Math 299 Notes

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1 Zorn's Lemma

1.1 Partially Ordered Sets

Definition (Partially Ordered Set, Chain). A **partially ordered set** is a set M on which there is defined a **partial ordering**, that is, a binary relation which is written $\leq$ and satisfies the conditions

(PO1) $a \leq a$ for every $a \in M$ (Reflexivity)

(P02) If $a \leq b$ and $b \leq a$, then $a = b$. (Antisymmetry)

(PO3) If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitivity)

- The term "partially" means that there may exists elements a and b such that neither $a \leq b$ nor $b \leq a$. When this is the case, we call the set M to be **incomparable**.
- On the other hand, we say that a and b are **comparable** if they satisfy $a \leq b$ or $b \leq a$ (or both).

Definition (Totally Ordered Set/Chain). We call a set M to be **totally ordered** or **chain** if it is a partially ordered set such that every two elements of the set are comparable.

Definition (Upper Bound). An **upper bound** of a subset W of a partially ordered set M is an element $u \in M$ such that

$$x \leq u \ \ \forall x \in W.$$

- Another way to think about a chain is that it does not contain any elements that are incomparable.
- Note that the depending on the properties of M and W, the existence of such an element may or may not exist.

Definition (Maximal Element). We call a number $m \in M$ a **maximal element** of M if

$$m \leq x \implies m = x.$$

Similarly, M may or may not have maximal elements and that they need not be an upper bound.

Theorem (Zorn's Lemma). Let $M \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subseteq M$ has an upper bound. Then M has at least one maximal element.

The above is to be taken as an axiom.

1.2 Applications

Theorem (Existence of a Hamel Basis). Every vector space $X \neq \{0\}$ contains a Hamel basis.

Proof. Let M be the set of all linearly independent subsets of X. Since $X \neq \{0\}$, there exists an element $x \neq 0$ and $\{x\} \in M$ such that $M \neq \emptyset$. Set inclusion defines a partial ordering on M. Every chain $C \subseteq M$ has an upper bound, namely, the union of all subsets of X which are elements of C. By Zorn's Lemma, M contains a maximal element B.

Now, we will show that B is a Hamel Basis for X. Let $Y = \text{span}B$. Then Y is a subspace of X, and $Y = X$ since otherwise $B \cup \{z\}$ where $z \in X$ and $z \notin Y$, would be a linearly independent set containing B as a proper subset, contrary to the maximality of B. ■

Before we go over the second example pertaining to Orthonormal sets, we recall some terms used within the proof.

Definition (Total Orthonormal Sets). A **total set** (or **fundamental set**) in a normed space X is a subset $M \subseteq X$ whose span is **dense** in X. Accordingly, an orthonormal set (or sequence or family) in an inner product space X which is total in X is called a **total orthonormal set** (or a sequence or family, respectively) in X.

Definition (Total Orthonormal Set). In every Hilbert space $H \neq \{0\}$, there exists a total orthonormal set.

Theorem (Totality (Theorem 3.6-2)). Let M be a subset of an inner product space X. Then:

(a) If M is total in X, then there does not exist a nonzero $x \in X$ which is orthogonal to every element of M; briefly,
$$x \perp M \implies x = 0.$$

(b) If X is complete, that condition is also sufficient for the totality of M in X.

Proof. Let M be the set of all orthonormal subsets of H. Since $H \neq \{0\}$, it contains an element $x \neq 0$, and an orthonormal subset of H is $\{y\}$, where $y = \|x\|^{-1}x$. Thus, $M \neq \emptyset$ and that set inclusion defines a partial ordering on M. Every chain $C \subseteq M$ has an upper bound, namely, the union of all subsets of X which are elements of C. By Zorn's Lemma, M contains a maximal element F.

We will show that F is total in H. Suppose for sake of contradiction that F is NOT total. Then using Theorem 3.6-2 (see book for details), there exists a nonzero $z \in H$ such that $z \perp F$. Hence, $F_1 = F \cup \{e\}$, where $e = \|z\|^{-1}z$ is orthonormal, and F is a proper subset of F_1. This contradicts the maximality of F. ■

2 Hahn-Banach Theorem

2.1 What is the Hahn-Banach Theorem?

- It is an extension theorem for linear functionals in normed spaces (in a real vector space).
- It guarantees the abundance of bounded linear functionals on a normed space.
- It characterizes the extent to which values of a linear functional can be pre-assigned.

Roughly speaking, when we talk about extending an object, we usually refer to preserving desired properties from one space to another. More specifically, the object of interest in the Hahn-Banach theorem is a linear functional f that is defined on a subspace Z of a vector space X and satisfies a certain boundedness property which will be represented in terms of a **sublinear functional**.

Definition (Sublinear Functional)**.** We say that p defined on a vector space X is a **sublinear functional** if

(1) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$; that is, p is **subadditive**

(2) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ in $\mathbb{R}$ and $x \in X$; that is, p is **positive-homogeneous**.

- We assume that the functional f to be extended is majorized (to be bounded) by a functional p (that is defined on X) that satisfies the above properties.
- We will extend f to a functional $\tilde{f}$ which will retain the boundedness properties that f has on X instead of the subset of X.
- This version of the theorem will assume that X will be a real vector space.

Theorem (Hahn-Banach Theorem (Extension of linear functionals))**.** Let X be real vector space and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \leq p(x) \quad \forall x \in Z.$$

Then f has a linear extension $\tilde{f}$ from Z to X satisfying

$$\tilde{f}(x) \leq p(x) \quad \forall x \in X; \tag{*}$$

that is, $\tilde{f}$ is a linear functional on X, satisfies (*) on X and $\tilde{f}(x) = f(x)$ for every $x \in Z$.

Proof. We will proceed using the following steps below:

(a) The set E of all linear extensions of g of f satisfying $g(x) \leq p(x)$ on their domain $D(g)$ can be partially ordered and Zorn's lemma yields a maximal element of $\tilde{f}$ of E.

(b) $\tilde{f}$ is defined on the entire space X.

(c) An auxiliary relation which was used in (b).

To start, we will prove (a). Let E be the set of all linear extensions g of f for which

$$g(x) \leq p(x) \quad \forall x \in D(g).$$

Note that $E \neq \emptyset$ since $f \in E$ by assumption. On E we can define a partial ordering by $g \leq h$ meaning h is an extension of g, that is, by definition, $D(h) \supseteq D(g)$ and $h(x) = g(x)$ for every $x \in D(g)$. For any chain $C \subseteq E$, we now define $\hat{g}$ by

$$\hat{g}(x) = g(x) \quad \text{if } x \in D(g) \qquad (g \in C)$$

where it can be proven relatively easily that $\hat{g}$ is a linear functional with the domain being

$$D(\hat{g}) = \bigcup_{g \in C} D(g)$$

which is a vector space since C is a chain. The definition of $\hat{g}$ is well-defined. Indeed, for $x \in D(g_1) \cap D(g_2)$ with $g_1, g_2 \in C$, we have $g_1(x) = g_2(x)$ since C is a chain so that $g_1 \leq g_2$ or $g_2 \leq g_1$. Clearly, $g \leq \hat{g}$ for all $g \in C$. Hence, $\hat{g}$ is an upper bound of C. Since $C \subseteq E$ was arbitrary, it follows from Zorn's lemma that E contains a maximal element $\tilde{f}$. By the definition of E, this is a linear extension of f which satisfies

$$\tilde{f}(x) \leq p(x) \quad . \qquad (x \in D(\tilde{f}))$$

Now, we will show that $D(\tilde{f})$ is all of X. Suppose for contradiction that $D(\tilde{f}) \neq X$. Then we can choose a $y_1 \in X \setminus D(\tilde{f})$ and consider the subspace $Y_1 = \text{span}(D(\tilde{f}) \cap \{y_1\})$. Note that $y_1 \neq 0$ since $0 \in D(\tilde{f})$. Any $x \in Y_1$ can be written as

$$x = y + \alpha y_1.. \qquad (y \in D(\tilde{f}))$$

Note that this representation is unique. Indeed, $y + \alpha y_1 = \tilde{y} + \beta y_1$ with $\tilde{y} \in D(\tilde{f})$ implies that

$$y - \tilde{y} = (\beta - \alpha)y_1.$$

Since $y_1 \notin D(\tilde{f})$, the only solution to the equation above is for $y - \tilde{y} = 0$ and $\beta - \alpha = 0$. This tells us now that our representation is unique.

A functional g_1 on Y_1 is defined by

$$g_1(y + \alpha y_1) = \tilde{f}(y) + \alpha c$$

where c is any real constant. It is not difficult to see that g_1 is linear. Furthermore, for $\alpha = 0$, we have $g_1(y) = \tilde{f}(y)$. Hence, g_1 is a proper extension of $\tilde{f}$, that is, an extension such that $D(\tilde{f})$ is a proper subset of $D(g_1)$. Thus, proving that $g_1 \in E$ by showing that $g_1(x) \leq p(x)$ for all $x \in D(g_1)$ will contradict the maximality of $\tilde{f}$ and so the fact that $D(\tilde{f}) = X$ must be true.

Indeed, we must show that this is the case for a suitable c that satisfies the above desired result. We consider any y and z in $D(\tilde{f})$. We have

$$\begin{aligned}
\tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y - z) &\leq p(y - z) \\
&= p(y + y_1 - y_1 - z) \\
&\leq p(y + y_1) + p(-y_1 - z).
\end{aligned}$$

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