Math 230B Lecture Notes

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Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Topics

- The derivative
- Continuity and Differentiability
- Differentiability Rules

Definition (Differentiability). (*) Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $c \in I$. We say f is **differentiable** at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number).

(*) In this case, the quantity $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is called the derivative of f at c and is denoted by

$$f'(c), \frac{df}{dx}(c), \left.\frac{df}{dx}\right|_{x=c}$$

(*) If $f : I \to \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. The following are equivalent characterizations of the differentiability:

$$\begin{aligned}
f'(c) = L &\iff \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = L \\
&\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon \\
&\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |h| < \delta \text{ then } \left| \frac{f(c + h) - f(c)}{h} - L \right| < \varepsilon \\
&\iff \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} = L
\end{aligned}$$

Theorem (Differentiability Implies Continuous). Let $I \subseteq \mathbb{R}$, $c \in I$, and $f : I \to \mathbb{R}$ is differentiable at c. Then f is continuous at c.

Proof. It suffices to show that $\lim_{x\to c} f(x) = f(c)$. Note that

$$\begin{aligned}\lim_{x\to c}(f(x)-f(c)) &= \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right](x-c)\\ &= \left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\left[\lim_{x\to c}(x-c)\right]\\ &= (f'(c))(0)\\ &= 0.\end{aligned}$$

So, we have

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}[f(x)-f(c)+f(c)]\\ &= \lim_{x\to c}(f(x)-f(c)) + \lim_{x\to c} f(c)\\ &= 0 + \lim_{x\to c} f(c)\\ &= 0 + f(c)\\ &= f(c).\end{aligned}$$

■

Corollary. If $f : I \to \mathbb{R}$ is NOT continuous at $c \in I$, then f is NOT differentiable at c.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(i) Prove that f is continuous at 0.
Proof. Our goal is to show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } |x| < \delta \text{ then } |f(x) - f(c)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Note that if $x \notin \mathbb{Q}$,

$$|f(x)| = |0| < \varepsilon.$$

Otherwise, we have $|f(x)| = |x^2| = |x|^2$. IN this case, we claim that $\delta = \sqrt{\varepsilon}$ will work. Indeed, if $|x| < \delta$, then we have

$$|f(x)| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

■

(ii) Prove f is discontinuous at all $x \neq 0$.
Proof. Let $c \neq 0$. Our goal is to show that f is discontinuous at c. By the sequential criterion for continuity, it suffices to find a sequence (a_n) such that $a_n \to c$ but $f(a_n) \not\to f(c)$. We will consider two cases; that is, we could either have $c \notin \mathbb{Q}$ or $c \in \mathbb{Q}$.

Suppose $c \notin \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rational numbers (r_n) such that $r_n \to c$. Note that $f(r_n) = r_n^2 \to c^2 \neq 0$, but $f(c) = 0$. Clearly, $f(r_n) \not\to f(c)$ and so f must be discontinuous at c.

Suppose $c \in \mathbb{Q}$. Since the set of irrational numbers is also dense in $\mathbb{R}$, we can find a sequence (s_n) such that $s_n \to c$. Note that $f(s_n) = 0$, but $f(c) = c^2 \neq 0$. Thus, $f(s_n) \not\to f(c)$. Therefore, f must be discontinuous at c. ■

(iii) Prove that f is nondifferentiable at all $x \neq 0$.
Proof. Let $c \neq 0$. Since f is discontinuous at c, we can conclude that f is not differentiable at c. ■

(iv) Prove that $f'(0) = 0$.

Proof. We need to show

$$\lim_{x \to c} \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = 0.$$

■

Theorem (Algebraic Differentiability Theorem)**.** Assume that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$ where (I is an interval on $\mathbb{R}$). Then

(i) For all $k \in \mathbb{R}$, kf is differentiable at c, and

$$(kf)'(c) = kf'(c)$$

(ii) $f + g$ is differentiable at c, and

$$(f + gk)'(c) = f'(c) + g'(c)$$

(iii) fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) $\frac{f}{g}$ is differentiable at c provided that $g(c) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

Chapter 2

Week 2

Chapter 3

Week 3

Chapter 4

Week 4

4.1 Lecture 6

4.2 Lecture 6

4.2.1 Topics

(1) The definition of Riemann-Stieltjes integral

(2) Refinement of partitions

Definition (Almost Disjoint Intervals). We say that two intervals I and J are **almost disjoint** if either $I \cap J$ is empty or $I \cap J$ has exactly one point.

Definition (Partition). A partition P of an interval $[a, b]$ is a finite set of points in $[a, b]$ that includes both a and b. We always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in an increasing order; so,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Remark. A partition of P of an interval $[a, b]$ is a finite collection of almost disjoint (nonempty) compact intervals whose union is $[a, b]$:

$$P = I_1, I_2, \ldots, I_n$$

where

$$I_1 = [x_0, x_1], \quad I_2 = [x_1, x_2], \quad \cdots \quad I_n = [x_{n-1}, x_n].$$

Again, we denote $x_0 = a$ and $x_n = b$.

Definition (Lower Sum, Upper Sum). Let $f : [a, b] \to \mathbb{R}$ be bounded, $\alpha : [a, b] \to \mathbb{R}$ be increasing, and $P = \{x_0, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. Let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$.

(i) The **Lower Riemann-Stieltjes Sum** of f with respect to the integrator α for the partition P is defined by

$$L(f, \alpha, P) = \sum_{k=1}^{n} m_k(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^{n} m_k \Delta\alpha_k.$$

(ii) The upper **Riemann-Stieltjes sum** of f with respect to the integrator α for the partition P

is defined by

$$U(f,\alpha,P) = \sum_{k=1}^{n} M_k(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^{n} M_k \Delta\alpha_k.$$

Definition (Upper R.S Integral, Lower R.s Integral). Let $f : [a,b] \to \mathbb{R}$ be bounded, $\alpha : [a,b] \to \mathbb{R}$ be increasing. Then

(i) The **Upper R.S integral** of f with respect to α (on $[a,b]$) is defined by

$$U(f,\alpha) = \inf_{P\in\Pi} U(f,\alpha,P).$$

Note that the set $\{U(f,\alpha,P) : P \in \Pi\}$ is bounded below by $m(\alpha(b) - \alpha(a))$. So the infimum above is a real number.

(ii) The **Lower R.S Integral** of f with respect to α (on $[a,b]$) is defined by

$$L(f,\alpha) = \sup_{P\in\Pi} L(f,\alpha,P).$$

Note that the set $\{L(f,\alpha,P) : P \in \Pi\}$ the lower sums is bounded above by $M(\alpha(b) - \alpha(a))$. So, the supremum above is a real number.

Definition (Riemann-Stieltjes integrable functions). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A function $f : [a,b] \to \mathbb{R}$ is said to be **Riemann-Stieltjes integrable** (on $[a,b]$) if

(i) f is bounded

(ii) $L(f,\alpha) = U(f,\alpha)$.

In this case, the R.S integral of f with respect to α, denoted by

$$\int_a^b f\, d\alpha \quad \text{or} \quad \int_a^b f(x)\, d\alpha(x) \quad \text{or} \quad \int_{[a,b]} f\, d\alpha$$

is the common value of $L(f,\alpha)$ and $U(f,\alpha)$. That is,

$$\int_a^b f\, d\alpha = L(f,\alpha) = U(f,\alpha).$$

4.3 Lecture 8-9-10

Theorem (Rudin 6.4). Let $f : [a,b] \to \mathbb{R}$ be bounded, $\alpha : [a,b] \to \mathbb{R}$ is increasing, P is a partition of $[a,b]$, and Q is a refinement of P. Then

(1) $L(f,\alpha,P) \le L(f,\alpha,Q)$

(2) $U(f,\alpha,P) \ge U(f,\alpha,Q)$

Proof. Here we will prove (1). The proof of (2) is completely analagous. We proceed via induction on $\ell = \operatorname{card}(Q \setminus P)$ (the number of points in $Q \setminus P$). Let $P = \{x_0, x_1, \ldots, x_n\}$.

If $\ell = 0$, then $P \subseteq Q$ and $\operatorname{card} Q = \operatorname{card} P$ implies that $P = Q$. Thus, $L(f,\alpha,P) = L(f,\alpha,Q)$.

If $\ell = 1$, then Q has exactly one extra point. Let's call this point z. So, $\{z\} = Q \setminus P$. Note that

$z \in [a, b]$ and P is a partition of $[a, b]$. Hence, there exists $1 \le i \le n$ such that $z \in (x_{i-1}, x_i)$. Let

$$m_i' = \inf_{x \in [x_{i-1}, z]} f(x)$$
$$m_i'' = \inf_{x \in [z, x_i]} f(x)$$

Recall that if $A \subseteq B$, then $\inf A \ge \inf B$. Hence, $m_i' \ge m_i$ and $m_i'' \ge m_i$. We have

$$\begin{aligned}
L(f, \alpha, P) &= \sum_{k=1}^{n} m_k(\alpha(x_k)) \\
&= \Big[\sum_{k \ne i} m_k(\alpha(x_k) - \alpha(x_{k-1}))\Big] + m_i(\alpha(x_i) - \alpha(z) + \alpha(z) - \alpha(x_{i-1})) \\
&= \Big[\sum_{k \ne i} m_k(\alpha(x_k) - \alpha(x_{k-1}))\Big] + m_i(\alpha(z) - \alpha(x_{i-1})) + m_i(\alpha(x_i) - \alpha(z)) \\
&\le \Big[\sum_{k \ne i} m_k(\alpha(x_k) - \alpha(x_{k-1}))\Big] + m_i'(\alpha(z) - \alpha(x_{i-1})) + m_i''(\alpha(x_i) - \alpha(z)) \\
&= L(f, \alpha, Q).
\end{aligned}$$

So, we have $L(f, \alpha, P) \le L(f, \alpha, Q)$.

Now, suppose the claim is true for $\ell = r \ge 1$. Our goal is to show that the claim holds for $\ell = r + 1$. Suppose $\text{card}(Q \setminus P) = r + 1$. Let

$$Q \setminus P = \{z_1, z_2, \dots, z_r, z_{r+1}\}.$$

Let $\hat{Q} = P \cup \{z_1, z_2, \dots, z_r\}$. We have

$$L(f, \alpha, P) \le L(f, \alpha, \hat{Q}) \le L(f, \alpha, Q)$$

where the first inequality holds due to our induction hypothesis and the second inequality holds because $Q \setminus \hat{Q}$ contains only one point. So, we have

$$L(f, \alpha, P) \le L(f, \alpha, Q).$$

■

Theorem. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $\alpha : [a, b] \to \mathbb{R}$ is increasing. Let P_1 and P_2 are any two partition of $[a, b]$. Then

$$L(f, \alpha, P_1) \le U(f, \alpha, P_2).$$

Proof. Let $Q = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Applying the previous theorem, we can see that $P_1 \subseteq P_1 \cup P_2$ and $P_2 \subseteq P_1 \cup P_2$ implies

$$L(f, \alpha, P_1) \le L(f, \alpha, Q) \le U(f, \alpha, Q) \le U(f, \alpha, P_2)$$

■

For the following theorem, we will use the lemma below.

Lemma. Suppose A and B are nonempty subsets of $\mathbb{R}$. If

$$\forall a \in A \ \forall b \in B \quad a \le b$$

then $\sup A \le \inf B$.

Theorem (Rudin 6.5)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\alpha : [a, b] \to \mathbb{R}$ is an increasing function. Then $L(f, \alpha) \leq U(f, \alpha)$.

Proof. Let $A = \{L(f, \alpha, P) : P \in \Pi\}$ and $B = \{U(f, \alpha, P) : P \in \Pi\}$. Using the lemma above and Theorem 2, we can see that for all $a \in A$ and for all $b \in B$, it follows that $\sup A \leq \inf B$; that is, $L(f, \alpha) \leq U(f, \alpha)$. ∎

Theorem (Cauchy Criterion for Riemann-Stieltjes Integrability Rudin 6.6)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. Then

$$f \in R_\alpha[a, b] \iff \forall \varepsilon > 0 \ \exists P_\varepsilon \in \Pi[a, b] \text{ such that } U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) < \varepsilon.$$

Proof. ($\Longleftarrow$) Our goal is to show that $L(f, \alpha) = U(f, \alpha)$. Note that $L(f, \alpha) \leq U(f, \alpha)$ implies $U(f, \alpha) - L(f, \alpha) \geq 0$. Hence, it suffices to show that for all $\varepsilon > 0$,

$$U(f, \alpha) - L(f, \alpha) < \varepsilon.$$

Let $\varepsilon > 0$ be given. By assumption, there exists $P_\varepsilon \in \Pi$ such that

$$U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) < \varepsilon.$$

We have

$$\begin{aligned} U(f, \alpha) &= \inf_{P \in \Pi} U(f, \alpha, P) \leq U(f, \alpha, P_\varepsilon) \\ L(f, \alpha) &= \sup_{P \in \Pi} L(f, \alpha, P) \geq L(f, \alpha, P_\varepsilon) \end{aligned}$$

Using Rudin 6.5, we can see that

$$L(f, \alpha, P_\varepsilon) \leq L(f, \alpha) \leq U(f, \alpha) \leq U(f, \alpha, P_\varepsilon).$$

So, the interval $[L(f, \alpha), U(f, \alpha)]$ is contained in the interval $[L(f, \alpha, P_\varepsilon), U(f, \alpha, P_\varepsilon)]$. Thus,

$$U(f, \alpha) - L(f, \alpha) \leq U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) < \varepsilon$$

as desired.

($\Longrightarrow$) Our goal is to show that for any $\varepsilon > 0$, there exists a partition $P_\varepsilon \in \Pi$ such that

$$U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) < \varepsilon.$$

Note that

$$\begin{aligned} U(f, \alpha) = \inf_{P \in \Pi} U(f, \alpha, P) &\Longrightarrow \exists P_1 \in \Pi \text{ such that } U(f, \alpha, P_1) < U(f, \alpha) + \frac{\varepsilon}{2} \\ L(f, \alpha) = \sup_{P \in \Pi} L(f, \alpha, P) &\Longrightarrow \exists P_2 \in \Pi \text{ such that } L(f, \alpha) - \frac{\varepsilon}{2} < L(f, \alpha, P_2) \end{aligned}$$

Let $P_\varepsilon = P_1 \cup P_2$ (we claim that this partition can be used as the one that we were looking for).

$$L(f, \alpha) - \frac{\varepsilon}{2} < L(f, \alpha, P_2) \leq L(f, \alpha, P_\varepsilon) \leq U(f, \alpha, P_\varepsilon) \leq U(f, \alpha, P_1) < U(f, \alpha) + \frac{\varepsilon}{2}.$$

Thus, we have

$$\begin{aligned} U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) &< \left[\left(U(f, \alpha) + \frac{\varepsilon}{2}\right) - \left(L(f, \alpha) - \frac{\varepsilon}{2}\right)\right] \\ &= U(f, \alpha) - L(f, \alpha) + \varepsilon \\ &= 0 + \varepsilon = \varepsilon \end{aligned}$$

as desired. ∎

Theorem (Rudin 6.7). Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $\alpha : [a, b] \to \mathbb{R}$ is an increasing function, fix $\varepsilon > 0$, $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, and

$$U(f, \alpha, P) - L(f, \alpha, P) < \varepsilon.$$

Then

(1) If Q is any refinement of P, then $U(f, \alpha, Q) - L(f, \alpha, Q) < \varepsilon$.

(2) If for every $1 \le k \le n$, t_k and s_k are arbitrary points in $[x_{k-1}, x_k]$, then

$$\sum_{k=1}^{n} |f(s_k) - f(t_k)|\Delta\alpha_k < \varepsilon.$$

(3) If $f \in R_\alpha[a, b]$ and for each $1 \le k \le n$, s_k is a point in $[x_{k-1}, x_k]$, then

$$\Big|\sum_{k=1}^{n} f(s_k)\Delta\alpha_k - \int_a^b f \; d\alpha\Big| < \varepsilon.$$

Proof. (1) We have

$$L(f, \alpha, P) \le L(f, \alpha, Q) \le U(f, \alpha, Q) \le U(f, \alpha, P).$$

Therefore,

$$U(f, \alpha, Q) - L(f, \alpha, Q) \le U(f, \alpha, P) - U(f, \alpha, P) < \varepsilon.$$

(2) For each $1 \le k \le n$, we have

$$\begin{aligned} m_k &\le f(s_k) \le M_k \\ m_k &\le f(t_k) \le M_k \Longrightarrow -M_k \le -f(t_k) \le -m_k. \end{aligned}$$

So, we have

$$m_k - M_k \le f(s_k) - f(t_k) \le M_k - m_k.$$

That is,

$$-(M_k - m_k) \le f(s_k) - f(t_k) \le M_k - m_k.$$

Therefore,

$$|f(s_k) - f(t_k)| \le M_k - m_k.$$

Hence, we have

$$\sum_{k=1}^{n} |f(s_k) - f(t_k)|\Delta\alpha_k \le \sum_{k=1}^{n} (M_k - m_k)\Delta\alpha_k = U(f, \alpha, P) - L(f, \alpha, P) < \varepsilon.$$

(3) For all $1 \le k \le n$, we have

$$m_k \le f(s_k) \le M_k.$$

So,

$$\sum_{k=1}^{n} m_k\Delta\alpha_k \le \sum_{k=1}^{n} f(s_k)\Delta\alpha_k \le \sum_{k=1}^{n} M_k\Delta\alpha_k.$$

Therefore,

$$L(f, \alpha, P) \le \sum_{k=1}^{n} f(s_k) \le U(f, \alpha, P) \tag{I}$$

Also, note that

$$L(f, \alpha, P) \le \int_a^b f \; d\alpha \le U(f, \alpha, P). \tag{II}$$

Hence,

$$\Big|\sum_{k=1}^{n} f(s_k)\Delta\alpha_k - \int_a^b f \; d\alpha\Big| \le U(f, \alpha, P) - L(f, \alpha, P) < \varepsilon$$

as desired.

■

Lemma. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Then

$$\forall 1 \leq k \leq n \quad \sup_{s,t \in [x_{k-1},x_k]} |f(s) - f(t)| = M_k - m_k.$$

Proof. Let $k \in \{1, 2, \dots, n\}$. We need to show

(1) $\forall s,t \in [x_{k-1}, x_k]$ $|f(s) - f(t)| \leq M_k - m_k$.

(2) $\forall \varepsilon > 0$, $\exists \hat{s}\hat{t} \in [x_{k-1}, x_k]$ such that $M_k - m_k - \varepsilon < |f(\hat{s}) - f(\hat{t})|$.

Note that we have already shown (1) in our discussion of Theorem 6.7.

Let $\varepsilon > 0$ be given. Then we have

$$m_k = \inf_{t \in [x_{k-1},x_k]} f(t) \Longrightarrow \hat{t} \in [x_{k-1}, x_k] \text{ such that } f(\hat{t}) < m_k + \frac{\varepsilon}{2}$$
$$M_k = \sup_{t \in [x_{k-1},x_k]} f(t) \Longrightarrow \hat{s} \in [x_{k-1}, x_k] \text{ such that } M_k - \frac{\varepsilon}{2} < f(\hat{s}).$$

Adding the inequalities above, we get

$$M_k - m_k - \varepsilon < f(\hat{s}) - f(\hat{t}) \leq |f(\hat{s}) - f(\hat{t})|.$$

■

Theorem (Rudin 6.8)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\alpha : [a,b] \to \mathbb{R}$ is an increasing function. Then $f \in R_\alpha[a,b]$.

Proof. Since $f : [a,b] \to \mathbb{R}$ is a continuous function and $[a,b]$ is compact, it follows from the Extreme Value Theorem that f is bounded on $[a,b]$. Now, according to the Cauchy Criterion for Riemann-Stieltjes integrability, it suffices to show that

$$\forall \varepsilon > 0 \ \exists P \in \Pi \text{ such that } U(f,\alpha,P) - L(f,\alpha,P) < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. By the same reasoning that showed f is bounded on $[a,b]$, it follows that f is uniformly continuous on $[a,b]$. For the given ε, there exists a $\delta > 0$ such that for all $s,t \in [a,b]$:

$$\text{if } |s - t| < \delta \text{ then } |f(s) - f(t)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a) + 1]}.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ such that $\|P\| < \delta$. We claim (*) holds for such a partition. Indeed, for all $k \in \{1, 2, \dots, n\}$ and for all $s,t \in [x_{k-1}, x_k]$, if $|s - t| < \delta$, then

$$|f(s) - f(t)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a) + 1]}.$$

Hence,

$$\sup_{s,t \in [x_{k-1},x_k]} |f(s) - f(t)| \leq \frac{\varepsilon}{2[\alpha(b) - \alpha(a) + 1]}.$$

Thus,

$$M_k - m_k \leq \frac{\varepsilon}{2[\alpha(b) - \alpha(a) + 1]}.$$

Therefore,

$$\begin{aligned}U(f,\alpha,P)-L(f,\alpha,P) &= \sum_{k=1}^{n}(M_k-m_k)\Delta\alpha_k \\ &\le \sum_{k=1}^{n}\frac{\varepsilon}{2[\alpha(b)-\alpha(a)+1]}\Delta\alpha_k \\ &= \frac{\varepsilon}{2[\alpha(b)-\alpha(a)+1]}\sum_{k=1}^{n}\Delta\alpha_k \\ &= \frac{\varepsilon}{2[\alpha(b)-\alpha(a)+1]}\cdot[\alpha(b)-\alpha(a)] \\ &\le \frac{\varepsilon}{2} \\ &< \varepsilon\end{aligned}$$

as desired.

■

Lemma. Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing and continuous function and $\alpha(a) < \alpha(b)$. Then for each $n \in \mathbb{N}$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ such that

$$\forall 1 \le k \le n \quad \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = \frac{\alpha(b)-\alpha(a)}{n}.$$

Proof. Let $n \in \mathbb{N}$. Divide the interval $[\alpha(a), \alpha(b)]$ into n subintervals of equal length: $\frac{\alpha(b)-\alpha(a)}{n}$. For each $1 \le k \le n$, we have $y_k \in (\alpha(a), \alpha(b))$. Hence, the Intermediate Value Theorem implies that

$$\exists x_k \in (a,b) \text{ such that } y_k = \alpha(x_k).$$

Since α is increasing, we have

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

This tells us that $P = \{x_0, x_1, \dots, x_n\}$ will be partition of $[a,b]$ such that

$$\forall 1 \le k \le n \quad \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = y_k - y_{k-1} = \frac{\alpha(b)-\alpha(a)}{n}.$$

■

Theorem (Rudin 6.9)**.** Let $\alpha : [a,b] \to \mathbb{R}$ be increasing and continuous. Then

(1) If $f : [a,b] \to \mathbb{R}$ is increasing, then $f \in R_\alpha[a,b]$.

(2) If $f : [a,b] \to \mathbb{R}$ is increasing

Proof. Here we will prove (1). The proof of (2) is analogous. First, note that

$$\forall x \in [a,b] \quad f(a) \le f(x) \le f(b) \Longrightarrow f \text{ is bounded on } [a,b].$$

If $\alpha(a) = \alpha(b)$, then we previously proved $f \in R_\alpha[a,b]$ and $\int_a^b f\, d\alpha = 0$. So, it remains to prove the claim for the case where $\alpha(a) \ne \alpha(b)$. According to the Cauchy Criterion for integrability, in order to show that $f \in R_\alpha[a,b]$, it suffices to show that

$$\forall \varepsilon > 0\ \exists P \in \Pi \text{ such that } U(f,\alpha,P) - L(f,\alpha,P) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Choose $n \in \mathbb{N}$ be large enough so that $\frac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)] < \varepsilon$. Let $\tilde{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$ such that

$$\forall 1 \le k \le n \quad \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = \frac{\alpha(b)-\alpha(a)}{n}.$$

We claim that $\tilde{P}$ can be used as the P that we were looking for. Now, since f is increasing, we know that for each $1 \le k \le n$

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = f(x_k)$$

and

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = f(x_{k-1}).$$

Hence, we see that

$$\begin{aligned} U(f, \alpha, \tilde{P}) - L(f, \alpha, \tilde{P}) &= \sum_{k=1}^{n} (M_k - m_k) \Delta \alpha_k \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \\ &= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \varepsilon \end{aligned}$$

as desired. ■

Theorem (Rudin 6.10). Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f has only finitely many points of discontinuity

$$y_1 < y_2 < \cdots < y_m$$

and $\alpha : [a, b] \to \mathbb{R}$ is increasing and α is continuous at $y_1, y_2, \ldots, y_m$. Then $f \in R_\alpha[a, b]$.

Proof. According to the Cauchy Criterion, it suffices to show that

$$\forall \varepsilon > 0 \; \exists P \in \Pi \text{ such that } U(f, \alpha, P) - L(f, \alpha, P) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Let $\tilde{M} = \sup_{x \in [a,b]} |f(x)|$. Let

$$\hat{\varepsilon} = \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2\tilde{M} + 1]}.$$

We will make the following two claims:

(1) There exists many disjoint intervals $[u_1, v_1], \ldots, [u_m, v_m]$ such that

  (I) $\forall 1 \le j \le m \; y_j \in [u_j, v_j]$.

  (II) $\forall 1 \le j \le m$ if $y_j \notin \{a, b\}$, then $y_j \in (u_j, v_j)$

  (III) $\forall 1 \le j \le m \; \alpha(v_j) - \alpha(u_j) < \frac{\hat{\varepsilon}}{m}$ and so

$$\sum_{j=1}^{m} \alpha(v_j) - \alpha(u_j) < \hat{\varepsilon}.$$

(2) Let $K = [a, b] \setminus \bigcup_{j=1}^{m} (u_j, v_j)$. Then f is uniformly continuous on K.

The two claims above will be proven as lemmas after the proof of this theorem. For now, we will assume that the two claims hold.

By claim 2, we know there exists $\delta > 0$ such that for all $s, t \in K$ if $|s - t| < \delta$, then

$$|f(s) - f(t)| < \hat{\varepsilon}.$$

Now, we form a partition $\tilde{P}$ of $[a, b]$ as follows:

(i) $\forall 1 \le j \le m \; u_j, v_j \in \tilde{P}$.

(ii) $\forall 1 \le j \le m$ no point of the segment (u_j, v_j) is in $\tilde{P}$

(iii) If $1 \le k \le m$ is such that $x_{k-1} \notin \{u_1, \ldots, u_m\}$, then we will choose x_k such that $x_k - x_{k-1} < \delta$.

We claim that this $\tilde{P}$ can be used as the P that we were looking for. Indeed, define the two sets

$$A = \{k : x_{k-1} \notin \{u_1, \dots, u_m\}\} \quad \text{and} \quad B = \{1, \dots, n\} \setminus A.$$

For the case that $k \in A$, $x_k - x_{k-1} < \delta$, so for all $s, t \in [x_{k-1}, x_k]$, if $|s - t| < \delta$, then $|f(s) - f(t)| < \hat{\varepsilon}$. Then taking the supremum, we have

$$\sup_{s,t \in [x_{k-1}, x_k]} |f(s) - f(t)| \leq \hat{\varepsilon}$$

and so from lemma 2, we have

$$M_k - m_k \leq \hat{\varepsilon}.$$

If $k \in B$, then

$$M_k - m_k = \sup_{s,t \in [x_{k-1}, x_k]} |f(s) - f(t)| \leq 2\tilde{M}.$$

Therefore,

$$\begin{aligned}
U(f, \alpha, P) - L(f, \alpha, P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta\alpha_k \\
&= \sum_{k \in A} (M_k - m_k)\Delta\alpha_k + \sum_{k \in B} (M_k - m_k)\Delta\alpha_k \\
&\leq \sum_{k \in A} \hat{\varepsilon}\Delta\alpha_k + 2\tilde{M} \sum_{k \in B} \Delta\alpha_k \\
&\leq \hat{\varepsilon}[\alpha(b) - \alpha(a)] + 2\tilde{M}\hat{\varepsilon} \\
&= [\alpha(b) - \alpha(a) + 2\tilde{M}]\hat{\varepsilon} \\
&< \varepsilon.
\end{aligned}$$

■

Lemma. There exists finitely many disjoint intervals

$$[u_1, v_1], \dots, [u_m, v_m]$$

in $[a, b]$ such that

(1) $\forall 1 \leq j \leq m$ $y_j \in [u_j, v_j]$;

(2) $\forall 1 \leq j \leq m$ if $y_j \notin \{a, b\}$ then $y_j \in (u_j, v_j)$;

(3) $\forall 1 \leq j \leq m$ $\alpha(v_j) - \alpha(u_j) < \frac{\hat{\varepsilon}}{m}$ and so

$$\sum_{j=1}^{m} [\alpha(v_j) - \alpha(u_j)] < \hat{\varepsilon}.$$

Proof. Since for each $1 \leq j \leq m$, α is continuous at y_j, we can choose $\delta_j > 0$ such that

$$\text{if } |y - y_j| < \delta_j, \text{ then } |\alpha(y) - \alpha(y_j)| < \frac{\hat{\varepsilon}}{2m}.$$

Now, let

$$\tilde{\delta} = \frac{1}{4} \min\{\delta_1, \delta_2, \dots, \delta_m, y_2 - y_1, y_3 - y_2, \dots, y_m - y_{m-1}\}.$$

For each $1 \leq j \leq m$, we define

(1) If $y_j \notin \{a, b\}$, then $[u_j, v_j] = [y_j - \hat{\delta}, y_j + \hat{\delta}]$

(2) If $y_j = a$, then $[u_j, v_j] = [a, a + \hat{\delta}]$

(3) If $y_j = b$, then $[u_j, v_j] = [b - \hat{\delta}, b]$.

Clearly, there intervals satisfy all the requirements, in particular,

$$\begin{aligned}\alpha(v_j) - \alpha(u_j) &= |\alpha(v_j) - \alpha(u_j)| \\ &\leq |\alpha(v_j) - \alpha(y_j)| + |\alpha(y_j) - \alpha(u_j)| \\ &< \frac{\hat{\varepsilon}}{2m} + \frac{\hat{\varepsilon}}{2m} \\ &= \frac{\hat{\varepsilon}}{m}\end{aligned}$$

where $|v_j - y_j| \leq \hat{\delta} < \delta_j$ and $|u_j - y_j| \leq \hat{\delta} < \delta_j$. ∎

Lemma (Claim 2). Let $K = [a, b] \setminus \bigcup_{j=1}^{m}(u_j, v_j)$. Then f is uniformly continuous on K.

Proof. Note that $\bigcup_{k=1}^{m}(u_j, v_j)$ is open. Hence,

$$K = [a, b] \setminus \bigcup_{j=1}^{m}(u_j, v_j) = [a, b] \cap \Big[\bigcup_{j=1}^{m}(u_j, v_j)\Big]^c$$

is closed. Since $K \subseteq [a, b]$, K is closed, and $[a, b]$ is compact, it follows from the fact that closed subsets of a compact set are compact that K is compact. Since $f : K \to \mathbb{R}$ is continuous and K is compact, we can conclude that f is uniformly continuous on K. ∎

Remark (Why is $f : K \to \mathbb{R}$ is continuous?). We will consider four claims:

(1) Suppose f is continuous at a and b. In this case by removing $\bigcup_{j=1}^{m}(u_j, v_j)$, the discontinuities of f will be removed.

(2) Since f is discontinuous at a, but continuous at b. In this case, by removing $\bigcup_{j=1}^{m}(u_j, v_j)$ all discontinuities will be removed except a. In this case, removing (u_1, v_1) makes a an isolated point of K. Every function is continuous at every is isolated point of its domain.

(3) Suppose f is continuous at a and discontinuities at b.

(4) Suppose f is both discontinuous at a and b.

Case (3) and (4) follows similarly from case (2).

Theorem (Rudin 6.11). Let $f \in R_\alpha[a, b]$, for all $x \in [a, b]$ $m \leq f(x) \leq M$, $\varphi : [m, M] \to \mathbb{R}$ is continuous. Then $h : \varphi \circ f : [a, b] \to \mathbb{R}$, then $h \in R_\alpha[a, b]$.

Proof. Firs note that a composition of bounded functions is bounded. So $h : \varphi \circ f$ is a bounded function on $[a, b]$. According to the Cauchy criterion, in order to show $h \in R_\alpha[a, b]$, it suffices to show that for all $\varepsilon > 0$, there exists $P \in \Pi$ such that

$$U(f, \alpha, P) - L(h, \alpha, P) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Let $\tilde{M} = \sup_{x\in[a,b]} |h(x)|$. Let

$$\hat{\varepsilon} = \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2\tilde{M} + 1]}.$$

We have

(I) Since φ is continuous in $[m, M]$ and $[m, M]$ is compact, it follows that φ is uniformly continuous on $[, m, M]$. So,

$$\exists 0 < \delta < \hat{\varepsilon} \text{ such that } \forall s, t \in [m, M] \text{ if} |s - t| < \delta \text{ then } |\varphi(s) - \varphi(t)| < \hat{\varepsilon}.$$

(II) Since $f \in R_\alpha[a,b]$, we know from the CauchyCriterion that

$$\exists \tilde{P} \in \Pi \text{ such that } U(f,\alpha,\tilde{P}) - L(f,\alpha,\tilde{P}) = \sum_{k=1}^{n}(M_k - m_k)\Delta\alpha_k < \delta^2.$$

We claim that this $\tilde{P}$ can be used as the P that we were looking for. Indeed, let for all $1 \le k \le n$

$$m_k^* = \inf_{x\in[x_{k-1},x_k]} h(x) \quad \text{and} \quad M_k^* = \sup_{x\in[x_{k-1},x_k]} h(x).$$

Note that

$$U(h,\alpha,\tilde{P}) - L(h,\alpha,\tilde{P}) = \sum_{k=1}^{n}(M_k^* - m_k^*)\Delta\alpha_k.$$

In what follows, we will show that the sum above is less than ε. Divide the indices $1,\ldots,n$ in two classes, namely

$$A = \{k : M_k - m_k < \delta\} \quad \text{and} \quad B = \{k : M_k - m_k \ge \delta\}.$$

We have

$$U(h,\alpha,\tilde{P}) - L(h,\alpha,\tilde{P}) = \sum_{k=1}^{n}(M_k^* - m_k^*)\Delta\alpha_k = \sum_{k\in A}(M_k^* - m_k^*)\Delta\alpha_k + \sum_{k\in B}(M_k^* - m_k^*)\Delta\alpha_k. \tag{1}$$

(*) If $k \in A$, then for all $x, y \in [x_{k-1}, x_k]$, we have

$$\begin{aligned} M_k - m_k < \delta &\implies \sup_{x,y\in[x_{k-1},x_k]} |f(x) - f(y)| \\ &\implies |f(x) - f(y)| < \delta \\ &\implies |\varphi(f(x)) - \varphi(f(y))| < \hat{\varepsilon} \\ &\implies |h(x) - h(y)| < \hat{\varepsilon} \\ &\implies \sup_{x,y\in[x_{k-1},x_k]} |h(x) - h(y)| \le \hat{\varepsilon} \\ &\implies M_k^* - m_k^* \le \hat{\varepsilon}. \end{aligned} \tag{2}$$

(*) For $k \in B$,

$$\begin{aligned} \delta \sum_{k\in B} \Delta\alpha_k = \sum_{k\in B} \delta\Delta\alpha_k &\le \sum_{k\in B}(M_k - m_k)\Delta\alpha_k \\ &\le \sum_{k=1}^{n}(M_k - m_k)\Delta\alpha_k = U(f,\alpha,\tilde{P}) - L(f,\alpha,\tilde{P}) < \delta^2. \end{aligned} \tag{3}$$

It follows from (1), (2), and (3) that

$$\begin{aligned} \sum_{k=1}^{n}(M_k^* - m_k^*)\Delta\alpha_k &= \sum_{k\in A}(M_k^* - m_k^*)\Delta\alpha_k + \sum_{k\in B}(M_k^* - m_k^*)\Delta\alpha_k \\ &\le \sum_{k\in A} \hat{\varepsilon}\Delta\alpha_k + \sum_{k\in B} 2\tilde{M}\Delta\alpha_k \\ &= \hat{\varepsilon}\sum_{k=1}^{n} \Delta\alpha_k + 2\tilde{M}\hat{\varepsilon} \\ &= \hat{\varepsilon}[\alpha(b) - \alpha(a)] + 2\tilde{M}\hat{\varepsilon} \\ &= [\alpha(b) - \alpha(a) + 2\tilde{M}]\hat{\varepsilon} \\ &= [\alpha(b) - \alpha(a) + 2\tilde{M}] \cdot \frac{\varepsilon}{\alpha(b) - \alpha(a) + 2\tilde{M} + 1} < \varepsilon \end{aligned}$$

as desired.

■

Chapter 5

Week 5

5.1 Lectures 11-12

5.1.1 Plan

(1) Sequential Criterion for integrability;

(2) Algebraic properties of R.S integral;

(3) Order properties of R.S integrals;

(4) Mean Value Theorem and Generalized Mean Value Theorem for integrals;

(5) Additivity for R.S integrals.

Theorem (Sequential Criterion for R.S Integrability). Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\alpha : [a, b] \to \mathbb{R}$ is an increasing function. Then

(1) If $f \in R_\alpha[a, b]$, then there exists a sequence of partitions $(P_n)_{n\geq 1}$ in $\Pi[a, b]$ such that $\lim_{n\to\infty}[U(f, \alpha, P_n)- L(f, \alpha, P_n)] = 0$.

(2) If there exists a sequence of partitions $(P_n)_{n\geq 1}$ in $\Pi[a, b]$ such that $\lim_{n\to\infty}[U(f, \alpha, P_n)-L(f, \alpha, P_n)] = 0$, then $f \in R_\alpha[a, b]$, and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty kU(f,\alpha,P_n)} = \lim_{n\to\infty} L(f, \alpha, P_n).$$

Proof. (1) Using the Cauchy Criterion, we see that $f \in R_\alpha[a, b]$ if and only if for all $\varepsilon > 0$, there exists $P_\varepsilon \in \Pi[a, b]$ such that

$$U(f, \alpha, P_\varepsilon) - L(f, \alpha, P_\varepsilon) < \varepsilon.$$

In particular, we can inductively construct a sequence of partitions $(P_n)_{n\geq 1}$ in the following way: for all $n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$. Then there exists $P_n \in \Pi$ such that

$$0 \leq U(f, \alpha, P_n) - L(f, \alpha, P_n) < \frac{1}{n}.$$

From the squeeze theorem, it follows that

$$\lim_{n\to\infty}[U(f, \alpha, P_n) - L(f, \alpha, P_n)] = 0.$$

(2) According to Cauchy Criterion, it suffices to show that

$$\forall \varepsilon > 0 \; \exists P \in \Pi[a, b] \text{ such that } U(f, \alpha, P) - L(f, \alpha, P) < \varepsilon.$$

Since $\lim_{n\to\infty}[U(f,\alpha,P_n) - L(f,\alpha,P_n)] = 0$, there exists an $N \in \mathbb{N}$ such that

$$\forall n > N \quad U(f,\alpha,P_n) - L(f,\alpha,P_n) < \varepsilon.$$

In particular, P_{N+1} can be used as the P that we were looking for. It remains to show that

$$\int_a^b f \; d\alpha = \lim_{n\to\infty} U(f,\alpha,P_n) = \lim_{n\to\infty} L(f,\alpha,P_n).$$

We have for all $n \geq 1$,

$$0 \leq U(f,\alpha,P_n) - U(f,\alpha) \leq U(f,\alpha,P_n) - L(f,\alpha) \leq U(f,\alpha,P_n) - L(f,\alpha,P_n).$$

Using the squeeze theorem on the inequality above, we have

$$\lim_{n\to\infty}[L(f,\alpha) - L(f,\alpha,P_n)] = 0.$$

So,

$$\lim_{n\to\infty} L(f,\alpha,P_n) = L(f,\alpha) = \int_a^b f \; d\alpha.$$

■

Theorem (Algebraic Properties of R.S Integral)**.** Assume $f, g \in R_\alpha[a,b]$. Then

(i) $\forall k \in \mathbb{R}$, $kf \in R_\alpha[a,b]$ with
$$\int_a^b kf \; d\alpha = k\int_a^b f \; d\alpha$$
;

(ii) $f + g \in R_\alpha[a,b]$ with
$$\int_a^b f + g \; d\alpha = \int_a^b f \; d\alpha + \int_a^b g \; d\alpha.$$

(iii-1) $f^2 \in R_\alpha[a,b]$;

(iii-2) $fg \in R_\alpha[a,b]$;

(iv-1) if $g \neq 0$ on $[a,b]$ and $\frac{1}{g}$ is bounded on $[a,b]$, then $\frac{1}{g} \in R_\alpha[a,b]$;

(iv-2) if $g \neq 0$ on $[a,b]$ and $\frac{1}{g}$ is bounded on $[a,b]$, then $\frac{1}{g} \in R_\alpha[a,b]$.

Lemma (lemma 3)**.** Let A be a subset of $\mathbb{R}$ and $f, g : A \in \mathbb{R}$ be two bounded functions. Then

(i) $\sup_A(f+g) \leq \sup_A f + \sup_A g$;

(ii) $\inf_A(f+g) \geq \inf_A f + \inf_A g$;

(iii-1) $\forall k \geq 0$, $\sup_A(kf) = k\sup_A f$;

(iii-2) $\forall k \geq 0$ $\inf_A kf = k\inf_A f$;

(iv-1) $\forall k < 0$ $\sup_A kf = k\inf_A f$;

(iv-2) $\forall k < 0$ $\inf_A kf = k\sup_A f$;

(v) $\sup_{x,y\in A}|f(x) - f(y)| = \sup_A f - \inf_A f$;

(vi) If there exists a constant $k > 0$ such that

$$\forall z, w \in A \quad |f(z) - f(w)| \leq k|g(z) - g(w)|,$$

then

$$\sup_A f - \inf_A f \leq k[\sup_A g - \inf_A g].$$

Lemma (lemma 4)**.** Let $f, g : [a,b] \to \mathbb{R}$ be two bounded functions, $\alpha : [a,b] \to \mathbb{R}$ is an increasing function, and $P \in \Pi[a,b]$. Then

(i) $U(f+g,\alpha,P) \leq U(f,\alpha,P) + U(g,\alpha,P)$;

(ii) $L(f+g,\alpha,P) \geq L(f,\alpha,P) + U(g,\alpha,P)$;

(iii-1) $\forall k \geq 0$ $U(kf,\alpha,P) = kU(f,\alpha,P)$

(iii-2) $\forall k \geq 0$, $L(kf,\alpha,P) = kL(f,\alpha,P)$;

(iv-1) $\forall k < 0$ $U(f,\alpha,P) = kL(f,\alpha,P)$

(iv-2) $\forall k < 0$ $L(kf,\alpha,P) = kU(f,\alpha,P)$.

Theorem (Order Properties of R.S Integral)**.** Assume $f, g \in R_\alpha[a,b]$. Then

(i) If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f\ d\alpha \leq M(\alpha(b) - \alpha(a)).$$

(ii) If $f \leq g$ on $[a,b]$, then

$$\int_a^b f\ d\alpha \leq \int_a^b g\ d\alpha.$$

Proof. (i) Note that for any $P \in \Pi[a,b]$, we have

$$\int_a^b f\ d\alpha = L(f,\alpha) \geq L(f,\alpha,P)$$
$$\int_a^b f\ d\alpha = U(f,\alpha) \leq U(f,\alpha,P).$$

In particular, for the partition $P = \{a,b\}$, we have

$$\int_a^b f\ d\alpha \geq L(f,\alpha,P) = \Big(\inf_{x\in[a,b]} f(x)\Big)(\alpha(b) - \alpha(a)) \geq m(\alpha(b) - \alpha(a)) \tag{1}$$
$$\int_a^b f\ d\alpha \leq U(f,\alpha,P) = \Big(\sup_{x\in[a,b]} f(x)\Big)(\alpha(b) - \alpha(a)) \leq M(\alpha(b) - \alpha(a)). \tag{2}$$

Using (1) and (2), we obtain our desired result.

(ii) Let $h = g - f$. We have $h \geq 0$, so, by part (i), we have

$$0(\alpha(b) - \alpha(a)) \leq \int_a^b h\ d\alpha.$$

Therefore,

$$0 \leq \int_a^b h\ d\alpha = \int_a^b g - f\ d\alpha = \int_a^b g\ d\alpha - \int_a^b f\ d\alpha$$
$$\implies \int_a^b f\ d\alpha \leq \int_a^b g\ d\alpha.$$

∎

Theorem (Triangle Inequality of Integrals). Assume $f \in R_\alpha[a,b]$. Then

(i) $|f| \in R_\alpha[a,b]$;

(ii) $\left|\int_a^b f \, d\alpha\right| \le \int_a^b |f| \, d\alpha$.

Proof. (i) Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = |x|$ which is clearly continuous on $\mathbb{R}$. Since $f \in R_\alpha[a,b]$, it follows from Rudin 6.11 that $\varphi \circ f \in R_\alpha[a,b]$. Hence, we have $|f| \in R_\alpha[a,b]$.

(ii) Recall that

$$|t| \le s \Longleftrightarrow -s \le t \le s.$$

So, our goal is to show that

$$-\int_a^b |f| \, d\alpha \le \int_a^b f \, d\alpha \le \int_a^b |f| \, d\alpha.$$

Also, we have

$$-|f(x)| \le f(x) \le |f(x)| \quad \forall x \in [a,b].$$

So,

$$-\int_a^b |f(x)| \, d\alpha \le \int_a^b f(x) \, d\alpha \le \int_a^b |f(x)| \, d\alpha$$

as desired.

■

Theorem (Mean Value Theorem for Integrals). Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha : [a,b] \to \mathbb{R}$ is an increasing function and $\alpha(a) \ne \alpha(b)$. Then there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f \, d\alpha.$$

Proof. Since f is continuous on $[a,b]$ and $[a,b]$ is a compact interval in $\mathbb{R}$, it follows from the Extreme Value Theorem that f attains its max and min on $[a,b]$. Let

$$m = \min_{x\in[a,b]} f(x) \quad \text{and} \quad M = \max_{x\in[a,b}]f(x).$$

We have, for all $x \in [a,b]$, $m \le f(x) \le M$. Thus,

$$m(\alpha(b) - \alpha(a)) \le \int_a^b f(x) \, d\alpha \le M(\alpha(b) - \alpha(a)).$$

Hence,

$$m \le \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f \, d\alpha \le M.$$

Using the Intermediate Value Theorem, we see from the assumption that f being continuous on $[a,b]$ that

$$\exists c \in [a,b] \text{ such that } f(c) = \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f \, d\alpha.$$

■

Theorem (Generalized Mean Value Theorme for Integrals). Let $f : [a,b] \to \mathbb{R}$ be continuous, $\alpha : [a,b] \to \mathbb{R}$ is increasing, and $g \in R_\alpha[a,b]$ and either $g \ge 0$ on $[a,b]$ or $g \le 0$ on $[a,b]$. Then

$$\exists c \in [a,b] \text{ such that } \int_a^b fg \, d\alpha = f(c) \int_a^b g \, d\alpha.$$

Theorem (Additivity for R.S Integrals). Let $f : [a, b] \to \mathbb{R}$ be continuous, $\alpha : [a, b] \to \mathbb{R}$ is increasing, $c \in (a, b)$. Then

$$f \in R_\alpha[a, b] \iff (f \in R_\alpha[a, c] \text{ and } f \in R_\alpha[c, b]).$$

In this case, we have

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.$$

5.2 Lectures 13-14

5.2.1 Topics

- Theorem: For "nice" α we have $\displaystyle\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$;
- Theorem (change of variable)
- The Fundamental Theorem of Calculus
- Integration By Parts
- Unit step function, representing sums by R.S integrals

Lemma. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $\alpha : [a, b] \to \mathbb{R}$ is an increasing function, $P = \{x_0, x_2, \ldots, x_n\}$ is a partition of $[a, b]$ and $R \in \mathbb{R}$. Then

(1) If for all tags $(s_k)_{1\le k\le n}$ of P, we have $\sum_{k=1}^n f(s_k)\Delta\alpha_k \le R$, then $U(f, \alpha, P) \le R$.

(2) If for all tags $(s_k)_{1\le k\le n}$ of P, we have $R \le \sum_{k=1}^n f(s_k)\Delta\alpha_k$, then $R \le L(f, \alpha, P)$.

Proof. (1) If α is constant, then

$$\sum_{k=1}^n f(s_k)\Delta\alpha_k = 0$$

which implies

$$U(f, \alpha, P) = \sum_{k=1}^n M_k\Delta\alpha_k = 0.$$

So, we may assume that $\alpha(a) \ne \alpha(b)$. It is suffices to show that

$$\forall\varepsilon > 0 \;\; U(f, \alpha, P) \le R + \varepsilon.$$

Let $\varepsilon > 0$ be given. For each $k \in \{1, \ldots, n\}$, we have

$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x) \implies \exists s_k \in [x_{k-1}, x_k] \text{ such that } M_k - \frac{\varepsilon}{\alpha(b) - \alpha(a)} < f(s_k).$$

We have

$$\begin{aligned} U(f, \alpha, P) - \sum_{k=1}^n M_k\Delta\alpha_k &< \sum_{k=1}^n \Big[f(s_k) + \frac{\varepsilon}{\alpha(b) - \alpha(a)}\Big]\Delta\alpha_k \\ &= \sum_{k=1}^n f(s_k)\Delta\alpha_k + \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_{k=1}^n \Delta\alpha_k \\ &\le R + \varepsilon \end{aligned}$$

as desired.

(2) Completely analogous to (1).

■

Theorem (Rudin 6.17). Let $f : [a,b] \to \mathbb{R}$ be a bounded function, $\alpha : [a,b] \to \mathbb{R}$ be an increasing function, and $\alpha' \in R[a,b]$. Then

$$f \in R_\alpha[a,b] \iff f\alpha' \in R[a,b]$$

and in this case,

$$\int_a^b f \; d\alpha = \int_a^b f(x)\alpha'(x) \; dx.$$

Proof. It suffices to show that

$$\begin{aligned} U(f,\alpha) &= U(f\alpha'), \\ L(f,\alpha) &= L(f\alpha') \end{aligned}$$

Indeed, if we prove (*), then

$$\begin{aligned} f \in R_\alpha[a,b] &\iff U(f,\alpha) = L(f,\alpha) \\ &\iff U(f\alpha') = L(f\alpha') \\ &\iff f\alpha' \in R[a,b]. \end{aligned}$$

Moreover, (*) would imply that

$$\int_a^b f \; d\alpha = U(f,\alpha) = U(f\alpha') = \int_a^b f(x)\alpha'(x) \; dx.$$

In what follows, we will prove $U(f,\alpha) = U(f\alpha')$. The proof of $L(f,\alpha) = L(f\alpha')$ is analogous. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$. Let (s_k) be any tag of P. Note that by the Mean Value Theorem, we can find a $t_k \in (x_{k-1}, x_k)$ for all $1 \le k \le n$ such that

$$\begin{aligned} \Delta\alpha_k &= \alpha(x_k) - \alpha(x_{k-1}) \\ &= \alpha'(t_k)(x_k - x_{k-1}). \end{aligned}$$

We have

$$\begin{aligned} &\Big|\sum_{k=1}^n f(s_k)\Delta\alpha_k - \sum_{k=1}^n f(s_k)\alpha'(s_k)\Delta x_k\Big| \\ &= \Big|\sum_{k=1}^n f(s_k)\alpha'(t_k)\Delta x_k - \sum_{k=1}^n f(s_k)[\alpha'(t_k) - \alpha'(s_k)]\Delta x_k\Big| \\ &\le \sum_{k=1}^n |f(s_k)||\alpha'(t_k) - \alpha'(s_k)|\Delta x_k \\ &\le \hat{M}\sum_{k=1}^n |\alpha'(t_k) - \alpha'(s_k)|\Delta x_k && (\hat{M} = \sup_{x\in[a,b]} |f(x)|) \\ &\le \hat{M}\sum_{k=1}^n [\sup_{I_k} \alpha' - \inf_{I_k} \alpha']\Delta x_k && \text{(lemma 1)} \\ &= \hat{M}[U(\alpha', P) - L(\alpha', P)]. \end{aligned}$$

Hence, we have

$$\Big|\sum_{k=1}^n f(s_k)\Delta\alpha_k - \sum_{k=1}^n f(s_k)\alpha'(s_k)\Delta x_k\Big| \le \hat{M}[U(\alpha', P) - L(\alpha', P)].$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n} f(s_k)\Delta\alpha_k &\le \sum_{k=1}^{n} f(s_k)\alpha'(s_k)\Delta x_k + \hat{M}[U(\alpha',P) - L(\alpha',P)]\\ &\le U(f\alpha',P) + \hat{M}[U(\alpha',P) - L(\alpha',P)]. \qquad (1)\end{aligned}$$

By Lemma 5, we have

$$U(f\alpha',P) \le U(f,\alpha,P) + \hat{M}[U(\alpha',P) - L(\alpha',P)]. \qquad (3)$$

It follows from (1) and (2) that

$$|U(f,\alpha,P) - U(f\alpha',P)| \le \hat{M}[U(\alpha',P) - L(\alpha',P)].$$

Note that

$$\begin{aligned}U(f,\alpha) = \inf_{P\in\Pi} U(f,\alpha,P) &\implies \exists (P_n^{(1)}) \subseteq \Pi \text{ such that}\\ U(f\alpha') = \inf_{P\in\Pi} U(f\alpha',P) &\implies \exists (P_n^{(2)}) \subseteq \Pi \text{ such that } \lim_{n\to\infty} U(f\alpha',P_n^{(2)}) = U(f\alpha'). \qquad (2)\end{aligned}$$

Since $\alpha' \in R[a,b]$, there exists $(P_n^{(3)}) \subseteq \Pi$ such that

$$\lim_{n\to\infty} [U(\alpha',P_n^{(3)}) - L(\alpha',P_n^{(3)})] = 0.$$

Now, for each $n \in \mathbb{N}$, let $P_n = P_n^{(1)} \cup P_n^{(2)} \cup P_n^{(3)}$. We have

$$\begin{aligned}&\forall n \ge 1 \;\; U(f,\alpha) \le U(f,\alpha,P_n) \le U(f,\alpha,P_n^{(1)}) \implies \lim_{n\to\infty} U(f,\alpha,P_n) = U(f,\alpha) \qquad (4)\\ &\forall n \ge 1 \;\; U(f\alpha') \le U(f\alpha',P_n) \le U(f\alpha',P_n^{2}) \implies \lim_{n\to\infty} U(f\alpha',P_n) = U(f\alpha'). \qquad (5)\end{aligned}$$

Since P_n is a refinement of $P_n^{(3)}$, we have

$$\begin{aligned}&0 \le [U(\alpha',P_n) - L(\alpha',P_n)] \le U(\alpha',P_n^{(3)} - L(\alpha' P_n^{(3)}))\\ &\implies \lim_{n\to\infty} U(\alpha',P_n) - L(\alpha',P_n) = 0. \qquad (6)\end{aligned}$$

It follows from (3) that

$$\forall n \ge 1 \;\; 0 \le U(f,\alpha,P_n) - U(f\alpha',P_n) \le \hat{M}[U(\alpha',P_n) - L(\alpha',P_n)]$$

Applying the squeeze theorem as $n \to \infty$ to both sides of the inequality above, we have

$$\begin{aligned}&\lim_{n\to\infty} |U(f,\alpha,P_n) - U(f\alpha',P_n)| = 0\\ &\implies \left|\lim_{n\to\infty} (U(f,\alpha,P_n) - U(f\alpha',P_n))\right| = 0\\ &\implies |U(f,\alpha) - U(f\alpha')| = 0\\ &\implies U(f,\alpha) - U(f\alpha') = 0\\ &\implies U(f,\alpha) = U(f\alpha')\end{aligned}$$

■

Theorem (Rudin 6.19; Change of Variable). Let $f \in R_\alpha[a,b]$ and $\varphi : [A,B] \to [a,b]$ be an onto and strictly increasing function. If we let $g = f \circ \varphi$ and $\beta = \alpha \circ \varphi$, then

$$g \in R_\beta[A,B] \quad \text{and} \quad \int_a^b f\, d\alpha = \int_A^B g\, d\beta.$$

Proof. Since $\alpha : [a,b] \to \mathbb{R}$ is increasing and $\varphi : [A,B] \to [a,b]$ is an increasing, we see that $\beta = \alpha \circ \varphi$ is also increasing on $[A,B]$. Also, note that, there is a one-to-one correspondence between $\Pi[a,b]$ and $\Pi[A,B]$:

$$H : \Pi[a,b] \to \Pi[A,B]$$

where $P = \{x_0, x_1, \ldots, x_n\}$ corresponding to $[a,b]$ gets mapped to $Q = \{y_0, y_1, \ldots, y_n\}$ corresponding to $[A,B]$. Under this 1-1 correspondence, we have

$$\forall 1 \le k \le n \quad \varphi([y_{k-1}, y_k]) = [x_{k-1}, x_k].$$

and

$$\forall 1 \le k \le n \quad \Delta\beta_k = \beta(y_k) - \beta(y_{k-1}) = \alpha(\varphi(y_k)) - \alpha(\varphi(y_{k-1})) = \alpha(x_k) - \alpha(x_{k-1}) = \Delta\alpha_k.$$

and

$$\begin{aligned} \forall 1 \le k \le n \quad M_k^{(g)} = \sup_{y \in [y_{k-1}, y_k]} g(y) &= \sup_{y \in [y_{k-1}, y_k]} f \circ \varphi(y) \\ &= \sup_{x \in [x_{k-1}, x_k]} f(x) \\ &= M_k^{(f)} \end{aligned}$$

Thus, under the correspondence above, we have ■

Chapter 6

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Chapter 8

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Chapter 9

Week 9

9.1 Lectures 13-14

Theorem (Integration by Parts)**.** Let $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ are differentiable and let $u' \in R[a, b]$ and $v' \in R[a, b]$. Then we have

(1) $uv' \in R[a, b]$

(2) $u'v \in R[a, b]$

(3) $\displaystyle\int_a^b uv' \, dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v \, dx.$

Proof. (1) Since $u : [a, b] \to \mathbb{R}$ is differentiable, we have $u \in C[a, b]$. So, we have $u \in R[a, b]$. By assumption, $v' \in R[a, b]$ and so we can conclude that $uv' \in R[a, b]$.

(2) Using the same argument above, we have $uv' \in R[a, b]$.

(3) By the product rule, we have

$$(uv)' = u'v + uv'.$$

In particular, since $(uv)'$ is a sum of integrable functions, it belongs to $R[a, b]$. Now, we integrate both sides

$$\int_a^b (uv)' \, dx = \int_a^b u'v \, dx + \int_a^b uv' \, dx. \tag{I}$$

According to FTC I, we have

$$\int_a^b (uv)' \, dx = [uv]_{x=a}^{x=b} = u(b)v(b) - u(a)v(a). \tag{II}$$

Hence, we have (I) and (II) imply that

$$u(b)v(b) - u(a)v(a) = \int_a^b u'v \, dx + \int_a^b uv' \, dx$$

which further implies that

$$\int_a^b uv' \, dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v \, dx.$$

■

9.2 Lectures 15-16

9.2.1 Topics

In general, does it necessarily hold that $(xy)^n = x^n y^n$? Consider the following proposition.