Math 230B Homework

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Problem 1. If E is nonempty subset of a metric space (X, d), define the distance from $x \in X$ to E by

$$\text{dist}(x, E) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\text{dist}(x, E) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that if E is compact, then the infimum in the definition above is attained, that is, if $x \in X$ and E is compact, then there exists $a \in E$ such that $\text{dist}(x, E) = d(x, a)$.

(c) Prove that if $x = \mathbb{R}^n$ and if E is closed, then the in the definition above is attained, that is, if $x \in \mathbb{R}^n$ and E is closed, then there exists $a \in E$ such that $\text{dist}(x, E) = d(x, a)$.

(d) Prove that $\text{dist}(x, E) = \text{dist}(x, \overline{E})$.

(e) Prove that $d_E : X \to \mathbb{R}$ defined by $d_E(x) = \text{dist}(x, E)$ is uniformly continuous function on X, by showing that

$$|d_E(x) - d_E(y)| \leq d(x, y) \quad \forall x \in X, y \in X.$$

Proof. (1-a) ($\Longrightarrow$) Suppose $\text{dist}(x, E) = 0$. Our goal is to show that $x \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$ be given. Since $\text{dist}(x, E) = \inf_{z \in E} d(x, z) = 0$, there exists $z_1 \in E$ such that

$$d(x, z_1) < \text{dist}(x, E) + \varepsilon = 0 + \varepsilon = \varepsilon.$$

Thus, $z_1 \in N_\varepsilon(x)$. Since we also have $z_1 \in E$, it follows that $N_\varepsilon(x) \cap E \neq \emptyset$ as desired.

($\Longleftarrow$) Suppose $x \in \overline{E}$. Our goal is to show that $\text{dist}(x, E) = 0$; that is, we need to show that $\inf_{z \in E} d(x, z) = 0$. To this end, it suffices to prove that

$$\forall z \in E \ \ d(x, z) \geq 0 \tag{i}$$

and

$$\forall \varepsilon > 0 \ \ \exists z \in E \ \text{ such that } \ d(x, z) < 0 + \varepsilon \tag{ii}$$

We see that (i) follows immediately because d defines a metric on X. To show (ii), let $\varepsilon > 0$ be given. Since $x \in \overline{E}$, $N_\varepsilon(x) \cap E \neq \emptyset$. So, there exists z_1 such that $z_1 \in E$ and $z_1 \in N_\varepsilon(x)$. Hence, $z_1 \in E$ such that $d(x, z_1) < \varepsilon$. Note that z_1 is the same z we were looking for. This conclude the proof for the backwards direction.

(1-b) We know that if $A \subseteq \mathbb{R}$ is a nonempty set that is bounded below, then $\inf A \in \overline{A}$ and so there exists a sequence (a_n) in A such that $a_n \to \inf A$. We have $\text{dist}(x, E) = \inf_{z \in E} d(x, z)$. So, there exists a sequence (z_n) in E such that $d(x, z_n) \to \text{dist}(x, E)$. Now, since E is compact, (z_n) contains a subsequence (z_{n_k}) that converges to a point $a \in E$. Thus, we have

$$z_{n_k} \to a \Longrightarrow d(x, z_{n_k}) \to d(x, a)$$

and

$$d(x, z_n) \to \text{dist}(x, E) \Longrightarrow d(x, z_{n_k}) \to \text{dist}(x, E)$$

imply that

$$\operatorname{dist}(x, E) = d(x, a)$$

by the uniqueness of limits.

(1-c) Recall that in $\mathbb{R}^n$ every closed and bounded set is compact. Pick any point $p \in E$. Let $r = d(x, p)$. Let $S = \overline{N_r(x)} \cap E$ (clearly, $p \in S$ and since $S \subseteq \overline{N_r(x)}$ and $\operatorname{dist}(x, S) \leq r$).

In what follows, we will show that $\operatorname{dist}(x, S) = \operatorname{dist}(x, E)$.

> **Remark.** Note that since S is the intersection of closed sets, it is closed. Also,
>
> $$S \subseteq \overline{N_r(p)} = \{z \in X : d(x, z) \leq r\} \subseteq N_{2r}(p).$$
>
> So, S is bounded. Since S is closed and bounded, it is compact. Thus, by (1-b), there exists $z \in S$ such that $d(x, z) = \operatorname{dist}(x, S)$. Since $\operatorname{dist}(x, S) = \operatorname{dist}(x, E)$, the claim in proved.

First, note that

$$\operatorname{dist}(x, S) = \inf_{z \in S} d(x, z) \underbrace{\geq}_{S \subseteq E} \inf_{z \in E} d(x, z) = \operatorname{dist}(x, E).$$

Hence, $\operatorname{dist}(x, S) \geq \operatorname{dist}(x, E)$. From here, we just need to prove that $\operatorname{dist}(x, E) \geq \operatorname{dist}(x, S)$. Our goal is to show that

$$\forall z \in E \quad d(x, z) \geq \operatorname{dist}(x, S).$$

Let $z \in E$ be given. If $z \in S$, then $d(x, z) \geq \inf_{w \in S} d(x, w) = \operatorname{dist}(x, S)$. If $z \notin S = \overline{N_r(x)} \cap E$, then since $z \in E$, we can conclude that $z \notin \overline{N_r(x)}$ and so $d(x, z) \geq r \geq \operatorname{dist}(x, S)$ as desired.

(1-d) First note that $E \subseteq \overline{E}$ (in genreal, if $A \subseteq B$, then $\inf A \geq \inf B$). So, we have

$$\operatorname{dist}(x, E) = \inf_{z \in E} d(x, z) \geq \inf_{z \in E} d(x, z) = \operatorname{dist}(x, \overline{E}).$$

It suffices to show that $\operatorname{dist}(x, \overline{E}) \geq \operatorname{dist}(x, E)$, that is,

$$\inf_{z \in \overline{E}} d(x, z) \geq \operatorname{dist}(x, E).$$

That is, our goal is to show that

$$\forall z \in \overline{E} \quad d(x, z) \geq \operatorname{dist}(x, E).$$

Let $z \in \overline{E}$ be given. By definition, we have

$$\forall \varepsilon > 0 \quad N_\varepsilon(z) \cap E \neq \emptyset.$$

Hence, there exists $p_\varepsilon \in N_\varepsilon(z) \cap E$ and so

$$\operatorname{dist}(x, E) \leq d(x, p_\varepsilon) \leq d(x, z) + d(z, p_\varepsilon) < d(x, z) + \varepsilon.$$

That is,

$$\forall \varepsilon > 0 \quad d(x, z) + \varepsilon > \operatorname{dist}(x, E).$$

Thus,

$$d(x, z) \geq \operatorname{dist}(x, E).$$

(1-e) Recall that $d_E : X \to \mathbb{R}$ is uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ if $d(x, y) < \delta$, then

$$|d_E(x) - d_E(y)| < \varepsilon. \tag{*}$$

If we prove that

$$\forall x, y \in X \quad |d_E(x) - d_E| \leq d(x, y), \tag{**}$$

then (*) will hold by setting $\delta = \varepsilon$ (or any positive nymber less than ε). So, it suffices to show that (**) holds. Let $x, y \in X$ be given. We have

$$d_E(x) = \inf_{z \in E} d(x, z) \Longrightarrow \forall z \in E \quad d_E(x) \leq d(x, z).$$

Then we have

$$\forall z \in E \quad d_E(x) \leq d(x,y) + d(y,z)$$

which can be further rewritten into

$$\forall z \in E \quad d_E(x) - d(x,y) \leq d(y,z).$$

This tells us that $d_E(x) - d(x,y)$ is a lower bound for the set

$$\{d(y,z) : z \in E\}.$$

Hence, we have that

$$d_E(x) - d(x,y) \leq \inf_{z \in E} d(y,z) = d_E(y)$$

and so

$$d_E(x) - d_E(y) \leq d(x,y). \tag{1}$$

Switching the roles of x and y in the argument above, we can derive a similar result; that is,

$$-(d_E(x) - d_E(y)) = d_E(y) - d_E(x) \leq d(y,x) = d(x,y). \tag{2}$$

Thus, (1) and (2) imply that

$$|d_E(x) - d_E(y)| \leq d(x,y)$$

which proves that d_E is a uniformly continuous function on X as desired. ■

Problem 2. Let A and B be nonempty subsets of a metric space (X,d). The distance between A and B is defined as follows:

$$\operatorname{dist}(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

(Note that in case $A = \{x\}$, $\operatorname{dist}(\{x\},B) = \operatorname{dist}(x,B)$ which was introduced in the previous exercise.) Prove that

$$\operatorname{dist}(A,B) = \inf_{x \in A} \operatorname{dist}(x,B) = \inf_{y \in B} \operatorname{dist}(y,A).$$

Proof. Here we will prove a more general claim: Let A and B be any two nonempty sets (not necessarily in a metric space) and let $F : A \times B \to \mathbb{R}$ be a function that is bounded below; that is, the set $\{F(x,y) : (x,y) \in A \times B\}$ is bounded below. Let

$$G : A \to \mathbb{R}, G(x) = \inf_{y \in B} F(x,y)$$
$$H : B \to \mathbb{R}, H(y) = \inf_{x \in A} F(x,y).$$

Then

(1) $\inf_{(x,y) \in A \times B} F(x,y) = \inf_{x \in A} G(x)$;

(2) $\inf_{(x,y) \in A \times B} F(x,y) = \inf_{y \in B} H(y)$.

Here we will prove (1). The proof of (2) is analogous. Let $L = \inf_{(x,y) \in A \times B} F(x,y)$. Our goal is to show that $L = \inf_{x \in A} G(x)$. To this end, it suffices to show that

(i) $L \leq G(x)$ for all $x \in A$

(ii) $\forall \varepsilon > 0$, $\exists x \in A$ such that $G(x) < L + \varepsilon$.

Indeed, let $x \in A$. Then we have

$$\begin{aligned}
\forall y \in B \quad (x,y) \in A \times B &\Longrightarrow \forall y \in B \quad L \leq F(x,y) \\
&\Longrightarrow L \text{ is a lower bounded of } \{F(x,y) : y \in B\} \\
&\Longrightarrow L \leq \inf_{y \in B} F(x,y) = G(x).
\end{aligned}$$

This proves (i). Now, we will show (ii). Let $\varepsilon > 0$ be given. Then

$$L = \inf_{(x,y)\in A\times B} F(x,y) \Longrightarrow \exists (x_0, y_0) \in A \times B \text{ such that } F(x_0, y_0) < L + \varepsilon.$$

Thus, we have

$$G(x_0) = \inf_{y\in B} F(x_0, y) \leq F(x_0, y_0) < L + \varepsilon.$$

From this, we can see that x_0 is the same x we were looking for. ■

Problem 3. Let (X, d) be a metric space. Prove that if A and B are two nonempty disjoint sets in X such that A is **compact** and B is **closed**, then $\text{dist}(A, B) > 0$.

Proof. Assume for contradiction that $\text{dist}(A, B) = 0$. We have

$$0 = \text{dist}(A, B) = \inf_{x\in A} d_B(x). \qquad \text{(See Exercise 2)}$$

In exercise 1, we proved that $d_B : X \to \mathbb{R}$ is uniformly continuous. As a consequence, $d_B : A \to \mathbb{R}$ is continuous. Since A is compact, it follows from the Extreme Value Theorem that

$$\exists a \in A \text{ such that } \inf_{x\in A} d_B(x) = d_B(a).$$

Since $\inf_{x\in A} d_B(x) = \text{dist}(A, B) = 0$, we can conclude that

$$d_B(a) = 0.$$

It follows from part (a) of exercise 1 that $a \in \overline{B}$. Since B is closed, we have $\overline{B} = B$ and so $a \in B$. Thus, $A \cap B \neq \emptyset$ since $a \in A$ and $a \in B$ which is a contradiction! ■

Problem 4. Let E be a nonempty subset of $\mathbb{R}^n$. Let $t > 0$ be a fixed positive number. Let $A = \{x \in \mathbb{R}^n : \text{dist}(x, E) \geq t\}$. Prove that

$$\circ A = \{x \in \mathbb{R}^n : \text{dist}(x, E) > t\}.$$

Homework 1

Problem 5. Let $m, n \in \mathbb{N}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^m \sin \frac{1}{x^n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(i) Prove that f is differentiable at all $x \neq 0$.

(ii) Prove that $m > 1$, then f is differentiable at 0.

(iii) Prove that if $m > 1 + n$, then f' is continuous at 0.

(iv) Prove that if $m > 2 + n$, then f' is differentiable on $\mathbb{R}$.

(v) Prove that if $m > 2 + 2n$, then f'' is continuous at 0.

(vi) Prove that if $2 + n < m \leq 2 + 2n$, then f'' is not continuous at 0.

Proof. (i) Suppose $x \neq 0$. Notice that

(1) x^m is a polynomial that is differentiable for any $x \neq 0$.

(2) $\frac{1}{x^n}$ is a rational function which is differentiable for any $x \neq 0$.

(3) $\sin x$ is a trigonometric function which is differentiable for any $x \neq 0$.

By (1), (2), and (3), we conclude via a combination of the Chain Rule and Algebraic Differentiability Theorem that f is differentiable for all $x \neq 0$.

(ii) Suppose $m > 1$. Our goal is to show that f is differentiable at 0; that is, we will show that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^m \sin \frac{1}{x^n}}{x} = \lim_{x\to 0} x^{m-1} \sin \frac{1}{x^n}. \tag{*}$$

Since $|\sin \frac{1}{x^n}| \leq 1$, we can see that

$$0 \leq |x^{m-1} \sin \frac{1}{x^n}| = |x^{m-1}| \left| \sin \frac{1}{x^n} \right| \leq |x^{m-1}|. \tag{**}$$

Since $m-1 > 0$ and x^{m-1} is a polynomial that is continuous everywhere on $\mathbb{R}$, we have $\lim_{x\to 0} x^{m-1} = 0$. As a consequence, we also have $\lim_{x\to 0} |x^{m-1}| = 0$. By applying the Squeeze Theorem for functions to the inequality in (**), we conclude that

$$\lim_{x\to 0} |x^{m-1} \sin \frac{1}{x^n}| = 0 \iff \lim_{x\to 0} x^{m-1} \sin \frac{1}{x^n} = 0.$$

But this implies that the limit in (*) exists and so f is differentiable at 0.

(iii) Computing f', we have

$$f'(x) = \begin{cases} mx^{m-1} \sin \frac{1}{x^n} - x^{m-n-1} \cos \frac{1}{x^n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

We wills show that both

$$mx^{m-1} \sin \frac{1}{x^n} \text{ and } x^{m-n-1} \cos \frac{1}{x^n}$$

approach 0 as $x \to 0$. Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we have

$$0 \leq |mx^{m-1} \sin \frac{1}{x^n}| \leq |mx^{m-1}|. \tag{I}$$

Since $m - 1 > n > 0$ and x^{m-1} is a polynomial which is continuous at 0, the Algebraic Continuity Theorem implies that

$$\lim_{x\to 0} mx^{m-1} = 0 \iff \lim_{x\to 0} |mx^{m-1}| = 0.$$

Using the Squeeze Theorem on (I), we conclude that

$$\lim_{x\to 0} |mx^{m-1} \sin \frac{1}{x^n}| = 0 \iff \lim_{x\to 0} mx^{m-1} \sin \frac{1}{x^n} = 0.$$

Using a similar argument, we can prove that

$$x^{m-n-1} \cos \frac{1}{x^n}$$

is continuous at 0. Indeed, we have

$$0 \leq |x^{m-n-1} \cos \frac{1}{x^n}| \leq |x^{m-n-1}|. \qquad (|\cos x| \leq 1 \; \forall x \in \mathbb{R})$$

Notice that $m - n - 1 > 0$ and that x^{m-n-1} is a polynomial which is continuous everywhere on $\mathbb{R}$, we have $\lim_{x\to 0} x^{m-n-1} = 0$. Hence,

$$\lim_{x\to 0} |x^{m-n-1}| = 0.$$

Applying the Squeeze Theorem, we have

$$\lim_{x\to 0} |x^{m-n-1} \cos \frac{1}{x^n}| = 0 \iff \lim_{x\to 0} x^{m-n-1} \cos \frac{1}{x^n} = 0.$$

Using the Algebraic Limit theorem for functions, we can conclude that as $x \to 0$

$$f'(x) = mx^{m-1} \sin \frac{1}{x^n} - x^{m-n-1} \cos \frac{1}{x^n} \to 0 = f'(0)$$

and so $f'(x)$ is at continuous at 0.

(iv) Our goal is to show that f' is differentiable on $\mathbb{R}$. Let $c \in \mathbb{R}$. Suppose $c = 0$. We will show that the limit

$$\begin{aligned}\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} &= \lim_{x\to 0} \frac{mx^{m-1}\sin\frac{1}{x^n} - x^{m-n-1}\cos\frac{1}{x^n}}{x} \\ &= \lim_{x\to 0}\left[mx^{m-2}\sin\frac{1}{x^n} - x^{m-n-2}\cos\frac{1}{x^n}\right] \\ &= 0.\end{aligned}$$

To prove this, we will use the same approach we used in part (iii); that is, we will show that each corresponding function of the above limit exists and equals 0. For te first function, notice that

$$0 \le |mx^{m-2}\sin\frac{1}{x^n}| \le |mx^{m-2}|. \tag{I}$$

Since x^{m-2} (note that $m - 2 > 0$ by assumption) is a polynomial that continuous everywhere, we have $\lim_{x\to 0} x^{m-2} = 0$. This holds if and only if $\lim_{x\to 0} |x^{m-2}| = 0$. By applying the Squeeze Theorem to the inequality in (I), we have

$$\lim_{x\to 0} |mx^{m-2}\sin\frac{1}{x^n}| = 0 \Longleftrightarrow \lim_{x\to 0} mx^{m-2}\sin\frac{1}{x^n} = 0.$$

Now, we will show that

$$\lim_{x\to 0} x^{m-n-2}\cos\frac{1}{x^n} = 0.$$

Again, with a similar argument used in part (iii), we have

$$0 \le |x^{m-n-2}\cos\frac{1}{x^n}| \le |x^{m-n-2}|. \qquad (|\cos x| \le 1\ \forall x \in \mathbb{R})$$

Since x^{m-n-2} is a polynomial which is continuous everywhere on $\mathbb{R}$, we have $\lim_{x\to 0} x^{m-n-2} = 0$. This holds if and only if

$$\lim_{x\to 0} |x^{m-n-2}| = 0.$$

Using the Squeeze Theorem on the inequality above, we have

$$\lim_{x\to 0} |x^{m-n-2}\cos\frac{1}{x^n}| = 0 \Longleftrightarrow \lim_{x\to 0} x^{m-n-2}\cos\frac{1}{x^n} = 0.$$

Using the Algebraic limit theorem for functions, we can conclude that f' is differentiable for $c = 0$. Suppose $c \neq 0$. From part (iii), we have

$$f'(x) = mx^{m-1}\sin\frac{1}{x^n} - x^{m-n-1}\cos\frac{1}{x^n}.$$

Note the following:

(1) x^{m-1} and x^{m-n-1} are polynomials which are differentiable everywhere on $\mathbb{R}$.

(2) $\sin x$ and $\cos x$ are trigonometric functions which are differentiable everywhere on $\mathbb{R}$.

(3) $\frac{1}{x^n}$ is a rational function that is differentiable for every $x \neq 0$ in $\mathbb{R}$.

(4) $\sin(\frac{1}{x^n})$ is differentiable for every $x \neq 0$ in $\mathbb{R}$ by the chain rule.

From (1), (2), (3), and (4) we can use the Algebraic differentiability theorem to conclude that $f'(x)$ is indeed differentiable.

(v) Using a combination of the product rule and chain rule, we have

$$f''(x) = \begin{cases} [m(m-1)x^{m-2} - nx^{m-2n-2}]\sin\frac{1}{x^n} & \\ -[mnx^{m-n-2} + n(m-n-1)x^{m-n-2}\cos\frac{1}{x^n} & \text{if } x \neq 0] \\ 0 & \text{if } x = 0 \end{cases}$$

By assumption, we have that

$$m > 2 + 2n \implies m - 2n - 2 > 0$$

and similarly

$$m - n - 2 > n > 0 \text{ and } m - 2 > 0.$$

Hence, we can see that x^{m-2} and x^{m-2n-2} are polynomials which are continuous everywhere on $\mathbb{R}$. In particular, these two polynomials are continuous at 0. Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we have

$$0 \leq \left| [m(m-1)x^{m-2} - nx^{m-2n-2}] \sin \frac{1}{x^n} \right| \leq |m(m-1)x^{m-2} - nx^{m-2n-2}|. \tag{*}$$

Since x^{m-2} and x^{m-2n-2} are both continuous at 0, the Algebraic Continuity Theorem implies that

$$\lim_{x \to 0} m(m-1)x^{m-2} - nx^{m-2n-2} = 0$$

which holds if and only if

$$\lim_{x \to 0} |m(m-1)x^{m-2} - nx^{m-2n-2}| = 0.$$

Then using the Squeeze Theorem on (*), we can conclude that

$$\lim_{x \to 0} \left| [m(m-1)x^{m-2} - nx^{m-2n-2}] \sin \frac{1}{x^n} \right| = 0$$

which holds if and only if

$$\lim_{x \to 0} [m(m-1)x^{m-2} - nx^{m-2n-2}] \sin \frac{1}{x^n} = 0. \tag{1}$$

Analogously, we can prove that

$$\lim_{x \to 0} [mnx^{m-n-2} + n(m-n-1)x^{m-n-2}] \cos \frac{1}{x^n} = 0. \tag{2}$$

Using the Algebraic Limit Theorem for functions, we can conclude that

$$\lim_{x \to 0} f''(x) = f''(0).$$

(vi) Computing f'', we have

$$f''(x) = \begin{cases} m(m-1)x^{m-2} \sin\left(\frac{1}{x^n}\right) + n^2 x^{m-2n-2} \sin\left(\frac{1}{x^n}\right) & \\ -n(2m-n-2)x^{m-n-2} \cos\left(\frac{1}{x^n}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Define $a_k = \frac{1}{\sqrt[n]{2\pi k + \frac{\pi}{2}}}$. We can see immediately that $a_k \to 0$ and that

$$\sin \frac{1}{(a_k)^n} = \sin\left(2\pi k + \frac{\pi}{2}\right) = 1,$$
$$\cos \frac{1}{(a_k)^n} = \cos\left(2\pi k + \frac{\pi}{2}\right) = 0$$

Now, for all $k \geq 1$, we can see that

$$\lim_{k \to \infty} f''(a_k) = \lim_{k \to \infty} \frac{1}{(a_k)^{(2+2n)-m}}.$$

Note that if $m = 2 + 2n$, we just have $\lim_{k \to +\infty} f''(a_k) = -n^2$. Otherwise, $\lim_{k \to +\infty} \frac{1}{(a_k)^{(2+2n)-m}} = \infty$. That is,

$$\lim_{k \to \infty} f''(a_k) \neq 0 = f''(0).$$

So, we conclude that f'' is not continuous at 0.

■

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t < 0 \\ 0 & \text{if } t \leq 0 \end{cases}.$$

Prove that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Proof. Our goal is to show that f is infinitely differentiable at 0. We will consider two cases for this. Note that for $t \leq 0$, one can immediately see that via induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Thus, it remains to be shown that if $t > 0$, then $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. First, we will show that for all $n \in \mathbb{N}$

$$f^{(n)}(t) = e^{-1/t}\frac{P_n(t)}{t^{2n}}.$$

Using a combination of the chain rule and product rule, we have that

$$f'(t) = \frac{e^{-1/t}}{t^2}.$$

Suppose that the result in (*) holds for $n = k$. We will show that it also holds for $n = k+1$ case. Indeed, we see that

$$\begin{aligned} f^{(k+1)}(t) &= \frac{d}{dt}[f^{(k)}(t)] = \frac{d}{dt}\left[P_k(t) \cdot \frac{e^{-1/t}}{t^{2k+2}}\right] \\ \implies f^{(k+1)}(t) &= \frac{d}{dt}[P_k(t)]\frac{e^{-1/t}}{t^{2k}} + t^2 P_k(t)\frac{e^{-1/t}}{t^{2k}} - 2kP_k(t)\frac{e^{-1/t}}{t^{2k+1}} \\ &= \underbrace{\left[\frac{1}{t}\frac{d}{dt}[P_k(t)] + tP_k(t) - 2kP_k(t)\right]}_{\text{polynomial of degree at most} n+1}\frac{e^{-1/t}}{t^{2k+1}} \\ &= P_{k+1}(t)\frac{e^{-1/t}}{t^{2k+1}}. \end{aligned}$$

This implies that

$$f^{(k+1)}(t) = P_{k+1}(t)\frac{e^{-1/t}}{t^{2k+1}}.$$

Our goal is to show that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. We proceed via induction on $n \in \mathbb{N}$. We will start with proving that f is differentiable once. Indeed,

$$\begin{aligned} \lim_{t \to 0} \frac{f(t) - f(0)}{t} &= \lim_{t \to 0} \frac{e^{\frac{-1}{t}} - 0}{t} \\ &= \lim_{t \to 0} \frac{1}{t} e^{-\frac{1}{t}} \\ &= \lim_{t \to 0} \frac{1}{te^{1/t}} \\ &= 0. \end{aligned}$$

Hence, $f'(0)$ exists and so $f'(0) = 0$ by the above. Suppose for our induction hypothesis that f is differentiable at 0 $k = n$ times. Our goal is to show that f is differentiable $n = k + 1$ times. Note that where $P_n(t)$ is a polynomial of at most degree n. Since $f^{(k+1)}(t)$ exists, we have

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t - 0} \\ &= \lim_{t \to 0} \frac{1}{t} \cdot \frac{e^{-1/t}P_k(t)}{t^{2k}} \\ &= \lim_{t \to 0} \left[P_k(t) \cdot \frac{e^{-1/t}}{t^{2k+1}}\right] \\ &= \lim_{t \to 0} P_k(t) \cdot \lim_{t \to 0} \frac{e^{-1/t}}{t^{2k+1}} \end{aligned}$$

Note that $\lim_{t\to 0} \frac{e^{-1/t}}{t^{2k+1}} = 0$ by L'Hopital's rule. Indeed, let's induct on $n \in \mathbb{N}$ to show that this is true. Let our base case be $n = 1$.

$$\lim_{t\to 0^+} \frac{t^{-1}}{e^{1/t}} \underbrace{=}_{\frac{\infty}{\infty}} \lim_{t\to 0^+} \frac{-t^{-2}}{-t^{-2}e^{1/t}} = \lim_{t\to 0^+} \frac{1}{e^{1/t}} = 0.$$

Suppose the result holds for $n = k$; that is,

$$\lim_{t\to 0^+} \frac{t^{-k}}{e^{1/t}} = 0.$$

Our goal is to show that the result holds $n = k + 1$ (where $k \geq 1$). Indeed, using L'Hopital's Rule, we have

$$\lim_{t\to 0^+} \frac{t^{-(k+1)}}{e^{1/t}} \underbrace{=}_{\frac{\infty}{\infty}} -(k+1) \lim_{t\to 0^+} \frac{t^{-k}}{-e^{1/t}} = 0.$$

Hence, we conclude that

$$f^{(k+1)}(0) = 0.$$

■

Problem 7. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Let $c \in I$. Recall that in class we proved that f is differentiable at c if and only if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. Use this result to prove that f is differentiable at c if and only if

$$\exists L \in \mathbb{R} \text{ such that } \lim_{h\to 0} \frac{f(c+h)-f(c)-Lh}{h} = 0.$$

Proof. Suppose that f is differentiable at c. Then

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists}$$

and so

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = L$$

for some $L \in \mathbb{R}$. Now, the right-hand side can be written in the following way:

$$L = \lim_{h\to 0} L = \lim_{h\to 0} \frac{Lh}{h}.$$

Note that the quantity $\frac{h}{h}$ holds because of the $\varepsilon - \delta$ definition of the derivative. Now, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = L &\implies \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{Lh}{h} \\
&\implies \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \lim_{h\to 0} \frac{Lh}{h} = 0.
\end{aligned}$$

By the Algebraic Limit Theorem for functions, we conclude that

$$\lim_{h\to 0} \frac{f(c+h)-f(c)-Lh}{h} = 0.$$

($\Longrightarrow$) Suppose that there exists an $L \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(c+h)-f(c)-Lh}{h} = 0.$$

Our goal is to show that f is differentiable at c; that is,

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists.}$$

Then we have

$$\begin{aligned}
\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - L + L\right] \\
&= \lim_{h\to 0}\left[\frac{f(c+h)-f(c)-Lh}{h} + L\right] \\
&= \lim_{h\to 0}\frac{f(c+h)-f(c)-Lh}{h} + \lim_{h\to 0} L \qquad \text{(ALT for Functions)} \\
&= 0 + L \\
&= L.
\end{aligned}$$

Hence, we can see that

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \text{ exists}$$

and so we conclude that f is differentiable at c. ■

Problem 8. Let $g : A \to \mathbb{R}$ where A is a nonempty subset of $\mathbb{R}$. Suppose 0 is an interior point of A. Use the $\varepsilon - \delta$ definition of limit to prove that $\lim_{h\to 0} g(h) = L$, then $\lim_{h\to 0} g(-h) = L$.

Proof. Our goal is to show that $\lim_{h\to 0} g(-h) = L$; that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |h| < \delta$, we have

$$|g(-h) - L| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim_{h\to 0} g(h) = L$, we can find a $\hat{\delta} > 0$ such that whenever $0 < |y| < \hat{\delta}$,

$$|g(y) - L| < \varepsilon.$$

Since 0 is an interior point of A, there exists an $\tilde{\delta} > 0$ such that $N_{\tilde{\delta}}(0) \subseteq A$; that is, we have

$$(-\tilde{\delta}, \tilde{\delta}) \subseteq A.$$

Set $\delta = \min\{\hat{\delta}, \tilde{\delta}\} > 0$. We claim that this δ is the same δ we were looking for. Observe that $|h| = |-(-h)| = |-h|$. Thus, if $h \in A$, then $0 < |h| < \delta$ implies that $0 < |-h| < \delta$. By setting $y = -h$, we can write

$$|g(-h) - L| < \varepsilon$$

which is our desired result. ■

Problem 9. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Let c be an interior point of I. Assume f is differentiable at c.

(a) Recall that $f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$. Use this and the result of Exercise 4 to show that

$$f'(c) = \lim_{h\to 0}\frac{f(c)-f(c-h)}{h}.$$

(b) Use the result of (a) to prove that

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}.$$

Proof. (a) Define $g : I \to \mathbb{R}$ by

$$g(h) = \frac{f(c+h)-f(c)}{h}.$$

Notice that

$$g(-h) = \frac{f(c-h)-f(c)}{-h} = \frac{f(c)-f(c-h)}{h}.$$

By exercise 4, we can see that

$$f'(c) = \lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h)$$

Hence, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}.$$

(b) For h sufficiently small, we have

$$\begin{aligned}\frac{f(c+h) - f(c-h)}{2h} &= \frac{f(c+h) - f(c)}{2h} + \frac{f(c) - f(c-h)}{2h} \\ &= \frac{1}{2}\cdot\frac{f(c+h) - f(c)}{h} + \frac{1}{2}\cdot\frac{f(c) - f(c-h)}{h}\end{aligned}$$

Now, taking the limit as $h \to 0$, we have by part (a) (and using the Algebraic Limit Theorem for functions) that

$$\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} = \frac{1}{2}f'(c) + \frac{1}{2}f'(c) = f'(c).$$

■

Problem 10. Recall that in one of the homework assignments of Math 230A we proved that $\sin x$ and $\cos x$ are continuous functions on $\mathbb{R}$. We also proved that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(i) Use this result to show that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

(ii) Use (i) to show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is differentiable at all points $c \in \mathbb{R}$ and $f'(c) = \cos c$ for all $c \in \mathbb{R}$.

Proof. (i) Suppose $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Then for for a sufficiently small neighborhood of zero, we may write

$$\begin{aligned}\frac{\cos h - 1}{h} &= \Big(\frac{\cos h - 1}{h}\Big)\Big(\frac{\cos h + 1}{\cos + 1}\Big) \\ &= \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \frac{-\sin^2 h}{h(\cos h + 1)} \\ &= \frac{\sin h}{h}\cdot\frac{-\sin h}{\cos h + 1}.\end{aligned}$$

Note that the first term of the product in the last equality above exists by assumption and the second term exists because

$$\lim_{h\to 0} \frac{-\sin h}{\cos h + 1} = 0.$$

Indeed, $\sin h$ and $\cos h$ are both continuous functions, and so $\lim_{h\to 0}(-\sin h) = -\sin 0 = 0$ and $\lim_{h\to 0}(\cos h + 1) = 2$ along with the Algebraic Continuity Theorem implies that the above limit holds. Now, using the Algebraic Limit Theorem for functions, we can write that

$$\begin{aligned}\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0}\Big(\frac{\sin h}{h}\cdot\frac{-\sin h}{\cos h + 1}\Big) \\ &= \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{-\sin h}{\cos h + 1} \\ &= 1\cdot 0 \\ &= 0\end{aligned}$$

which is our desired result.

(ii) By the summation trigonometric identity

$$\begin{aligned}\frac{\sin(c+h)-\sin h}{h} &= \frac{[\sin c\cos h+\cos c\sin h]-\sin c}{h}\\ &= \frac{\sin c(1-\cos h)+\cos c\sin h}{h}\\ &= \sin c\cdot\frac{1-\cos h}{h}+\cos c\cdot\frac{\sin h}{h}.\end{aligned}$$

Using part (i) along with the fact that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, we have

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(c+h)-\sin h}{h} &= \lim_{h\to 0}\left(\sin c\cdot\frac{1-\cos h}{h}\right)+\lim_{h\to 0}\left(\cos c\cdot\frac{\sin h}{h}\right)\\ &= \sin c\cdot\lim_{h\to 0}\frac{1-\cos h}{h}+\cos c\cdot\lim_{h\to 0}\frac{\sin h}{h}\\ &= \sin c\cdot 0+\cos c\cdot 1\\ &= \cos c.\end{aligned}$$

Clearly, we can see that the limit above does exist. Now, we can conclude that

$$f'(c)=\cos c.$$

■

Problem 11. Prove the following theorem.

Theorem (Generalized Mean Value Theorem)**.** If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point $c\in(a,b)$ where

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c).$$

Proof. Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Our goal is to show that there exists a point $c\in(a,b)$ where

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)f'(c).]$$

To this end, define the function $h:[a,b]\to\mathbb{R}$ by

$$h(x)=[f(b)-f(a)]g(x)-[g(b)-g(a)]f(x).$$

Our goal is to show that h is continuous on $[a,b]$ and differentiable on the open interval (a,b). Indeed, knowing that f and g are continuous on $[a,b]$ implies, by the Algebraic Continuity Theorem, that $h(x)$ is continuous. Furthermore, f and g are differentiable on (a,b), and so $h(x)$ must also be differentiable by the Algebraic differentiability Theorem. Also, we have

$$\begin{aligned}h(b)-h(a) &= [f(b)-f(a)]g(b)-[g(b)-g(a)]f(b)\\ &\quad-\Big([f(b)-f(a)]g(a)-[g(b)-g(a)]f(b)\Big)\\ &= f(b)g(b)-f(a)g(b)-g(b)f(b)+g(a)f(b)\\ &\quad-f(b)g(a)+f(a)g(a)+g(b)f(b)-g(a)f(b)\\ &= 0\end{aligned}$$

Thus, we have $h(b)=h(a)$ and so, the Rolle's Theorem implies that there exists a $c\in(a,b)$ such that $h'(c)=0$. Hence, we have

$$h'(c)=[f(b)-f(a)]g'(c)-[g(b)-g(a)]f'(c)=0$$

and so

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c)$$

which is our desired result. ■

Problem 12. Prove the following theorem.

Theorem. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Prove that

$$\forall x \in I, f'(x) > 0 \implies f \text{ is strictly increasing on } I.$$

Proof. Suppose that for all $x \in I$, we have $f'(x) > 0$. Our goal is to show that f is strictly increasing on I; that is, for all $x_1, x_2 \in I$ with $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Since f is differentiable on I, we must also have that f is continuous on I. Consider the open interval (x_1, x_2) in I. Then f must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, we can see that $f'(c) > 0$. Since $x_2 - x_1 > 0$, we can see that

$$f(x_2) - f(x_1) > 0 \iff f(x_2) > f(x_1) \ \forall x_1, x_2 \in I.$$

■

Problem 13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $C > 0$.

(i) Suppose $|f(u) - f(v)| \leq C|u - v|$ for all $u, v \in \mathbb{R}$. Prove that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$.

(ii) Suppose $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. Prove that $|f(u) - f(v)| \leq C|u - v|$ for all $u, v \in \mathbb{R}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $C > 0$.

(i) Our goal is to show that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. To this end, let $x \in \mathbb{R}$. To show the result, we must show that

$$-C \leq \lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C. \tag{*}$$

By assumption, we can see that

$$\begin{aligned} |f(\hat{y}) - f(x)| \leq C|\hat{y} - x| &\iff \left| \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \right| \leq C \\ &\iff -C \leq \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C. \end{aligned}$$

Since f is differentiable on $\mathbb{R}$, we can see that

$$\lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \text{ exists.}$$

Applying the Order Limit Theorem for functions on the above inequality implies that

$$-C \leq \lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C$$

which tells us further that

$$|f'(x)| \leq C.$$

(ii) Suppose $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. Our goal is to show that

$$|f(u) - f(v)| \leq C|u - v| \ \forall u, v \in \mathbb{R}.$$

Let $u, v \in \mathbb{R}$. Consider the closed interval $[u, v] \subseteq \mathbb{R}$. Since f is continuous on $\mathbb{R}$, it follows immediately that f must also be continuous on $[u, v]$ (since f is differentiable on $\mathbb{R}$). Furthermore,

f is differentiable on the open interval (u, v) since f is differentiable on $\mathbb{R}$. By the Mean Value Theorem, there exists a $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(u) - f(v)}{u - v}.$$

By assumption, we can see that $|f'(\xi)| \leq C$ and so

$$\left|\frac{f(u) - f(v)}{u - v}\right| = |f'(\xi)| \leq C.$$

Thus, we have

$$|f(u) - f(v)| \leq C|u - v|$$

which is our desired result.

■

> **Problem 14.** Let $f : \mathbb{R} \to \mathbb{R}$ be given $f(x) = x^5 + x^3 - x^2 + 5x + 3$.
>
> (i) Prove that there exists a solution to the equation $f(x) = 0$.
>
> (ii) Prove that there cannot be more than one solution to the equation $f(x) = 0$.

Proof. (i) We proceed by using the Intermediate Value Theorem to show the result. Since f is continuous everywhere (because f is a polynomial), we can just consider a closed interval $[-1, 1]$. We will show that $f(-1) < 0$ and $f(1) > 0$. Indeed, we have

$$\begin{aligned} f(-1) &= (-1)^5 + (-1)^3 - (-1)^2 + 5(-1) + 3 \\ &= -5 < 0 \end{aligned}$$

and

$$\begin{aligned} f(1) &= (1)^5 + (1)^3 - (1)^2 + 5(1) + 3 \\ &> 9 > 0. \end{aligned}$$

Thus, the intermediate value theorem implies that there exists $\hat{c} \in [-1, 1]$ such that $f(\hat{c}) = 0$.

(ii) Suppose for sake of contradiction that there exists more than one solution $c_1, c_2 \in [-1, 1]$ such that $f(c_1) = 0$ and $f(c_2) = 0$. Thus, there exists $\tilde{c} \in (-1, 1)$ such that

$$f'(\tilde{c}) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = 0.$$

Since f is also differentiable everywhere on $\mathbb{R}$, we have

$$f'(x) = 5x^4 + 3x^2 - 2x + 5.$$

But note that $3x^2 - 2x + 5$ is a positive quadratic for all $x \in \mathbb{R}$. Hence, $f'(x) > 0$ for all $x \in \mathbb{R}$ which contradicts the fact that $f'(\tilde{c}) = 0$. ■

> **Problem 15.** In class,we gave a proof of L'Hopital's Rule. If we add the following three assumptions to the hypotheses of the corresponding theorem, then we can give a shorter proof of H'opital's Rule:
>
> (i) $f'(a)$ and $g'(a)$ exist.
>
> (ii) $g'(a) \neq 0$.
>
> (iii) f' and g' are continuous at a.

Here is the shorter proof:

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

Solution. The first equality

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

holds because of (iii) and (ii). The third equality holds because of (i) and by definition of the derivative. Since are the referring to limits of functions, we can justify multiplying and dividing by $x - a$. The last equality holds because $f(a) = 0$ and $g(a) = 0$ from our original set of assumptions. ■

Problem 16. Let $n \in \mathbb{N}$ and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function for which the equation $f'(x) = 0$ has at most $n - 1$ solutions. Prove that the equation $f(x) = 0$ has at most n solutions.

Proof. Suppose for sake of contradiction that $f(x) = 0$ has at least n solutions. Denote the roots by

$$x_1 < x_2 < \cdots < x_n < x_{n+1}.$$

Now, notice that

$$f(x_1) = f(x_2) = \cdots = f(x_n) = f(x_{n+1}) = 0.$$

Since f is differentiable on $\mathbb{R}$, we can find an $c_i \in (x_i, x_i)$ for $1 \leq i \leq n$ such that $f'(c_i) = 0$ by the Mean Value Theorem. This implies that $f'(x)$ has n solutions which contradicts our assumption that $f'(x)$ has $n - 1$ solutions. ■

Problem 17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(i) Prove that f is differentiable at all $x \neq 0$.

(ii) Prove that $f'(0) = \frac{1}{2}$.

(iii) Prove that f is NOT increasing on any open interval containing 0.

Proof. (i) Note that x and x^2 are polynomials which is differentiable for all $x \neq 0$ in $\mathbb{R}$, $\sin x$ is differentiable for all $x \neq 0$ in $\mathbb{R}$, and $\frac{1}{x}$ is a rational function which is also differentiable for all $x \neq 0$ in $\mathbb{R}$. By the algebraic differentiability theorem, we have that $f(x)$ is a differentiable function for all $x \neq 0$.

(ii) Observe that

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \left[\frac{1}{2} + x \sin\frac{1}{x}\right] \\ &= \lim_{x\to 0} \left[\frac{1}{2} + \frac{\sin\frac{1}{x}}{\frac{1}{x}}\right] \\ &= \frac{1}{2} + 0 && \text{(Algebraic Limit Theorem)} \\ &= \frac{1}{2} \end{aligned}$$

where

$$\lim_{x\to 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 0.$$

Indeed, we have

$$0 \le |x \sin \frac{1}{x}| \le |x|.$$

Applying the squeeze theorem for functions to the inequality above, we have that

$$\lim_{x\to 0} \left| x \sin \frac{1}{x} \right| = 0$$

which further implies that

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

(iii) Define $a_n = \frac{1}{2\pi n} \to 0$. Our goal is to show that $f'(a_n) < 0$. Computing $f'(x)$ for $x \neq 0$, we have

$$f'(x) = \frac{1}{2} + \left[2x \sin \frac{1}{x} - \cos \frac{1}{x}\right].$$

Then we have

$$\begin{aligned} f'(a_n) &= \frac{1}{2} + \left[\frac{1}{\pi} n \cdot \sin 2\pi n - \cos 2\pi n\right] \\ &= \frac{1}{2} + [0 - 1] \\ &= \frac{1}{2} - 1 \\ &= \frac{-1}{2} < 0. \end{aligned}$$

Hence, $f'(a_n) < 0$ and so we conclude that f is NOT increasing on any open interval containing zero.

■

Problem 18. Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ be a differentiable function.

(a) Show that if there exists some $L \ge 0$ such that $|f'(x)| \le L$ for all $x \in I$, then f is uniformly continuous.

(b) Is the converse true? Prove it or give a counterexample.

Proof. (a) Suppose that there exists some $L \ge 0$ such that $|f'(x)| \le L$. Our goal is to show that f is uniformly continuous; that is, we need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Let $x, y \in \mathbb{R}$ and let $\varepsilon > 0$. Suppose, without loss of generality, that $x < y$. Since f is differentiable on I and $(x, y) \subseteq I$, we can see that f is also differentiable on (x, y). Furthermore, f being differentiable on I implies that f is continuous on I and so f is continuous on $[x, y]$. By the Mean Value Theorem, we can find an $\ell \in (x, y)$ such that

$$f'(\ell) = \frac{f(x) - f(y)}{x - y}.$$

By assumption, we can see that for $L > 0$ we have

$$\begin{aligned} |f'(\ell)| \le L &\implies \left| \frac{f(x) - f(y)}{x - y} \right| \le L \\ &\implies |f(x) - f(y)| \le L|x - y|. \end{aligned}$$

Now, choose $\delta = \frac{\varepsilon}{L}$. Then whenever $|x - y| < \delta$, we can see that

$$|f(x) - f(y)| \le L|x - y| < L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Hence, we conclude that f must be uniformly continuous on $\mathbb{R}$.

(b) Consider the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. We claim that this function is uniformly continuous on $\mathbb{R}$ but its derivative $f'(x) = \frac{1}{2}x^{-1/2}$ is not bounded above for some $L \geq 0$. Indeed, f is differentiable on $(0, \infty)$ since the following limit

$$\begin{aligned} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \\ &= \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} \\ &= \frac{1}{2\sqrt{c}} \end{aligned}$$

exists.

To show that f is uniformly continuous, let $x, y \in (0, \infty)$ and let $\varepsilon > 0$. Choose $\delta = \sqrt{\varepsilon}$. Since $x, y \in (0, \infty)$, we have

$$|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|.$$

Observe that if $|x - y| < \delta$, then we have

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \sqrt{\varepsilon}$$

which further implies that

$$|\sqrt{x} - \sqrt{y}| < \varepsilon.$$

Thus, $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$.

Now, we want to show that the derivative of f is NOT bounded above. Note that

$$f'(x) = \frac{1}{2}x^{-1/2}.$$

We need to show that for all $M > 0$ such that there exists $\hat{x} \in (0, \infty)$ such that $|f'(x)| > M$. To this end, let $\varepsilon > 0$. Choose $\hat{x} = \frac{1}{64M^2} > 0$. Then we have

$$\begin{aligned} |f'(\hat{x})| = \frac{1}{2\sqrt{\hat{x}}} &= \frac{1}{2} \cdot \frac{1}{\sqrt{1/64M^2}} \\ &= \frac{1}{2} \cdot \frac{1}{1/8M} \\ &= 4M \\ &> M. \end{aligned}$$

Hence, f is not bounded above. ■

Problem 19. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) with $|f'(x)| \leq M$ for $x \in (a, b)$ and $M \geq 0$. Prove that

$$\lim_{x \to b^-} f(x)$$

exists.

Proof. Suppose $f : (a, b) \to \mathbb{R}$ is a differentiable function on (a, b) with $|f'(x)| \leq M$ for all $x \in (a, b)$. By Exercise 14, f must be uniformly continuous on (a, b). By Exercise 16 of Homework 10 from Math 230A, we can find a continuous function $F : [a, b] \to \mathbb{R}$ such that $F|_{(a,b)} = f$. As a consequence, we have $\lim\limits_{x \to b} F(x) = F(b)$. Since we are only referring to the limit

$$\lim_{x \to b^-} f(x)$$

and $F|_{(a,b)} = f$, it follows that

$$\lim_{x \to b} f(x) = F(b)$$

exists. Clearly, if this holds, then

$$\lim_{x \to b^-} f(x) = F(b)$$

must also hold. ■

Problem 20. Let $f : (0, 1] \to \mathbb{R}$ be differentiable with $0 < f'(x) < 1$ for all $x \in (0, 1]$. Define a sequence (a_n):

$$a_n = f\left(\frac{1}{n}\right)$$

Prove that $\lim_{n\to\infty} a_n$ exists.

Proof. Our goal is to show that $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, it suffices to show that $a_n = f\left(\frac{1}{n}\right)$ is a Cauchy sequence; that is, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n, m > N$

$$|a_n - a_m| < \varepsilon.$$

To this end, let $\varepsilon > 0$ be given. Since $0 < f'(x) < 1$ for all $x \in (0, 1]$, we have

$$|a_n - a_m| = \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right)\right| \leq \left|\frac{1}{n} - \frac{1}{m}\right|$$

by exercise 9. Choose $\hat{N} = \frac{2}{\varepsilon}$. Then for any $n, m > N$, we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we see that $a_n = f\left(\frac{1}{n}\right)$ is a Cauchy sequence. ■

Problem 21. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Prove that, if $\lim_{x\to+\infty} f(x) = M \in \mathbb{R}$, then there exists a sequence (x_n) such that $|f'(x_n)| \to 0$.

Proof. Suppose $\lim_{x\to\infty} f(x) = M$. Our goal is to show that $|f'(x_n)| \to 0$. By assumption, let $\varepsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then there exists $\zeta_n > 0$ such that for any $x > \zeta_n$, we have

$$|f(x) - M| < \frac{1}{2n}.$$

In particular, we have $\zeta_n + 2 > \zeta_n + 1 > \zeta_n$. Since f is differentiable on $(0, \infty)$, f must also be continuous on $(0, \infty)$. Consider the open interval $(\zeta_n + 2, \zeta_n + 1)$ for all $n \in \mathbb{N}$. Since f is continuous on $(0, \infty)$, f is also continuous on $[\zeta_n + 1, \zeta_n + 2]$. By the Mean Value Theorem, we can find an $x_n \in (\zeta_n + 1, \zeta_n + 2)$ such that

$$f'(x_n) = \frac{f(\zeta_n + 2) - f(\zeta_n + 1)}{\zeta_n + 2 - (\zeta_n + 1)} = f(\zeta_n + 2) - f(\zeta_n + 1).$$

By the triangle inequality, we have

$$0 < |f'(x_n)| \leq |f(\zeta_n + 2) - M| + |M - f(\zeta_n + 1)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

By using the Squeeze Theorem for sequential limits on the inequality above, we conclude that $|f'(x_n)| \to 0$.

■

Problem 22. Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Show that if $f'(x) \neq 1$ for all $x \in (0, 1)$, then there exists a unique $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

Proof. By exercise 16 of Homework 10 of 230A, we have that $f : [0, 1] \to [0, 1]$ being continuous on $[0, 1]$ implies that there exists a $c \in [0, 1]$ such that $f(c) = c$. Now, we want to show that this element $c \in [0, 1]$ is unique. Suppose for sake of contradiction that there exists $c_1, c_2 \in [0, 1]$ such that $f(c_1) = c_1$ and $f(c_2) = c_2$. Now, f must be differentiable on $(0, 1)$ implies that f is differentiable on (c_1, c_2). By the Mean Value Theorem, there exists $\hat{c} \in (0, 1)$ such that

$$f'(\hat{c}) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

But this contradicts the fact that $f(x) \neq 1$ for all $x \in [0,1]$. Hence, the element $c \in [0,1]$ must be unique. ■

Problem 23. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ with $f(0) = 0$. Assume that f is differentiable on $(0, \infty)$ with f' is increasing on $(0, \infty)$. Let $g : (0, \infty) \to \mathbb{R}$ be defined as $g(x) = \frac{f(x)}{x}$. Prove that g is increasing on $(0, \infty)$.

Proof. Our goal is to show that g is increasing on $(0, \infty)$; that is, for all $x, yin(0, \infty)$ with $x < y$, we have

$$g(x) < g(y).$$

Let $x, y \in (0, \infty)$. Since f and g are differentiable on $(0, \infty)$ and continuous on $[0, \infty)$, there exists $\hat{x} \in (0, x)$ and $\hat{y} \in (0, y)$ such that

$$f'(\hat{x}) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

and

$$f'(\hat{y}) = \frac{f(y) - f(0)}{y - 0} = \frac{f(y)}{y}.$$

Since f' is increasing on $(0, \infty)$ and $\hat{x} < \hat{y}$, we have

$$f'(\hat{x}) < f'(\hat{y}) \iff \frac{f(x)}{x} < \frac{f(y)}{y} \iff g(x) < g(y).$$

Hence, we can conclude that g is increasing. ■

Problem 24. Let $f : [0, +\infty) \to \mathbb{R}$ be continuous function, which is differentiable on $(0, +\infty)$.

(a) Prove that if $\lim_{x \to +\infty} f'(x) = 0$, then f is uniformly continuous on $[0, \infty)$.

(b) Give an example of such a function with unbounded derivative.

Proof. (a) Suppose $\lim_{x \to +\infty} f'(x) = 0$. Our goal is to show that f is uniformly continuous on $[0, +\infty)$. Let $x, y \in [0, +\infty)$. Since $\lim_{x \to \infty} f'(x) = 0$, we know that for any $\varepsilon > 0$, there exists $R > 0$ such that for any $x \geq R$, we have

$$|f'(x)| < 1.$$

Now, for any $x \in [R, +\infty)$, we have $|f'(x)| < 1$. By exercise 14, f is uniformly continuous on $[R, \infty)$. Next, consider the closed and bounded interval $[0, R] \subseteq \mathbb{R}$. By Heine-Borel, $[0, R]$ is a compact set. Since f is continuous on $[0, +\infty)$, f is also continuous $[0, R]$; that is, f is uniformly continuous on $[0, R]$. Our goal is to, for any given $x, y[0, +\infty)$, find a $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Since f is uniformly continuous on $[0, R]$, we can find a $\delta_1 > 0$ such that whenever $|x - y| < \delta_1$, we have

$$|f(x) - f(y)| < \varepsilon. \tag{1}$$

Similarly, since f is uniformly continuous on $[R, +\infty)$, we can find a $\delta_2 > 0$ such that whenever $|x - y| < \delta_2$, we have

$$|f(x) - f(y)| < \varepsilon. \tag{2}$$

Lastly, f being uniformly continuous on $[R - 1, R + 1]$ implies that there exists $\delta_3 > 0$ such that whenever $|x - y| < \delta_3$, we have

$$|f(x) - f(y)| < \varepsilon. \tag{3}$$

Now, choose $\delta = \min\{\delta_1, \delta_2, \delta_3, 1\}$ and suppose $x, y \in [0, \infty)$. We will consider three cases.

(I) Let $x, y \in [0, R]$. Then suppose $|x - y| < \delta < \delta_1$. From (1), we can conclude that f is uniformly continuous.

(II) Let $x, y \in [R, +\infty)$. Then suppose $|x - y| < \delta < \delta_2$. From (3), we can conclude that f is uniformly continuous.

(III) Let $x \in [0, R]$ and $y \in [R, +\infty)$. Then whenever $|x - y| < \delta < 1$, we can conclude that $x, y \in [R - 1, R + 1]$. Then $|x - y| < \delta < \delta_3$ implies that

$$|f(x) - f(y)| < \varepsilon.$$

Hence, f is uniformly continuous.

(b) Refer to the example in exercise 14 ($f(x) = \sqrt{x}$). We see that $\lim\limits_{x \to +\infty} f'(x) = 0$ and f is uniformly continuous on $[0, \infty)$, but f is unbounded near zero.

■

Problem 25. Let f be differentiable on (a, b) and let $c \in (a, b)$. Suppose $f'(c) > 0$. Prove that there exists some $\delta > 0$ such that $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(x) > f(c)$ for $x \in (c, c + \delta)$.

Proof. Suppose that f is differentiable on (a, b). By definition, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \text{ exists.}$$

Hence, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

In particular, let $\varepsilon = \frac{f'(c)}{2}$. Then

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < 1 \tag{*}$$

which further implies that

$$\frac{f'(c)}{2} + f'(c) < \frac{f(x) - f(c)}{x - c} < \frac{f'(c)}{2} + f'(c) \Longrightarrow \frac{f'(c)}{2}(x - c) < f(x) - f(c) < \frac{3f'(c)}{2}(x - c).$$

Suppose $x \in (c - \delta, c)$. It immediately follows that $x < c$ and so $x - c < 0$. Then we have

$$f(x) - f(c) < \underbrace{\frac{3f'(c)}{2}}_{>0} \underbrace{(x - c)}_{<0} < 0.$$

Hence, we have $f(x) < f(c)$. Now, suppose $x \in (c, c + \delta)$. Similarly, we have $x > c$ which implies that $x - c > 0$. Then we have

$$f(x) - f(c) > \underbrace{\frac{f'(c)}{2}}_{>0} \underbrace{(x - c)}_{x>0} > 0.$$

Hence, we have $f(x) > f(c)$. ■

Consider the following problem: Let f be continuous on $[a, b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt \ \forall x \in [a, b].$$

Prove that $f(x) = 0$ on $[a, b]$.

Homework 4

Problem 26. Prove the following theorem.

Theorem (Generalize Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ is integrable and either $g \geq 0$ on $[a, b]$ or $g \leq 0$ on $[a, b]$. Then there exists a $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g. \tag{*}$$

Proof. Our goal is to find a $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

Since $f : [a, b] \to \mathbb{R}$ is a continuous function and $[a, b]$ is a compact set, f attains its maximum and minimum on $[a, b]$. Then

$$m = \sup_{x \in [a,b]} f(x) \text{ and } m = \inf_{x \in [a,b]} f(x)$$

and hence,

$$m \leq f(x) \leq M \implies mg(x) \leq f(x)g(x) \leq Mg(x) \ \forall x \in [a, b]. \tag{†}$$

Now, note that the inequality in follows immediately if $g(x) = 0$ for all $x \in [a, b]$. Hence, suppose that $g(x) \neq 0$ on $[a, b]$. By the Order Theorem for Integrals, (†) implies

$$\int_a^b mg \leq \int_a^b fg \leq \int_a^b Mg$$

which further implies (by the algebraic theorem for integrals) that

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g. \tag{1}$$

From (1), we may consider two cases; that is, either $\int_a^b g \neq 0$ or $\int_a^b g = 0$. If $\int_a^b g \neq 0$, then we can divide by $\int_a^b g$ on (1). So, we have

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

But note that f is continuous on $[a, b]$ and so, by the Intermediate Value Theorem, there exists a $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b fg}{\int_a^b g} \implies \int_a^b fg = f(c) \int_a^b g$$

which is our desired result.

If $\int_a^b g = 0$, then it follows from (1) that

$$\int_a^b fg = 0$$

and so

$$\int_a^b fg = f(c) \int_a^b g$$

for all $c \in [a, b]$. ■

Problem 27. Prove the following theorem.

Theorem (Rudin, Theorem 6.15). If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f \ d\alpha = f(s).$$

Proof. Our goal is to show that $\int_a^b f \ d\alpha = f(s)$. Define the partition

$$P_n = \left\{a, s - \frac{1}{n}, s + \frac{1}{n}, b\right\}.$$

Since $f \in R_\alpha[a, b]$, we have

$$\int_a^b f \ d\alpha = \lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n). \tag{*}$$

So, it suffices to show that

$$\lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n) = f(s).$$

Note that

$$\alpha(x) = I(x - s) = \begin{cases} 1 & \text{if } x > s \\ 0 & \text{if } x \leq s \end{cases}$$

By definition of α, we have

$$\begin{aligned} k = 1; \Delta\alpha_1 &= \alpha(s - \delta) - \alpha(a) = 0 - 0 = 0 \\ k = 2; \Delta\alpha_2 &= \alpha(s + \delta) - \alpha(s - \delta) = 1 - 0 = 1 \\ k = 3; \Delta\alpha_3 &= \alpha(b) - \alpha(s + \delta) = 1 - 1 = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} U(f, \alpha, P_n) &= \sum_{k=1}^{3} M_k \Delta\alpha_k \\ &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3 \\ &= 0 + M_2 \cdot 1 + 0 \\ &= M_2 \cdot 1 \\ &= \sup_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x) \end{aligned}$$

and similarly,

$$L(f, \alpha, P_n) = \sum_{k=1}^{3} m_k \Delta\alpha_k = m_2 = \inf_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x).$$

Because f is continuous at s, we know by exercise 4 of homework 3 that

$$\lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} \sup_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x) = f(s)$$

and

$$\lim_{n\to\infty} L(f, \alpha, P_n) = \lim_{n\to\infty} \inf_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x) = f(s)$$

which is our desired result. ∎

Problem 28. Prove the following theorem.

Theorem (Rudin, Theorem 6.16). (a) Let $N \in \mathbb{N}$. Let $c_1, \ldots, c_N$ be nonnegative numbers. Suppose $s_1, \ldots, s_N$ are distinct points in (a, b), and let $\alpha(x) = \sum_{n=1}^{N} c_n I(x - s_n)$. Let f be continuous at s. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{N} c_n f(s_n).$$

(b) Suppose $c_n \geq 0$ for $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} c_n$ converges, (s_n) is a sequence of distinct points in (a, b), and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous at s. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof. (i) Denote $\alpha_n = I(x - s_n)$ for $1 \leq n \leq N$. Note that we can easily show that each α_n is an increasing function by definition of $I(x - s_n)$. Since f is continuous on $[a, b]$, we see that $f \in R_{\alpha_n}[a, b]$ for each $1 \leq n \leq N$. Furthermore, we can easily show, through an induction argument and by an exercise 5 from homework 3, that

$$\int_a^b f \, d\Big(\sum_{n=1}^{N} c_n a_n\Big) = \sum_{n=1}^{N} \int_a^b f \, d(c_n a_n) \tag{1}$$

Also, note that for all $1 \leq n \leq N$

$$\int_a^b f \, d\alpha_n = f(s_n) \tag{2}$$

by applying the previous exercise for each $1 \leq n \leq N$. Then (1) and (2) imply that

$$\begin{aligned}
\int_a^b f \, d\alpha &= \int_a^b \, d\Big(\sum_{n=1}^{N} c_n \alpha_n\Big) \\
&= \sum_{n=1}^{N} \int_a^b f \, d(c_n \alpha_n) \\
&= \sum_{n=1}^{N} c_n \int_a^b f \, d\alpha_n && \text{(Exercise 5 from HW3)} \\
&= \sum_{n=1}^{N} c_n f(s_n).
\end{aligned}$$

Hence, we have that

$$\int_a^b f \, d\alpha = \sum_{n=1}^{N} c_n f(s_n).$$

(ii) Our goal is to show that

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

where $c_n \geq 0$ for all $n \in \mathbb{N}$. That is, we want to show that for any $\varepsilon > 0$, there exists K such that for any $N > K$ that

$$\Big| \sum_{n=1}^{N} c_n f(s_n) - \int_a^b f \, d\alpha \Big| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Note that $I(x - s_n) \leq 1$ for all $x \in [a, b]$ and so we have

$$0 \leq c_n I(x - s_n) \leq c_n.$$

Since $\sum c_n$ is a convergent series, it follows from applying the comparison test to the above inequality that $\sum c_n I(x - s_n)$ converges. Observe that α is also an increasing function. Indeed, we can easily see that this is the case by using the fact that $\sum c_n I(x - s_n)$ is a convergent series and applying the order limit theorem for any $x < y$ in $[a, b]$. Since α is an increasing function on $[a, b]$ and f is a continuous function on $[a, b]$, we have that $f \in R_\alpha[a, b]$. Now, consider $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ where

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n) \quad \text{and} \quad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n).$$

Also, note that by an exercise in homework 3, we have

$$\int_a^b f \, d\alpha = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2. \tag{1}$$

By part (a), we see that

$$\int_a^b f \, d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n).$$

Hence, the left-hand side of (*) can be written as (by using (1) and the equality above)

$$\begin{aligned} \Big| \sum_{n=1}^{N} c_n f(s_n) - \int_a^b f \, d\alpha \Big| &= \Big| \int_a^b f \, d\alpha_2 - \int_a^b f \, d\alpha \Big| \\ &= \Big| \int_a^b f \, d\alpha_2 \Big|. \end{aligned}$$

Since f is bounded on $[a, b]$ (because it is continuous on the compact interval $[a, b]$), we have that for some $\tilde{M} > 0$, we have $|f(x)| \le \tilde{M}$ for all $x \in [a, b]$. By the triangle inequality for integrals, we can see that

$$\Big| \int_a^b f \, d\alpha_2 \Big| \le \int_a^b |f| \, d\alpha_2 \le \tilde{M}(\alpha_2(b) - \alpha_2(a))$$

by a Theorem proven in lecture. Note that on the above inequality, we see that $|f| \in R_{\alpha_2}[a, b]$ since $f \in R_{\alpha_2}[a, b]$. Next, consider the difference $\alpha_2(b) - \alpha_2(a)$. Then observe that

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^{\infty} c_n$$

by definition of $I(x - s_n)$. Indeed, we see that $\alpha_2(a) = 0$ since $a < s_n$ and $\alpha_2(b) = 1$ since $a > s_n$. Since $\sum c_n$ converges, it follows from an exercise done in 230A that

$$\lim_{N \to \infty} \sum_{n=N+1}^{N} c_n = 0.$$

As a consequence, we can find a $\hat{K}$ such that for any $n > \hat{K}$

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^{\infty} c_n < \frac{\varepsilon}{\tilde{M}}.$$

Note that we dropped the absolute value on the above quantity because of the fact that c_n is nonnegative for all $n \in \mathbb{N}$. We claim that this is the desired K we were looking for. Indeed, for any $n > \hat{K}$, we have

$$\begin{aligned} \Big| \sum_{n=1}^{N} c_n f(s_n) - \int_a^b f \, d\alpha \Big| &= \Big| \int_a^b f \, d\alpha_2 \Big| \\ &\le \tilde{M}(\alpha_2(b) - \alpha_2(a)) \\ &< \tilde{M} \cdot \frac{\varepsilon}{\tilde{M}} \\ &= \varepsilon \end{aligned}$$

which is our desired result.

■

Problem 29. Let $p, q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

4-1) Prove that if $f \in R_\alpha[a,b]$ and $g \in R_\alpha[a,b]$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \; d\alpha = 1 = \int_a^b g^q \; d\alpha,$$

then $\int_a^b fg \; d\alpha \leq 1$.

4-2) Prove that if $f \in R_\alpha[a,b]$ and $g \in R_\alpha[a,b]$, then

$$\int_a^b |fg| \; d\alpha \leq \Big[\int_a^b |f|^p \; d\alpha \Big]^{\frac{1}{p}} \Big[\int_a^b |g|^q \; d\alpha \Big]^{\frac{1}{q}}.$$

Proof. 4-1) Since $f \geq$, $g \geq 0$, and $p, q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}.$$

Note that if $f, g = 0$ on $[a,b]$, then the result immediately holds. So, suppose $f, g > 0$ on $[a,b]$. Since $f \in R_\alpha[a,b]$ and $g \in R_\alpha[a,b]$, we can use the Order Theorem for Integrals and the Algebraic Theorem for integrals to write

$$\begin{aligned}
\int_a^b |fg| \; d\alpha &= \int_a^b fg \; d\alpha && (fg > 0) \\
&\leq \int_a^b \Big[\frac{f^p}{p} + \frac{g^q}{q} \Big] \; d\alpha \\
&= \int_a^b \frac{f^p}{p} \; d\alpha + \int_a^b \frac{g^q}{q} \; d\alpha \\
&= \frac{1}{p} \int_a^b f^p \; d\alpha + \frac{1}{q} \int_a^b g^q \; d\alpha \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Hence, we have

$$\int_a^b |fg| \; d\alpha \leq 1.$$

4-2) Our goal is to show that

$$\int_a^b |fg| \; d\alpha \leq \Big(\int_a^b |f|^p \; d\alpha \Big)^{\frac{1}{p}} \Big(\int_a^b |g|^q \; d\alpha \Big)^{\frac{1}{q}}.$$

Let

$$A = \Big(\int_a^b |f|^p \; d\alpha \Big)^{\frac{1}{p}} \quad \text{and} \quad B = \Big(\int_a^b |g|^q \; d\alpha \Big)^{\frac{1}{q}}.$$

We will consider two cases:

(i) $(A, B \neq 0)$ Let

$$\tilde{f(x)} = \frac{f(x)}{A} = \frac{f(x)}{\Big(\int_a^b |f|^p \; d\alpha \Big)^{\frac{1}{p}}}$$

and

$$\tilde{g(x)} = \frac{g(x)}{B} = \frac{g(x)}{\left(\int_a^b |g|^q \, d\alpha\right)^{\frac{1}{q}}}.$$

From here, it suffices to show that

$$\int_a^b |\tilde{f}||\tilde{g}| \, d\alpha \leq 1.$$

Note that

$$\begin{aligned}\int_a^b |\tilde{f}|^p \, d\alpha &= \int_a^b \frac{|f|^p}{\int_a^b |f|^p \, d\alpha} \, d\alpha \\ &= \frac{1}{\int_a^b |f|^p \, d\alpha} \int_a^b |f|^p \, d\alpha = 1.\end{aligned}$$

Similarly, we have

$$\int_a^b |\tilde{g}|^q \, d\alpha = 1.$$

So, by (4-1), we have

$$\int_a^b |\tilde{f}||\tilde{g}| \, d\alpha \leq 1.$$

(ii) ($A = 0$ or $B = 0$) From this case, we can see that

$$\int_a^b |f|^p \, d\alpha = 0 \quad \text{or} \quad \int_a^b |g|^q \, d\alpha = 0.$$

Our goal is to show that

$$\int_a^b |fg| \, d\alpha \leq \left(\int_a^b |f|^p \, d\alpha\right)^{\frac{1}{p}} \left(\int_a^b |g|^q \, d\alpha\right)^{\frac{1}{q}}.$$

It suffices to show that the left-hand side of the above inequality is zero. Suppose that

$$\int_a^b |f|^p \, d\alpha = 0.$$

The proof for the other case follows analogously. Indeed, by using Young's Inequality and a linearity property of the R.S integral, we have

$$\begin{aligned}0 \leq \int_a^b |fg| \, d\alpha &\leq \int_a^b \left(\frac{|f|^p}{p} + \frac{|g|^q}{q}\right) d\alpha \\ &= \frac{1}{p}\int_a^b |f|^p \, d\alpha + \frac{1}{q}\int_a^b |g| \, d\alpha \\ &= \frac{1}{q}\int_a^b |g|^q \, d\alpha.\end{aligned}$$

Hence,

$$0 \leq \int_a^b |fg| \, d\alpha \leq \frac{1}{q}\int_a^b |g|^q \, d\alpha.$$

Note that for all $r > 0$, we have

$$0 \leq \int_a^b |r \cdot fg| \, d\alpha \leq \frac{1}{q}\int_a^b r^q |g|^q \, d\alpha.$$

That is,

$$0 \leq r\int_a^b |fg| \, d\alpha \leq \frac{1}{q} r^q \int_a^b |g|^q \, d\alpha \Longrightarrow 0 \leq \int_a^b |fg| \, d\alpha \leq \frac{r^{q-1}}{q}\int_a^b |g|^q \, d\alpha.$$

If we let $r \to 0$, we will get that

$$\frac{r^{q-1}}{q} \int_a^b |g|^q \, d\alpha \to 0$$

and similarly, the left-hand side of the above inequality also goes to 0 as $r \to 0$. Hence, the Squeeze Theorem implies that

$$\int_a^b |fg| \, d\alpha \to 0$$

as $r \to 0$. That is,

$$\int_a^b |fg| \, d\alpha = 0.$$

■

Problem 30. Suppose $f \in C^1[a,b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)\, dx = 1$. Prove that

$$\int_a^b x f(x) f'(x) \, dx = \frac{-1}{2}$$

and

$$\int_a^b [f'(x)]^2 \, dx \cdot \int_a^b x^2 f^2(x) \, dx \geq \frac{1}{4}.$$

Proof. Using Integration by Parts, we have

$$\begin{aligned}\int_a^b x f(x) f'(x) \, dx &= x f^2(x)\Big]_a^b - \int_a^b f(x)[f(x) + x f'(x)] \, dx \\ &= x f^2(x)\Big]_a^b - \int_a^b f^2(x) \, dx - \int_a^b x f(x) f'(x) \, dx \\ &= [b f^2(b) - a f^2(a)] - 1 - \int_a^b x f(x) f'(x) \, dx \\ &= -1 - \int_a^b x f(x) f'(x) \, dx.\end{aligned}$$

Hence, we have

$$2 \int_a^b x f(x) f'(x) \, dx = -1 \Longrightarrow \int_a^b x f(x) f'(x) \, dx = \frac{-1}{2}.$$

From our result, we can see that

$$\int_a^b -(x f(x) f'(x)) \, dx = \frac{1}{2}.$$

As a consequence of the above equality, we have

$$\Big|\frac{-1}{2}\Big| = \Big|\int_a^b x f(x) f'(x) \, dx\Big| \leq \int_a^b |x f(x) f'(x)| \, dx$$

by the Triangle Inequality for integrals. Let $p = q = 2$. Then $\frac{1}{p} + \frac{1}{q} = 1$. Let $u = f'(x)$ and $v = x f(x)$. By applying Holder's Inequality for Integrals, we have

$$\int_a^b |x f(x) f'(x)| \, dx \leq \Big(\int_a^b (f'(x))^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_a^b x^2 f^2(x) \, dx\Big)^{\frac{1}{2}} \tag{2}$$

Using (1) and (2), we conclude that

$$\frac{1}{2} \leq \Big(\int_a^b (f'(x))^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_a^b x^2 f^2(x) \, dx\Big)^{\frac{1}{2}}$$

and so

$$\frac{1}{4} \leq \int_a^b (f'(x))^2 \, dx \cdot \int_a^b x^2 f^2(x) \, dx$$

as our desired result. ■

Problem 31. (a) Let f be continuous on $[a,b]$ such that for any subinterval $[c,d] \subseteq [a,b]$

$$\int_c^d f(t)\ dt = 0.$$

Prove that $f(x) = 0$ on $[a,b]$.

(b) Let f be continuous on $[a,b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt,\ \forall x \in [a,b].$$

Proof. (a) Let f be continuous on $[a,b]$ such that for any subinterval $[c,d] \leq [a,b]$

$$\int_c^d f(t)\ dt = 0.$$

Suppose for sake of contradiction that $f(c) \neq 0$ for some $c \in [a,b]$. Since f is continuous on $[a,b]$, it follows that f is continuous at c. We have two cases to consider; that is, either

(1) $f(c) > 0$

(2) $f(c) < 0$

We will prove the case where $f(c) > 0$. The other case will follow analogously. Indeed, since f is continuous at c, we have

$$f(c) > 0 \Longrightarrow \exists \delta > 0 \text{ such that } \forall x \in (c-\delta, c+\delta)\ f(x) > 0.$$

By the Order Theorem for Integrals, we have

$$f(x) > 0 \Longrightarrow \int_{c-\delta}^{c+\delta} f(x)\ dx > 0.$$

But this contradicts our assumption. Hence, it must follow that $f(x) = 0$ for all $x \in [a,b]$.

(b) Let f be continuous on $[a,b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt\ \forall x \in [a,b].$$

Denote $F(x) = \int_a^x f(t)\ dt$ and $G(x) = \int_x^b f(t)\ dt = -\int_b^x f(t)\ dt$. Suppose for sake of contradiction that there exists a $c \in [a,b]$ such that $f(c) \neq 0$. Since f is continuous on $[a,b]$ and, in particular, f is continuous at $c \in [a,b]$, then by FTC II, we have $F'(c) = f(c)$ and $G'(c) = -f'(c)$. But by assumption, $f(c) = -f(c)$ which implies that

$$2f(c) = 0 \Longrightarrow f(c) = 0$$

which contradicts our assumption that $f(c) \neq 0$. Hence, it must be the case that $f(x) = 0$ for all $x \in [a,b]$.

■

Problem 32 (A Substitution Formula For Ordinary Riemann Integrals)**.** Suppose that

- $g \in C^1[a,b]$ and $f \in C^0[c,d]$.
- The range of $g, g([a,b]) = \{g(x) : x \in [a,b]\}$, is contained in $[c,d]$ (so that the composition $f \circ g$ is defined).

7-1) Explain why f is the derivative of some function.

7-2) Explain why $(f \circ g)g'$ is a derivative of some function.

7-3) Prove the substitution formula:

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Proof. 7-1) Define the function $F : [c,d] \to \mathbb{R}$ by

$$F(x) = \int_c^x f(t)\ dt.$$

The above integral holds because $f \in R[c,d]$ by assumption. Since f is continuous for any $x \in [c,d]$ it follows that $F'(x) = f(x)$ for any $x \in [g(a), g(b)]$ by the Second Fundamental Theorem of Calculus.

7-2) Note that $(f \circ g)g'$ is the derivative of $F \circ g$.

7-3) Our goal is to show that

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Note that we can use (7-2) to write $((F \circ g)(x))' = f(g(x))g'(x)$ for all $x \in [a,b]$. Since $[g(a), g(b)] \subseteq [c,d]$, we can use the First Fundamental Theorem of Calculus to get

$$\begin{aligned}
\int_a^b f(g(x))g'(x)\ dx &= \int_a^b ((F \circ g)(x))'\ dx \\
&= (F \circ g)(b) - (F \circ g)(a) \\
&= F(g(b)) - F(g(a)) \\
&= \int_{g(a)}^{g(b)} F'(u)\ du \\
&= \int_{g(a)}^{g(b)} f(u)\ du
\end{aligned}$$

which is our desired result.

■

Problem 33. Prove the following integration by parts for "improper" Riemann Integrals:

Theorem (Integration by Parts of Improper Riemann Integrals)**.** Let $a \in \mathbb{R}$, $u : [a,\infty) \to \mathbb{R}$ and $v : [a,\infty) \to \mathbb{R}$ are differentiable, $\forall b > a\ u', v' \in R[a,b]$. Additionally, assume that $\int_a^\infty vu'\ dx$ exists in $\mathbb{R}$ and

$$\lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] \text{ exists (in } \mathbb{R}).$$

Then $\int_a^\infty uv'\ dx$ exists in $\mathbb{R}$ and

$$\int_a^\infty uv'\ dx = \lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] - \int_a^\infty vu'\ dx.$$

Proof. By assumption u' exists and so $u \in C[a,b]$. Hence, $u \in R[a,b]$ as an immediate consequence. Since $v' \in R[a,b]$, we can conclude that the product $v'u \in R[a,b]$. By the Ordinary Riemann Integration by Parts, we have that

$$\int_a^b u(x)v'(x)\ dx = [u(b)v(b) - u(a)v(b)] - \int_a^b u'(x)v(x)\ dx.$$

By assumption, the limit as $b \to \infty$ of each term on the right-hand side of the above equation holds. Thus, we have that

$$\lim_{b\to\infty} \int_a^b uv'\ dx \text{ exists.}$$

Hence, we have that

$$\lim_{b\to\infty}\int_a^b u(x)v'(x)\ dx = \lim_{b\to\infty}\left[(u(b)v(b)-u(a)v(a)) - \int_a^b u'(x)v(x)\ dx\right]$$
$$= \lim_{b\to\infty}[u(b)v(b)-u(a)v(a)] - \lim_{b\to\infty}\int_a^b u'(x)v(x)\ dx \qquad \text{(ALT for Functions)}$$

which can be re-written into

$$\int_a^\infty u'v\ dx = \lim_{b\to\infty}[u(b)v(b)-u(a)v(a)] - \int_a^\infty uv'\ dx$$

as our desired result. ■

Problem 34. Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a,b]$ for every $b > a$. Let $c > a$. Prove that the improper integral $\int_a^\infty f(x)\ dx$ converges if and only if the improper integral $\int_c^\infty f(x)\ dx$ converges.

Proof. ($\Longrightarrow$) Suppose that $\int_a^\infty f(x)\ dx$ converges. Our goal is to show that

$$\int_c^\infty f(x)\ dx \text{ exists}$$

Without loss of generality, suppose that $b < c$ Suppose $b < c$. Then by the segment addition property of the Riemann Integral, we have that

$$\int_a^c f = \int_a^b f + \int_b^c f = \int_a^b f - \int_c^b f.$$

Hence, we have

$$\int_c^b f = \int_a^b f - \int_a^c f.$$

By the Algebraic Limit Theorem for Functions, we have that

$$\int_c^\infty f = \lim_{b\to\infty}\int_c^b f = \lim_{b\to\infty}\left[\int_a^b f - \int_a^c f\right]$$
$$= \lim_{b\to\infty}\int_a^b f - \lim_{b\to\infty}\int_a^c f \qquad \text{(ALT for Functions)}$$
$$= \int_a^\infty f - \int_a^c f.$$

Hence, we see that $\int_c^\infty f$ exists. ($\Longleftarrow$) Suppose $\int_c^\infty f(x)\ dx$ converges. Then by the segment addition property of integration, we have that

$$\int_a^x f(t)\ dt = \int_a^c f(t)\ dt + \int_c^x f(t)\ dt.$$

Then applying the limit as $x \to \infty$ on both sides, we have

$$\lim_{x\to\infty}\int_a^x f(t)\ dt = \lim_{x\to\infty}\left[\int_a^c f(t)\ dt + \int_c^x f(t)\ dt\right]$$
$$= \lim_{x\to\infty}\int_a^c f(t)\ dt + \lim_{x\to\infty}\int_c^x f(t)\ dt$$
$$= \int_a^c f(t)\ dt + \int_c^\infty f(t)\ dt$$

Note that the first term is just a constant so the limit always exists and the second exists by assumption. Hence, we can conclude that

$$\int_a^\infty f(t)\ dt \text{ converges.}$$

■

Problem 35. Let $a > 0$. Prove that $\displaystyle\int_a^\infty \frac{1}{x^p}\ dx \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}.$

Proof. We will consider three cases:

(i) ($p = 1$) Consider the integral $\int_a^t \frac{1}{x}\ dx$. Since $(\ln x)' = \frac{1}{x}$ on $[a, t]$ for any $t > 0$, we can see by FTC I that

$$\int_a^t \frac{1}{x}\ dx = \ln(t) - \ln(a).$$

Since $\lim_{t\to\infty} \ln(t) = \infty$, it follows that when we apply the limit as $t \to \infty$ to the above equation that $\int_a^\infty \frac{1}{x}\ dx$ diverges.

(ii) ($p > 1$) Consider the integral $\int_a^t \frac{1}{x^p}\ dx$. Note that

$$\frac{d}{dx}\Big[\frac{1}{1-p}x^{1-p}\Big] = \frac{1}{x^p}$$

for all $x \in [a, t]$ where $t > a$. Then by FTC I, we can see that

$$\int_a^t \frac{1}{x^p}\ dx = \frac{1}{1-p}t^{1-p} + \frac{1}{1-p}a^{t-p}. \tag{*}$$

Since $1-p < 0$ by assumption, we can see that $\lim_{t\to\infty} t^{1-p} = 0$ by the fact given to us and so applying the limit as $t \to \infty$ to (*), we can see that

$$\int_a^\infty \frac{1}{x^p}\ dx = 0 - \frac{1}{1-p}a^{t-p} = -\frac{1}{1-p}a^{t-p}.$$

by the Algebraic Limit Theorem.

(iii) ($p < 1$) From the fact that was given to us, we can see that $1 - p > 0$ implies $\lim_{t\to\infty} t^{1-p} = \infty$. Hence, from the equality in (*), we can see that $\int_a^\infty \frac{1}{x^p}\ dx$ diverges.

■

Problem 36 (Cauchy Criterion For Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\displaystyle\int_a^\infty f(x)\ dx$ converges if and only if

$$\forall \varepsilon > 0\ \exists M > a \text{ such that } \forall A, B > M\quad \Big|\int_A^B f(x)\ dx\Big| < \varepsilon.$$

Proof. ($\Longrightarrow$) Suppose $\int_a^\infty f(x)\ dx$ converges. Our goal is to show that for all $\varepsilon > 0$, there exists $M > a$ such that for all $A, B > M$

$$\Big|\int_A^B f(x)\ dx\Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Without loss of generality, assume that $A < B$. Since $\int_a^\infty f(x)\ dx$ converges, it follows from our given ε that there exists an $M_1 > B$ such that for any $t > M_1$, we have

$$\Big|\int_B^t f(x)\ dx - L\Big| < \frac{\varepsilon}{2} \tag{1}$$

Similarly, there exists an $M_2 > A$ such that for any $t > M_2$, we have

$$\Big|\int_A^t f(x)\ dx - L\Big| < \frac{\varepsilon}{2}. \tag{2}$$

Using the segment addition property of integration, we have

$$\begin{aligned}\int_A^B f(x)\ dx &= \int_A^t f(x)\ dx + \int_t^B f(x)\ dx \\ &= \int_A^t f(x)\ dx - \int_B^t f(x)\ dx.\end{aligned}$$

Let $M = \max\{M_1, M_2\}$. Then we have for any $B > A > M$, (1) and (2) imply

$$\begin{aligned}\Big|\int_A^B f(x)\ dx\Big| &= \Big|\int_A^t f(x)\ dx - \int_B^t f(x)\ dx\Big| \\ &\leq \Big|\int_A^t f(x)\ dx - L\Big| + \Big|L - \int_B^t f(x)\ dx\Big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

which is our desired result.

($\Longleftarrow$) Suppose that for any $\varepsilon > 0$, there exists $M > a$ such that for all $A, B > M$,

$$\Big|\int_A^B f(x)\ dx\Big| < \varepsilon.$$

Consider the sequence (c_n) in $\mathbb{R}$ defined by

$$c_n = \int_a^n f(x)\ dx.$$

First, we will show that c_n converges to some $L \in \mathbb{R}$. To this end, we will show that c_n is a Cauchy sequence. We claim that the same M can be used to do this. Then for any $n > m > M$, we have

$$\begin{aligned}|c_n - c_m| &= \Big|\int_a^n f(x)\ dx - \int_a^m f(x)\ dx\Big| \\ &= \Big|-\int_n^a f(x)\ dx - \int_a^m f(x)\ dx\Big| \\ &= \Big|\int_n^m f(x)\ dx\Big| \\ &< \varepsilon.\end{aligned}$$

Hence, we see that c_n is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $c_n \to L$ for some $L \in \mathbb{R}$. As a consequence, we need to show that

$$\lim_{x\to\infty} \int_a^x f(t)\ dt = L.$$

Our goal is to show that for any $\varepsilon > 0$, there exists an $M > a$ such that for any $x > M$, we have

$$\Big|\int_a^x f(t)\ dt - L\Big| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Observe by segment addition property of the integral, we can see that

$$\int_a^x f(t)\ dt = \int_a^{\lfloor x\rfloor} f(t)\ dt + \int_{\lfloor x\rfloor}^x f(t)\ dt.$$

Since $\lim\limits_{n\to\infty} \int_a^n f(x)\ dx = L$, we know there exists an $M_1 > a$ (with our given ε) such that for any $n > M_1$, we have that

$$\Big|\int_a^n f(t)\ dt - L\Big| < \frac{\varepsilon}{2}.$$

In particular, since $\lfloor x\rfloor \in \mathbb{N}$ and $\lfloor x\rfloor > M_1$, we have that

$$\Big|\int_a^{\lfloor x\rfloor} f(t)\ dt - L\Big| < \frac{\varepsilon}{2}. \tag{1}$$

By assumption, there also exists an $M_2 > a$ such that for any $A > B > M_2$, we have that

$$\Big| \int_A^B f(t)\ dt \Big| < \frac{\varepsilon}{2}.$$

In particular, since $x > \lfloor x \rfloor > M_2$, we have that

$$\Big| \int_{\lfloor x \rfloor}^x f(t)\ dt \Big| < \frac{\varepsilon}{2}. \tag{2}$$

Now, let $M = \max\{M_1, M_2\}$. Then using (1) and (2), we have that for any $x > M$,

$$\begin{aligned}
\Big| \int_a^x f(t)\ dt - L \Big| &= \Big| \int_a^{\lfloor x \rfloor} f(t)\ dt + \int_{\lfloor x \rfloor}^x f(t)\ dt - L \Big| \\
&\leq \Big| \int_a^{\lfloor x \rfloor} f(t)\ dt - L \Big| + \Big| \int_{\lfloor x \rfloor}^x f(t)\ dt - L \Big| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we conclude that

$$\lim_{x \to \infty} \int_a^x f(t)\ dt = L.$$

■

Problem 37 (Absolute Convergence Implies Convergence)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\int_a^\infty |f(x)|\ dx$ converges, then the improper integral $\int_a^\infty f(x)\ dx$ also converges.

Proof. Suppose $f \in R[a, b]$ for every $b > a$. Our goal is to show that for all $\varepsilon > 0$ such that there exists $M > a$ such that for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\int_a^\infty |f(x)|\ dx$ converges, there exists an $\hat{M} > a$ such that for all $A, B > \hat{M}$

$$\Big| \int_A^B |f(x)|\ dx \Big| < \varepsilon.$$

We claim that $\hat{M}$ can be used as the same M we were looking for. Let $M = \hat{M}$. Indeed, we have for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| \leq \int_A^B |f(x)|\ dx = \Big| \int_A^B |f(x)|\ dx \Big| < \varepsilon \tag{*}$$

by the triangle inequality for integrals. Note that the second equality holds because $|f(x)| \geq 0$ and that

$$\int_A^B |f(x)|\ dx \geq 0$$

by another theorem proven in class. Hence, we see that (*) is our desired result. ■

Problem 38 (Comparison Test For Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Prove that the improper integral $\int_a^\infty g(x)\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.

Proof. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges given that $\int_a^\infty g(x)\ dx$ converges; that is, we want to show that for all $\varepsilon > 0$, there exists an $M > a$ such that for all $A, B > M$, we have

$$\Big|\int_A^B f(x)\ dx\Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Then $\int_a^\infty g(x)\ dx$ converges implies that there exists an $\hat{M} > a$ such that for any $A, B > \hat{M}$, we have

$$\Big|\int_A^B g(x)\ dx\Big| < \varepsilon.$$

Also, there exists a $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $M = \max\{k, \hat{M}\} + 1$. Note that with this constructed M, we have, by the order property of the integral that

$$0 \leq \int_A^B f(x)\ dx \leq \int_A^B g(x)\ dx.$$

Then for any $A, B > M$, we have that

$$\Big|\int_A^B f(x)\ dx\Big| = \int_A^B f(x)\ dx \leq \int_A^B g(x)\ dx = \Big|\int_A^B g(x)\ dx\Big| < \varepsilon$$

which is our desired result. ■

Problem 39 (Limit Comparison Test for Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $L = \lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$. Prove that

(i) If $0 < L < \infty$, then $\int_a^\infty f(x)\ dx$ converges if and only if $\int_a^b g(x)\ dx$ converges.

(ii) If $L = \infty$ and $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$ converges.

(iii) If $L = 0$ and $\int_a^\infty g(x)\ dx$ converges, then $\int_a^\infty f(x)\ dx$ converges.

Proof. Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$, we know that for all $\varepsilon > 0$, there exists $\hat{M} > 0$ such that for any $x \geq M$, we have

$$\Big|\frac{f(x)}{g(x)} - L\Big| < \varepsilon$$

which can be further written as

$$-\varepsilon < \frac{f(x)}{g(x)} - L < \varepsilon$$

and so

$$L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon. \tag{*}$$

If $L = 0$, then the above inequality can be re-written as

$$-\varepsilon < \frac{f(x)}{g(x)} < \varepsilon. \tag{**}$$

If $L = \infty$, then for all $C > 0$, there exists $\kappa > 0$ such that for all $x > \kappa$, we have

$$\frac{f(x)}{g(x)} > C. \tag{***}$$

(i) Suppose $0 < L < \infty$. ($\Longrightarrow$) Our goal is to show that $\int_a^\infty g(x)\ dx$ converges given that $\int_a^\infty f(x)\ dx$ converges. From the left-hand side of (*), we can see that for all $x > \hat{M}$, we have

$$g(x) < \frac{1}{L - \varepsilon} f(x).$$

In particular, if $\varepsilon = \frac{L}{2}$, then we have

$$g(x) < \frac{2}{L} f(x).$$

Since there exists a $K \in \mathbb{R}$ such that for all $x \geq K$, $f(x) \geq 0$ and $g(x) \geq 0$. So, if we take $M = \max\{\hat{M}, k\}$, then for any $x > M > a$, we have

$$0 < g(x) < \frac{2}{L} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, we can see that $\int_a^\infty \frac{2}{L} f(x)\ dx$ converges. By the Comparison Test for Integrals, we have $\int_a^\infty g(x)\ dx$ converges.

($\Longleftarrow$) Suppose $\int_a^\infty g(x)\ dx$ converges. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges. From the right-hand side of (*), we can see that for any $x \geq \hat{M}$, we have

$$f(x) < (L + \varepsilon) g(x).$$

In particular, if $\varepsilon = L$, then

$$f(x) < 2Lg(x).$$

Since there exists $K \in \mathbb{R}$ such that $f(x) \geq 0$ and $g(x) \geq 0$, define $M = \max\{K, \hat{M}\}$. Then for any $x \geq M > a$, we have

$$0 < f(x) < 2Lg(x).$$

Now, since $\int_a^\infty g(x)\ dx$ converges, it follows that $\int_a^\infty 2Lg(x)\ dx$ converges. Hence, the Comparison Test implies that $\int_a^\infty f(x)\ dx$ converges.

(ii) Our goal is to show that if $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$ converges. From (**), we can see that

$$g(x) < \frac{1}{C} f(x).$$

Define $\tilde{M} = \max\{\kappa, \hat{M}\}$. Then we have for any $x > \tilde{M}$

$$0 < g(x) < \frac{1}{C} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, also have that $\int_a^\infty \frac{1}{C} f(x)\ dx$ converge. By the Comparison Test, we can see that $\int_a^\infty g(x)\ dx$ converges.

(iii) Suppose $L = 0$. Then by (**), we can see that for any $x > \hat{M}$ that

$$f(x) < \varepsilon g(x).$$

In this case, if we let $\varepsilon = 1$, we have that

$$f(x) < g(x)$$

for all $x > \hat{M}$. Since there exists $K \in \mathbb{R}$ such that for any $x \geq K$, we have $f(x) \geq 0$ and $g(x) \geq 0$. Note that in this case, we require that $g(x) > 0$. Otherwise, the ratio above will not be defined. So, using the same M from part (i), we have that for any $x > M$,

$$0 < f(x) < g(x).$$

Since $\int_a^\infty g(x)\ dx$ converges, it follows from the Comparison Test that $\int_a^\infty f(x)\ dx$ also converges.

■

Problem 40. Determine all values α and β for which $\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx$ is convergent.

Proof. We will consider three cases.

(i) ($\alpha = 1$) Consider the improper integral below

$$\int_2^\infty \frac{1}{x(\ln x)^\beta}\ dx. \tag{*}$$

Using a change of variables (using $u = \ln x$ and so $u' = \frac{1}{x}$), we have that

$$\int_2^\infty \frac{1}{x(\ln x)^\beta}\ dx = \int_2^\infty \frac{1}{u^\beta}\ du.$$

From Exercise 10, we can see that the improper integral above converges if $\beta > 1$ and diverges if $\beta \le 1$. In this case, if $\alpha = 1$ and $\beta > 1$, then the improper integral in (*) converges.

(ii) ($\alpha < 1$) Choose $p \in (1, \alpha)$. Using Exercise 10 again, we have that

$$\int_2^\infty \frac{1}{x^p}\ dx = \lim_{t\to\infty} \int_2^t \frac{1}{x^p}\ dx$$

converges if $p > 1$ and diverges if $p \le 1$. Define the function $f : [2, t] \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^\alpha (\ln x)^\beta}$$

and $g : [2, t] \to \mathbb{R}$ by

$$g(x) = \frac{1}{x^p}.$$

Since $p - \alpha < 0$, it follows from the fact given to us that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{p-\alpha}}{(\ln x)^\beta} = 0.$$

By the Limit Comparison Test, since $\int_2^\infty g(x)\ dx$ converges, we can conclude that $\int_2^\infty f(x)\ dx$ converges. That is, if $\alpha > 1$, then regardless of the value of β, the improper integral

$$\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx$$

converges.

(iii) ($\alpha > 1$) Let $p \in (\alpha, 1)$. Since $p > 1$, it follows that $\int_2^\infty g(x)\ dx$ diverges. Since $p - \alpha > 0$, we see that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{p-\alpha}}{(\ln x)^\beta} = \infty$$

By the Limit Comparison Test, we can conclude that

$$\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx$$

diverges.

■

Problem 41. (a) Prove that $\int_0^\infty \frac{\sin x}{(1+x)^2}\ dx$ is absolutely convergent.

(b) Use the result of Exercise 8 to prove that

$$\int_0^\infty \frac{\cos x}{1+x}\ dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\ dx.$$

Proof. (a) Our goal is to show that $\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ is absolutely convergent; that is, we need to show that $\int_a^\infty |\frac{\sin x}{(1+x)^2}|\,dx$ converges. Please note that every $b > 0$

$$\Big|\frac{\sin x}{(1+x)^2}\Big| = \frac{|\sin x|}{(1+x)^2} \in C[0,b]$$

and so it must be contained in $R[0,b]$. Furthermore, for all $x \in [1,\infty)$

$$\frac{|\sin x|}{(1+x)^2} \leq \frac{1}{(1+x)^2} \leq \frac{1}{x^2} \tag{*}$$

Since $p = 2 > 1$, we have that the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges}$$

by exercise 10. Using the comparison on the inequality on (*), we conclude that

$$\int_1^\infty \Big|\frac{\sin x}{(1+x)^2}\Big|\,dx \text{ converges}$$

and so the improper integral

$$\int_1^\infty \frac{\sin x}{(1+x)^2}\,dx$$

converges absolutely.

(b) By part (a), we see that

$$\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx \tag{1}$$

converges and that

$$\lim_{b\to\infty}\Big[\frac{-\sin x}{1+x}\Big]_0^b = \frac{\sin(0)}{1+0} - \lim_{b\to\infty}\frac{\sin b}{1+b} = 0 - 0 = 0. \tag{2}$$

From (1) and (2), we can use Exercise 8 to write

$$\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx = \lim_{b\to\infty}\Big[\frac{-\sin x}{1+x}\Big]_0^b - \int_0^\infty \frac{-\cos x}{1+x}\,dx$$

which implies that

$$\begin{aligned}\int_0^\infty \frac{-\cos x}{1+x}\,dx &= \lim_{b\to\infty}\Big[\frac{-\sin x}{1+x}\Big]_0^b - \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx \\ &= -\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.\end{aligned}$$

Hence, we see that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

∎

Homework 5

Problem 42. Let A be a nonempty set of $\mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, $f_n : A \to \mathbb{R}$ is a *uniformly continuous* function on A. Prove that if (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is *uniformly continuous* on A.

Proof. Suppose that $f_n \to f : A \to \mathbb{R}$ uniformly. Our goal is to show that f is uniformly continuous; that is, we want to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x, y \in A$. Since $f_n \to f$ uniformly, there exists an $N \in \mathbb{N}$ such that for any $x \in A$ and for any $n > N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}. \tag{1}$$

Since f_n is uniformly continuous on A for all $n \in \mathbb{N}$. In particular, f_n is uniformly if $n = N + 1$; that is, there exists a$\hat{\delta} > 0$ such that for any $|x - y| < \hat{\delta}$, we have

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}. \tag{2}$$

We claim that $\hat{\delta}$ can be used as the same δ we were looking for. Indeed, whenever $|x - y| < \hat{\delta}$, (1) and (2) imply that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

as our desired result. ■

Problem 43. Let A be a nonempty set and suppose $(f_n : A \to \mathbb{R})_{n \geq 1}$ is a sequence of functions. Suppose $f : A \to \mathbb{R}$ is a function. Prove that the following statement are equivalent:

(i) (f_n) converges uniformly to $f : A \to \mathbb{R}$.

(ii) $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ $\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon$.

(iii) $\lim\limits_{n \to \infty} (\sup_{x \in A} |f_n(x) - f(x)|) = 0$.

Proof. ($(i) \implies (ii)$) Suppose that $f_n \to f$ uniformly. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$

$$\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly, there exists an $\tilde{N} \in \mathbb{N}$ such that for all $x \in A$, for all $n > \tilde{N}$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Note that $\varepsilon/2$ is an upper bound for the set

$$\{|f_n(x) - f(x)| : \forall x \in A \ \ \forall n > \tilde{N}\}.$$

We claim that $\tilde{N}$ is the same N we were looking for. Taking the supremum of the inequality above, we have

$$\sup_{x \in A} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

for any $n > \tilde{N}$ which is our desired result.

($(ii) \implies (iii)$) Suppose that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon.$$

Our goal is to show that $\lim\limits_{n \to \infty} \left(\sup_{x \in A} |f_n(x) - f(x)| \right) = 0$. By assumption, we can let $\varepsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$ such that there exists an $\kappa_n \in \mathbb{N}$ such that for any $n > \kappa_n$, we have k

$$0 \leq \sup_{x \in A} |f_n(x) - f(x)| < \frac{1}{n}.$$

Clearly, we see that $\frac{1}{n} \to 0$ as $n \to \infty$. Applying the squeeze theorem to the inequality above as $n \to \infty$, we have that

$$\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

($(iii) \implies (i)$) Suppose that $\lim\limits_{n\to\infty} \Big(\sup_{x\in A} |f_n(x) - f(x)| \Big) = 0$. Our goal is to show that $f_n \to f : A \to \mathbb{R}$ uniformly; that is, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $x \in A$, for any $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \Big(\sup_{x\in A} |f_n(x) - f(x)| \Big) = 0$, (with our given ε) there exists an $\tilde{N} \in \mathbb{N}$ such that for any $n > \hat{N}$

$$|\sup_{x\in A} |f_n - f(x)|| < \varepsilon;$$

that is, for any $n > \hat{N}$

$$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon.$$

Note that $|f_n(x) - f(x)| \leq \sup_{x\in A} |f_n(x) - f(x)|$ for all $x \in A$. We claim that $\hat{N}$ is the same N we were looking for. Hence, for any $n > \hat{N}$, we have

$$|f_n(x) - f(x)| < \varepsilon$$

Hence, $f_n \to f$ uniformly. ■

Problem 44. Suppose (a_n) and (b_n) are two sequences of real numbers and $a_n \geq b_n$ for all $n \in \mathbb{N}$. Suppose $\lim\limits_{n\to\infty} b_n > 0$. Explain in one line why it follows from the order limit theorem that $\lim\limits_{n\to\infty} a_n$ cannot be zero.

Proof. There exists $n_o \in \mathbb{N}$ such that $a_n \geq b_n$ for all $n \geq n_0$, $\lim\limits_{n\to\infty} a_n \geq \lim\limits_{n\to\infty} b_n > 0$. ■

Problem 45 (4-1). For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^2}{n^4+x^2}$.

Proof. Clearly, f_n converges to f pointwise. Indeed, for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{x^2}{n^4 + x^2} \leq \frac{x^2}{n^4}.$$

Consider the right-hand side of the above inequality, we have $\lim\limits_{n\to\infty} \frac{x^2}{n^4} = x^2 \lim\limits_{n\to\infty} \frac{1}{n^4} = x^2 \cdot 0 = 0$. By applying the Squeeze theorem as $n \to \infty$ to the inequality above, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{n^4 + x^2} = 0$$

pointwise.

Now, we want to show that $f_n \to 0$ is NOT uniform. Immediately, we see that

$$\sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^4 + x^2} \right| \geq |f_n(x)|$$

for all $x \in \mathbb{R}$. In particular, if we let $x = n^2$, then we have

$$\sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^4 + x^2} \right| \geq |f_n(n^2)| = \frac{1}{2}.$$

Clearly, if we define $b_n = f(n^2)$, we have

$$\lim_{n\to\infty} f(n^2) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} > 0.$$

■

Problem 46 (4-2)**.** For each $n \in \mathbb{N}$, let $f_n : [0, 1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$. It is easy to show that $f_n \to 0$ pointwise. Prove that the convergence is NOT uniform.

Proof. We can easily show that $f_n \to 0$ pointwise. We will show that the convergence is NOT uniform. Define $b_n = \left(1 - \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Indeed, we see that

$$\sup_{x\in[0,1)} |f_n(x)| = \sup_{x\in[0,1)} |x^n| = \sup_{x\in[0,1)} \geq \left(1 - \frac{1}{n}\right)^n \tag{1}$$

and

$$\lim_{n\to\infty} \left(1 - \frac{1}{kn}\right)^n = \frac{1}{\varepsilon} > 0. \tag{2}$$

Thus, (1) and (2) imply that $f_n \to 0$ does NOT converge uniformly. ■

Problem 47. Suppose that $A = G \cup H$ where G and H are nonempty sets. Prove that if (f_n) converges uniformly to f on both G and H, then (f_n) converges uniformly to f on A.

Proof. Our goal is to show that for any $\varepsilon > 0$ be given and for any $x \in A$, for any $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $A = G \cup H$, we either have $x \in G$ or $x \in H$. If $x \in G$, then we can use the fact that $f_n \to f$ uniformly on G, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

On the other hand, if $x \in H$, then using the fact that $f_n \to f$ uniformly on H, there exists an $N_2 \in \mathbb{N}$, with our given ε, such that for any $n > N_2$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Hence, in both cases $f_n \to f$ on A uniformly. ■

Problem 48. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and suppose that $a_n \to a$ in $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. For each $n \in \mathbb{N}$,define $f_n : A \to \mathbb{R}$ by $f_n(x) = f(x) + a_n$. Prove that (f_n) converges uniformly to the function $f + a$ on the set A.

Proof. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in A$ and for all $n > N$, we have

$$|f_n(x) - (f(x) + a)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $a_n \to a$, there exists $\tilde{N} \in \mathbb{N}$ such that for any $n > \tilde{N}$, we have

$$|a_n - a| < \varepsilon.$$

We claim that $\tilde{N}$ is the same N we were looking for. Hence, for any $n > \tilde{N}$, we have

$$\begin{aligned} |f_n - (f(x) + a)| &= |(f(x) + a_n) - (f(x) + a)| \\ &= |a_n - a| \\ &< \varepsilon. \end{aligned}$$

Hence, $f_n \to f + a$ uniformly. ■

Problem 49. Suppose that (g_k) converges uniformly to g on the nonempty st A. Use the Cauchy Criterion for uniform convergence of sequences to prove that the sequence (h_k) where $h_k = g_{k+1} - g_k$ converges uniformly to zero on A.

Proof. Our goal is to show that $h_k \to 0$ uniformly on A; that is, we want to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $k > N$, we have

$$|h_k| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $g_n \to g$ converges uniformly on A, the Cauchy Criterion implies that there exists an $\tilde{N} \in \mathbb{N}$ such that for any $m > n > \tilde{N}$, we have

$$|g_n(x) - g_m(x)| < \varepsilon. \tag{*}$$

We claim that $\tilde{N}$ can be used as the same N we were looking for. Indeed, for any $k+1 > k > \tilde{N}$, (*) implies that

$$|g_{k+1}(x) - g_k(x)| < \varepsilon.$$

But we have $h_k = g_{k+1} - g_k$ for all $k \in \mathbb{N}$, we have

$$|h_k| = |g_{k+1} - g_k| < \varepsilon$$

as desired. ■

Problem 50. Complete the following the proof presented.

Proof. Because $|\sin(n_{\hat{i}} x_0) - \sin(n_{N+1} x_0)| \geq 1$ and $b_n = \sin(n_{\hat{i}} x_0) > 0$ for all $n \in \mathbb{N}$, we have found a subsequence of $(\sin(nx))_{n \geq 1}$ that does not converge uniformly on the interval $[0, 2\pi]$ which is a contradiction. ■

Problem 51. For all $n \geq 1$ define $f_n : [0,1] \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} 1 & \text{if } n!x \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $f_n \to f$ pointwise where $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{I} \\ 1 & \text{if} x \in \mathbb{Q}. \end{cases}$$

Proof. (1) If $x \in [0,1]$ is irrational, then $f_n(x) = 0$. Clearly, we see that $f_n \to f$.

(2) If $x \in [0,1]$ is a rational number, then $c = \frac{p}{q}$ for some nonnegative $p, q \in \mathbb{Z}$. Then for all $n > q$, $n!x \in \mathbb{Z}$. Hence, for all $n > q$ $f_n(c) = 1$. From here, it immediately follows that $f_n \to f$.

Hence, (1) and (2) imply that $f_n \to f$ pointwise. ■

Problem 52 (i). For all $n \geq 1$ define $f_n : [0, \infty) \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } 0 \leq x \leq n \\ 0 & \text{if} x > n \end{cases}$$

Prove that $f_n \to f$ uniformly where $f : [0, \infty) \to \mathbb{R}$ is defined by $f \equiv 0$.

Proof. If $x > n$, $f(x) = 0$. Clearly, $f_n \to 0$ uniformly. Otherwise, assume that $0 \leq x \leq n$. Then $f_n(x) = \frac{1}{n}$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Hence, for any $n > N$, we have

$$|f_n(x) - 0| = |f_n(x)| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Hence, $f_n \to 0$ uniformly. ■

Problem 53 (ii)**.** Show that $\lim_{n\to\infty} \int_0^\infty f_n\ dx \neq \int_0^\infty f\ dx$.

Proof. Note that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty f_n(x)\ dx &= \lim_{n\to\infty}\int_0^\infty \frac{1}{n}\ dx\\ &= \lim_{n\to\infty}\left[\lim_{R\to\infty}\int_0^R \frac{1}{n}\ dx\right]\\ &= \lim_{n\to\infty}\left[\lim_{R\to\infty}\frac{R}{n}\right] = \infty.\end{aligned}$$

On the other hand, we have

$$\int_0^\infty f\ dx = \lim_{R\to\infty}\left[\int_0^R (0)\ dx\right] = \lim_{R\to\infty}(0) = 0.$$

Clearly, we have that

$$\lim_{n\to\infty}\int_0^\infty f_n\ dx \neq \int_0^\infty f\ dx.$$

■

Problem 54 ((i))**.** For all $n \geq 1$ define $f_n : [-1,1] \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+n^2x^2}$. Prove that f_n converges uniformly to $f : [-1,1] \to \mathbb{R}$ defined by $f \equiv 0$.

Proof. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in [-1,1]$ and for all $n > N$, we have

$$|f_n(x) - 0| < \varepsilon.$$

Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2N} < \frac{1}{N} < \varepsilon.$$

Then from our hint, we can see that for any $n > N$, we have

$$\begin{aligned}|f_n(x) - 0| &= \left|\frac{x}{1+n^2x^2}\right|\\ &= \frac{|x|}{1+n^2x^2}\\ &\leq \frac{|x|}{2n|x|}\\ &= \frac{1}{2n}\\ &< \frac{1}{2N}\\ &< \varepsilon.\end{aligned}$$

hence, we can see that $f_n \to f$ uniformly on $[-1,1]$. ■

Problem 55. Prove that f_n' converges pointwise to $g : [-1,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 0\\ 0 & \text{if } 0 < |x| \leq 1\end{cases}.$$

Proof. Since each f_n is differentiable, we have that

$$\begin{aligned} f_n'(x) &= \frac{1}{1+n^2x^2} - \frac{x}{(1+n^2x^2)^2} \cdot 2n^2x \\ &= \frac{(1+n^2x^2) - 2n^2x^2}{(1+n^2x^2)^2}. \end{aligned}$$

Hence, we have

$$f_n'(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}$$

and note that

$$f_n'(0) = \frac{1}{1} = 1.$$

Clearly, if $x = 0$, then $f_n'(0) \to g(0)$. Otherwise, suppose $0 < |x| \leq 1$. Then we have

$$\begin{aligned} |f_n'(x) - 0| &= |f_n'(x)| \\ &= \left|\frac{1-n^2x^2}{(1+n^2x^2)^2}\right| \\ &= \frac{|1-n^2x^2|}{(1+n^2x^2)^2} \\ &\leq \frac{2(1-n|x|)}{4n^2|x|^2} \\ &= \frac{1}{n^2|x|^2} - \frac{1}{2n|x|} \\ \underbrace{\to}_{\text{ALT}} 0 + 0 = 0. \end{aligned}$$

Using the Squeeze Theorem, we have that as $n \to \infty$, we have

$$|f_n'(x)| \to 0.$$

Clearly, the convergence above depends on x. Thus, the $f_n' \to 0$ pointwise. ∎

Problem 56 (iii). Does f_n' converge uniformly to g.

Solution. No, because the pointwise limit of f_n' in part (ii) is NOT a continuous function. ∎

Problem 57. Prove the following theorem.

Theorem. Assume that for each $n \in \mathbb{N}$, $f_n : [a,b] \to \mathbb{R}$ is differentiable, there exists $x_0 \in [a,b]$ such that $(f_n(x_0))_{n\geq 1}$ converges, and (f_n') converges uniformly on $[a,b]$. Then (f_n) converges uniformly on $[a,b]$.

Proof. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$ and $\forall x \in [a,b]$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Without loss of generality $x_0 < x$. By the Mean Value Theorem, there exists an $\hat{x} \in (x_0, x)$ such that

$$f_n'(\hat{x}) = \frac{f_n(x) - f_n(x_0)}{x - x_0} \Longrightarrow f_n(x) - f_n(x_0) = f_n'(\hat{x})(x - x_0)$$

and similarly, we have

$$f_m'(\hat{x}) = \frac{f_m(x) - f_m(x_0)}{x - x_0} \Longrightarrow f_m(x) - f_m(x_0) = f_m'(\hat{x})(x - x_0).$$

Subtracting these two quantities gives us

$$\begin{aligned}
&f_n(x) - f_n(x_0) - (f_m(x) - f_m(x_0)) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) \\
\implies &f_n(x) - f_m(x)q - (f_n(x_0) - f_m(x_0)) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) \\
\implies &f_n(x) - f_m(x) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) + f_n(x_0) - f_m(x_0).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |(f_n'(\hat{x})) - f_m'(\hat{x})(x - x_0) + f_n(x_0) - f_m(x_0)| \\
&\leq |f_n'(\hat{x}) - f_m'(\hat{x})||x - x_0| + |f_n(x_0) - f_m(x_0)|.
\end{aligned}$$

Since each f_n is differentiable on $[a, b]$, we know that each f_n is continuous on $[a, b]$. Therefore, f_n is continuous at $x_0 \in [a, b]$. That is, there exists a $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|f_n(x) - f_n(x_0)| < \varepsilon.$$

Furthermore, we can see by our assumption that if $(f_n(x_0))_{n\geq 1}$ converges, we have that $(f_n(x_0))_{n\geq 1}$ is a Cauchy sequence. That is, there exists an $N_1 \in \mathbb{N}$ such that for any $n > m > N_2$, we have

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Since (f_n') converges uniformly, there exists an $N_2 \in \mathbb{N}$ such that for any $n > m > N_1$ (given $\hat{x} \in [a, b]$)

$$|f_n'(\hat{x}) - f_m'(\hat{x})| < \frac{\varepsilon}{2\delta}.$$

Let $N = \max\{N_1, N_2\}$. Then for any $n > m > N$, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n'(\hat{x}) - f_m'(\hat{x})||x - x_0| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\varepsilon}{2 \cdot \delta} \cdot \delta + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, f_n converges uniformly on $[a, b]$ by the Cauchy Criterion. ■

$\int_a^b f(x)\ dx$

Consider the linear operator $T : \ell^\infty \to \ell^\infty$ defined by $T(x) = y$, where $x = (x_j)$ and $y = (y_j)$ and $y_j = \frac{x_j}{j}$.

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\vec{x} = A\vec{x}$. Is it correct that $\|A\| = \sqrt{6}$.

Homework 6

Problem 58. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Our goal is to show that there exists an M such that for all $n \geq 1$ and for all $x \in A$, we have

$$|f_n(x)| \leq M.$$

Since $f_n \to f$ uniformly, we know that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $m, n > N$ and for all $x \in A$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

In particular, if $\varepsilon = 1$, then

$$|f_n(x) - f_m(x)| < 1 \iff |f_n(x)| < |f_m(x)| + 1 \quad \forall n, m > N. \tag{*}$$

Since each f_n is bounded, it follows that there exists an R_n such that

$$|f_n(x)| \leq R_n$$

for all $n \in \mathbb{N}$ and for all $x \in A$. Let $R = \max\{R_1, R_2, \ldots, R_m\}$. Then from (*), we can see that

$$|f_n(x)| < |f_m(x)| + 1 \leq R_m + 1 \leq R + 1$$

for any $x \in A$ and for any $n \in \mathbb{N}$ where $M = R + 1$ is the desired M we were looking for. Hence, (f_n) is a uniformly bounded sequence of functions. ■

> **Problem 59.** If (f_n) and (g_n) converge uniformly on a set A, prove that $(f_n + g_n)$ converges uniformly on A. Also,

Proof. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in A$, we have

$$|(f_n + g_n)(x) - (f_m + g_m)(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since (f_n) converges uniformly on A, it follows from the Cauchy Criterion for uniform convergence that, with our given ε, there exists an $N_1 \in \mathbb{N}$ such that for any $n, m > N_1$ and for any $x \in A$, we have

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}. \tag{1}$$

Similarly, the uniform convergence of (g_n) on A implies that there exists an $N_2 \in \mathbb{N}$ such that for any $n, m > N_2$ and for any $x \in A$ that

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2}. \tag{2}$$

Then for any $n, m > \max\{N_1, N_2\} + 1$ and for any $x \in A$, we have

$$\begin{aligned} |(f_n + g_n)(x) - (f_m + g_m)(x)| &\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which is our desired result. ■

> **Problem 60.** If (f_n) and (g_n) are two sequences of bounded functions that converge uniformly on a set A, prove that $(f_n g_n)$ converges uniformly on A.

Proof. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in A$, we have

$$|(f_n g_n)(x) - (f_m g_m)(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. By problem 1, it follows from the uniform convergence of both bounded sequences (f_n) and (g_n) that there exists an $M_1, M_2 > 0$ such that for any $x \in A$ and for any $n \in \mathbb{N}$ that

$$|f_n(x)| \leq M_1 \quad \text{and} \quad |g_n(x)| \leq M_2,$$

respectively. Since (f_n) converges uniformly on A, it follows that there exists an $N_1 \in \mathbb{N}$ such that for any $x \in A$ and for any $n, m > N_1$ that

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2M_2}. \tag{1}$$

Similarly, the uniform convergence of (g_n) implies that there exists an $N_2 \in \mathbb{N}$ such that for any $x \in A$ and for any $n, m > N_2$, we have

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2M_1}. \tag{2}$$

Now, for any $x \in A$ and for any $n, m > \max\{N_1, N_2\} + 1$, we have

$$\begin{aligned}
|(f_n g_n)(x) - (f_m g_m)(x)| &\leq |f_n g_n(x) - f_m g_n(x) + f_m g_n(x) - f_m g_m(x)| \\
&\leq |f_n g_n(x) - f_m(x) g_n(x)| + |f_m(x) g_n(x) - f_m(x) g_m(x)| \\
&= |g_n(x)||f_n(x) - f_m(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
&\leq M_2 |f_n(x) - f_m 9x| + M_1 |g_n(x) - g_m(x)| \\
&< M_2 \cdot \frac{\varepsilon}{2M_2} + M_1 \cdot \frac{\varepsilon}{2M_1} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

■

Problem 61. Consider the sequences $(f_n : \mathbb{R} \to \mathbb{R})$ and $(g_n : \mathbb{R} \to \mathbb{R})$ defined as follows:

$$f_n(x) = 2 + \frac{5}{n} \quad g_n(x) = x + \frac{2}{n}$$

Prove that both (f_n) and (g_n) converge uniformly on the set $\mathbb{R}$, but $(f_n g_n)$ does not converge uniformly on $\mathbb{R}$.

Proof. Clearly, we see that $f_n \to f = 2$ uniformly. Also, it is not difficult to see that $g_n \to g$ where $g(x) = x$ for all $x \in \mathbb{R}$. We will show that this convergence is uniform. Let $\varepsilon > 0$ be given. Choose $N = \frac{5}{\varepsilon}$ and observe that for any $x \in \mathbb{R}$ and for any $n > N$, we have

$$|g_n(x) - g(x)| = \left| \left(x + \frac{5}{n} \right) - x \right| = \frac{5}{n} < \frac{5}{N} = \varepsilon.$$

Hence, we see that $g_n \to g$ uniformly on $\mathbb{R}$. Note that

$$\begin{aligned}
f_n g_n(x) &= \left(2 + \frac{5}{n}\right)\left(x + \frac{2}{n}\right) \\
&= \left(2 + \frac{5}{n}\right)x + \frac{4}{n} + \frac{10}{n^2}.
\end{aligned}$$

Since we see that $f_n g_n \to 2x$, we have

$$f_n g_n(x) - 2x = \frac{5}{n} x + \frac{6}{n}.$$

Define $b_n = f_n g_n(n) - 2n$. Then it follows that

$$b_n = 5 + \frac{6}{n}$$

which implies that $\lim_{n\to\infty} b_n = 5 > 0$ and that

$$\sup_{x \in \mathbb{R}} |f_n g_n(x) - 2x| \geq b_n.$$

Hence, $f_n g_n(x) \to 2x$ for any $x \in \mathbb{R}$ is NOT uniform. ■

Problem 62. Let $A \subseteq (X, d)$. Let $(f_n : A \to \mathbb{R})$ be a sequence of continuous functions which converges uniformly to a function f on the set A. Let (x_n) be a sequence in A such that $x_n \to x \in A$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Proof. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, we have

$$|f_n(x_n) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly where each f_n is a continuous function, we also have that f is a continuous function by a theorem proven in class. Since $x_n \to x \in A$ and f is continuous on A, it follows from the sequential criterion of continuity that $f(x_n) \to f(x)$ on A. With our given ε, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}. \tag{1}$$

Since $f_n \to f$ uniformly, there exists an $N_2 \in \mathbb{N}$ such that for any $n > N_2$ and any $x \in A$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}. \tag{2}$$

Choose $N = \max\{N_1, N_2\} + 1$. Because $x_n \in A$ for all $n \in \mathbb{N}$, it follows that for any $n > N_1$, we have

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$$

from (1). For any $n > N$, it follows that

$$\begin{aligned}
|f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we conclude that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

■

Problem 63. Let $A \subseteq (X, d)$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that if $(f_n : A \to \mathbb{R})_{n\geq 1}$ is a sequence of bounded functions that converges uniformly to $f : A \to \mathbb{R}$, then $(g \circ f_n)_{n\geq 1}$ converges uniformly to $g \circ f$.

Proof. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$, we have

$$|(g \circ f_n)(x) - (g \circ f)(x)| < \varepsilon.$$

That is, we want to find an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$,

$$|g(f_n(x)) - g(f(x))| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Since (f_n) is a sequence of bounded functions that converges uniformly to $f : A \to \mathbb{R}$, we have that (f_n) is uniformly bounded by problem 1. Hence, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in A$. As a consequence, we see that $|f(x)| \leq M$ for all $x \in A$. Consider the compact interval $[-M, M]$ and $g|_{[-M,M]}$. Since g is continuous and $[-M, M]$ is compact, it follows that g is uniformly continuous. Hence, there exists (with our given ε) a $\delta > 0$ such that for all $s, t \in [-M, M]$ whenever $|s - t| < \delta$, we have

$$|g(s) - g(t)| < \varepsilon.$$

Since $f_n \to f$ uniformly, we can find an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$ and for any $x \in A$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

We claim that this $\hat{N}$ can be used as the same N we were looking for. Indeed, if we take $\varepsilon = \delta$, then if $|f_n(x) - f(x)| < \delta$, then (*) will hold for any $n > \hat{N}$ and we are done. ■

Problem 64. For each $n \in \mathbb{N}$, let $f_n : (0,1) \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{nx+1}$.

1. Explain in one line why $f_n \to f$ pointwise where $f \equiv 0$.

   Proof. Note that for all $x \in (0,1)$ and for all $n \in \mathbb{N}$, we have

   $$0 \le \frac{1}{nx+1} \le \frac{1}{nx} \to 0.$$

   Thus, the Squeeze Theorem implies that $\frac{1}{nx+1} \to 0$ pointwise. ■

2. Explain in one line why each f_n is continuous.

   Proof. Since 1 is a constant function and $nx+1$ is a polynomial function which are both continuous function where $nx+1 \neq 0$, it follows from the Algebraic Continuity Theorem that each f_n is continuous. ■

3. Explain why for each $n \in \mathbb{N}$, we have $f_{n+1} \le f_n$.

   Proof. It immediately follows that for all $n \in \mathbb{N}$, $1+nx$ is an increasing function. Define $\hat{f}_n(x) = 1+nx$. Then from our observation $\hat{f}_n \le \hat{f}_{n+1}$ for all $n \in \mathbb{N}$. Dividing we get

   $$\frac{1}{\hat{f}_{n+1}} \le \frac{1}{\hat{f}_n} \implies f_{n+1} \le f_n \quad \forall n \in \mathbb{N}.$$

   Thus, f_n is a decreasing sequence of functions. ■

4. Explain why $f_n \to f$ is NOT uniform.

   Solution. Since f_n is defined over a non-compact interval $(0,1)$, it follows from Dini's Theorem that $f_n \to f$ is NOT uniform. ■

5. Explain why this example does not contradict the following theorem.

   Solution. This does not contradict the theorem because we still have pointwise convergence of $f_n \to f$. ■

Problem 65. (a) Prove that $(f_n : A \to \mathbb{R})_{n\ge1}$ converges uniformly to 0 if and only if $(|f_n|)_{n\ge1}$ converges uniformly to 0.

(b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function and assume that $f(1) = 0$. Prove that $(x^n f(x))$ converges uniformly on $[0,1]$.

Proof. (a) ($\Longrightarrow$) Our goal is to show that $(|f_n|)_{n\ge1}$ converges uniformly to 0. It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $m, n > N$ and for any $x \in A$, we have

$$\big||f_n(x)| - |f_m(x)|\big| < \varepsilon.$$

Let $\varepsilon > 0$. Since $(f_n)_{n\ge1}$ converges uniformly on A, it follows from our given ε that there exists an $\hat{N} \in \mathbb{N}$ such that for any $n, m > \hat{N}$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

We claim that we can use $\hat{N}$ as the N we were looking for. Indeed, we can see that for any $n, m > \hat{N}$ and $x \in A$ that

$$\big||f_n(x)| - |f_m(x)|\big| \le |f_n(x) - f_m(x)| < \varepsilon$$

which is our desired result.

($\Longleftarrow$) Our goal is to show that $(f_n : A \to \mathbb{R})_{n\ge1}$ converges uniformly to 0. It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$, we have

$$|f_n(x)| < \varepsilon.$$

Let $\varepsilon > 0$. Clearly, since $|f_n|$ converges uniformly to 0, then there exists an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$ and $x \in A$,

$$|f_n(x)| < \varepsilon$$

with $\hat{N}$ as the same N we were looking for as desired. Hence, $f_n \to 0$ uniformly.

(b) We will show that f_n defined by $f_n(x) = x^n f(x)$ for all $x \in [0,1]$ converges to 0 uniformly using Dini's theorem. In what follows, we will show that each f_n satisfies the following conditions:

(1) $[0,1]$ is a compact set.

(2) For each $n \in \mathbb{N}$, $f_n : [0,1] \to \mathbb{R}$ is continuous.

(3) $f_n \to 0$ pointwise on K (Clearly, the zero function is continuous).

(4) For each $n \in \mathbb{N}$, we have $f_{n+1} \leq f_n$.

Clearly, (1) is satisfied by the Heine-Borel theorem on $\mathbb{R}$. Also, since f is continuous on $[0,1]$ and x^n is a polynomial function which is clearly continuous on $[0,1]$, it follows from the Algebraic Continuity Theorem that each f_n is a continuous function and so (2) is satisfied. Next, notice that for $x = 0$, $f_n \to 0$ immediately. Similarly, if $x = 1$, then it immediately follows that $f_n \to 0$. On the other hand, if $x \in (0,1)$, then $x^n \to 0$. Using the Algebraic Limit Theorem, it follows that $f(x)x^n \to 0$ for any $x \in (0,1)$. Thus, we see that (3) is satisfied with the pointwise limit being clearly continuous on $[0,1]$. Lastly, we see that for any $x \in [0,1]$, x^n is a decreasing function. Hence, $f_n(x) = f(x)x^n$ is a decreasing function and so (4) is satisfied.

By Dini's Theorem, we can conclude that $f_n \to 0$ uniformly on $[0,1]$.

■

Problem 66. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function, and for each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = f(x + \frac{1}{n})$. Prove that $(f_n)_{n\geq 1}$ converges uniformly to f on $\mathbb{R}$.

Proof. It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in \mathbb{R}$, we have

$$|f_n(x) - f_m(x)| < \varepsilon. \tag{*}$$

That is, we need to find an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in \mathbb{R}$, we have

$$\left|f\left(x + \frac{1}{n}\right) - f\left(x + \frac{1}{m}\right)\right| < \varepsilon.$$

Since f is uniformly continuous, it follows from our given ε that there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$ satisfying $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Notice that $x_n = x + \frac{1}{n}$ converges to $x \in \mathbb{R}$. Hence, (x_n) is a Cauchy sequence in $\mathbb{R}$. Thus, for any $\varepsilon > 0$, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n, m > \hat{N}$, we have

$$|x_n - x_m| < \varepsilon.$$

That is,

$$|x_n - x_m| = \left|\left(x - \frac{1}{n}\right) - \left(x - \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$$

for any $n, m > \hat{N}$. Using $\varepsilon = \delta$, it follows from (*) that whenever $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \delta$,

$$\left|f\left(x + \frac{1}{n}\right) - f\left(x + \frac{1}{m}\right)\right| < \varepsilon \quad \forall n, m > \hat{N}.$$

That is,

$$|f_n(x) - f_m(x)| < \varepsilon$$

for any $n, m > \hat{N}$. Hence, (f_n) converges to f uniformly. ■

Problem 67. For each case, determine whether the given sequence of functions converges pointwise. If it does, determine whether the convergence is uniform.

10-1) $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = n^3 x^n$.

Solution. f_n diverges since $n^3 \to \infty$ as $n \to \infty$. ■

10-2) $f_n : [0, \pi] \to \mathbb{R}$ defined by $f_n(x) = \sin^n(x)$.
Solution. Note that $|\sin x| \leq 1$ for all $x \in [0, \pi]$. Furthermore, f_n is a sequence of continuous functions (since $\sin x$ is a continuous function and x^n is a polynomial which is continuous so their composition is continuous) and decreasing. Also, note that the convergence of f_n to 0 is pointwise and $[0, 1]$ is a compact set in $\mathbb{R}$. Dini's theorem implies that $f_n \to 0$ uniformly. ■

10-3) $f_n : (0, 1) \to \mathbb{R}$ defined by $f_n(x) = 2nxe^{-n^2x^2}$.
Proof. Converges pointwise to 0 but convergence is not uniform since $(0, \infty)$ is not a compact set (By Dini's Theorem). ■

10-4) $f_n : (0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{n^2x}{(nx+1)^3}$.
Proof. Converges pointwise to 0 but convergence is not uniform since $(0, \infty)$ is not a compact set (By Dini's theorem). ■

10-5) $f_n : (0, 1) \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{nx+1}$.
Proof. Converges pointwise to 0 but convergence is not uniform since $(0, 1)$ is not a compact set (By Dini's theorem). ■

10-6) $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{nx^2+1}$.
Proof. Converges pointwise to 0 but convergence is not uniform since $\mathbb{R}$ is not a compact set (By Dini's theorem). ■

10-7) $f_n : [0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{nx}{n^3+x^3}$.
Solution. Note that f_n is a sequence of continuous functions (since it is a ratio of continuous functions) which converges to 0 pointwise, but not uniform since $[0, \infty)$ is not a compact set by Dini's Theorem. ■

10-8) $f_n : [0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{nx^2}{n^3+x^3}$.
Solution. Note that f_n is a sequence of continuous functions (since it is a ratio of continuous functions) which converges to 0 pointwise, but not uniform since $[0, \infty)$ is not a compact set by Dini's Theorem. ■

10-9) $f_n : [0, 1] \to \mathbb{R}$ with $g_n = f_n'$ where $f_n : [0, 1] \to \mathbb{R}$ is defined by $f_n(x) = \frac{\ln(1+nx)}{n}$.
Solution. Note that for all $x \in [0, 1]$, we have

$$g_n(x) = f_n'(x) = \frac{1}{nx + 1}$$

which is a sequence of continuous functions and that converges to 0 pointwise. Since $[0, 1]$ is compact and g_n is a decreasing sequence of functions, it follows from Dini's Theorem that $g_n \to 0$ uniformly. Notice that for $x = 0$, the sequence $f_n(0) \to 0$. Hence, Exercise 12 from homework 5 implies that $f_n \to 0$ uniformly on $[0, 1]$. ■

Problem 68. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

Prove that the Weierstrass M-Test cannot be applied to establish the uniform convergence of $\sum_{n=1}^{\infty} f_n$. Nevertheless, show that this series converges uniformly.

Proof. Note that we cannot use the Weierstrass M-Test because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series which diverges. Hence, we will show via the Cauchy Criterion that $\sum f_n$ converges uniformly on $[0, 1]$.

Clearly, if $x \notin (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges. Suppose $x \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Then for all $n \in \mathbb{N}$, $f_n(x) = \frac{1}{n}$. Our goal is to show that for any $\varepsilon > 0$, there exists an

$N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|\sum_{k=m+1}^{n} f_n(x)| < \varepsilon.$$

Since at most one $f_n(x)$ is nonzero for any x, and $f_n(x) = \frac{1}{n}$, it follows that we can find an N large enough so that $\frac{1}{N} < \varepsilon$. Then for all $x \in [0,1]$ and for all $n, m > N$, we have

$$\Big|\sum_{k=m+1}^{n} f_n(x)\Big| \leq \sup_{n\in\mathbb{N}} \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Hence, f_n converges uniformly on $[0,1]$. ■

Problem 69. Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly on $[0,\infty)$.

Proof. Our goal is to show that the series $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly. It suffices to show that the series above satisfies the Weierstrass M-Test. Indeed, we see that for all $n \in \mathbb{N}$ and for all $x \in [0,\infty]$,

$$\Big|\frac{x}{1+n^4x^2}\Big| \leq \Big|\frac{x}{n^4x^2}\Big| = \Big|\frac{1}{n^4x}\Big| = \frac{1}{|n^4||x|} \leq \frac{1}{n^4}$$

where $M_n = \frac{1}{n^4}$. Clearly, the series $\sum_{n=1}^{\infty} M_n$ converges via the p-series test. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly. ■

Problem 70. (a) Use Taylor's Theorem with Lagrange remainder to prove that for all $x > 0$, we have $e^x > \frac{x^2}{2}$.

(b) Prove that $\sum_{n=1}^{\infty} x^2e^{-nx}$ converges uniformly on $[0,\infty)$.

Proof. (a) Clearly, we can see that e^x is differentiable $n+1$ times and so by Taylor's Theorem with Lagrange Remainder, it follows that

$$e^x = \sum_{k=1}^{n} \frac{x^k}{k!} > \frac{x^2}{2!} = \frac{x^2}{2}$$

for all $x > 0$.

(b) We proceed via the Weierstrass M-Test to prove that $\sum_{n=1}^{\infty} x^2e^{-nx}$ converges uniformly on $[0,\infty)$. From part (a), it follows for all $n \in \mathbb{N}$ that

$$|x^2e^{-nx}| = x^2e^{-nx} < 2e^x \cdot e^{-nx} = 2e^{x(1-n)} = \frac{2}{e^{x(n-1)}} < 2e \cdot \Big(\frac{1}{e}\Big)^n.$$

Note that $|r| = \frac{1}{e} < 1$ and so the series

$$\sum_{n=1}^{\infty} 2e \cdot \Big(\frac{1}{e}\Big)^n.$$

is geometric which converges. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} x^2e^{-nx}$ converges uniformly.

■

Problem 71. Let $a > 0$ be a fixed number. Prove that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a,\infty)$ and does not converge uniformly on $(0,\infty)$.

Proof. Our goal is to show that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a, \infty)$. We will do this via the Weierstrass M-Test. Note that for all $n \in \mathbb{N}$ and for all $x \in [a, \infty)$, we can use a result from homework 10 in Math 230A to write

$$\left|2^n \sin\left(\frac{1}{3^n x}\right)\right| \leq \frac{2^n}{|x|3^n} = \frac{1}{|x|} \cdot \left(\frac{2}{3}\right)^n \leq \frac{1}{a} \cdot \left(\frac{2}{3}\right)^n. \tag{*}$$

Observe that $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent series because it is geometric series ($|r| = \frac{2}{3} < 1$). Thus, the Algebraic Theorem for Series implies that $\sum_{n=1}^{\infty} \frac{1}{a} \cdot (\frac{2}{3})^n$ converges. Using the Weierstrass M-Test, it follows that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a, \infty)$.

Now, let us consider the same series on the interval $(0, \infty)$. From our inequality in (*), we see that the series defined on the following sequence term

$$\frac{1}{|x|} \cdot \left(\frac{2}{3}\right)^n$$

depends on $x \in (0, \infty)$ and is not a constant sequence. Hence, it follows that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ does not converge uniformly via the Weierstrass M-Test. ■

Problem 72. Let $a > 0$ be a fixed number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{nx}{1 + n^4 x^2}$$

converges uniformly on $[a, \infty)$ and does not converge uniformly on $[0, \infty]$.

Proof. Consider $\sum_{n=1}^{\infty} \frac{nx}{1+n^4x^2}$ over the interval $[a, \infty)$ where $a > 0$ is fixed. For all $n \in \mathbb{N}$, it follows that

$$\left|\frac{nx}{1 + n^4 x^2}\right| \leq \frac{n|x|}{|1 + n^4 x^2|} \leq \frac{1}{n^3 |x|} \leq \frac{1}{an^3}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series (via the P-Series Test), we know by the Algebraic Limit Theorem for Series that $\sum_{n=1}^{\infty} \frac{1}{an^3}$ is also a convergent series. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} \frac{nx}{1+n^4x^2}$ converges uniformly on $[a, \infty)$

Now, consider the same series over $[0, \infty)$. Clearly, the series converges if $x = 0$. Performing a similar set of computations, we obtain the following inequality

$$\left|\frac{nx}{1 + n^4 x^2}\right| \leq \frac{n|x|}{|1 + n^4 x^2|} \leq \frac{n|x|}{2n^2|x|} \leq \frac{1}{2n}$$

for any $x \in (0, \infty)$. Note that the series on the right-hand side of the above inequality diverges because it is a harmonic series. Hence, it follows from the Weierstrass M-Test that the series does NOT converge uniformly on the interval $(0, \infty)$. ■

Homework 7

Problem 73. Let $(V, \|\cdot\|)$ be an infinite dimensional normed space.

(i) Assume that $(V, \|\cdot\|)$ is Banach.

Problem 74 (Extra Credit). Let $(V, \|\cdot\|)$ be a normed space in which for any sequence (v_n) in V

$$\sum_{n=1}^{\infty} \|v_n\| < \infty \Longrightarrow \sum_{n=1}^{\infty} v_n \text{ converges in } V.$$

Prove that $(V, \|\cdot\|)$ is Banach.

Proof. Suppose that every absolutely convergent series is convergent. Our goal is to show that $(V, \|\cdot\|)$ is a Banach space. To do this, we will show that every Cauchy sequence in V converges. Let (v_n) be a Cauchy sequence in V. From here, our strategy is to find a subsequence (v_{n_k}) of (v_n) such that (v_{n_k}) converges in V (by the lemma). By definition, (v_n) being Cauchy implies that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$\|v_n - v_m\| < \varepsilon.$$

For $\varepsilon = 1$, there exists an $n_1 \in \mathbb{N}$ such that for any $n > m > n_1$, we have

$$\|v_n - v_m\| < 1.$$

Furthermore, if $\varepsilon = \frac{1}{2}$. So, there exists an $n_2 > n_1$ by the Archimedean Property such that for any $n > m > n_2$, we have

$$\|v_n - v_m\| < \frac{1}{2}.$$

In particular, if $\varepsilon = \frac{1}{2^{k-1}}$ for all $k \in \mathbb{N}$, then we can find an $n_k \in \mathbb{N}$ such that for any $n > m > n_k$, we have

$$\|v_n - v_m\| < \frac{1}{2^{k-1}}.$$

Moreover, by the Archimedean Property we can find an $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k > n_{k-1}$. Hence, it follows that (v_{n_k}) is a subsequence in V such that

$$0 \le \|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^{k-1}}. \tag{*}$$

Note that since $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ is a geometric series it follows from the Comparison Test that

$$\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\|$$

converges to some $v \in V$. By assumption, this tells us that

$$\sum_{k=1}^{\infty} (v_{n_{k+1}} - v_{n_k})$$

converges to some v in V. Now, observe that

$$\begin{aligned} v_{n_1} + \sum_{j=1}^{k-1} (v_{n_{j+1}} - v_{n_j}) &= v_{n_1} + (v_{n_2} - v_{n_1}) + (v_{n_3} - v_{n_2}) + \cdots + (v_{n_k} - v_{n_{k-1}}) \\ &= v_{n_k}. \end{aligned}$$

Taking the limit on both sides of the above equality, we see that

$$\begin{aligned} \lim_{k\to\infty} v_{n_k} &= \lim_{k\to\infty} \left[v_{n_k} + \sum_{j=1}^{k-1} (v_{n_{j-1}} - v_{n_j}) \right] \\ &= v_{n_1} + \lim_{k\to\infty} \sum_{j=1}^{k-1} (v_{n_{j+1}} - v_{n_j}) \\ &= v_{n_1} + v. \end{aligned}$$

Thus, we now see that (v_{n_k}) converges in V which tells us that (v_n) is a converges in V. Hence, ■

Lemma. Let $(V, \|\cdot\|)$ be a normed space. Suppose (v_n) is a Cauchy sequence, and some subsequence (v_{n_k}) converges to a point v in V. Then (v_n) converges to v in V.

Proof. Let $n > m$. Since (v_n) is a Cauchy sequence in V, it follows that

$$\|v_n - v_m\| \to 0$$

as $n, m \to \infty$. Also, (v_{n_k}) converges to some $v \in V$. So, for $k \to \infty$, we have

$$\|v_{n_k} - v\| \to 0.$$

Using the triangle inequality, it follows that

$$0 \leq \|v_n - v\| \leq \|v_n - v_{n_k}\| + \|v_{n_k} - v\| \to 0.$$

Using the Squeeze Theorem, we have

$$\|v_n - v\| \to 0$$

as $n \to \infty$ and we are done. ■