1 Handout-14

1.1 Plan

Discuss vector valued integrals and complex line integrals.

1.2 Recap

- Let's recall how we find anti-derivative or primitive of a function in calculus of one-variable.

  Question: Let $f : [a, b] \to \mathbb{R}$. Is it possible to find $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$?

  Answer: Yes, it is possible. A sufficient condition is to have f be Riemann integrable on $[a, b]$ such that
$$F(x) = \int_a^x f(t)\ dt.$$

  Question: If $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$, can we find $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all z?

  Answer: Not obvious. First note that $f = u + iv$. Thus, if we want to mimick the real case, then we need to be able to discuss on integral of a "vector" valued function over a line segment. But a careful set up will allows us to use some of the ideas from calculus and real analysis.

- **Digression:** Vector valued integrals: Let $f : [a, b] \to \mathbb{R}^2$, $f(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. We say that f is integrable on $[a, b]$ if both u and v are integrable in the sense of real analysis; that is,
$$\int_a^b |x(t)|\ dt \ \text{ and } \ \int_a^b |y(t)|\ dt \ \text{ exist.}$$
  We define
$$\int_a^b f(t)\ dt = \begin{pmatrix} \int_a^b x(t)\ dt \\ \int_a^b y(t)\ dt \end{pmatrix}.$$
  We can show
  (i) Linearity of the integral.
  (ii) Let $F : [a, b] \to \mathbb{R}^2$ such that $F'(t) = f(t)$ for all $t \in [a, b]$. Then
$$\int_a^b f(t)\ dt = F(b) - F(a) \in \mathbb{R}^2$$
  assuming all quantities exist.

- Integration of functions of the form $f : [a, b] \to \mathbb{C}$ with $f(t) = x(t) + iy(t)$ are integrable and define
$$\int_a^b f(t)\ dt = \int_a^b x(t)\ dt + i \int_a^b y(t)\ dt.$$

Example. Let $f : [0, 1] \to \mathbb{C}$ be defined by $f(t) = 3t^2 + 2it$. Then one can easily check that
$$\int_0^1 f(t)\ dt = 1 + i.$$

Let $\mathcal{F} = \{f : [a, b] \to \mathbb{C} \text{ such that } f \text{ is integrable}\}$.

(i) Then the map $I : \mathcal{F} \to \mathbb{C}$ defined by
$$I(f) = \int_a^b f(t)\ dt \text{ is } \mathbb{C}\text{−linear.}$$

Indeed, it follows immediately that for any $f_1, f_2 \in \mathcal{F}$, we have

$$I(f_1 + f_2) = I(f_1) + I(f_2).$$

For any $\alpha \in \mathbb{R}$ and $f \in \mathcal{F}$, we have

$$\begin{aligned}
I(\alpha f) &= \int_a^b [\Re(\alpha)x(t) - \Im(\alpha)y(t) + i(\Re(\alpha)y(t) + \Im(\alpha)x(t))]\ dt \\
&= \Re(\alpha)\int_a^b x(t)\ dt - \Im(\alpha)\int_a^b y(t)\ dt \\
&+ i[\Re(\alpha)\int_a^b y(t)\ dt + \Im(\alpha)\int_a^b x(t)\ dt] \\
&= \alpha \int_a^b f(t)\ dt.
\end{aligned}$$

(ii) Let $F : [a, b] \to \mathbb{C}$ such that $F'(t = f(t))$ for all $t \in [a, b]$. Then $\int_a^b f(t)\ dt = F(b) - F(a)$.

(iii) Let I_1, I_2 be two intervals in $\mathbb{R}$, $\varphi : I_1 \to I_2$ continuous differentiable, $f : I_2 \to \mathbb{C}$ integrable. Let $a, b \in I_1$, $a < b$. Then

$$\int_{\varphi(a)}^{\varphi(b)} f(s)\ ds = \int_a^b f(\varphi(t))\varphi'(t)\ dt.$$

(iv) Let $f, g : [a, b] \to \mathbb{C}$, f and g are continuously differentiable (i.e $\Re(f), \Im(f), \Re(g), \Im(g)$ are continuously differentiable). Then

$$\int_a^b f(t)g'(t)\ dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)\ dt.$$

The key point here is that if we have a complex valued function defined on $I \subseteq \mathbb{R}$, then the integral enjoys properties similar to that of a real valued function defined on I.

1.3 Complex Line Integrals

We would like to define the integral of a complex-valuedf function along a curve in $\mathbb{C}$ in the same way we defined a line integral of a function in multivariable calculus.

We will start by introducing some basic terminologies.

Definition (Curve)**.** A curve in $\mathbb{C}$ is a continuous map $\alpha : [a, b] \to \mathbb{C}$, where $[a, b]$ is an interval in $\mathbb{R}$. We call $\alpha(a)$ the starting point and $\alpha(b)$ the end point of α.

Definition (Trace of a Curve)**.** The set

$$\mathrm{tr}(\alpha) = \{\alpha(t) : t \in [a, b]\} \subseteq \mathbb{C}$$

is called the trace or the image of α.

Example. (i) Let $z, w \in \mathbb{C}$ and define $\alpha : [0, 1] \to \mathbb{C}$ by

$$\alpha(t) = (1 - t)z + tw, \quad \alpha(0) = z,\ \alpha(1) = w.$$

(ii) Let $\alpha : [0, 1] \to \mathbb{C}$ be defined by $\alpha(t) = e^{2\pi i t}$ with $\alpha(0) = 1$ and $\alpha(1) = 1$.

Definition (Smooth Curve)**.** A curve α is smooth if it is continuously differentiable.

Definition (Piecewise Smooth Curve). A curve $\alpha : [a, b] \to \mathbb{C}$ is **piecewise smooth** if there is a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $\alpha|_{[a_{i-1}, a_i]}$ is smooth for $i = 1, 2, \ldots, n$.

Definition (Integral of Curve in terms of its Parametrization). Let $\alpha : [a, b] \to \mathbb{C}$ be a smooth curve, $f : D \to \mathbb{C}$ continuous, $\alpha([a, b]) \subseteq D$. Then, we define

$$\int_\alpha f = \int_\alpha f(z)\ dz = \int_a^b f(\alpha(t))\alpha'(t)\ dt \tag{*}$$

where the integrand of the above equation is a complex valued function defined on $[a, b]$. We call (*) the **line integral of f along α.**

Definition (Integral of a Piecewise Smooth Curve). Assume $\alpha : [a, b] \to \mathbb{C}$ is a piecewise smooth curve with partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that $\alpha|_{[a_{i-1}, a_i]}$ smooth for $i = 1, 2, \ldots, n$. Let $f : D \to \mathbb{C}$ be a continuous function and $\alpha([a, b]) \subseteq D$. Then we define

$$\int_\alpha f(z)\ dz = \sum_{i=1}^{n} \int_{\alpha|_{[a_{i-1}, a_i]}} f(z)\ dz. \tag{**}$$

We call (**) the integral along the piecewise curve α.

Lemma. Let $\alpha : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. Let $a = a_0 < \cdots < a_n = b$ be a partition of $[a, b]$ such that $\alpha|_{[a_{i-1}, a_i]}$ is smooth for $i = 1, 2, \ldots, n$. Let $a = a'_0 < a'_1 < \cdots < a'_m = b$ be another partition of $[a, b]$ such that $\alpha|_{[a_{j-1}, a_j]}$ is smooth. Assume that $f : D \to \mathbb{C}$ continuous and $\alpha([a, b]) \subseteq \mathbb{C}$. Then

$$\sum_{j=1}^{m} \int_{\alpha|_{[a'_{j-1}, a'_j]}} f(z)\ dz = \sum_{i=1}^{n} \int_{\alpha|_{[a_{i-1}, a_i]}} f(z)\ dz.$$

Proof. Our goal is to show that for any smooth curve $\gamma : [c, d] \to \mathbb{C}$ and $c < f < d$, we have

$$\int_\gamma f(z)\ dz = \int_{\gamma|_{[c,f]}} f(z)\ dz + \int_{\gamma|_{[f,d]}} f(z)\ dz. \tag{†}$$

Note that (*) follows immediately from the properties of integrals of complex valued functions defined on an interval. Indeed, without loss of generality, assume that $m = n + 1$ and

$$a_0 = a'_0 < a'_1 < a_1 = a'_2 < a_2 = a'_3 < \cdots < a_{n-1} = a'_n < a_n = a'_{n+1}.$$

By (*), we have

$$\int_{\alpha|_{[a_0, a_1]}} f = \int_{\alpha|_{[a_0, a'_1]}} f + \int_{\alpha|_{[a'_1, a'_2]}} f.$$

Thus,

$$\sum_{i=1}^{n} \int_{\alpha|_{[a_{i-1}, a_i]}} f = \sum_{j=1}^{n} \int_{\alpha|_{[a'_{j-1}, a'_j]}} f.$$

■

The lemma above shows us that (*) is well-defined.

Definition (Arc Length)**.** Let $\alpha : [a, b] \to \mathbb{C}$ be a smooth curve. Then we define

$$\ell(\alpha) = \int_a^b |\alpha'(t)| \, dt.$$

If α is piecewise smooth then we define

$$\ell(\alpha) = \text{sum of arc lengths of smooth arcs of } \alpha.$$