Math 299 Exercises

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Contents

1 Week 1

1.1 Zorn's Lemma

Problem 1 (4.1.2)**.** Let X be the set of all real-valued functions on X on the interval $[0, 1]$, and let $x \leqq y$ to mean that $x(t) \leq y(t)$ for all $t \in [0, 1]$. Show that this defines a partial ordering. Is it total ordering? Does X have a maximal elements?

Proof. Define $F = \{\text{all real-valued functions on the interval } [0, 1]\}$. First, we show reflexivity. Let $x \in F$. Then for all $t \in [0, 1]$, $x(t) \leq x(t)$. Hence, $x \leqq x$ and so F contains the reflexive property. For antisymmetry, suppose $x \leqq y$ and $y \leqq x$. Then for all $t \in [0, 1]$, we have $x(t) \leq y(t)$ and $y(t) \leq x(t)$. Then $x(t) = y(t)$ (using the partial ordering of the real numbers) for all $t \in [0, 1]$. Then $x = y$ and so antisymmetry is satisfied. For transitivity, suppose $x \leqq y$ and $y \leqq z$. Then $x(t) \leq y(t)$ and $y(t) \leq z(t)$ for all $t \in [0, 1]$. Then $x(t) \leq z(t)$ for all $t \in [0, 1]$ using the transitivity of the real numbers. Hence, $x \leqq z$ and so transitivity is satisfied.

F is totally ordered, but has no maximal elements. ■

Problem 2 (4.1.5)**.** Prove that a finite partially ordered set A has at least one maximal element.

Proof. ■

1.2 Hahn-Banach Theorem

Problem 3 (Existence of a Sublinear Functional)**.** Show that a sublinear functionnal p satisfies $p(0) = 0$ and $p(-x) \geq -p(x)$.

Proof. Define $P : X \to \mathbb{R}$ by

$$p(x) = \|x\|$$

where X is some vector space. Then for $x = 0$, we have

$$p(0) = \|0\| = 0.$$

Note that for all $x \in X$, we have

$$p(-x) = \| - x\| = | - 1|\|x\| = 1 \cdot \|x\| = \|x\| > 0$$

and $p(-x) = p(x)$. Now, note that for all $x \in X$

$$-\|x\| \leq \|x\| \iff -p(x) \leq p(x) = p(-x).$$

Thus, for all $x \in X$, $p(-x) \geq -p(x)$. ■

Problem 4 (Convex Set). If p is a sublinear functional on a vector space X, show that $M = \{x : p(x) \leq \gamma, \gamma > 0 \text{ fixed}\}$, is a convex set.

Proof. Assume that p is a sublinear functional on a vector space X. Let $W = \{v = \alpha y + (1-\alpha)z : 0 \leq \alpha \leq 1\}$. Our goal is to show that $W \subseteq M$. To this end, let $v \in W$ be arbitrary. Then for some $y, z \in M$, we have $v = \alpha y + (1-\alpha)z$ where $\alpha \in \mathbb{R}$. Since $y, z \in M$, we have $p(y) \leq \gamma$ and $p(z) \leq \gamma$ where $\gamma > 0$ is fixed. Our goal is to show that $p(v) \leq \gamma$ for fixed γ. Using the sublinearity of p, we get that

$$\begin{aligned} p(v) &= p(\alpha y + (1-\alpha)z) \\ &\leq p(\alpha y) + p((1-\alpha)z) \\ &= |\alpha| p(y) + |(1-\alpha)| p(z) \\ &= \alpha p(y) + (1-\alpha) p(z) && (0 \leq \alpha \leq 1) \\ &\leq \alpha\gamma + (1-\alpha)\gamma \\ &= \gamma. \end{aligned}$$

Hence, we conclude that $W \subseteq M$ since v was an arbitrary element and so M is a convex set. ∎

Problem 5. Let p be a sublinear functinoal on a real vector space X. Let f be defined on $Z = \{x \in X : x = \alpha x_0,\ \alpha \in \mathbb{R}\}$ by $f(x) = \alpha p(x_0)$ with fixed $x_0 \in X$. Show that f is a linear functional on Z satisfying $f(x) \leq p(x)$.

Proof. First, we will show that f is a linear functional on Z. Let $u, v \in Z$. Then $u = \alpha_1 x_0$ and $v = \alpha_2 x_0$ for $\alpha_1, \alpha_2 \in \mathbb{R}$. Let $\delta \in \mathbb{R}$. Observe that

$$f(x) = \alpha p(x_0) = p(\alpha x_0).$$

Using this observation, we have

$$\begin{aligned} \delta f(u) + f(v) &= \delta\alpha_1 p(x_0) + \alpha_2 p(x_0) \\ &= (\delta\alpha_1 + \alpha_2) p(x_0) \\ &= p(\delta\alpha_1 x_0 + \alpha_2 x_0) \\ &= p(\delta u + v) \\ &= f(\delta u + v). \end{aligned}$$

Hence, f is a linear functional. Using our observation again, we can also see that for $\alpha > 0$, $f(x) = p(x)$ and so $f(x) \leq p(x)$ for all $x \in Z$. Clearly, the inequality holds if $\alpha = 0$. Now, suppose $\alpha < 0$. Then

$$f(x) = \alpha p(x_0) = |\alpha| p(x_0) = p(|\alpha| x_0) = p(x)$$

and so $f(x) \leq p(x)$ for all $x \in Z$. ∎

Problem 6. If p is a sublinear on a real vector space X, show that there exists a linear functional $\tilde{f}$ on X such that

$$-p(-x) \leq \tilde{f}(x) \leq p(x).$$

Proof. Define Z as in the set in the previous problem and define $f(x) = \alpha p(x_0)$ with fixed $x_0 \in X$. Using the same problem, we proved that f defines linear functional such that $f(x) \leq p(x)$ for all $x \in Z$. Since $Z \subseteq X$, we can find an extension (via the Hahn-Banach Theorem) $\tilde{f}$ that is also a linear functional from Z to X satisfying $\tilde{f}(x) \leq p(x)$ for all $x \in X$. All we need to show now is $-p(-x) \leq \tilde{f}(x)$. Since the bound in the previous statement holds for all $x \in X$, we have

$$-\tilde{f}(x) = \tilde{f}(-x) \leq p(-x).$$

Multiplying through by a negative, we now have

$$\tilde{f}(x) \geq -p(-x)$$

which completes our proof. ∎