# Math 234A: Homework 1

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1. (i) **(Parallelogram identity)** Let $z, w \in \mathbb{C}$. Show that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

**Proof.** Let $z, w \in \mathbb{C}$ with $z = x + iy$ and $w = u + iv$ with $x, y \in \mathbb{R}$ and $u, v \in \mathbb{R}$. Our goal is to show that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Consider $|z-w|^2$ and notice that

$$z - w = (x-u) + i(y-v).$$

By definition of the modulus, we have

$$\begin{aligned} |z-w|^2 &= (z-w)\overline{z-w} \\ &= ((x-u) + i(y-v))((x-u) - i(y-v)) \\ &= (x-u)^2 + (y-v)^2 \\ &= x^2 - 2xu + u^2 + y^2 - 2yv + u^2 \\ &= (x^2+y^2) - 2(xu+yv) + (u^2+v^2) \\ &= |z|^2 - 2(xu+yv) + |w|^2. \end{aligned}$$

Note that

$$z + w = (x+u) + i(y+v).$$

$$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} \\ &= ((x+u) + i(y+v))((x+u) - i(y+v)) \\ &= (x+u)^2 + (y+v)^2 \\ &= x^2 + 2xu + u^2 + y^2 + 2yv + v^2 \\ &= |z|^2 + 2(xu+yv) + |w|^2. \end{aligned}$$

Adding these two moduli together gives us

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 = 2(|z|^2 + |w|^2)$$

which is our desired result. ■

(ii) **(Binomial Expansion)**: Let $z, w \in \mathbb{C}$ and $n$ be a positive integer. Show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

**Proof.** Let $z, w \in \mathbb{C}$. We proceed via induction on $n \in \mathbb{Z}^+$ to show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Let $n = 1$ be our base case. Then we have

$$\begin{aligned}\sum_{k=0}^{1} \binom{n}{k} z^k w^{n-k} &= \binom{1}{0} z^0 w + \binom{1}{1} z^1 w^0 \\ &= (z+w)^1,\end{aligned}$$

which tells us that the result holds in our base case. Now, suppose the result holds for $n$th case. We will show the result holds for the $n+1$ case. By our induction hypothesis, we see that

$$\begin{aligned}(z+w)^{n+1} &= (z+w)(z+w)^n \\ &= (z+w) \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1}.\end{aligned}$$

Reordering indices in the first summation by setting $m = k+1$, we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1} &= \sum_{m=1}^{n+1} \binom{n}{m-1} z^m w^{(n+1)-m} \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} z^k w^{(n-k)+1}.\end{aligned}$$

Then separating the first and last term of each summation, respectively, we have

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{n} z^n w + \sum_{m=1}^{n} \binom{n}{m-1} z^m w^{(n-k)+1} + \sum_{k=1}^{n} \binom{n}{k} z^k w^{n-k+1} + \binom{n}{0} w^{n+1} \\ &= \binom{n}{n} z^n w + \sum_{k=1}^{n} \left[ \binom{n}{k} + \binom{n}{k-1} \right] z^k w^{(n+1)-k} + \binom{n}{0} w^{n+1}.\end{aligned}$$

Using the fact that **(Need to show this!)**

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

and collecting the first and last terms of the summation, we see that

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{0} z^0 w^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} z^k w^{(n+1)-k} + \binom{n}{n} z^{n+1} w^0 \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} z^k w^{(n+1)-k}\end{aligned}$$

which completes our induction argument. ■

2. For $z, w \in \mathbb{C}$. Define $\langle z, w \rangle = \Re(z\overline{w})$. (If we think of $\mathbb{C}$ as two dimensional real vector space, then $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{C}$).

   (i) Cauchy Schwarz Inequality: Show that $|\langle z, w \rangle|^2 \leq |z|^2 |w|^2$ for all $z, w \in \mathbb{C}$.

**Proof.** First, we would like to show that for any $z, w \in \mathbb{C}$

$$\langle z, w\rangle^2 + \langle iz, w\rangle^2 = |z|^2|w|^2.$$

Let $z, w \in \mathbb{C}$. By definition of $\langle z, w\rangle$, we see that

$$\langle z, w\rangle^2 + \langle iz, w\rangle^2 = (xu + yv)^2 + (xv - uy)^2. \tag{1}$$

Observe that

$$(xu + yv)^2 = (xu)^2 + 2xuyv + (yv)^2 \tag{2}$$

and

$$(xv - uy)^2 = (xv)^2 - 2xuyv + (uy)^2. \tag{3}$$

Then adding (2) and (3) results in

$$\begin{aligned}
\langle z, w\rangle^2 + \langle iz, w\rangle^2 &= (xu)^2 + (yv)^2 + (xv)^2 + (uy)^2 \\
&= x^2u^2 + y^2v^2 + x^2v^2 + u^2y^2 \\
&= u^2(x^2 + y^2) + v^2(x^2 + y^2) \\
&= (x^2 + y^2)(u^2 + v^2) \\
&= |z|^2|w|^2.
\end{aligned}$$

Now, we need to show that $|\langle z, w\rangle|^2 \leq |z|^2|w|^2$. By using the result above, we can write

$$|\langle z, w\rangle|^2 \leq |\langle z, w\rangle|^2 + |\langle iz, w\rangle|^2 = |z|^2|w|^2$$

and we are done. ∎

(ii) Triangle Inequalities: Show

$$|z + w| \leq |z| + |w|$$

and

$$||z| - |w|| \leq |z - w|$$

for all $z, w \in \mathbb{C}$.

**Proof.** Let $z, w \in \mathbb{C}$. We will first show that $|z + w| \leq |z| + |w|$. First, we will show the following results:

$$|z + w|^2 = |z|^2 + 2|\langle z, w\rangle| + |w|^2 \tag{1}$$

and

$$|z - w|^2 = |z|^2 - 2|\langle z, w\rangle| + |w|^2. \tag{2}$$

Let $z = x + iy$ and $w = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that

$$z + w = (x + u) + i(y + v)$$

and

$$z - w = (x - u) + i(y - v).$$

Using the definition of the modulus, we see that

$$\begin{aligned}
|z + w|^2 = (z + w)\overline{(z + w)} &= ((x + u) + i(y + v))((x + u) - i(y + v)) \\
&= (x + u)^2 + (y + v)^2 \\
&= x^2 2xu + u^2 + y^2 + 2yv + v^2 \\
&= (x^2 + y^2) + 2(xu + yv) + (y^2 + v^2) \\
&\qquad\qquad (\text{since } \Re(z\overline{w}) = xu + yv) \\
&= |z|^2 + 2\langle z, w\rangle + |w|^2. \qquad (\text{since } \langle z, w\rangle = \Re(z\overline{w}))
\end{aligned}$$

Similarly, we have

$$\begin{aligned}|z-w|^2 = (z-w)\overline{(z-w)} &= ((x-u)+i(y-v))((x-u)-i(y-v)) \\ &= (x-u)^2 + (y-v)^2 \\ &= x^2 - 2xu + u^2 + y^2 - 2yv + v^2 \\ &= x^2 + y^2 - 2(xu+yv) + v^2 \\ &= (x^2+y^2) - 2\langle z, w\rangle + (v^2+u^2) \\ &= |z|^2 - 2\langle z, w\rangle + |w|^2.\end{aligned}$$

Now, let us prove that $|z+w| \leq |z| + |w|$. Consider $|z+w|^2$. By part (a), we see that

$$\begin{aligned}|z+w|^2 &= |z|^2 + 2\langle z, w\rangle + |w|^2 \\ &\leq |z|^2 + 2zw + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 \\ &= (|z|+|w|)^2.\end{aligned}$$

By taking the square root of both sides, we see that

$$|z+w| \leq |z| + |w|.$$

To show the second inequality, consider $|z-w|^2$. Then using part (a) again, we have

$$\begin{aligned}|z-w|^2 &= |z|^2 - 2\langle z, w\rangle + |w|^2 \\ &\geq |z|^2 - 2|z||w| + |w|^2 \\ &= (|z|-|w|)^2.\end{aligned}$$

By taking the square root of both sides, we see that

$$|z-w| \geq ||z| - |w||.$$

■

3. **(Lagrange Identity)** Let $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Show that

$$\left|\sum_{k=1}^n z_k w_k\right|^2 = \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2 - \sum_{1 \leq i < j \leq n} |z_i \overline{w_j} - z_j \overline{w_i}|^2.$$

Use this to deduce that

$$\left|\sum_{k=1}^n z_k w_k\right|^2 \leq \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2.$$

**Proof.** ■

4. Express the following complex number in the form $\alpha + i\beta$:

(i) $(1+i)^{-1}$

**Solution.** ■

(ii) $(1+i)/2i$

**Solution.** ■

(iii) $(5+5i)^{10}$

**Solution.** ■

(iv) $\left(\frac{2+i}{3-2i}\right)^2$

**Solution.** ■

(v) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

**Solution.** ■

5. Let $z \in \mathbb{C}^{\cdot}$ and $z = r\cos\varphi + i\sin\left(\frac{\varphi+2\pi k}{n}\right)$ where $k \in \mathbb{Z}$. Show that $w^n = z$.

**Proof.** ■

6. **(Computing fourth roots)**: Find your distinct complex numbers $w$ such that $w^4 = z$ for

(i) $z = i$.

**Solution.** ■

(ii) $z = -i$.

**Solution.** ■

(iii) $z = 1$.

**Solution.** ■

(iv) $z = -1$.

**Solution.** ■