Using Variational Calculus To Model Paths Around Obstacles

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September 13, 2023

Chapter 1

Problem Statement

There are N number of robots located at position $x_i \in \mathbb{R}^2, i = 1, ..., N$. They need to be transferred to some given target position $y_i \in \mathbb{R}^2, i = 1, ..., N$. There are M circular obstacles in the domain of radius $R > 0$ and centers $c_i, i = 1, ..., M$, which the robots must avoid. Additionally, the robots should not collide with each other. Find optimal paths for the robots, so that their velocities over the trajectory is minimized with these objectives in mind.

Abstract

In order to model this situation as an optimization problem, we develop a vector valued cost functional. The minimization of this cost functional should produce optimal trajectories for the robots according to the conditions of the problem. In order to motivate this cost function, we divide it's derivation into three sections, one for each of the problem's conditions. The final functional is a sum of these three sub-parts. Then, we go on to calculate the first order necessary conditions for optimality, followed by numerically solving the resulting differential equations. Lastly, we consider a few different scenarios to verify that our model provides a reasonable representation of the trajectories.

Notation

- $P_j(t)$:
  - Position of robot j at time t.
  - $P_j(t) = (x_j(t), y_j(t))$ for $j = 1, .., N$.
  - Assume $t \in [0, 1]; t_f = 1$
- P_j^0 :

- Initial position of robot j.
- $P_j^0 = (x_j^0, y_j^0)$ for $j = 1, .., N$.
- Note that: $(x_j^0, y_j^0) = (x_j(0), y_j(0))$.

- P_j^F :
  - Final position of robot j.
  - $P_j^F = (x_j^F, y_j^F)$ for $j = 1, .., N$.
  - Note that $P_j^F = (x_j^F, y_j^F) = (x_j(1), y_j(1))$.

- C_ℓ :
  - Center of Obstacle ℓ where $C_\ell = (a_\ell, b_\ell)$ for $\ell = 1, ..., M$

- $P(t) = (P_1(t), P_2(t), ..., P_N(t))$

- $C = (C_1, C_2, ..., C_M)$

Chapter 2

Theory

2.1 Modelling the Trajectory travelled by each robot

We know that the length of the trajectory travelled by the robot can be represented by the following integral:

$$
\begin{aligned}
L_j &= \int_0^1 |P_j'(t)| \; dt \text{ for } j = 1, ..., N \\
&= \int_0^1 \sqrt{\left(\frac{dx_j}{dt}\right)^2 + \left(\frac{dy_j}{dt}\right)^2}.
\end{aligned}
$$

We can use the equation above to find the total length of the trajectories travelled by N robots. Hence, we have that

$$
\begin{aligned}
L(P_1, .., P_N) &= \sum_{j=1}^{N} L_j = \sum_{j=1}^{N} \left[\int_0^1 |P_j'(t)| \; dt \right] \\
\Rightarrow L(P_1, .., P_N) &= \int_0^1 \left[\sum_{j=1}^{N} |P_j'(t)| \right] \; dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L(P_1, .., P_N) &= \int_0^1 |P'(t)| \; dt. \\
\text{where } |P'(t)| &= \sum_{j=1}^{N} \left| \frac{dP_j}{dt} \right| \\
&= \sum_{j=1}^{N} \sqrt{\left(\frac{dx_j}{dt}\right)^2 + \left(\frac{dy_j}{dt}\right)^2}.
\end{aligned}
$$

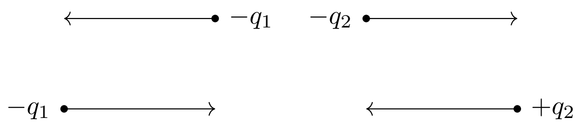
Minimizing this functional will create the shortest trajectory for each robot. Given that the robots must travel the distance within the time period [0,1], it follows that the shortest trajectories will also result in the smallest required velocities. This is because the longer the path, the faster the robot must move to get from start point to end point in the set time.

2.2 Modelling Repulsion Between Robots

In this section, we will use Coulomb's law to model our situation.

2.2.1 Background on Coulomb's Law

- Let q_1 and q_2 be two charges. They experience an electrostatic force of attraction or repulsion.



- The magnitude of the electrostatic force is inversely proportional to the square of the distance between them:

$$|F| = K\frac{|q_1||q_2|}{r^2}$$

  where $|F|$ represents the magnitude of force, K is Coulomb's constant, q_1, q_2 are the particle charges, and r is the distance between q_1 and q_2.

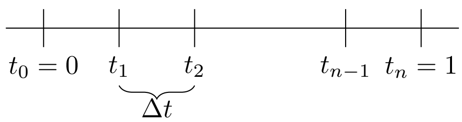
Note that the closer the two particles are, the larger the force. We incorporate this idea into our situation by viewing each robot as a moving particle with charge q_j. We want to minimize the force between them which results in maximizing the distance between them, therefore preventing collision.

Our assumption is that all the robots are identical. Therefore, they have the same charge q. Hence, at a fixed time t, the magnitude of the force F_j between each robot i and j is:

$$|F_{ij}| = K\frac{Q^2}{|P_i(t) - P_j(t)|^2}.$$

But, we need to evaluate the force F_{ij} over continuous time.

We consider the following equal subdivision of the travel time interval $[0, 1]$:



where we have n subintervals $[t_\ell, t_{\ell+1}]$ and $\Delta t = \frac{1-0}{n} = \frac{1}{n}$. We assume that every time step Δt is small enough that the force is constant for it's duration. So, we have,

$$F_{ij}(t) = K\frac{Q^2}{|P_i(t_\ell) - P_j(t_\ell)|^2} \text{ for all } t \in [t_\ell, t_{\ell+1})].$$

Hence, the total force over all the intervals is:

$$\begin{aligned} F_{ij} &= \sum_{\ell=0}^{n} F_{ij}(t_\ell)\Delta t \\ &= \frac{1}{n}\sum_{\ell=0}^{N} F_{ij}(t_\ell). \end{aligned}$$

Observe that as that as $N \to +\infty$, this Riemann sum converges to

$$\frac{1}{n}\sum_{\ell=0}^{N} F_{ij}(t_\ell) \to \int_0^1 F_{ij}(t)\ dt.$$

Hence, the magnitude of the total force between robot i and j during their trip is given by:

$$F_{ij} = KQ^2 \int_0^1 \frac{dt}{|P_i(t) - P_j(t)|^2}.$$

To make calculations much simpler, we set the $KQ^2 = 1$ since they are just constants. Hence, the total force between each robot is given by:

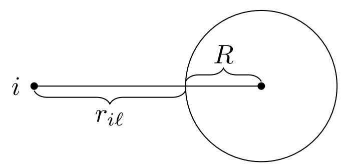
$$F = \frac{1}{2}\left(\sum_{i,j=1}^{N} F_{ij}\right)$$

where $i \neq j$. Hence,

$$F(P) = F(P_1, ..., P_N) = \frac{1}{2}\int_0^1 \sum_{i,j=1}^{N} F_{ij}\ dt.$$

2.3 Modelling Repulsion Between Robots and Obstacles

In this section, we will create a functional that models repulsion of robots from obstacles. Our goal is to minimize the likelihood that any robot will hit obstacles. Following the principles presented in the previous section, we assume that all the obstacles have the same charge



where $r_{i\ell}$ represents the distance between robot and outside of obstacle and R is the radius of obstacle. Hence,

$$|r_{i\ell}(t)| = \sqrt{(x_i(t) - a_\ell)^2 + (y_i(t) - b_\ell)^2} - R$$

and using a similar procedure to the previous section, we arrive at:

$$G = \int_0^1 \left(\sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{1}{|r_{i\ell}(t)|^2} \right) dt.$$

So, in order to avoid hitting obstacles, we want to minimize the functional

$$G(P) = G(P_1, ..., P_N) = \int_0^1 \left(\sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{1}{|r_{i\ell}(t)|^2} \right) dt.$$

Chapter 3

Derivations

Our goal is to minimize the functional $\tilde{L}(P)$ such that:

$$\min_{P \in D} \tilde{L}(P)$$

where:

$$\tilde{L}(P) = L(P) + F(P) + G(P)$$

in which $L(P)$ represents the total length of trajectories, $F(P)$ represents the total force between each robot, and $G(P)$ represents the total force between each robot and each obstacle. To determine the appropriate space D, we evaluate the Gateaux Derivative of $\tilde{L}(P)$.

The Gateaux derivative of $\delta\tilde{L}(P;V)$ is

$$\delta\tilde{L}(P;V) = \delta L(P;V) + \delta F(P;V) + \delta G(P;V)$$

where $V = (v_1, ..., v_n)$ and $v_j = (u_j, w_j)$.

We will compute the Gateaux derivatives $\delta L(P;V), \delta F(P;V), \delta G(P;V)$ separately.

3.0.1 The Gateaux Derivative of $L(P)$

Proposition 1. The Gateaux Derivative of $L(P)$ is

$$\delta L(P;v) = \int_0^1 \tilde{P'} \cdot V' \, dt.$$

where

$$\tilde{P'} = \left(\frac{P_1'}{|P_1'|}, \frac{P_2'}{|P_2'|}, ..., \frac{P_N'}{|P_N'|} \right).$$

Proof. Expand $L(P)$ to get

$$L(P) = \int_0^1 |P'(t)|\ dt = \int_0^1 \sum_{j=1}^{N} |P_j'(t)|\ dt.$$

Define $g(\varepsilon) = L(P + \varepsilon V)$. We can write $g(\varepsilon)$ as

$$\begin{aligned} g(\varepsilon) &= \int_0^1 \sum_{j=1}^{N} |P_j'(t) + \varepsilon V_j'|\ dt \\ &= \int_0^1 \sum_{j=1}^{N} \sqrt{(x_j' + \varepsilon u_j')^2 + (y_2' + \varepsilon w_j')^2}\ dt. \end{aligned}$$

Computing $g'(\varepsilon)$, then gives us

$$\begin{aligned} g'(\varepsilon) &= \frac{d}{d\varepsilon} \int_0^1 \sum_{j=1}^{N} \sqrt{(x_j' + \varepsilon u_j')^2 + (y_j' + \varepsilon w_j')^2}\ dt \\ &= \int_0^1 \sum_{j=1}^{N} \frac{\partial}{\partial \varepsilon} \sqrt{(x_j' + \varepsilon u_j')^2 + (y_j' + \varepsilon w_j')^2}\ dt \\ &= \int_0^1 \sum_{j=1}^{N} \frac{(x_j' + \varepsilon u_j')u_j' + (y_j' + \varepsilon w_j')w_j'}{\sqrt{(x_j' + \varepsilon u_j')^2 + (y_j' + \varepsilon w_j')^2}}\ dt. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$, then gives us

$$\begin{aligned} g'(0) &= \int_0^1 \sum_{j=1}^{N} \frac{x_j' u_j' + y_j' w_j'}{\sqrt{(x_j')^2 + (y_j')^2}}\ dt \\ &= \int_0^1 \sum_{j=1}^{N} \frac{P_j' V_j'}{|P_j'|}\ dt \end{aligned}$$

where $x_j' u_j' + y_j' w_j' - P_j' \cdot V_j'$ and $|P_j'| = \sqrt{(x_j')^2 + (y_j')^2}$. Hence, applying the Gateaux Derivative to the total length of all possible trajectories is

$$\delta L(P; v) = \int_0^1 \tilde{P}' \cdot V'\ dt.$$

where

$$\tilde{P}' = \left(\frac{P_1'}{|P_1'|}, \frac{P_2'}{|P_2'|}, \ldots, \frac{P_N'}{|P_N'|} \right).$$

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3.0.2 The Gateaux Derivative of F(P)

Proposition 2. The Gateaux Derivative of the total force between each robot is

$$\delta F(P;V) = \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{2(P_i - P_j) \cdot (v_i - v_j)}{|P_i - P_j|^4} \, dt.$$

Proof. First we expand $F(p)$ to get

$$\begin{aligned}
F(P) &= \frac{1}{2} \int_0^1 \sum_{i,j=1}^{N} F_{ij} \, dt \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} F_{ij} \, dt \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{dt}{|P_i(t) - P_j(t)|^2}.
\end{aligned}$$

Define $g(\varepsilon) = F(P + \varepsilon V)$, we compute

$$\begin{aligned}
g(\varepsilon) &= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{dt}{|(P_i + \varepsilon v_i) - (P_j + \varepsilon v_j)|^2} \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{dt}{[(x_i + \varepsilon v_i) - (x_j + \varepsilon u_j)]^2 + [(y_i + \varepsilon w_i) - (y_j + \varepsilon w_j)]^2} \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{dt}{[(x_i - x_j) + \varepsilon(u_i - u_j)]^2 + [(y_i - y_j) + \varepsilon(w_i - w_j)]^2}.
\end{aligned}$$

We can take the derivative with respect to ε in the integrand will give us the following expression

$$\begin{aligned}
g'(\varepsilon) &= \frac{d}{d\varepsilon} \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{dt}{[(x_i - x_j) + \varepsilon(u_i - u_j)]^2 + [(y_i - y_j) + \varepsilon(w_i - w_j)]^2} \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{\partial}{\partial \varepsilon} \left[\frac{1}{[(x_i - x_j) + \varepsilon(u_i - u_j)]^2 + [(y_i - y_j) + \varepsilon(w_i - w_j)]^2} \right] dt \\
&= \int_0^1 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{2[(x_i - x_j) + \varepsilon(u_i - u_j)](u_i - u_j) + 2[(y_i - y_j) + \varepsilon(w_i - w_j)](w_i - w_j)}{\Big[[(x_i - x_j) + \varepsilon(u_i - u_j)]^2 + [(y_i - y_j) + \varepsilon(w_i - w_j)]^2\Big]^2} \, dt.
\end{aligned}$$

Letting $\varepsilon \to 0$ and recalling that $v_i = (u_i, w_i)$ and $v_j = (u_j, w_j)$, we have that

$$\begin{aligned}\lim_{\varepsilon\to 0} g'(\varepsilon) &= \int_0^1 \sum_{i=2}^{N}\sum_{j=1}^{i-1} \frac{2(x_i - x_j)(u_i - u_j) + 2(y_i - y_j)(w_i - w_j)}{[(x_i - x_j)^2 + (y_i - y_j)^2]^2}\, dt \\ &= \int_0^1 \sum_{i=2}^{N}\sum_{j=1}^{i-1} \frac{2(P_i - P_j)\cdot(v_i - v_j)}{|P_i - P_j|^4}\, dt.\end{aligned}$$

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Hence, the Gateaux Derivative of the total force $F(p)$ is

$$\delta F(P;V) = \int_0^1 \sum_{i=2}^{N}\sum_{j=1}^{i-1} \frac{2(P_i - P_j)\cdot(v_i - v_j)}{|P_i - P_j|^4}\, dt.$$

3.0.3 Gateaux Derivative of $G(P)$

Proposition 3. Show that the Gateaux Derivative of $G(P)$ is

$$\delta G(P;v) = \int_0^1 \sum_{i=1}^{N}\sum_{\ell=1}^{M} \frac{-2(P_i - C_\ell \cdot V_i)}{|r_{i\ell}|^3 |P_i(t) - C_\ell|}\, dt.$$

Proof. Expanding $G(P)$, we get

$$\begin{aligned}G(P) &= \int_0^1 \sum_{i=1}^{N}\sum_{\ell=1}^{M} \frac{dt}{|r_{i\ell}(t)|^2} \\ &= \int_0^1 \sum_{i=1}^{N}\sum_{\ell=1}^{M} \frac{dt}{(\sqrt{(x_i(t) - a_\ell)^2 + (y_i(t) - b_\ell)^2} - R)^2}.\end{aligned}$$

Define $g(\varepsilon) = G(P + \varepsilon V)$ and write

$$g(\varepsilon) = \int_0^1 \sum_{i=1}^{N}\sum_{\ell=1}^{M} \frac{dt}{(\sqrt{((x_i + \varepsilon u_i) - a_\ell)^2 + ((y_i + \varepsilon w_i) - b_\ell)^2} - R)^2}.$$

with the Gateaux derivative being

$$g'(\varepsilon) = \int_0^1 \sum_{i=1}^{N}\sum_{\ell=1}^{M} \left[-2\frac{2((x_i + \varepsilon u_i) - a_\ell)u_i + 2((y_i + \varepsilon w_i) - b_\ell)w_i}{|r_{i\ell}(t)|^3 2\sqrt{((x_i + \varepsilon u_i) - a_\ell)^2 + ((y_i + \varepsilon w_i) - b_\ell)^2}} \right] dt.$$

Letting $\varepsilon \to 0$, we can write

$$\begin{aligned}\lim_{\varepsilon \to 0} g'(\varepsilon) &= \int_0^1 \sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{-2[(x_i - a_\ell)u_i + (y_i - b_\ell)w_i]}{|r_{i\ell}|^3 |P_i(t) - C_\ell|} \, dt \\ &= \int_0^1 \sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{-2(P_i - C_\ell) \cdot V_i}{|r_{i\ell}|^3 |P_i(t) - C_\ell|} \, dt.\end{aligned}$$

Hence, we have

$$\delta G(P; V) = \int_0^1 \sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{-2(P_i - C_\ell) \cdot V_i}{|r_{i\ell}|^3 |P_i(t) - C_\ell|} \, dt.$$

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3.0.4 Result

Putting everything together, we end up with

$$\begin{aligned}\delta L(\tilde{P}; V) &= \delta L(P; V) + \delta F(P; V) + \delta G(P; V) \\ &= \int_0^1 \left[\sum_{j=1}^{N} \frac{P_j' \cdot V_j'}{|P_j'|} + \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{2(P_i - P_j) \cdot (V_i - V_j)}{|P_i - P_j|^4} - \sum_{i=1}^{N} \sum_{\ell=1}^{M} \frac{2(P_i - C_\ell) \cdot V_i}{\left|r_{i\ell}\right|^3 \left|P_i(t) - C_l\right|} \right] dt.\end{aligned}$$

Chapter 4

First Order Necessary Conditions for Optimality