Stat 215A Homework 1

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Proposition (A.1.1)**.** For all sets $A, B, C \subseteq \Omega$.

(1) Union and intersection commutative and distributive:

  (i) $A \cup B = B \cup A$
  (ii) $A \cap B = B \cap A$
  (iii) $(A \cup B) \cup C = A \cup (B \cup C)$
  (iv) $(A \cap B) \cap C = A \cap (A \cap C)$
  (v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  (vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $(A^c)^c = A$, $\emptyset^c = \Omega$, and $\Omega^c = \emptyset$;

(3) $\emptyset \subseteq A$;

(4) $A \subseteq A$;

(5) $A \subseteq B$ and $B \subseteq A$ implies $A \subseteq C$;

(6) $A \subseteq B$ if and only if $B^c \subseteq A^c$;

(7) $A \cup A = A = A \cap A$;

(8) $A \cup \Omega = \Omega$ and $A \cap \Omega = A$;

(9) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Part (1)

(i) Our goal is to show that $A \cup B = B \cup A$. It suffices to show the following two containments:

$$A \cup B \subseteq B \cup A \tag{*}$$

and

$$B \cup A \subseteq A \cup B. \tag{**}$$

We will first show (*). Let $x \in A \cup B$ be arbitrary. Then either $x \in A$, $x \in B$, or in both. If $x \in A$, then $x \in B \cup A$. If $x \in B$, then $x \in B \cup A$. If x is in both A and B, then $x \in B \cup A$. Hence, in all three cases, $x \in B \cup A$ and so $A \cup B \subseteq B \cup A$, satisfying (*). To show (**), let $x \in B \cup A$ be arbitrary. Then either $x \in B$, $x \in A$, or x is in both A and B. If $x \in B$, then $x \in B \cup A$ by definition. If $x \in A$, then $x \in A \cup B$. If x is in both A and B, then $x \in A \cup B$. Thus, in all three cases, $x \in A \cup B$.

(ii) Our goal is to show that $A \cap B = B \cap A$. It suffices to show the following two containments:

$$A \cap B \subseteq B \cap A \tag{*}$$

and

$$B \cap A \subseteq A \cap B. \tag{**}$$

To show (*), let $x \in A \cap B$ be arbitrary. Then this holds if and only if $x \in A$ and $x \in B$. That is, $x \in B$ and $x \in A$. Thus, $x \in B \cap A$. Hence, $A \cap B \subseteq B \cap A$, proving (*). Let $x \in B \cap A$ be arbitrary. Then both $x \in B$ and $x \in A$. Thus, $x \in A$ and $x \in B$. Therefore, $x \in A \cap B$ and so $B \cap A \subseteq A \cap B$, proving (**). From (*) and (**), we get $A \cap B = B \cap A$.

(iii) Our goal is to show that $A \cap (B \cap C) = (A \cap B) \cap C$. We will show the following two containments:

$$A \cup (B \cup C) \subseteq (A \cup B) \cup C \tag{*}$$

and

$$(A \cup B) \cup C \subseteq A \cup (B \cup C). \tag{**}$$

To show (*), let $x \in A \cup (B \cup C)$ be arbitrary. Then either $x \in A$, $x \in B \cup C$, or in both. If $x \in A$, then $x \in A \cup B$. Hence, $x \in (A \cup B) \cup C$, by definition. If $x \in B \cup C$, then either $x \in B$, $x \in C$, or $x \in B \cup C$. If $x \in B$, then $x \in A \cup B$ and thus, $x \in (A \cup B) \cup C$. If $x \in C$, then $x \in (A \cup B) \cup C$ by definition. If x is in both, then immediately $x \in (A \cup B) \cup C$. Now, if $x \in A$ and $x \in B \cup C$, then we also have $x \in (A \cup B) \cup C$. Thus, we have $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

To show (**), let $x \in (A \cup B) \cup C$ be arbitrary. Then either $x \in A \cup B$, $x \in C$ or both. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$. If $x \in B$, then $x \in B \cup C$. So, $x \in A \cup (B \cup C)$. Now, if $x \in C$, then $x \in B \cup C$. By definition, this tells us that $x \in A \cup (B \cup C)$. If x is in both, then immediately we have $x \in A \cup (B \cup C)$ (since it is in all of them and we only require x to be in one of them at least). Thus, we have $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

(iv) Our goal is to show that $(A \cap B) \cap C = A \cap (A \cap C)$. We will show the following two containments:

$$(A \cap B) \cap C \subseteq A \cap (B \cap C) \tag{*}$$

and

$$A \cap (B \cap C) \subseteq (A \cap B) \cap C. \tag{**}$$

To show (*), let $x \in (A \cap B) \cap C$ be arbitrary. Then $x \in A \cap B$ and $x \in C$. Thus, $x \in A$, $x \in B$ and $x \in C$. Thus, $x \in A$ and $x \in B \cap C$. By definition, $x \in A \cap (B \cap C)$. Thus, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$, proving (*).

To show (**), let $x \in A \cap (B \cap C)$ be arbitrary. Then $x \in A$ and $x \in B \cap C$. Thus, $x \in A$, $x \in B$, and $x \in C$. Now, $x \in A \cap B$ and $x \in C$ and so $x \in (A \cap B) \cap C$, by definition. Therefore, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$, proving (**). Thus, (*) and (**) implies that $(A \cap B) \cap C = A \cap (B \cap C)$.

(v) Our goal is to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$. We will show the following two containments:

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \tag{*}$$

and

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C). \tag{**}$$

Starting with (*), let $x \in A \cap (B \cup C)$ be arbitrary. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, then either $x \in B$ or $x \in C$. Now, if $x \in B$, then since $x \in A$ as well, we have $x \in A \cap B$. But now x lies in at least one of the sets in the union $(A \cap B) \cup (A \cap C)$. Hence, $x \in (A \cap B) \cup (A \cap C)$ and so $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Likewise, if $x \in C$, then since $x \in A$ as well, we have $x \in A \cap C$. By definition of union, $x \in (A \cap B) \cup (B \cup C)$. Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, proving (*).

With (**), let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$ or both. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \in B$, it must lie in $B \cup C$ because it is contained in at least one of the sets within that union. Thus, we have $x \in A$ and $x \in B \cup C$ and so $x \in A \cap (B \cup C)$. Therefore, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, it follows that $x \in B \cup C$ by the same reasoning as before. So, $x \in A$ and $x \in B \cup C$. Then $x \in A \cap (B \cup C)$ and so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, proving (**).

From (*) and (**), we have our desired result.

(vi) Our goal is to show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. It suffices to show the following two containments:

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (B \cup C) \tag{*}$$

and

$$(A \cup B) \cap (B \cup C) \subseteq A \cup (B \cap C). \tag{**}$$

Starting with (*), let $x \in A \cup (B \cap C)$ be arbitrary. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ because x is contained in at least one of the sets in $A \cup B$ (of course, it is A). But we also have that $x \in A \cup C$ by the same reasoning. Hence, $x \in A \cup B$ and $x \in A \cup C$. So, $x \in (A \cup B) \cap (A \cup C)$ and so $A \cup (B \cap C) \subseteq (A \cup B) \cap (B \cup C)$, proving (*).

With (**), let $x \in (A \cup B) \cap (B \cup C)$ be arbitrary. Then $x \in A \cup B$ and $x \in A \cup C$. Then $x \in A$ or $x \in B$ or x is in both and $x \in A$ or $x \in C$ or x is in both.

Part (3)

Proof. Our goal is to show that $\emptyset \subseteq A$. Let $x \in \emptyset$ be arbitrary. Since $x \in A$ is a vacuously true statement (by definition of the emptyset), it follows that $\emptyset \subseteq A$. ■

Part (2)

Proof. Our goal is to show that

$$(A^c)^c \subseteq A \tag{1}$$

and

$$(A^c)^c \supseteq A. \tag{2}$$

Let $x \in (A^c)^c$ be arbitrary. Since $A^c = \Omega \setminus A$, we have

$$(A^c)^c = \Omega \setminus A^c.$$

Hence, $x \in \Omega$, but $x \notin A^c$. However, $x \notin A^c$ implies that $x \notin \Omega$ or $x \in A$. Note that the former yields a contradiction because $x \in \Omega$ from an earlier statement. Thus, it must be the case that $x \in A$. Hence, $(A^c)^c \subseteq A$.

For the containment in (2), assume for contradiction that $A \not\subseteq (A^c)^c$. Hence, there exists an $x \in A$ such that $x \notin (A^c)^c$. By definition of complement with respect to Ω, we have $(A^c)^c = \Omega \setminus A^c$. Since $x \notin (A^c)^c$, then either $x \notin \Omega$ or $x \in A^c$. If $x \notin \Omega$, then we have a contradiction because we assumed that $x \in A$ earlier. If $x \in A^c = \Omega \setminus A$, then we are also saying that $x \in \Omega$ but $x \notin A$ which contradicts our earlier assumption that $x \in A$. Hence, we must have that $A \subseteq (A^c)^c$. ■

Proof. Our goal is to show that $\emptyset^c = \Omega$. Note that the complement of $\emptyset$ with respect to Ω is $\emptyset^c = \Omega \setminus \emptyset = \Omega$. Hence, $\emptyset^c = \Omega$. ■

Proof. Note that, from part (3), we have $\emptyset \subseteq \Omega^c$. Let $x \in \Omega^c$. Then $x \in \Omega \setminus \Omega$. Hence, $x \in \Omega$, but $x \notin \Omega$. This tells us that $x \in \emptyset$. Hence, we conclude that $\Omega^c = \emptyset$. ■

Part (4)

Proof. Let $x \in A$ be arbitrary. Since $A \subseteq \Omega$ and $\Omega \neq \emptyset$, we have that $x \in A$. Hence, $A \subseteq A$. ■

Part (5)

Proof. Suppose $A \subseteq B$ and $B \subseteq C$. Our goal is to show that $A \subseteq C$; that is, for all $x \in A$, $x \in C$. To this end, let $x \in A$. Since $A \subseteq B$, we have $x \in B$. Since $B \subseteq C$, we have $x \in C$. Thus, $A \subseteq C$. ■

Part (6)

Proof. ($\Longrightarrow$) Suppose $A \subseteq B$. Our goal is to show that $B^c \subseteq A^c$. Suppose for contradiction that $B^c \not\subseteq A^c$. Then there exists an $x \in B^c$ such that $x \notin A^c$. Since $x \notin A^c$, it follows that $x \in (A^c)^c$. But from part (2), we have $(A^c)^c = A$. Thus, $x \in A$. Since $A \subseteq B$, we have $x \in B$ which is a contradiction.

($\Longleftarrow$) Suppose $B^c \subseteq A^c$. Our goal is to show that $A \subseteq B$. Suppose for contradiction that $A \not\subseteq B$. Then there exists an $x \in A$ such that $x \notin B$. Then $x \in B^c$. But $B^c \subseteq A^c$, and so $x \in A^c$. Thus, $x \notin A$ which is a contradiction. Thus, $A \subseteq B$. ■

Part (7)

Proof. Our goal is to show that $A \cup A = A = A \cap A$. First, we will show that $A \cup A = A$. We will show the following containments; $A \cup A \subseteq A$ and $A \subseteq A \cup A$. Starting with the first containment, let $x \in A \cup A$ be arbitrary. Then $x \in A$ or $x \in A$ or x in both. In either case, $x \in A$ and so $A \cup A \subseteq A$ because $A \subseteq A$ in part (4). If x is in both, then $x \in A$ by using the same fact. Hence, $A \cup A \subseteq A$. For the second containment, let $x \in A$ be arbitrary. Immediately, $x \in A$ or $x \in A$ since $A \subseteq A$ and x lies in all the sets in the union $A \cup A$. Thus, $x \in A \cup A$. Hence, $A \cup A = A$.

Second, we will show that $A \cap A = A$. Let $x \in A \cap A$ be arbitrary. Then $x \in A$ and $x \in A$. Hence, $x \in A$ since $A \subseteq A$ in part (4) and so $A \cap A \subseteq A$. Let $x \in A$ be arbitrary. Then immediately $x \in A$ and $x \in A$ by using part (4) again. Hence, $x \in A \cap A$ and so $A \subseteq A \cap A$. ■

Part (8)

Proof. Our goal is to show that $A \cup \Omega = \Omega$ and $A \cap \Omega = A$. Starting with the first equation, it suffices to show that

$$A \cup \Omega \subseteq \Omega \tag{1}$$

and

$$\Omega \subseteq A \cup \Omega. \tag{2}$$

For (1), let $x \in A \cup \Omega$ be arbitrary. Then either $x \in A$ or $x \in \Omega$ or x is in both. If $x \in A$, we have

$$A \subseteq \Omega \implies x \in \Omega.$$

Clearly, we see that $x \notin \emptyset$ because both A and Ω are non-empty sets. So, $A \cup \Omega \subseteq \Omega$. On the other hand, if $x \in \Omega$, we are done. If x is in both, then we have $x \in \Omega$ and $x \in A$. Since $A \subseteq \Omega$, we have

$$A \cup \Omega \subseteq \Omega \cup \Omega = \Omega$$

by part (4). Thus, $A \cup \Omega \subseteq \Omega$.

For (2), let $x \in \Omega$ be arbitrary. Since $\Omega \subseteq \Omega$, it follows that x is contained in the union $A \cup \Omega$ containing Ω. Hence, $x \in A \cup \Omega$ and so $A \subseteq A \cup \Omega$.

Now, we will show $A \cap \Omega = A$. We will first show $A \cap \Omega \subseteq A$. Let $x \in A \cap \Omega$. Then $x \in A$ and $x \in \Omega$. Since $x \in A$, we have $A \cap \Omega \subseteq A$. Let $x \in A$ be arbitrary. Since $A \subseteq \Omega$, we have $x \in \Omega$. Since $x \in A$ and $x \in \Omega$, we have $x \in A \cap \Omega$. Thus, $A \subseteq A \cap \Omega$. ■

Part (9)

Proof. Our goal is to show the following two equations:

$$A \cup \emptyset = A \tag{1}$$

and

$$A \cap \emptyset = \emptyset \tag{2}$$

First, we show (1). It suffices to show that

$$A \cup \emptyset \subseteq A \tag{*}$$

and

$$A \subseteq A \cup \emptyset. \tag{**}$$

To show the first containment, we use the fact that $A \cup A = A$, $A \subseteq A$ and $\emptyset \subseteq A$ to get

$$A \cup \emptyset \subseteq A \cup A = A.$$

Hence, the first containment is proved.

To show the second containment, suppose for contradiction that $A \not\subseteq A \cup \emptyset$. Then there exists an $x \in A$ such that $x \notin A \cup \emptyset$. Then $x \in (A \cup \emptyset)^c$. That is, $x \in A^c \cap \emptyset^c$. But from part (2), $\emptyset^c = \Omega$. Hence, $x \in A^c = \Omega \setminus A$, but $x \in \Omega$. That is, $x \notin A$, but $x \in \Omega$ which contradicts the assumption that $x \in A$. Therefore, we must have $A \subseteq A \cup \emptyset$.

■