Math 241 Lecture Notes

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1.1 Plan

1.2 Learning Objectives

- Basics of Metric Spaces
- Discuss topological aspects of metric spaces

1.3 Metric Spaces

On $\mathbb{R}$, we have the usual notion of distance between two real numbers x and y defined by the function $d(x, y) = |x - y|$. We learned that this function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ enjoys certain properties:

(i) $d(x, y) = 0$ if and only if $x = y$; (Nondegeneracy)

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}$; (Symmetricity)

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}$. (Triangle Inequality)

We would like to extend this idea and define a notion of distance in a general situation by using these properties.

Definition (Metric)**.** Let X be a non-empty set. A metric d on X is a function $d : X \times X \to \mathbb{R}$ such that

(i) $d(x, y) = 0$ if and only if x and y are equal;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example. Assume that d is a metric on X. Show that $d(x, y) \geq 0$ for all $x, y \in X$.

Example. Let $X = \{a, b\}$. Is it possible to define a metric on X? **Yes** with the discrete metric.

Definition (Metric Spaces)**.** A **metric space** is a pair (X, d) where X is a non-empty set and d is a metric on X.

Example. (i) Let X be a non-empty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Then d is a metric on X (called the **discrete metric**) and (X, d) is a metric space.

(ii)