Hasse Diagrams of Groups.

We have discussed before what are the Hasse Diagrams of Groups but for concreteness let us describe them.

Definition: Let G be a group. A Hasse Diagram (for the subgroups of G) is a drawing that will contain the subgroups of G distributed as follows:

→ It is layered by rows labeled 1, 2, 3, ...

- The row layered n contains exactly the subgroups of G of order n.
- If H_1, H_2 are two subgroups of G, there is a line from H_1 to H_2 if and only if $H_1 \leq H_2$ but there is not another subgroup H of G with $H_1 \leq H \leq H_2$.

The Hasse Diagram is an organizational tool that serves to read many properties of a group. However, it is more useful for finite groups where we can actually see it.

Example: We saw in a previous lecture the Hasse Diagram for C_{12}. We reconstruct it here.

By **Lagrange's Theorem** if $H \leq C_{12}$ then $|H| \,\Big|\, |C_{12}|$, that is, $|H|$ divides 12.

The divisors of 12 are

$$1, 2, 3, 4, 6, 12.$$

Thus the Diagram has at most 6 layers.

We say at most because potentially there are some of these divisors who are not the orde of any subgroup.

We have proven in Homework that in a cyclic group there is exactly one subgroup of each possible order and that they are all cyclic.

$\therefore$ We have $C_1, C_2, C_3, C_4, C_6, C_{12}$ as the subgroups of C_{12}.

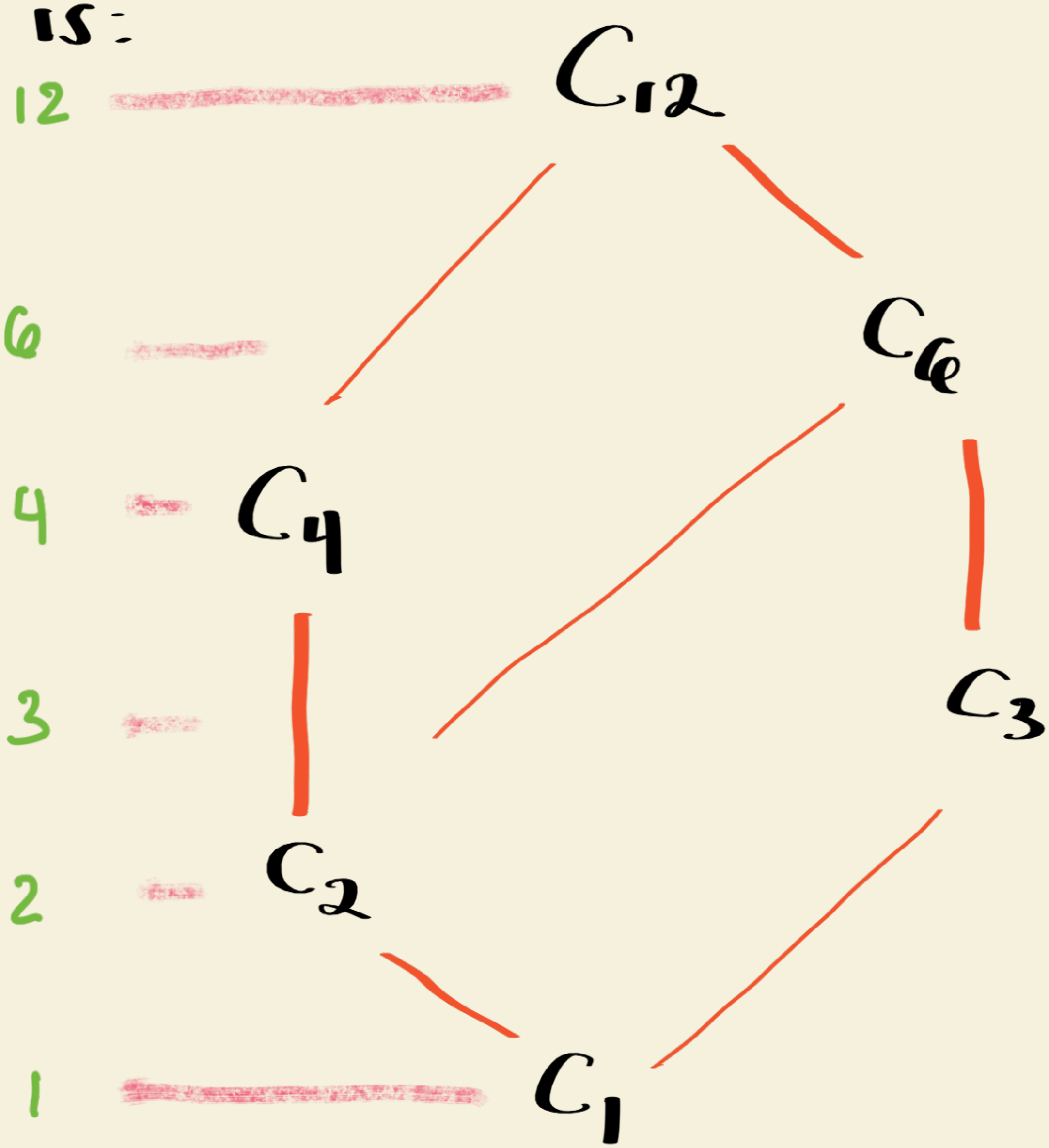
Notice that

$$C_m \leq C_n \iff m \mid n$$

Again by Lagrange & the result from Homework.

Make sure you agree!!

Thus our Hasse Diagram of C_{12} is:



Notice we can say who is each subgroup:

$C_{12} = \{0, 1, 2, \ldots, 11\}$

$C_6 = \{0, 2, 4, 6, 8, 10\}$

$C_4 = \{0, 3, 6, 9\}$

$C_3 = \{0, 4, 8\}$

$C_2 = \{0, 6\}$

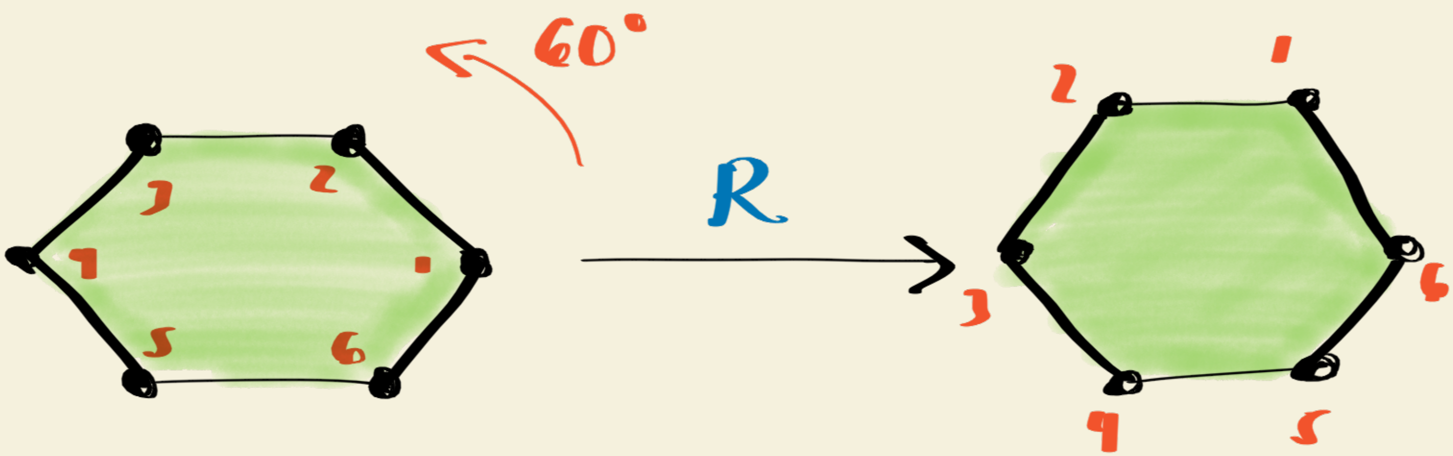
$C_1 = \{0\}$

The shaded element is a generator.

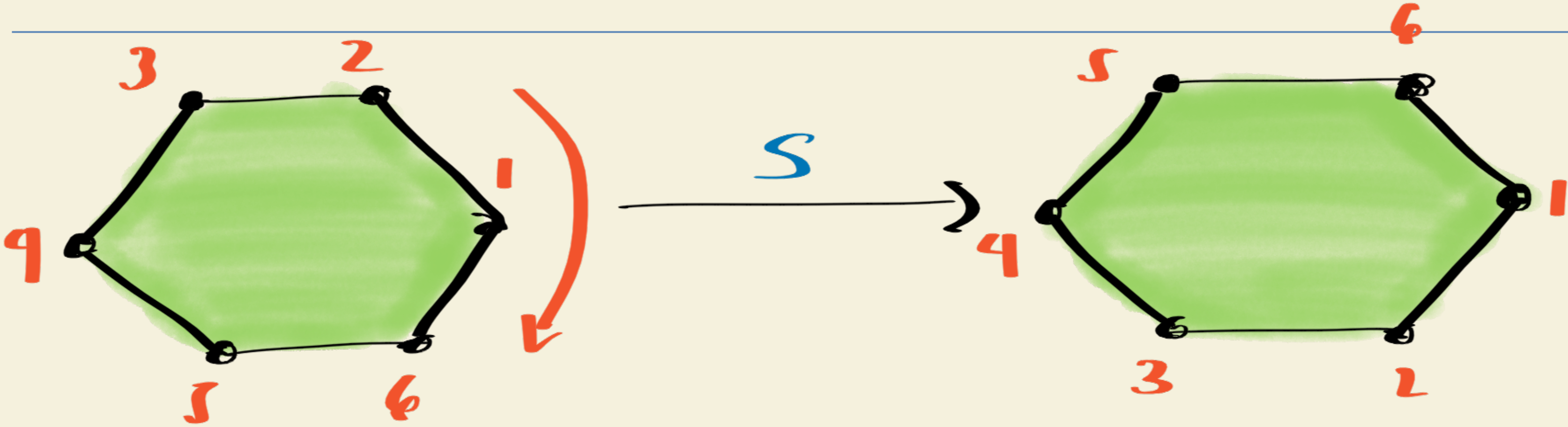
WED FEB 7, 2024

Our task in these notes is to build the Hasse Diagram of D_6, i.e. the dihedral group associated to the hexagon.

You studied this group on homework 1. Let us just fix the two generators:



Rotation counterclockwise by 60° is called R.



Reflection on the x-axis is called S.

You have proven in homework that all elements of D_{12} are:

$1, R, R^2, R^3, R^4, R^5, S, SR, SR^2, SR^3, SR^4, SR^5$

and that $R^6 = 1$, $S^2 = 1$, $SRS = R^5$.

To build the Hasse Diagram we need the divisors of $|D_6| = 12$. Hence, we have they are

$$1, 2, 3, 4, 6, 12.$$

The same as for C_{12} but now this is not cyclic so we don't know immediately what the subgroups are!

First task: What cyclic subgroups are there in D_6?

The easiest subgroups to find are the cyclic groups because they correspond to the order of elements. Hence, we can simply see the elements.

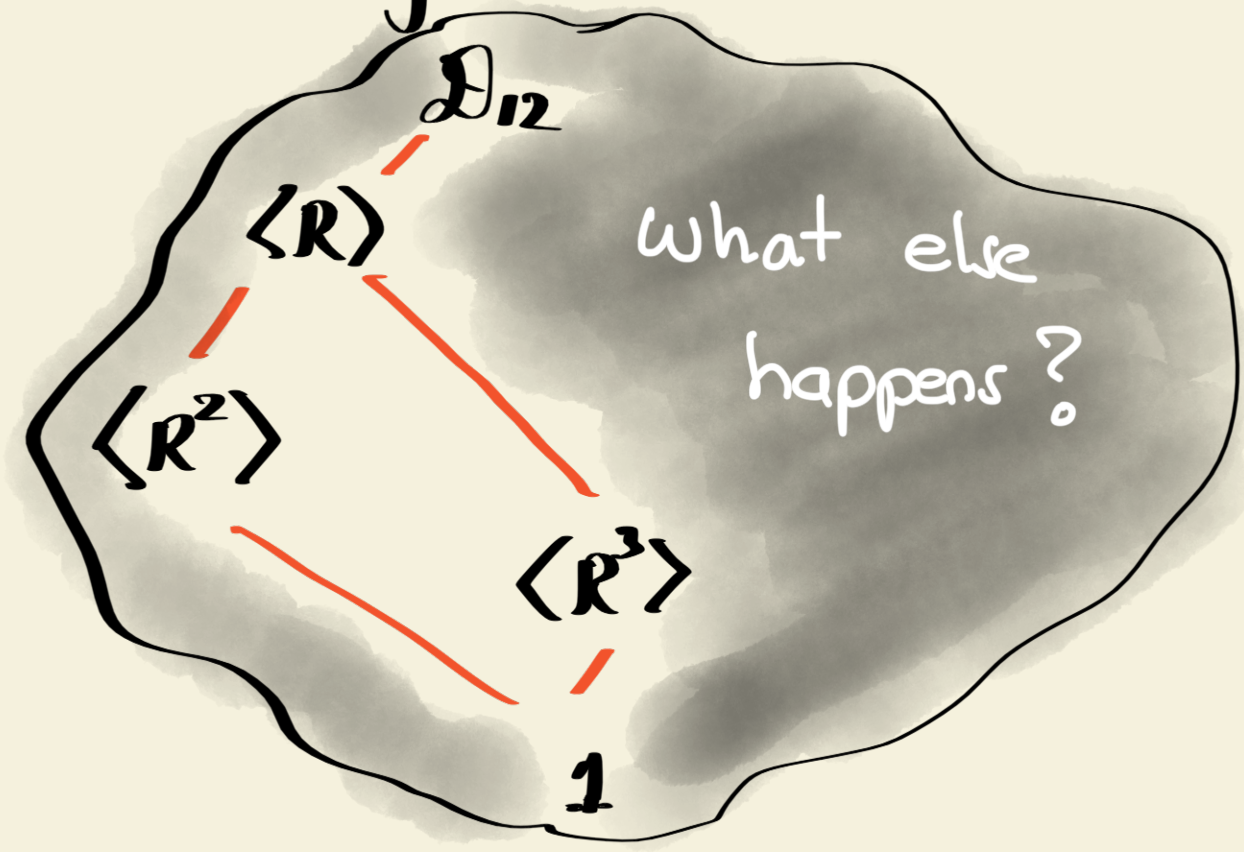
There are two types of elements: those with an S and those without an S.

Those without are $1, R, R^2, R^3, R^4, R^5$ which are the powers of R and form a C_6 generated by R.

Thus we know everything of these elements:

element	1	R	R^2	R^3	R^4	R^5
order	1	6	3	2	3	6

This subgroup alone contributes like this to the Hasse Diagram



We now check what happens with the elements with an S:

$$S,\ SR,\ SR^2,\ SR^3,\ SR^4,\ SR^5$$

We claim all of them have order 2:

- $S^2 = 1$ we already knew.
- $SRSR = R^5 \cdot R = R^6 = 1$
- $SR^2SR^2 = R^{10}R^2 = R^{12} = 1$
- $SR^3SR^3 = R^{15}R^3 = R^{18} = 1$
- $SR^4SR^4 = R^{20}R^4 = R^{24} = 1$

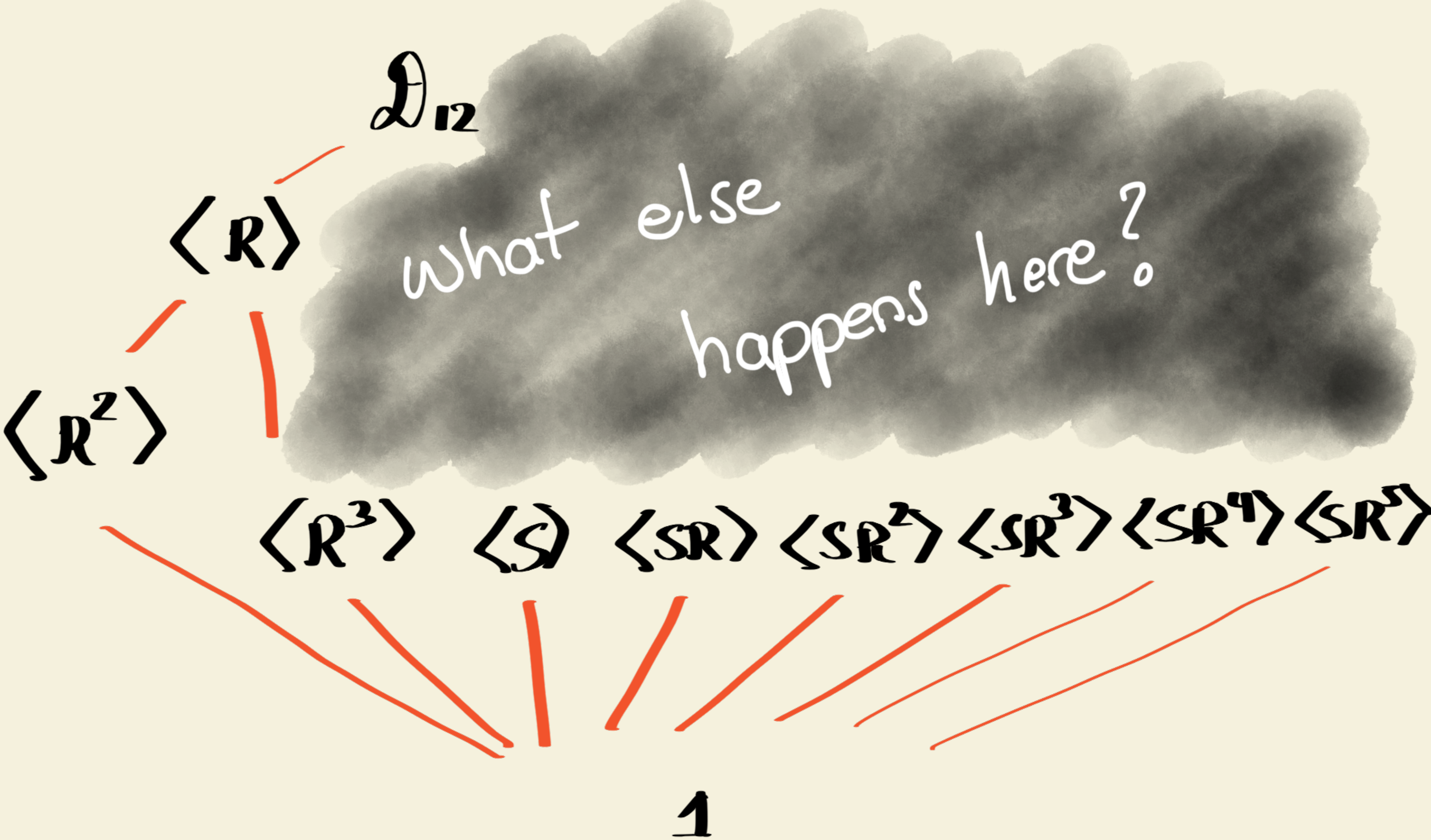
$$SR^5SR^5 = R^{25}R^5 = R^{30} = 1$$

We used

$$SR^kS = (SRS)^k = (R^5)^k = R^{5k}$$

conjugation by S!!

Thus each of them produces different cyclic groups of order 2. So we have



WED FEB 7, 2024

We have now found all cyclic subgroups. Also for prime orders, since the only groups of prime order are cyclic, we have found all possible subgroups of that order.

Order of Subgroup	Status
12	Only D_6 ; ALL FOUND
6	$\langle R \rangle$; Maybe noncyclic missing
4	NONE FOUND YET
3	ALL FOUND, ONLY $\underline{1}$: $\langle R^2 \rangle$
2	ALL FOUND; 7 in total
1	ALL FOUND; only $\{1\}$

Peculiar!

Second task: Are there normal subgroups?

① → Those of order 6 are normal because their index is 2.

② → The one of order 3 is normal because it is unique. Indeed,

$$g\langle R^2\rangle g^{-1}$$

must be another subgroup of order 3... but there is no other! Thus $g\langle R^2\rangle g^{-1}=\langle R^2\rangle$.

For all g, so normal!!

How do I use this? Well, it changes group by group but let us see some ideas.

→ Let $H \leq D_6$ be of order 6.

∴ H is normal in D_6 and thus there must exist an homomorphism

$$\phi : D_6 \longrightarrow \{1, -1\}$$

This is the image because the index is 2

with $\ker \phi = H$.

We don't know ϕ explicitly but we know it exists because of normality.

Now let $x \in D_6$ with order m. Then

$$x^m = 1.$$

Take ϕ:

$$\phi(x)^m = 1$$

If m is odd then $\phi(x) = 1$!!

We conclude: all elements of odd order are in the kernel of ϕ!!

That is, all elements of odd order are in H.

But the elements of odd order are exactly

$$\{1, R^2, R^4\} = \langle R^2 \rangle$$

which was the only subgroup of order 3 and thus normal.

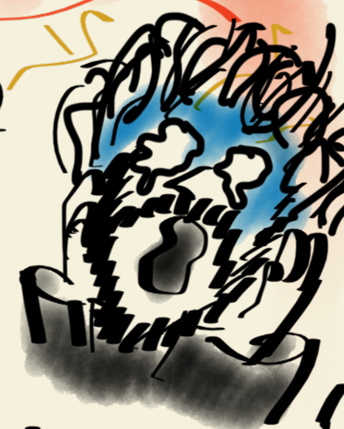
Thus we have proven: let $H \leq D_6$ be of order 6. Then

$$\langle R^2 \rangle \leq H \leq D_6 .$$

normal in D_6!!

So now we **TAKE THE QUOTIENT**

(without fear!! Only by losing fear of quotients can we move on!!)



Quotiening by $\langle R^2 \rangle$ means making "R^2" trivial.

So our new words are

1	R	R^2	R^3	R^4	R^5	S	SR	SR^2	SR^3	SR^4	SR^5
$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$

And the new elements are $1, \bar{R}, \bar{S}, \overline{SR}$.

The bar means "coset".

WED FEB 7, 2024

In the table we are seeing the cosets of $\{1, R^2, R^4\}$:

1	R	R^2	R^3	R^4	R^5	S	SR	SR^2	SR^3	SR^4	SR^5
$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$

The bar means "the coset that it represents" and the same shaded elements form a coset.

$\bar{1} = \{1, R^2, R^4\}$

$\bar{R} = R\{1, R^2, R^4\}$

$\bar{S} = S\{1, R^2, R^4\}$

$\overline{SR} = SR\{1, R^2, R^4\}$

$\times$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\overline{SR}$
$\bar{1}$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\overline{SR}$
$\bar{R}$	$\bar{R}$	$\bar{1}$	$\overline{SR}$	$\bar{S}$
$\bar{S}$	$\bar{S}$	$\overline{SR}$	$\bar{1}$	$\bar{R}$
$\overline{SR}$	$\overline{SR}$	$\bar{S}$	$\bar{R}$	$\bar{1}$

Multiplication table of $C_2 \times C_2$!!

Its Hasse Diagram is

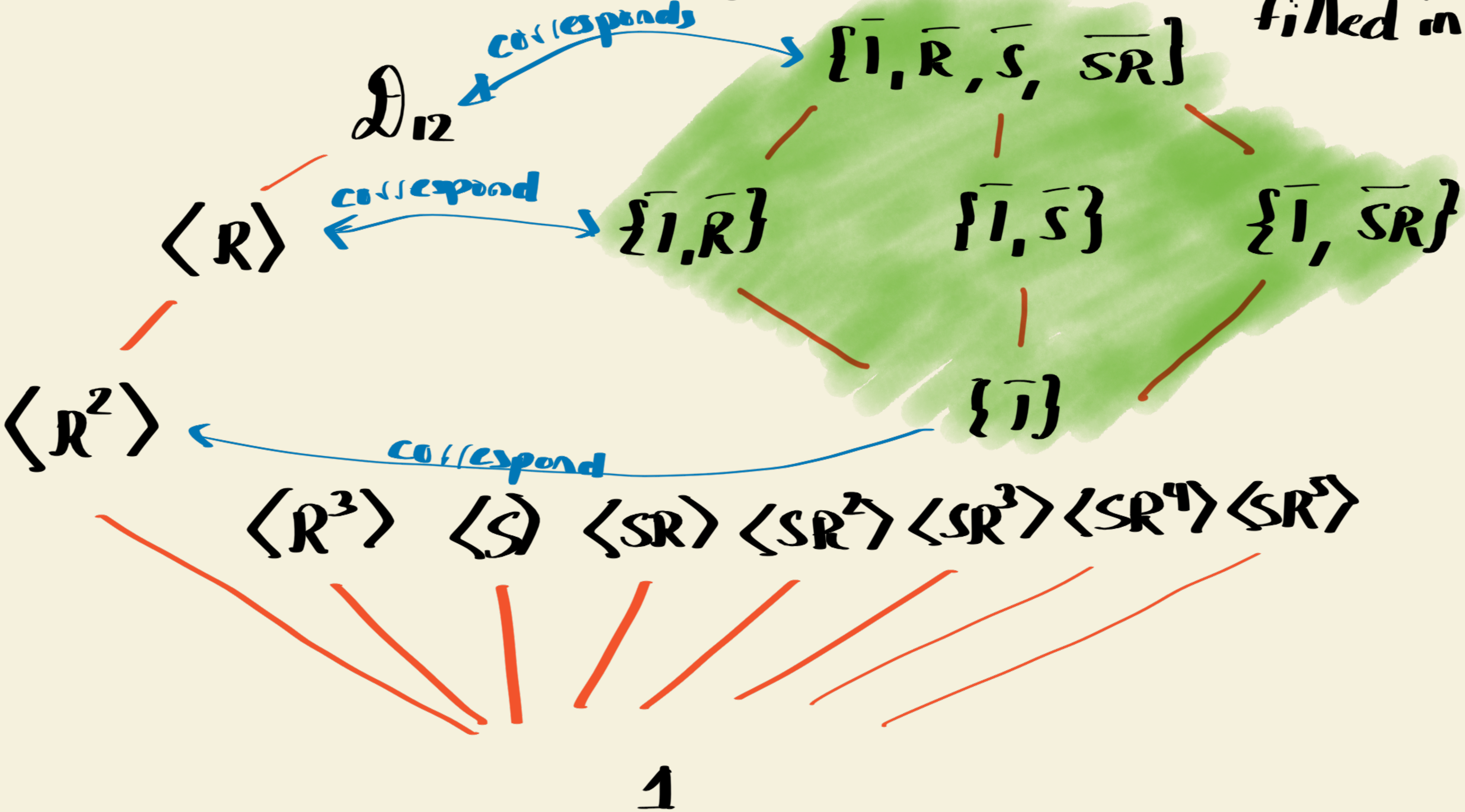
$$\begin{tikzcd}[arrows=-]
 & \{\bar{1}, \bar{R}, \bar{S}, \overline{SR}\} \arrow[dl] \arrow[d] \arrow[dr] & & \text{order } 4 \\
\{\bar{1}, \bar{R}\} \arrow[dr] & \{\bar{1}, \bar{S}\} \arrow[d] & \{\bar{1}, \overline{SR}\} \arrow[dl] & \text{order } 2 \\
 & \{\bar{1}\} & & \text{order } 1
\end{tikzcd}$$

We now invoke the correspondence theorem which states that the subgroups

$$\langle R^2 \rangle \leq H \leq D_6$$

organis themselves as a copy of the Hasse Diagram of $D_6 / \langle R^2 \rangle$.

So in the Hasse Diagram a new region can be filled in



The question is how to describe the new groups we have found by correspondence.

It is by taking preimages:

1	R	R^2	R^3	R^4	R^5	S	SR	SR^2	SR^3	SR^4	SR^5
$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$	$\bar{S}$	$\overline{SR}$

$$
\begin{tikzcd}[column sep=tiny]
 & \langle R^2\rangle \cup R\langle R^2\rangle \cup S\langle R^2\rangle \cup SR\langle R^2\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle R^2\rangle \cup R\langle R^2\rangle \arrow[dr, dash] & \langle R^2\rangle \cup S\langle R^2\rangle \arrow[d, dash] & \langle R^2\rangle \cup SR\langle R^2\rangle \arrow[dl, dash] \\
 & \langle R^2\rangle &
\end{tikzcd}
$$

correspondence

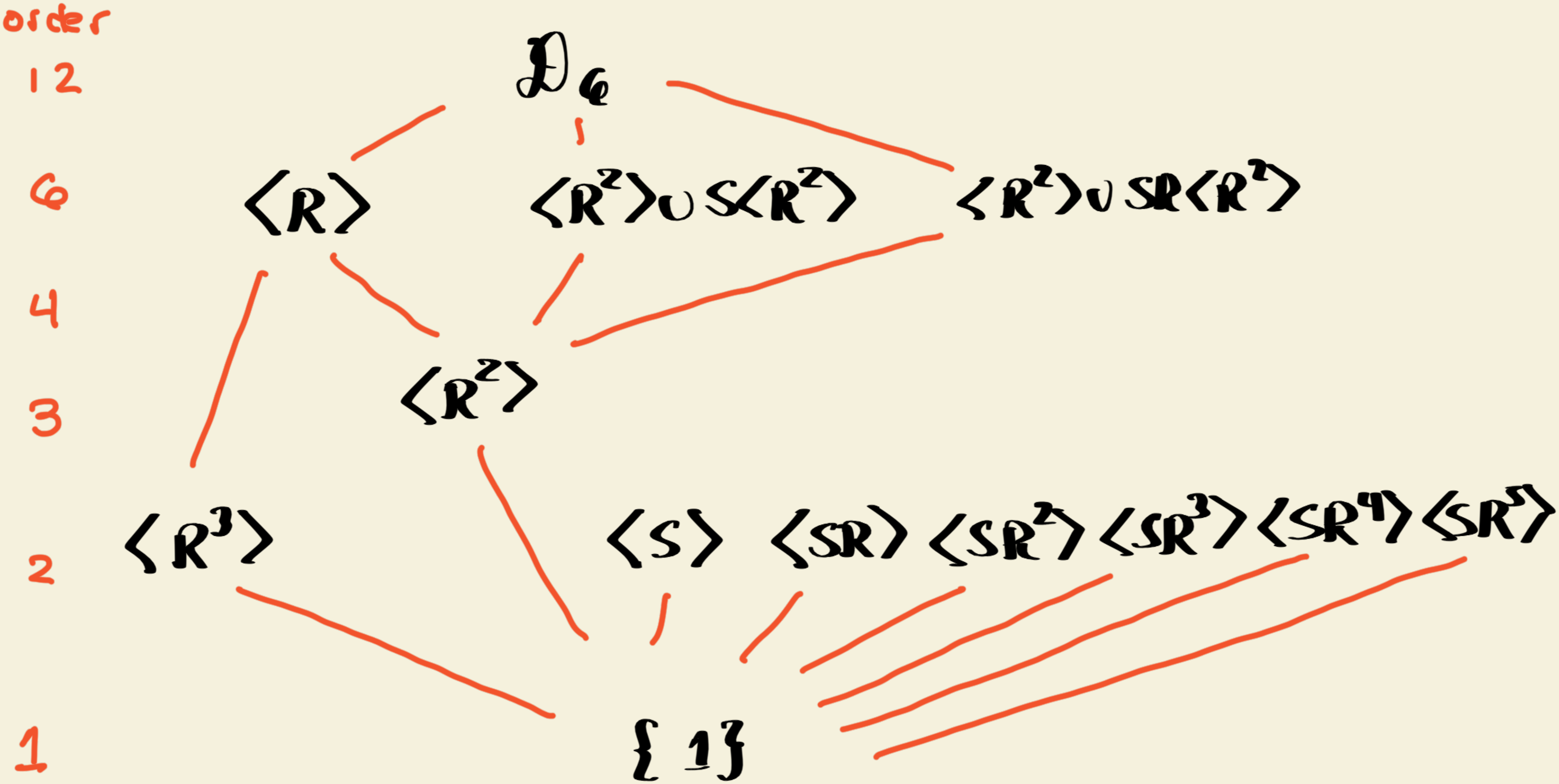
$$
\begin{tikzcd}
 & \{1, \bar{R}, \bar{S}, \overline{SR}\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle \bar{R}\rangle \arrow[dr, dash] & \langle \bar{S}\rangle \arrow[d, dash] & \langle \overline{SR}\rangle \arrow[dl, dash] \\
 & \langle \bar{1}\rangle &
\end{tikzcd}
$$

This is how it manifests on the Hasse of D_6.

This is Hasse of $C_2 \times C_2$.

THU FEB 8, 2024

So the Hasse looks like



And our task table looks like:

Order of Subgroup	Status	Total
12	ALL FOUND	1
6	ALL FOUND	3
4	NONE FOUND YET	?.
3	ALL FOUND	1
2	ALL FOUND	7
1	ALL	1

<u>Third</u> <u>Task</u>: Find the order 4 subgroups.

They cannot be cyclic because no element of D_6 has order 4.

Hence if we have such subgroups they must be isomorphic to $C_2 \times C_2$, but who are they?

Observation: $C_2 \times C_2$ contains three elements of order 2.

In our case only one element of order 2 does not have an S. Thus if some $H \leq D_6$ has $|H| = 4$ then at least two have an S. Say

$$SR^i, SR^j, \quad i \neq j$$

$$H \ni SR^i \cdot SR^j = (SR^i S)R^j = R^{5i+j}$$

This must be the third element of order 2. But it has no S!! It is R^3.

We have proven: If $H \leq D_6$ has order 4 then

$$\langle R^3 \rangle \leq H \leq D_6.$$

Thus we might be able to invoke correspondence if $\langle R^3 \rangle$ is normal in D_6.

There are two approaches here:

(1) Prove $\langle R^3 \rangle \trianglelefteq D_6$ by finding an homomorphism it is a kernel of.

(2) Verify the generators of D_6 conjugate $\langle R^3 \rangle$ to itself.

We do the second one: since $\langle R^3 \rangle$ is cyclic it is enough to check the generator. (Make sure you agree!!)

We have

$$R \cdot R^3 \cdot R^{-1} = R^3,$$

$$S R^3 S = R^3.$$

It works, so $\langle R^3 \rangle \trianglelefteq D_6$!!

Challenge: Find a nice homomorphism whose kernel is $\langle R^3 \rangle$.

Thus we can do correspondence. So now...

TAKE THE QUOTIENT

1	R	R^2	R^3	R^4	S	SR	SR^2	SR^3	SR^4	SR^5	R^5
$\bar{1}$	$\bar{R}$	$\bar{R}^2$	$\bar{1}$	$\bar{R}$	$\bar{S}$	$\bar{S}\bar{R}$	$\bar{S}\bar{R}^2$	$\bar{S}$	$\bar{S}\bar{R}$	$\bar{S}\bar{R}^2$	$\bar{R}^2$

We now have 6 cosets and thus the quotient group has 6 elements.

Challenge: who is this group?

This is not cyclic (not even abelian) so it is S_3.

We are looking for subgroups of order 4:

$$\langle R^2 \rangle \leq H \leq D_6$$

and thus

$$\{\bar{1}\} \leq H/\langle R^3 \rangle \leq D_6/\langle R^3 \rangle$$

and

$$\left| H/\langle R^3 \rangle \right| = |H|/\langle R^3 \rangle = 4/2 = 2$$

And viceversa. So subgroups of order 4 correspond to subgroups of the quotient of order 2!!

Let us look for them:

$\overline{1}$ has order 1

$\overline{R} \cdot \overline{R} = \overline{R^2} \neq \overline{1}$

$\overline{R^2} \cdot \overline{R^2} = \overline{R^4} = \overline{R} \neq \overline{1}$

These actually have order 3!

$\overline{S} \cdot \overline{S} = \overline{S^2} = \overline{1}$

$\overline{SR} \cdot \overline{SR} = \overline{SRSR} = \overline{1}$

$\overline{SR^2}\ \overline{SR^2} = \overline{SR^2SR^2} = \overline{1}$

There are three elements of order 2 and the preimages of what they generate are the subgroups of order 4.

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Thus the subgroups are

1	R	R^2	R^3	R^4	S	SR	SR^2	SR^3	SR^4	SR^5	R^5
$\overline{1}$	$\overline{R}$	$\overline{R}^2$	$\overline{1}$	$\overline{R}$	$\overline{S}$	$\overline{SR}$	$\overline{SR}^2$	$\overline{S}$	$\overline{SR}$	$\overline{SR}^2$	$\overline{R}^2$

so they are

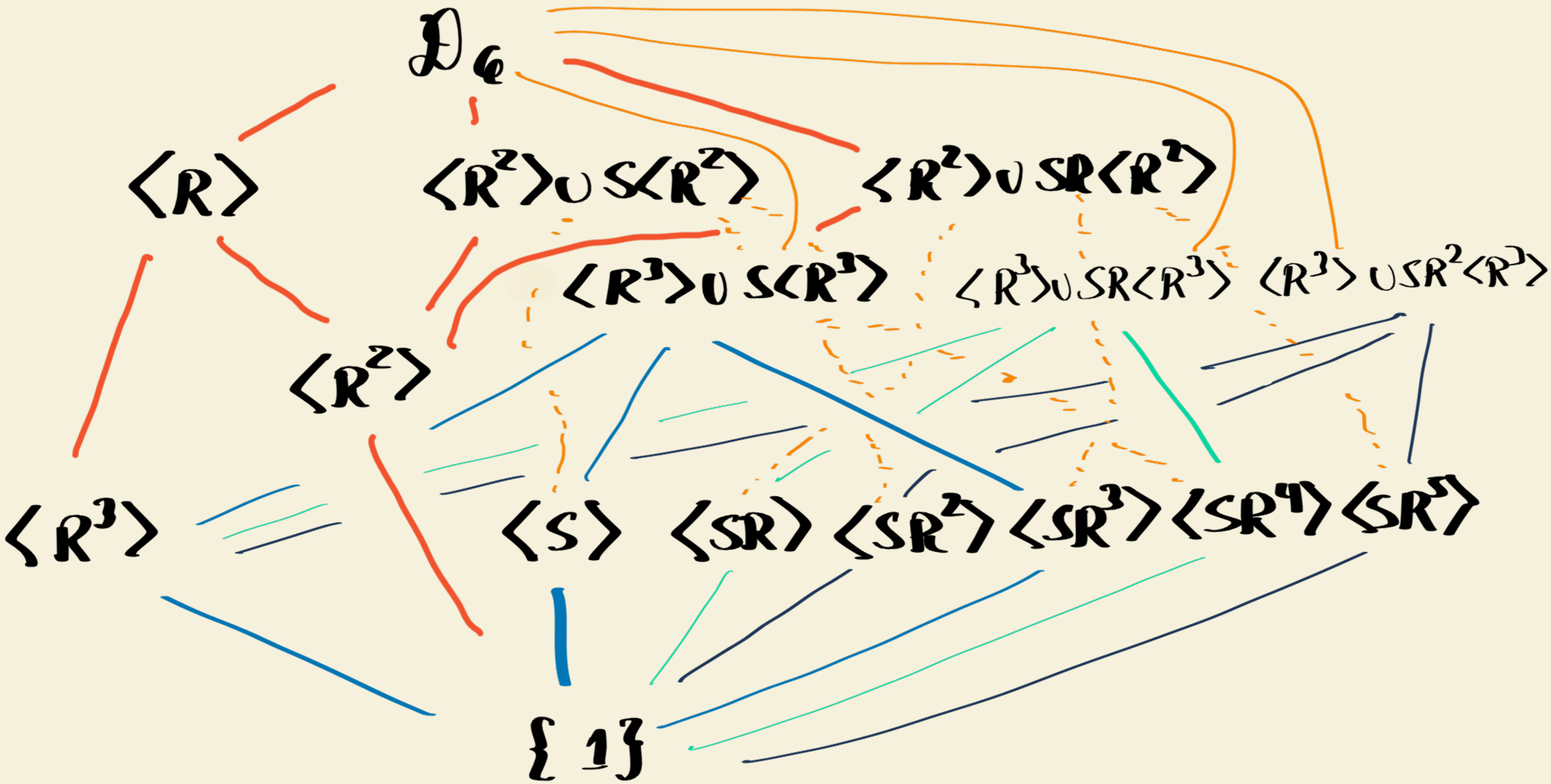
$$\overline{S} \longleftrightarrow \{1, R^3, S, SR^3\}$$

$$\overline{SR} \longleftrightarrow \{1, R^3, SR, SR^4\}$$

$$\overline{SR^2} \longleftrightarrow \{1, R^3, SR^2, SR^5\}$$

Thus we have 3 subgroups of order 4!

The Hasse Diagram looks like:



Slightly convoluted because there are many intersections, but everything is there.

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Our final table of subgroups is:

Order of Subgroup	Status	Total
12	ALL FOUND	1
6	ALL FOUND	3
4	ALL FOUND	3
3	ALL FOUND	1
2	ALL FOUND	7
1	ALL FOUND	1