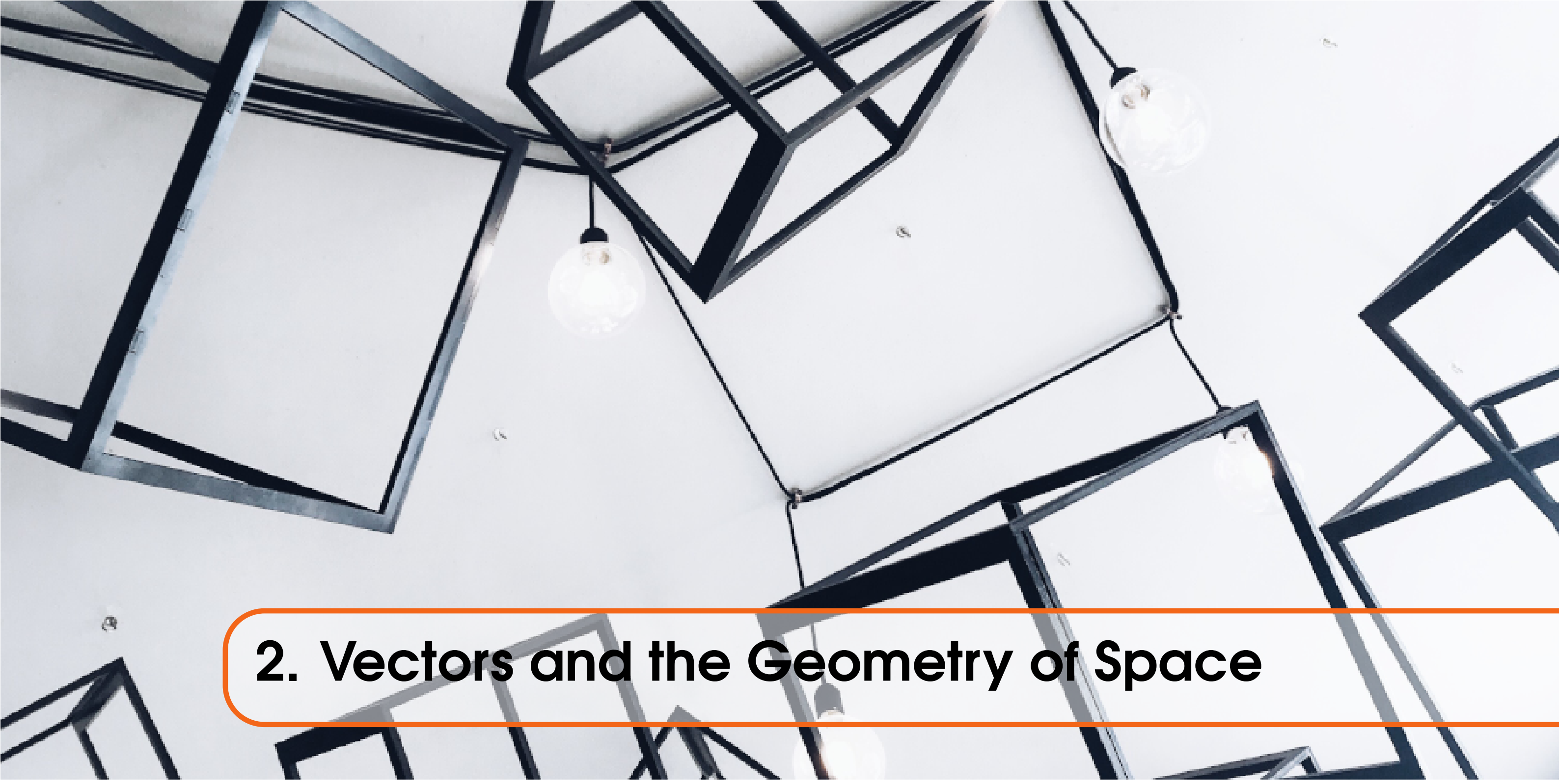
1. $x = 2y^2$
2. $\dfrac{x^2}{2} + \dfrac{y^2}{9} = 1$
3. $x^2 + 2y^2 = 4$
4. $x^2 + 3y^2 + 2x - 12y + 10 = 0$
5. $y^2 - x^2 = 1$
6. $\dfrac{(x-2)^2}{4} - \dfrac{(y+2)^2}{9} = 1$

■

Exercise 1.2 Sketch the following surfaces.

1. $x^2 + y^2 = 1$
2. $z = x^2 + y^2$
3. $z^2 = x^2 + y^2$
4. $x^2 + y^2 + \dfrac{z^2}{4} = 1$
5. $z = \left(\sqrt{x^2 + y^2} - 1\right)^2$
6. $x^2 + y^2 + z^2 = 2z$

■



2. Vectors and the Geometry of Space

2.1 Vector and Matrix

A point in 2D needs 2 coordinates. It is typically written as, for example:

$$\vec{a} = (1,2) \quad \text{or} \quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This is the point $x = 1$ and $y = 2$ on the 2D xy-plane.

A point in 3D needs 3 coordinates. It is typically written as, for example:

$$\vec{b} = (0,2,1) \quad \text{or} \quad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

This is the point $x = 0$, $y = 2$, and $z = 1$ in 3D space.

The point can also be interpreted as a *vector*, written in the same way. We can also think of a vector as an *arrow* from the origin to the point.

Typically, it doesn't matter if we write the vector horizontally or vertically.

Definition 2.1.1 — Matrix. A *matrix* is a block of numbers written in a rectangle (or square) in a specific order.

For example, A is a 2×3 (2 by 3) matrix, with 2 rows and 3 columns:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

We may think of vectors as $1 \times n$ or $n \times 1$ matrix (for $n = 2$ or 3). We can add matrices, and multiply matrices by a scalar (number).

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

- $r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb & rc & rd \end{bmatrix}$

Standard arithmetic rules apply. For matrices A, B, and scalar c:

- $A+B=B+A$
- $cA=Ac$
- $c(A+B)=cA+cB$

2.2 Determinant

The determinant of a **square** matrix is a number.

Starting at the first row, first position, write down this value, and remove this row and this column from the original matrix. Multiply this value by the determinant of the remaining matrix.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} \dots$$

Now, move right, write down this value, and remove this row and this column from the original matrix. Multiply this value to the determinant of the remaining matrix with a **minus sign**.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

Now, move right again and repeat until we have reached the last column. The positive and negative signs need to **alternate**.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Each time we apply the algorithm, we end up with several new determinants to calculate, but with matrices of *smaller sizes*.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad = bc$$

Determinant does not behave well with addition and scalar multiplication:

$$\det(A+B) \neq \det(A) + \det(B) \qquad \det(cA) \neq c \cdot \det(A)$$

Exercise 2.1 Find the determinant.

1. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

■

2.3 Dot product

2 vectors can form a *dot product*, and the result is a scalar (number).

$$\vec{x}\cdot\vec{y}=\begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix}\cdot\begin{bmatrix}y_1\\y_2\\\vdots\end{bmatrix}=x_1y_1+x_2y_2+\cdots$$

Note that this dot product is different from scalar multiplication, as we are multiplying 2 vectors together, not a scalar with a vector.

The *length* (absolute value, or *norm*) of a vector using the Pythagoras Theorem:

$$|\vec{x}|=\sqrt{\vec{x}\cdot\vec{x}}=\sqrt{(x_1)^2+(x_2)^2+\cdots}$$

Similarly, define the distance between two points using pythagoras Theorem:

$$d(\vec{x},\vec{y})=|\vec{x}-\vec{y}|=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots}$$

Going from a point a to point c, is always shorter than going to some other point b first and then back to c. This is called the *Triangle Inequality*:

$$d(\vec{a},\vec{c})\le d(\vec{a},\vec{b})+d(\vec{b},\vec{c})$$

$$|\vec{x}+\vec{y}|\le|\vec{x}|+|\vec{y}|$$

Let θ be the angle between $\vec{x}$ and $\vec{y}$, the dot product is also given by

$$\vec{x}\cdot\vec{y}=|\vec{x}||\vec{y}|\cos\theta$$

Thus, 2 vectors are *orthogonal* (perpendicular) if the dot product is zero.

The dot product have the usual properties of multiplication:

- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- $c(\vec{a}\cdot\vec{b})=(c\vec{a})\cdot\vec{b}=a\cdot(c\vec{b})$

Note that since the dot product needs two vectors and produce a number, a quantity such as $\vec{a}\cdot\vec{b}\cdot\vec{c}$ is not well defined.

Exercise 2.2 Find the length of the vectors. Find $\vec{a}\cdot\vec{b}$ and the angle between them.

1. $\vec{a}=(4,3)$, $\vec{b}=(2,-1)$
2. $\vec{a}=(4,0,2)$, $\vec{b}=(2,-1,0)$

■

2.4 Cross product

2 vectors (in 3D) can form a *cross product*, and the result is a *vector* (in 3D).

Let $\vec{x}=(x_1,x_2,x_3)$, $\vec{y}=(y_1,y_2,y_3)$ in 3D.

We typically write the cross product using determinant:

$$\vec{x}\times\vec{y}=\det\begin{bmatrix}i&i&k\\x_1&x_2&x_3\\y_1&y_2&y_3\end{bmatrix}$$

$$\vec{x}\times\vec{y}=i(x_2y_3-x_3y_2)-j(x_1y_3-x_3y_1)+k(x_1y_2-x_2y_1)$$

$$\vec{x} \times \vec{y} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$$

The notations $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ are very easy to use, where the quantity attached to i is the first component of the vector, and the quantity attached to j would be the second, and k would be the third.

The vector produced by the cross product, $\vec{x} \times \vec{y}$, would be orthogonal to both vectors $\vec{x}$ and $\vec{y}$. Notice that in most cases, there would be 2 such vectors with this property. They are exactly

$$\vec{x} \times \vec{y} \qquad \text{and} \qquad -\vec{x} \times \vec{y}$$

In fact, we have

$$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$$

Notice that the order of a dot product does not matter, but the order of cross product matters up to a negative sign.

The other usual properties of multiplication hold:

- $(a\vec{a} \times \vec{b}) = c(\vec{a} \times \vec{b}) = a \times (c\vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Similar to the dot product, we can also relate the angle between the 2 vectors,

$$|\vec{x} \times \vec{y}| = |\vec{x}||\vec{y}| \sin\theta$$

Note that since $\vec{x} \times \vec{y}$ is a vector, we need to take its absolute value. So 2 vectors are *parallel* if the cross product is zero.

Exercise 2.3 Let $\vec{x} = (3,-2,1)$, $\vec{y} = (1,-1,1)$. Find $\vec{x} \times \vec{y}$. ■

2.5 Lines and Planes

For a *line* to be defined, we need a direction $\vec{v}$ and a point on the line $\vec{r}_0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} := \vec{r} = t\vec{v} + \vec{r}_0$$

The vector $\vec{r}$ and the value t do not need to be determined [1].

For a *plane* to be defined, we need a **normal vector** $\vec{n} = (a,b,c)$ and a point on the plane $\vec{r}_0$.

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$ax + by + cz = \vec{n} \cdot \vec{r}_0$$

where $\vec{r}$ is the same as above and does not need to be determined. The *normal vector* $\vec{n}$ is orthogonal to the plane [2].

Plane Defined by 3 Points

Consider 3 points $\vec{u}$, $\vec{v}$, and $\vec{w}$. We can connect any 2 pairs of points together to create 2 direction vectors which are inside the plane. For example, we may take $\vec{x} = \vec{u} - \vec{v}$, and$\vec{y} = \vec{w} - \vec{v}$. From there we can form the cross product to attain a vector that is orthogonal to both vectors inside the plane, so the cross product would be the normal vector, which is orthogonal to the plane.

[1] $\vec{v}$ and $\vec{r}_0$ are not unique, and any correct pair would give the right equation.

[2] Similar to lines, $\vec{v}$ and $\vec{r}_0$ are not unique.

Exercise 2.4 Find the equation of the line or plane.

1. The line through $(-8,0,4)$ and $(3,-2,4)$.
2. The plane through the origin and perpendicular to the vector $(1,5,2)$.
3. The plane through $(2,4,6)$ parallel to the plane $x+y-z=5$.
4. The plane through $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.
5. The plane equidistant from point $(3,1,5)$ and $(-2,0,0)$.

■

3. Partial Derivatives

3.1 Multivariable Function

In first year, we study 1D functions: *functions of one variable*, as $f(x)$. This can be plotted in 2D plane, with $y = f(x)$. Now, we can study functions of multiple variables.

Definition 3.1.1 — 2D Function. A *2D function*, $f(x,y)$, has 2 inputs being x, y, and 1 output.

Sometimes, we can label the output as $z = f(x,y)$.

We can plot this 2D function in 3D space, where the height z above the point (x,y) on the xy-plane is equal to $f(x,y)$.

This would give a graph, which is a surface.

Alternatively, $z = f(x,y)$ can be thought of as an equation which involves x, y, and z, which would also give a surface in 3D space.

However, similar to 1D functions, given an equation involving x, y, and z, it is not always possible to isolate z into $z = f(x,y)$.

Examples of 2D functions include:

$$f(x,y) = x^2 + 2y \qquad f(x,y) = xe^{xy} \qquad f(x,y) = \sin(xy^2)$$

Definition 3.1.2 — 3D Function. A *3D function*, $f(x,y,z)$, has 3 inputs being x, y, and z, and 1 output.

It is possible to label the output as $w = f(x,y,z)$, but this is typically not useful as this introduces a 4th variable. The "plot" of this function would be in 4D space, which does not exist.

Examples of 3D functions include:

$$f(x,y,z = x^2 + 2z - 1) \qquad f(x,y,z) = ye^{xz}$$

Definition 3.1.3 — Level Set. The *level set* of f at k is given by setting f to be equal to the constant k. This will reduce the dimension of f by 1.

Exercise 3.1
- Draw the level set of $f(x,y) = 4x^2 + y^2 + 1$ for $k = 2$ and $k = 5$.
- Draw the level set of $f(x,y,z) = x^2 + y^2 - z$ for $k = 0$ and $k = 2$.

■

3.2 Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if

$$\lim_{(x,y)\to\vec{a}} f(x,y) = f(\vec{a})$$

Every standard function we know are continuous in their domains. Any function transformations (such as $+$, $-$, $\times$, $/$, composition) of continuous functions, are also continuous (except division by 0).

As most functions are continuous, most limits can be obtained by putting $(x,y) = \vec{a}$ in the formula of $f(x,y)$.

Different from 1D functions, computing limits in 2D is much more difficult. Most limit evaluation techniques from 1D functions does not work for 2D functions. In particular, there is no L'Hôpital's rule Rule for 2D functions.

Fortunately, since most functions are continuous except when dividing by 0, we typically only need to focus on the point where the function is dividing by 0 (and state the function is continuous everywhere else).

It turns out that it is much easier to show a limit does not exist.

Test for limit that does not exist in $\mathbb{R}^2$

Given $f(x,y)$, compute the limit as (x,y) approaches $\vec{a} = (0,0)$.

1. Replace y with a easy curve $y = g(x)$, passing through $\vec{a} = (0,0)$, (such as $y = 0$, $y = x$, $y = x^2$), then let x approach 0 to attain a (1D) limit.
2. Replace x with a easy curve $x = h(y)$, passing through $\vec{a} = (0,0)$, (such as $x = 0$, $x = y^2$), then let y approach 0 to attain a (1D) limit.
3. Try with several different $g(x)$ and $h(y)$ to attain many (1D) limits.
4. If you find 2 **different** (1D) limits generated by 2 different curves, then the (2D) limit of (x,y) approaches $\vec{a} = (0,0)$ of $f(x,y)$ does not exist.

Note that if all the 1D limits are the same, that is **not** enough for you to conclude the 2D limit exists. To show the limit exists, we typically must use *Squeeze Theorem*.

Since most limit evaluation techniques does not work for 2D functions, it is a very fortunate fact that Squeeze Theorem does work for 2D functions. The formulation is almost the same as the 1D version.

Theorem 3.2.1 — Squeeze Theorem. To attain $\lim_{(x,y)\to\vec{a}} f(x,y)$, we can try to find $g(x,y)$ and $h(x,y)$ such that

1. $g(x,y) \le f(x,y) \le h(x,y)$ near the point $\vec{a}$
2. $\lim_{(x,y)\to\vec{a}} g(x,y) = L = \lim_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y) = L$.

Use Squeeze Theorem to show limit exist

In practice, we typically want to show $\lim_{(x,y)\to\vec{a}} f(x,y) = 0$.

Start at $|f(x,y)|$, create a *chain of inequalities*, and simplify $|f(x,y)|$ to attain $|g(x,y)|$, which has limit 0 as (x,y) approaches $\vec{a}$:

$$|f(x,y)| \leq \cdots \leq |g(x,y)| \to 0 \quad \text{i.e.} \quad \lim_{(x,y)\to\vec{a}} |g(x,y)| = 0$$

Then we conclude $\lim_{(x,y)\to\vec{a}} f(x,y) = 0$.

The typical strategy in constructing the above inequality is to remove positive quantities from the denominator of $f(x,y)$.

Exercise 3.2

$$f(x,y) = \frac{x^2 y}{x^4 + y^2} \quad \text{and} \quad f(0,0) = 0$$

Show the limit at $(x,y) = (0,0)$ does not exist. ■

Exercise 3.3

$$f(x,y) = \frac{xy}{\sqrt{x^4 + y^2}} \quad \text{and} \quad f(0,0) = 0$$

Show the function **continuous** at $(0,0)$. ■

Exercise 3.4 Find $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{2x^4 + y^2}$ ■

3.3 Derivative

Consider multivariable function $f(x,y)$ or $f(x,y,z)$.

Define the *derivative* to be:

$$\text{For } f(x,y) \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\text{For } f(x,y,z) \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

where ∇f is called *'del' f*, or *gradient* of f.

We can interpret ∇f as a vector, even though $f(x,y)$ or $f(x,y,z)$ is a scalar.

The quantities in ∇f such as $\frac{\partial f}{\partial x}$ are called *partial derivatives*. When taking partial derivatives with respect to a variable, we regard **all other variables** as *constants* and take the derivative as usual. Sometimes, we use the notations

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y \qquad \frac{\partial f}{\partial z} = f_z$$

■ **Example 3.1** Let $f(x,y) = x^2 y^3 + x$.

To take the partial derivative with respect to x, $\frac{\partial f}{\partial x}$, we view y as constant.

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

To take the partial derivative with respect to y, $\frac{\partial f}{\partial y}$, we view x as constant.

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

■

■ **Example 3.2** Let $f(x,y,z) = x^2z + yz^2$.

To take the partial derivative with respect to x, $\frac{\partial f}{\partial x}$, we view **both** y and z as constants. Similarly, for other partial derivatives,

$$\frac{\partial f}{\partial x} = 2xz \qquad \frac{\partial f}{\partial y} = z^2 \qquad \frac{\partial f}{\partial z} = x^2 + 2yz$$

■

Exercise 3.5 Computer ∇f.

1. $f(x,y) = (x^2-1)(y+1)$
2. $f(x,y) = (xy-2)^2$
3. $f(x,y) = \dfrac{2}{x+3y}$
4. $f(x,y) = x^{x+4y}$
5. $f(x,y,z) = x^2+2z-1$
6. $f(x,y,z) = ye^{xz}$

■

3.4 Directional Derivative

Remember one dimensional derivatives in first year?

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

where f is defined on the real line.

We start at the point $x = a$, and we ask, if we move just a tiny bit away from $x = a$, how much does the function change?

We generalize this idea of moving a tiny bit away in higher dimensions. In the real line, we can move only left or right. In 2D, we can move in any direction we like (on the plane).

We can describe the direction of movement by a *unit vector*, similar to the direction vector of the equation of a line. The *directional derivative* toward the direction $\vec{u}$ at the point $\vec{a}$ [1]:

$$D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u})-f(\vec{a})}{h}$$

This is also the *rate of change* of the function in the direction $\vec{u}$ at the point $\vec{a}$.

Given a vector $\vec{v}$, we can turn it into a *unit vector* $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$.

Definition 3.4.1 — Unit Vector. A *unit vector* is a vector whose length (norm) is exactly 1.

Some useful facts:

1. Partial derivatives are directional derivatives with $\vec{u}$ being in a *coordinate direction*. For example, for $f(x,y)$,

[1] This is a 1D limit, as h is a number (the length of the movement).

- $\dfrac{\partial f}{\partial x} = D_{\vec{u}} f$ with $\vec{u} = (1,0)$;
- $\dfrac{\partial f}{\partial y} = D_{\vec{u}} f$ with $\vec{u} = (0,1)$;

2. If f is **differentiable**, then all directional derivative exist and

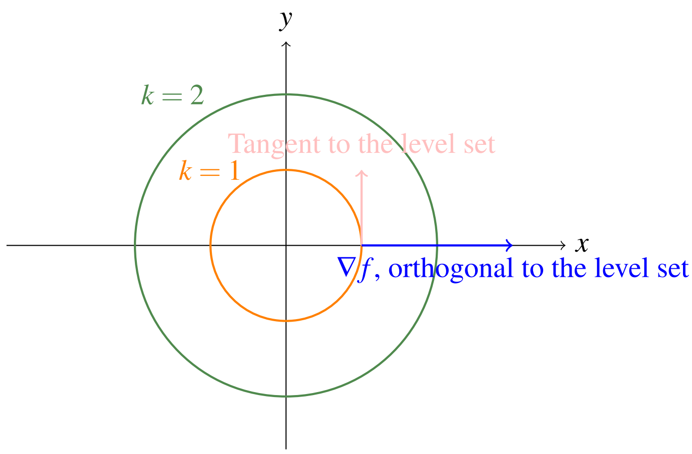
$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

where $\vec{u}$ is a **unit vector**.
This formula gives an easy way to calculate directional derivative.

3. We can interpret $\nabla f(\vec{a})$ as a vector.
If we evaluate ∇f at a point $\vec{a}$, and if we interpret the vector $\nabla f(\vec{a})$ as a direction, then the directional derivative in the direction of $\nabla f(\vec{a})$ at the point $\vec{a}$ is the largest, and the value of this maximum directional derivative is equal to $|\nabla f(\vec{a})|$.

4. Recall equation of plane: $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
For $f(x,y,z) = k$ for some constant k, gives a *level set* surface. Define the *tangent plane* at some fixed point $\vec{r}_0 = (x_0, y_0, z_0)$ by the **normal vector** $\vec{n} = \nabla f(x_0, y_0, z_0)$.

For example, consider $f(x,y) = x^2 + y^2$.
$\nabla f = (2x, 2y)$. At $(1,0)$, $\nabla f = (2,0)$.



Exercise 3.6 $f(x,y,z) = e^{2x} y + y^2 + 4$. At the point $(0,0)$, find the directional derivative along the direction given by $\vec{u} = (1,3)$ (first turn $\vec{v}$ into a unit vector). ■

Exercise 3.7 $f(x,y,z) = x^2 y + z$. At the point $(2,2,1)$, find the maximum rate of change, and the direction with the maximum (the direction with maximum rate of change is ∇f, and the value of this maximum rate of change is $|\nabla f|$). ■

Exercise 3.8 Define $f(x,y,z) = xy^2 e^z$. Let $f(x,y,z) = e$.
At the point $(1,1,1)$, find the equation of the tangent plane. ■

Exercise 3.9 Find all directional derivatives at $(0,0)$ (including the partials), if they exist. Is the function **continuous** at $(0,0)$?

a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

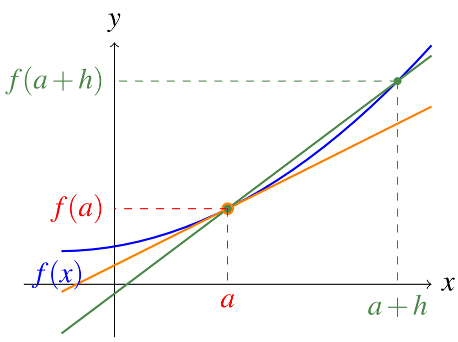
b) $f(x,y) = \sqrt{|xy|}$

Strategy:

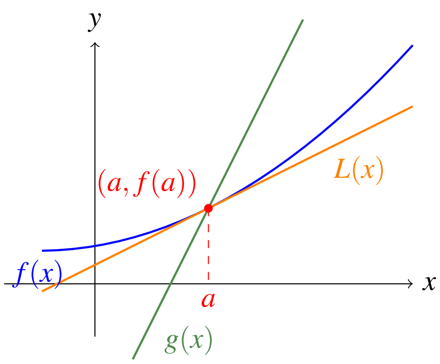
When the function has division by 0, or some other problems at the origin (such as absolute value or square root), we must use the definition of directional derivative, because f **may not be differentiable** at $(0,0)$ so the formula $\nabla f \cdot \vec{u}$ does not work. Set $\vec{a} = (0,0)$ and $\vec{u} = (a,b)$ where $a^2+b^2 = 1$.

3.5 Differentiation

In first year, we defined the tangent line to be $\dfrac{\text{rise}}{\text{run}}$, and we calculate the limit as the *run* approaches 0, where the secant line becomes the tangent line. This gives the slope of the tangent line of the function at $x = a$.



There is another way to interpret the tangent line, however. It is the *best approximation of the function* using a line. Let $L(x)$ be the equation of the tangent line, and $g(x)$ be the equation of a random line that passes through $(a, f(a))$. Why is $L(x)$ the tangent line and not $g(x)$?



Clearly, the error made from $L(x)$, $f(x) - L(x)$ goes to 0 as $x \to a$. However, $\lim_{x\to a} g(x)$ is also 0. It is not sufficient to say that the error goes to 0, but we also want to check the rate at which the error goes to 0. So what makes $L(x)$ better than $g(x)$? While both are approaching 0, the error for

$L(x)$ approaches 0 **faster** than that of $g(x)$ [2]. For the tangent line, we want $L(x)$ to approach $f(a)$ the fastest, in particular, faster than how x approached a. Thus, we have $\lim\limits_{x\to a}\dfrac{f(x)-L(x)}{x}=0$.

Recall that the equation of the tangent line is $y-f(a)=f'(a)(x-a)$, so $L(x)=f(a)+f'(a)(x-a)$ [3], where $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ as before. Take $x=a+h$, we have alternatively

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-L(a+h)}{h-a} &= 0\\
\lim_{h\to 0}\frac{f(a+h)-\left[f(a)+f'(a)\cancelto{h}{(x-a)}\right]}{a} &= 0\\
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}-\frac{f'(a)\cdot\cancel{h}}{\cancel{h}} &= 0\\
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= f'(a)
\end{aligned}$$

This happens to be the equation we have before.

We can expand this result to higher dimensions. If we have a 2D function, we should have a *tangent plane* that allows the error between the function and the tangent plane to approach 0 faster than any other planes.

Given the tangent plane $w=L(x,y)$, we want $\lim\limits_{\vec{x}\to\vec{a}}f(x,y)-L(x,y)=0$, and approaches 0 the fastest. That is, $\lim\limits_{\vec{x}\to\vec{a}}\dfrac{f(x,y)-L(x,y)}{|\vec{x}-\vec{a}|}=0$ [4].

To get the equation of the tangent plane, we can think of the original function, $f(x,y)$, as the level set of the 3D function $F(x,y,z)=f(x,y)-z$ at $k=0$. Then, by the result from the previous section, $\nabla F(x_0,y_0,z_0)$ is the normal vector to the tangent plane at the given point (x_0,y_0,z_0). That is, $\vec{n}=\nabla F=\left(\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y},\dfrac{\partial F}{\partial z}\right)=\left(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y},-1\right)$.

Note that at $\vec{a}=(x_0,y_0)$, we obtain the point $(x_0,y_0,f(\vec{a}))$ on the graph of $f(x,y)$. Denote this point $\vec{r}_0$, and this allows us to get the formula for the tangent plane.

$$\begin{aligned}
\vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}_0\\
\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)\cdot\vec{r} &= \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)\cdot(x_0,y_0,f(\vec{a}))\\
\frac{\partial f}{\partial x}\cdot x+\frac{\partial f}{\partial y}\cdot y-z &= \frac{\partial f}{\partial x}\cdot x_0+\frac{\partial f}{\partial y}\cdot y_0-f(\vec{a})\\
z &= f(\vec{a})+\frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)\\
z &= f(\vec{a})+\nabla f(\vec{a})\cdot(x-x_0,y-y_0)
\end{aligned}$$

To further simplify this equation, take $(x,y)=\vec{x}=\vec{a}+\vec{h}$. Then, $\vec{h}=\vec{x}-\vec{a}=(x-x_0,y-y_0)$. Now, $z=f(\vec{a})+\nabla f(\vec{a})\cdot\vec{h}$.

---

[2] To denote "$F(x)$ goes to 0 faster than $G(x)$", we use $\lim\limits_{x\to 0}\dfrac{F(x)}{G(x)}=0$.

[3] This is in fact the first order Taylor expansion of $f(x)$ at a.

[4] Similar as before, this means we want the error to approach 0 faster than $\vec{x}$ approaches $\vec{a}$.

Substitute the equation for $L(x,y)$ back into $\lim_{\vec{x}\to\vec{a}} \dfrac{f(x,y)-L(x,y)}{|\vec{x}-\vec{a}|} = 0$, we obtain

$$\lim_{\vec{x}\to\vec{a}} \frac{f(x,y)-L(x,y)}{|\vec{x}-\vec{a}|} = 0$$
$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{a}+\vec{h})-\left(f(\vec{a})+\nabla f(\vec{a})\cdot\vec{h}\right)}{|\vec{h}|} = 0$$
$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{a}+\vec{h})-f(\vec{a})-\nabla f(\vec{a})\cdot\vec{h}}{|\vec{h}|} = 0$$

Definition 3.5.1 — Differentiable. f is *differentiable* if the gradient vector $\nabla f(\vec{a})$ satisfies

$$\lim_{\vec{h}\to\vec{0}} = \frac{f(\vec{a}+\vec{h})-f(\vec{a})-\nabla f(\vec{a})\cdot\vec{h}}{|\vec{h}|} = 0$$

Note that this is a *2D limit*. This limit is typically used when $\vec{a}=(0,0)$, so we may set $\vec{h}=(x,y)$ and show the limit is zero using Squeeze Theorem.

Some useful facts:

1. If f is C^1 at $\vec{a}$ [5], then f is differentiable at $\vec{a}$ (∇f exists).
2. Being differentiable is a stronger condition than having directional derivative. If f is differentiable, then all directional derivative is given by $D_{\vec{u}}f = \nabla f\cdot\vec{u}$.
3. If a function f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.

Exercise 3.10 Define

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Show the directional derivatives exist at $(0,0)$, but f is **not differentiable** at $(0,0)$.

Strategy:
There are 2 ways to approach the problem.

1. If the function is **not continuous** at $(0,0)$, then it is **not differentiable** at $(0,0)$. So we may try to show it is not continuous at $(0,0)$. However, some functions **are continuous** at $(0,0)$, so we can attain no conclusion in such cases.
2. Compute all the directional derivatives. If f is differentiable, then every directional derivative should satisfy $D_{\vec{u}} = \nabla f\cdot\vec{u}$.

   Check this equation for $\vec{u}=(1,0)$, $\vec{u}=(0,1)$, and $\vec{u}=\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$ (easy unit vectors). ■

Exercise 3.11 Show whether the function is differentiable at $(0,0)$.

a) $f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

[5] This means all partial derivatives exist near $\vec{a}$ and at $\vec{a}$, and the partial derivatives are **continuous** (as functions) at $\vec{a}$.

b) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

3.6 Higher Order Derivative

We may take (partial) derivatives on top of partial derivatives.

In 2 dimensions, for $f(x,y)$, there are 4 ways of taking second derivatives. We put them into the *Hessian Matrix*:

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial xy} \\ \dfrac{\partial^2 f}{\partial yx} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) & \dfrac{\partial f}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) \\ \dfrac{\partial f}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) & \dfrac{\partial f}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) \end{bmatrix} = \begin{bmatrix} f_{xx} & fxy \\ fyx & fyy \end{bmatrix}$$

If f is C^2 at $\vec{a}$ [6], then **mixed partial derivatives commute**:

The order of taking derivatives does not change the final answer: $f_{xy} = f_{yx} \rightarrow$ Taking derivative in x then y is the same as taking in y then x.

In other words, if f is C^2, then **the Hessian Matrix is symmetric**.

We can also construct the Hessian for $f(x,y,z)$, which will be a 3×3 matrix. We can also construct higher order derivatives, such as 3rd order f_{xyz} or f_{xyy}. In general, if f is C^k, then mixed partial derivatives commute (up to order k).

Exercise 3.12 Find the Hessian Matrix.

1. $f(x,y) = 3x^2 + 4xy + 5y^2$
2. $f(x,y) = \cos(x+2y)$
3. $f(x,y) = x^{2x+y}$
4. $f(x,y) = e^{x^2+y}$
5. $f(x,y) = e^x \sin(y)$
6. $f(x,y,z) = x^2y + xz + z^2$

Exercise 3.13

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find f_x and f_y at $(x,y) = (0,0)$ **and** $(x,y) \neq (0,0)$. Then show $f_{xy} \neq f_{yx}$ at $(0,0)$. i.e. Taking 2nd derivatives in different orders give different answers.

Is f C^2 at $(0,0)$ (and have we reacher a contradiction)?

3.7 Critical Points

In first year, we found critical points of functions by setting the derivative to zero. Then we check their second derivative to conclude whether critical points are max or min. In higher dimensions, it is very similar.

Let $f(x,y)$ be a 2D function (which is C^2).

We set $\nabla f(\vec{x}) = \vec{0}$. Solve $\vec{x}$, these are the *critical points*.

[6] This means all the 2nd order partial derivatives exist near $\vec{a}$ and at $\vec{a}$, and the 2nd order partial derivatives are continuous (as functions) at $\vec{a}$ (In practice, a function can fail to be continuous when there is division by zero).

Then we check second derivatives: **the Hessian Matrix**.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Note that H can contain x and y, so H would be different for different points. For each critical point $\vec{a}$,

- if $\det(H(\vec{a})) > 0$, look at the top left entry
  - if $\frac{\partial^2 f}{\partial x^2}(\vec{a}) > 0$, then it is a *local minimum*.
  - if $\frac{\partial^2 f}{\partial x^2}(\vec{a}) < 0$, then it is a *local maximum*.
- if $\det(H(\vec{a})) < 0$, it is a *saddle point*.
- if $\det(H(\vec{a})) = 0$, it is inconclusive.

To find the (absolute) maximum and minimum of f in a region D,

1. Find all local extrema inside D using critical points
2. Find all extrema on the **boundary** of D
3. Compare all the function values to attain the absolute extrema

Exercise 3.14 Find and classify all critical points.

1. $f(x,y) = 3x^2 + 2xy + 5y^2$
2. $f(x,y) = x^2 + 3y^4 + 4y^3 - 12y^2$
3. $f(x,y) = (x-1)(x^2+y^2)$
4. $f(x,y) = (x^2+2y^2)e^{-x^2-y^2}$

■

Exercise 3.15 Let $f(x,y) = x^2 + y^2 - 4x$

Find the max and min indide the triangle D defined by $(0,-3)$, $(0,3)$, and $(3,0)$. ■

3.8 Lagrange Multiplier

Sometimes, we want to maximim/minimize a function on some specific curve, or there are some constraints on the variables that the function takes in.

- Take the function we want to maximize/minimize to be f.
- Take the equation that describes the constraint to be $g = 0$ (perhaps we have more than 1 constraint, so the second constraint is $h = 0$).
- The *Lagrange Multiplier Formula* is

$$\nabla f = \lambda \nabla g (+\mu \nabla h + \cdots)$$

Each constraint will create an extra ∇ term on the right side with some constant in front (we need every ∇ term on the right to be non-zero).

λ and μ are constants that you need to solve for, along with n variables of f. The equation is an equation for vectors, so each of the components are equal. So, there are n equations in the formula since ∇f has n components. For each constant like λ or μ, there is a equation of constraint for each. So if there are 2 constraints, there are $n+2$ variables that you need to solve, and you have $n+2$ equations to do it.

You are **not allowed to divide by zero** when you cancel terms. When attempting to divide a quantity x, you need to split into 2 cases:

- When $x \neq 0$, you can divide
- when $x = 0$, you cannot

Exercise 3.16 Find the max and min of the function $f(x,y)$ subject to constraints given.
$f(x,y) = 2x - 6y$ with constraint $4x^2 + 2y^2 = 25$. ■

Exercise 3.17 Consider the curve in $\mathbb{R}^2$, $C : y = x^2$.
Find the point p on C so that the distance between p and $(0,-1)$ is smallest.
Strategy: We minimize $f(x,y) = \text{distance}^2 = (x-0)^2 + (y-(-1))^2$. ■

Exercise 3.18 Consider 2 curves in $\mathbb{R}^2$, $C_1 : y = x^2$ and $C_2 : y = x - 1$
Find points p on C_1 and q on C_2 so that the distance between them is smallest. ■

3.9 Chain Rule in 1D

We typically think of chain rule as a rule for computation:

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

where we are given a function inside another function such as $y = \sin(e^x)$, and we take the derivative of the outside (not touching the inside), then multiply the derivative of the inside.

Alternatively, we can define a **variable dependence**. For example, define the variables as $y = y(t)$, and $t = t(x)$. (In the above example, we would set $y = \sin t$ and $t = e^x$.) Then we may express the chain rule (in 1D) as:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

Intuitively, $\frac{dy}{dx}$ asks how much would y change, if there is a small change in x. Given the variable dependence above, a small change in x would cause a small change in t, which in turn would cause a small change in y. Hence, we have the 2 "fractions" ($\frac{dy}{dt}$ and $\frac{dt}{dx}$) on the right.

2nd Derivative

We may use the above form of the chain rule to calculate the 2nd derivative.

We apply another differential operator $\frac{d}{dx}$ on top of $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Plug in the above form of the chain rule,

$$= \frac{d}{dy}\left(\frac{dy}{dt}\frac{dt}{dx}\right)$$

We want to take the derivative of this product, so we need *product rule*,

$$= \frac{d}{dx}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} + \frac{dy}{dt} \cdot \frac{d}{dx}\left(\frac{dt}{dx}\right)$$

Notice that there are 2 types of derivatives here.

The 2nd term involves the expression $\frac{d}{dx}\left(\frac{dt}{dy}\right)$. Recall that we assumed $t = t(x)$ (such as $t = e^x$), so $\frac{dt}{dx}$ is also **a function of x**. We want to take the derivative of such function **against x**. This results in the 2nd derivative of $t = t(x)$ against x:

$$\frac{d}{dx}\left(\frac{dt}{dx}\right) = \frac{d^2t}{dx^2}$$

The 1st term involves the expression

$$\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

This term is more difficult (and requires an **additional expansion**).

Recall that we assumed y = y(t) (such as y = sin t), so $\frac{dy}{dt}$ is also **a function of t** (such as $dy = \cos t$).

We want to take the derivative of such function **against x**, on an expression involving t.

Notice that $\frac{dy}{dt}$ is an expression involving t, similar to $y = y(t)$. So to take the derivative against x, on an expression involving t, we know that for $y = y(t)$:

$$\frac{d}{dx}(y(t)) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{d}{dx}(y(t))\frac{dt}{dx}$$

Since $\frac{dy}{dt}$ is also an expression involving t, it would **behave in the same way** as $y(t)$ when taking derivative **against x**.

We can substitute $y(t)$ with $\frac{dy}{dx}$ in the above expression:

$$\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

In conclusion, to take **the 2nd derivative involving chain rule**:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\frac{dt}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx} + \frac{dy}{dt}\cdot\frac{d}{dx}\left(\frac{dt}{dx}\right)$$

Taking an **additional expansion** in the first term:

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}\cdot\frac{dt}{dx} + \frac{dy}{dt}\cdot\frac{d^2t}{dx^2}$$
$$= \frac{d^2y}{dt^2}\frac{dt}{dx}\cdot\frac{dt}{dx} + \frac{dy}{dt}\cdot\frac{d^2t}{dx^2}$$

Notice that we get a square of $\frac{dt}{dx}$ in the first term. Thus **the chain rule formula for 2nd derivative** is:

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\cdot\frac{d^2t}{dx^2}$$ [7]

[7] Note $\left(\frac{dt}{dx}\right) \neq \frac{d^2t}{dx^2}$.

Of course, given a function $y = y(x)$, one would not go through this long and complicated process just to take 2nd derivative. It is much easier to just take the derivative (twice) using the computation rules.

However, this formula shines when **at least one of the function is unknown**. This is exactly the case for **ODE**, where $y = y(x)$ is unknown. We may use this formula to carry out a **change of variables** and turn a complicated ODE into a simple one.

ODE - Euler Equation

Consider $y = y(x)$. For numbers a, b, c,

$$ax^2y'' + bxy' + cy = 0$$

This is a *2nd order linear equation*, but with non-constant coefficients. In general, this type of equation has no solution. But here, notice that we have exactly x^2 in front of y'', and x in front of y'.

Due to this, we can guess the solution to be $y(x) = x^n$. We plug in $y(x) = x^n$ into the ODE, and solve for n.

Typically, there will be 2 solutions for n. The general solution will be a linear combination of the 2 solutions (with constants $c1$,$c2$ in front).

■ Example 3.3

$$x^2y'' - xy' - 3y = 0$$

Take $y = x^n$, then $y' = nx^{n-1}$ and $y'' = n(n-1)x^{n-2}$.
Plugging into the ODE,

$$x^2n(n-1)x^{n-2} - xnx^{n-1} - 3x^n = 0$$

Notice that we can divide both sides by x^n.

$$\begin{aligned} n(n-1) - n - 3 &= 0 \\ n^2 - 2n - 3 &= 0 \end{aligned}$$

We will always get a quadratic of this form. In this case, we get

$$(n-3)(n+1) = 0$$

Thus the general solution is given by

$$y(x) = c_1x^3 + c_2^{-1}$$

■

However, notice that given an equation with different numbers, the quadratic we get may have only one (repeated) solution. In this case, we would be "missing" the second solution. We can instead find the solution by using the chain rule formula for 2nd derivative.

■ Example 3.4

$$x^2y'' - 3xy' + 4y = 0$$

If we proceed using the previous approach, we would get $(n-2)^2 = 0$, giving one (repeated) solution.

Instead, we can perform a change of variable, $t = \ln x$. We get

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$$

$$y'' = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}^2\right) + \frac{dy}{dt}\cdot\frac{d^2t}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{1}{x}\right)^2 + \frac{dy}{dt}\cdot\frac{-1}{x^2}$$

Plugging the above into the ODE,

$$x^2y'' - 3xy' + 4y = 0$$

$$x^2\left(\frac{d^2y}{dt^2}\left(\frac{1}{x}\right)^2 + \frac{dy}{dt}\cdot\frac{-1}{x^2}\right) - 3x\left(\frac{dy}{dt}\frac{1}{x}\right) + 4y = 0$$

Notice that all the x cancels out.

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 4y = 0$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$$

This is a *2nd order linear equation with constant coefficients*. The solution is given by $y(t) = e^{rt}$. Plugging into the equation,

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

This is a **repeated** root solution. Thus the general solution for $y(t)$ is given by

$$y(t) = c_1e^{2t} + c_2te^{2t}$$

Since $t = \ln x$, we get the general solution for $y(x)$ is given by

$$\begin{aligned} y(t) &= c_1e^{2\ln x} + c_2\ln x e^{2\ln x} \\ &= c_1\left(e^{\ln x}\right)^2 + c_2\ln x\left(e^{\ln x}\right)^2 \\ &= c_1x^2 + c_2(\ln x)x^2 \end{aligned}$$

■

In other words, the "missing" second solution from the repeated root of $n = 2$, is attained by multiplying an extra factor of $\ln x$ in front of x^n.

Exercise 3.19 Find the general solution to the ODE, where $y = y(x)$.

1. $x^2y'' - 3xy' + 4y = 0$
2. $2x^2y'' - 5xy' + 5y = 0$
3. $x(1-x^2)^2y'' - (1-x^2)(1+4x^2)y' + 2x^3y = 0$ with the change of variable $t = \dfrac{-1}{2}\ln(1-x^2)$ ■

II

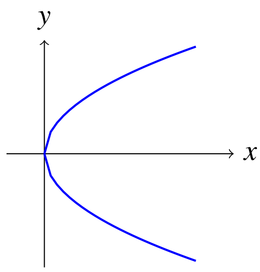
Appendices

Solutions to Exercise Questions

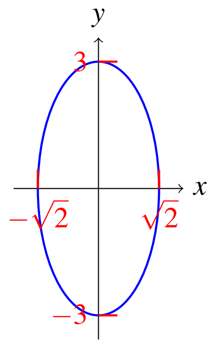
Chapter 1

Exercise 1.1

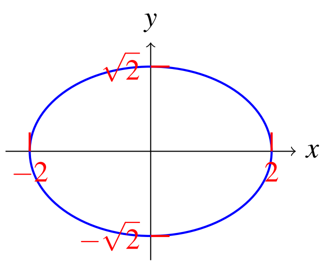
1. $x = 2y^2$



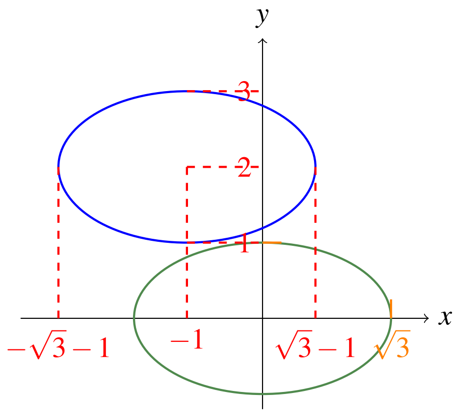
2. $$\frac{x^2}{2} + \frac{y^2}{9} = 1$$
$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$



3. $$x^2 + 2y^2 = 4$$
$$\frac{x^2}{4} + \frac{2y^2}{4} = 1$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$



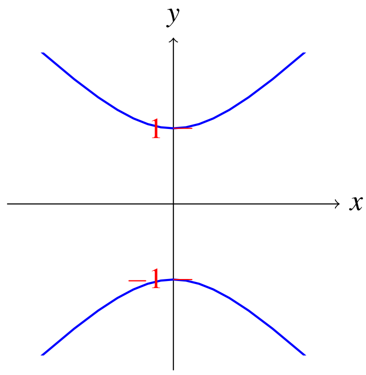
4. $$x^2+3y^2+2x-12y+10=0$$
$$(x^2+2x+1)+3(y^2-4y+4)-3=0$$
$$(x+1)^2+3(y-2)^2=3$$
$$\left(\frac{x+1}{\sqrt{3}}\right)^2+\left(\frac{y-2}{1}\right)^2=1$$
Ellipse before the shift (in green):
$$\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y}{1}\right)^2=1$$
Then, shift 1 unit left and 2 units up.



5. $y^2-x^2=1$

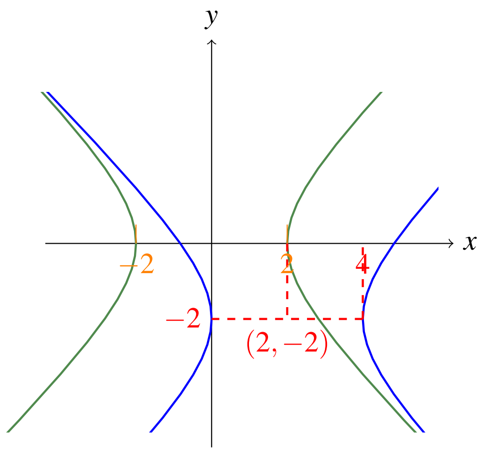
If $y=0$, then $-x^2=1$, not possible. Thus, the graph must not cross the horizontal axis, and the hyperbola opens **up and down**.

If $x=0$, then $y^2=1$, $y=\pm1$.



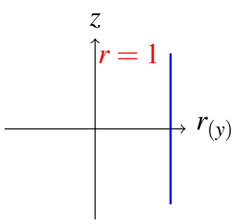
6. $$\frac{(x-2)^2}{4}-\frac{(y+2)^2}{9}=1$$
$$\left(\frac{x-2}{2}\right)^2-\left(\frac{y+2}{3}\right)^2=1$$
Hyperbola before the shift (in green):
$$\left(\frac{x}{2}\right)^2-\left(\frac{y}{3}\right)^2=1$$
If $x=0$, then $-y^2=1$, not possible. Thus, the graph must not cross the vertical axis, and the hyperbola opens **left and right**.

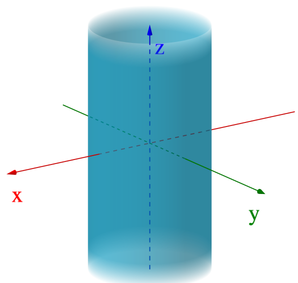
If $y=0$, then $\left(\frac{x}{2}\right)^2=1$, $x=\pm2$.



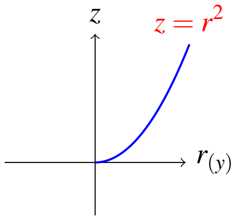
Exercise 1.2

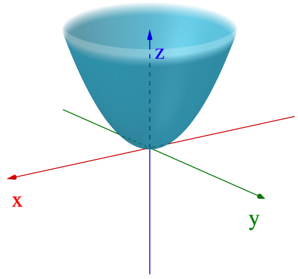
1. $x^2+y^2=1$
$$r^2=1$$
$$r=1$$



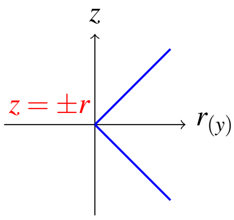


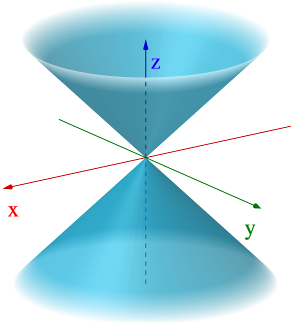
2. $z = x^2 + y^2$
$z = r^2$



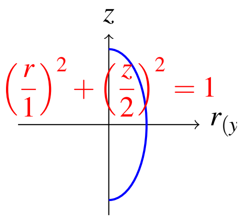


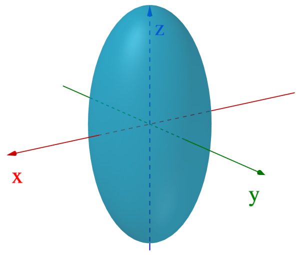
3. $z^2 = x^2 + y^2$
$z^2 = r^2$
$z = \pm r$



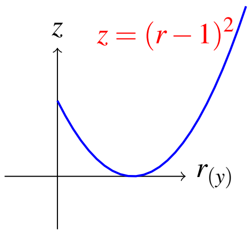


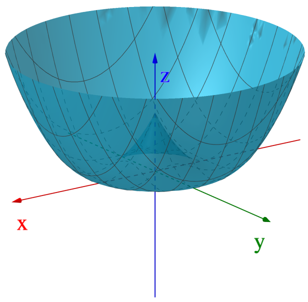
4. $$x^2 + y^2 + \frac{z^2}{4} = 1$$
$$r^2 + \frac{z^2}{4} = 1$$
$$\left(\frac{r}{1}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$



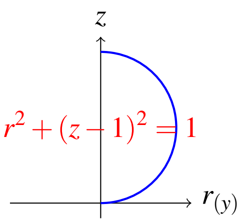


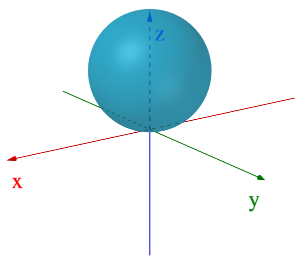
5. $$z = \left(\sqrt{x^2 + y^2} - 1\right)^2$$
$$z = (r - 1)^2$$





6. $$x^2 + y^2 + z^2 = 2z$$
$$r^2 + z^2 - 2z + 1 = 1$$
$$r^2 + (z - 1)^2 = 1$$





Chapter 2

Exercise 2.1

1. $$\det\begin{pmatrix}3 & -1\\ 2 & 5\end{pmatrix} = 3\times 5 - 2\times(-1)$$
$$= 15+2$$
$$= 17$$

2. $$\det\begin{pmatrix}1 & 3 & -3\\ -3 & -5 & 2\\ -4 & 4 & -6\end{pmatrix} = 1\cdot\det\begin{pmatrix}-5 & 2\\ 4 & -6\end{pmatrix} - 3\cdot\det\begin{pmatrix}-3 & 2\\ -4 & -6\end{pmatrix} + (-3)\cdot\det\begin{pmatrix}-3 & -5\\ -4 & 4\end{pmatrix}$$
$$= 1\cdot 22 - 3\cdot 26 + (-3)\cdot(-32)$$
$$= 22 - 78 + 96$$
$$= 40$$

3. $$\det\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 4\\ 0 & 0 & 4\end{pmatrix} = 0\cdot\det\begin{pmatrix}1 & 0\\ 3 & 4\end{pmatrix} - 0\cdot\det\begin{pmatrix}3 & 0\\ 1 & 4\end{pmatrix} + 4\cdot\det\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} \quad \text{since} \begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$
$$= 0 - 0 + 4\cdot 8$$
$$= 32$$

Exercise 2.2

1. $$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{4^2+3^2} = 5$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{2^2+(-1)^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = 4\cdot 2 + 3\cdot(-1) = 5$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{5\cdot\sqrt{5}}$$
$$\theta = \arccos\left(\frac{1}{\sqrt{5}}\right)$$

2. $$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{4^2+0^2+2^2} = 2\sqrt{5}$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{2^2+(-1)^2+0^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = 4\cdot 2 + 0\cdot(-1) + 2\cdot 0 = 8$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{8}{2\sqrt{5}\cdot\sqrt{5}}$$
$$\theta = \arccos\left(\frac{4}{5}\right)$$

Exercise 2.3

$$\begin{aligned}\vec{x}\times\vec{y} &= \det\begin{pmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & -1 & 1\end{pmatrix} \\ &= i\cdot\det\begin{pmatrix}-2 & 1 \\ -1 & 1\end{pmatrix} - j\cdot\det\begin{pmatrix}3 & 1 \\ 1 & 1\end{pmatrix} + k\cdot\begin{pmatrix}3 & -2 \\ 1 & -1\end{pmatrix} \\ &= (-1,-2,-1)\end{aligned}$$

Exercise 2.4

1. $$\begin{aligned}\vec{v} &= (3,-2,4)-(-8,0,4) \\ &= (11,-2,0)\end{aligned}$$
   $$\vec{r} = t\begin{bmatrix}11 \\ 2 \\ 0\end{bmatrix} + \begin{bmatrix}-8 \\ 0 \\ 4\end{bmatrix}$$
2. $$\begin{cases}\vec{r}_0 = (0,0,0) \\ \vec{n} = (1,5,2)\end{cases}$$
   $$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$
   $$\begin{bmatrix}1 \\ 5 \\ 2\end{bmatrix}\cdot\vec{r} = \begin{bmatrix}1 \\ 5 \\ 2\end{bmatrix}\cdot\begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix}$$
   $$\begin{bmatrix}1 \\ 5 \\ 2\end{bmatrix}\cdot\vec{r} = 0$$
3. $$\begin{cases}\vec{r}_0 = (2,4,6) \\ \vec{n} \ = (1,1,-1)\end{cases}$$
   $$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$
   $$\begin{bmatrix}1 \\ 1 \\ -1\end{bmatrix}\cdot\vec{r} = \begin{bmatrix}1 \\ 1 \\ -1\end{bmatrix}\cdot\begin{bmatrix}2 \\ 4 \\ -6\end{bmatrix}$$
   $$\begin{bmatrix}1 \\ 1 \\ -1\end{bmatrix}\cdot\vec{r} = 0$$
4. $\vec{x} = (1,0,1)-(0,1,1) = (1,-1,0)$, and $\vec{y} = (1,1,0)-(0,1,1) = (1,0,-1)$.
   $$\begin{aligned}\vec{n} &= \vec{x}\times\vec{y} \\ &= \det\begin{pmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1\end{pmatrix} \\ &= i\cdot\det\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix} - j\cdot\det\begin{pmatrix}1 & 0 \\ 1 & -1\end{pmatrix} + k\cdot\det\begin{pmatrix}1 & -1 \\ 1 & 0\end{pmatrix} \\ &= \begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix}\end{aligned}$$

$$\begin{cases} \vec{r}_0 = (0,1,1) \\ \vec{n} \ = (1,1,1) \end{cases}$$

$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{r} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{r} = 0$$

5. $$\begin{aligned}\vec{r}_0 &= \frac{(3,1,5)+(-2,0,0)}{2} \\ &= \frac{1}{2}(1,1,5)\end{aligned}$$

$$\begin{aligned}\vec{n} &= (-2,0,0)-(3,1,5) \\ &= (-5,-1,-5)\end{aligned}$$

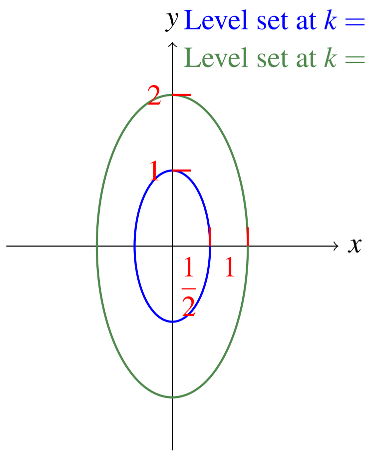
$$\begin{aligned}\vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}_0 \\ (-5,1,-5)\cdot\vec{r} &= (-5,1,-5)\cdot\frac{1}{2}(1,1,5) \\ -5x-y-5z &= \frac{1}{2}(-5+1-25) \\ &= \frac{-31}{2}\end{aligned}$$

Chapter 3

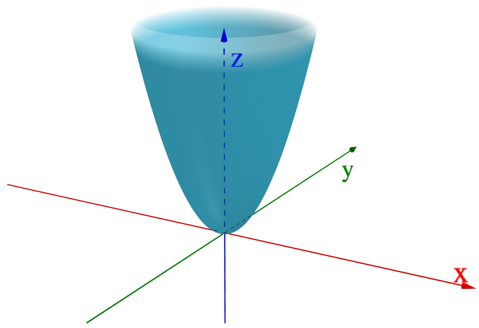
Exercise 3.1

1. At $k=2$,
$$\begin{aligned}4x^2+y^2+1 &= 2 \\ 4x^2+y^2 &= 1 \\ \left(\frac{x}{\frac{1}{2}}\right)^2+\left(\frac{y}{1}\right)^2 &= 1\end{aligned}$$

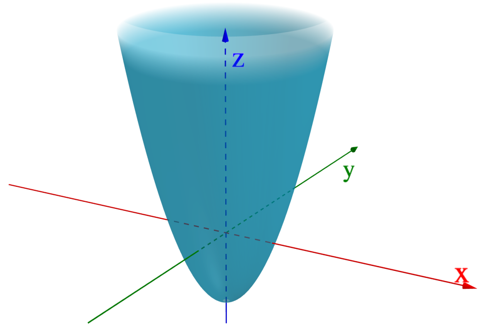
At $k=5$,
$$\begin{aligned}4x^2+y^2+1 &= 5 \\ 4x^2+y^2 &= 4 \\ x^2+\frac{y^2}{4} &= 1 \\ \left(\frac{x}{1}\right)^2+\left(\frac{y}{2}\right)^2 &= 1\end{aligned}$$



2. At $k=0$, $x^2+y^2-z=0$
$$x^2+y^2=z$$
$$r^2=z$$



At $k=2$, $x^2+y^2-z=2$
$$x^2+y^2-2=z$$
$$r^2-2=z$$



Exercise 3.2

- For $x=0$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{0}{0+y^2} = 0.$
- For $y=x$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2x}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x+1} = 0.$
- For $y=0$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{0}{x^4+0} = 0.$
- For $y=x^2$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$

Thus, the limit at $(a,y)=(0,0)$ does not exist.

Exercise 3.3

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^4+y^2}}\right| \leq \left|\frac{xy}{\sqrt{y^2}}\right| = \left|\frac{xy}{y}\right| = |x| \to 0$$

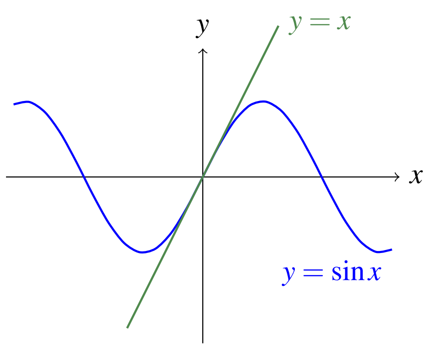
Thus, by the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^4+y^2}} = 0 = f(0,0).$

Thus, the function is continuous at $(0,0)$.

Exercise 3.4

By removing $2x^4$ from the denominator, we obtain $\left|\frac{x^2\sin^2 y}{2x^4+y^2}\right| \leq \left|\frac{x^2\sin^2 y}{y^2}\right|.$

Note that $|\sin\theta| \leq |\theta|$, and that as $\theta \to 0$, $\sin\theta \approx \theta$. This is the *small angle approximation*.



So $\left|\frac{x^2\sin^2 y}{y^2}\right| \approx \left|\frac{x^2\cdot y^2}{y^2}\right| = |x^2| \to 0.$

Then, by the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{2x^4+y^2} = 0.$

Exercise 3.5

1. $\frac{\partial f}{\partial x} = 2x(y+1)$ and $\frac{\partial f}{\partial y} = x^2-1.$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x(y+1), x^2-1).$
2. $\frac{\partial f}{\partial x} = 2y(xy-2)$ and $\frac{\partial f}{\partial y} = 2x(xy-2).$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2y(xy-2), 2x(xy-2)).$
3. $\frac{\partial f}{\partial x} = \frac{-2}{(x+3y)^2}$ and $\frac{\partial f}{\partial y} = \frac{-6}{(x+3y)^2}.$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{-2}{(x+3y)^2}, \frac{-6}{(x+3y)^2}\right).$
4. $\frac{\partial f}{\partial x} = e^{x+4y}$ and $\frac{\partial f}{\partial y} = 4e^{x+4y}.$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^{x+4y}, 4e^{x+4y}).$
5. $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 2.$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 0, 2).$
6. $\frac{\partial f}{\partial x} = yze^{xz}$, $\frac{\partial f}{\partial y} = e^{xy}$, and $\frac{\partial f}{\partial z} = xye^{xz}.$
   So $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (yze^{xz}, e^{xz}, xye^{xz}).$

Exercise 3.6

$$\begin{aligned}\vec{u} &= \frac{\vec{v}}{|\vec{v}|}\\ &= \frac{(1,3)}{\sqrt{1^2+3^2}}\\ &= \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\ &= \begin{bmatrix}2ye^{2x}\\ e^{2x}+2y\end{bmatrix}\\ \nabla f(0,0) &= \begin{bmatrix}2\cdot 0\cdot e^{2\cdot 0}\\ e^{2\cdot 0}+2\cdot 0\end{bmatrix}\\ &= \begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{So } D_{\vec{u}}f &= \nabla f\cdot\vec{u}\\ &= \begin{bmatrix}0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}\\ &= \frac{3}{\sqrt{10}}\end{aligned}$$

Exercise 3.7

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \begin{bmatrix}2xy\\ x^2\\ 1\end{bmatrix}.$$

The directional vector, $\nabla f(2,2,1) = \begin{bmatrix}2\cdot 2\cdot 2\\ 2^2\\ 1\end{bmatrix} = \begin{bmatrix}8\\4\\1\end{bmatrix}.$

The maximum rate of change is $|\nabla f(2,2,1)| = \sqrt{8^2+4^2+1} = 9$.

Thus, the maximum rate of change at $(2,2,1)$ is 9 in the $\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$ direction.

Exercise 3.8

$\nabla f = \left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} \right) = \left(y^2 e^z, 2xye^z, xy^2 e^z \right)$

So $\vec{n} = \nabla f(1,1,1) = \begin{bmatrix} 1^2 \cdot e^1 \\ 2 \cdot 1 \cdot 1 \cdot e^1 \\ 1 \cdot 1^2 \cdot e^1 \end{bmatrix} = \begin{bmatrix} e \\ 2e \\ e \end{bmatrix}$.

For the equation of the tangent plane, $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$

$$\begin{aligned} \begin{bmatrix} e \\ 2e \\ e \end{bmatrix} \cdot \vec{r} &= \begin{bmatrix} e \\ 2e \\ e \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ ex + 2ey + ez &= e + 2e + e \\ &= 4e \\ x + 2y + z &= 4 \end{aligned}$$

Exercise 3.9

a) Let $\vec{u} = (a,b)$ such that $a^2 + b^2 = 0$ (that is, $\vec{u}$ is a unit vector).
We focus on $\vec{a} = (0,0)$.

$$\begin{aligned} \text{Now, } D_{\vec{u}} f(\vec{a}) &= \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h \to 0} \frac{f(ha,hb) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{ha \cdot hb}{(ha)^2 + (hb)^2} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{h^2 ab}{h^2(a^2+b^2)} \\ &= \lim_{h \to 0} \frac{ab}{h} \\ &= \begin{cases} 0 & b = 0 \ (\vec{u} = (1,0)) \\ 0 & a = 0 \ (\vec{u} = (0,1)) \\ \text{DNE} & \text{otherwise} \end{cases} \end{aligned}$$

To check for continuity, we want $\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0$.

- For $x = 0$, $\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{0 \cdot y}{0^2 + y^2} = 0$.
- For $y = 0$, $\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{y \to 0} \frac{x \cdot 0}{x^2 + 0^2} = 0$.
- For $y = x$, $\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq 0$.

Thus, this function is not continuous.

b) Let $\vec{u} = (a,b)$ such that $a^2 + b^2 = 0$ (that is, $\vec{u}$ is a unit vector).

We focus on $\vec{a} = (0,0)$.

$$\begin{aligned}\text{Now, } D_{\vec{u}} f(\vec{a}) &= \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h\to 0} \frac{f(ha,hb) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \sqrt{|ha\cdot hb|} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \sqrt{h^2|a\cdot b|} \\ &= \frac{|h|}{h}\sqrt{|a\cdot b|} \\ &= \begin{cases} 0 & b = 0\ (\vec{u} = (1,0)) \\ 0 & a = 0\ (\vec{u} = (0,1)) \\ \text{DNE} & \text{otherwise} \end{cases}\end{aligned}$$

Since there is no division by 0, this function is continuous.

Exercise 3.10

Solution 1

Let $\vec{u} = (a,b)$.

We focus on $\vec{a} = (0,0)$.

$$\begin{aligned}\text{Now, } D_{\vec{u}} f(\vec{a}) &= \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - \cancelto{0}{f(\vec{a})}}{h} \\ &= \lim_{h\to 0} \frac{f(ha,hb)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(ha)^2\cdot hb}{(ha)^4 + (hb)^2} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^3\cdot ab}{h^2(h^2a^4 + b^2)} \\ &= \lim_{h\to 0} \frac{a^2b}{h^2a^4 + b^2} \\ &= \begin{cases} 0 & b = 0\ (\vec{u} = (1,0)) \\ \dfrac{a^2}{b} & \text{otherwise} \end{cases}\end{aligned}$$

Thus, the directional derivatives exist at $(0,0)$.

For f to be differentiable, we need $\displaystyle\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h}}{|\vec{h}|} = 0$.

Note that $\nabla f(\vec{a}) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = \begin{bmatrix} D_{(1,0)}f(\vec{a}) \\ D_{(0,1)}f(\vec{a}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Let $(x,y) = \vec{x} = \vec{a}+\vec{h} = \vec{h}$ since $\vec{a} = (0,0)$.

Thus, $\lim\limits_{\vec{h}\to\vec{0}} \dfrac{f(\vec{a}+\vec{h}) - \cancelto{0}{f(\vec{a})} - \cancelto{0}{\nabla f(\vec{a})\cdot\vec{h}}}{|\vec{h}|} = \lim\limits_{\vec{h}\to\vec{0}} \dfrac{f(\vec{a}+\vec{h})}{|\vec{h}|}$

$$= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x^2y}{x^4+y^2}$$

- For $x=0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to0} \dfrac{1}{\sqrt{0^2+y^2}} \cdot \dfrac{0^2\cdot y}{0^4+y^2} = 0.$
- For $y=0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to0} \dfrac{1}{\sqrt{x^2+0^2}} \cdot \dfrac{x^2\cdot 0}{x^4+0^2} = 0.$
- For $y=x$, $\lim\limits_{\vec{h}\to\vec{0}} \dfrac{f(\vec{a}+\vec{h}) - \cancelto{0}{f(\vec{a})} - \cancelto{0}{\nabla f(\vec{a})\cdot\vec{h}}}{|\vec{h}|} = \lim\limits_{(x,y)\to(0,0)} \dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{x^2y}{x^4+y^2}$

$$= \lim_{x\to0} \frac{1}{\sqrt{x^2+x^2}} \cdot \frac{x^2\cdot x}{x^4+x^2}$$

$$= \lim_{x\to0} \frac{1}{\sqrt{2x^2}} \cdot \frac{\cancel{x^2}\cdot x}{\cancel{x^2}(x^2+1)}$$

$$= \lim_{x\to0} \frac{x}{\sqrt{2}|x|} \cdot \frac{1}{x^2+1}$$

$$= \pm\frac{1}{\sqrt{2}}$$

Thus, $\lim\limits_{\vec{h}\to\vec{0}} \dfrac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h}}{|\vec{h}|}$ does not exist. f is not differentiable.

Solution 2

Take $\vec{a}=(0,0)$ and $\vec{u}=(a,b)$ [8], $D_{\vec{u}}f(\vec{a}) = \lim\limits_{h\to0} \dfrac{f(\vec{a}+h\vec{u}) - \cancelto{0}{f(\vec{a})}}{h}$

$$= \lim_{h\to0} \frac{f(ha,hb)}{h}$$

$$= \lim_{h\to0} \frac{1}{h} \cdot \frac{(ha)^2\cdot hb}{(ha)^4+(hb)^2}$$

$$= \lim_{h\to0} \frac{1}{h} \cdot \frac{h^3\cdot ab}{h^2(h^2a^4+b^2)}$$

$$= \lim_{h\to0} \frac{a^2b}{h^2a^4+b^2}$$

$$= \begin{cases} 0 & b=0\ (\vec{u}=(1,0)) \\ \dfrac{a^2}{b} & \text{otherwise} \end{cases}$$

Assume f is differentiable at $(0,0)$. Then, $D_{\vec{u}}f = \nabla f\cdot\vec{u}$.
Let $\vec{u}=(a,b)$, $\nabla f(\vec{a}) = (c,d)$.

- When $a=1$, $b=0$, $D_{\vec{u}}f = a|b| = 1\cdot 0 = 0$, and $\nabla f\cdot\vec{u} = (c,d)\cdot(1,0) = c$. Thus, $c=0$.
- When $a=0$, $b=1$, $D_{\vec{u}}f = a|b| = 0\cdot 1 = 0$, and $\nabla f\cdot\vec{u} = (c,d)\cdot(0,1) = d$. Thus, $d=0$.
- When $a=\dfrac{1}{\sqrt{2}}$, $b=\dfrac{1}{\sqrt{2}}$, $D_{\vec{u}}f = \dfrac{b}{a^3} = \dfrac{0}{0^3}$ DNE.

---

[8] Here $\vec{u}$ is a always unit vector, so we have $a^2+b^2=1$.

Exercise 3.11

a) Take $\vec{a}=(0,0)$ and $\vec{u}=(a,b)$,
$$\begin{aligned}
D_{\vec{u}}f &= \lim_{h\to 0}\frac{f(\vec{a}+h\vec{u})-f(\vec{a})}{h}\\
&= \lim_{h\to 0}\frac{f(ha,hb)-\cancelto{0}{f(0,0)}}{h}\\
&= \lim_{h\to 0}\frac{1}{\cancel{h}}\cdot\frac{\cancel{h}\cdot a\cdot|hb|}{\sqrt{(ha)^2+(hb)^2}}\\
&= \lim_{h\to 0}\frac{a|hb|}{\sqrt{h^2\cancelto{-1}{(a^2+b^2)}}}\\
&= \lim_{h\to 0}\frac{a|hb|}{|h|}\\
&= \lim_{h\to 0}\frac{a\cdot\cancel{|h|}}{\cancel{|h|}}\\
&= \lim_{h\to 0}a|b|\\
&= a|b|
\end{aligned}$$
Assume f is differentiable at $(0,0)$. Then, $D_{\vec{u}}f=\nabla f\cdot\vec{u}$.
Let $\vec{u}=(a,b)$, $\nabla f(\vec{a})=(c,d)$.

- When $a=1$, $b=0$, $D_{\vec{u}}f=a|b|=1\cdot 0=0$, and $\nabla f\cdot\vec{u}=(c,d)\cdot(1,0)=c$. Thus, $c=0$.
- When $a=0$, $b=1$, $D_{\vec{u}}f=a|b|=0\cdot 1=0$, and $\nabla f\cdot\vec{u}=(c,d)\cdot(0,1)=d$. Thus, $d=0$.
- When $a=\frac{1}{\sqrt{2}}$, $b=\frac{1}{\sqrt{2}}$, $D_{\vec{u}}f=a|b|=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\frac{1}{2}$, but $\nabla f\cdot\vec{u}=(c,d)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=(0,0)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=0$

This is a contradiction. Thus, f must not be differentiable at $(0,0)$.

b) Take $\vec{a}=(0,0)$ and $\vec{u}=(a,b)$,
$$\begin{aligned}
D_{\vec{u}}f &= \lim_{h\to 0}\frac{f(\vec{a}+h\vec{u}-f(\vec{a}))}{h}\\
&= \lim_{h\to 0}\frac{f(ha,hb)-\cancelto{0}{f(0,0)}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(ha)^4+(hb)^4}{(ha)^2+(hb)^2}\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{h^4(a^4+b^4)}{h^2\cancelto{1}{(a^2+b^2)}}\\
&= \lim_{h\to 0}h(a^4+b^4)\\
&= 0
\end{aligned}$$

To show f is differentiable, we want $\lim_{\vec{h}\to\vec{0}}\frac{f(\vec{a}+\vec{h})-\cancelto{0}{f(\vec{a})}-\nabla f(\vec{a})\cdot\vec{h}}{|h|}=0$.
Take $(x,y)=\vec{x}=\vec{a}+\vec{h}$. Since $\vec{a}=\vec{0}$, $\vec{h}=\vec{x}=(x,y)$.
Then, $\frac{\partial f}{\partial x}=D_{(1,0)}f(\vec{a})=0$ and $\frac{\partial f}{\partial x}=D_{(0,1)}f(\vec{a})=0$. Thus, $\nabla f(\vec{a})=(0,0)$.
Thus, $\lim_{(x,y)\to(0,0)}\frac{f(\vec{a}+\vec{h})-\cancelto{0}{\nabla f(\vec{a})\cdot\vec{h}}}{|h|}=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x^4+y^4}{x^2+y^2}$.

$$
\begin{aligned}
\left|\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x^4+y^4}{x^2+y^2}\right| &\leq \left|\frac{x^4}{\sqrt{x^2+\cancel{y^2}}(x^2+\cancel{y^2})}\right| + \left|\frac{y^4}{\sqrt{\cancel{x^2}+y^2}(\cancel{x^2}+y^2)}\right| \\
&= \left|\frac{x^4}{\sqrt{x^2}\cdot x^2}\right| + \left|\frac{y^4}{\sqrt{y^2}\cdot y^2}\right| \\
&= |x|+|y| \to 0
\end{aligned}
$$

Thus, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{|\vec{h}|} = 0$. That is, f is differentiable at $(0,0)$.

Exercise 3.12

1. $\nabla f = (6x+4y, 4x+10y)$

   $H = \begin{bmatrix} 6 & 4 \\ 4 & 10 \end{bmatrix}$

2. $\nabla f = (-\sin(x+2y), -2\sin(x+2y))$

   $H = \begin{bmatrix} -\cos(x+2y) & -2\cos(x+2y) \\ -2\cos(x+2y) & -4\cos(x+2y)) \end{bmatrix}$

3. $\nabla f = (2e^{2x+y}, e^{2x+y})$

   $H = \begin{bmatrix} 4e^{2x+y} & 2e^{2x+y} \\ 2e^{2x+y} & e^{2x+y} \end{bmatrix}$

4. $\nabla f = (2xe^{x^2+y}, e^{x^2+y})$

   $H = \begin{bmatrix} 4x^2e^{x^2+y}+2e^{x^2+y} & 2xe^{x^2+y} \\ 2xe^{x^2+y} & e^{x^2+y} \end{bmatrix}$

5. $\nabla f = (e^x\sin(y), e^x\cos(y))$

   $H = \begin{bmatrix} e^x\sin(y) & e^x\cos(y) \\ e^x\cos(y) & -e^x\sin(y) \end{bmatrix}$

6. $\nabla f = (2xy+z, x^2, x+2z)$

   $H = \begin{bmatrix} 2y & 2x & 1 \\ 2x & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Exercise 3.13

Solution 1

Take $\vec{a} = (0,0)$ and $\vec{u} = (a,b)$,
$$
\begin{aligned}
D_{\vec{u}}f(\vec{a}) &= \lim_{h\to 0}\frac{f(\vec{a}+h\vec{u}) - \cancelto{0}{f(\vec{a})}}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\cdot f(ha+hb) \\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(ha)^3\cdot hb - ha\cdot(hb)^3}{(ha)^2+(hb)^2} \\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{h^4(a^3b-ab^3)}{h^2\cancelto{1}{(a^2+b^2)}} \\
&= \lim_{h\to 0} h(a^3b-ab^3) \\
&= 0
\end{aligned}
$$

In particular, $\nabla f = (0,0)$. That is, $\dfrac{\partial f}{\partial x} = 0$ (for $\vec{u} = (1,0)$) and $\dfrac{\partial f}{\partial y} = 0$ (for $\vec{u} = (0,1)$).

Let $g(x,y) = \dfrac{\partial f}{\partial x} = \begin{cases} \dfrac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-xy^3)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Then, $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial g}{\partial y} = D_{\vec{u}}g(\vec{a})$ at $\vec{a} = (0,0)$ when $\vec{u} = (0,1)$.

$$\begin{aligned}
D_{\vec{u}}f(\vec{a}) &= \lim_{h\to 0} \frac{g(\vec{a}+h\vec{u}) - \cancelto{0}{g(\vec{a})}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot g(0,h) \\
&= \lim_{h\to 0} \frac{1}{h}\cdot \frac{(-h^2)(h^2)}{(h^2)^2} \\
&= \lim_{h\to 0} -1 \\
&= -1
\end{aligned}$$

Let $k(x,y) = \dfrac{\partial f}{\partial y} = \begin{cases} \dfrac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-xy^3)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Then, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial k}{\partial x} = D_{\vec{u}}k(\vec{a})$ at $\vec{a} = (0,0)$ when $\vec{u} = (1,0)$.

$$\begin{aligned}
D_{\vec{u}}k(\vec{a}) &= \lim_{h\to 0} \frac{k(\vec{a}+h\vec{u}) - \cancelto{0}{k(\vec{a})}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot k(h,0) \\
&= \lim_{h\to 0} \frac{1}{h}\cdot \frac{(h^3)(h^2)}{(h^2)^2} \\
&= \lim_{h\to 0} 1 \\
&= 1
\end{aligned}$$

Thus, $f_{xy} = 1 \neq -1 = f_{yx}$, f is not C^2 at $(0,0)$.

Solution 2

Show f is not C^2 at $(0,0)$ by trying to show $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ is not continuous at $(0,0)$.

Let $g(x,y) = \dfrac{\partial f}{\partial x} = \begin{cases} \dfrac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-xy^3)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

That is, after simplification, $g(x,y) = \begin{cases} \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

$$\begin{aligned}
\text{Then, } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial g}{\partial y} &= \frac{(x^4+12x^2y^2-5y^4)(x^2+y^2)^2 - 4y(x^2+y^2)(x^4y+4x^2y^3-y^5)}{[(x^2+y^2)^2]^2} \\
&= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}
\end{aligned}$$

- For $x=0$, $\dfrac{\partial g}{\partial y} = \dfrac{0^6+9\cdot 0^4\cdot y^2-9\cdot 0^2\cdot y^4-y^6}{(0^2+y^2)^3} = -1$
- For $y=0$, $\dfrac{\partial g}{\partial y} = \dfrac{x^6+0\cdot x^4\cdot 0^2-9\cdot x^2\cdot 0^4-0^6}{(x^2+0^2)^3} = 1$

Thus, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}$ DNE, $\dfrac{\partial g}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ is not continuous at $(0,0)$. That is, f is not C^2 at $(0,0)$.

Exercise 3.14

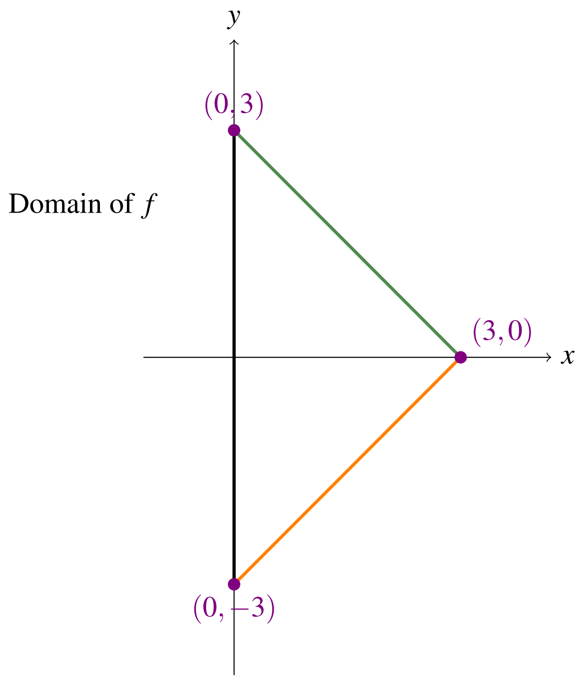
1. $\nabla f = (6x+4y, 4x+10y)$.

   For critical points, $\nabla f = \vec{0}$. That is, $\begin{cases} 6x+4y=0 \\ 4x+10y=0 \end{cases} \implies \begin{cases} x=0 \\ y=0 \end{cases}$.

   Thus, $(0,0)$ is the only critical point.

2.
3.
4.

Exercise 3.15



Critical Points

$\nabla f = (2x-4, 2y)$.

For critical points, $\nabla f = \vec{0}$. That is, $\begin{cases} 2x-4=0 \\ 2y=0 \end{cases} \implies \begin{cases} x=2 \\ y=0 \end{cases}$

Thus, $(2,0)$ is the only critical point.

$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $\det H = 4 > 0$.

Since $H_{00} = 2 > 0$, $(2,0)$ is a local minimum.

Boundary Points

Boundary points: $\begin{cases} (0,-3) \\ (-3,0) \\ (3,0) \end{cases}$

Boundary Curves

- The curve is $x = 0$.
  Then, $f(x,y) = f(0,y) = y^2$. That is, $f(y) = y^2$.
  $f'(y) = 2y$.
  For critical points on this curve, $f'(y) = 0 \implies y = 0$.
  Thus, $(0,0)$ is a possible point for absolute minimum / maximum for the function.
- The curve is $y = x - 3$.
  Then, $f(x,y) = f(x,x-3) = 2x^2 - 10x + 9$. That is, $f(x) = 2x^2 - 10x + 9$.
  $f'(x) = 4x - 10$.
  For critical points on this curve, $f'(x) = 0 \implies x = \frac{5}{2}$.
  Now, $y = x - 3 = -\frac{1}{2}$.
  Thus, $(\frac{5}{2}, -\frac{1}{2})$ is a possible point for absolute minimum / maximum for the function.
- The curve is $y = -x + 3$.
  Then, $f(x,y) = f(x,-x+3) = 2x^2 - 10x + 9$. That is, $f(x) = 2x^2 - 10x + 9$.
  $f'(x) = 4x - 10$.
  For critical points on this curve, $f'(x) = 0 \implies x = \frac{5}{2}$.
  Now, $y = x - 3 = \frac{1}{2}$.
  Thus, $(\frac{5}{2}, \frac{1}{2})$ is a possible point for absolute minimum / maximum for the function.

$(x,_y)$	$f(x,y)$	
(2, 0)	-4	min
(0, 0)	0	
(3, 0)	-3	
(0, 3)	9	max
(0, -3)	9	max
$(\frac{5}{2}, \frac{1}{2})$	$-\frac{7}{2}$	
$(\frac{5}{2}, -\frac{1}{2})$	$-\frac{7}{2}$	

Exercise 3.16

Consider the level set $f(x,y) = k$

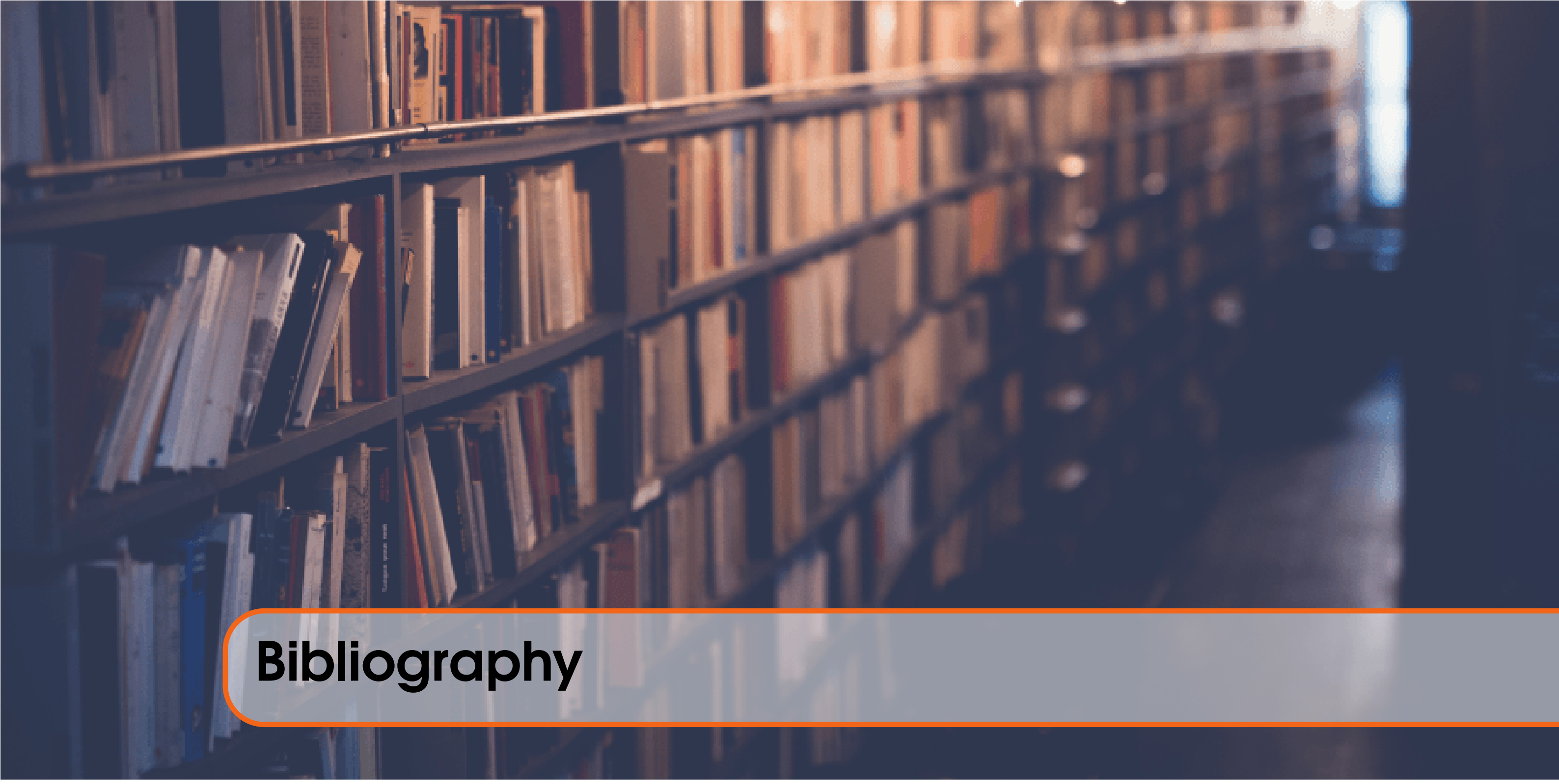
$$3x - 6y = k$$

$$y = \frac{1}{2}x - \frac{1}{6}k$$

Set $g(x,y) = 4x^2 + 2y - 25$. Consider the level set $g(x,y) = 0$.
Then, $\nabla g(x,y) = (8x, 4y)$.

Exercise 3.17

Exercise 3.18



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