Homework 4

p 1

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p 2

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p 3

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p 6

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Problem 1

(i) Let $D \subseteq \mathbb{C}$ $f: D \to \mathbb{C}$ be a function, $a \in \mathbb{C}$ an accumulation point of D, and let $l \in \mathbb{C}$. Prove that the following statements are equivalent.

(a) $\lim\limits_{z \to a} f(z)$ exists and $\lim\limits_{z \to a} f(z) = l$.

(b) Define $\tilde{f}: D \cup \{a\} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ l & \text{if } z = a \end{cases}$$

Then f is continuous at D.

proof

(a) $\Rightarrow$ (b)

Suppose $\lim_{z \to a} f(z)$ exists and $\lim_{z \to a} f(z) = l$.

Our goal is to show that $\tilde{f} : D \cup \{a\} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ l & \text{if } z = a \end{cases}$$

is continuous at D. To this end, let $z \in D \cup \{a\}$ and let $\varepsilon > 0$. Since $z \in D \cup \{a\}$, we have two cases to consider; that is, either $z \in D$ or $z \in \{a\}$ (for which $z = a$). Let $z \in D$. Then, by definition of $\tilde{f}$, we have $\tilde{f}(z) = f(z)$. Our goal is to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - a| < \delta$,

$$|\tilde{f}(z) - \tilde{f}(a)| < \varepsilon .$$

since $\tilde{f}(z) = f(z)$ and $\tilde{f}(a) = l$, we can use the assumption that $\lim_{z \to a} f(z) = l$ to conclude that

$$|\tilde{f}(z) - \tilde{f}(a)| = |f(z) - l| < \varepsilon.$$

Hence, $\tilde{f}$ is continuous for any $z \in D$. Now, if $z = a$, then clearly

$$|\tilde{f}(z) - \tilde{f}(a)| < \varepsilon.$$

Hence, $\tilde{f}$ is continuous at D.

<u>(b) $\Rightarrow$ (a)</u>

Now, suppose $\tilde{f}$ is continuous at D. Our goal is to show that $\lim_{z - a} f(z)$ exists and $\lim_{z \to a} f(z) = l$.

Note that if $z = a$, then the result immediately holds. Suppose $z \neq a$. Let $\varepsilon > 0$. Suppose there exists $\delta > 0$ such that $0 < |z - a| < \delta$.

By definition of $\tilde{f}$ and using the assumption that $\tilde{f}$ is continuous at D, we see that

$$|f(z) - \ell| = |\tilde{f}(z) - \tilde{f}(a)| < \varepsilon .$$

Hence, we conclude that $\lim_{z \to a} f(z)$ exists

and that $\lim_{z \to a} f(z) = \ell$.

$\blacksquare$

(ii) Let $D \subseteq \mathbb{C}$, $f: D \to \mathbb{C}$ be a function, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$, and let $l \in \mathbb{C}$. Use (i) to prove the following statements are equivalent.

(a) f is complex differentiable at a and $f'(a) = l$.

(b) Define $g: D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ l & \text{if } z = a \end{cases}$$

Then g is continuous at a

When f is complex differentiable then $g(a) = f'(a)$. Use this to establish that f is complex differentiable at a if and only if there exists a function $g: D \to \mathbb{C}$ such that g is continuous at a

if and only if there exists a function $g: D \to \mathbb{C}$ such that g is continuous at a and

$f(z) = f(a) + (z-a)g(z)$ for all $z \in D$.

(ii)

(a) $\Rightarrow$ (b)

Suppose f is complex differentiable at a and $f'(a) = l$. Define $g: D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ l & \text{if } z = a \end{cases}$$

Our goal is to show that g is continuous at a; that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - a| < \delta$, we have

$$|g(z) - g(a)| < \varepsilon.$$

Notice that if $z = a$, then the result immediately holds. Thus, assume that $z \neq a$. Since f is complex differentiable and $f'(z) = l$, we know there exists $\delta > 0$ such that whenever $|z - a| < \delta$, we have

$$|g(z)-g(a)| = \left|\frac{f(z)-f(a)}{z-a} - l\right| < \varepsilon.$$

Hence, we see that g is continuous at a.

<u>(b) $\Rightarrow$ (a)</u>

Now, suppose g is continuous at a. Our goal is to show that f is complex differentiable at a and $f'(a) = l$; that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |z-a| < \delta$, we have

$$\left|\frac{f(z)-f(a)}{z-a} - l\right| < \varepsilon.$$

Clearly, if $z = a$, then the result immediately holds. So, assume $z \neq a$. Let $\varepsilon > 0$ and suppose there exists $\delta > 0$ such that $0 < |z-a| < \delta$. Since g is continuous at a, we see that

$$\left| \frac{f(z) - f(a)}{z - a} - l \right| = |g(z) - g(a)| < \varepsilon .$$

Hence, f is complex differentiable at a and $f'(a) = l$.

$\blacksquare$

Now, we see that if f is complex differentiable, then $g(a) = f'(a)$.

- Use this to establish that f is complex differentiable at a if and only if there exists a function $g: D \to \mathbb{C}$ such that g is continuous at a and $f(z) = f(a) + (z-a)g(z)$ for all $z \in D$.

<u>proof</u>

($\Rightarrow$) Suppose f is complex differentiable at a. Define $g: D \to \mathbb{C}$ as follows:

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ l & \text{if } z = a \end{cases}$$

.

Since f is complex differentiable, we know that g is continuous at a by part (b) and that if $z \neq a$, we have

$$g(z) = \frac{f(z) - f(a)}{z - a} \iff f(z) = f(a) + (z-a) g(z) \quad \forall z \in D. \\ z \neq a$$

Clearly, if $z = a$, then g is continuous at a.

($\Leftarrow$) Suppose there exists a function $g: D \to \mathbb{C}$ such that g is continuous at a, and

$$f(z) = f(a) + (z-a) g(a) \qquad \forall z \in D. \qquad (*)$$

Our goal is to show that f is complex differentiable at a; that is, we need to show that

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists}.$$

Notice by $(*)$, we have

$$g(z) = \frac{f(z) - f(a)}{z - a}. \qquad (**)$$

Since g is continuous at a, we have

$$\lim_{z \to a} g(z) \text{ exists.}$$

and furthermore, $\lim_{z \to a} g(z) = g(a)$. By (**),

$$\lim_{z \to a} g(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists.}$$

which tells us that f is complex differentiable at a and thus

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z \to a} g(z) = g(a)$$

$\Rightarrow f'(a) = g(a)$.

$\blacksquare$

Problem-2: Use problem-1 (ii) in the following problems:

Let $D \subseteq \mathbb{C}$ and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$.

(i) Suppose $f, g : D \to \mathbb{C}$ are complex differentiable at a and

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

(ii) Suppose that $f : D \to \mathbb{C}$ is complex differentiable at a and $f(z) \neq 0$ for all $z \in D$. Show that $\frac{1}{f}$ is also complex differentiable at a and

$$\left(\frac{1}{f}\right)'(a) = \frac{-f'(a)}{(f(a))^2}.$$

proof of (i)

Suppose $f, g : D \to \mathbb{C}$ are complex differentiable at a. We will show the following statements:

(1) fg is complex differentiable

(2) $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.

Starting with (1), notice that since f and g are differentiable, we can find continuous functions $\lambda : D \to \mathbb{C}$ and $\varphi : D \to \mathbb{C}$, respectively, such that

$$f(z) = f(a) + \lambda(z)(z-a) \qquad (*)$$

and

$$g(z) = g(a) + \varphi(z)(z-a). \qquad (**)$$

Multiplying $(*)$ and $(**)$, we have

$$f(z)g(z) = [f(a) + \lambda(z)(z-a)][g(a) + \psi(z)(z-a)]$$
$$= f(a)g(a) + \lambda(z)g(a)(z-a) + \psi(z)f(a)(z-a) + \lambda(z)\psi(z)(z-a)^2$$
$$= f(a)g(a) + [\lambda(z)g(a) + \psi(z)f(a) + \lambda(z)\psi(z)](z-a).$$

Set $\phi(z) = \lambda(z)g(a) + \psi(z)f(a) + \lambda(z)\psi(z)$.

Note $\phi : D \to \mathbb{C}$ is continuous since λ and ψ are continuous. Thus, fg is differentiable at a.

Now, our goal is to show that

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

Since fg is differentiable at a, we see that

$$(fg)'(a) = \lim_{z \to a} \frac{(fg)(z) - (fg)(a)}{z-a} = \lim_{z \to a} \frac{f(z)g(z) - f(a)g(a)}{z-a}$$

$$= \lim_{z \to a} \frac{f(z)g(z) - f(z)g(a) + f(z)g(a) - f(a)g(a)}{z-a}$$

$$= \lim_{z \to a} \left[f(z) \cdot \frac{g(z)-g(a)}{z-a} + g(a) \cdot \frac{f(z)-f(a)}{z-a} \right]$$

$$= \lim_{z \to a} f(z) \cdot \lim_{z \to a} \frac{g(z)-g(a)}{z-a} + g(a) \lim_{z \to a} \frac{f(z)-f(a)}{z-a}$$

$$= f(a) \cdot g'(a) + g(a) \cdot f'(a)$$

$$= f'(a)g(a) + g'(a)f(a)$$

∎

proof of (ii)

Suppose that $f: D \to \mathbb{C}$ is complex differentiable at a and $f(z) \neq 0$, for all $z \in D$. Our goal is to show that $f(a) \neq 0$

(1) $1/f$ is differentiable at a

(2) $\left(1/f\right)'(a) = \dfrac{-f'(a)}{(f(a))^2}$.

Starting with (1), we see that, by assumption, there exists a continuous function $\lambda : D \to \mathbb{C}$ such that

$$f(z) = f(a) + \lambda(z)(z-a)$$

since f is complex differentiable at a.

Hence, observe that

$$\frac{1}{f(z)} - \frac{1}{f(a)} = \frac{-[f(z) - f(a)]}{f(z)f(a)}$$

$$= \frac{-\lambda(z)(z-a)}{f(z)f(a)}$$

$$= \frac{-\lambda(z)}{f(z)f(a)}(z-a).$$

$$\Rightarrow \left(\frac{1}{f}\right)(z) = \left(\frac{1}{f}\right)(a) + \left[\frac{-\lambda(z)}{f(z)f(a)}\right](z-a)$$

and set $\varphi(z) = \dfrac{-\lambda(z)}{f(z)f(a)}$ which is

continuous since λ and f are continuous at $a \in D$.

Hence, $\dfrac{f(a)\cdot\lambda}{f}$ is complex differentiable at $a \in D$

Now, with (2), observe that for $f(z) = 0 \ \forall z \in D_r$ we see that

$$(1/f)'(a) = \lim_{z \to a} \frac{(1/f)(z) - (1/f)(a)}{z - a}$$

$$= \lim_{z \to a} \frac{-(f(z) - f(a))/(z-a)}{f(z)f(a)}$$

$$= \frac{-\lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a}}{f(a) \lim\limits_{z \to a} f(z)} = \frac{-f'(a)}{f(a) \cdot f(a)} = \frac{-f'(a)}{(f(a))^2}$$

$$\Rightarrow \quad (1/f)'(a) = \frac{-f'(a)}{(f(a))^2}.$$

∎

Problem-3:

Let $D \subseteq \mathbb{C}$, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$ and $f: D \to \mathbb{C}$ be a function. Let $D' \subseteq \mathbb{C}$ such that $f(D) \subseteq D'$, $f(a)$ is an accumulation point of $D' \setminus \{f(a)\}$.

Let $g: D' \to \mathbb{C}$. Assume that f is complex differentiable at a and g is complex differentiable at a and g is complex differentiable at a and g is complex differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

proof

Our goal is to show the following results:

(1) $g \circ f$ is complex differentiable,

(2) $(g \circ f)'(a) = g'(f(a)) f'(a)$.

Starting with (1), we know that, by assumption, that f and g being differentiable implies that there exists continuous maps $\lambda : D \to \mathbb{C}$ and $\varphi : D' \to \mathbb{C}$ such that

$$f(z) = f(a) + \lambda(z)(z-a) \qquad (*)$$

and

$$g(z) = g(a) + \varphi(z)(z-a) \qquad (**).$$

By $(*)$ and $(**)$, we see that

$$(g \circ f)(z) = g(f(z))$$

$$= g(f(a)) + \varphi(f(a))(f(z) - f(a)) \qquad (*)$$

$$= g(f(a)) + \varphi(f(a))\,\lambda(z)\,(z-a)$$

$$= (g \circ f)(a) + \alpha(z)(z-a)$$

where $\alpha(z) = \varphi(f(a))\,\lambda(z)$ is continuous since $\lambda(z)$ is continuous. Thus, we conclude that $g \circ f$ is indeed complex differentiable at a. Now, we will show (2). By definition of complex differentiability, we see that

$$(g\circ f)'(a) = \lim_{z\to a} \frac{(g\circ f)(z) - (g\circ f)(a)}{z-a}$$

$$= \lim_{z\to a} \frac{g(f(z)) - g(f(a))}{z-a}$$

$$= \lim_{z\to a} \left[\frac{g(f(z)) - g(f(a))}{f(z) - f(a)} \cdot \frac{f(z) - f(a)}{z-a} \right]$$

$$= \underbrace{\lim_{z\to a} \frac{g(f(z)) - g(f(a))}{f(z) - f(a)}}_{g \text{ complex diff at } f(a)} \cdot \underbrace{\lim_{z\to a} \frac{f(z) - f(a)}{z-a}}_{f \text{ complex diff at } a}$$

$$= g'(f(a))f'(a) =$$

$$\Rightarrow (g\circ f)'(a) = g'(f(a))f'(a)$$

$\blacksquare$

Problem -4:

(i) Assume that $D \subseteq \mathbb{C}$, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. Let $f, g : D \to \mathbb{C}$ such that both f and g are complex differentiable at a; $f(a) = 0$, $g(a) = 0$; and $g'(a) \neq 0$.

Show that $\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}$.

(ii) Compute $\lim_{z \to i} \frac{z^2 + 1}{z^4 - 1}$ and

$$\lim_{z \to i} \frac{z^3 + (1-3i) z^2 + (i-3) z + (2+i)}{z - i}.$$

proof of (4)

Let $f, g : D \to \mathbb{C}$ such that both f and g are complex differentiable at a; $f(a) = 0$, $g(a) = 0$; and $g'(4) \neq 0$. By assumption, we see that

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z \to a} \frac{f(z)}{z - a} \qquad (*)$$

and

$$g'(a) = \lim_{z \to a} \frac{g(z) - g(a)}{z - a} = \lim_{z \to a} \frac{g(z)}{z - a} \qquad (**) .$$

Thus, we see that

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{z \to a} \frac{f(z)}{z - a}}{\lim_{z \to a} \frac{g(z)}{z - a}} = \lim_{z \to a} \frac{f(z)}{g(z)}$$

$$\Rightarrow \lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)} .$$

$\blacksquare$

(ii)

Label $f(z) = z^2+1$, $g(z) = z^4-1$

and note that $a = i$. So, we have

$f(i) = i^2+1 = -1+1 = 0$,

$g(i) = i^4-1 = (-1)^2-1 = 1-1 = 0$,

$g'(a) = 4z^3 \Big|_{a=i} = 4(-1) \neq 0$.

Hence, $\lim\limits_{z \to i} \dfrac{z^2+1}{z^4-1} = \dfrac{f'(i)}{g'(i)} = \dfrac{2(i)}{4(i)^3}$

$= \dfrac{1}{2i^2}$

$= -\dfrac{1}{2}$.

(ii) continued

$$\lim_{z \to i} \frac{z^3 + (1-3i)z^2 + (i-3)z + (2+i)}{z - i}$$

Label

$$f(z) = z^3 + (1-3i)z^2 + (i-3)z + (2+i),$$

$$g(z) = z - i,$$

$$g'(a) = 1 \neq 0$$

and notice that

$$f(i) = i^3 + (1-3i)i^2 + (i-3)i + (2+i)$$

$$= \cancel{-i} - (1-3i) - 1 - 3i + 2 + \cancel{i}$$

$$= -1 + \cancel{3i} - 1 - \cancel{3i}$$

$$= -2 \neq 0$$

$g(i) = i - i = 0.$

so, we have

$$\lim_{z \to i} \frac{f(z)}{g(z)} = \frac{f'(i)}{g'(i)}$$

$$= \frac{3(i)^2 + (1-3i)(2i) + (i-3)}{1}$$

$$= -3 + 2i + 6 + i - 3$$

$$= -6 + 2i + 6 + i$$

$$= 3i$$

$$\Rightarrow \lim_{z \to i} \frac{f(z)}{g(z)} = 3i.$$

Problem - 5

(i) Let $P, Q : \mathbb{C} \to \mathbb{C}$ be polynomial functions of degree m and n respectively, where m and n are positive integers. Define $A = \{ z \in \mathbb{C} : Q(z) = 0 \}$. Let $D = \mathbb{C} \setminus A$. Show that $f(z) = \dfrac{P(z)}{Q(z)}$ is complex differentiable on D.

(ii) Determine the largest $D \subseteq \mathbb{C}$ on which the following functions are complex differentiable

(a) $z \mapsto \dfrac{1}{z^3+1}$ (b) $z \mapsto z^2 + \dfrac{1+i}{z}$

(c) $z \mapsto \dfrac{1}{e^z - 1}$

(proof of (i))

Since P and Q are just polynomials over $\mathbb{C}$, and that polynomials are differentiable over $\mathbb{C}$, it must also be the case that P and Q are differentiable over $D = \mathbb{C} \setminus A$. Since $f(z) = \frac{p(z)}{q(z)}$, the quotient $\frac{p}{q}$ must also be differentiable, so f must also be differentiable.

$\blacksquare$

(ii) The following maps are complex differentiable on

(a) $z \to \frac{1}{z^3+1}$; $D = \{ z \in \mathbb{C} : z \neq 1 \}$

(b) $z \to z^2 + \frac{1+i}{z}$; $D = \{ z \in \mathbb{C} : z \neq 0 \}$

(c) $z \to \frac{1}{e^z - 1}$; $D = \{ z \in \mathbb{C} : z \neq \ln|1|, 2\pi i \}$

Problem - 6:

Let $D \subseteq \mathbb{C}$ and $f: D \to \mathbb{C}$ be a function.

Let $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. Assume that f is complex differentiable at a.

Define $D^* = \{ z : \bar{z} \in D \}$ and $g^*: D^* \to \mathbb{C}$ by $g(z) = \overline{f(\bar{z})}$. Show that g is complex differentiable at $\bar{a} \in D^*$ and $g'(\bar{a}) = \overline{f'(a)}$

proof

Our goal is to show that g is complex differentiable at $\bar{a} \in D^*$ and $g'(\bar{a}) = \overline{f'(a)}$.

Since f is complex differentiable at a, we know there must exists $\lambda: D \to \mathbb{C}$ such that

$$f(z) = f(a) + \lambda(z)(z-a) \qquad (*)$$

Thus, observe that setting

$\psi(\bar{z}) = \overline{\lambda(z)}$, we see that

$$g(\bar{z}) = \overline{f(z)} = \overline{f(a) + \lambda(z)(z-a)}$$

$$= \overline{f(a)} + \overline{\lambda(z)}\,\overline{(z-a)}$$

$$= g(\bar{a}) + \psi(\bar{z})(\bar{z} - \bar{a})$$

$$\Rightarrow g(\bar{z}) = g(\bar{a}) + \psi(\bar{z})(\bar{z} - \bar{a})$$

which tells us that g is complex differentiable at $\bar{a} \in D^*$. Now, we will show that $g'(\bar{a}) = \overline{f'(a)}$. Hence, observe that

$$g'(\bar{a}) = \lim_{\bar{z} \to \bar{a}} \frac{g(\bar{z}) - g(\bar{a})}{\bar{z} - \bar{a}}$$

$$= \lim_{z \to a} \frac{\overline{f(z)} - \overline{f(a)}}{\overline{z-a}}$$

$$= \lim_{z \to a} \frac{\overline{f(z) - f(a)}}{\overline{z-a}}$$

$$= \lim_{z \to a} \overline{\left(\frac{f(z)-f(a)}{z-a}\right)}$$

$$= \overline{f'(a)}$$

$$\Rightarrow \quad g'(\bar{a}) = \overline{f'(a)}$$

$\blacksquare$

Problem 7:

(i) Let $\mathbb{C}_- = \mathbb{C}^\bullet - \{z \in \mathbb{C} : z < 0\}$

Show that for any $z \in \mathbb{C}_-$

$$\operatorname{Arg}(z) = \begin{cases} \cos^{-1}\left(\frac{\operatorname{Re}(z)}{|z|}\right) & \text{if } \operatorname{Im}(z) \geq 0 \\ -\cos^{-1}\left(\frac{\operatorname{Re}(z)}{|z|}\right) & \text{if } \operatorname{Im}(z) < 0 \end{cases}$$

(ii) Show that $\operatorname{Arg} : \mathbb{C}_- \to \mathbb{R} \subseteq \mathbb{C}$ is continuous.

(iii) Show that $\operatorname{Log} : \mathbb{C}_- \to \mathbb{C}$ is continuous.

proof of (i) Consider the two sets
$\mathbb{H}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{H}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$
Let $z \in \mathbb{C}_-$ and suppose $\mathrm{Im}(z) > 0$.
Then notice that $(z \in \mathbb{H}^+)$

$$\cos(\mathrm{Arg}(z)) = \frac{\mathrm{Re}(z)}{|z|}.$$

Then $\mathrm{Arg}(z) = \cos^{-1}\left(\frac{\mathrm{Re}(z)}{|z|}\right)$.

Now, suppose $\mathrm{Im}(z) < 0$. $(z \in \mathbb{H}^-)$ Notice that

$$\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z) \Leftrightarrow \mathrm{Arg}(z) = -\mathrm{Arg}(\bar{z}).$$

so, we have

$$\cos(-\mathrm{Arg}(z)) = \cos(\mathrm{Arg}(\bar{z}))$$

$$= \frac{\mathrm{Re}(\bar{z})}{|\bar{z}|} = \frac{\mathrm{Re}(z)}{|z|}$$

$$\Rightarrow -\mathrm{Arg}(z) = \cos^{-1}\left(\frac{\mathrm{Re}(z)}{|z|}\right)$$

$$\Rightarrow \mathrm{Arg}(z) = -\cos^{-1}\left(\frac{\mathrm{Re}(z)}{|z|}\right).$$

$\blacksquare$

proof of (ii)

Observe that $\operatorname{Arg}(z) = \cos^{-1}\left(\frac{\operatorname{Re}(z)}{|z|}\right)$.

Since $\cos^{-1}(z)$ and $\frac{\operatorname{Re}(z)}{|z|}$ are continuous on $\mathbb{H}^+$ functions and that the composition of these functions is continuous on $\mathbb{H}^+$. Therefore, $\operatorname{Arg}(z)$ is continuous on $\mathbb{H}^+$ and similarly on $\mathbb{H}^-$. We will now show that $\operatorname{Arg}(z)$ is continuous on the positive real axis. First, observe that $\operatorname{Arg}(\mathbb{R}_+) = \{0\}$ (where $\mathbb{R}_+$ is the positive real axis); that is, Arg is constant on $\mathbb{R}_+$.

Now, let $x_0 \in \mathbb{R}_+$ and let $(z_n) \subseteq \mathbb{C}$ s.t $(z_n) \to x_0$. Since $\operatorname{Arg}(z)$ is on the positive real axis, we have

$$\operatorname{Arg}(z_n) = \tan^{-1}\left(\frac{\operatorname{Im}(z_n)}{\operatorname{Re}(z_n)}\right) \xrightarrow[\substack{z_n \to x_0 \\ n \to \infty}]{} \tan^{-1}\left(\frac{\operatorname{Im}(x_0)}{x_0}\right) = 0$$

Hence, we see that $\mathrm{Arg}(z)$ is continuous on the positive real axis ($\mathbb{R}_+$).

<u>proof of (iii)</u>

Our goal is to show that $\mathrm{Log}: \mathbb{C}_- \to \mathbb{C}$ is continuous. Note that for any $z \in \mathbb{C}_-$, we see that

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

Our goal is to show that $Re(Log(z))$ and $Im(Log(z))$ is continuous so that $Log(z)$ is continuous. Clearly, $\ln|z|$ is a continuous function on $\mathbb{C}_-$. By part (ii), $Arg(z)$ is continuous on $\mathbb{C}_-$. Thus, $Log(z)$ must be continuous.

$\blacksquare$

Problem 8

$$z' = f(a) \neq 0$$

(i) Let $D, D' \subseteq \mathbb{C}$ be open, $f: D \to \mathbb{C}$ and $g: D' \to \mathbb{C}$ continuous. Moreover, assume $f(D) \subseteq D'$ and $g(f(z)) = z$ for all $z \in D$. Let $a \in D$ and $b = f(a) \in D'$. Show that if g is complex differentiable at b and $g'(b) \neq 0$, then f is complex differentiable at a and $f'(a) = \frac{1}{g'(b)}$.

(ii) Show that $\mathrm{Log}: \mathbb{C}_- \to \mathbb{C}$ is complex differentiable and $(\mathrm{Log}\, z)' = \frac{1}{z}$.

proof of (i)

Assume that g is differentiable at $b = f(a) \in D'$. Set $w = f(z)$. By assumption, there exists a continuous map $\lambda: D \to \mathbb{C}$ at $b = f(a) \in D'$ such that

$$g(w) = g(f(a)) + \lambda(w)(w - f(a))$$

Then we have

$$g(f(z)) = g(f(a)) + \lambda(f(z))(f(z) - f(a))$$

$$= g(f(a)) + \lambda(f(z))f(z) - \lambda(f(z))f(a)$$

$$\Rightarrow g(f(z)) - g(f(a)) = \lambda(f(z))f(z) - \lambda(f(z))f(a) \quad (*)$$

By assumption, we also see that $\lambda(f(z)) \neq 0$.
Furthermore, $g(f(z)) = z$. So,

$$\overset{(*)}{\Rightarrow} (z-a) + \lambda(f(z))f(a) = \lambda(f(z))f(z).$$

$$\Rightarrow f(z) = f(a) + \frac{1}{\lambda(f(z))}(z-a)$$

Since $\lambda : D \to \mathbb{C}$ and $f : D \to \mathbb{C}$ are both continuous at $a \in D$, we have $\lambda \circ f$ is also continuous at $a \in D$. Together with the fact that $(\lambda \circ f)(z) \neq 0$, we can see that $\frac{1}{\lambda \circ f}$ is continuous at $a \in D$. Thus, set $\varphi(z) = \frac{1}{(\lambda \circ f)(z)}$.

So, we conclude that f is indeed differentiable at $a \in D$.

Since f is differentiable at a and g is differentiable at $b = f(a)$, we see that

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z \to a} \frac{f(z) - f(a)}{g(f(z)) - g(\underbrace{f(a)}_{b})}$$

$$= \lim_{f(z) \to b} \frac{1}{\dfrac{g(f(z)) - g(b)}{f(z) - b}}$$

$$= \frac{1}{\displaystyle\lim_{f(z) \to b} \frac{g(f(z)) - g(b)}{f(z) - b}}$$

$$= \frac{1}{g'(b)}$$

$$\Rightarrow f'(a) = \frac{1}{g'(b)}$$

Proof of (ii)

We will show the following results:

(1) $\mathrm{Log}(z)$ is complex differentiable on $\mathbb{C}^{\circ}$

(2) $(\mathrm{Log}\, z)' = \dfrac{1}{z}$.

Consider $\exp : \mathbb{C} \to \mathbb{C}^{\circ}$ and note that $\exp(\mathrm{Log}(z)) = z$. Let $b = \mathrm{Log}(a) \in \mathbb{C}^{\circ}$ be arbitrary ($a \in \mathbb{C}^{\circ}$). Notice that $\exp$ is differentiable and continuous on b. Furthermore, $\mathrm{Log}(\mathbb{C}^{\circ}) \subseteq \mathbb{C}$ and $\mathbb{C}^{\circ} \subseteq \mathbb{C}$. Thus, it follows that Log must be differentiable at $a \in \mathbb{C}^{\circ}$ by part (a) which proves (1). From part (a), we can also see that

$$(\mathrm{Log}\, z)' = \frac{1}{\exp(w)} \qquad (w = \mathrm{Log}\, z)$$

which implies further that

$$(\mathrm{Log}\, z)' = \frac{1}{\exp(\mathrm{Log}(z))} = \frac{1}{z}$$

which proves (2).

$\blacksquare$