Stat 215A Lecture Notes

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1 Lecture 1

Definition (Indicator Function)**.** Given a set $A \subseteq \Omega$, we define the indicator function $I_A : \Omega \to \mathbb{R}$ as

$$I_A(W) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{otherwise} \end{cases}$$

Example. For any set $A \subseteq C$, $I_A = 1 - I_{A^c}$.

Proof. Let A be any set. Let $x \in A$. Then $x \notin A^c$ and so $I_A(x) = 1$ and $I_{A^c}(x) = 0$. Thus,

$$I_A = 1 = 1 - 0 = 1 - I_{A^c}(x).$$

Let $y \in A^c$. Then $y \notin A$. Then $I_{A^c} = 1$, and $I_A(y) = 0$. Hence,

$$I_A = 0 = 1 - 1 = 1 - I_{A^c}(y).$$

■

An alternative proof to the above result goes something like:

$$\begin{aligned} (1 - I_{A^c}) &= \begin{cases} 1 - 1 & \text{if } w \in A^c \\ 1 - 0 & \text{if} w \notin A^c \end{cases} \\ &= \begin{cases} 0 & \text{if } w \notin A \\ 1 & \text{if } w \in A \end{cases} \\ &= I_A. \end{aligned}$$

Example. If $A \subseteq B$, then $I_A \subseteq I_B$.

Proof. If $w \in A$, then $w \in B$ and so $I_a(w) = 1$ and $I_B(w) = 1$. Thus, $I_A(w) \leq I_B(w)$. If there exists $w \in B \cap A^c$, then $I_A(w) = 0$ and $I_B(w) = 1$. Thus, we have $I_A(w) \leq I_B(w)$. If there exists $w \in \Omega \cap B^c$, then $I_A(w) = 0$ and $I_B(w) = 0$. Then $I_A(w) \leq I_B(w)$. ■

Definition. For a sequence of sets A_n, $n \in \mathbb{N}$, we define

$$\inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$
$$\sup_{k \geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$
$$\lim_{n\to\infty} \inf A_k = \bigcup_{n\in\mathbb{N}} \inf_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k$$
$$\lim_{n\to\infty} \sup A_k = \bigcap_{n\in\mathbb{N}} \sup_{k\geq n} A_k = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k.$$

For $\lim\inf$, we have a set of nondecreasing sets. On the other hand, for $\lim\sup$, we have a set of nonincreasing sets.

Proposition (De Morgan's Laws for $\lim\sup$ and $\lim\inf$)**.**

$$(\lim_{n\to\infty} \inf A_k)^c = \lim_{n\to\infty} \sup A_n^c$$

Proof. We have

$$\begin{aligned}(\lim_{n\to\infty} \inf_{k\geq n} A_n)^c &= \Big(\bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k\Big)^c \\ &= \bigcap_{n\in\mathbb{N}} \Big(\bigcup_{k=n}^{\infty} A_k\Big)^c \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k^c.\end{aligned}$$

■

Definition. If $\lim\limits_{n\to\infty} \inf A_n = \lim\limits_{n\to\infty} \sup A_n$, then we define the limit of A_n as

$$\lim_{n\to\infty} A_n. \tag{*}$$

Example. Let $A_k = \left[0, \frac{k}{k+1}\right]$ for all $k \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \inf A_k = [0,1)$ and $\lim\limits_{n\to\infty} \sup A_k = [0,1)$, so $A_k \to [0,1)$ as $n \to \infty$.

Proposition. Let A_n for $n \in \mathbb{N}$ be a sequence of subsets in Ω. Then

$$\lim_{n\to\infty} \sup A_n = \{w \in \Omega : \sum_{n\in\mathbb{N}} I_{A_n}(w) = \infty\}$$
$$\lim_{n\to\infty} \inf A_n = \{w \in \Omega : \sum_{n\in\mathbb{N}} I_{A_n}(w) < \infty\}.$$

Roughly speaking, the $\lim\sup$ is the set of $w \in \Omega$ that appear infinitely often, and $\lim\inf$ is the set $w \in \Omega$ that appear except for finitely many times.

Proof. We will prove the first equation. The second uses a similar argument. Let $w \in \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k$. Then

$$w \in \bigcup_{k=n}^{\infty} A_k \quad \forall n \in \mathbb{N}.$$

Indeed, for $w \in \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k = \lim\limits_{n\to\infty} \sup A_n$. Then there exists $k_1 \in \mathbb{N}$ such that $w \in A_{k_1}$. Similarly,

$$w \in \bigcup_{k=k_1+1}^{\infty} A_{k_1} \implies \exists k_1 \in \mathbb{N} \text{ such that } w \in A_{k_1}.$$

Also,

$$w \in \bigcup_{k=k_2+1}^{\infty} A_{k_1} \implies \exists k_3 > k_2 \text{ such that } w \in A_{k_3}.$$

We can continue this process inductively to get

$$I_{A_n} = 1$$

for all $n \in \mathbb{N}$. Hence,

$$\sum_{i=1}^{\infty} I_{A_{k_i}}(w) = \infty.$$

Thus, we see that $w \in \lim_{n\to\infty} \sup A_n$. ■

End of Lecture 1

2 Lecture 2

Proposition. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, then $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$.

Proof. Note that

$$\bigcap_{k\geq 1} A_k = A_1 \cap A_2 \cap \cdots = A_1$$

$$\bigcap_{k\geq 2} A_k = A_2 \cap A_3 \cap \cdots = A_2$$

$$\vdots$$

$$\bigcap_{k\geq n} A_k = A_n \cap A_{n+1} \cap \cdots = A_n$$

Thus, we see that

$$\lim_{n\to\infty} \inf A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

Similarly, we have

$$\bigcup_{k\geq 1} A_k = A_1 \cup A_2 \cup \cdots = \bigcup_{n\in\mathbb{N}} A_n$$

$$\bigcup_{k\geq 2} A_k = A_2 \cup A_3 \cup \cdots = \bigcup_{n\in\mathbb{N}} A_n$$

$$\vdots$$

$$\bigcup_{k\geq n} A_k = A_n \cup A_{n+1} \cup \cdots = \bigcup_{n\in\mathbb{N}} A_n$$

and so, we have

$$\lim_{n\to\infty} \sup A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

This tells us that

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

■

Corollary. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence (of sets). Then $B_n = \bigcup_{k\geq n} A_n$ and $C_n = \bigcap_{k\geq n} A_n$ are monotone, and $\lim_{n\to\infty} \inf A_n = \lim_{n\to\infty} \inf_{k\geq n} A_n$ and $\lim_{n\to\infty} \sup A_n = \lim_{n\to\infty} \sup_{k\geq n} A_n$

Proof. ■

Proposition. Let G_1 and G_2 be open sets. Then $G_1 \cap G_2$ is an open set.

Proof. Let $x \in G_1 \cap G_2$. Our goal is to show that $B_{r_x} \subseteq G_1 \cap G_2$; that is,

$$G_1 \cap G_2 = \bigcup_{x \in G_1 \cap G_2} B_{r_x}(x).$$

Since $x \in G_1$ and $G_1 = \bigcup_{\alpha \in \Lambda} B_{r_\alpha}(x_\alpha)$, we have

$$B_{r_1}(x_1) \subseteq G_1.$$

Here r_1 can be found because G_1 is an open set. Similarly, $x \in G_2$ and $G_2 = \bigcup_{\beta \in \Lambda} B_{r_\beta}(x_\beta)$ implies that $x \in B_{r_2}(x_2) \subseteq G_2$. Like G_1, r_2 can be found because G_2 is an open set. Now, take $r_x = \min\{r_1 - d(x_1, x), r_2 - d(x_2, x)\}$. Let $y \in B_{r_x}(x)$. Then

$$d(x, y) < r_x < r_1 - d(x_1, x) < r_1 \tag{1}$$

and

$$d(x, y) < r_x < r_2 - d(x_2, x) < r_2 \tag{2}$$

Now, (1) and (2) imply that $y \in G_1 \cap G_2$ (because $B_{r_2}(x_2) \subseteq G_2$ and $B_{r_1}(x_1) \subseteq G_1$). Hence, $G_1 \cap G_2$ is an open set. ■

End of Lecture 2

3 Lecture 3

Proposition. Let $\{G_i\}_{1 \le i \le n}$ be a collection of open sets. Then $\bigcap_{i=1}^n G_i$ is an open set.

Proof. Our goal is to show that there exists a $\delta > 0$ such that

$$B_\delta(x) \subseteq \bigcap_{i=1}^n G_i$$

for all $x \in \bigcap_{i=1}^n G_i$. Let $x \in \bigcap_{i=1}^n G_i$. Then for all $1 \le i \le n$, $x \in G_i$. Then for all $11 \le i \le n$, (because G_i is an open set), there exists a $\delta_i > 0$ such that $B_{\delta_i}(x) \subseteq G_i$. Take

$$\hat{\delta} = \frac{1}{2} \min\{\delta_i : 1 \le i \le n\}.$$

It can be seen immediately that this δ is the same δ we were looking for. Indeed, for all $y \in B_\delta(x)$, we have

$$d(x, y) < \delta < \delta_i \quad \forall 1 \le i \le n.$$

Hence, $y \in B_{\delta_i}$ for all $1 \le i \le n$. ■

End of Lecture 3

4 Lecture 4-4-24

Definition (Outer Measure)**.** Let P be a probability measure on an algebra ξ. Define an **outer measure** $P^*(A) = \inf \sum_{n \in \mathbb{N}} P(A_n)$ where $A \subseteq \bigcup A_n$.

Proposition. The outer probability P^* of a probability measure P has the following properties:

(1) $P^*(\emptyset) = 0$

(2) $P^*(A) \ge 0$

(3) $A \subseteq B \Longrightarrow P^*(A) \le P^*(B)$ (monotonicity)

(4) $P^*(\bigcup_{n \in \mathbb{N}} A_n) \le \sum_n P^*(A_n)$ (subcountable additivity)

Proof. (1) Note that $P^*(\emptyset) = \inf \sum_n P(A_n)$ where $\emptyset \subseteq \bigcup_{n \in \mathbb{N}} A_n$. Because $\emptyset \subseteq \emptyset$ implies that $\emptyset$ is a cover of itself, and $P(\emptyset) = 0$, and so $P^*(\emptyset) = 0$.

(2) Since P is a nonnegative function, $P^*(a) = \inf P(A) \geq 0$.

(3) Let $A \subseteq B$. Then any cover of B is also a cover of A, so $P^*(A) \leq P^*(B) = \inf \sum P(A_c)$.

(4) ($\Longleftarrow$) Let A_n where $n \in \mathbb{N}$ is a sequence of sets of Ω. For each A_n, let $\varepsilon > 0$ and for $\frac{\varepsilon}{2^n}$, obtain a cover $\bigcup_{k \in \mathbb{N}} B_k^n$ such that

$$\sum_n P(B_k^n) \leq P^*(A_n) + \frac{\varepsilon}{2^n}.$$

Otherwise, if no such cover exists, then $P^*(A_n) \neq \inf \sum_n P(A_n)$.

($\Longrightarrow$) We have

$$\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} \Big(\bigcup_{k \in \mathbb{N}} B_k^n \Big).$$

From the monotonicity property (3) of P^*, we have

$$P^*\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) \leq P^*\Big(\bigcup_{n \in \mathbb{N}} \Big(\bigcup_{k \in \mathbb{N}} B_k^n \Big) \Big). \tag{*}$$

However,

$$\bigcup_{n \in \mathbb{N}} \Big(\bigcup_{k \in \mathbb{N}} B_k^n \Big)$$

is a particular cover of $\bigcup_{n \in \mathbb{N}} A_n$. Hence,

$$P^*\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) \leq \sum_{n \in \mathbb{N}} P\Big(\bigcup_{k \in \mathbb{N}} B_k^n \Big). \tag{**}$$

Also, since $\bigcup_{k \in \mathbb{N}} B_k^n$ is a particular cover of A_n, so

$$P^*\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) \leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} P(B_k^n).$$

■

End of Lecture 4-4-24