Measure Theory Axler Notes

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Contents

1 Section 2A

1.1 Basics/Definitions

Definition (Length of Open Interval; $\ell(I)$)**.** The **length** $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b-a & \text{if } I = (a,b) \text{ for some } a, b \in \mathbb{R} \text{ with } a < b \\ 0 & \text{if } I = \emptyset \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R} \\ \infty & \text{if } I = (-\infty, \infty) \end{cases}$$

Definition (Outer Measure; $|A|$)**.** The **outer measure** $|A|$ of a set $A \subseteq \mathbb{R}$ is defined by

$$|A| = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

1.2 Good Properties of Outer Measure

Proposition (Countable Sets Have Outer Measure 0)**.** Every countable subset of $\mathbb{R}$ has outer measure 0.

Proposition (Outer Measure Preserves Order). Suppose A and B are subsets of $\mathbb{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Proof. ■

Definition (Translation; $t + A$). If $t \in \mathbb{R}$ and $A \subset \mathbb{R}$, then the **translation** $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

Proposition (Outer Measure is Translation Invariant). Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$. Then $|t + A| = |A|$.

Proposition (Countable Subadditivity of Outer Measure). Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\Big|\bigcup_{k=1}^{\infty} A_k\Big| \leq \sum_{k=1}^{\infty} |A_k|.$$

Proof. ■

Definition (Open Cover). Suppose $A \subseteq \mathbb{R}$.

- A collection $\{O_\alpha\}_{\alpha \in \Lambda}$ of open subsets of $\mathbb{R}$ is called an **open cover** of A if A is contained in the union of all the sets in $\{O_\alpha\}_{\alpha \in \Lambda}$.
- An open $\{O_\alpha\}_{\alpha \in \Lambda}$ of A is said to have a **finite subcover** if A is contained in the union of some finite list of sets in $\{O_\alpha\}_{\alpha \in \Lambda}$.

Proposition (Heine-Borel Theorem). Every open cover of a closed bounded subset of $\mathbb{R}$ has a finite subcover.

1.3 Outer Measure of Closed Bounded Interval

Proposition (Outer Measure of a Closed Interval). Suppose $a, b \in \mathbb{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proposition (Nontrivial Intervals are Uncountable). Every interval in $\mathbb{R}$ that contains at least two distinct elements is uncountable.

1.4 Outer Measure is Not Additive

Proposition (Nonadditivity of Outer Measure). There exist disjoint subsets A and B of $\mathbb{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

2 Section 2B

2.1 Nonexistence of Extension of Length to All Subsets of $\mathbb{R}$

Proposition (Nonexistence of Extension of Length to All Subsets of $\mathbb{R}$)**.** There does not exist a function μ with all the following properties:

(a) μ is a function from the set of subsets of $\mathbb{R}$ to $[0, \infty]$,

(b) $\mu(I) = \ell(I)$ for every open interval I of $\mathbb{R}$,

(c) $\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of $\mathbb{R}$,

(d) $\mu(t + A) = \mu(A)$ for every $A \subseteq \mathbb{R}$ and every $t \in \mathbb{R}$.

2.2 σ-Algebra

Definition (σ-Algebra)**.** Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a σ-algebra on X if the following three conditions are satisfied:

- $\emptyset \in \mathcal{S}$;
- if $E \in S$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in S$.

Proposition (σ-algebras are Closed Under Countable Intersection)**.** Suppose $\mathcal{S}$ is a σ-algebra on a set X. Then

(a) $X \in \mathcal{S}$;

(b) if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;

(c) if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

Definition (Measureable Space; Measurable Set)**.**

- A **measurable space** is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of S is called an **$\mathcal{S}$-measurable set**, or just a **measurable set** if $\mathcal{S}$ is clear from the context.

2.3 Borel Subsets of $\mathbb{R}$

Proposition (Smallest σ-algebra containing a collection of subsets)**.** Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algbras on X that contain $\mathcal{A}$ is a σ-algebra on X.

Definition (Borel Set)**.** The smallest σ-algebra on $\mathbb{R}$ containing all open susbets of $\mathbb{R}$ is called the collection of **Borel subsets of** $\mathbb{R}$. An element of this σ-algebra is called a **Borel set**.

Definition (Inverse Image; $f^{-1}(A)$)**.** If $f : X \to Y$ is a function and $A \subset Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Proposition (Algebra of Inverse Images). Suppose $f : X \to Y$ is a function. Then

(a) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for every $A \subset Y$;

(b) $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$ for every $\mathcal{A}$ of subsets of Y;

(c) $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$ for every $\mathcal{A}$ of subsets of Y.

Proposition (Inverse Image of a Composition). Suppose $f : X \to Y$ and $g : Y \to W$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \quad \forall A \subset W.$$

2.4 Measurable Functions

Proposition (Condition for Measurable Function). Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is a function such that

$$f^{-1}((a, \infty)) \in \mathcal{S} \quad \forall a \in \mathbb{R}.$$

Then f is an $\mathcal{S}$-measurable function.

Definition (Borel Measurable Function). Suppose $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called **Borel measurable** if $f^{-1}(B)$ is a Borel set for every $B \subset \mathbb{R}$.

Proposition (Every Continuous Function is Borel Measurable). Every continuous real-valued function defined on a Borel subset of $\mathbb{R}$ is a Borel measurable function.

Definition (Increasing Function). Suppose $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function.

- f is called **increasing** if $f(x) \leq f(y)$ for all $x, y \in X$ with $x < y$.
- f is called **strictly increasing** if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

Proposition (Every Increasing Function is Borel Measurable). Every increasing function defined on a Borel subset of $\mathbb{R}$ is a Borel measurable function.

Proposition (Composition of Measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is an $\mathcal{S}$-measurable function. Suppose g is a real-valued Borel measurable function defined on a subset of $\mathbb{R}$ that includes the range of f. Then $g \circ f : X \to \mathbb{R}$ is an $\mathcal{S}$-measurable function.

Proposition (Algebraic Operations with Measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f, g : X \to \mathbb{R}$ are $\mathcal{S}$-measurable. Then

(a) $f + g, f - g$, and fg are $\mathcal{S}$-measurable functions;

(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

Proposition (Limit of $\mathcal{S}$-measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbb{R}$. Suppose $\lim\limits_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f : X \to \mathbb{R}$ by
$$f(x) = \lim_{k\to\infty} f_k(x).$$
Then f is an $\mathcal{S}$-measurable function.

Definition (Borel Subsets). A subset of $[-\infty, \infty]$ is called a **Borel set** if its intersection with $\mathbb{R}$ is a Borel set.

Definition (Measurable Function). Suppose $(X, \mathcal{S})$ is a measurable space. A function $f : X \to [-\infty, \infty]$ is called $\mathcal{S}$-measurable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \subset [-\infty, \infty]$.

Proposition (Condition for Measurable Function). Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to [-\infty, \infty]$ is a function such that
$$f^{-1}((a, \infty]) \in \mathcal{S} \quad \forall a \in \mathbb{R}.$$
Then f is an $\mathcal{S}$-measurable function.

Proposition (Infimum and Supremum of a Sequence of $\mathcal{S}$-measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $[-\infty, \infty]$. Define $g, h : X \to [-\infty, \infty]$ by
$$g(x) = \inf_{k\in\mathbb{Z}^+} f_k(x) \text{ and } h(x) = \sup_{k\in\mathbb{Z}^+} f_k(x).$$
Then g and h are $\mathcal{S}$-measurable functions.

3 Section 2C

3.1 Definition of Measures

Definition (Measure). Suppose X is a set and $\mathcal{S}$ is a σ-algebra on X. A **measure** on $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and
$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$$
for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$.

Definition (Measure Space). A **Measure Space** is an ordered triple $(X, \mathcal{S}, \mu)$, where X is a set, $\mathcal{S}$ is a $\sigma-$algebra on X, and μ is a measure on $(X, \mathcal{S})$.

3.2 Properties of Measures

Proposition (Measure Preserves Order; Measure of a Set Difference). Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $D, E \in \mathcal{S}$ are such that $D \subset E$. Then

(a) $\mu(D) \leq \mu(E)$;

(b) $\mu(E \setminus D) = \mu(E) - \mu(D)$ provided that $\mu(D) < \infty$.

Proposition (Countable Subadditivity). Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, E_2, \dots \in \mathcal{S}$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Proposition (Measure of an Increasing Union). Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k).$$

Proposition (Measure of a Decreasing Intersection). Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{S}$, with $\mu(E_1) < \infty$. Then

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k).$$

Proposition (Measure of Union). Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $D, E \in \mathcal{S}$ with $\mu(D \cap E) < \infty$. Then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

4 Section 2D

4.1 Additivity of Outer Measure on Borel Sets

Proposition (Additivity of Outer Measure if One of the Sets is Open). Suppose A and G are disjoint subsets of $\mathbb{R}$ and G is open. Then

$$|A \cup G| = |A| + |G|.$$

Proposition (Additivity of Outer Measure if One of the Sets is Closed). Suppose A and F are disjoint subsets of $\mathbb{R}$ and F is closed. Then

$$|A \cup F| = |A| + |F|.$$

Proposition (Approximation of Borel Sets from Below by Closed Sets). Suppose $B \subset \mathbb{R}$ is a Borel set. Then for every $\varepsilon > 0$, there exists a closed set $F \subset B$ such that $|B \setminus F| < \varepsilon$.

Proposition (Additivity of Outer Measure if One of the Sets is a Borel Set). Suppose A and B are disjoint subsets of $\mathbb{R}$ and B is a Borel set. Then

$$|A \cup B| = |A| + |B|.$$

Proposition (Existence of a subset of $\mathbb{R}$ is not a Borel set)**.** There exists a set $B \subset \mathbb{R}$ such that $|B| < \infty$ and B is not a Borel set.

Proposition (Outer Measure is a Measure on Borel Sets)**.** Outer measure is a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra on Borel subsets of $\mathbb{R}$.

Definition (Lebesuge Measure)**.** **Lebesgue Measure** is the measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$, that assigns to each Borel set its outer measure.

4.2 Lebesgue Measurable Sets

Definition (Lebesgue Measurable Set)**.** A set $A \subset \mathbb{R}$ is called **Lebesgue Measurable** if there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

Proposition (Equivalence for being a Lebesgue measurable set)**.** Suppose $A \subset \mathbb{R}$. Then the following are equivalent:

(a) A is Lebesgue measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $F \subset A$ with $|A \setminus F < \varepsilon|$.

(c) There exist closed sets $F_1, F_2, \ldots$ contained in A such that

$$\Big|A \setminus \bigcup_{k=1}^{\infty} F_k\Big| = 0.$$

(d) There exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

(e) For each $\varepsilon > 0$, there exists an open set $G \supset A$ such that $|G \setminus A| < \varepsilon$.

(f) There exist open sets $G_1, G_2, \ldots$ containing A such that

$$\Big|\Big(\bigcup_{k=1}^{\infty} G_k\Big) \setminus A\Big| = 0.$$

(g) There exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$.

Proposition (Outer Measure is a measure on Lebesgue Measurable sets)**.** (a) The set $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra on $\mathbb{R}$.

(b) Outer measure is a measure on $(\mathbb{R}, \mathcal{L})$.

Definition (Lebesgue Measure)**.** **Lebesgue Measure** is the measure on $(\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebsegue measurable subsets of $\mathbb{R}$, that assigns to each Lebesgue measurable set its outer measure.