Homework 6: 241A

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Problem 1. (a) Let $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ has derivates of all order}\}$. Then V is a vector space over $\mathbb{R}$. Define a linear operator $(D(T), T, R(T) \subseteq V)$ by setting $T(f) = f'$.

(b) Let $V = \ell^\infty$. Define a linear operator $(D(T), T, R(T) \subseteq V)$ by

$$T((x_n)) = (y_n)$$

where $y_n = x_{n+1}$ for all $n \in \mathbb{N}$.

(a) (i) Show that V is infinite dimensional.

Proof. To show that V is infinite dimensional, it suffices to show that T is an unbounded linear operator. Let $f_n(x) = \sin(nx)$. Then $f_n'(x) = n\cos(nx)$. Clearly, we see that $\|f_n(x)\|_\infty = 1$, but

$$\|f_n'\|_\infty = n \to \infty$$

Hence, V is infinite dimensional. ■

(ii) Show that $R(T) = V$.

Proof. It suffices to show that $V \subseteq R(T)$. Let $f \in V$. Our goal is to show that there exists some $g \in V$ such that $T(g) = f$. Since $f \in V$, f is differentiable of all order. In particular, f is clearly differentiable once. Hence, f must be continuous on any $[a, b] \subseteq \mathbb{R}$. Define

$$F(x) = \int_a^x f \; dt.$$

By the Second Fundamental Theorem of Calculus, it follows that $T(F) = F' = f$ for any $[a, b] \subseteq \mathbb{R}$. Hence, $f \in R(T)$ and we conclude that $V \subseteq R(T)$. ■

(iii) Show that T^{-1} does not exist.

Proof. Since $\dim(V) = \dim(R(T))$ and $\dim(V) = \infty$, it follows from a theorem proven in class that T^{-1} does not exists. ■

(iv) Compare this with Homework-5 problem 5(b) and comment on it.

Solution. Because V is infinite dimensional, regardless of the fact that $R(T) = V$, T^{-1} does not guarantee that it exists. ■

(b) (i) Show that $R(T) = V$.

Proof. It suffices to show that $V \subseteq R(T)$. Let $y \in V = \ell^\infty$. Define $x_n = y_{n-1}$ for all $n \in \mathbb{N}$. Then apply T we obtain

$$T((x_n) = (x_{n+1}) = (y_n).$$

Hence, $y \in R(T)$ and we conclude that $V = R(T)$. ■

(ii) Show that T^{-1} does not exist.

Proof. Since $\dim(V) = \infty$ and $\dim(R(T)) = \dim(V)$ from the previous part, it follows from a theorem proven in class that T^{-1} does not exist. ■

(iii) Compare this with Homework-5 problem 5(b) and comment on it.

Proof. Since V is infinite dimensional, we see that regardless of the fact that $\dim(V) = \dim(R(T))$, T^{-1} still cannot exist. ■

> **Problem 2.** Consider the linear operator $T : \ell^\infty \to \ell^\infty$ defined by $T(x) = y$, where $x = (x_j)$ and $y = (y_j)$ and $y_j = \frac{x_j}{j}$.

(i) Show that T is a bounded linear operator.
Proof. Our goal is to show that there exists a $C > 0$ such that $\|T(x)\|_\infty \leq C\|x\|_\infty$ for all $x \in \ell^\infty$.

$$\|T(x)\|_\infty = \|y\|_\infty = \sup_{j\in\mathbb{N}} \frac{|x_j|}{j}.$$

Note that as $j \to \infty$, we see that $\frac{1}{j} \to 0$. Hence, for all $j \in \mathbb{N}$, $\frac{1}{j} \leq R$ for some $R > 0$. Now, observe that

$$|y_j| = \left|\frac{x_j}{j}\right| = \frac{|x_j|}{j} \leq R|x_j| \leq R\|(x_j)\|_\infty \quad \forall j \in \mathbb{N}.$$

Taking the supremum of the left-hand side, we see that

$$\|T((x_j))\|_\infty \leq R\|(x_j)\|_\infty \quad \text{for some} R > 0.$$

Thus, we conclude that T is a bounded linear operator. ■

(ii) Show that T is injective.
Proof. Let $x^{(1)}, x^{(2)} \in \ell^\infty$. Suppose $T(x^{(1)}) = T(x^{(2)})$. Then by definition, we see that

$$\begin{aligned} y_j^{(1)} = y_j^{(2)} &\iff \frac{x_j^{(1)}}{j} = \frac{x_j^{(2)}}{j} \\ &\iff x_j^{(1)} = x_j^{(2)} \quad \forall j \in \mathbb{N}. \end{aligned}$$

Hence, T is injective. ■

(ii) Let $W = R(T)$. Then $T : \ell^\infty \to R(T)$ is injective and onto and hence T^{-1} exists and $(R(T), T^{-1}, \ell^\infty)$ is a linear operator. Compute T^{-1} explicitly and prove that T^{-1} is not bounded.
Proof. Since $y_j = \frac{x_j}{j}$ for all $j \in \mathbb{N}$, it follows that $x_j = jy_j$ for all $j \in \mathbb{N}$. So, we have $T^{-1}(y_j) = jx_j$..
To prove that T^{-1} is unbounded, let $y_j^{(n)} = (1)$ for all $j \in \mathbb{N}$. Using T^{-1}, we obtain $T^{-1}(y_j^{(n)}) = j \cdot 1$. Immediately, we see that

$$\|T^{-1}(y_j^{(n)})\|_\infty = \|j\|_\infty \to \infty$$

as $j \to \infty$. Thus, T^{-1} is unbounded. ■

> **Problem 3.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed spaces and $T : V \to W$ be a bounded linear operator that is onto. Assume that there is $B > 0$ such that $B\|v\|_V \leq \|T(v)\|_W$.

(i) Show that T is injective.
Proof. Our goal is to show that $N(T) = \{0\}$. Suppose $T(x) = 0$ for all $x \in V$. We will show that $x = 0$. By assumption, there exists a $B > 0$ such that

$$\begin{aligned} &\|T(x)\|_W \geq B\|x\|_V \geq 0 \\ \implies &\|0\|_W \geq B\|x\|_V \geq 0 \\ \implies &0 \geq B\|x\|_V \geq 0. \end{aligned}$$

Hence, we have $\|x\|_V = 0$ which implies that $x = 0$. Hence, we see that $N(T) = \{0\}$ and so T is injective. ■

(ii) Show that T^{-1} exists and is also bounded linear operator.
Proof. Since T is onto (by assumption) and injective (by part (a)), we see that T must be bijective. Hence, T^{-1} must exist. Since T is a linear operator, it immediately follows that T^{-1} is also linear. By assumption, we know there exists a $B > 0$ such that

$$B\|v\|_V \leq \|T(v)\|_W \quad \forall v \in V. \tag{*}$$

Since T is a bijective map, we have $T^{-1}T = I_V$ and $TT^{-1} = I_W$. Our goal is to show that T^{-1} is bounded; that is, for any $y \in W$, there exists a $C > 0$ such that

$$\|T^{-1}(y)\|_V \leq C\|y\|_W.$$

Since T is also onto, we have that for any $y \in Y$, $T(x) = y$ for some $x \in V$. Thus, (*) implies that

$$\|T^{-1}(T(x))\|_V \leq \frac{1}{B}\|T(x)\|_W.$$

Hence, we have

$$\|T^{-1}(y)\|_V \leq \frac{1}{B}\|T(x)\|_W$$

where $C = \frac{1}{B}$. Thus, T^{-1} is a bounded linear operator. ■

(iii) Compare this problem with 2-(iii).
Solution. ■

Problem 4. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed spaces and $T : V \to W$ be a bounded linear operator. Show that

$$\|T\| = \inf\{C : \|T(v)\|_W \leq C\|v\|_V \quad \forall v \in V\}.$$

Proof. It suffices to show the following inequalities

$$\|T\| \leq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\} \tag{1}$$

and

$$\|T\| \geq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}. \tag{2}$$

By definition, we have

$$\|T\| = \sup_{v \in V \setminus \{0\}} \frac{\|T(v)\|_W}{\|v\|_V}.$$

Since T is a bounded linear operator, we have that

$$\exists C > 0 \text{ such that } \forall v \in V \setminus \{0\} \ \|T(v)\|_W \leq C\|v\|_V$$

and so we have

$$\frac{\|T(v)\|_W}{\|v\|_V} \leq C.$$

Since the left-hand side of the above inequality is a lower-bound of the set

$$\{C : \|T(v)\|_W \leq C\|v\|_V\},$$

it follows that

$$\frac{\|T(v)\|_W}{\|v\|_V} \leq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}.$$

Since the right-hand side of the above is an upper-bound, we can take the supremum to obtain

$$\sup_{v \in V \setminus \{0\}} \frac{\|T(v)\|_W}{\|v\|_V} \leq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}.$$

Thus,

$$\|T\| \leq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}$$

which establishes (1).

Now, suppose for sake of contradiction that

$$\|T\| < \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}. \tag{*}$$

In what follows, we will show that

$$\|T\| \geq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\} - \varepsilon$$

for any $\varepsilon > 0$, contradicting the above statement. Let $\varepsilon > 0$. Using the definition of the supremum, it follows from our given ε that there exists a $\hat{V} \in V \setminus \{0\}$ such that

$$\frac{\|T(\hat{v})\|_W}{\|v\|_W} > \|T\| - \varepsilon.$$

By (*) and the fact that $\|T\| \geq \frac{\|T(v)\|_W}{\|v\|_V}$ for all $v \in V \setminus \{0\}$, it follows that

$$\|T\| \geq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\} - \varepsilon.$$

Hence, we have

$$\|T\| \geq \inf\{C : \|T(v)\|_W \leq C\|v\|_V\}$$

which contradicts (*). Thus, (2) is established. ■

Problem 5. Let $A = (a_{ij})$. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.

(i) Show that T is a bounded linear operator.
Proof. Observe that since addition and matrix multiplication are linear operations, it follows from our definition of T that T is a linear operator. Denote $x = (x_j)$ and $y = (y_j)$ as the column vectors with n and m components, respectively. By matrix multiplication, it follows that

$$y_j = \sum_{k=1}^{n} a_{jk} x_k \tag{*}$$

Note that the norm for $\mathbb{R}^n$ is given by

$$\|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}$$

and

$$\|y\| = \Big(\sum_{k=1}^{n} y_k^2\Big)^{\frac{1}{2}}.$$

Using (*), it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\|T(x)\|^2 = \sum_{j=1}^{m} y_j^2 &= \sum_{j=1}^{m} \Big[\sum_{k=1}^{n} a_{jk} x_k\Big]^2 \\
&\leq \sum_{j=1}^{m} \Big[\Big(\sum_{k=1}^{n} a_{jk}^2\Big)^{1/2} \Big(\sum_{\ell=1}^{n} x_\ell^2\Big)^{1/2}\Big]^2 \\
&= \|x\|^2 \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2.
\end{aligned}$$

Denote

$$c^2 = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2$$

which is just a constant since it does not depend on x. Hence, it follows that

$$\|T(x)\|^2 \leq c^2 \|x\|^2 \Longrightarrow \|T(x)\| \leq c\|x\|.$$

Hence, we conclude that T is a bounded linear operator on ■

(ii) Show that $\|T\| = \sqrt{\text{largest eigenvalue of } A^T A}$.

(a) Consider an eigenbasis $\{v_1, v_2, \ldots, v_n\}$ of $A^T A$ such that $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and $\|\vec{v}i\|^2 = 1$ for all $1 \leq i \leq n$. Let

$$v = \sum_{j=1}^{n} a_j v_j \in \mathbb{R}^n. \tag{1}$$

Show that

$$\|T(v)\|^2 = \sum_{j=1}^{n} \lambda_j^2 a_j^2. \tag{*}$$

Proof. Observe that

$$\begin{aligned}
\|T(v)\|^2 &= \Big\|T\Big(\sum_{i=1}^{n} a_i v_i\Big)\Big\|^2 \\
&= \Big\|\sum_{i=1}^{n} a_i T(v_i)\Big\|^2 && \text{(Linearity of } T\text{)} \\
&= \Big\langle \sum_{i=1}^{n} a_i T(v_i), \sum_{j=1}^{n} a_j T(v_j)\Big\rangle \\
&= \sum_{i=1}^{n} a_i \Big\langle T(v_i), \sum_{j=1}^{n} a_j T(v_j)\Big\rangle && \text{(Linearity of Inner Product)} \\
&= \sum_{i=1}^{n} a_i \Big(\sum_{j=1}^{n} \overline{a_j} \langle T(v_i), T(v_j)\rangle\Big) && \text{(Conjugate Linearity of Inner Product)} \\
&= \sum_{i=1}^{n} a_i \Big(\sum_{j=1}^{n} \overline{a_j} \langle Av_i, Av_j\rangle\Big) \\
&= \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} a_j \langle A^T Av_i, v_j\rangle\Big) && (a_i \in \mathbb{R}) \\
&= \sum_{i=1}^{n} a_i \Big(\sum_{j=1}^{n} a_j \langle \lambda_i v_i, v_j\rangle\Big) && (A^T Av_i = \lambda_i v_i) \\
&= \sum_{i=1}^{n} \lambda_i a_i \Big(\sum_{j=1}^{n} a_j \langle v_i, v_j\rangle\Big) \\
&= \sum_{j=1}^{n} \lambda_j a_j^2 \langle v_j, v_j\rangle && (\langle v_i, v_j\rangle = 0 \text{ if } i \neq j) \\
&= \sum_{j=1}^{n} \lambda_j a_j^2 \|v_j\|^2 \\
&= \sum_{j=1}^{n} \lambda_j a_j^2. && (\|v_j\|^2 = 1)
\end{aligned}$$

Hence, we conclude that

$$\|T(v)\|^2 = \sum_{j=1}^{n} \lambda_j a_j^2.$$

■

(b) Let $\lambda = \max\{\lambda_1, \ldots, \lambda_n\}$. Show that $\|T(v)\|^2 \leq \lambda \|v\|^2$.
Proof. Using our result from (a), we can see that

$$\|T(v)\|^2 = \sum_{j=1}^{n} \lambda_j a_j^2 \leq \lambda \sum_{j=1}^{n} a_j^2 \|v_j\|^2 = \Big\|\sum_{j=1}^{n} a_j v_j\Big\|^2 = \|v\|^2.$$

Hence, we have

$$\|T(v)\|^2 \leq \lambda \|v\|^2.$$

■

(c) Show that $\|T\|^2 = \lambda$.

Proof. It suffices to show the following inequalities:

$$\|T\|^2 \leq \lambda \tag{1}$$

and

$$\|T\|^2 \geq \lambda. \tag{2}$$

To show (1), note that from (b) we have

$$\|T(v)\|^2 \leq \lambda \|v\|^2 \Longrightarrow \frac{\|T(v)\|^2}{\|v\|^2} \leq \lambda \Longrightarrow \frac{\|T(v)\|}{\|v\|} \leq \sqrt{\lambda}$$

where $\|v\| \neq 0$. Taking the supremum of the left-hand side, it follows that

$$\|T\| = \sup_{v \in V \setminus \{0\}} \frac{\|T(v)\|}{\|v\|} \leq \sqrt{\lambda}.$$

Hence, $\|T\|^2 \leq \lambda$ which establishes (1).

To prove (2), we notice that by definition of λ, there exists a $1 \leq k \leq n$ such that $\lambda_k = \max_{1 \leq i \leq n} \lambda_i = \lambda$. Now, note that

$$\begin{aligned}
\|T(v_k)\|^2 &= \langle T(v_k), T(v_k) \rangle \\
&= \langle A v_k, A v_k \rangle \\
&= \langle A^T A v_k, v_k \rangle \\
&= \langle \lambda v_k, v_k \rangle \\
&= \lambda \langle v_k, v_k \rangle \\
&= \lambda.
\end{aligned}$$

Now, we can see from the fact that

$$\|T(v_k)\|^2 \leq \|T\|^2$$

it follows that $\lambda \leq \|T\|^2$ which establishes (2). Hence, (1) and (2) imply that

$$\|T\|^2 = \lambda.$$

■

(iii) Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$. Compute $\|A\|$.
Solution. Computing A^T, we obtain

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then

$$C = A^T A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Computing the eigenvalues of this matrix, we get

$$\begin{aligned}
\det(C - I\lambda) &= (1 - \lambda)(5 - \lambda) - 4 \\
&= 5 - 6\lambda + \lambda^2 - 4 \\
&= \lambda^2 - 6\lambda + 1.
\end{aligned}$$

Computing the roots of the above polynomial, we get $\lambda_{1,2} = 3 \pm 2\sqrt{2}$. Hence, $\|A\| = 3 \pm 2\sqrt{2}$. ■