Understanding Analysis Notes

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Contents

Chapter 1

The Axiom of Completeness

Theorem 1.0.1 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition 1.0.1. We call a set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Otherwise, a set is *bounded below* if there exists a $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for every $a \in A$.

Definition 1.0.2. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it satisfies the following criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We denote the least upper bound of a set A by calling it the *supremum* of A i.e $\sup(A)$. Similarly, we denote the greatest upper bound of set A by calling it the *infimum* of A i.e $\inf(A)$.

Note that a set can have many upper/lower bounds. But there can only exist one supremum and one infimum. In other words, these bounds are unique. Furthemore, the infimum and supremum need not be in the set.

Consider the following set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

This set is bounded above and below. In addition, we can see that $\sup(A) = 1$ and $\inf(A) = 0$ (this is because each subsequent number in the sequence gets smaller and smaller).

Definition 1.0.3. We say that $a_0 \in \mathbb{R}$ is a *maximum* of the set A if $a_0 \in A$ and $a \leq a_0$ for all $a \in A$. Likewise, we say that $a_1 \in \mathbb{R}$ is a *minimum* of A if $a_1 \in A$ and $a \geq a_1$ for every $a \in A$.

If we have an open set $(0, 2)$ then the end points of this set are the infimum and supremum of the set respectively. Note that the maximum and the minimum do not exists because the infimum and the supremum are not in the set. If this set were to be closed, then the supremum and infimum would be in the set which implies that the max and min exists.

Now consider the Example

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

Notice that when we try and search for the supremum for this set, we cannot find one since we can always find a smaller number for an upper bound. One might say that $r = \sqrt{2}$ is the supremum of S but this is false since $r \notin \mathbb{Q}$ and is irrational.

Example. Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and is bounded above. Let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Prove that $\sup(c + A) = c + \sup(A)$

Proof. We use defintion 0.2 to prove this proposition. First, we need to prove that this $\sup(c + A)$ is an upper bound. We have $\sup(A) = s$ for some $s \in A$ if $s \geq a$ for all $a \in A$. We find that adding $c \in \mathbb{R}$ gives us

$$c + s \geq c + a.$$

Hence, we have that $c + s$ is an upper bound for the set $c + A$.

Next, we prove that $\sup(c + A) = c + s$ is the *least upper bound*. We know that $c + s \geq c + a$ for all $a \in A$. Suppose we have another upper bound $b \in A$ such that $c + a \leq b$ for all $a \in A$. Another manipulation gives us $a \leq b - c$ for all $a \in A$. Since $\sup(A) = s$ is the least upper bound for A, it follows that $s \leq b - c$. Hence, we have

$$c + s \leq b \implies \sup(c + A) = c + \sup(A).$$

■

There is another way to restate part (ii) of defintion 0.2 i.e

Lemma 1.0.1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

Proof. For the forward direction, suppose that $s = \sup A$ and consider $s - \epsilon$. Since s is an upper bound, we have that $s - \epsilon < s$. This means that $s - \epsilon$ is not an upper bound. Hence, we can find an element $a \in A$ such that $s - \epsilon < a$ because otherwise $s - \epsilon$ would be an upper bound. This concludes the forward direction.

For the backwards direction, assume s is an upper bound. We must satisfy part (ii) of defintion 0.2. Let $\epsilon > 0$, then $\epsilon = s - b$. But since any number smaller than s is not an upper bound, we have that $s \leq b$ if b is any other upper bound for S. Hence, $s = \sup A$. ■

Exercises

Exercise 1.3.3

(a) Let $A \neq \emptyset$ and bounded below, and define

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\},$$

Show that $\inf A = \sup B$.

Proof. Our goal is to show that both $\inf A \leq \sup B$ and $\inf A \geq \sup B$. Since $B \neq \emptyset$ and bounded above, we have that the $\sup B$ exists. First we want to show that $\inf A \leq \sup B$. By definition of $\sup B$, it is the greatest lower bound of B. Since $A \neq \emptyset$ and bounded below, we have that the $\sup B$ is greater than any lower bound of A. Hence, we have that $\inf A \leq \sup B$. Now we want to show that $\inf A \geq \sup B$. Suppose for sake of contradiction that $\inf A < \sup B$. Since $A \neq \emptyset$ and bounded below, we have that

$$a - \epsilon \geq \inf A \tag{1}$$

for some $a \in A$. Our goal is to show that there exists that some $a \in A$ is less than $\sup B$. Hence, choose $\epsilon = \sup B + a$ such that (1) and $\inf A < \sup B$

implies that

$$\begin{aligned} a - \epsilon &< \sup B \\ a - (\sup B + 3a) &< \sup B \\ a &< \sup B. \end{aligned}$$

But this is a contradiction since every element in A has to be bigger than B i.e $b > a$ for all $b \in B$. Hence, it must be the case that $\inf A \geq \sup B$. Hence, we have that $\inf A = \sup B$. ■

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. There is no need to assert that greatest lower bounds exist as part of the axiom because we can always seperate a set A that is bounded below into a set B that just consists of lower bounds from A. Since the infimum is just the greatest lower bound, it is equivalent to taking the supremum of a set of lower bounds. We can do this because every element in B is bounded above by every element in A which is permitted by the Axiom of Completeness. ■

Exercise 1.3.4

Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

1. Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\cup_{k=1}^{n} A_k\right)$.

   Solution. For $\sup(A_1 \cup A_2)$, we have

   $$\sup(A_1 \cup A_2) = \sup\{A_1, A_2\}$$

   and for $\sup\left(\cup_{k=1}^{n} A_k\right)$, we have

   $$\sup\left(\cup_{k=1}^{n} A_k\right) = \sup\{A_k\}$$

   for $k \in \mathbb{N}$. ■

2. Consider $\sup\left(\cup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

   Solution. No, because then $\cup_{k=1}^{\infty} A_k$ would be an unbounded set. ■

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set

$$cA = \{ca : a \in A\}.$$

(a) If $c \geq 0$, show that $\sup A(cA) = c \sup A$.

Proof. Suppose $c \geq 0$. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Denote $\alpha = \sup A$. By definition, we have that $\alpha \geq a$ for all $a \in A$. Multiplying by $c \geq 0$, we have that

$$\begin{aligned} c\alpha &\geq ca \\ c \sup A &\geq ca \end{aligned}$$

for all $a \in A$. This shows that $c \sup A$ is an upper bound for cA.

Now we want to show that this upper bound is the least upper bound in cA. Hence, take any upper bound in $b \in A$ such that $ca \leq b$. This implies that $a \leq b/c$. Since $\alpha = \sup A$ is the least upper bound for A, we have that $\sup A \leq b/c$ which further implies that $c \sup A \leq b$ showing that it is the least upper bound in cA. Hence, we have that $\sup A(cA) = c \sup A$. ∎

(b) Postulate a similar type of statement for $\sup(cA)$ for the cases $c < 0$.

Postulate. For cases $c < 0$, we have $\sup(cA) = c \inf A$. ∎

Exercise 1.3.6

Suppose that $A, B \neq \emptyset$ and bounded above. Prove that

$$\sup(A + B) = \sup A + \sup B$$

We prove this proposition using two methods. One deals with direct application of the definition and the other deals with using lemma 1.3.8.

Proof. Our goal is to show that

$$\sup(A + B) = \sup A + \sup B$$

We know that since $A, B \neq \emptyset$ and bounded above, we have that $\sup A, \sup B$ exists. we denote the supremums by the following

$$\begin{aligned} \sup A &= \alpha, \\ \sup B &= \beta. \end{aligned}$$

It suffices to show that following

$$\sup(A+B) \leq \sup A + \sup B \tag{1.1}$$

and

$$\sup(A+B) \geq \sup A + \sup B \tag{1.2}$$

We first show (2) first then we will show (1) next. Suppose we have abirtrary $x \in A$ and $y \in B$. Because $A, B \neq \emptyset$ and bounded above, we know that the set $A + B$ is also non-empty and bounded above which means its supremum $\sup(A + B)$ also exists. Hence, we know that

$$x + y \leq \sup(A+B)$$

Subtracting $y \in B$ to the other side of this inequality will yield

$$x \leq \sup(A+B) - y$$

But we know that since $x \in A$ and $\sup A \geq a$ for all $a \in A$, we have that

$$\sup A \leq \sup(A+B) - y.$$

Likewise, we isolate $y \in B$ to the other side and note that $b \leq \sup B$ for all $b \in B$. Then we get the following:

$$\begin{aligned} y &\leq \sup(A+B) - \sup A \\ \sup B &\leq \sup(A+B) - \sup A \end{aligned}$$

But this implies that

$$\sup A + \sup B \leq \sup(A+B)$$

Now we show (1). By lemma 1.3.8, we know that for all $\epsilon > 0$, we have that

$$\begin{aligned} \sup A - \frac{\epsilon}{2} &< a \\ \sup B - \frac{\epsilon}{2} &< b \end{aligned}$$

for some $a \in A$ and $b \in B$. Adding these two together we have that

$$\sup A + \sup B - \epsilon < a + b$$

But we also know that a and b are bounded above by their respective supremums so

$$\sup A + \sup B - \epsilon < a + b \leq \sup A + \sup B$$

Setting $\epsilon = \sup A + \sup B - \sup(A + B)$. Hence, we have that

$$\sup(A + B) \leq \sup A + \sup B.$$

Since we have (1) and (2), we see that

$$\sup(A + B) = \sup A + \sup B$$

■

Exercise 1.3.7

Prove that if a is an upper bound for A, and $a \in A$, then $\sup A = a$.

Proof. We want to show that $a \leq \sup A$ and $a \geq \sup A$. We start with the former. Since $A \neq \emptyset$ and bounded above, we have that the $\sup A$ exists. Label this supremum as $\sup A = \beta$. For every $\epsilon > 0$, we have that there exists $b \in A$ such that $\sup A - \epsilon \leq b$. Choose $\epsilon = 2\sup A - a - b$ such that

$$\begin{aligned} \sup A - 2\sup A + a + b &\leq b \\ -\sup A + a &\leq 0 \\ \implies a &\leq \sup A \end{aligned}$$

Now for the latter case, since $\sup A = \beta$ is the least upper bound of A and $a \in A$, it follows immediately that $a \leq \sup A$ for all $a \in A$. Hence, $\sup A = a$

■

Exercise 1.3.8

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Suppose $\sup A < \sup B$. Since we have $\sup B$, by lemma 1.3.8 we can say that for every $\epsilon > 0$, there exists $b \in B$ such that

$$\sup B - \epsilon < b \tag{1}$$

Choose $\epsilon = \sup B - \sup A$. We can do this because $\sup A < \sup B$. Hence, (1) implies

$$\begin{aligned} \sup B - \epsilon &< b \\ \sup B - (\sup B - \sup A) &< b \\ \sup A &< b. \end{aligned}$$

By definition, $\sup A$ is the least upper bound for A. Since $\sup A \geq a$ for all $a \in A$, it follows that from (1) that $a < b$ for all $a \in A$. Hence, for some $b \in B$, b is an upper bound for A.

$\blacksquare$

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Exercise 1.3.10 (Cut Property)

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

Proof. Suppose A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$. By Axiom of Completeness, A and B are bounded above and below respectively. This implies that their supremum and infimums exists.

Firstly, we want to show that there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$. Since $a < b$ for all $a \in A$ and $b \in B$, every $b \in B$ is an upper bound for A. Denote B as the set of upper bounds for A. Hence, there must exist $c \in B$ such that c is the **least upper bound** for A due to the Axiom of Completeness. Furthermore, note that $\sup A \in B$ and not in A since $A \cap B = \emptyset$ which means $\sup A \in \mathbb{R}$. Hence, $\sup A \leq b$. But $x \in A$ so $x \leq \sup A$.

Now we want to show there exists $c \in \mathbb{R}$ such that $x \geq c$. Since every $a \in A$ is a lower bound for B and that $B \neq \emptyset$, there must exist an element in A such that it is the **greatest lower bound** for B. Denote this element as $c = \inf B$. Hence, $\inf B \geq a$ for all $a \in A$. Furthermore, $\inf B \in A$ and not in B since $A \cap B = \emptyset$ so $\inf B \in \mathbb{R}$ when we union A and B together. Since $x \in B$, we have that $\inf B \leq x$.

Furthermore, B is nonempty and bounded below and A is the set of lower bounds for B, we have that $\inf B = \sup A = c \in \mathbb{R}$. ■

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove that $\sup E$ exists.

Proof. Assume $\mathbb{R}$ possesses the Cut Property and let $E \neq \emptyset$ that is bounded above. Suppose we have that $E \subseteq \mathbb{R}$. Since $\mathbb{R}$ possesses the cut property, we can find $c \in \mathbb{R}$ such that $x \leq c$ if $x \in E$. Since $A \cap B = \emptyset$, $c \in A \cup B = \mathbb{R}$. Hence, either $c \in A$ or $c \in B$. If $c \in A$, then c is not an upper bound for E since every $a \in A$ is less than every $b \in B$. Furthermore, if $c \in A$ and A is the set of lower bounds for B, then it would contradict that c is an upper bound for E. Thus, we must have $c \in B$. Since $c \in B$, B is the set of upper bounds for E, and $E \neq \emptyset$ and bounded above, $c \in B$ is the smallest element in B which makes it the **least upper bound** for E. Hence, $c = \sup E$ exists. ■

1.1 Consequences of Completeness

The first application of the Axiom of Completeness is a result that says that the real line contains no gaps.

Theorem 1.1.1. *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Our goal is to produce a real number x such that this element is in every closed interval I_n for every $n \in \mathbb{N}$. Using the Axiom of Completeness, we can denote the following sets

$$A = \{a_n : n \in \mathbb{N}\}$$
$$B = \{b_n : n \in \mathbb{N}\}$$

where A and B consists of the left-hand and right-hand endpoints respectively. Since every closed interval are nested, we know that every b_n serves as an upper bound for A. By the Axiom of completeness, we can say that a supremum exists for A and we can label this supremum as $x = \sup A$. By definition, this is an upper bound for A. Hence, we have that $a_n \leq x$. But since x is the least upper bound and every $b_n \in B$ is an upper bound for every $a_n \in A$, we have that $x \leq b_n$. Hence, we have that $a_n \leq x \leq b_n$ which means that $x \in I_n$ for all $n \in \mathbb{N}$. This precisely means that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. ■

1.1.1 The Density of the Rationals

Theorem 1.1.2. *(Archimedean Property)*

- *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$*
- *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$*

Before we head on to the proof, it is important to notice that $\mathbb{N}$ is not bounded above and we shall not prove this fact since we are taking this property of the set to be a given just like all the properties that are contained in $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Proof. Assume for sake of contradiction that $\mathbb{N}$ is bounded above. Using the Axiom of Completeness, $\mathbb{N}$ contains a supremum, say, $\sup \mathbb{N} = \alpha$. Using lemma 1.3.8, we know that there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n. \qquad (\epsilon = 1)$$

This impplies that

$$\alpha < n + 1$$

but this shows that $n + 1 \in \mathbb{N}$ which is a contradiction because we assumed that $\alpha \geq n$ for all $n \in \mathbb{N}$ thereby rendering α to no longer be an upper bound for $\mathbb{N}$. Hence, we have that there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$. The second part of this theorem follows immediately by setting $x = 1/y$. ■

Theorem 1.1.3. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) For every two $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. Our goal is to choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b \qquad (1)$$

The idea is to choose a denominator large enough so that when we increment by size $\frac{1}{n}$ that it will be too big to increment over the open interval (a, b). Using the (2) of the Archimedean Property, we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a. \tag{2}$$

We now need to choose an $m \in \mathbb{Z}$ such that na is smaller than this chosen number. A diagram for choosing such a number is helpful. Hence,

Judging from our diagram, we can see that

$$m - 1 \leq na < m.$$

Focusing on the left part of the inequality, we can solve (2) for a and say that

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - 1/n) + 1 \\ &= nb \end{aligned}$$

This implies that $m < nb$ and consequently $na < m < nb$ which is equivalent to (1).

■

1.2 The Existence of Square Roots

Theorem 1.2.1. *There exists $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We need to show that $\alpha^2 = 2$. Hence, we need to show cases where $\alpha^2 < 2$ and $\alpha^2 > 2$. The idea behind these cases is to produce a contradiction that will show that having either one of these cases will violate the fact that $\alpha$is an upper bound for T and α is the least upper bound respectively.

Assume the first case, $\alpha^2 < 2$. We know that α is an upper bound for T. We need to construct an element that is larger than α. Hence, we construct

$$\alpha + \frac{1}{n} \in T \tag{1}$$

Squaring (1) we have that

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

We can use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

Rearranging we get that

$$\frac{2\alpha + 1}{n_0} < 2 - \alpha^2$$

and consequently

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2$$

But this means that $\alpha + 1/n_0 \in T$ showing that α is not an upper bound for T contradicting our assumption.

Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound.

■

1.2.1 Exercises

Exercise 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

Proof. Suppose $a, b \in \mathbb{Q}$. Then $p, q, m, n \in \mathbb{Z}$ such that $n, q \neq 0$. Hence, $a = \frac{p}{q}$ and $b = \frac{m}{n}$. Adding $a + b$ will give us

$$\begin{aligned} a + b &= \frac{p}{q} + \frac{m}{n} \\ &= \frac{pn + mq}{qn}. \end{aligned}$$

Since $pq + mn, qn \in \mathbb{Z}$ with $q, n \neq 0$, we have that $a + b \in \mathbb{Q}$. Now we multiply a and b together. Then we have

$$\begin{aligned} ab &= \frac{p}{q} \cdot \frac{m}{n} \\ &= \frac{pm}{qn}. \end{aligned}$$

Since $pm, qn \in \mathbb{Z}$ and $q, n \neq 0$, we have that $ab \in \mathbb{Q}$. ■

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Suppose for sake of contradiction that $at = r$ where $r \in \mathbb{Q}$. Solving for t, we have that $t = \frac{r}{a}$. But this tells us that $t \in \mathbb{Q}$ since $r, a \in \mathbb{Q}$ which is a contradics our assumption that $t \in \mathbb{I}$.

■

(c) Part (a) can be summarised by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. We can say that $s+t$ is an irrational number while st can either be rational or irrational depending if $s=t$ or $s \neq t$. If $s=t$, then st is rational and if $s \neq t$, then st is irrational. ■

Exercise 1.4.2

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. **Show that** $s = \sup A$.

Proof. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Since $s + \frac{1}{n}$ for all $n \in \mathbb{N}$ is an upper bound for A, we have that

$$\sup A \leq s + \frac{1}{n} \tag{1}$$

for all $n \in \mathbb{N}$. On the other hand, $s - \frac{1}{n}$ is a lower bound for A. Hence,

$$\sup A > s - \frac{1}{n} \tag{2}$$

for all $n \in \mathbb{N}$. We have (1) and (2) imply

$$s - \frac{1}{n} < \sup A \leq s + \frac{1}{n}. \tag{3}$$

This means that either $\sup A < s, \sup A > s$, or $\sup A = s,$. If $\sup A < s$, then $s - \sup A > 0$. Using the Archimedean Property, we can find an $n \in \mathbb{N}$ such that

$$s - \sup A > \frac{1}{n}$$

but this means that $\sup A < s - \frac{1}{n}$ which contradicts (3). On the other hand, if $\sup A > s$, then $\sup A - s > 0$. Using the Archimedean property again, we can find an $n \in \mathbb{N}$ such that

$$\sup A - s > \frac{1}{n}$$

but this means that $\sup A > s + \frac{1}{n}$ which is a contradiction since $\sup A < s + \frac{1}{n}$ from (3). Hence, it must be that $\sup A = s$. ■

Exercise 1.4.3

Prove that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion for the theorem to hold.

Proof. Suppose $x \in (0, \frac{1}{n})$, then $x > 0$. By the Archimedean Property, we can find an $N \in \mathbb{N}$ that is sufficiently large such that $x > \frac{1}{N}$. But this means that $x \in (0, 1/n)$ for all $n \in \mathbb{N}$. Hence, $x \notin \cap_{n=1}^{\infty}(0, \frac{1}{n})$ and then

$$\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset.$$

■

Exercise 1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup T = b$.

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Hence, we use lemma 1.3.8 and the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to state that for every $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $b - \epsilon < r < b$. But this means that $r \in T$ and $b - \epsilon$ is not an upper bound for T. Hence, $\sup T = b$.

■

Another proof for this:

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Since $a < b$ where $a, b \in \mathbb{R}$, we can find $x \in \mathbb{Q}$ such that $a < x < b$. Since $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ with $n \neq 0$, we have that $na < m < nb$. But note that nb is another upper bound for T for n sufficiently large and $nb > b$ implying that b is the least upper bound of T. Hence, $\sup T = b$. ■

Exercise 1.4.5

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. Consider the real numbers $a - \sqrt{p}$ and $b - \sqrt{p}$ where p is any prime number. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that

$$a - \sqrt{p} < r < b - \sqrt{p}$$

for some $r \in \mathbb{Q}$. Adding $\sqrt{p}$ to both sides, we have that

$$a < r + \sqrt{p} < b.$$

But know that $r + \sqrt{p} \in \mathbb{I}$ by (c) of Exercise 1.4.1. Hence, $t = r + \sqrt{p}$. We can follow the same procedure for trancendental numbers and make this conclusion. ■

Exercise 1.4.7

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Proof. Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound. ■

Exercise 1.4.6

Recall that a set B is dense in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set $\{r \in \mathbb{Q} : q \leq 10\}$

Solution. Yes, since $a < \frac{p}{10} < \frac{p}{q} < b$.

■

(b) The set of all rationals p/q such that q is a power of 2.

Proof. Yes since $a < \frac{p}{2^n} < b$ for $n \in \mathbb{N}$. ■

(c) The set of all rationals p/q with $10|p| \geq q$

Proof. ■

1.3 Cardinality

1.3.1 Correspondence

Definition 1.3.1. A function $f : A \to B$ is *one-to-one* if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, there exists an element $a \in A$ for which $f(a) = b$.

An equivalent defintion for a function to be one-to-one is the following:

Definition 1.3.2. A function $f : A \to B$ is *one-to-one* if $f(a_1) \neq f(a_2)$ implies that $a_1 = a_2$.

A function that is both one-to-one and onto is said to be bijective. Meaning that we have a one-to-one correspodence between the sets A and B. Another way to explain a function being injective is to say that no two elements from A can map to the same element in B (think of the function x^2). And a function being onto can be explained as every element in A has to be mapped to an element in B.

From an algebraic perspective, we can denote a function being bijective to mean the same thing as two sets having the same cardinality i.e we can say that

Definition 1.3.3. Two sets A and B have the same cardinality if there exists $f : A \to B$ that is both one-to-one and onto. We can denote this symbolically as $A \sim B$

Example. Some examples of bijective maps are

1. Let the following map $f : \mathbb{N} \to \mathbf{E}$ be defined as $f(n) = 2n$. We can see that $\mathbb{N} \sim \mathbf{E}$. It's true that $\mathbf{E}$ is indeed a subset of $\mathbb{N}$, but do not conclude that it is a smaller set than $\mathbb{N}$ since they have the same cardinality or isomorphic to each other.

2. We can show this again. This time let us have a map $f : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ is odd.} \\ -n/2 \text{ if } n \text{ is even.} \end{cases}$$

   We have that $\mathbb{N} \sim \mathbb{Z}$ indeed.

1.3.2 Countable Sets

Definition 1.3.4. A set A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an *uncountable set*.

Theorem 1.3.1. *Let* $\mathbb{Q}, \mathbb{R}$. *Then*

- *The set* $\mathbb{Q}$ *is countable.*
- *The set* $\mathbb{R}$ *is uncountable.*

Proof. 1. Suppose we define A_n to be split into two sets. When $n = 1$, define A_n to be

$$A_1 = \{0\}$$

and define A_n when $n \geq 2$ as

$$A_n = \left\{ \pm \frac{p}{q} : \text{ where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

We can observe here that for every $n \in \mathbb{N}$ we can find every element of $\mathbb{Q}$ exactly once in the sets we have defined. So we can conclude that our map is onto. Since we designed our sets so that each rational numer appears once and the fact that for $n = 1$ and $n \geq 2$ produces two disjoint sets, we can see that our map is also one-to-one.

2. We can prove that second statement of theorem by contradiction. Assume for the sake of contradiction that there exists a *one-to-one* and *onto* function where $f : \mathbb{N} \to \mathbb{R}$. Letting $x_1 = f(1)$ and $x_2 = f(2)$ and so on, then we can enumerate each element of $\mathbb{R}$ i.e

$$\mathbb{R} = \{x_1, x_2, x_3, ...\}.$$

Using the Nested Interval Property, we will now produce a real number that is not in this set. Let I_n be a closed interval which does not contain x_n but contains x_{n+1}. Furthermore, I_{n+1} is contained within I_n. Note that within I_n there are two sets which are disjoint and x_{n+1} can be in either one of these sets. Now consider the following intersection $\cap_{n=1}^{\infty} I_n$. Using our construction that every $x_n \notin I_n$, then we can say that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

But this is a contradiction because the nested interval property asserts that this intersection is nonempty meaning that every $x \in \mathbb{R}$ is contained in the above set. Hence, we cannot emumerate every single element x_n of $\mathbb{R}$. Therefore, $\mathbb{R}$ is an *uncountable* set.

$\blacksquare$

This gives us three insights:

1. The smallest type of infinite set is the countable set.
2. We can create another set by deleting or inserting elements into it.
3. Anything smaller than a countable set is either finite or countable.

We can create $\mathbb{R}$ by taking the union of $\mathbb{Q}$ and $\mathbb{I}$. Since $\mathbb{R}$ is not countable and $\mathbb{Q}$ is, this would mean that the set of irrational numbers $\mathbb{I}$ would be uncountable. This tells us that $\mathbb{I}$ is a bigger subset of $\mathbb{R}$ than $\mathbb{Q}$.

We can summarize these results in the follow two theorems:

Theorem 1.3.2. *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Theorem 1.3.3. *1. If $A_1, A_2, ...A_n$ are each countable sets, then the union of*

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.3.3 Exercises

Exercise 1.5.1

Finish the following proof for Theorem 1.5.7.

Proof. Assume B is a countable set. So there exists a map $f : \mathbb{N} \to B$ such that f is surjective and injective. Let $A \subseteq B$ be an infinite subset of B. We want to show that A is countable. That is, A is both

1. injective
2. surjective.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Let $g : \mathbb{N} \to A$ be the map defined by

$$g(1) = f(n_1).$$

To show injectivity of g, we proceed via induction on the index $i \in \mathbb{N}$. Let the base case be $i = 2$. Then suppose $g(1) = g(2)$. By definition of g and injectivity of f, we have that

$$\begin{aligned} g(1) &= g(2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

But this means that $n_2 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Hence, g is injective. Now for the inductive step, assume this holds for every $1 \leq i \leq k-1$. We want to show that this holds for $i = k$. Suppose that

$$g(1) = g(k).$$

By defintion of g and injectivity of f, we have that

$$\begin{aligned} f(n_1) &= f(n_k) \\ n_1 &= n_k. \end{aligned}$$

But this also means that $n_k = \min\{n \in \mathbb{N} : f(n) \in \mathbb{N}\}$. Hence, g is injective.

Now we want to show that g is surjective. Note that we have

$$g(i) = A \cap \{f(n_1), f(n_2), f(n_3), ..., f(n_k)\}.$$

Then by definition of g, we have that $g(i) = f(n_i)$. Since f is surjective, there exists some $b \in B$ such that $f(n_i) = b$. But since $n_i = \{n_i \in \mathbb{N} : f(n_i) \in A\}$, we have that $f(n_i) \in A$ so g is surjective as well. Hence, we have that g is both injective and surjective which means that $\mathbb{N} \sim A$. Therefore, A is countable.

■

Exercise 1.5.2

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable.

The proposition is: $\mathbb{Q}$ is uncountable.

Proof. Assume for contradiction that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3\}$ and as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while the nested interval property implies that this intersection is nonempty. This contradiction implies $\mathbb{Q}$ must therefore be uncountable. ■

Thoughts. I think the main issue with this proof is when the author assumed that the set of rationals are closed. Since $\mathbb{Q}$ contains irrantional numbers within each subset of the $\mathbb{Q}$ as well as real numbers, $\mathbb{Q}$ cannot be closed. Hence, we cannot apply the nested interval property here. ■

Exercise 1.5.3

Prove theorem 1.5.8

Theorem 1.3.4. *1. If $A_1, A_2, ...A_n$ are each countable sets, then the union of*

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1. First, prove statement (i) for two countable sets, A_1 and A_2.

Proof. Suppose A_1 and A_2 are countable sets. Then $\mathbb{N} \sim A_1$ and $\mathbb{N} \sim A_2$. Furthermore, we have that the maps $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to A_2$ are bijective. Our goal is to show the union $A_1 \cup A_2$ is also countable i.e we need to show that the map $h : \mathbb{N} \to A_1 \cup A_2$ is bijective. Before we proceed, let us replace A_2 with the following set B_2 defined as

$$B_2 = A_2 \setminus A_1 = \{h(n) \in A_2 : h(n) \notin A_1\}.$$

Now our following map is $h : \mathbb{N} \to A_1 \cup B_2$ (this is equivalent to $A_1 \cup A_2$)and define it as follows

$$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k + 1 \in B_2 \end{cases}$$

Suppose we have $n_1, n_2 \in \mathbb{N}$ and $h(n_1) = h(n_2)$. Since f and g are injective, we have

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

This shows that h is injective (the same process can be applied to g when $h \in B_2$). Note that $A_1 \cap B_2 = \emptyset$ because otherwise h would not be well defined. Now we need to show that h is surjective. Since f and g are surjective, there exists either $x \in A_1$ or $x \in B_2$ such that $h(n) = f(n) = x$ or $h(n) = g(n) = x$. Hence, we have that h is surjective. Since h is a bijective map, we now have that $\mathbb{N} \sim A_1 \cup B_2$.

Suppose we use induction on the index $i \in \mathbb{N}$. Since we have already proven the base case for two countable sets, let us assume $A_1, A_2, ..., A_k$ are all countable sets such that for $i \leq k - 1$, the union $A_1 \cup A_2 ... \cup A_{k-1}$ is countable. Let's set $A' = A_1 \cup A_2 ... \cup A_{k-1}$. Our goal is to show that the union $A' \cup A_k$ is countable. Let's define the map $h : \mathbb{N} \to A' \cup B'$ such that

$$B' = A_k \setminus A' = \{h(n) \in A_k : x \notin A'\}.$$

and

$$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k + 1 \end{cases}$$

Let $n_1, n_2 \in \mathbb{N}$. Since A' and A_k are countable sets, we have that

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

Hence, h is injective. Now we want to show that h is surjective. If either $h(n) \in A'$ or $h(n) \in A_k$, then since $f : \mathbb{N} \to A'$ and $g : \mathbb{N} \to A_k$ are surjective functions, we have that there exists $x \in A_k$ or $x \in A'$ such that $h(n) = x$. Hence, h is surjective as well. Since h is now a bijective function, we conclude that the union $A_1 \cup A_2 ... \cup A_k$ is countable.

■

2. Explain why induction cannot be used to prove part (1) of Theorem 1.5.8 from part (2)

   Solution. We cannot use induction on part (2) of theorem 1.5.8 because the index itself $n \in \mathbb{N}$ is infinite and induction only works only finite n. ■

Proof of the second part of theorem

Proof. Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of countable sets. Define the union

$$S = \bigcup_{n \in \mathbb{N}} S_n.$$

for all $n \in \mathbb{N}$. Assume each S_n is disjoint. Otherwise, let S_1 such that for each $n \geq 1$, define

$$S'_{n+1} = S_{n+1} \setminus S_n = \{x \in S_{n+1} : x \notin S_n\}.$$

This is to ensure that our following map is well-defined. let F_n denote the set of all injections from $S_n \to \mathbb{N}$ Let $\varphi : S \to \mathbb{N} \times \mathbb{N}$ be the map that is defined by

$$\varphi(x) = (n, f_n(x))$$

where $n \in \mathbb{N}$ smallest guranteed by the Well-Ordering Principle. Since each f_n is an injection, it follows that φ is also an injection. Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists an injection $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Composing the two functions φ and α, we have that $\alpha \circ \varphi : S \to \mathbb{N}$ is an injection. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, we know that α is also surjective. Hence, the composition $\alpha \circ \varphi$ is also surjective. Therefore, we have that S is countable.

■

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Proof. Let (a, b) any interval. Then define the function $f : (a, b) \to \mathbb{R}$ as

$$f(x) = x^2$$

Our objective is to show that f is injective and surjective. To show that f is injective, we need to let $x_1, x_2 \in (a, b)$. Then suppose

$$f(x_1) = f(x_2).$$

Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^2 &= x_2^2 \\ x_1 &= x_2. \end{aligned}$$

This shows that f is injective. Now we want to show that f is surjective. Then there exists $\sqrt{y} \in (a, b)$. Let

$$x = \sqrt{y}.$$

Then we have that

$$\begin{aligned} x^2 &= y \\ f(x) &= y. \end{aligned}$$

Hence, f is surjective. Since $f : (a, b) \to \mathbb{R}$ is a bijective function, we have that $(a, b) \sim \mathbb{R}$. ■

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof. Let $(0, \infty) = \{x : x > 0\}$. Our goal is to show that $(a, \infty) \sim \mathbb{R}$. To show this, we need to show the map $f : (a, \infty) \to \mathbb{R}$ is bijective. Define $f$as the following:

$$f(x) = \ln(x).$$

Then suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in (a, \infty)$. Then

$$\begin{aligned} \ln(x_1) &= \ln(x_2) \\ x_1 &= x_2 \end{aligned}$$

Hence, we have that f is an injective function. Now we want to show that f is surjective. Then let $e^y = x \in (0, \infty)$. Then taking the natural log of both sides, we have that $\ln(x) = y$. Hence, we have that f is a surjective function. Since f is a bijective function, we know that $(0, \infty) \sim \mathbb{R}$. ∎

(c) Using open intervals makes it more covenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Proof. We want to show that $[0, 1) \sim (0, 1)$. Define the map $f : [0, 1) \to (0, 1)$ as

$$f(x) = \frac{1}{x - 1}$$

Our goal is to show that this map is bijective. Hence, we need to show that this map is both injective and surjective.

To show that f is injective. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in [0, 1)$. Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{1}{x_1 - 1} &= \frac{1}{x_2 - 1} \\ x_1 - 1 &= x_2 - 1 \\ x_1 &= x_2. \end{aligned}$$

Hence, f is injective.

To show that f is surjective, suppose we have $x - 1 = \frac{1}{y}$. Then

$$y = \frac{1}{x - 1}.$$

But we have that $f(x) = \frac{1}{x-1}$ so we have

$$f(x) = \frac{1}{x - 1} = y.$$

Hence, f is surjective.

Since f is bijective, we have that $[0, 1) \sim (0, 1)$. ∎

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

Solution. $A \sim A$ because A is a bijection onto itself (same elements map to the same elements of the same set). ∎

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Solution. If $A \sim B$, then the map $f : A \to B$ is a bijection. Meaning we can map unique eleemnts from A to unique elements to B. Since there is unique mapping of elements from $A \to B$ thenn we would expect to see the same thing when we map the same elements from $B \to A$. ∎

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation.*

Proof. Suppose we have three sets A, B, and C. Suppose $A \sim B$ and $B \sim C$ then we have two maps $f : A \to B$ and $g : B \to C$ that are bijective. Composing the two functions we get $g \circ f : A \to C$. We want to show that this mapping is also bijective. Let $x_1, x_2 \in A$ then suppose $g \circ f(x_1) = g \circ f(x_2)$. By definition of composition, we have

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) \\ f(x_1) &= f(x_2) && (\text{ } g \text{ is injective}) \\ x_1 &= x_2. && (\text{ } f \text{ is injective}) \end{aligned}$$

Hence, $g \circ f$ is an injective function. Now we want to show that $g \circ f$ is a surjective mapping. Since f is surjective, there exists a $y \in B$ such that $f(x) = y$. Since g is also surjective, there exists a $z \in C$ such that $g(y) = c$. Hence, we have that $g(f(x)) = c$ which means $g \circ f$ is a surjective mapping. Therefore, $A \sim C$. ∎

Exercise 1.5.11

[Shroder-Bernstein Theorem] Assume there exists an injective function $f : X \to Y$ and another injective function $g : Y \to X$. Show that $X \sim Y$. The strategy is to partition X and Y into components

$$\begin{aligned} X &= A \cup A' \\ Y &= B \cup B' \end{aligned}$$

with $A \cup A' = \emptyset$ and $B \cup B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

Solution. Taking disjoint sets prevents the problem of an element from either map mapping to two elements onto it's image. Thus, allowing us to have a well-defined function. Having two injective maps also would lead to the $X \sim Y$ because composing these two functions would allow us take a unique mapping from one element from each other. ∎

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is similar collection in Y.

Proof. We set proceed by induction on $n \in \mathbb{N}$ and let $P(n)$ be the statement that $\{A_n : n \in \mathbb{N}\}$ and $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint. Define

$$A_{n+1} = g(f(A_n))$$

and for each $n \in \mathbb{N}$. Note that $g(f(A_{n+1})) = A_{n+2}$.

Let our base case be $n = 1$. Then $A_2 = g(f(A_1))$. By definition of A_1, we have that $x \in X$ but not in $g(Y)$. If $x \notin g(Y)$ then $x \notin g(f(A_1))$ as well. Hence, we have that $A_1 \cap A_2 = \emptyset$ Now assume $P(n)$ holds for $n \leq k-1$. Define

$$A_{n+1} = A_{n+1} \setminus g(f(A_{n+1})) = \{x \in A_{n+1} : x \notin g(f(A_{n+1}))\}$$

Since A_n is pairwise disjoint for each $n \leq k-1$, we also have that

$$A_{k-1} = A_{k-1} \setminus g(f(A_{k-1})) = \{x \in A_{k-1} : x \notin g(f(A_{k-1}))\}.$$

But notice that $g(f(A_{k-1})) = A_k$ which tells us that $A_{k-1} \cap A_k$ are also pairwise disjoint. Thus, A_n for each $n \in \mathbb{N}$ is pairwise disjoint.

∎

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

Proof. ∎

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'

Proof. ■

Chapter 2

Sequences and Series

2.1 The Limit of a Sequence

Understanding infinite series depends on understanding sequences that make up sequences of partial sums.

Definition 2.1.1. A sequence is a function whose domain is $\mathbb{N}$.

A way we describe sequences is to assign each $n \in \mathbb{N}$, use a mapping rule, and then have an output for the nth term. Mathematically we can describe it as a map $f : \mathbb{N} \to \mathbb{R}$.

Example. Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$
2. $\{\frac{1+n}{n}\}_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ...)$
3. (a_n), where $a_n = 2^n$ for each $n \in \mathbb{N}$,
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

It should not be confused that in some instances, the index n will start at $n = 0$ or $n = n_0$ for some other $n_0 > 1$. It is important to keep in mind that sequences are just infinite lists of real numbers. The main point of our analysis deals with what happens at the ”tail” end of a given sequence.

Definition 2.1.2 (Convergence of a Sequence)**.** A sequence (a_n) *converges* to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Furthermore, the convergence of a sequence (a_n) to a is denoted by

$$\lim_{n \to \infty} a_n = a.$$

To understand the last part of this definition, namely, $|a_n - a| < \epsilon$, we can think of it as a neighborhood where a given value will be located in.

Definition 2.1.3. Given $a \in \mathbb{R}$ and $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the *ϵ-neighborhood of a.*

We can think of $V_\epsilon(a)$ as an interval where

$$a - \epsilon < a < a + \epsilon.$$

Another way is to think of it as a ball with radius $\epsilon > 0$ centered at a. we can also think about the convergence of a sequence to a point with the following definition.

Definition 2.1.4. A sequence (a_n) converges to a if, given any $\epsilon-$neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every $\epsilon-$neighborhood contains all but a finite number of the terms of (a_n).

The main idea here is that for some $n \in \mathbb{N}$ along a sequence (a_n), all the points of the sequence converge to some point within a certain $\epsilon-$neighborhood. Note that when increase the value of $n \in \mathbb{N}$, the smaller this $\epsilon-$neighborhood has to be and vice versa.

Example. Consider the sequence (a_n), where $a_n = \frac{1}{\sqrt{n}}$. From our regular understanding of calculus, one can see that the limit of this sequence goes to zero.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{e^2}.$$

We now proceed by verifying that this choice $N \in \mathbb{N}$ has the desired property that $a_n \to 0$. Let $n \geq N$ such that $n > \frac{1}{\epsilon^2}$. Hence, we have

$$\frac{1}{\sqrt{n}} < \epsilon.$$

But this implies that $|a_n - 0| < \epsilon$ and hence our sequence contains the desired property. ■

The main idea of these convergence proofs is to find an $N \in \mathbb{N}$ such that the value we want can be "hit" within some range that we specify with any number $\epsilon > 0$.

Quantifiers

The phrase

"For all $\epsilon > 0$", there exists $N \in \mathbb{N}$ such that ..."

means that for every positive integer I give you, there exists some index or natural number that contains some property that allows the sequence to converge to some value that we desire and as long as we satisfy this rule, then we can say that the sequence converges to our desired value. The template for our subsequent covergence proof will follow the steps below:

- "Let $\epsilon > 0$" be arbitrary."
- Demonstrate that a specific choice of $N \in \mathbb{N}$ leads to the desired property. Note that finding this N often involves working backwards from $|a_n - a| < \epsilon$.
- Show that this N actually works.
- Now assume $n \geq N$.
- With this choice of $\mathbb{N}$, you can work towards the property that $|a_n - a| < \epsilon$

Example. Show

$$\lim \left(\frac{n+1}{n}\right) = 1.$$

In other words, show that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|a_n - 1| < \epsilon$$

where

$$a_n = \frac{n+1}{n}.$$

To obtain our choice of $N \in \mathbb{N}$, we must work backwards from our conclusion. Hence, we have

$$\begin{aligned} a_n - 1 &< \epsilon \\ \frac{n+1}{n} - \frac{n}{n} &< \epsilon \\ \iff \frac{1}{n} &< \epsilon \\ \iff \frac{1}{\epsilon} &< n. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ is $N = 1/\epsilon$. Now for the actual proof.

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N = 1/\epsilon$ such that

$$N > \frac{1}{\epsilon}.$$

Let $n \geq N$. Then we proceed by showing that this choice of $N \in \mathbb{N}$ leads to the desired property. Hence,

$$\begin{aligned} n &> \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{n} \\ \epsilon &> \frac{n+1}{n} - \frac{n}{n} \\ \epsilon &> \frac{n+1}{n} - 1 \\ \epsilon &> |a_n - 1|. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ leads to $a_n \to 1$. We can now conclude that

$$\lim_{n\to\infty} a_n = 1.$$

■

Theorem 2.1.1 (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Proof. Suppose we have $(a_n) \subseteq \mathbb{R}$. Suppose $a_n \to a$ and $a_n \to a'$. We want to show that

$$a = a'.$$

By definition, we have that

$$|a_n - a| < \epsilon/2 \text{ for some } n_1 \in \mathbb{N}$$
$$|a_n - a'| < \epsilon/2 \text{ for some } n_2 \in \mathbb{N}.$$

We can show that $a = a'$ by showing that $|a - a'| < \epsilon$. Hence, choose $N = \min\{n_1, n_2\}$ such that

$$\begin{aligned} |a - a'| &< |a - a_n + a_n - a'| \\ &< |a - a_n| + |a_n - a'| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Hence, we have that $a = a'$ showing that our limit is unique. ■

2.1.1 Divergence

We can study the divergence of sequences by negating the definition we have above.

Example. Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5} ...\right)$$

We can prove that this sequence does not converge to zero. Why? When we choose an $\epsilon = 1/10$, there is none of the term of the sequence converge within the neighborhood $(-1/10, 1/10)$ since the sequence oscillates between $-1/5$ and $1/5$. There is no $N \in \mathbb{N}$, that satisfies $a_n \to 0$. We can also give a counter-example in which we disprove the claim that (a_n) converges to $1/5$. Choose $\epsilon = 1/10$. This produces the neighborhood $(1/10, 3/10)$. We can see that the sequence does in fact converge to $1/5$, but it does so in an oscillating fashion. Furthermore, the sequence does not stay within the neighbor we specified where we expect all the terms of the sequence to converge towards the value. Hence, there is no such $N \in \mathbb{N}$ where the property can be satisfied.

Definition 2.1.5. A sequence that does not converge is said to diverge.

2.1.2 Exercises

Exercise 2.2.1

What happens if we reverse the order of the quantifiers in our convergence definition?

> **Definition 2.1.6** (Reversed)**.** A sequence x_n converges to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ we have for $n \geq N$ such that
>
> $$|x_n - x| < \epsilon.$$

Give an example of a convergent sequence. Is there an example of a convergent sequence that is divergent? Can a sequence converge to two different values? What exactly is being described in this strange definition.

(a) When we reverse the quantifiers, the definition now requires us to construct such an ϵ such that any choice of $N \in \mathbb{N}$ will satisfy the property.

(b) An example of a convergent sequence is $x_n = 1/n$. It can be easily shown that $x_n \to 0$.

(c) Based on our definition and the fact that we can choose any $N \in \mathbb{N}$ suggest that we can have two different values for which the sequence can converge to.

(d) There is a specific contruction of an ϵ such that all x_n clusters converges towards a point determined by any choice of $N \in \mathbb{N}$.

Exercise 2.2.2

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1. $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$ Let $x_n = \frac{2n+1}{5n+4}$. We want to work backwards from our conclusion

$$|x_n - \frac{2}{5}| < \epsilon$$

to find our choice of $N \in \mathbb{N}$. Hence,

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon$$
$$\frac{3}{5(5n+4)} < \epsilon.$$

Solving for n, we get that

$$n > \frac{3/\epsilon - 20}{25}.$$

This only holds for all $0 < \epsilon < 3/20$. Hence, our choice of $N \in \mathbb{N}$ is

$$N = \frac{3/\epsilon - 20}{25}.$$

Proof. Let $0 < \epsilon < 3/20$. Choose $N = \frac{3/\epsilon - 20}{25}$ such that $N > \frac{3/\epsilon - 20}{25}$. Suppose $n \geq N$. We want to show that

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon.$$

So we have the folowing manipulations

$$\begin{aligned} n &> \frac{3/\epsilon - 20}{25} \\ 25n\epsilon &> 3 - 20\epsilon \end{aligned}$$

so we have

$$\epsilon(25n + 20) > 3.$$

Hence, we have

$$\epsilon > \frac{3}{25n + 20}$$

which satisfies our given property that

$$\lim x_n = 2/5.$$

■

2. $\lim \frac{2n^2}{n^3+3} = 0$ Let $x_n = \frac{2n^2}{n^3+3}$. We want to produce an $N \in \mathbb{N}$ from

$$|x_n - 0| < \epsilon.$$

Observe that

$$\frac{2n^2}{n^3+3} < \epsilon$$

Notice that it is somewhat difficult to solve for n so we need to upper bound and lower bound the numerator and the denominator separately. Furthermore, we notice that (x_n) is bounded by $\frac{2n^2}{n^3} = \frac{2}{n}$. Then we lower bound the denominator. Observe that $n^3 + 3 \geq n^3$. Hence, we can estimate x_n to have the following form:

$$\frac{2n^2}{n^3+3} \leq \frac{2}{n} < \epsilon$$

which implies that

$$n > \frac{2}{\epsilon}$$

for $n > 2$.

Proof. Let $\epsilon > 0$. Choose $N = \min\{2, \frac{2}{\epsilon}\}$ and suppose $n \geq N$. Then observe that

$$\epsilon > \frac{2}{n} \geq \frac{2n^2}{n^3+3}.$$

Hence, we have

$$\frac{2n^2}{n^3+3} < \epsilon$$

and our property is satisfied. ■

3. $\lim \frac{\sin(n^2)}{n^{1/3}} = 0$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N = 1/\epsilon^3 \in \mathbb{N}$ and assume $n > N$. Then observe that

$$\frac{\sin(n^2)}{n^{1/3}} \leq \frac{1}{n^{1/3}} < \epsilon$$

   since $\sin(n^2) \leq 1$. Hence, we have that

$$\left|\frac{\sin(n^2)}{n^{1/3}} - 0\right| < \epsilon.$$

   Hence, the property is satisfied. ■

Definition 2.1.7 (Greatest Integer)**.** For all $x \in \mathbb{R}$, if for all $k \in \mathbb{Z}$, $r \in \mathbb{Z}$ where $k > r$ such that $k \leq x < k+1$ and $r \leq x < r+1$ then we say that $\max(k, r)$ is the greatest integer less than or equal to x and denote it as

$$k = [[x]].$$

Exercise 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. Find $\lim a_n$ and supply proofs for each conlusion if

(a) $a_n = [[1/n]]$,

Proof. We claim that the limit of $a_n = [[1/n]]$ is equal to zero. We want to show that for all $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that for every $n \geq N$

$$|a_n - 0| < \epsilon.$$

We proceed by choosing $N > 1$. Suppose $n \geq N$. Our goal is to show that following property above. Since for every $N > 1$ such that $a_n = 0$, we have $n \geq N$

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Hence, our $N \in \mathbb{N}$ shows that $\lim a_n = 0$. ■

(b) $a_n = [[(10 + n)/2n]]$.

Proof. We claim that $\lim a_n = 0$. Our goal is to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - 0| < \epsilon$$

Choose $N > 10$. Suppose $n \geq N$ then we have

$$|a_n - 0| = |0 - 0| < \epsilon.$$

Hence, we have $\lim a_n = 0$. ■

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the $\epsilon-$neighborhood, the larger N may have to be."

Exercise 2.2.6

Prove the uniqueness of limits theorem. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Proof. Suppose $a_n \to a$ and $a_n \to b$. Then for every $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for every $n \geq N_1$ and $n \geq N_2$

$$|a_n - a| < \epsilon/2,$$
$$|a_n - b| < \epsilon/2.$$

Choose $N = \min\{N_1, N_2\}$ and assume $n \geq N$. We want to show that $a = b$ by showing that

$$|a - b| < \epsilon.$$

Hence, we have

$$\begin{aligned} |a-b| &< |a - a_n + a_n - b| \\ &< |a - a_n| + |a_n - n| && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\ a_n \to a,\ a_n \to b) \\ &= \epsilon. \end{aligned}$$

Therefore, $|a - b| < \epsilon$ and thus $a = b$.

■

Exercise 2.2.7

Here are two useful definitions

Definition 2.1.8. A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

and

Definition 2.1.9. A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

Solution. The sequence $(-1)^n$ is frequently in the set $\{1\}$ since for every $n > 0$, the sequence oscillates between two values in the set $\{-1, 1\}$. ■

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

Solution. The first definition is stronger because it implies that any sequence (x_n) will eventually converge to a point in some set $A \subseteq \mathbb{R}$ whereas the second definition explains how a point is constantly being "hit" but not letting all the terms of x_n settle within $A \subseteq \mathbb{R}$ past some $N \in \mathbb{N}$. ■

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

Solution. We can rephrase definition 2.2.3B (Convergence of a Sequence: Topological Version) by replacing every instance of the word *converge* with the phrase "eventually settling into" and rephrasing the $\epsilon-$neighborhood as a set $A \subseteq \mathbb{R}$ that a sequence x_n "eventually settles into to".

∎

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. Since $(x_n) = 2$ for all $n \in \mathbb{N}$, x_n is frequently in the interval $(1.9, 2.1)$.

∎

2.2 The Algebraic and Order Limit Theorems

The goal of having a rigorous definition of convergence in Analysis is to prove statements about sequences in general like the notion of "boundedness" which we will define below.

Definition 2.2.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n). This naturally leads us to the point that all convergent sequences are bounded i.e

Theorem 2.2.1. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit ℓ. This means that given $\epsilon = 1$, we can find an $N \in \mathbb{N}$ such that for every $n \geq N$, we can say that

$$\begin{aligned} &\implies |x_n - \ell| < 1 \\ &\iff -1 < x_n - \ell < 1 \\ &\iff \ell - 1 < x_n < \ell + 1. \end{aligned}$$

Note the terms of the sequence (x_n) can be found in the open interval $(\ell - 1, \ell + 1)$. Since $\ell \in \mathbb{R}$ can either be positive or negative, we can conclude that

$$|x_n| < |\ell| + 1$$

for all $n \geq N$ where

$$M = \max\{|x_1|, |x_2|, ..., |\ell| + 1\}.$$

Hence, it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. ■

Theorem 2.2.2 (Algebraic Limit Theorem). *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$ *for all* $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$ *provided that* $a \neq 0$.

Proof of (i). We begin by proving part (i). Suppose $a_n \to a$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - a| < \epsilon/|c|. \tag{1}$$

In order to show (i), we need to show that

$$|ca_n - ca| < \epsilon.$$

Hence, observe that

$$\begin{aligned} |ca_n - ca| &< |c(a_n - a)| \\ &< |c||a_n - a| \\ &< |c|\frac{\epsilon}{|c|} \\ &= \epsilon. \end{aligned}$$

If $c = 0$, then our sequence (ca_n) reduces to the sequence $\{0, 0, 0, ..., 0\}$ which is clearly converging to $ca = 0$. Hence, we have attained our desired property that $\lim(ca_n) = ca$. The parts are left to you to prove. ■

Proof of (ii). To show part (ii), it suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n + b_n - (a+b)| < \epsilon.$$

Hence, we start with the left side of (ii). Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}
|a_n + b_n - (a+b)| &< |(a_n - a) + (b_n - b)| \\
&< |a_n - a| + |b_n - b| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(a_n + b_n) = a + b$ as required.

■

proof of (iii). To show part (iii), it suffices to show for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - ab| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}
|a_n b_n - ab| &< |a_n b_n - a_n b + a_n b - ab| \\
&< |a_n(b_n - b) + b(a_n - a)| \\
&< |a_n(b_n - b)| + |b(a_n - a)| \\
&< |a_n||b_n - b| + |b||a_n - a| \\
&< M\frac{\epsilon}{2M} + |b|\frac{\epsilon}{2|b|} && (\text{ } a_n \text{ is bounded}) \\
&< \epsilon
\end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = ab$. ■

Proof of (iv). To show part (iv), it suffices to show for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$ with $b \neq 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $n \geq N_1, N_2$, we can have

$$\begin{aligned} |a_n - a| &< M\epsilon/2, \\ |b_n - b| &< \frac{|b|}{|a|} \cdot \frac{M\epsilon}{2}. \end{aligned}$$

we can choose $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\ &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\ &= \left|\frac{a_n b - ab + ab - b_n a}{b_n b}\right| \\ &= \left|\frac{b(a_n - a) + (b - b_n)a}{b_n b}\right| \\ &< \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b|} \cdot \frac{|b_n - b|}{|b_n|} \\ &< \frac{M\epsilon}{2M} + \frac{|a|}{|b|} \cdot \frac{|b|M\epsilon}{|a|2M} && (\ b_n \text{ bounded}) \\ &= \epsilon. \end{aligned}$$

Hence, it follows that $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ provided that $b \neq 0$.

■

Theorem 2.2.3 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $a \geq 0$.

(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $a \leq b$.

(iv) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$*, for all* $n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$*, then* $a \leq c$.

(i) *Proof.* We proceed by contradiction by assuming that $a < 0$. Suppose $a_n \geq 0$ and $a_n \to a$. Let $\epsilon = |a|$ and suppose $n \geq N$. Then

$$|a_n - a| < |a| = -a.$$

But this means that $a_N < 0$ which is a contradiction since $a_N \geq 0$. ■

(ii) *Proof.* We can ensure that the sequence $b_n - a_n$ converges to $b - a$ by the Algebraic Limit Theorem. Since $b_n - a_n \geq 0$, we can use (i) to write $b - a \geq 0$. Hence, $a \leq b$. ■

(iii) *Proof.* Suppose there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$. Suppose $a_n = c$ then using (ii) yields $c \leq b$. Suppose $a_n \leq c$ for all $n \in \mathbb{N}$ then setting $b_n = c$ and using (ii) again yields $a \leq c$. ■

2.2.1 Exercises

Exercise 2.3.1

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

Proof. Suppose $x_n \geq 0$ and $x_n \to 0$. In order to show that $\sqrt{x_n} \to 0$, it suffices to show that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|\sqrt{x_n} - 0| < \epsilon.$$

Choose $N \in \mathbb{N}$. Suppose $x_n = 0$ for all $n \in \mathbb{N}$, then $(\sqrt{x_n}) = 0$ for all $n \geq N$ which means that $(\sqrt{x_n}) \to 0$. Suppose $x_n > 0$ for all $n \in \mathbb{N}$, then observe that since $(x_n) \to 0$ and (x_n) bounded, we have

$$\begin{aligned}
|\sqrt{x_n} - 0| &= |\sqrt{x_n}| \\
&= \left|\frac{x_n}{\sqrt{x_n}}\right| \\
&= \left|\frac{x_n - 0}{\sqrt{x_n}}\right| \\
&= \frac{|x_n - 0|}{\sqrt{x_n}} \\
&< \sqrt{M}\frac{\epsilon}{\sqrt{M}} \\
&= \epsilon
\end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to 0$. ■

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $(x_n) \to x$. We want to show that $(\sqrt{x_n}) \to x$. Suppose $x_n = 0$ and suppose $N \in \mathbb{N}$ such that for every

$n \geq N$, then we have the first case above where $x = 0$ and $(\sqrt{x_n}) \to 0$. Now suppose $x_n > 0$ and choose $N \in \mathbb{N}$ such that for every $n \geq N$, then observe that since $(x_n) \to x$ and (x_n) is bounded by an integer $M > 0$, we have that

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} \\ &< (\sqrt{M} + \sqrt{x}) \frac{\epsilon}{(\sqrt{M} + \sqrt{x})} \\ &= \epsilon. \end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to \sqrt{x}$. ■

Exercise 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

Proof. Suppose $(x_n) \to 2$. Our goal is to show that property above. It suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left| \frac{2x_n - 1}{3} - 1 \right| < \epsilon.$$

Choose $N \in \mathbb{N}$ and suppose $n \geq N$

$$\begin{aligned} \left| \frac{2x_n - 1}{3} - 1 \right| &= \left| \frac{2x_n - 4}{3} \right| \\ &= \left| \frac{2}{3}(x_n - 2) \right| \\ &= \left| \frac{2}{3} \right| |x_n - 2| \\ &< \frac{2}{3} \cdot \frac{3\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, it follows that

$$\left(\frac{2x_n - 1}{3} \right) \to 1.$$

■

(b) $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$.

Proof. We want to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have
$$\left|\frac{1}{x_n} - \frac{1}{2}\right| < \epsilon.$$
Choose $N \in \mathbb{N}$ and assume $n \geq N$. Since $(x_n) \to 2$, we can write
$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &= \frac{|x_n - 2|}{2|x_n|}. \end{aligned} \tag{1}$$

Since $(x_n) \to 2$, we can set $\epsilon = 1$ so that we can lower bound the denominator of (1) using
$$2 - \epsilon < |x_n| \implies 1 < |x_n|.$$
Then we can set $N = \max\{1, \epsilon/2\}$ so that
$$\frac{|x_n - 2|}{2|x_n|} < \frac{2\epsilon}{2} = \epsilon$$
which satisifes our desired property. ■

Exercise 2.3.3

Show

> **Theorem 2.2.4** (Squeeze Theorem). *If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = \ell$, then $\lim y_n = \ell$.*

Proof. Suppose $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and suppose $\lim x_n = \lim z_n = \ell$. We want to show that $\lim y_n = \ell$. By the Order Limit Theorem, we have $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ implies that $\ell \leq y_n \leq \ell$ for all $n \in \mathbb{N}$. But this means that $y_n = \ell$ for all $n \in \mathbb{N}$. Hence, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$
$$|y_n - \ell| = |\ell - \ell| = 0 < \epsilon.$$
Hence, it follows that $\lim y_n = \ell$. ■

Exercise 2.3.4

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined).

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\
&= \frac{\lim 1 + \lim(2a_n)}{\lim 1 + \lim(3a_n) - \lim(4a_n^2)} \\
&= \frac{1+2\cdot 0}{1+3\cdot 0+4\cdot 0^2} \\
&= 1.
\end{aligned}$$

$\blacksquare$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim \left(\frac{(a_n+2)^2-4}{a_n} \right) &= \lim \left(\frac{a_n^2+4a_n}{a_n} \right) \\
&= \lim \Big(a_n + 4 \Big) \\
&= \lim a_n + \lim 4 \\
&= 0 + 4 \\
&= 4.
\end{aligned}$$

$\blacksquare$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$.

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right) \\
&= \frac{\lim 2 + \lim(3a_n)}{\lim 1 + \lim(5a_n)} \\
&= \frac{2+3\cdot 0}{1+5\cdot 0} \\
&= 2.
\end{aligned}$$

■

Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence

$$(x_1, y_1, x_2, y_2, ..., x_n, y_n).$$

For the forwards direction, assume (z_n) is a convergent sequence. We want to show that $\lim x_n = \lim y_n$. It suffices to show that given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|x_n - y_n| < \epsilon.$$

Suppose $(x_n) \to x$ and $(y_n) \to y$, then we can write

$$\begin{aligned}
|x_n - y_n| &= |x_n - z_n + z_n - y_n| \\
&< |x_n - z_n| + |z_n - y_n| \\
&= |x_n - z + z - z_n| + |z_n - z + z - y_n| \\
&< |x_n - x| + |x - z_n| + |z_n - y| + |y - y_n|. \qquad (1)
\end{aligned}$$

By definition, (z_n) is a shuffled sequence and convergent. Hence, $z_n \to x$ and $z_n \to y$. But by the uniqueness of limits, $x = y$ so we have that

$$|x_n - y_n| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

which means $\lim(x_n - y_n) = \lim x_n - \lim y_n = 0$.

Now for the backwards direction, assume $\lim x_n = \lim y_n$. We want to show (z_n) converges i.e for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|z_n - z| < \epsilon.$$

Exercise 2.3.6

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Proof. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Assume $(1/n) \to 0$ and $\sqrt{x_n} \to \sqrt{x}$. Then taking the limit of b_n, we have

$$\begin{aligned}
\lim b_n &= \lim(n - \sqrt{n^2 + 2n}) \\
&= \lim \frac{-2n}{n + \sqrt{n^2 + 2n}} \\
&= \lim \frac{-2}{1 + \sqrt{1 + 2/n}} \\
&= \frac{\lim(-2)}{\lim(1 + \sqrt{1 + 2/n})} \\
&= \frac{\lim(-2)}{\lim(1) + \lim(\sqrt{1 + 2/n})} \\
&= \frac{-2}{1 + 1 + 0} && ((1/n) \to 0, (\sqrt{x_n}) \to \sqrt{x}) \\
&= -1.
\end{aligned}$$

Hence, we have $\lim b_n = -1$. Now we can show that b_n does reach this limit.

Let $\epsilon > 0$. Then choose

$$N = \frac{2}{\sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1}.$$

Then assumme $n \geq N$. Our goal is to show that

$$|b_n + 1| < \epsilon.$$

Then

$$\begin{aligned}
n &> \frac{2}{\sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1} \\
\implies \sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1 &> \frac{2}{n}
\end{aligned}$$

Then we have

$$\begin{aligned}\sqrt{1+2/n} &< \frac{1+\epsilon}{1-\epsilon}\\ (1-\epsilon)\sqrt{1-2/n} &< 1+\epsilon\\ (1-\epsilon)\sqrt{1-2/n}-1 &< \epsilon.\end{aligned}$$

Then we get

$$-1+\sqrt{1+2/n} < \epsilon(1+\sqrt{1+2/n})$$

and then

$$\begin{aligned}\frac{-1+\sqrt{1+2/n}}{1+\sqrt{1+2/n}} &< \epsilon\\ \frac{-2n}{n+\sqrt{n^2+2n}}+\frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}} &< \epsilon\\ n-\sqrt{n^2+2n}+1 &< \epsilon.\end{aligned}$$

Hence, it follows that $|b_n+1| < \epsilon$.

■

Exercise 2.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

Proof. Let $(x_n) \to x$ and let $p(x)$ be a polynomial. Let

$$p(x) = \sum_{i=0}^{m} a_i x^i$$

and

$$p(x_n) = \sum_{i=0}^{m} a_i x_n^i.$$

Our goal is to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|p(x_n) - p(x)| < \epsilon.$$

Then by part (i) of the Algebraic Limit Theorem, we have

$$\begin{aligned}
|p(x_n) - p(x)| &= \Big|\sum_{i=0}^{m} a_i x_n^i - \sum_{i=0}^{m} a_i x^i\Big| \\
&= \Big|\sum_{i=0}^{m} a_i (x_n^i - x^i)\Big| \\
&< \sum_{i=0}^{m} |a_i x_n^i - a_i x^i| && \text{(T.I)} \\
&< \sum_{i=0}^{m} \frac{\epsilon}{m} && (x_n \to x) \\
&= \frac{\epsilon}{m} \cdot m \\
&= \epsilon.
\end{aligned}$$

Hence, we have $p(x_n) \to p(x)$.

■

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Exercise 2.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

Proof. Let (a_n) be a bounded but not necessarily convergent sequence, and assume $\lim b_n = 0$. We want to show that $\lim(a_n b_n) = 0$. It suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - 0| < \epsilon. \tag{1}$$

Since (a_n) bounded, there exists an $M > 0$ such that $|a_n| < M$. Starting with the left side of (1), choose $N \in \mathbb{N}$ such that for every $n \geq N$

$$\begin{aligned}
|a_n b_n - 0| &= |a_n||b_n| \\
&< M \cdot \frac{\epsilon}{M} && (b_n \to 0) \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(a_n b_n) \to 0$. We cannot use the Algebraic Limit Theorem here because (a_n) does not necessarily have a defined limit even though it is bounded. ■

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

Solution. It would simply not converge. ■

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Proof. Suppose $a_n \to a$ where $a = 0$ and $b_n \to b$. Our goal is to show that $\lim(a_n b_n) = 0$. Let $\epsilon > 0$, then choose $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\begin{aligned} |a_n b_n - 0| &< |a_n||b_n| \\ &< \frac{\epsilon}{M} \cdot M && (a_n \to 0, b_n \to b) \\ &< \epsilon \end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = 0$. ■

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

Counterexample. Suppose $a_n = \frac{n}{2n+1}$ and $b_n = \frac{n}{2n+5}$. We have $\lim a_n = \lim b_n$ but $\lim(a_n - b_n) \neq 0$. ■

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

Proof. Let $\epsilon > 0$. Consider $||b_n| - |b||$. Assume $n \geq N$ then

$$||b_n| - |b|| < |b_n - b| < \epsilon$$

by reverse triangle inequality and $(b_n) \to b$. ■

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

Proof. Assume $(a_n) \to a$ and $(b_n - a_n) \to 0$. Let $\epsilon > 0$. By assumption,

$$\begin{aligned} |a_n - a| &< \frac{\epsilon}{2}, \ \ n \geq N_1 \\ |b_n - a_n| &< \frac{\epsilon}{2}, \ \ n \geq N_2. \end{aligned}$$

Hence, choose $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned}|b_n - a| &= |b_n - a_n + a_n - a| \\ &< |b_n - a_n| + |a_n - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence, $(b_n) \to a$. ∎

(d) If $(a_n) \to a$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Proof. Let $\epsilon > 0$. Choose N so that $a_n \to 0$. Then consider $|b_n - b|$ and observe that

$$|b_n - b| \leq a_n < \epsilon,$$

Hence, it follows that $(b_n) \to b$. ∎

Exercise 2.3.13(Iterated Limits).

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* Limits

$$\lim_{n\to\infty}\Big(\lim_{m\to\infty} a_{mn}\Big) \text{ and } \lim_{m\to\infty}\Big(\lim_{n\to\infty} a_{mn}\Big).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

Proof. ∎

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exists? How do these three values compate? Answer these same questions for $a_{mn} = mn/(m^2 + n^2)$.

Proof. ∎

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but neither iterated limit can be computed.

Solution. ∎

1. Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

   Proof. Suppose $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. We want to show that $\lim_{m\to\infty} b_m = a$. Consider $|b_m - a|$. Then fix $m \in \mathbb{N}$ such that for any $m, n \geq N$, we have that

$$\begin{aligned}|b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |b_m - a_{mn}| + |a_{mn} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

   Hence, $(b_m) \to a$. ■

2.3 The Monotone Convergence Theorem

As we have seen in the last section, convergent sequences are bounded while the converse is not true. But if a sequence is monotone then surely it is convergent.

Definition 2.3.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.3.1 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be *monotone* and *bounded*. We need to show that (a_n) converges to some value s. Let our set of points a_n be defined as

$$A = \{a_n : \text{ for all } n \in \mathbb{N}\}$$

and because we have a bounded sequence, we must have an upper bound s which can be defined as out supremum i.e

$$s = \sup\{a_n : \text{ for all } n \in \mathbb{N}\}.$$

Let $\epsilon > 0$. We need to show that

$$|a_n - s| < \epsilon$$

Since $s - \epsilon$ is not an upper bound of A, there exists $N \in \mathbb{N}$ such that

$$s - \epsilon < a_N.$$

Let's assume that (a_n) is an increasing sequence. By assuming $n \geq N$, we can say that $a_n \geq a_N$. Since $s + \epsilon$ is an upper bound and s is the least upper bound, then we can say that

$$s - \epsilon < a_N \leq a_n < s \leq s + \epsilon$$

which imply that

$$\begin{aligned} s - \epsilon < a_n < s + \epsilon \\ \implies |a_n - s| < \epsilon. \end{aligned}$$

Hence, it follows that any *monotone* and *bounded* sequence converges. ■

The key takeaway from this theorem is that we don't actually need to specify a value for a limit in order to show that it converges. As long as we have a monotone sequence and that we know it is bounded then we know for sure that the sequence converges.

Definition 2.3.2 (Convergence of a Series)**.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots.$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + ... + b_m = \sum_{i=1}^{m} s_i,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges* to B if the sequence (s_m) converges to B. In this case, we write

$$\sum_{n=1}^{\infty} b_n = B.$$

Example. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Because the terms in the sum are all positive, the sequence of partial sums are given by

$$s_m = \sum_{k=1}^{m} \frac{1}{k^2}$$

is increasing. Our goal is to show that this sequence is convergent so that the series converges. We proceed by using the Monotone Convergence Theorem to do this. Since we already have a monotone sequence of partial sums, only we need to do now find an upper bound for s_m. Observe that

$$\begin{aligned}
s_m &= 1 = \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + ... + \frac{1}{m^2} \\
&< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + ... + \frac{1}{m(m-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{(m-1)} - \frac{1}{m}\right) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2.
\end{aligned}$$

The third second equality is found by taking the partial fractions of the line before it. Thus, we find that 2 is an upper bound for the sequence of partial sums, so we can conclude that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Example (Harmonic Series)**.** Let's consider the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums is defined as follows

$$s_m = \sum_{k=1}^{m} \frac{1}{k}.$$

Like our last example, we expect these sequence of terms to be bounded by 2 but upon further inspection, we have

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

which is not true. Similarly, we find that $s_8 > 2\frac{1}{2}$, and we can see that in general we have that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^k} + ... + \frac{2}{2^k}\right) \\
&= 1 + ... + \left(2^{k-1}\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2} \\
&= 1 + k\frac{1}{2}.
\end{aligned}$$

This shows that our sequence is unbounded because we found $M = 1+k\left(\frac{1}{2}\right) > 0$ such that $s_k > M$. Despite how slow the sequence of partial of sums may be at reaching this point, it does end up surpassing every number on the postive real line. Since we have an unbounded sequence of partial sums, we conclude that the Harmonic series as divergent.

Theorem 2.3.2 (Cauchy Condensation Test)**.** *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series*

$$\sum_{n=0}^{\infty} b_n$$

converges if and only if

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

converges.

Proof. For the forwards direction, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. This means that the sequence of partial sums

$$t_k = b_1 + 2b_2 + ... + 2^k b_{2k}$$

are bounded. Hence, there exists $M > 0$ such that $t_k \leq M$ for all $k \in \mathbb{N}$. Our goal is to show that the sequence of partial sums for the series

$$\sum_{n=0}^{\infty} b_n.$$

Since $b_n \geq 0$ and that for all $n \in \mathbb{N}$ b_n decreasing, we have that the partial sums t_k is monotone. Our goal is to show that

$$s_m = \sum_{k=0}^{m} b_k$$

is bounded. Hence, fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$ and hence $s_m \leq s_{2^{k+1}-1}$ which imply that

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\ &= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\ &= t_k \end{aligned}$$

Hence, we have $s_m \leq s_{2^{k+1}-1} < t_k \leq M$ which means that (s_m) is bounded. By the Monotone Convergence Theorem, it follows that the series $\sum_{n=1}^{\infty} b_n$ converges. For the forwards direction, we proceed with contrapostive. Hence, assume for sake of contradiction that the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

is a divergent series. We want to show that the series

$$\sum_{n=0}^{\infty} b_n$$

is also a divergent series. ∎

2.3.1 Exercises

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$ converges.

Proof. Let (x_n) be the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$. Our goal is to show that (x_n) is convergent. It is sufficient to show that (x_n) is both *monotone* and *bounded*. We first show that (x_n) is *monotone*. We claim that (x_n) is a *decreasing* sequence. Hence, we will show that for all $n \in \mathbb{N}$, we have $x_n > x_{n+1}$. We proceed by inducting on n. Let the base case be $n = 1$. Then we have that

$$x_1 = 3 > x_2 = \frac{1}{4-3} = 1.$$

Hence, we have $x_1 > x_2$. Now we assume that (x_n) is decreasing for all $1 < n \leq k-1$. We want to show that $x_n > x_k$ for all $n < k$. Since $n \leq k-1$, we have $x_{k-1} \leq x_n$ by inductive hypothesis. Consider x_k. By definition, we have that $x_k = 1/(4 - x_{k-1})$. Since $x_{k-1} \leq x_n$, then for all $n \in \mathbb{N}$ we have

$$x_k = \frac{1}{4 - x_{k-1}} < \frac{1}{4 - x_n}.$$

Hence, $x_k < x_n$ for all $n \in \mathbb{N}$. This is equivalent to showing $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, (x_n) is a *monotone* sequence. Now we show that (x_n) is *bounded*. Since $3 = x_1 \geq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_{n+1} = 1/(4 - x_n) > 0$, we have that

$$0 < x_n \leq 3.$$

Hence, (x_n) is bounded. Since (x_n) is *monotone* and *bounded*, we have that (x_n) is a convergent sequence by the Monotone Convergence theorem. ■

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Solution. Since (x_n) is *monotone* and *bounded*, then (x_{n+1}) is also *monotone* and *bounded*. By the Monotone Convergence Theorem, we have that (x_{n+1}) is also convergent. Hence, $\lim x_{n+1}$ also exists. ■

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution. Since $\lim x_n = \lim x_{n+1}$, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim \frac{1}{4-x_n} \\ &= \frac{\lim 1}{\lim(4-x_n)} \\ &= \frac{1}{\lim(4) - \lim x_n} \\ &= \frac{1}{4-x}. && (\lim x_n = x) \end{aligned}$$

Then we have

$$x = \frac{1}{4-x}$$

and then

$$x^2 - 4x + 1 = 0$$

which we can solve via the quadratic formula. Hence, we have $x = 2 + \sqrt{3}$. ■

Exercise 2.4.3

Following the model of Exercise 2.4.2, show that the sequence defined by $y_1 = 1$ and $y_{n+1} = 2 - \frac{1}{y_n}$ converges and find the limit.

Proof. Let (y_n) be the sequence defined by $y_1 = 1$ and

$$y_{n+1} = 4 - \frac{1}{y_n}.$$

for all $n \in \mathbb{N}$. We want to show that (y_n) converges. Hence, our goal is to show that (y_n) is *monotone* and *bounded*. We claim that (y_n) is increasing. Hence, we show this by inducting on $n \in \mathbb{N}$. Our goal is to show that $y_n \leq y_{n+1}$ for all $n \in \mathbb{N}$. Let the base case be $n = 1$. Then observe that

$$y_1 = 1 < y_2 = 4 - \frac{1}{1} = 3$$

Hence, we have $y_1 < y_2$.

Now assume that (y_n) is increasing for all $1 \leq n \leq k-1$. Hence, $y_n \leq y_{k-1}$. Our goal now is to show that $y_n \leq y_k$ for all $n \in \mathbb{N}$. Let's consider y_k. Then by definition of (y_n), we have

$$y_k = 4 - \frac{1}{y_{k-1}}.$$

Since $y_n \leq y_{k-1}$, we have

$$y_k = 4 - \frac{1}{y_{k-1}} \geq 4 - \frac{1}{y_n}$$

This shows that $y_k \geq y_n$ for any $n \in \mathbb{N}$. Hence, it follows that y_n is an increasing sequence and, therefore, *monotone*. Now

Now we want to show that (y_n) is *bounded*. Observe that $1 < y_n$ for all $n \in \mathbb{N}$ which means (y_n) contains a lower bound. Furthermore, for each $n \in \mathbb{N}$ we also have that $y_{n+1} = 4 - 1/y_n < 4$ which means that (y_n) also contains an uppper bound. Hence, it follows that

$$1 < y_n < 4$$

for all $n \in \mathbb{N}$. Hence, we have (y_n) is *bounded*. By the Monotone Convergence Theorem, it follows that (y_n) is a convergent sequence.

By last exercise, we know that $\lim y_n = \lim y_{n+1}$. Let's assume $(y_n) \to y$. Our goal is to compute $\lim y_n$. By the Algebraic Limit Theorem, we have

$$\begin{aligned} y = \lim y_n &= \lim \left(4 - \frac{1}{y_n}\right) \\ &= \lim(4) - \lim \left(\frac{1}{y_n}\right) \\ &= 4 - \frac{\lim(1)}{\lim y_n} \\ &= 4 - \frac{1}{y}. \end{aligned}$$

Hence, we have

$$y = 4 - \frac{1}{y}$$

which yields the following quadratic equation set to zero

$$y^2 - 4y + 1 = 0.$$

Solving for y using the quadratic formula yields $y = 2 + \sqrt{3}$ ■

Exercise 2.4.5 (Calculating Square Roots)

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big).$$

(a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

Proof. Our first goal is to show that $x_n^2 > 2$ for all $n \in \mathbb{N}$. We proceed by inducting on $n \in \mathbb{N}$. Let our base case be $n = 1$. Then

$$\begin{aligned} x_1 = 2 < x_1^2 \\ &= 4 \\ &< \frac{9}{4} \\ &= \frac{1}{4}\Big(x_1^2 + \frac{4}{x_1^2} + 4\Big) \\ &= x_2^2 \end{aligned}$$

which implies that $2 < x_1^2 < x_2^2$. Now suppose $x_{k-1}^2 > 2$ for all $n \leq k-1$. We want to show that $x_k^2 > 2$ for all $n \in [1, k)$. Consider x_k and then by definition, we have

$$\begin{aligned} x_k^2 &= \frac{1}{4}\Big(x_{k-1}^2 + \frac{4}{x_{k-1}^2} + 4\Big) \\ &> \frac{1}{4}(2 + 2 + 4) \\ &= \frac{8}{4} \\ &= 2. \end{aligned}$$

Hence, $x_k^2 > 2$ for all $n \in \mathbb{N}$. Now we want to show that $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Consider $x_n - x_{n+1}$ then observe that since $x_n^2 > 2$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{x_n^2 - 2}{2x_n} \\ &> \frac{2-2}{2\sqrt{2}} \\ &= 0. \end{aligned}$$

Furthermore, when $x^2 = 2$ we get that $x_n - x_{n+1} = 0$. Hence, we have $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, we get that (x_n) is a convergent sequence. Since $\lim x_n = \lim x_{n+1}$, we can show that $\lim x_n = \sqrt{2}$. By the Algebraic Limit Theorem, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim \Big(\frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big)\Big) \\ &= \frac{1}{2} \lim \Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(\lim x_n + \lim \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(x + \frac{2}{x}\Big) \\ &= \frac{1}{2}x + \frac{1}{x} \\ &= \frac{x^2+2}{2x} \end{aligned}$$

which implies that

$$\begin{aligned} x^2 &= 2 \\ \implies x &= \sqrt{2}. \end{aligned}$$

Hence, we have $\lim x_n = \sqrt{2}$ ■

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution. Let the sequence (x_n) be defined recursively as $x_1 = c$ and

$$x_{n+1} = \frac{1}{c}\Big(x_n + \frac{c}{x_n}\Big).$$

Assume $x_n^2 > c$ for all $n \in \mathbb{N}$ and $x_n - x_{n+1} \geq 0$, then we have $\lim x_n = \sqrt{c}$. ■

2.4 Subsequences and Bolzano-Weierstrass

In the last section, we observed that the convergence of partial sums of a particular series can be determined by the behavior of a subsequence of the partial sums.

Definition 2.4.1. Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \dots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

A few remarks about subsequences:

(a) The order of the subsequence is the same as in the original sequence.

Example. If we have the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$

then the subsequences

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right)$$

and

$$\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots\right)$$

are permitted.

(b) Repetitions and swapping are not allowed.

Example. Like

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, \dots\right)$$

and

$$\left(1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots\right)$$

Since subsequences have the same ordering as the original sequence, one can conjecture about them converging to the same limit.

Theorem 2.4.1. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Let $(a_n) \to a$ and let (a_{n_k}) be a subsequence for (a_n). We want to show (a_{n_k}) converges to a as well. Since $(a_n) \to a$, there exists an N such that for any $n \geq N$, we have $|a_n - a| < \epsilon$.

We claim that $n_k \geq k$ for any $k \in \mathbb{N}$. Let us proceed by inducting on k. Let the base case be $k = 1$. Since n_k is an *increasing* sequence of natural numbers, we see that $n_1 \geq 1$. Now let us assume $n_{k-1} \geq k - 1$. Since (a_{n_k}) in *increasing*, we have $a_k \geq a_{k-1} \geq k - 1$ which implies that $n_k \geq k$.

Since any choice of $n \geq N$, we can say that $n_k \geq k \geq N$. Hence, we have

$$|a_{n_k} - a| < \epsilon$$

which is what we desired. ■

Example. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > ... > 0,$$

the sequence (b^n) is *decreasing* and *bounded* below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some ℓ satisfying $0 \leq \ell < b$. To compute ℓ, notice that (b^{2n}) is a subsequence, so $b^{2n} \to \ell$ by Theorem 2.5.2. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $b^{2n} \to \ell \cdot \ell = \ell^2$. Because limits are unique (Theorem 2.2.7), $\ell^2 = \ell$, and thus $\ell = 0$.

Example. Suppose we have an oscillating sequence of numbers

$$\Big(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\Big)$$

Note that this sequence does not converge to any proposed limit yet if we take a subsequence of it, weget a sequence that converges! Observe, that the subsequence

$$\Big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, ...\Big)$$

and

$$\Big(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, ...\Big)$$

converge to $1/5$ and $-1/5$ respectively. Since we have two subsequences that converge to two different limits, we immediately conclude that the original sequence diverges.

This leads us to our next theorem that states that

Theorem 2.4.2 (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a *bounded* sequence. Then there exists $M > 0$ such that $a_n \in [-M, M]$. Suppose we divide this interval in half for k times: that is, let the length of the intervals be defined by the sequence $M(1/2)^{k-1}$. We claim that a subsequence (a_{n_k}) lies in either one of these intervals: that is, let $n_k > n_{k-1}$ for all $k \in \mathbb{N}$ such that $a_{n_k} \in I_k$.

Let us induct on k. Then let our base case be $k = 1$. Since we have an increasing sequence of natural numbers n_k, we have that $n_2 > n_1$ which means that $a_{n_2} \in I_2$ as well as $a_{n_1} \in I_1$. Now let us assume that this holds for all $k \leq \ell - 1$. We want to show that this holds for $k < \ell$. By the monotonicity of n_k, we have that $n_\ell > n_{\ell-1} > n_k > n_1$ which implies that $a_{n_\ell} \in I_\ell$ for all $\ell \in \mathbb{N}$. Furthermore, the sets

$$I_1 \subseteq I_2 \subseteq I_3 ...$$

form a nested sequence of closed intervals.

By the *Nested Interval Property*, we can conclude that there exists an $x \in I_k$ for all $k \in \mathbb{N}$ such that $\bigcup_{k=1}^{\infty} I_k \neq \emptyset$. Let $\epsilon > 0$. Since $a_{n_k}, x \in I_k$ for all $k \in \mathbb{N}$ and $M(1/2)^{k-1} \to 0$ by the Algebraic Limit Theorem, we can choose an $N \in \mathbb{N}$ such that for any $k \geq n_k \geq N$, we have

$$|a_{n_k} - x| < \epsilon.$$

Hence, $(a_{n_k}) \to x$. ■

Definitions

Definition 2.4.2. Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \dots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.4.3. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Theorem 2.4.4 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

2.4.1 Exercises

Exercise 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Solution. The subsubsequence of the bounded subsequence must converge by the *Bolzano-Weierstrass* theorem. ■

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Solution. Let (a_n) be a sequence defined by

$$a_n = \begin{cases} \frac{1}{2n} \text{ if } n = 2k \\ \frac{1}{2n} + 1 \text{ if } n = 2k+1. \end{cases}$$

Note that $1, 0 \notin (a_n)$ but if we take the subsequences $(a_{2k}) = 1/4k$ and $(a_{2k+1}) = 1/(4k+2)+1$, and take their limit, then we end up with the former converging to 0 and the latter converging to 1.

■

(c) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

Solution. Let's define the infinite set

$$A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

and define a subsequence such that we can make a subsequence for each $n \in \mathbb{N}$ where (a_n) hits every value of A_n. ■

(d) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\},$$

and no subsequences converging to points outside of this set.

Solution. This is not possible. There exists such a subsequence that does go to 0 but it is not within the infinite set. ■

Exercise 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

Solution. If every proper subsequence of (x_n) converges to x, then $(x_2, x_3, x_4, ...)$ also converges to x. Hence, $(x_n) \to x$ by the *uniqueness of limits.* ■

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

Solution. This is just the contrapositve of the statement:

"If (x_n) converges then every subsequence of (x_n) converges as well."

■

(c) If (x_n) contains a divergent subsequence, then there exists two subsequences of (x_n) that converge to different limits.

Solution. This is false. we can find an (x_n) that is not bounded such that we cannot find a subsequence that converges to a limit. ■

(d) If (x_n) is *monotone* and contains a convergent subsequence, then (x_n) converges.

Proof. Assume (x_n) is *monotone* and contains a convergent *subsequence* (x_{n_k}). It suffices to show that (x_n) is *bounded*. Since (x_{n_k}) is convergent, it is also *bounded*. Hence, there exists $M > 0$ such that for all $n_k \in \mathbb{N}$, we have $|x_{n_k}| \leq M$. Since (x_n) *monotone* then either $n \leq n_k$ or $n \geq n_k$ for all $n \in \mathbb{N}$. Hence, we can write either $-M \leq x_n$ or $x_{n_k} \leq M$. But this means that (x_n) is also *bounded*. Since (x_n) both *bounded* and *monotone*, (x_n) is convergent by the *Monotone Convergence Theorem*. Also, (x_n) and (x_{n_k}) converge to the same limit by the *Uniqueness of Limits*. ■

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative propeprty holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + ...$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + ... + a_{n_1}) + (a_1 + a_2 + ... + a_{n_2}) + (a_1 + a_2 + ... + a_{n_3}) + ...$$

leads to a series that also converges to L.

Proof. Our goal is to show that the associative property for a converging infinite series holds. Let us define the terms of the subsequence

$$\begin{aligned} b_1 &= a_1 + a_2 + a_3 + ... + a_{n_1} \\ b_2 &= a_{n_1+1} + a_{n_1+2} + a_{n_1+3} + \cdots + a_{n_2} \\ &\vdots \\ b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}. \end{aligned}$$

Our goal is to show that the subsequence (b_m) converges to L as well. Suppose $\lim s_n = L$. Let the partial sums (t_m) be regrouped in terms of the subsequence above

$$\begin{aligned} t_m &= b_1 + b_2 + \cdots + b_m \\ &= (a_1 + a_2 + a_3 + ... + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \ldots \\ &+ (a_{n_{m-1}+1} + \cdots + a_{n_m}). \end{aligned}$$

Since $\lim s_n = L$, its sequence of partial sums also converge to L. But this means every subsequence of (t_k) also converges to L. Hence, (b_m) converges to L as well. ■

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to associative. Why doesn't our proof in (a) apply to this example?

Solution. We cannot have infinite series be associative if the sequence of partial sums diverges. This means we cannot regroup the terms of our partial sums into a subsequence that converges, since there are divergent subsequences. ■

Exercise 2.5.5

Assume (a_n) is *bounded* sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose for sake of contradiction that $(a_n) \not\to a$. Then there exists $\epsilon_0 > 0$ such that $|a_n - a| \geq \epsilon_0$ for all $N \in \mathbb{N}$. Since (a_n) is *bounded*, we can find a subsequence (a_{n_k}) that converges to some $\ell \in \mathbb{R}$. Since $(a_n) \not\to a$, then $(a_{n_k}) \to \ell$ where $\ell \neq a$. Yet we assumed every convergent subsequence of (a_n) converges to the same limit a but $a \neq \ell$ which is a contradiction. Hence, it must be the case that $\lim a_n = a$. ■

Exercise 2.5.6

Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed)

Proof. Let $b \geq 0$. Our goal is to show that $\lim b^{1/n}$ exists. We observe that

$$b > b^{1/2} > b^{1/3} > b^{1/4} > ... > b^{1/n} \geq 0$$

and conclude by induction that $b^{1/n}$ is a *decreasing* sequence. Since $0 \leq b^{1/n} < b$, we can also conclude that $(b^{1/n})$ is a *bounded* sequence. Hence, $(b^{1/n})$ is a convergent sequence. But note that $(b^{1/n}) \to 0$ for all $b^{1/n} \geq 0$ by exercise 2.3.1. Hence, $\lim b^{1/n} = 0$. ■

Exercise 2.5.7

Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof. Suppose $\lim b^n = 0$. Let $\epsilon = 1$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $-1 < b^n < 1$. Then we have $-1 < b < 1$. Hence, $|b| < 1$.

Now let us show the converse. Assume $|b| < 1$; that is, $-1 < b < 1$. Since $0 \leq b < 1$ holds by Example 2.5.3, we can write $\lim b^n = 0$. Suppose $-1 < b < 0$. We observe that

$$b < b^2 < b^3 < b^4 < ... < 0$$

impying that b^n is an *increasing* sequence for all $n \in \mathbb{N}$ for $b^n \in (-1, 0)$. Furthermore, (b^n) is *bounded* since $1 < b^n < 0$. Hence, (b^n) is a convergent sequence by the *Monotone Convergence Theorem*. Hence, $(b^{2n}) \to \ell$ satisfying $b < \ell \leq 0$. Suppose $\lim(b^n) = \ell$. Let (b^{2n}) be a subsequence, then (b^{2n}) also converges to the same limit

b. Hence, we have

$$\begin{aligned}\ell = \lim b^n &= \lim b^{2n} \\ &= \lim(b^n \cdot b^n) \\ &= \lim b^n \cdot \lim b^n \\ &= \ell^2\end{aligned}$$

Then by the same process in Example 2.5.3, we have $\lim(b^n) = 0$.

■

Exercise 2.5.6

Let (a_n) be a *bounded* sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Proof. Let (a_n) be a *bounded* sequence. We observe that $S \neq \emptyset$ and S *bounded above* since there exists $M > 0$ such that $|a_n| \leq M$ where $x < a_n \leq M$. By the *Axiom of Completeness*, $s = \sup S$ exists. Then by lemma 1.3.8, let $\epsilon = 1/n_k$ such that for some $a_{n_k} \in S$, we have

$$s - \frac{1}{n_k} \leq a_{n_k} \leq s$$

Note that we can write $\lim(s - 1/n_k) = s$ by the *Algebraic Limit Theorem*. By the *Squeeze Theorem*, it follows that $a_{n_k} \to s = \sup S$. ■

2.5 The Cauchy Criterion

Definition 2.5.1. A sequence (a_n) is called a *Cauchy Sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|a_n - a_m| < \epsilon.$$

In the regular convergence defintion, we are given any $\epsilon > 0$ where there is a point in the sequence $N \in \mathbb{N}$ such that past this point, all of our terms fall within an ϵ

range around some limit point. In the Cauchy Criterion defintion, we begin with the same conditions but this time, all the terms of the sequence are all tightly packed together wthin the $\epsilon > 0$ range we were given. It turns out, that these two definitions are equivalent: that is, *Cauchy sequences* are convergent sequences and convergent sequences are *Cauchy sequences*.

Theorem 2.5.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, there must existsa point $N \in \mathbb{N}$ after which we can conclude that

$$|x_n - x_m| < \epsilon.$$

Let $\epsilon > 0$. Since $(x_n) \to x$, we can choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|x_n - x| < \frac{\epsilon}{2}, \tag{1}$$

$$|x_m - x| < \frac{\epsilon}{2}. \tag{2}$$

Consider $|x_n - x_m|$. Then (1) and (2) imply that

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &< |x_n - x| + |x - x_n| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence.* ■

We can prove the other direction, by using either the *Bolzano Weierstrass Theorem* or the *Monotone Convergence Theorem*. This is a little bit more difficult since we need to have a proposed limit for the sequence to converge to.

Lemma 2.5.1. *Cauchy sequences are bounded.*

Proof. Given $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < 1$ for amm $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$ (just substituted $m = N$ here). Hence, define

$$M = \max\{|x_1|, |x_1|, |x_1|, ..., |x_{N-1}|, |x_N| + 1\}.$$

Therefore, $|x_n| < M$ for all $n \in \mathbb{N}$ Hence, the *Cauchy sequence* (x_n) is *bounded.* ∎

Theorem 2.5.2. *A sequence converges if and only if it is a Cauchy sequence.*

Proof. ($\Rightarrow$) This direction is just Theorem 2.6.2 which we have proved above.

($\Leftarrow$) Suppose (x_n) is a *Cauchy sequence.* Let $\epsilon > 0$. Since (x_n) is a *bounded* sequence, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$ by the *Bolzano Weierstrass Theorem.* Let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, every $n_k \geq N$ has the property

$$|x_{n_k} - x| < \epsilon.$$

∎

Our goal now is to show that $(x_n) \to x$. Hence, consider $|x_n - x|$. Then for every $n, n_k \geq N$, we have

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &< |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $(x_n) \to x$.

2.5.1 Completeness Revisited

We can summarize all of our results thus far in the following way

$$\text{AOC} \begin{cases} \text{NIP} \implies \text{BW} \implies \text{CC} \\ \text{MCT} \end{cases}$$

where AOC is our defining axiom to base all our reults on and giving us the notion that an ordered field contains no holes. We could also take the MCT to be our defining axiom and gives us the notion of least upper bounds by proving NIP. In addition, we could also take NIP to be our starting point but we need to have an extra hypothesis; that is, the Archimedean Property to prove all our results above (This is unavoidable).

It could be possible to assume the Arcimedean property holds, suppose one of the results we have proven is true, and derive the others yet this is sort of limited since $\mathbb{Q}$ contains a set that is not complete.

Below is the least of implications we can prove based on which theorem we would like to select asour defining axiom. Hence, we have

$$\text{NIP} + \text{Archimedean Property} \implies \text{AOC}$$

and

$$\text{BW} \implies \text{MCT} \implies \text{Archimedean Property}$$

2.5.2 Exercises

Exercise 2.6.1

Prove that every convergent sequence is *Cauchy*.

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, we must have for every $\epsilon > 0$, there must exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we must have

$$|x_n - x_m| < \epsilon.$$

Let $\epsilon > 0$. Since $(x_n) \to x$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we must have

$$|x_n - x| < \frac{\epsilon}{2}.$$

Since (x_n) converges to x, every subsequence (x_{n_k}) of (x_n) converges to x. This means for $n_k \geq N$, we also have

$$|x_{n_k} - x| < \frac{\epsilon}{2}.$$

Now consider $|x_n - x_{n_k}|$ and assume $n, n_k \geq N$. Then by the *triangle inequality*, we can write

$$\begin{aligned}
|x_n - x_{n_k}| &= |x_n - x + x - x_{n_k}| \\
&< |x_n - x| + |x - x_{n_k}| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence*. ■

Exercise 2.6.2

Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not *monotone*.

Solution. This is possible. Suppose (x_n) is defined such that

$$x_n = \frac{(-1)^n}{n}.$$

We have (x_n) is Cauchy and thus convergent but it is not monotonic. ∎

Takeaway: Just because a sequence is convergent does not generally imply that it is monotonic.

(b) A cauchy sequence with an unbounded subsequence.

Solution. This is not possible since cauchy sequences must be convergent and convergent sequences are bounded which means every subsequence is bounded as well. ∎

(c) A divergent monotone sequence with a Cauchy subsequence.

Solution. This is not possible. A divergent monotone sequence must contain divergent subsequences. Thus, these subsequences cannot be Cauchy by the Cauchy Criterion. ∎

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. This is possible. Let's define the following sequence

$$(1, 2, 1, 4, 1, 6, 1, 8, ...)$$

where

$$x_n = \begin{cases} 1 \text{ if } n \text{ odd} \\ \text{even if } n \text{ even} \end{cases}.$$

is an *unbounded* sequence. As we can see, if we take the subsequence (x_{2k+1}), then we find the subsequence

$$(1, 1, 1, 1, 1, ...)$$

converges to 1. ∎

Exercise 2.6.3

If (x_n) and (y_n) are *Cauchy* sequences, then one easy way to prove that $(x_n + y_n)$ is *Cauchy* is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the *Algebraic Limit Theorem* then implies $(x_n + y_n)$ is convergent and hence *Cauchy*.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algbraic Limit Theorem.

Proof. Suppose (x_n) and (y_n) are *Cauchy Sequences*. Our goal is to show that $(x_n + y_n)$ is also a *Cauchy* sequence. Since (x_n) is *Cauchy*, let $\epsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ for every $m, n \geq N$, we have

$$|x_n - x_m| < \frac{\epsilon}{2}.$$

Likewise, there exists $N_2 \in \mathbb{N}$ such that for every $m, n \geq N$, we have

$$|y_n - y_m| < \frac{\epsilon}{2}.$$

Our goal is to show that

$$|(x_n + y_n) - (x_m + y_m)| < \epsilon$$

Now choose $N = \max\{N_1, N_2\}$ such that $n, m \geq N$ and using the triangle inequality, we write

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &< |x_n - x_m| + |y_n - y_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have $(x_n + y_n)$ is also *Cauchy*. ∎

(b) Do the same for the product $(x_n y_n)$.

Proof. Suppose (x_n) and (y_n) are both *Cauchy*. This implies (x_n) and (y_n) are *bounded* as well as their subsequences (x_{n_k}) and (y_{b_k}). Hence, there exists $N_1, N_2 \in \mathbb{N}$ such that for every $n_k \geq N_1, N_2$, Since (x_{n_k}) converges and (y_n) are Cauchy sequences, it follows that they are also *bounded*. Hence, there exists

$M_1, M_2 > 0$ such that $|y_n| < M_1$ and $|x_{n_k}| < M_2$ for all n, n_k. Our goal is to show

$$|x_n y_n - x_{n_k} y_{n_k}| < \epsilon.$$

Choose $N = \max\{N_1, N_2\}$ such that for every $n, n_k \geq N$

$$\begin{aligned}
|x_n y_n - x_{n_k} y_{n_k}| &= |x_n y_n - x_{n_k} y_n + x_{n_k} y_n - x_{n_k} y_{n_k}| \\
&= |y_n(x_n - x_{n_k}) + x_{n_k}(y_n - y_{n_k})| \\
&< |y_n||x_n - x_{n_k}| + |x_{n_k}||y_n - y_{n_k}| \\
&< M_1 \cdot \frac{\epsilon}{2M_1} + M_2 \cdot \frac{\epsilon}{2M_2} \\
&= \epsilon.
\end{aligned}$$

Hence, $(x_n y_n)$ is a *Cauchy sequence.* ■

Exercise 2.6.4

Let (a_n) and (b_n) be *Cauchy sequences.* Decide whether each of the following sequences is a *Cauchy sequence*, justifying each conclusion.

(a) $c_n = |a_n - b_n|$.

Solution. We claim that (c_n) is a *Cauchy* sequence. Let $\epsilon > 0$. We want to show that given some $N \in \mathbb{N}$, if $n, m \geq N$, then we have

$$|c_n - c_m| < \epsilon.$$

Then, by the *Reverse Triagle Inequality*

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)| \\
&= |(a_n - a_m) + (b_n - b_m)| \\
&< |a_n - a_m| + |b_n - b_m| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, (c_n) is *Cauchy sequence.* ■

(b) $c_n = (-1)^n a_n$.

Solution. This is false. Consider the *Cauchy sequence*

$$a_n = (1, 1, 1, , 1, ...)$$

If we take $c_n = (-1)^n a_n$, then (c_n) is not *Cauchy sequence* since

$$(1, -1, 1, -1, 1, ...)$$

is not. ■

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution. This is false. Consider (a_n) defined by the alternating sequence

$$a_n = \frac{(-1)^n}{n}.$$

This sequence is *Cauchy* but (c_n) is not because we have for all $n \in \mathbb{N}$

$$c_n = \left[\left[\frac{(-1)^n}{n}\right]\right] = \begin{cases} 0 \text{ if } n = 2k \\ -1 \text{ if } n = 2k+1. \end{cases}$$

which diverges. ■

Exercise 2.6.5

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

Solution. False. Take the sequence $a_n = n$ and note that $|a_{n+1} - a_n| < \epsilon$ given any $\epsilon > 0$, but a_n is *unbounded.* ■

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution. Suppose (x_n) and (y_n) are *pseudo-Cauchy.* We want to show that $(x_n + y_n)$ is also *pseudo-Cauchy.* Let $\epsilon > 0$. Choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we have

$$\begin{aligned}|(x_{n+1}+y_{n+1})-(x_n+y_n)| &= |(x_{n+1}-x_n)+(y_{n+1}-y_n)| \\ &< |x_{n+1}-x_n|+|y_{n+1}-y_n| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

by the *Triangle Inequality.* Hence, (x_n+y_n) are *pseudo-Cauchy.* ■

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness ans the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without manking any appeal to the Archimedean Property. This shows that BW, AOC, and MCT are all equivalent.

Proof. Suppose (x_n) is a *bounded* and *monotone* sequence. Our goal is to show that $(x_n) \to x$. By assumption, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for every $n_k \geq N$, we have

$$|x_{n_k} - x| < \epsilon.$$

Since (x_n) is *monotone*, then either $n_k \geq n$ or $n \geq n_k$. If $n \geq n_k \geq N$ for all $n \in \mathbb{N}$, then $|x_n - x| < \epsilon$. If $n_k \geq n$, then for any choice of $n \geq N$, we observe that

$$|x_n - x| \leq |x_{n_k} - x| < \epsilon.$$

Hence, we conclude that (x_n) is a convergent sequence. ■

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

Proof. Assume the *Cauchy Criterion* holds. We want to show that there exists (x_{n_k}) such that $(x_{n_k}) \to x$. Since (x_n) *bounded above* and *non-empty*, $x =$

$\sup(x_n)$ exists. Furthermore, $(x_n) \to x$ since (x_n) is *Cauchy*. Since n_k is an *increasing* set of natural numbers and (x_n) is *bounded above*, we have that

$$x_n - \frac{1}{n_k} \leq x_{n_k} \leq x.$$

By the *Squeeze Theorem*, we have $(x_{n_k}) \to x$. ■

2.6 Properties of Infinite Series

We have learned the convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n) where

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim s_n = A.$$

We called (s_n) the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$. Just like the *Algebraic Limit Theorem* for sequences, we can also do the same thing for series.

Theorem 2.6.1 (Algebraic Limit Theorem for Series). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$*, then*
(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$*,*
(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof. Suppose $\sum_{k=1}^{\infty} a_k = A$ and let $c \in \mathbb{R}$. Define the sequence of partial sums of $\sum_{k=1}^{\infty} ca_k$ as

$$t_k = cs_n = ca_1 + ca_2 + ca_3 + ... + ca_n.$$

By the *Algebraic Limit Theorem*, we know that $\lim cs_n = cA$. Hence,

$$\sum_{k=1}^{\infty} ca_k = cA.$$

To prove the addition rule, suppose $\sum_{k=1}^{\infty} b_k = B$. We want to show that

$$\sum_{k=1}^{\infty} (a_k + b_k) = A + B.$$

Define the sequence of partial sums for the two series as the following:

$$t_k = a_1 + a_2 + ... + a_n,$$
$$u_k = b_1 + b_2 + ... + b_n$$

Since $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, their sequence of partial sums also converges to the same value. Hence, let $\lim t_k = A$ and $\lim u_k = B$. By the *Algebraic Limit Theorem*, the sum of these two limits also converges i.e

$$\lim(t_k + u_k) = \lim t_k + \lim u_k = A + B.$$

Hence,

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B$$

■

We can summarize this theorem by keeping in mind that we can perform distribution over infinite addition and that we can add two infinite series together.

Theorem 2.6.2 (Cauchy Criterion for Series)**.** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Proof. Let $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Suppose $\sum_{k=1}^{\infty} a_k$ converges. This is true if and only if the sequence of partial sums (t_k) converges. This is true if and only if (s_k) is *Cauchy* by the *Cauchy Criterion*. Hence, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$

$$|s_n - s_m| < \epsilon.$$

Note that

$$\begin{aligned}|s_n - s_m| &= |\sum_{k=m+1}^{\infty} a_k - \sum_{k=m}^{m} a_k| \\ &= |\sum_{k=m+1}^{n} a_k| \\ &= |a_{m+1} + ... + a_n| < \epsilon\end{aligned}$$

■

This gives us the opportunity to prove some basic facts about series.

Theorem 2.6.3. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Proof. From the last theorem, we note that for every $\epsilon > 0$ such that whenever $n \geq m \geq N$, we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{\infty} a_k - 0 \right| < \epsilon$$

implies that $(a_n) \to 0$. ■

Keep in mind that the converse of this statement is not true! Just because (a_k) tends to 0 does not immediately imply that the series converges!

Theorem 2.6.4 (Comparison Test)**.** *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then we have*
(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let us show part (i). Suppose $\sum_{k=1}^{\infty} b_k$ converges. We want to show that $\sum_{k=1}^{\infty} a_k$ converges. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n > m \geq N$ and the fact that $a_k \leq b_k$ for all $k \in \mathbb{N}$

$$\begin{aligned}\left| \sum_{k=m+1}^{n} a_k \right| &\leq \left| \sum_{k=m+1}^{n} b_k \right| \\ &< \epsilon.\end{aligned}$$

Hence, a_k converges as well.

Note that part (ii) is just the contrapositive of part (i) which is also true. ■

Note that the convergence of sequences and series are relatively immutable when it comes to changes in some finite number of initial terms: that is, the behavior of sequences and series can be found past some choice of $N \in \mathbb{N}$. In order for the above test to be of any use to us, it is important to have a few examples under our belt i.e any $p > 1$ implies that

$$\sum_{n=1}^{\infty} 1/n^p \text{ converges if and only if } p > 1.$$

Example. A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots .$$

If $r = 1$ and $a \neq 0$, the series diverges. We can use the following algebraic identity, for $r \neq 1$, to write the following:

$$(1-r)(1+r+r^2 \dots + r^{m-1}) = 1 - r^m$$

which allows us to rewrite the partial sum (s_m) of the above series to say that

$$s_m = a + ar + ar^2 + ar^3 + \dots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

where $s_m = at_m$ where

$$t_m = 1 + r + r^2 + \dots + r^{m-1}$$

is a convergent sequence. Using the *Algebraic Limit Theorem*, therefore, allows us to say that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

The next theorem is a modification of the *Comparison Test* to handle series that contain negative terms.

Theorem 2.6.5 (Absolute Convergence Test)**.** *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Proof. Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. We want to show that $\sum_{n=1}^{\infty} a_n$ converges as well. Let $\epsilon > 0$. By the *Cauchy Criterion* for series, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have

$$\begin{aligned} \Big|\sum_{k=m+1}^{n} a_k\Big| &\leq \sum_{k=m+1}^{n} |a_k| \\ &< \epsilon. \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges. ■

Note that the converse of the above statement is false as taking the absolute value of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

produces the regular harmonic series which *diverges*.

Theorem 2.6.6 (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,*
(i) $a_1 \geq a_2 \geq a_3 \ldots \geq a_n \geq a_{n+1} \geq \ldots$ and
(ii) $(a_n) \to 0$.
Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof. See exercise 2.7.1 for proof ■

Definition 2.6.1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.* If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

We can chart a few examples of some *conditionally convergent* series and *absolutely convergent* series.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \implies$ *conditionally convergent*
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \sum_{n=1}^{\infty} \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \implies$ *converges absolutely*

This tells us that any convergent series with positive terms must converge absolutely.

2.6.1 Rearrangements

We can obtain a rearrangement of an infinite series by permuting terms in the sum in some other order. In order for a sum to be a valid rearrangement, all the terms must appear and there should be no repeats.

Definition 2.6.2. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

We can now explain the weird behavior for why the *harmonic series* converges to a different limit when rearranging the terms; that is, it is because the *harmonic series* is a *conditionally convergent* series which leads us to the next theorem.

Theorem 2.6.7. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ *converges absolutely* to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let us define the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ as

$$s_n = \sum_{k=1}^{n} a_k$$

and the sequence of partial sums for the rearranged series $\sum_{n=1}^{\infty} b_n$ as

$$t_m = \sum_{k=1}^{m} b_k.$$

Since $\sum_{n=1}^{\infty} a_n$ *converges absolutely*, let $\epsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|s_n - A| < \frac{\epsilon}{2}$$

as well some $N_2 \in \mathbb{N}$ such that whenever $n > m \geq N_2$, we have

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}.$$

All that is left to do is to set a point in the sequence of the rearranged series where our ultimate goal is to have $|t_m - A| < \epsilon$. Hence, define

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

Let $m \geq M$ such that, when using the *triangle inequality*, we get

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have that $\sum_{n=1}^{\infty} b_n$ converges to A. ■

2.6.2 Definitions

Theorem 2.6.8 (Algebraic Limit Theorem for Series)**.** *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*
(i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$,
(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Theorem 2.6.9 (Cauchy Criterion for Series)**.** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Theorem 2.6.10. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Theorem 2.6.11 (Comparison Test)**.** *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then we have*
(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Theorem 2.6.12 (Absolute Convergence Test)**.** *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Definition 2.6.3. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

Definition 2.6.4. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Theorem 2.6.13. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Theorem 2.6.14 (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 ... \geq a_n \geq a_{n+1} \geq ...$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

2.6.3 Exercises

Exercise 2.7.1

Proving the *Alternating Series Test* amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - ... \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n). Different characterizations of completeness lead to different proofs.

(a) Prove the *Alternating Series Test* by showing that (s_n) is a *Cauchy Sequence*.

Proof. Let (a_n) be a *decreasing sequence* and suppose $(a_n) \to 0$. We want to show that the *Alternating series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ meets the *Cauchy Criterion*.

We first need to show that for every $n > m$, we have the property

$$0 \leq |a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n| \leq |a_{m+1}|$$

Hence, we proceed by induction on k. Note that

$$\sum_{k=m+1}^{n}(-1)^{k+1}a_k = a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n.$$

Let our base case be $P(1)$. Then $a_{m+1} \geq 0$. For $P(2)$, we have $a_{m+1} \geq a_{m+2}$ for all m since (a_n) is a *decreasing sequence*. Suppose this holds for all $m \leq k-1$. We want to show that this holds for $P(k)$. Since (a_n) is *decreasing*, we have that $a_{k-1} \geq a_k$. Hence, $a_{k-1} - a_k \geq 0$. Since $P(k-1)$ holds where

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_{k-1} \leq a_{m+1}.$$

But this means that every term leading up to a_k is bounded by a_{m+1}. Hence,

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_k \leq a_{m+1}.$$

Let $\epsilon > 0$. All is left to show is that

$$\Big|\sum_{k=1}^{n}(-1)^{k+1}a_k\Big| < \epsilon.$$

Hence, for some $N \in \mathbb{N}$, let $n > m \geq N$ and $(a_n) \to 0$ such that

$$\begin{aligned}\Big|\sum_{k=1}^{\infty}(-1)^{k+1}a_k\Big| &\leq |a_{m+1}| \\ &< \epsilon.\end{aligned}$$

Hence, the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ meets the *Cauchy Criterion*. ■

(b) Supply another proof for this result using the Nested Interval Property.

Proof. Suppose (a_n) is *decreasing* sequence and $(a_n) \to 0$. Our goal is to show the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Since (a_n) is *decreasing*, we can use the *Nested Interval Property* to construct closed intervals $I_n = [s_n, s_{n+1}]$ such that the length of these intervals is $|s_n - s_{n+1}| \leq a_n$. The *Nested Interval Property* gurantees the following property that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

where $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Hence, $S \in \mathbb{R}$ can be our candidate limit since $S \in I_n$ for all n. Let $\epsilon > 0$. Since $(a_n) \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$

$$|s_n - S| \leq a_n < \epsilon.$$

■

Hence, $(s_n) \to S$.

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the *Monotone Convergence Theorem* leads to a third proof for the *Alternating Series Test*.

Proof. Define the subsequence of partial sums (s_{2n}) as

$$\sum_{k=1}^{n}(-1)^{2k}a_{2k}.$$

Since (a_n) is a *decreasing sequence*, we have that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
s_1 &= a_2 \geq 0 \\
s_2 &= a_2 + a_4 \geq s_1 \\
s_3 &= a_2 + a_4 + a_6 \geq s_2 \\
&\vdots \\
s_n &= a_2 + a_4 + a_6 + \ldots + a_{2n}.
\end{aligned}$$

We can see that s_{2n} is an *increasing sequence*. Also, $|s_{2n}| < M$ since (a_n) is a *bounded sequence*. Hence, we can conclude that the subsequence of partial sums (s_{2n}) is converges to some $S \in \mathbb{R}$.

We can show that (s_{2n+1}) converges to S as well. Since $s_{2n+1} = s_{2n} + a_{2n+1}$, we can use the *Algebraic Limit Theorem* to say that

$$\begin{aligned}
\lim(s_{2n+1}) &= \lim(s_{2n} + a_{2n+1}) \\
&= \lim(s_{2n}) + \lim(a_{2n+1}) \\
&= S + 0 \\
&= S.
\end{aligned}$$

Since $(s_{2n}) \to S$ and $(s_{2n+1}) \to S$, we have $(s_n) \to S$ as well. ■

Exercise 2.7.4

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the *Cauchy Criterion* for Series.

Proof. Suppose (a_k) and (b_k) are sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Assume $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to show that $\sum_{k=1}^{\infty} a_k$ converges. Define the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ as

$$t_n = \sum_{k=1}^{n} a_k.$$

Let $\epsilon > 0$. Since $a_k \leq b_k$ and $\sum_{n=1}^{\infty} b_n$ converges, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, we have

$$\begin{aligned} |t_n - t_m| &= \Big| \sum_{k=m+1}^{n} a_k \Big| \\ &\leq \Big| \sum_{k=m+1}^{n} b_k \Big| \\ &< \epsilon. \end{aligned}$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ converges. Note that part (ii) is just the contrapositive of part (i). Hence, it is also true. ■

(b) Give another proof for the *Comparison Test*, this time using the *Monotone Convergence Theorem*.

Proof. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to use the *Monotone Convergence Theorem* to show that $\sum_{n=1}^{\infty} a_n$ converges i.e our goal is to show that the sequence of partial sums $t_n = \sum_{k=1}^{n} a_n$ is *bounded* and *monotone.*

Since the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$ are *bounded* and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, it follows that we have $|t_n| \leq M$ as well.

Now we want to show that (t_n) is a *decreasing sequence.* Since $\sum_{n=1}^{\infty} b_n$ is convergent, we know that $b_n \to 0$. Since $a_n \geq 0$ and $(b_n) \to 0$, the terms (t_n) must also be *decreasing.* Hence, $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$.

Since (t_n) is both *decreasing* and *bounded*, it follows that $\sum_{n=1}^{\infty} a_n$ is a convergent ■

Exercise 2.7.4

Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

Solution. Take $\sum x_n = (-1)^n$ and $\sum y_n = 1/n$. These two series diverge but $\sum x_n y_n = (-1)^n/n$ converges. ■

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

Solution. Take the convergent series $\sum 1/n^2$ and the bounded sequence $y_n = \sin(n)$. We have $\sum x_n y_n = \sum \sin(n)/n^2$ is divergent by the comparison test. ■

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converges but $\sum y_n$ diverges.

Solution. This is impossible. By the Algebraic Series Theorem, we cannot have $\sum (x_n + y_n)$ converge without $\sum y_n$ converging as well. ■

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution. By the comparison test, $\sum (-1)^n x_n$ diverges. ■

Exercise 2.7.5

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Corollary 2.6.14.1. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

Proof. We start with the backwards direction. Suppose $p > 1$. Our goal is to show that $\sum_{n=1}^{\infty} 1/n^p$ converges. Notice that $b_n = 1/n^p$ where $b_n \geq 0$ and b_n *decreasing*. By the *Cauchy Condensation Test*, we can prove that

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^p}\right)^n.$$

converges. Since $p > 1$, we have that

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^{p-1} = \sum_{n=0}^{\infty} \left(\frac{1}{2^p}\right)^n.$$

Since $|r| = |1/2^p| < 1$, we know that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ is a *Geometric Series*. By the *Cauchy Condensation Test*, we can say that $\sum_{n=1}^{\infty} b_n$ converges.

For the forwards direction, since $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges, the only reasonable choice of p is when $p > 1$ or else it is *Harmonic Series* which diverges. ■

Exercise 2.7.6

Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about *subvergent* series:

(a) If (a_n) is *bounded*, then $\sum a_n$ *subverges.*

Solution. This is a valid proposition since the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ are bounded which implies that the sequence of partial sums contains a subsequence partial sums that is convergent. Hence, we can say that $\sum a_n$ is a *subvergent* series. ■

(b) All convergent series are *subvergent.*

Solution. This is valid since the sequence of partial sums for a convergent series converges and hence all of the possible subsequence of partial sums for the series converges to the same limit. ■

(c) If $\sum |a_n|$ *subverges*, then $\sum a_n$ *subverges* as well.

Solution. This is not valid. ■

(d) If $\sum a_n$ *subverges*, then (a_n) has a convergent subsequence.

Solution. This is not valid. ■

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

Proof. Suppose for sake of contradiction that $\sum a_n$ converges. Hence, $(a_n) \to 0$. This means that $\lim(na_n) = 0$ but this contradicts our assumption that $\lim(na_n) = l \neq 0$. Hence, the series $\sum a_n$ must diverge. ■

Another why is to use the limit assumption directly.

Proof. Suppose $a_n > 0$ and $\lim(na_n) = l$. We want to show that $\sum a_n$ diverges. Since $\lim(na_n) = l \neq 0$, let $\epsilon = 1$ such that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ for all n, we have

$$|na_n - l| < 1 \iff a_n < \frac{1+l}{n}.$$

This implies that

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n}.$$

Note that $\sum \frac{1+l}{n}$ is not a *p-series* since n^p where $p = 1$. Hence, the series $\sum \frac{1+l}{n}$ diverges. Hence, we have that $\sum a_n$ is also a divergent series by the comparison test. ■

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Proof. Suppose $a_n > 0$ and $\lim(n^2 a_n)$ exists. Suppose $\lim(n^2 a_n) = L$ for some $L \in \mathbb{R}$. Let $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|n^2 a_n - L| < \epsilon.$$

Hence, we have

$$n^2 a_n - L < 1 \iff a_n < \frac{1+l}{n^2} \tag{1}$$

Our goal is to show via *comparison test* that the series $\sum a_n$ converges. From (1), we have

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n^2}.$$

Observe that the series $\sum \frac{1+l}{n^2}$ is a *p-series* test which converges. Hence, the series $\sum a_n$ converges by the *Comparison test.* ■

Exercise 2.7.8

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ *converges absolutely*, then $\sum a_n^2$ also *converges absolutely.*

Proof. Since $\sum a_n$ converges absolutely, then we have the series $\sum |a_n|$ converges. In order for $\sum a_n^2$ to converge absolutely, we need to show that $\sum |a_n^2|$

converges. Furthermore, (a_n) is a *bounded* sequence. Hence, there exists $M > 0$ such that $|a_n| \leq M$. Since there exists $N \in \mathbb{N}$, for any $n \geq N$, we can write

$$\begin{aligned}
\sum |a_n^2| &= \sum |a_n \cdot a_n| \\
&= \sum |a_n| \cdot |a_n| \\
&\leq \sum M \cdot |a_n| \\
&= M \sum |a_n|
\end{aligned}$$

We know by the *Algebraic Limit Theorem* for series that $M \sum |a_n|$ converges. Hence, the series $\sum a_n^2$ converges absolutely by the *Comparison Test*. ∎

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

Proof. Since (b_n) converges, we have that (b_n) is also *bounded*. Hence, there exists $M > 0$ such that for all n we have $b_n \leq M$. Hence, we have

$$\sum a_n b_n \leq M \sum a_n.$$

By the *Algebraic Limit Theorem* for series, we have that $M \sum a_n$ converges. Since $a_n b_n \leq M a_n$, we have that the series $\sum a_n b_n$ also converges by the *Comparison test*. ∎

(c) If $\sum a_n$ *converges conditionally*, then $\sum n^2 a_n$ diverges.

Solution. This is false. Consider the series $\sum \frac{(-1)^n}{n^2}$ which *converges conditionally* but note that $\sum n^2 \frac{(-1)^n}{n^2} = \sum (-1)^n$ diverges. ∎

Exercise 2.7.9 (Ratio Test).

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the *Ratio Test* states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

Proof. There exists $N \in \mathbb{N}$ such that $n \geq N$ because $\lim |\frac{a_{n+1}}{a_n}| = r$. This means that $|\frac{a_{n+1}}{a_n}|$ is *bounded*. Hence, we have that $|\frac{a_{n+1}}{a_n}| \leq r'$ which means that $|a_{n+1}| \leq r'|a_n|$. ■

(b) Why does $|a_N| \sum (r')^n$ converge?

Proof. The series $|a_N| \sum (r')^n$ converges because $|r'| < 1$ which means that $|a_N| \sum (r')^n$ is a *geometric series* which converges. ■

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Proof. Consider the series $\sum |a_n|$ and the fact that

$$\sum |a_n| \leq |a_N| \sum (r')^n$$

for all $n \geq N$. Since the right hand series is *geometric* which converges, we can conclude that $\sum |a_n|$ also converges by the comparison test. Hence, the series $\sum a_n$ converges *absolutely* and thus the series $\sum a_n$ converges.

■

2.7 Double Summations and Products

We discovered in an earlier section that given any doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbb{N}\}$, it can be an ambiguous task to define

$$\sum_{i,j=1}^{\infty} a_{ij}. \tag{1}$$

We also observed that performing *iterated summations*can lead to different summations. Of course, this can be avoided completely if we were to define the partial sum of (1) in the following way

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

for $m, n \in \mathbb{N}$. In order for the sum of (1) to converge we have to have the following hold:

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n \to \infty} s_{mn}$$

Exercise 2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{mn}$. How does this value compare to the two iterated values for the sum already computed?

Proof. ■