Linear Algebra Notes

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November 22, 2023

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Chapter 1

Vector Spaces

1.1 Linear Dependence and Linear Independence

1.1.1 Motivation

Suppose V is a vector space over a field F and that $W \subseteq V$ is a subspace of V.

- Suppose we have a set S that will generate every vector in V as a linear combination of vectors in S.
- It is desirable to find a subset of S that is as small as possible or rather just enough to generate each vector in V.
- The reason for this is to reduce the amount of computations done to represent a vector in V.
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in S.
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in S.
- Doing this, guarantees that we can write any one of the vectors in S as a linear combination of the others.

Definition 1 (Linearly Dependence)**.** A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors in $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

In this case, we also say that the vectors of S are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $v_1, v_2, \ldots, v_n \in S$ where for all $1 \leq i \leq n$, we have $a_i = 0$.
- This tells us that our definition of **Linear Dependence** implies that 0 is a non-trivial linear combination.
- Any subset of V that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is, $1 \cdot 0 = 0$.

1.1.2 Examples of Linearly Dependent Sets

Example. Consider a subset in $\mathbb{R}^4$ defined by

$$S = \{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4),(-1,0,1,0)\}.$$

To show that S is a linear dependent set, we need to find scalars $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that a_i for all $1 \le i \le 4$ not all zero such that

$$a_1(1,3,-3,2) + a_2(2,2,-4,0) + a_3(1,-3,2,-4) + a_4(-1,0,1,0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients $a_1 = 4, a_2 = -3, a_3 = 2$, and $a_4 = 0$. Thus, we have that S is linearly dependent subset of $\mathbb{R}^4$ and hence we can write any vector in S as a linear combination of the other vectors contained in S.

Example. Define a subset of $M_{2\times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} \}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients a_1, a_2, a_3 such that $a_1 = 5, a_2 = 3$, and $a_3 = -2$ where

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 2 (Linear Independence)**.** A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

1.1.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is, $\{v\}$ is linearly independent.
- If $\{v\}$ is linearly dependent, then it the singleton has to be the zero vector 0. This is because $au = 0$
$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

Example. It can be shown that the set

$$S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of 0 represent the trivial representation; that is, we have scalars $a_1 = a_2 = a_3 = a_4 = 0$.