

Cardinality

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Chapter 1

Cardinality

1.0.1 Correspondence

Definition 1.0.1. A function $f : A \rightarrow B$ is *one-to-one* if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B . The function f is *onto* if, given any $b \in B$, there exists an element $a \in A$ for which $f(a) = b$.

An equivalent definition for a function to be one-to-one is the following:

Definition 1.0.2. A function $f : A \rightarrow B$ is *one-to-one* if $f(a_1) \neq f(a_2)$ implies that $a_1 = a_2$.

A function that is both one-to-one and onto is said to be bijective. Meaning that we have a one-to-one correspondence between the sets A and B . Another way to explain a function being injective is to say that no two elements from A can map to the same element in B (think of the function x^2). And a function being onto can be explained as every element in B has to be mapped to an element in A .

From an algebraic perspective, we can denote a function being bijective to mean the same thing as two sets having the same cardinality i.e we can say that

Definition 1.0.3. Two sets A and B have the same cardinality if there exists $f : A \rightarrow B$ that is both one-to-one and onto. We can denote this symbolically as $A \sim B$

Example. Some examples of bijective maps are

1. Let the following map $f : \mathbb{N} \rightarrow \mathbf{E}$ be defined as $f(n) = 2n$. We can see that $\mathbb{N} \sim \mathbf{E}$. It's true that \mathbf{E} is indeed a subset of \mathbb{N} , but do not conclude that it is a smaller set than \mathbb{N} since they have the same cardinality or isomorphic to each other.
2. We can show this again. This time let us have a map $f : \mathbb{N} \rightarrow \mathbb{Z}$ such that

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd.} \\ -n/2 & \text{if } n \text{ is even.} \end{cases}$$

We have that $\mathbb{N} \sim \mathbb{Z}$ indeed.

1.0.2 Countable Sets

Definition 1.0.4. A set A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an *uncountable set*.

Theorem 1.0.1. Let \mathbb{Q}, \mathbb{R} . Then

- The set \mathbb{Q} is countable.
- The set \mathbb{R} is uncountable.

Proof. 1. Suppose we define A_n to be split into two sets. When $n = 1$, define A_n to be

$$A_1 = \{0\}$$

and define A_n when $n \geq 2$ as

$$A_n = \left\{ \pm \frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

We can observe here that for every $n \in \mathbb{N}$ we can find every element of \mathbb{Q} exactly once in the sets we have defined. So we can conclude that our map is onto. Since we designed our sets so that each rational number appears once and the fact that for $n = 1$ and $n \geq 2$ produces two disjoint sets, we can see that our map is also one-to-one.

2. We can prove that second statement of theorem by contradiction. Assume for the sake of contradiction that there exists a *one-to-one* and *onto* function

where $f : \mathbb{N} \rightarrow \mathbb{R}$. Letting $x_1 = f(1)$ and $x_2 = f(2)$ and so on, then we can enumerate each element of \mathbb{R} i.e

$$\mathbb{R} = \{x_1, x_2, x_3, \dots\}.$$

Using the Nested Interval Property, we will now produce a real number that is not in this set. Let I_n be a closed interval which does not contain x_n but contains x_{n+1} . Furthermore, I_{n+1} is contained within I_n . Note that within I_n there are two sets which are disjoint and x_{n+1} can be in either one of these sets. Now consider the following intersection $\cap_{n=1}^{\infty} I_n$. Using our construction that every $x_n \notin I_n$, then we can say that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

But this is a contradiction because the nested interval property asserts that this intersection is nonempty meaning that every $x \in \mathbb{R}$ is contained in the above set. Hence, we cannot enumerate every single element x_n of \mathbb{R} . Therefore, \mathbb{R} is an *uncountable* set. ■

This gives us three insights:

1. The smallest type of infinite set is the countable set.
2. We can create another set by deleting or inserting elements into it.
3. Anything smaller than a countable set is either finite or countable.

We can create \mathbb{R} by taking the union of \mathbb{Q} and \mathbb{I} . Since \mathbb{R} is not countable and \mathbb{Q} is, this would mean that the set of irrational numbers \mathbb{I} would be uncountable. This tells us that \mathbb{I} is a bigger subset of \mathbb{R} than \mathbb{Q} .

We can summarize these results in the follow two theorems:

Theorem 1.0.2. *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Theorem 1.0.3. 1. If A_1, A_2, \dots, A_n are each countable sets, then the union of

$$A_1 \cup A_2 \cup \dots \cup A_n$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.0.3 Exercises

Exercise 1.5.1

Finish the following proof for Theorem 1.5.7.

Proof. Assume B is a countable set. So there exists a map $f : \mathbb{N} \rightarrow B$ such that f is surjective and injective. Let $A \subseteq B$ be an infinite subset of B . We want to show that A is countable. That is, A is both

1. injective
2. surjective.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Let $g : \mathbb{N} \rightarrow A$ be the map defined by

$$g(1) = f(n_1).$$

To show injectivity of g , we proceed via induction on the index $i \in \mathbb{N}$. Let the base case be $i = 2$. Then suppose $g(1) = g(2)$. By definition of g and injectivity of f , we have that

$$\begin{aligned} g(1) &= g(2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

But this means that $n_2 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Hence, g is injective. Now for the inductive step, assume this holds for every $1 \leq i \leq k-1$. We want to show that this holds for $i = k$. Suppose that

$$g(1) = g(k).$$

By definition of g and injectivity of f , we have that

$$\begin{aligned} f(n_1) &= f(n_k) \\ n_1 &= n_k. \end{aligned}$$

But this also means that $n_k = \min\{n \in \mathbb{N} : f(n) \in \mathbb{N}\}$. Hence, g is injective.

Now we want to show that g is surjective. Note that we have

$$g(i) = A \cap \{f(n_1), f(n_2), f(n_3), \dots, f(n_k)\}.$$

Then by definition of g , we have that $g(i) = f(n_i)$. Since f is surjective, there exists some $b \in B$ such that $f(n_i) = b$. But since $n_i = \{n_i \in \mathbb{N} : f(n_i) \in A\}$, we have that $f(n_i) \in A$ so g is surjective as well. Hence, we have that g is both injective and surjective which means that $\mathbb{N} \sim A$. Therefore, A is countable. ■

Exercise 1.5.2

Review the proof of Theorem 1.5.6, part (ii) showing that \mathbb{R} is uncountable, and then find the flaw in the following erroneous proof that \mathbb{Q} is uncountable.

The proposition is: \mathbb{Q} is uncountable.

Proof. Assume for contradiction that \mathbb{Q} is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3\}$ and as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while the nested interval property implies that this intersection is nonempty. This contradiction implies \mathbb{Q} must therefore be uncountable. ■

Thoughts. I think the main issue with this proof is when the author assumed that the set of rationals are closed. Since \mathbb{Q} contains irrational numbers within each subset of the \mathbb{Q} as well as real numbers, \mathbb{Q} cannot be closed. Hence, we cannot apply the nested interval property here. ■

Exercise 1.5.3

Prove theorem 1.5.8

Theorem 1.0.4. 1. If A_1, A_2, \dots, A_n are each countable sets, then the union of

$$A_1 \cup A_2 \cup \dots \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1. First, prove statement (i) for two countable sets, A_1 and A_2 .

Proof. Suppose A_1 and A_2 are countable sets. Then $\mathbb{N} \sim A_1$ and $\mathbb{N} \sim A_2$. Furthermore, we have that the maps $f : \mathbb{N} \rightarrow A_1$ and $g : \mathbb{N} \rightarrow A_2$ are bijective. Our goal is to show the union $A_1 \cup A_2$ is also countable i.e we need to show that the map $h : \mathbb{N} \rightarrow A_1 \cup A_2$ is bijective. Before we proceed, let us replace A_2 with the following set B_2 defined as

$$B_2 = A_2 \setminus A_1 = \{h(n) \in A_2 : h(n) \notin A_1\}.$$

Now our following map is $h : \mathbb{N} \rightarrow A_1 \cup B_2$ (this is equivalent to $A_1 \cup A_2$) and define it as follows

$$h(n) = \begin{cases} f(k) & \text{if } n = 2k \\ g(k) & \text{if } n = 2k + 1 \in B_2 \end{cases}$$

Suppose we have $n_1, n_2 \in \mathbb{N}$ and $h(n_1) = h(n_2)$. Since f and g are injective, we have

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

This shows that h is injective (the same process can be applied to g when $h \in B_2$). Note that $A_1 \cap B_2 = \emptyset$ because otherwise h would not be well defined. Now we need to show that h is surjective. Since f and g are surjective, there exists either $x \in A_1$ or $x \in B_2$ such that $h(n) = f(n) = x$ or $h(n) = g(n) = x$. Hence, we have that h is surjective. Since h is a bijective map, we now have that $\mathbb{N} \sim A_1 \cup B_2$.

Suppose we use induction on the index $i \in \mathbb{N}$. Since we have already proven the base case for two countable sets, let us assume A_1, A_2, \dots, A_k are all countable sets such that for $i \leq k - 1$, the union $A_1 \cup A_2 \dots \cup A_{k-1}$ is countable. Let's set $A' = A_1 \cup A_2 \dots \cup A_{k-1}$. Our goal is to show that the union $A' \cup A_k$ is countable. Let's define the map $h : \mathbb{N} \rightarrow A' \cup B'$ such that

$$B' = A_k \setminus A' = \{h(n) \in A_k : x \notin A'\}.$$

and

$$h(n) = \begin{cases} f(k) & \text{if } n = 2k \\ g(k) & \text{if } n = 2k + 1 \end{cases}$$

Let $n_1, n_2 \in \mathbb{N}$. Since A' and A_k are countable sets, we have that

$$h(n_1) = h(n_2)$$

$$f(n_1) = f(n_2)$$

$$n_1 = n_2.$$

Hence, h is injective. Now we want to show that h is surjective. If either $h(n) \in A'$ or $h(n) \in A_k$, then since $f : \mathbb{N} \rightarrow A'$ and $g : \mathbb{N} \rightarrow A_k$ are surjective functions, we have that there exists $x \in A_k$ or $x \in A'$ such that $h(n) = x$. Hence, h is surjective as well. Since h is now a bijective function, we conclude that the union $A_1 \cup A_2 \dots \cup A_k$ is countable. ■

2. Explain why induction cannot be used to prove part (1) of Theorem 1.5.8 from part (2)

Solution. We cannot use induction on part (2) of theorem 1.5.8 because the index itself $n \in \mathbb{N}$ is infinite and induction only works only finite n . ■

Proof of the second part of theorem

Proof. Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of countable sets. Define the union

$$S = \bigcup_{n \in \mathbb{N}} S_n.$$

for all $n \in \mathbb{N}$. Assume each S_n is disjoint. Otherwise, let S_1 such that for each $n \geq 1$, define

$$S'_{n+1} = S_{n+1} \setminus S_n = \{x \in S_{n+1} : x \notin S_n\}.$$

This is to ensure that our following map is well-defined. let F_n denote the set of all injections from $S_n \rightarrow \mathbb{N}$ Let $\varphi : S \rightarrow \mathbb{N} \times \mathbb{N}$ be the map that is defined by

$$\varphi(x) = (n, f_n(x))$$

where $n \in \mathbb{N}$ smallest guaranteed by the Well-Ordering Principle. Since each f_n is an injection, it follows that φ is also an injection. Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists an injection $\alpha : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Composing the two functions φ and α , we have that $\alpha \circ \varphi : S \rightarrow \mathbb{N}$ is an injection. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, we know that α is also surjective. Hence, the composition $\alpha \circ \varphi$ is also surjective. Therefore, we have that S is countable. ■

Exercise 1.5.4

- (a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b) .

Proof. Let (a, b) any interval. Then define the function $f : (a, b) \rightarrow \mathbb{R}$ as

$$f(x) = x^2$$

Our objective is to show that f is injective and surjective. To show that f is injective, we need to let $x_1, x_2 \in (a, b)$. Then suppose

$$f(x_1) = f(x_2).$$

Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^2 &= x_2^2 \\ x_1 &= x_2. \end{aligned}$$

This shows that f is injective. Now we want to show that f is surjective. Then there exists $\sqrt{y} \in (a, b)$. Let

$$x = \sqrt{y}.$$

Then we have that

$$\begin{aligned} x^2 &= y \\ f(x) &= y. \end{aligned}$$

Hence, f is surjective. Since $f : (a, b) \rightarrow \mathbb{R}$ is a bijective function, we have that $(a, b) \sim \mathbb{R}$. ■

- (b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as \mathbb{R} as well.

Proof. Let $(0, \infty) = \{x : x > 0\}$. Our goal is to show that $(a, \infty) \sim \mathbb{R}$. To show this, we need to show the map $f : (a, \infty) \rightarrow \mathbb{R}$ is bijective. Define f as the following:

$$f(x) = \ln(x).$$

Then suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in (a, \infty)$. Then

$$\begin{aligned} \ln(x_1) &= \ln(x_2) \\ x_1 &= x_2 \end{aligned}$$

Hence, we have that f is an injective function. Now we want to show that f is surjective. Then let $e^y = x \in (0, \infty)$. Then taking the natural log of both sides, we have that $\ln(x) = y$. Hence, we have that f is a surjective function. Since f is a bijective function, we know that $(0, \infty) \sim \mathbb{R}$. ■

- (c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Proof. We want to show that $[0, 1) \sim (0, 1)$. Define the map $f : [0, 1) \rightarrow (0, 1)$ as

$$f(x) = \frac{1}{x-1}$$

Our goal is to show that this map is bijective. Hence, we need to show that this map is both injective and surjective.

To show that f is injective. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in [0, 1)$. Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{1}{x_1-1} &= \frac{1}{x_2-1} \\ x_1-1 &= x_2-1 \\ x_1 &= x_2. \end{aligned}$$

Hence, f is injective.

To show that f is surjective, suppose we have $x-1 = \frac{1}{y}$. Then

$$y = \frac{1}{x-1}.$$

But we have that $f(x) = \frac{1}{x-1}$ so we have

$$f(x) = \frac{1}{x-1} = y.$$

Hence, f is surjective.

Since f is bijective, we have that $[0, 1) \sim (0, 1)$. ■

Exercise 1.5.5

- (a) Why is
- $A \sim A$
- for every set
- A
- ?

Solution. $A \sim A$ because A is a bijection onto itself (same elements map to the same elements of the same set). ■

- (b) Given sets
- A
- and
- B
- , explain why
- $A \sim B$
- is equivalent to asserting
- $B \sim A$
- .

Solution. If $A \sim B$, then the map $f : A \rightarrow B$ is a bijection. Meaning we can map unique elements from A to unique elements to B . Since there is unique mapping of elements from $A \rightarrow B$ then we would expect to see the same thing when we map the same elements from $B \rightarrow A$. ■

- (c) For three sets
- A, B
- , and
- C
- , show that
- $A \sim B$
- and
- $B \sim C$
- implies
- $A \sim C$
- . These three properties are what is meant by saying that
- \sim
- is an
- equivalence relation*
- .

Proof. Suppose we have three sets A, B , and C . Suppose $A \sim B$ and $B \sim C$ then we have two maps $f : A \rightarrow B$ and $g : B \rightarrow C$ that are bijective. Composing the two functions we get $g \circ f : A \rightarrow C$. We want to show that this mapping is also bijective. Let $x_1, x_2 \in A$ then suppose $g \circ f(x_1) = g \circ f(x_2)$. By definition of composition, we have

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) \\ f(x_1) &= f(x_2) && (g \text{ is injective}) \\ x_1 &= x_2. && (f \text{ is injective}) \end{aligned}$$

Hence, $g \circ f$ is an injective function. Now we want to show that $g \circ f$ is a surjective mapping. Since f is surjective, there exists a $y \in B$ such that $f(x) = y$. Since g is also surjective, there exists a $z \in C$ such that $g(y) = z$. Hence, we have that $g(f(x)) = z$ which means $g \circ f$ is a surjective mapping. Therefore, $A \sim C$. ■

Exercise 1.5.11

[Shroder-Bernstein Theorem] Assume there exists an injective function $f : X \rightarrow Y$ and another injective function $g : Y \rightarrow X$. Show that $X \sim Y$. The strategy is to partition X and Y into components

$$\begin{aligned} X &= A \cup A' \\ Y &= B \cup B' \end{aligned}$$

with $A \cup A' = \emptyset$ and $B \cup B' = \emptyset$, in such a way that f maps A onto B , and g maps B' onto A'

- (a) Explain how achieving this would lead to a proof that $X \sim Y$.

Solution. Taking disjoint sets prevents the problem of an element from either map mapping to two elements onto its image. Thus, allowing us to have a well-defined function. Having two injective maps also would lead to the $X \sim Y$ because composing these two functions would allow us take a unique mapping from one element from each other. ■

- (b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X , while $\{f(A_n) : n \in \mathbb{N}\}$ is similar collection in Y .

Proof. We set proceed by induction on $n \in \mathbb{N}$ and let $P(n)$ be the statement that $\{A_n : n \in \mathbb{N}\}$ and $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint. Define

$$A_{n+1} = g(f(A_n))$$

and for each $n \in \mathbb{N}$. Note that $g(f(A_{n+1})) = A_{n+2}$.

Let our base case be $n = 1$. Then $A_2 = g(f(A_1))$. By definition of A_1 , we have that $x \in X$ but not in $g(Y)$. If $x \notin g(Y)$ then $x \notin g(f(A_1))$ as well. Hence, we have that $A_1 \cap A_2 = \emptyset$ Now assume $P(n)$ holds for $n \leq k - 1$. Define

$$A_{n+1} = A_{n+1} \setminus g(f(A_{n+1})) = \{x \in A_{n+1} : x \notin g(f(A_{n+1}))\}$$

Since A_n is pairwise disjoint for each $n \leq k - 1$, we also have that

$$A_{k-1} = A_{k-1} \setminus g(f(A_{k-1})) = \{x \in A_{k-1} : x \notin g(f(A_{k-1}))\}.$$

But notice that $g(f(A_{k-1})) = A_k$ which tells us that $A_{k-1} \cap A_k$ are also pairwise disjoint. Thus, A_n for each $n \in \mathbb{N}$ is pairwise disjoint. ■

- (c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B .

Proof. ■

- (d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'

Proof.

