Density of The Rationals

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1 The Density of the Rationals

Theorem 1.1. *(Archimedean Property)*
- *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$*
- *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$*

Before we head on to the proof, it is important to notice that $\mathbb{N}$ is not bounded above and we shall not prove this fact since we are taking this property of the set to be a given just like all the properties that are contained in $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Proof. Assume for sake of contradiction that $\mathbb{N}$ is bounded above. Using the Axiom of Completeness, $\mathbb{N}$ contains a supremum, say, $\sup \mathbb{N} = \alpha$. Using lemma 1.3.8, we know that there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n. \tag{$\epsilon = 1$}$$

This impplies that

$$\alpha < n + 1$$

but this shows that $n+1 \in \mathbb{N}$ which is a contradiction because we assumed that $\alpha \geq n$ for all $n \in \mathbb{N}$ thereby rendering α to no longer be an upper bound for $\mathbb{N}$. Hence, we have that there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$. The second part of this theorem follows immediately by setting $x = 1/y$. ■

Theorem 1.2. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) For every two $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. Our goal is to choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b \tag{1}$$

The idea is to choose a denominator large enough so that when we increment by size $\frac{1}{n}$ that it will be too big to increment over the open interval (a, b). Using the (2) of the Archimedean Property, we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a. \tag{2}$$

We now need to choose an $m \in \mathbb{Z}$ such that na is smaller than this chosen number. A diagram for choosing such a number is helpful. Hence,

Judging from our diagram, we can see that

$$m - 1 \leq na < m.$$

Focusing on the left part of the inequality, we can solve (2) for a and say that

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - 1/n) + 1 \\ &= nb \end{aligned}$$

This implies that $m < nb$ and consequently $na < m < nb$ which is equivalent to (1).

■

2 The Existence of Square Roots

Theorem 2.1. *There exists $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We need to show that $\alpha^2 = 2$. Hence, we need to show cases where $\alpha^2 < 2$ and $\alpha^2 > 2$. The idea behind these cases is to produce a contradiction that will show that having either one of these cases will violate the fact that $\alpha$is an upper bound for T and α is the least upper bound respectively.

Assume the first case, $\alpha^2 < 2$. We know that α is an upper bound for T. We need to construct an element that is larger than α. Hence, we construct

$$\alpha + \frac{1}{n} \in T \tag{1}$$

Squaring (1) we have that

$$\begin{aligned} \left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}. \end{aligned}$$

We can use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}.$$

Rearranging we get that

$$\frac{2\alpha+1}{n_0} < 2-\alpha^2$$

and consequently

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2-\alpha^2) = 2$$

But this means that $\alpha + 1/n_0 \in T$ showing that α is not an upper bound for T contradicting our assumption.

Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2-2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound.

■

3 Exercises

3.1 Exercise 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

Proof. Suppose $a, b \in \mathbb{Q}$. Then $p, q, m, n \in \mathbb{Z}$ such that $n, q \neq 0$. Hence, $a = \frac{p}{q}$ and $b = \frac{m}{n}$. Adding $a + b$ will give us

$$\begin{aligned} a + b &= \frac{p}{q} + \frac{m}{n} \\ &= \frac{pn + mq}{qn}. \end{aligned}$$

Since $pq + mn, qn \in \mathbb{Z}$ with $q, n \neq 0$, we have that $a + b \in \mathbb{Q}$. Now we multiply a and b together. Then we have

$$\begin{aligned} ab &= \frac{p}{q} \cdot \frac{m}{n} \\ &= \frac{pm}{qn}. \end{aligned}$$

Since $pm, qn \in \mathbb{Z}$ and $q, n \neq 0$, we have that $ab \in \mathbb{Q}$. ■

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Suppose for sake of contradiction that $at = r$ where $r \in \mathbb{Q}$. Solving for t, we have that $t = \frac{r}{a}$. But this tells us that $t \in \mathbb{Q}$ since $r, a \in \mathbb{Q}$ which is a contradics our assumption that $t \in \mathbb{I}$.

■

(c) Part (a) can be summarised by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. We can say that $s + t$ is an irrational number while st can either be rational or irrational depending if $s = t$ or $s \neq t$. If $s = t$, then st is rational and if $s \neq t$, then st is irrational. ■

3.2 Exercise 1.4.2

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. **Show that** $s = \sup A$.

Proof. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Since $s + \frac{1}{n}$ for all $n \in \mathbb{N}$ is an upper bound for A, we have that

$$\sup A \leq s + \frac{1}{n} \tag{1}$$

for all $n \in \mathbb{N}$. On the other hand, $s - \frac{1}{n}$ is a lower bound for A. Hence,

$$\sup A > s - \frac{1}{n} \tag{2}$$

for all $n \in \mathbb{N}$. We have (1) and (2) imply

$$s - \frac{1}{n} < \sup A \leq s + \frac{1}{n}. \tag{3}$$

This means that either $\sup A < s, \sup A > s$, or $\sup A = s$,. If $\sup A < s$, then $s - \sup A > 0$. Using the Archimedean Property, we can find an $n \in \mathbb{N}$ such that

$$s - \sup A > \frac{1}{n}$$

but this means that $\sup A < s - \frac{1}{n}$ which contradicts (3). On the other hand, if $\sup A > s$, then $\sup A - s > 0$. Using the Archimedean property again, we can find an $n \in \mathbb{N}$ such that

$$\sup A - s > \frac{1}{n}$$

but this means that $\sup A > s + \frac{1}{n}$ which is a contradiction since $\sup A < s + \frac{1}{n}$ from (3). Hence, it must be that $\sup A = s$. ■

3.3 Exercise 1.4.3

Prove that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion for the theorem to hold.

Proof. Suppose $x \in (0, \frac{1}{n})$, then $x > 0$. By the Archimedean Property, we can find an $N \in \mathbb{N}$ that is sufficiently large such that $x > \frac{1}{N}$. But this means that $x \in (0, 1/n)$ for all $n \in \mathbb{N}$. Hence, $x \notin \cap_{n=1}^{\infty}(0, \frac{1}{n})$ and then

$$\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset.$$

■

3.4 Exercise 1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup T = b$.

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Hence, we use lemma 1.3.8 and the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to state that for every $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $b - \epsilon < r < b$. But this means that $r \in T$ and $b - \epsilon$ is not an upper bound for T. Hence, $\sup T = b$.

■

Another proof for this:

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Since $a < b$ where $a, b \in \mathbb{R}$, we can find $x \in \mathbb{Q}$ such that $a < x < b$. Since $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ with $n \neq 0$, we have that $na < m < nb$. But note that nb is another upper bound for T for n sufficiently large and $nb > b$ implying that b is the least upper bound of T. Hence, $\sup T = b$. ■

3.5 Exercise 1.4.5

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$.

Proof. Consider the real numbers $a-\sqrt{p}$ and $b-\sqrt{p}$ where p is any prime number. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that

$$a-\sqrt{p} < r < b-\sqrt{p}$$

for some $r \in \mathbb{Q}$. Adding $\sqrt{p}$ to both sides, we have that

$$a < r+\sqrt{p} < b.$$

But know that $r+\sqrt{p} \in \mathbb{I}$ by (c) of Exercise 1.4.1. Hence, $t = r+\sqrt{p}$. We can follow the same procedure for trancendental numbers and make this conclusion. ■

3.6 Exercise 1.4.7

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Proof. Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound. ■

3.7 Exercise 1.4.6

Recall that a set B is dense in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set $\{r \in \mathbb{Q} : q \leq 10\}$

Solution. Yes, since $a < \frac{p}{10} < \frac{p}{q} < b$.

■

(b) The set of all rationals p/q such that q is a power of 2.

Proof. Yes since $a < \frac{p}{2^n} < b$ for $n \in \mathbb{N}$. ■

(c) The set of all rationals p/q with $10|p| \geq q$

Proof. ■

3.8 Extra Exercises

1. Show that $\sup A = r$ where $r \in \mathbb{R}$ and

$$A = \{q \in \mathbb{Q} : q < r\}.$$

Proof. We know that the $\sup A$ exists since the Archimedean Property implies that $A \neq \emptyset$ and r is an upper bound of A by definition of A. Now we need to show that r is the least upper bound of A. By definition, $\sup A \leq r$. Suppose for sake of contradiction that $\sup A < r$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a $q \in \mathbb{Q}$ such that

$$\sup A < q < r.$$

But this means that $q \in A$ implying that $\sup A$ is not an upper bound for A which is a contradiction. Hence, it must be that $\sup A \leq r$. Hence, $\sup A = r$. ■

2. Assume that A, B are non-empty sets of reals that are bounded above and $A \subseteq B$. Show that $\sup A \leq \sup B$.

Proof. Suppose $A, B \neq \emptyset$ where $A, B \subseteq \mathbb{R}$ and bounded above. Furthermore, $A \subseteq B$. By Axiom of Completeness, $\sup A$ and $\sup B$ exists. Using lemma 1.3.8, we can say that for every $\epsilon > 0$

$$\sup A - \epsilon \leq \alpha \tag{1}$$

for some $\alpha \in A$. Since $A \subseteq B$, $\alpha \in B$ so by definition of $\sup B$, we have that $\alpha \leq \sup B$. Hence, (1) implies that

$$\sup A - \epsilon \leq \sup B.$$

Of course, it follows immediately that $\sup A \leq \sup B$. ■

3. Given nonempty subsets A and B of positive real numbers, define

$$A \cdot B = \{z = x \cdot y : x \in A, y \in B\}$$

Show that $\sup(A \cdot B) = \sup A \cdot \sup B$

Proof. Since $A, B \neq \emptyset$ and bounded above, we can say that $\sup A$ and $\sup B$ exists. Label these supremum by the following $\sup A = \alpha$ and $\sup B = \beta$. By definition, we have that

$$x \leq \alpha, \text{ for all } x \in A$$
$$y \leq \beta, \text{ for all } y \in B.$$

Multiplying these inequalities together, we have that

$$xy \leq \alpha\beta \text{ for all } x \in A, y \in B.$$

Hence, we have that $\alpha\beta \in A \cdot B$ is an upper bound. Now we want to show that $\alpha\beta \in A \cdot B$ is the least upper bound. For every $\epsilon > 0$, we have

$$\alpha - \epsilon < a \leq \alpha$$
$$\beta - \epsilon < b \leq \beta$$

for some $a \in A$ and for some $b \in B$. Multiplying these two quantities together we have that

$$\alpha\beta - \epsilon(\alpha + \beta + \epsilon) \leq \alpha\beta.$$

Since we can make $\epsilon' = \epsilon(\alpha + \beta + \epsilon) > 0$ abirtrarly small so that $\alpha\beta - \epsilon'$ is not an upper bound of $A \cdot B$. Hence, we have that

$$\sup(A \cdot B) = \sup A \cdot \sup B.$$

■

4. Let $A \subseteq \mathbb{R}$ be a nonempty set. Define the following set as

$$-A = \{x : -x \in A\}.$$

Show that

$$\sup(-A) = -\inf A$$
$$\inf(-A) = -\sup A.$$

Proof. Since $-A \neq \emptyset$ and $-A$ is bounded above by every $a \in A$, we know that for every $\epsilon > 0$ we have that

$$\sup(-A) - \epsilon \leq -\alpha$$

for some $-\alpha \in -A$. Multiplying by a negative, we have that

$$\alpha \leq -\sup(-A) + \epsilon.$$

But this is just the lemma for the infimum so we have that $\inf A = -\sup(-A)$ so we have that

$$\sup(-A) = -\inf A.$$

Now we want to show (2). We can do the same process as we did above and use lemma 1.3.8 to say that for every $\epsilon > 0$, we have that

$$-\beta \leq \inf(-A) + \epsilon$$

for some $-\beta \in -A$. Multiplying by a negative again, we get that

$$-\inf(-A) - \epsilon \leq \beta.$$

Note that this is some $\beta \in A$. So by lemma 1.3.8, we have that $-\inf(-A) = \sup A$ and hence

$$\inf(-A) = -\sup A.$$

■

5. Let $A, B \subseteq \mathbb{R}$ be nonempty. Define the following sets as the following:

$$\sup(A - B) = \sup A - \inf B.$$

Proof. Since $A, B \neq \emptyset$, $\sup A$ and $\sup B$ exist by the axiom of completeness. In order to show (1), we would need to show that following:

$$\sup(A - B) \leq \sup A - \inf B,$$
$$\sup A - \inf B \leq \sup(A - B).$$

To show the first inequality, suppose we have $x - y \in A - B$ such that this bounds $\sup(A - B)$.Hence, we have

$$\sup(A - B) \leq x - y.$$

If we add $y \in B$ to both sides, then we get the following

$$\sup(A - B) + y \leq x.$$

But since $x \in A$ is bounded by $\sup A$, we have that

$$\sup(A - B) + y \leq \sup A.$$

Now we want to isolate $y \in B$ to get

$$\sup(A - B) - \sup A \leq -y$$

Since $-y \in -B$, we have that $-y \leq \sup(-B)$ for all $-y \in -B$. Hence,

$$\sup(A - B) - \sup A \leq \sup(-B).$$

But we know from problem 1 that $\sup(-B) = -\inf B$ so we have

$$\sup(A - B) \leq \sup A - \inf B.$$

Now we need to show the other inequality. By lemma 1.3.8, we have that for every $\epsilon > 0$, there exists some $\alpha \in A$ and $-\beta \in -B$ such that

$$\sup A - \epsilon/2 \leq \alpha,$$
$$\sup(-B) - \epsilon/2 \leq -\beta.$$

Adding these two inequalities, we have

$$\sup A + \sup(-B) - \epsilon \leq \alpha - \beta.$$

But this means we can take $\epsilon > 0$ abitrarly small to make the left-hand side not an upper bound. Hence, we have that

$$\sup A + \sup(-B) \leq \sup(A - B)$$

which implies that

$$\sup A - \inf B \leq \sup(A - B).$$

■

6. Given two sets of positive real numbers $A, B \neq \emptyset$ that are bounded, define

$$\frac{1}{A} = \left\{ z = \frac{1}{x} : x \in A \right\}.$$

Prove that

$$\sup \left(\frac{1}{A} \right) = \frac{1}{\inf A}$$

and prove that

$$\sup \left(\frac{1}{A} \right) = +\infty$$

if $\inf A > 0$.

Proof. Since $A \subseteq \mathbb{R}^+$ is bounded, we know that $x \geq \inf A$ for all $x \in A$. Since $x > 0$ and $\inf A > 0$, we have that

$$\frac{1}{x} \leq \frac{1}{\inf A}.$$

for all $z = \frac{1}{x} \in \frac{1}{A}$. This shows that $\frac{1}{\inf A}$ is an upper bound for $\frac{1}{A}$. Now we want to show that $\frac{1}{\inf A}$ is the least upper bound. Using lemma 1.3.8, we have for every $\epsilon > 0$

$$\alpha \leq \inf A + \epsilon$$

for some $\alpha \in A$. Since $\alpha > 0$ and $\inf A > 0$, we get that

$$\frac{1}{\inf A} - \epsilon' \leq \frac{1}{\alpha}$$

which holds true for all $e' = \frac{\epsilon}{\inf A \cdot \alpha} > 0$ for some $\frac{1}{\alpha} \in \frac{1}{A}$. Using lemma 1.3.8, we have, indeed, $\frac{1}{\inf A}$ is the least upper found of $\frac{1}{A}$. Hence, we obtain

$$\sup \left(\frac{1}{A} \right) = \frac{1}{\inf A}.$$

■