Homework-7-Math-241

Problem-1: Let $(V, \|\cdot\|_V)$ be a finite dimensional normed space.

(i) Let W be a subspace of V such that $\dim(W) < \dim(V)$. Let $f: W \to \mathbb{F}$ be a linear functional. Show that there exists a linear functional $\tilde{f}: V \to \mathbb{F}$ such that $\tilde{f}(v) = f(v)$ for all $v \in W$.

(ii) Let $v \in V \setminus \{0\}$. Show that there is $f \in V^*$ such that $f(v) = 1$ and $\|f\| = \|v\|$.

Problem-2: Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Let $B(V, W) = \{T: V \to W \,/\, T \text{ is bounded and linear}\}$

We proved that $B(V, W)$ is a normed space.

Assume that $(W, \|\cdot\|_W)$ is Banach. The goal of this exercise is to show $B(V, W)$ is a Banach space.

(i) Let (T_n) be a Cauchy sequence in $B(V, W)$. Let $v \in V$. Show that $(T_n v)$ converges in W.

(ii) Use (i) to define $T: V \to W$ and show that T is linear.

(iii) Prove that for large n, $T_n - T \in B(V, W)$ and $\|T_n - T\| \to 0$ as $n \to \infty$.

(iv) Prove that $T \in B(V, W)$ and conclude that $B(V, W)$ is complete.

Problem 3: Let $p > 1$. Prove that $((\ell^p)', \|\cdot\|)$ is isomorphic to $(\ell^q, \|\cdot\|_q)$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Problem-4: Let V be a vector space.

(i) Let $\mathcal{B}$ be a basis for V. Show that for each $b \in \mathcal{B}$ there is $f_b \in V^*$ such that $f_b(b) = 1$

(ii) Let $u \in V \setminus \{0\}$. Show that there is $f \in V^*$ such that $f(u) \neq 0$.

(iii) Use (ii) to prove the canonical map $C: V \to V^{**}$ is injective.

Problem-5: Let V be an infinite dimensional normed space. Note that V' is a subspace of V^*. Prove that there is $f \in V^*$ such that $f \notin V'$.
(In other words $V' \subsetneq V^*$).

(Hint: some ideas in 4 can be useful here.)

Problem-6: Let $(V, \|\cdot\|)$ be an infinite dimensional normed space.

(i) Assume that $(V, \|\cdot\|)$ is Banach.

Let (v_n) be a sequence in V. Assume that $\sum_{n=1}^{\infty} \|v_n\|$ converges in $\mathbb{R}$.

Prove that $\sum_{n=1}^{\infty} v_n$ converges in V.

Hint: Write $S_n - S_m$ explicitly.

(ii) Let $V \subseteq \ell^1$ that consists of sequences $x = (x_n)$ such that $x_n = 0$ for all $n \geq N$ for some N, i.e. V consists of all sequences for which all terms are zero after some n^{th} term.

Define $y^{(n)} \in \ell^{\infty}$ by setting

$$y^{(n)}_j = \begin{cases} \frac{1}{2^n} & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

i.e. n^{th} term of $y^{(n)}$ is $\frac{1}{2^n}$ and all other terms are zero.

Then $y^{(n)} \in V$ for all n.

Show that (a) $\sum_{n=1}^{\infty} \|y^{(n)}\|_1 < \infty$.

(b) $\sum_{n=1}^{\infty} y^{(n)}$ does not converge in V.

(Recall from Quiz-2 that V is NOT complete)

This example shows that $\sum_{n=1}^{\infty} \|v_n\|$ converges does not guarantee that $\sum_{n=1}^{\infty} v_n$ converge.

(iii) (Challenge problem, do not use any outside resources, but you are more than welcome to ask me in the class or other time)

Let $(V, \|\cdot\|)$ be a normed space in which for any sequence (v_n) in V

$$\sum_{n=1}^{\infty} \|v_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} v_n \text{ converge}$$

in V. Prove that $(V, \|\cdot\|)$ is Banach.

(Absolute convergence $\Rightarrow$ convergence can be guaranteed only in complete spaces)

Hint: Let (v_n) be a Cauchy sequence in V. Using $\epsilon = \frac{1}{2^j}$, construct a subsequence (v_{n_j}) of (v_n) that converges in V.