Homework 4

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Problem 1. If E is nonempty subset of a metric space (X, d), define the distance from $x \in X$ to E by
$$\text{dist}(x, E) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\text{dist}(x, E) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that if E is compact, then the infimum in the definition above is attained, that is, if $x \in X$ and E is compact, then there exists $a \in E$ such that $\text{dist}(x, E) = d(x, a)$.

(c) Prove that if $x = \mathbb{R}^n$ and if E is closed, then the in the definition above is attained, that is, if $x \in \mathbb{R}^n$ and E is closed, then there exists $a \in E$ such that $\text{dist}(x, E) = d(x, a)$.

(d) Prove that $\text{dist}(x, E) = \text{dist}(x, \overline{E})$.

(e) Prove that $d_E : X \to \mathbb{R}$ defined by $d_E(x) = \text{dist}(x, E)$ is uniformly continuous function on X, by showing that
$$|d_E(x) - d_E(y)| \leq d(x, y) \quad \forall x \in X, y \in X.$$

Proof. (1-a) ($\Longrightarrow$) Suppose $\text{dist}(x, E) = 0$. Our goal is to show that $x \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,
$$N_\varepsilon(x) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$ be given. Since $\text{dist}(x, E) = \inf_{z \in E} d(x, z) = 0$, there exists $z_1 \in E$ such that
$$d(x, z_1) < \text{dist}(x, E) + \varepsilon = 0 + \varepsilon = \varepsilon.$$

Thus, $z_1 \in N_\varepsilon(x)$. Since we also have $z_1 \in E$, it follows that $N_\varepsilon(x) \cap E \neq \emptyset$ as desired.

($\Longleftarrow$) Suppose $x \in \overline{E}$. Our goal is to show that $\text{dist}(x, E) = 0$; that is, we need to show that $\inf_{z \in E} d(x, z) = 0$. To this end, it suffices to prove that
$$\forall z \in E \;\; d(x, z) \geq 0 \tag{i}$$
and
$$\forall \varepsilon > 0 \;\; \exists z \in E \;\; \text{such that} \;\; d(x, z) < 0 + \varepsilon \tag{ii}$$

We see that (i) follows immediately because d defines a metric on X. To show (ii), let $\varepsilon > 0$ be given. Since $x \in \overline{E}$, $N_\varepsilon(x) \cap E \neq \emptyset$. So, there exists z_1 such that $z_1 \in E$ and $z_1 \in N_\varepsilon(x)$. Hence, $z_1 \in E$ such that $d(x, z_1) < \varepsilon$. Note that z_1 is the same z we were looking for. This conclude the proof for the backwards direction.

(1-b) We know that if $A \subseteq \mathbb{R}$ is a nonempty set that is bounded below, then $\inf A \in \overline{A}$ and so there exists a sequence (a_n) in A such that $a_n \to \inf A$. We have $\text{dist}(x, E) = \inf_{z \in E} d(x, z)$. So, there exists a sequence (z_n) in E such that $d(x, z_n) \to \text{dist}(x, E)$. Now, since E is compact, (z_n) contains a subsequence (z_{n_k}) that converges to a point $a \in E$. Thus, we have
$$z_{n_k} \to a \Longrightarrow d(x, z_{n_k}) \to d(x, a)$$
and
$$d(x, z_n) \to \text{dist}(x, E) \Longrightarrow d(x, z_{n_k}) \to \text{dist}(x, E)$$

imply that

$$\operatorname{dist}(x, E) = d(x, a)$$

by the uniqueness of limits.

(1-c) Recall that in $\mathbb{R}^n$ every closed and bounded set is compact. Pick any point $p \in E$. Let $r = d(x, p)$. Let $S = \overline{N_r(x)} \cap E$ (clearly, $p \in S$ and since $S \subseteq \overline{N_r(x)}$ and $\operatorname{dist}(x, S) \le r$).

In what follows, we will show that $\operatorname{dist}(x, S) = \operatorname{dist}(x, E)$.

> **Remark.** Note that since S is the intersection of closed sets, it is closed. Also,
>
> $$S \subseteq \overline{N_r(p)} = \{z \in X : d(x, z) \le r\} \subseteq N_{2r}(p).$$
>
> So, S is bounded. Since S is closed and bounded, it is compact. Thus, by (1-b), there exists $z \in S$ such that $d(x, z) = \operatorname{dist}(x, S)$. Since $\operatorname{dist}(x, S) = \operatorname{dist}(x, E)$, the claim in proved.

First, note that

$$\operatorname{dist}(x, S) = \inf_{z \in S} d(x, z) \underbrace{\ge}_{S \subseteq E} \inf_{z \in E} d(x, z) = \operatorname{dist}(x, E).$$

Hence, $\operatorname{dist}(x, S) \ge \operatorname{dist}(x, E)$. From here, we just need to prove that $\operatorname{dist}(x, E) \ge \operatorname{dist}(x, S)$. Our goal is to show that

$$\forall z \in E \quad d(x, z) \ge \operatorname{dist}(x, S).$$

Let $z \in E$ be given. If $z \in S$, then $d(x, z) \ge \inf_{w \in S} d(x, w) = \operatorname{dist}(x, S)$. If $z \notin S = \overline{N_r(x)} \cap E$, then since $z \in E$, we can conclude that $z \notin \overline{N_r(x)}$ and so $d(x, z) \ge r \ge \operatorname{dist}(x, S)$ as desired.

(1-d) First note that $E \subseteq \overline{E}$ (in genreal, if $A \subseteq B$, then $\inf A \ge \inf B$). So, we have

$$\operatorname{dist}(x, E) = \inf_{z \in E} d(x, z) \ge \inf_{z \in E} d(x, z) = \operatorname{dist}(x, \overline{E}).$$

It suffices to show that $\operatorname{dist}(x, \overline{E}) \ge \operatorname{dist}(x, E)$, that is,

$$\inf_{z \in \overline{E}} d(x, z) \ge \operatorname{dist}(x, E).$$

That is, our goal is to show that

$$\forall z \in \overline{E} \quad d(x, z) \ge \operatorname{dist}(x, E).$$

Let $z \in \overline{E}$ be given. By definition, we have

$$\forall \varepsilon > 0 \quad N_\varepsilon(z) \cap E \ne \emptyset.$$

Hence, there exists $p_\varepsilon \in N_\varepsilon(z) \cap E$ and so

$$\operatorname{dist}(x, E) \le d(x, p_\varepsilon) \le d(x, z) + d(z, p_\varepsilon) < d(x, z) + \varepsilon.$$

That is,

$$\forall \varepsilon > 0 \quad d(x, z) + \varepsilon > \operatorname{dist}(x, E).$$

Thus,

$$d(x, z) \ge \operatorname{dist}(x, E).$$

(1-e) Recall that $d_E : X \to \mathbb{R}$ is uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ if $d(x, y) < \delta$, then

$$|d_E(x) - d_E(y)| < \varepsilon. \tag{*}$$

If we prove that

$$\forall x, y \in X \quad |d_E(x) - d_E| \le d(x, y), \tag{**}$$

then (*) will hold by setting $\delta = \varepsilon$ (or any positive nymber less than ε). So, it suffices to show that (**) holds. Let $x, y \in X$ be given. We have

$$d_E(x) = \inf_{z \in E} d(x, z) \Longrightarrow \forall z \in E \quad d_E(x) \le d(x, z).$$

Then we have

$$\forall z \in E \quad d_E(x) \leq d(x,y) + d(y,z)$$

which can be further rewritten into

$$\forall z \in E \quad d_E(x) - d(x,y) \leq d(y,z).$$

This tells us that $d_E(x) - d(x,y)$ is a lower bound for the set

$$\{d(y,z) : z \in E\}.$$

Hence, we have that

$$d_E(x) - d(x,y) \leq \inf_{z\in E} d(y,z) = d_E(y)$$

and so

$$d_E(x) - d_E(y) \leq d(x,y). \tag{1}$$

Switching the roles of x and y in the argument above, we can derive a similar result; that is,

$$-(d_E(x) - d_E(y)) = d_E(y) - d_E(x) \leq d(y,x) = d(x,y). \tag{2}$$

Thus, (1) and (2) imply that

$$|d_E(x) - d_E(y)| \leq d(x,y)$$

which proves that d_E is a uniformly continuous function on X as desired. ■

Problem 2. Let A and B be nonempty subsets of a metric space (X,d). The distance between A and B is defined as follows:

$$\operatorname{dist}(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

(Note that in case $A = \{x\}$, $\operatorname{dist}(\{x\},B) = \operatorname{dist}(x,B)$ which was introduced in the previous exercise.) Prove that

$$\operatorname{dist}(A,B) = \inf_{x\in A} \operatorname{dist}(x,B) = \inf_{y\in B} \operatorname{dist}(y,A).$$

Proof. Here we will prove a more general claim: Let A and B be any two nonempty sets (not necessarily in a metric space) and let $F : A \times B \to \mathbb{R}$ be a function that is bounded below; that is, the set $\{F(x,y) : (x,y) \in A \times B\}$ is bounded below. Let

$$G : A \to \mathbb{R}, G(x) = \inf_{y\in B} F(x,y)$$
$$H : B \to \mathbb{R}, H(y) = \inf_{x\in A} F(x,y).$$

Then

(1) $\inf_{(x,y)\in A\times B} F(x,y) = \inf_{x\in A} G(x)$;

(2) $\inf_{(x,y)\in A\times B} F(x,y) = \inf_{y\in B} H(y)$.

Here we will prove (1). The proof of (2) is analogous. Let $L = \inf_{(x,y)\in A\times B} F(x,y)$. Our goal is to show that $L = \inf_{x\in A} G(x)$. To this end, it suffices to show that

(i) $L \leq G(x)$ for all $x \in A$

(ii) $\forall \varepsilon > 0$, $\exists x \in A$ such that $G(x) < L + \varepsilon$.

Indeed, let $x \in A$. Then we have

$$\begin{aligned}\forall y \in B \quad (x,y) \in A \times B &\Longrightarrow \forall y \in B \quad L \leq F(x,y)\\ &\Longrightarrow L \text{ is a lower bounded of } \{F(x,y) : y \in B\}\\ &\Longrightarrow L \leq \inf_{y\in B} F(x,y) = G(x).\end{aligned}$$

This proves (i). Now, we will show (ii). Let $\varepsilon > 0$ be given. Then

$$L = \inf_{(x,y)\in A\times B} F(x,y) \Longrightarrow \exists (x_0, y_0) \in A \times B \text{ such that } F(x_0, y_0) < L + \varepsilon.$$

Thus, we have

$$G(x_0) = \inf_{y\in B} F(x_0, y) \leq F(x_0, y_0) < L + \varepsilon.$$

From this, we can see that x_0 is the same x we were looking for. ■

> **Problem 3.** Let (X, d) be a metric space. Prove that if A and B are two nonempty disjoint sets in X such that A is **compact** and B is **closed**, then $\operatorname{dist}(A, B) > 0$.

Proof. Assume for contradiction that $\operatorname{dist}(A, B) = 0$. We have

$$0 = \operatorname{dist}(A, B) = \inf_{x\in A} d_B(x). \qquad \text{(See Exercise 2)}$$

In exercise 1, we proved that $d_B : X \to \mathbb{R}$ is uniformly continuous. As a consequence, $d_B : A \to \mathbb{R}$ is continuous. Since A is compact, it follows from the Extreme Value Theorem that

$$\exists a \in A \text{ such that } \inf_{x\in A} d_B(x) = d_B(a).$$

Since $\inf_{x\in A} d_B(x) = \operatorname{dist}(A, B) = 0$, we can conclude that

$$d_B(a) = 0.$$

It follows from part (a) of exercise 1 that $a \in \overline{B}$. Since B is closed, we have $\overline{B} = B$ and so $a \in B$. Thus, $A \cap B \neq \emptyset$ since $a \in A$ and $a \in B$ which is a contradiction! ■

> **Problem 4.** Let E be a nonempty subset of $\mathbb{R}^n$. Let $t > 0$ be a fixed positive number. Let $A = \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) \geq t\}$. Prove that
>
> $$\circ A = \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) > t\}.$$