

# Math 230A Lecture Notes

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# Chapter 1

## Week 1

### 1.1 Lecture 1

#### 1.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

and their results will all be rigorously proven and generalized.

#### 1.1.2 The Structure of the Real Numbers

The set  $\mathbb{R}$  is NOT just a boring collection of elements.  $\mathbb{R}$  is a set equipped with four defining properties.

- $\mathbb{R}$  is a **field**.
- $\mathbb{R}$  is an **ordered field**.
- $\mathbb{R}$  is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$  contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$  is a normed space and a metric space (these two are not equivalent).

#### 1.1.3 The First Defining Property

The set of real numbers is a field.

**Definition (Fields).** A field is a set  $F$  with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all  $x, y \in F$ , we have  $x + y \in F$ .

- (A2) For all  $x, y \in F$ , we have  $x + y = y + x$ .
- (A3) For all  $x, y, z \in F$ , we have  $(x + y) + z = x + (y + z)$ .
- (A4) There exists an element  $0 \in F$  such that for any  $x \in F$ ,  $x + 0 = x$ .
- (A5) If  $x \in F$ , then there exists an element  $-x \in F$  such that  $x + (-x) = 0$ .
- (M1) For all  $x, y \in F$ , we have  $xy \in F$ .
- (M2) For all  $x, y \in F$ , we have  $xy = yx$ .
- (M3) For all  $x, y, z \in F$ , we have  $(xy)z = x(yz)$ .
- (M4) For all  $x \in F$ , there exists an element  $1 \neq 0$  such that  $x \cdot 1 = x$ .
- (M5) If  $x \in F$  and  $x \neq 0$ , then there exists an element  $\frac{1}{x} \in F$  such that  $x \cdot \frac{1}{x} = 1$ .
- (D1) If  $x, y, z \in F$ , then  $x(y + z) = xy + xz$ .

### 1.1.4 The Second Defining Property

**Definition (Ordered Fields).** An **ordered field** is a field  $F$  equipped with a relation,  $<$ , with the following properties

- (i) If  $x \in F$  and  $y \in F$ , then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

- (ii) (**Transitive Property**) If  $x, y, z \in F$  and  $x < y$  and  $y < z$ , then  $x < z$ .
- (iii) If  $x, y, z \in F$  and  $y < z$ , then  $x + y < x + z$ .
- (iv) If  $x, y \in F$ , and  $x > 0$  and  $y > 0$ , then  $xy > 0$ .

**Remark.** We say that  $x$  is positive if  $x > 0$ , and negative if  $x < 0$ . Furthermore,  $x \leq y$  is equivalent to  $x = y$  or  $x < y$ .

The first two defining properties alone of  $\mathbb{R}$  do not uniquely specify it. For example,  $\mathbb{Q}$  is another field that satisfies the first two properties of  $\mathbb{R}$ .

**Definition (Upper Bounds).** Suppose  $F$  is an ordered field, and  $A \subseteq F$ . If there exists  $\beta \in F$  such that for all  $x \in A$ ,  $x \leq \beta$  for all  $x \in A$ . We call  $\beta$  an **upper bound of  $A$** .

**Remark.** We call the collection of upper bounds of  $A$  by  $\text{UP}(A)$ . If  $\text{UP}(A) \neq \emptyset$ , then we say that  $A$  is **bounded above**.

Similarly, we define the lower bounds of a set.

**Definition (Lower Bounds).** Suppose  $F$  is an ordered field, and  $A \subseteq F$ . If there exists  $\alpha \in A$  such that for all  $x \in A$ ,  $x \geq \alpha$ , then  $\alpha$  is called the **lower bound of  $A$** .

**Remark.** Similarly, we denote the set of lower bounds of  $A$  by  $\text{LO}(A)$ . We say that  $A$  is **bounded below** if  $\text{LO}(A) \neq \emptyset$ .

**Example.** Suppose we have  $A = [0, 1)$ . We have

$$\begin{aligned}\text{UP}(A) &= [1, \infty) \\ \text{LO}(A) &= (-\infty, 0].\end{aligned}$$

## 1.2 Lecture 2

### 1.2.1 Review of Least Upper Bound Property

**Definition (Supremum).** Suppose  $F$  is an ordered field, and  $A \subseteq F$ . Suppose there exists  $\beta \in F$  such that

- (i)  $\beta \in \text{UP}(A)$
- (ii) If  $\gamma \in F$  and  $\gamma < \beta$ , then  $\gamma \notin \text{UP}(A)$ .

We call  $\beta$  the **least upper bound** of  $A$  or the **supremum** of  $A$ . We denote the supremum of  $A$  as  $\beta = \sup A$ .

**Remark.** When we say THE supremum, we are implicitly stating that the supremum of  $A$  is unique.

**Definition (Infimum).** Suppose  $F$  is an ordered field, and  $A \subseteq F$ . Suppose there exists  $\alpha \in F$  such that

- (i)  $\alpha \in \text{LO}(A)$
- (ii) If  $\gamma \in F$  and  $\gamma > \alpha$ , then  $\gamma \notin \text{LO}(A)$ .

We call  $\alpha$  the **greatest lower bound** of  $A$  or the **infimum** of  $A$ , and write  $\alpha = \inf A$ .

**Definition (Least Upper Bound Property).** An ordered field  $F$  is said to have the **least-upper-bound property** if the following is true:

Every nonempty set  $A$  in  $F$  that is bounded above has a least upper bound in  $F$ .

That is, if  $A \neq \emptyset$  and  $\text{UP}(A) \neq \emptyset$ , then  $\sup(A)$  exists.

**Theorem.** There is exactly one ordered field that has the least-upper-bound bound property. The set  $\mathbb{R}$  is the unique ordered field that contains  $\mathbb{Q}$  as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$  is dedekind complete
- $\mathbb{R}$  satisfies the Axiom of Completeness.

**Remark.** Note that  $\mathbb{Q}$  being an ordered field does not immediately imply that  $\mathbb{Q}$  has the LUBP.

**Definition (Maximums and Infimums).** Let  $A \subseteq \mathbb{R}$ .

- If  $\sup A \in A$ , then we call,  $\sup A$ , the **maximum of**  $A$  and we denote this by  $\max A$ .
- If  $\inf A \in A$ , we call,  $\inf A$ , the **minimum of**  $A$  and we denote this by  $\min A$ .

**Lemma (Useful Fact for Supremum).** Let  $A \subseteq \mathbb{R}$ . Then  $\beta = \sup A$  if and only if

- (i)  $\beta \in \text{UP}(A)$  and
- (ii) For all  $\varepsilon > 0$ , there exists  $a \in A$  such that  $a > \beta - \varepsilon$ .

**Remark.** We can restate property (ii) above as "for all  $\varepsilon > 0$ ,  $\beta - \varepsilon \notin \text{UP}(A)$ ".

**Lemma** (Useful Fact for Infimums). Let  $A \subseteq \mathbb{R}$ . Then  $\alpha = \inf A$  if and only if

- (i)  $\alpha \in \text{LO}(A)$  and
- (ii) For all  $\varepsilon > 0$ , there exists  $a \in A$  such that  $a < \alpha + \varepsilon$ .

**Remark.** Similarly, we can restate property (ii) as "for all  $\varepsilon > 0$ ,  $\alpha + \varepsilon \notin \text{LO}(A)$ ".

**Theorem** (Greatest Lower Bound Property of  $\mathbb{R}$ ). Every nonempty subset  $A$  of  $\mathbb{R}$  that is bounded below has a **greatest upper bound** in  $\mathbb{R}$ .

Another way to say this is the following:

If  $A \neq \emptyset$  and  $\text{LO}(A) \neq \emptyset$ , then  $\inf A$  exists in  $\mathbb{R}$ .

### 1.2.2 Consequences of Least Upper Bound Property

**Theorem** (Archimedean Property). If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$  and  $x > 0$ , then there exists  $n \in \mathbb{Z}^+$  such that  $nx > y$ .

**Proof.** Let  $A = \{nx : n \in \mathbb{N}\}$ . Note that  $A \neq \emptyset$  since  $1 \cdot x \in A$ . Suppose for sake of contradiction that for all  $n \in \mathbb{Z}^+$ ,  $nx \leq y$ . This means that  $y$  is an upper bound of  $A$ . Let  $\beta = \sup A$ . By the first useful fact, we have that for all  $\varepsilon > 0$ , there exists an  $n \in \mathbb{N}$  such that  $\beta - \varepsilon < nx$ . Let  $\varepsilon = x$ . Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \implies \beta < x(n+1).$$

But this tells us that  $x(n+1) \in A$  ( $x \in A$  and  $n+1 \in \mathbb{N}$ ) and that  $\beta$  is NOT an upper bound which is a contradiction. Thus, it must be the case that  $nx > y$  for some  $n \in \mathbb{Z}^+$ . ■

**Remark.** The well ordering property of  $\mathbb{N}$  can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

**Corollary.** Let  $A$  be a nonempty subset of  $\mathbb{R}$  that consists of only integers.

- (i) If  $A$  is bounded above, then  $\sup(A) \in A$ .
- (ii) If  $A$  is bounded below, then  $\inf(A) \in A$ .

**Theorem** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). Let  $x, y \in \mathbb{R}$  with  $x < y$ , there exists a  $p \in \mathbb{Q}$  such that  $x < p < y$ .

**Proof.** Our goal is to find a  $p \in \mathbb{Q}$  such that

$$x < p < y$$

with  $p = \frac{m}{n}$  for  $m \in \mathbb{Z}$  and  $n \in \mathbb{N}$ ; that is, find  $m \in \mathbb{Z}$  and  $n \in \mathbb{N}$  such that

$$nx < m < ny.$$

First, notice that  $x < y$ . This implies that  $y - x > 0$ . By the Archimedean Property, there exists  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \quad (1)$$

Choose  $m \in \mathbb{Z}$  such that  $m$  to be the minimum element greater than  $nx$ ; that is, choose  $m \in \mathbb{Z}$  such that

$$m - 1 \leq nx < m. \quad (2)$$

Let  $A = \{k \in \mathbb{Z} : k > nx\}$  which is nonempty by the Archimedean Property. Furthermore,  $nx$  is a lower bound for  $A$ . By the Well-ordering property,  $A$  contains a minimum. Thus,  $m = \min A$ . Hence, we have

$$nx < m \implies x < \frac{m}{n}. \quad (3)$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \leq nx \implies m &\leq nx + 1 < n\left(y - \frac{1}{n}\right) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \quad (4)$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \iff x < p < y.$$

■



# Chapter 2

## Week 2

### 2.1 Lecture 3

#### 2.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

#### 2.1.2 Review, Existence of Roots

**Proposition.** There is no rational number whose square is 2.

**Theorem.** There is a unique positive real number  $\alpha$  satisfying  $\alpha^2 = 2$ .

**Proof.** (i) **Uniqueness:** Suppose there are two of them  $\alpha_1$  and  $\alpha_2$ . Prove that both  $\alpha_1 < \alpha_2$  and  $\alpha_1 > \alpha_2$  lead to a contradiction. Thus,  $\alpha_1 = \alpha_2$ .

(ii) **Existence:** Show that  $A$  is nonempty and bounded above. Let  $\alpha = \sup A$ . Prove that both  $\alpha^2 > 2$  and  $\alpha^2 < 2$  leads to a contradiction. Thus,  $\alpha^2 = 2$ . ■

**Remark.** A similar argument can be used to prove that if  $x > 0$  and  $m \in \mathbb{N}$ , then there exists a unique positive real number  $\alpha$  such that  $\alpha^m = x$ . We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

#### 2.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

**Definition** (Usual Way of Defining Functions). Let  $A$  and  $B$  be two sets. A **function** from  $A$  to  $B$  denoted by  $f : A \rightarrow B$ , is a rule that assigns each element  $x \in A$  a unique element  $f(x) \in B$ .

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

**Definition** (The Correct Way of Defining Functions). Let  $A$  and  $B$  be two sets. A function from  $A$  to  $B$  is a triple  $(f, A, B)$  where  $f$  is a **relation** from  $A$  to  $B$  satisfying

- (i)  $\text{Dom}(f) = A$
- (ii) If  $(x, y) \in f$  and  $(x, z) \in f$ , then  $y = z$ . (In this case,  $A$  is called the **domain** of  $f$  and  $B$  is called the **codomain** of  $f$ )

**Example.** Let  $A = \emptyset$  and  $B$  be any set. Clearly,  $\emptyset \times B = \emptyset$ . So, the only function from  $A = \emptyset$  to  $B$  is the empty function  $(f, \emptyset, B)$ .

- The empty function is one-to-one.
- The empty function is onto only when  $B = \emptyset$ .

**Definition** (Image, Range, Onto (Surjective)). Consider a function  $f : A \rightarrow B$ . Let  $E \subseteq A$ . Define the **image** of  $f$  as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of  $f$  as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If  $f(A) = B$ , then we say  $f$  is **Onto (Surjective)**.

**Definition** (Preimage). Consider a function  $f : A \rightarrow B$ . Let  $D \subseteq B$ . Then the **preimage** of  $D$  under  $f$  is denoted by

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

**Definition** (One-to-One (Injective)). Consider a function  $f : A \rightarrow B$ . We call  $f$  one-to-one if any of the following equivalent conditions hold:

- (i) For all  $x_1, x_2 \in A$ , if  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$ .
- (ii) For all  $x_1, x_2 \in A$ , if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ .
- (iii) For all  $y \in B$ , the set  $f^{-1}(\{y\})$  consists at most one element of  $A$ .

### 2.1.4 Equivalent Sets

**Definition.** Let  $A$  and  $B$  be two sets. We say that  $A$  and  $B$  have the same cardinal number, and we write  $A \sim B$ , if there is a function  $f : A \rightarrow B$  that is both injective and surjective.

**Remark.** • An injective and surjective mapping is a bijective mapping.

- $A$  and  $B$  have the same cardinal number

=  $A$  and  $B$  have the same cardinality

=  $A$  and  $B$  can be put in the **one-to-one correspondence**

=  $\text{card } A = \text{card } B$

=  $A$  and  $B$  are equivalent

=  $A$  and  $B$  are equipotent

**Example.** Consider  $\{1, 2, 3\} \sim \{a, b, c\}$ . Indeed, the function  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

**Example.**  $\mathbb{N} \sim \{2, 4, 6, \dots\}$ . Indeed, the function  $f : \mathbb{N} \rightarrow \{2, 4, 6, \dots\}$  defined by

$$f(n) = 2n$$

is a bijection.

**Example.**  $\mathbb{N} \sim \mathbb{Z}$ . Indeed,  $f : \mathbb{N} \rightarrow \mathbb{Z}$  defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

**Example.**  $(-\infty, \infty) \sim (0, \infty)$ . Indeed,  $f(x) = e^x$  is a bijection between  $(-\infty, \infty)$  and  $(0, \infty)$ .

**Example.**  $(0, \infty) \sim (0, 1)$ . Indeed, the function  $f : (0, \infty) \rightarrow (0, 1)$  defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

**Example.**  $[0, 1) \sim (0, 1)$ . Indeed, the function  $f : [0, 1) \rightarrow (0, 1)$  defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

**Definition** ( $\sim$  is an equivalence relation). Let  $A$  and  $B$  be two sets. Note that

- (i)  $A \sim A$  ( $\sim$  is reflexive)

- (ii) If  $A \sim B$ , then  $B \sim A$  ( $\sim$  is symmetric)
- (iii) If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$  ( $\sim$  is transitive).

Observe the following notation

$$\begin{aligned}\mathbb{N}_n &= \{1, 2, 3, \dots, n\} \\ \mathbb{N} &= \{1, 2, 3, \dots\}\end{aligned}$$

### 2.1.5 Finite, Infinite, Countable, At most countable

**Definition** (Finite, Infinite, Countable, At most countable). Let  $A$  be any set.

- (a) We say that  $A$  is **finite** if  $A \neq \emptyset$  or  $A \sim \mathbb{N}_n$  for some natural number  $n$ .
  - (\*) When  $A \sim \mathbb{N}_n$ , we say  $A$  has  $n$  elements and we write  $\text{card}(A) = n$ .
  - (\*) Also, we set  $\text{card}(\emptyset) = 0$ .
- (b) The set  $A$  is said to be **infinite** if it is not finite.
- (c) The set  $A$  is said to be **countable** if  $A \sim \mathbb{N}$ ; that is, there exists  $g : \mathbb{N} \rightarrow A$  is a bijection where  $A = \{g(1), g(2), g(3), \dots\}$ .
- (d) The set  $A$  is said to be **uncountable** if it is neither countable or finite.
- (e) The set  $A$  is said to be **at most countable** if it is either finite or countable.

**Remark.** Previously, we shared  $\mathbb{Z} \sim \mathbb{N}$ . Thus,  $\mathbb{Z}$  is countable. (Also, note that  $\mathbb{N}$  is a proper subset of  $\mathbb{Z}$ , nevertheless,  $\mathbb{N} \sim \mathbb{Z}$ )

Below are some Basic Theorems

- (i) Every countable set is infinite (There is no bijection  $\mathbb{N}_n \rightarrow \mathbb{N}$ ).
- (ii) Suppose  $A \sim B$ . Then

$$\begin{aligned}A \text{ is finite} &\iff B \text{ is finite} \\ A \text{ is countable} &\iff B \text{ is countable} \\ A \text{ is uncountable} &\iff B \text{ is uncountable}\end{aligned}$$

- (iii) The union of two finite sets is finite. If  $A$  is infinite and  $B$  is infinite, then  $A \setminus B$  is infinite.
- (iv) If  $A$  is **at most countable**, then there exists a 1 – 1 function  $f : A \rightarrow \mathbb{N}$ .

# Chapter 3

## Week 3

### 3.1 Lecture 4

#### 3.1.1 Topics

- (1) Sequences
- (2) Infinite subset of a countable set is countable.

#### 3.1.2 Sequences

**Definition (Sequence).** We call a **sequence**, we mean a function  $f$  on the set  $\mathbb{N}$ .

- We can let  $x_n = f(n)$ . Then it is customary to denote the sequence  $f$  by  $(x_n)_{n \geq 1}$  or  $x_1, x_2, \dots$
- Note that  $x_1, x_2, \dots$  need not be distinct.
- If for all  $n \in \mathbb{N}$ ,  $x_n \in A$ , then we say  $(x_n)_{n \geq 1}$  is a sequence in  $A$ .
- Sometimes it is convenient to replace  $\mathbb{N}$  in the definition above with  $\{0, 1, 2, \dots\}$  or  $\{-1, 0, 1, 2, \dots\}$ .

#### 3.1.3 Infinite subset of a countable set is countable

**Theorem.** Every infinite subset of a countable set is countable.

**Proof.** Let  $A$  be a countable set. Let  $E \subseteq A$  and  $E$  is infinite. Our goal is to show that  $E$  is countable. Since  $A$  is countable, there exists a bijective function  $g : \mathbb{N} \rightarrow A$ , so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with  $x_n = g(n)$  for all  $n \in \mathbb{N}$ . Now, let us construct the sequence  $n_1, n_2, \dots$  as follows:

- (1) Let  $n_1$  be the smallest positive integer such that  $x_{n_1} \in E$ .
- (2) Let  $n_2$  be the smallest positive integer greater than  $n_1$  such that  $x_{n_2} \in E$ .
- $\vdots$
- (k) Let  $n_k$  be the smallest integer greater than  $n_{k-1}$  such that  $x_{n_k} \in E$ .

Observe that the set

$$\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}$$

is both nonempty (since  $E$  is infinite) and bounded below (by the well ordering property of natural numbers). Thus, we know that

$$n_k = \min\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}.$$

Now, define the function  $f : \mathbb{N} \rightarrow E$  as follows:

$$f(k) = x_{n_k}.$$

We claim that this is a bijective map. If we can prove this fact, then surely  $E$  is countable. We need to prove two things:

- (1)  $f$  is injective and
- (2)  $f$  is surjective.

Starting with (1), suppose  $k_1 \neq k_2$ . Then we have  $n_{k_1} \neq n_{k_2}$  and thus,  $x_{n_{k_1}} \neq x_{n_{k_2}}$ . Hence, we see that  $f(k_1) \neq f(k_2)$ . Therefore,  $f$  is injective which proves (1).

With (2), let  $b \in E$ . Since  $E \subseteq A$ , we must have  $b \in A$  and thus there exists an  $m \in \mathbb{N}$  such that  $b = x_m$ . Hence, there exists  $1 \leq k \leq m$  such that  $x_{n_k} = x_m = b$ . Thus,  $f(k) = b$  which shows that  $f$  is surjective and so (2) is satisfied.

Therefore,  $f$  must be a bijective map and we can now conclude that  $E$  is countable. ■

**Remark.** Consider the contrapositive of the theorem above:

Let  $E \subseteq A$ . If  $E$  is not countable, then  $A$  is not countable.

When we say that a set is NOT countable, we do not necessarily mean that the set is uncountable. However, in the case that  $E$  is an infinite set, we can say that  $E$  being not countable is equivalent to saying that  $E$  is uncountable. So, only in the case in which  $E$  is infinite then the contrapositive is

Let  $E \subseteq A$  be infinite. If  $E$  is uncountable, then  $A$  is uncountable. By the theorem we just proved, we see that  $f(A) \subseteq S$  and  $S$  being countable implies that  $f(A)$  must be at most countable. Thus,  $A$  must be at most countable.

**Corollary.** Let  $A$  be any set and let  $S$  be a countable set. If there exists an injective mapping  $f : A \rightarrow S$ , then  $A$  is at most countable.

**Proof.** Let  $A$  be any set and let  $S$  be a countable set. Suppose there exists an injective mapping  $f : A \rightarrow S$ . We can restrict the codomain  $S$  of  $f$  to its range  $f(A)$ . Because  $f$  is injective, the mapping  $f : A \rightarrow f(A)$  will be bijective. Thus, we have that  $A \sim f(A)$ . ■

**Example** ( $\mathbb{N} \times \mathbb{N}$  is countable). (1) The function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$f(x, y) = 2^x 3^y$$

is injective. By the corollary we have just proved, we see that  $\mathbb{N} \times \mathbb{N}$  is at most countable.

- (2) Notice that  $g : \mathbb{N} \rightarrow \{1\} \times \mathbb{N}$  is a bijection  $g(a) = (1, a)$ . Hence,  $\{1\} \times \mathbb{N}$  is countable. Now, we have that

- (i)  $\{1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$
- (ii)  $\{1\} \times \mathbb{N}$  is countable implies that  $\mathbb{N} \times \mathbb{N}$  is infinite.

Thus, (i) and (ii) imply that  $\mathbb{N} \times \mathbb{N}$  is countable.

Proving that a set  $A$  is countable can be done by finding a subset  $B$  that is countable which forces  $A$  to be countable.

**Example** ( $\mathbb{Q}$  is countable). (1) The function  $f : \mathbb{Q} \rightarrow \mathbb{N}$  defined by

$$f(x) = \begin{cases} 2(2^p 3^q) & \text{if } x = \frac{p}{q} \ p \in \mathbb{N}, q \in \mathbb{N}, (p, q) = 1 \\ 2(2^p 3^q) + 1 & \text{if } x = -\frac{p}{q} \ p \in \mathbb{N}, q \in \mathbb{N}, (p, q) = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is an injective mapping. So,  $\mathbb{Q}$  is at most countable.

(2) Observe that  $\mathbb{N} \subseteq \mathbb{Q}$  and  $\mathbb{N}$  being a countable (and thus  $\mathbb{N}$  is infinite) set implies that  $\mathbb{Q}$  is infinite. Thus,  $\mathbb{Q}$  is countable.

### 3.1.4 Countable union of at most countable sets is at most countable

**Theorem.** Countable union of at most countable sets is at most countable.

**Proof.** Let  $\{A_n : n \in \mathbb{N}\}$  be a countable family of at most countable sets; that is, for each  $n \in \mathbb{N}$ ,  $A_n$  is at most countable. Our goal is to show that

$$K = \bigcup_{n \in \mathbb{N}} A_n \text{ is at most countable.}$$

To this end, it suffices to show that there exists an injective map  $f : K \rightarrow \mathbb{N} \times \mathbb{N}$ . Let

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1$$

$$B_3 = A_3 \setminus (A_1 \cup A_2)$$

$$\vdots$$

$$B_{n+1} = A_{n+1} \setminus \left( \bigcup_{k=1}^n A_k \right).$$

We leave as an exercise to show that

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \text{ are pairwise disjoint.}$$

Note that for all  $n \in \mathbb{N}$ , we have  $B_n \subseteq A_n$  and  $A_n$  is at most countable. Then for each  $n \in \mathbb{N}$ ,  $B_n$  must be at most countable. So, for each  $n \in \mathbb{N}$ , there exists an injective mapping  $f_n : B_n \rightarrow \mathbb{N}$ . Our goal is to show that

$$\bigcup_{n=1}^{\infty} B_n \text{ is at most countable.}$$

To this end, we define the function

$$f : \bigcup_{n=1}^{\infty} B_n \rightarrow \mathbb{N} \times \mathbb{N}$$

as follows:

For each  $x \in \bigcup_{n=1}^{\infty} B_n$ , there is exactly one  $n \in \mathbb{N}$  such that  $x \in B_n$ ; let us denote this  $n$  by  $n_x$ .

Thus, we define

$$f(x) = (n_x, f_{n_x}(x)).$$

This function is injective because

$$\begin{aligned} f(x) = f(y) &\implies (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y)) \\ &\implies n_x = n_y \wedge f_{n_x}(x) = f_{n_y}(y) \\ &\implies f_{n_x}(x) = f_{n_x}(y) \\ &\implies x = y \end{aligned} \quad (\text{Since } f_{n_x} \text{ is injective})$$

By the corollary, we see that the set

$$\bigcup_{n=1}^{\infty} B_n$$

is at most countable. ■

**Corollary.** A countable union of countable sets is countable.

**Proof.** Let  $\{A_n : n \in \mathbb{N}\}$  be a countable collection of countable sets. By the previous Theorem, we see that  $\bigcup_{n \in \mathbb{N}} A_n$  is at most countable. Note that  $A_1 \subseteq \bigcup_{n \in \mathbb{N}} A_n$  and that  $A_1$  is countable. Thus,  $A_1$  must be infinite and so

$$\bigcup_{n \in \mathbb{N}} A_n \text{ is infinite.}$$

Thus, we see that

$$\bigcup_{n \in \mathbb{N}} A_n$$

must be countable. ■

**Corollary.** If  $A$  and  $B$  are at most countable, then  $A \cup B$  at most countable.

**Proof.** Let  $A_1 = A$  and  $A_2 = B$  where  $A_3 = A_4 = \dots = A_n = \emptyset$ . Then  $A \cup B$  is at most countable by the previous Theorem. ■

**Theorem.** If  $A$  is countable, then  $A \times A$  is countable.

**Proof.** (1) Note that  $A \times A = \bigcup_{b \in A} (\{b\} \times A)$  for each  $b \in A$ .

(2) The function  $f : A \rightarrow \{b\} \times A$  defined by  $f(x) = (b, x)$  is bijective. So,  $A \sim \{b\} \times A$ . Hence,  $\{b\} \times A$  is countable.

Since a countable union of countable sets is countable, we see that  $A \times A$  must be countable by (1) and (2). ■

## 3.2 Lecture 5

### 3.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks



- Inequalities

### 3.2.2 Summary of Last Lecture

1. How to prove  $A$  is at most countable:
  - Show that  $A \subseteq S$  where  $S$  is countable.
  - Find an injective function  $f : A \rightarrow S$  where  $S$  is countable.
2. How to prove  $A$  is infinite:
  - Prove that  $A$  has a countable subset.
  - Find an injective function  $f : \mathbb{N} \rightarrow A$ .
  - Find an injective function  $f : A \rightarrow A$  that is not onto.
  - Find a proper subset  $B \subseteq A$  such that  $A \sim B$ .
3. How to prove that  $A \sim B$ .
  - find a bijective function  $f : A \rightarrow B$ .
  - (Shroder-Bernstein)
    - Find an injective function  $g : A \rightarrow B$
    - Find an injective function  $h : B \rightarrow A$ .
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

**Theorem (D).** Let  $A$  be the set of all sequences whose terms are the digits 0 and 1 (that is  $A$  is the collection of all binary sequences). This set  $A$  is uncountable.

**Proof.** First, notice that  $A$  is infinite. Let  $h : \mathbb{N} \rightarrow A$  be the function defined by

for all  $h(n) =$  The binary sequence whose  $n$ th term is the digit 1 and all other terms are zero.

Clearly,  $h$  is an injective map. Hence,  $h : \mathbb{N} \rightarrow h(\mathbb{N})$  is bijective. We have  $\mathbb{N} \sim h(\mathbb{N})$ , and so  $h(\mathbb{N})$  is infinite. Note that  $h(\mathbb{N}) \subseteq A$ . Thus,  $A$  is infinite

Suppose for sake of contradiction that  $A$  is NOT uncountable. Since  $A$  is infinite and not uncountable, this assumption tells us that  $A$  must be countable. Thus, there exists a bijective map  $f : \mathbb{N} \rightarrow A$ . So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each  $n \in \mathbb{N}$ ,  $f(n)$  is a binary sequence. Let

$$\begin{aligned} f(1) &= (a_1^1, a_2^1, \dots) \\ f(2) &= (a_1^2, a_2^2, \dots) \\ f(3) &= (a_1^3, a_2^3, \dots) \\ &\vdots \\ f(n) &= (a_1^n, a_2^n, \dots) \\ &\vdots \end{aligned}$$

The goal is to construct a binary sequence  $(b_1, b_2, \dots)$  that is not in this list! This will contradict the fact that  $A$  contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N} \quad b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence  $(b_n)$  is not the same as any of the sequences of the list above; that is, for all  $i \in \mathbb{N}$ ,  $b_i \neq f(i)$ . ■

### 3.2.3 Preliminary Remarks

- $\mathbb{R}$  is NOT just an ordered field, it has more extra structures.
- In  $\mathbb{R}$ , there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given  $a \in \mathbb{R}$ , the **size** of  $a$  is defined to be  $|a|$ . But note that the **absolute value of  $a$**  is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers  $a$  and  $b$  in  $\mathbb{R}$ , the **standard distance** between  $a$  and  $b$  is

$$\text{dist}(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as  $h$  gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

**Theorem** (Basic Properties of Standard Size in  $\mathbb{R}$ ). Let  $a \in \mathbb{R}$ . Then

- (i)  $|a| \geq 0$
- (ii)  $|a| = 0$  if and only if  $a = 0$ .
- (iii)  $|\alpha a| = |\alpha||a|$  for all  $\alpha \in \mathbb{R}$  and  $a \in \mathbb{R}$ .
- (iv)  $|a + b| \leq |a| + |b|$  for all  $a, b \in \mathbb{R}$ .

**Remark.** Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

- (i) Every  $a \in \mathbb{R}$ , we have  $-|a| \leq a \leq |a|$  and  $-|b| \leq b \leq |b|$ .

Is it possible to generalize the notion of "size"? **YES!**

**Definition.** Let  $V$  be a (real) vector space. A function from  $\|\cdot\| : V \rightarrow \mathbb{R}$  is called a **norm** on  $V$  if it satisfies the following properties:

- (i) For all  $x \in V$ ,  $\|x\| \geq 0$
- (ii) For all  $x \in V$ ,  $\|x\| = 0$  if and only if  $x = 0$ .
- (iii) For all  $\alpha \in \mathbb{R}$  and  $x \in V$ ,  $\|\alpha x\| = |\alpha| \|x\|$ .
- (iv) For all  $x, y \in V$ , we have

$$\|x + y\| \leq \|x\| + \|y\| \quad (\text{triangle inequality})$$

A vector space  $V$  equipped with notion of norm is called a normed space. (Sometimes, we write  $(V, \|\cdot\|)$  is a normed space). So, call  $d(x, y) = \|x - y\|$ .

**Theorem** (Basic Properties of Standard Distance in  $\mathbb{R}$ ). For all  $a, b \in \mathbb{R}$ .

- (i)  $\text{dist}(a, b) \geq 0$
- (ii)  $\text{dist}(a, b) = 0$  if and only if  $a = b$ .
- (iii)  $\text{dist}(a, b) = \text{dist}(b, a)$ .
- (iv) For all  $a, b, c \in \mathbb{R}$ , we have  $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$ .

### 3.2.4 Metric Spaces

**Definition** (Metric Spaces). Let  $X \neq \emptyset$ . A function  $d : X \times X \rightarrow \mathbb{R}$  is said to be a **distance function** or a **metric** if it satisfies the following properties:

- (i) For all  $x, y \in X$ ,  $d(x, y) \geq 0$ .
- (ii) For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$ .
- (iii) For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$ .
- (iv) For all  $x, y, z \in X$ ,  $d(x, y) \leq d(x, z) + d(z, y)$ .

A set  $X$  equipped with a metric  $d$  is called a **metric space**. (Sometimes we write  $(X, d)$  is a metric space)

**Remark.**  $X$  does **NOT** need to be a vector space. Furthermore,  $d$  is a more general notion of size than  $\|\cdot\|$ .

**Example.**  $(\mathbb{R}, d)$  where  $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  is defined by

$$d(x, y) = |x - y|.$$

(or  $d(x, y) = \alpha|x - y|$  where  $\alpha > 0$  is a fixed real number)

# Chapter 4

## Week 4

### 4.1 Lecture 6

#### 4.1.1 A few examples of Metrics

**Example.** Consider  $(\mathbb{R}, d)$  where  $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  is defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

This is a metric on  $(\mathbb{R}, d)$ . **Prove this on homework!**

**Remark.** If  $(X, D)$  is a metric space, then  $(X, d)$  is also a metric space where

$$d(x, y) = \frac{D(x, y)}{1 + D(x, y)}.$$

We can define a metric that will always be less than or equal to 1.

**Example (Taxi Cab Metric).** Consider  $(\mathbb{R}^2, d)$  where  $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$  is defined by

$$d((a, b), (x, y)) = |a - x| + |b - y|$$

We want to show that this is a metric.

- (i) For all  $(a, b), (x, y) \in \mathbb{R}^2$ , we have  $d((a, b), (x, y)) = |a - x| + |b - y| \geq 0$  by property of the absolute value  $|\cdot|$ .
- (ii) For all  $(a, b), (x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned} d((a, b), (x, y)) = 0 &\iff |a - x| + |b - y| = 0 \\ &\iff |a - x| = 0 \text{ and } |b - y| = 0 \\ &\iff a - x = 0 \text{ and } b - y = 0 \\ &\iff (a, b) = (x, y). \end{aligned}$$

- (iii) For all  $(a, b), (x, y) \in \mathbb{R}^2$ .

$$d((a, b), (x, y)) = |a - x| + |b - y| = |x - a| + |y - b| = d((x, y), (a, b)).$$

(iv) For all  $(a, b), (x, y), (t, s) \in \mathbb{R}^2$ , we want to show that

$$d((a, b), (x, y)) \leq d((a, b), (t, s)) + d((t, s), (x, y)).$$

We have

$$\begin{aligned} d((a, b), (t, s)) + d((t, s), (x, y)) &= |a - t| + |b - s| + |t - x| + |s - y| \\ &= (|a - t| + |t - x|) + (|b - s| + |s - y|) \\ &\geq |a - x| + |b - y| \\ &= d((a, b), (x, y)) \end{aligned}$$

**Example (Discrete Metric).** Consider  $X \rightarrow$  any nonempty set. Consider  $d : X \times X \rightarrow [0, \infty)$ ,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

We want to show that this is a metric on  $(X, \text{any non-empty set})$ .

- (i) For all  $x, y \in X$ , we have  $d(x, y) \geq 0$  because either the  $d(x, y) = 1$  or  $d(x, y) = 0$ .
- (ii) Let  $x, y \in X$ . Clearly, we have  $x = y$  if and only if  $d(x, y) = 0$  by definition of  $d(x, y)$ .
- (iii) Let  $x, y \in X$ . Clearly, we have  $d(x, y) = d(y, x)$  by definition.
- (iv) Let  $x, y, z \in X$ . We want to show that

$$d(x, y) \leq d(x, z) + d(z, y)$$

Let us consider two cases:

- (1)  $x = y$ . Thus, both sides of the triangle inequality clearly hold since  $d(x, y) = 0$ .
- (2)  $x \neq y$ . In this case, we have  $d(x, y) = 1$ . So, we need to show that

$$d(x, z) + d(z, y) \geq 1.$$

Since  $x \neq y$ , at least one of the statements  $z \neq y$  or  $z \neq x$  is true. If  $z \neq x$ , then  $d(z, x) = 1$  and so

$$d(x, z) + d(z, y) = 1 + d(z, y) \geq 1.$$

If  $z \neq y$ , then  $d(z, y) = 1$ , and so

$$d(x, z) + d(z, y) = d(x, z) + 1 \geq 1.$$

**Example.** Consider  $(V, \|\cdot\|) \rightarrow$  any normed vector space with the metric  $d : V \times V \rightarrow [0, \infty)$  and  $d(x, y) = \|x - y\|$ . We want to show that this is a metric on  $V$ .

- (i) Let  $x, y \in V$ . By the property of the norm, we have  $d(x, y) = \|x - y\| \geq 0$ .
- (ii) Let  $x, y \in V$ . By the 2nd property of the norm, we have

$$\begin{aligned} d(x, y) = 0 &\iff \|x - y\| = 0 \\ &\iff x - y = 0 \\ &\iff x = y. \end{aligned}$$

(iii) Let  $x, y \in V$ . We have

$$d(x, y) = \|x - y\| = \|(y - x)\| = |-1|\|y - x\| = \|y - x\| = d(y, x).$$

(iv) Let  $x, y, z \in V$ . We want to show that

$$d(x, y) \leq d(x, z) + d(z, y).$$

We have,

$$\begin{aligned} d(x, z) + d(z, y) &= \|x - z\| + \|z - y\| \\ &\geq \|(x - z) + (z - y)\| \\ &= \|x - y\| \\ &= d(x, y). \end{aligned}$$

This example illustrates how every time we have a norm, we immediately have a metric space. But the other around is not necessarily true!

**Example (Standard Distance in  $\mathbb{R}^n$ ).** Consider  $(\mathbb{R}^n, d)$  where  $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by for all  $x, y \in \mathbb{R}^n$  with  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$ , we have

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + \dots + |x_n - y_n|^2}$$

Note that if we define for all  $x \in \mathbb{R}^n$ , we have

$$\|x\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}.$$

Then

$$d(x, y) = \|x - y\|_2.$$

Proving that the above is a norm is enough to show that the original in question is a metric.

Let  $x, y \in \mathbb{R}^n$ .

(i) We have

$$\|x\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2} \geq 0.$$

(ii) We have

$$\begin{aligned} \|x\|_2 = 0 &\iff \sqrt{|x_1|^2 + \dots + |x_n|^2} = 0 \\ &\iff |x_1|^2 + \dots + |x_n|^2 = 0 \\ &\iff |x_1| = 0, |x_2| = 0, \dots, |x_n| = 0 \\ &\iff x_1 = 0, x_2 = 0, \dots, x_n = 0. \end{aligned}$$

Thus, we have  $x = 0$ .

(iii) For all  $\alpha \in \mathbb{R}$ , we have

$$\begin{aligned} \|\alpha x\|_2 &= \sqrt{(\alpha x_1)^2 + \dots + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2(x_1^2 + \dots + x_n^2)} \\ &= |\alpha| \sqrt{x_1^2 + \dots + x_n^2} \\ &= |\alpha| \|x\|_2. \end{aligned}$$

(iv) Now, we want to show that

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

That is, we want to show that

$$\sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} \leq \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}.$$

**We will show this later!**

**Example.** Consider  $(\mathbb{R}^n, d)$  where  $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by for all  $x, y \in \mathbb{R}^n$ , we have

$$d_p(x, y) = \left[ |x_1 - y_1|^p + \cdots + |x_n - y_n|^p \right]^{\frac{1}{p}}.$$

### 4.1.2 Inequalities

In what follows we will list several key inequalities that can be used in proving that, for example, a given expression is a metric or a norm.

(1) The triangle inequality for the standard norm in  $\mathbb{R}$ . We have for all  $x, y \in \mathbb{R}$ , we have

$$|x + y| \leq |x| + |y|.$$

More generally, we have for all  $x_1, \dots, x_n \in \mathbb{R}$ , we have

$$\left| \sum_{i=1}^n x_i \right| \leq \sum_{i=1}^n |x_i|.$$

(2) Reverse triangle inequality for the standard norm in  $\mathbb{R}$ ; that is, we have for all  $x, y \in \mathbb{R}$ ,

$$||x| - |y|| \leq |x - y|.$$

(3) For all  $a, b \geq 0$  and for all  $\rho > 0$ , we have

$$ab \leq \frac{1}{2} \left( \rho a^2 + \frac{1}{\rho} b^2 \right).$$

Note that we are dealing with real numbers here!

(4) Cauchy-Schwarz Inequality. For all  $x, y \in \mathbb{R}^n$ , we have

$$|x_1 y_1 + \cdots + x_n y_n| \leq \left( \sqrt{x_1^2 + \cdots + x_n^2} \right) \left( \sqrt{y_1^2 + \cdots + y_n^2} \right). \quad (*)$$

## 4.2 Lecture 7

### 4.2.1 Topics

- Review of Inequalities
- Neighborhood of a point
- Limit point of a set, isolated point of a set
- Closed set
- Interior point of a set

- Open set
- Bounded set
- Closure
- Dense

### 4.2.2 Inequalities

### 4.2.3 Minkowski

We want to show the triangle inequality for  $\|\cdot\|_2$  in  $\mathbb{R}^n$ ; that is, we want to show that

**Proposition.** Let  $\|\cdot\|_2$  be a norm in  $\mathbb{R}^n$ . Then for all  $x, y \in \mathbb{R}^n$ ,  $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ .

**Proof.** We see that  $x \cdot y \leq |x \cdot y| \leq \|x\|_2 \|y\|_2$  by the Cauchy Cauchy-Schwarz Inequality. Thus, we have

$$\begin{aligned} \|x + y\|_2^2 &= (x + y) \cdot (x + y) \\ &= x \cdot x + x \cdot y + y \cdot x + y \cdot y \\ &= \|x\|_2^2 + 2x \cdot y + \|y\|_2^2 \\ &\leq \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2 \\ &= (\|x\|_2 + \|y\|_2)^2. \end{aligned}$$

Hence, we have

$$\|x + y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2.$$

Therefore, we have

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

■

### 4.2.4 Minkowski for General p

For general  $p$ , we have Holder's Inequality which is

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

for any fixed real number  $p \geq 1$  and for any  $x, y \in \mathbb{R}^n$ .

### 4.2.5 $(x + 1)^n$

Recall that for all  $a, b \in \mathbb{R}$ , for all  $n \in \mathbb{N}$ , we have

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

So, in particular, if  $x \geq 0$  and  $n \in \mathbb{N}$ , then

$$\begin{aligned} (x + 1)^n &= \sum_{k=0}^n \binom{n}{k} x^k 1^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} x^k \\ &= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\ &\geq 1 + nx. \end{aligned}$$



Hence, we have for all  $x \geq 0$  and for all  $n \in \mathbb{N}$ , we have

$$(x + 1)^n \geq 1 + nx.$$

#### 4.2.6 $p$ -means

Let  $x_1, \dots, x_n$  be positive real numbers. Let  $p \in \mathbb{N} \cup \{0\}$ . By the  $p$ -mean of  $x_1, \dots, x_n$  denoted by  $A_p(x_1, \dots, x_n)$ , we mean

$$A_p(x_1, \dots, x_n) = \begin{cases} \sqrt[p]{\frac{x_1^p + \dots + x_n^p}{n}} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 \dots x_n} & \text{if } p = 0. \end{cases}$$

For example, if  $p = 1$ , we have

$$A_1(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

which is the Arithmetic Mean. If  $p = 2$ , we have

$$A_2(x_1, \dots, x_n) = \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}.$$

If  $p = 0$ , we have

$$A_0(x_1, \dots, x_n) = \sqrt[n]{x_1 \dots x_n}$$

which is the geometric mean. It can be shown that

$$A_0(x_1, \dots, x_n) \leq A_1(x_1, \dots, x_n) \leq A_2(x_1, \dots, x_n) \leq \dots$$

In particular, we have  $A_0 \leq A_1$ ; that is,

$$\sqrt[n]{x_1 \dots x_n} \leq \frac{x_1 + \dots + x_n}{n}. \quad (\text{AM-GM Inequality})$$

We can prove this inequality using differentiation. Without differentiation, we can prove the same inequality using **Cauchy Induction**.

#### 4.2.7 Jensen's Inequality

Suppose  $f : (a, b) \rightarrow \mathbb{R}$  is a convex function ( $f''(x) \geq 0$  for all  $x \in (a, b)$ ). Let  $x_1, \dots, x_n$  be points in  $(a, b)$ . Let  $\lambda_1, \dots, \lambda_n \geq 0$  such that  $\lambda_1 + \dots + \lambda_n = 1$ . Then

$$f(\lambda_1 x_1 + \dots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \dots + \lambda_n f(x_n).$$

We want to show this inequality holds for  $n = 2$ ; that is,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

where  $\lambda_1 + \lambda_2 = 1$ . Note that

$$f((1 - \lambda_2)x_1 + \lambda_2 x_2) \leq (1 - \lambda_2)f(x_1) + \lambda_2 f(x_2).$$

**Remark.** If we want to have an expression that defines a function that gives us any number in between two points  $e < h$ , we can have

$$f(\lambda) = (1 - \lambda)e + \lambda h$$

for any  $0 \leq \lambda \leq 1$ .

### 4.2.8 Neighborhood of a point

**Definition** (Neighborhood). Let  $(X, d)$  be a metric space. Let  $p \in X$ . For any  $\varepsilon > 0$ , we call

$$N_\varepsilon(p) = \{x \in X : d(p, x) < \varepsilon\}.$$

the **neighborhood of  $p$  of radius  $\varepsilon$** .

**Example.** Let  $(\mathbb{R}, d)$  and  $d(x, y) = |x - y|$ . The neighborhood of any  $p \in \mathbb{R}$  with radius  $\varepsilon > 0$  is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Note that  $|x - p| < \varepsilon$  is the same thing as  $p - \varepsilon < x < p + \varepsilon$  or that  $x \in (p - \varepsilon, p + \varepsilon)$ .

**Example.** Let  $(\mathbb{R}^2, d)$  with  $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$ . Let  $(a, b) \in \mathbb{R}^2$  with  $\varepsilon > 0$ . Then

$$\begin{aligned} N_\varepsilon((a, b)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (a, b)) < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \varepsilon^2\}. \end{aligned}$$

Thus,  $N_\varepsilon((a, b))$  consists of the points inside the circle of radius  $\varepsilon$  centered in  $(a, b)$ .

**Example.**  $(\mathbb{R}^2, d)$  with  $d((a, b), (x, y)) = |a - x| + |b - y|$  and let  $\varepsilon = 1$ . We have

$$\begin{aligned} N_1((0, 0)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x - 0| + |y - 0| < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \end{aligned}$$

If we graph this out in  $\mathbb{R}^2$ , then the shape of the neighborhood will take on a rhombus.

**Example.** Let  $(\mathbb{R}, d)$  with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let  $p \in \mathbb{R}$ . Let  $\varepsilon > 0$ . Let us consider two cases:

(1) Let  $\varepsilon \leq 1$ . Note that if

$$d(x, p) < \varepsilon \leq 1,$$

then  $d(x, p) < 1$ , and so  $d(x, p) = 0$ . Hence,  $x = p$ . Then the neighborhood is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{p\}.$$

(2) Let  $\varepsilon > 1$ . Clearly, for all  $x \in \mathbb{R}$ , we have  $d(x, p) \leq 1 < \varepsilon$ . So,

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \mathbb{R}.$$

### 4.2.9 Limit Points

**Definition (Limit Points, Isolated Points).** Let  $(X, d)$  is a metric space with  $E \subseteq X$ . Then we call

- (1) A point  $p \in X$  is said to be a **limit point of  $E$**  if for all  $\varepsilon > 0$ ,

$$N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset.$$

- (2) The **collection of all the limit points of  $E$**  is denoted by  $E'$ ; that is,

$$E' = \{p \in X : \text{for all } \varepsilon > 0, N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset\}.$$

- (3) A point  $p \in E$  is said to be an **isolated point of  $E$**  if  $p$  is NOT a limit point; that is,  $p \in E$  but  $p \notin E'$ . Another way of saying this is  $E \setminus E'$ .

- (4) If  $p$  is NOT a limit point, we have  $p \notin E'$  if and only if there exists  $\varepsilon > 0$  such that

$$N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset.$$

**Remark.** The statement for all  $\varepsilon > 0$ ,  $N_\varepsilon(p)$  is equivalent to for all  $N_\varepsilon(p)$ .

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$ . Note that  $0 \notin E$ . Also, recall that  $0 \in E'$  if and only if for all  $\varepsilon > 0$ ,  $N_\varepsilon(0) \cap (E \setminus \{0\}) \neq \emptyset$ . If  $0 \in E'$ , we just need to show that for all  $\varepsilon > 0$ ,  $N_\varepsilon(0) \cap E \neq \emptyset$ ; that is, we need to show that

$$\text{for all } \varepsilon > 0, (-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let  $\varepsilon > 0$ . By the Archimedean Property of  $\mathbb{R}$ , there exists  $m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ . Clearly,  $1/m \in (-\varepsilon, \varepsilon) \cap E$ .

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  and

$$E = (1, 2) \cup \{5\}.$$

Prove that 5 is an isolated point. Since  $5 \in E$ , it is enough to show that 5 is not in  $E'$ . Recall that  $5 \in E'$  if and only if there exists an  $\varepsilon > 0$  such that

$$N_\varepsilon(5) \cap (E \setminus \{5\}) = \emptyset.$$

Noticing that  $E \setminus \{5\} = (1, 2)$ , we can write  $5 \notin E'$  if and only if there exists an  $\varepsilon > 0$  such that  $(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = \emptyset$ . Clearly,  $\varepsilon = 1$  does the job and we are done; that is,

$$(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = (4, 6) \cap (1, 2) = \emptyset.$$

**Example.** Let  $(\mathbb{R}^2, d)$  and  $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$  and

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

What is  $E'$ ?

$$E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

For example, if  $(a, b)$  is such that  $a^2 + b^2 > 4$ , then  $(a, b) \notin E'$ . Let  $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$ . Clearly,  $N_\delta \cap (E \setminus \{p\}) = \emptyset$ .

**Definition (Closed Set).** Let  $(X, d)$  be a metric space,  $E \subseteq X$ . We say that  $E$  is **closed** if every limit point of  $E$  is contained within  $E$ ; that is,

$$E \text{ is closed} \iff E' \subseteq E.$$

**Example.** Let  $(\mathbb{R}, d)$ ,  $d(x, y) = |x - y|$  and  $E = \{1, 2, 3\}$ .

(i) What is  $E'$ ?

**Claim:**  $E' = \emptyset$ . Let  $p \in \mathbb{R}$ . Our goal is to show that  $p$  is not in  $E'$ . That is, we want to show that there exists an  $\varepsilon > 0$  such that  $N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset$  where  $N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon)$ .

We may consider the following cases:

- (1) If  $p < 1$ . Let  $\varepsilon = \frac{1-p}{2}$  works.
- (2) If  $p > 3$ , we have  $\varepsilon = \frac{p-3}{2}$  works.
- (3) If  $p \in \{1, 2, 3\}$ , then  $\varepsilon = \frac{1}{4}$  works.
- (4) If  $1 < p < 2$ , then let  $\varepsilon = \frac{1}{2} \min\{p - 1, 2 - p\}$  works.
- (5) If  $2 < p < 3$ , then  $\varepsilon = \frac{1}{2} \{\min p - 2, 3 - p\}$  works.

(ii) Is  $E$  closed? Since  $E' = \emptyset$ , we have  $E' \subseteq E$  and so  $E$  is closed.

**Remark.** Any finite set is closed!

**Definition (Interior Point).** Let  $(X, d)$  is a metric space and let  $E \subseteq X$ . We say that a point  $p \in E$  is said to be an **interior point** of  $E$  if there exists a neighborhood  $N_\varepsilon(x)$  such that  $N_\varepsilon(x) \subseteq E$ . The collection of all interior points of  $E$  is called the **interior** of  $E$  and is denoted by the set:

$$E^\circ = \{x \in E : \exists N_\varepsilon(x) \subseteq E\}.$$

**Remark.** Note that by definition we know that the interior of  $E$  is always contained within  $E$ ; that is,  $E^\circ \subseteq E$ . Also,

$$p \in E^\circ \text{ if and only if there exists a neighborhood } N_\delta(p) \text{ such that } N_\delta(p) \subseteq E.$$

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  with  $E = (1, 3]$ . What is  $E^\circ$ ? We claim that  $E^\circ = (1, 3)$ .

Let  $p \in (1, 3)$ . We want to show that

- (1) If  $p \in (1, 3)$ , then  $p$  is an interior point.
- (2) If  $p = 3$ , then  $p \notin E^\circ$ .

We proceed by showing each case above:

- (1) It suffices to show that there exists  $\delta > 0$  such that  $N_\delta(p) \subseteq E$ . Clearly, choose  $\delta = \frac{1}{2} \min\{p - 1, 3 - p\}$  and we are done.
- (2) Suppose  $p = 3$ . It suffices to show that

$$\forall \varepsilon > 0, N_\varepsilon(3) \not\subseteq E.$$

That is, we want to show that

$$\forall \varepsilon > 0, (3 - \varepsilon, 3 + \varepsilon) \cap E^c \neq \emptyset.$$

Clearly, for all  $\varepsilon > 0$ , we have  $3 + \frac{\varepsilon}{2} \in (3 - \varepsilon, 3 + \varepsilon)$ . Thus,  $3 + \frac{\varepsilon}{2} \in E^c$ . Hence, we have

$$(3 - \varepsilon, 3 + \varepsilon) \cap E^c \neq \emptyset.$$

To show that boundary points are not interior points, it suffices to show that intersection with each neighborhood and the complement of the set is nonempty.

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  and  $E = \{1, 2, 3\}$ . What is  $E^\circ$ ? We claim that  $E^\circ = \emptyset$ . The reason is as follows: Let  $p \in \{1, 2, 3\}$ ; that is, for all  $\varepsilon > 0$ , we have  $N_\varepsilon(p) \not\subseteq E$  but  $N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon)$  has infinitely many points. We proved that if  $p \in E$ , then  $p \notin E^\circ$ . So,  $E^\circ = \emptyset$ .

**Definition (Open Sets).** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say that  $E$  is **open** if every point of  $E$  is an interior point of  $E$ ; that is,

$$E \text{ is open} \iff E \subseteq E^\circ.$$

**Remark.** We know that, for any set,  $E^\circ \subseteq E$ . So, we can rewrite our definition as follows:

$$E \text{ is open} \iff E = E^\circ.$$

That is, the other inclusion holds!

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  and  $E = \{1, 2, 3\}$ . Is  $E$  open? Note that  $E^\circ = \emptyset$ . So,  $E^\circ \neq E$  and so  $E$  is NOT open.

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  and  $E = (1, 4)$ . Prove that  $E$  is open. It suffices to show that every point  $p \in E$  is an interior point. Let  $p \in E$ . That is, we want to show that there exists  $\delta > 0$  such that  $N_\delta(p) \subseteq E$ . If we choose  $\delta = \frac{1}{2}\{p - 1, 4 - p\}$  does the job and we are done.

**Definition (Bounded Sets).** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say that  $E$  is **bounded** if there exists  $\varepsilon > 0$  and  $q \in X$  such that  $E \subseteq N_\varepsilon(q)$ .

**Example.** Let  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$  and  $E = [0, \infty)$ . Is  $E$  bounded? **NO!** This is because for all  $q \in \mathbb{R}$  and  $\varepsilon > 0$ , we have

$$[0, \infty) \not\subseteq (q - \varepsilon, q + \varepsilon).$$

**Example.** Let  $(\mathbb{R}, d)$  with the discrete metric and  $E = [0, \infty)$ . Is  $E$  bounded? **YES!** For example, we have

$$E \subseteq N_{10}(0) = \mathbb{R}.$$

**Definition (Closure).** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . The **closure** of  $E$ , denoted by  $\overline{E}$ , is defined as follows:

$$\overline{E} = E \cup E'$$

that is, the closure of  $E$  is the union of the isolated points and limit points.

**Example.** Let  $(\mathbb{R}, d)$  and  $d(x, y) = |x - y|$ . What is  $\overline{\mathbb{Q}}$ ? Show within the homework that  $\mathbb{Q}' = \mathbb{R}$ . So,

**Definition (Dense).** Let  $(X, d)$  metric space and  $E \subseteq X$ . We say that  $E$  is dense in  $X$  if  $\overline{E} = X$ . (that is, every point of  $X$  is either in  $E$  or is a limit point of  $E$ ).

**Example.**  $\overline{\mathbb{Q}} = \mathbb{R}$ , so  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

# Chapter 5

## Week 5

### 5.0.1 Topics

- (1) Every neighborhood is an open set.
- (2) If  $p$  is a limit point of a set  $E$ , then every neighborhood of  $p$  contains infinitely many points of  $E$ .
- (3) Theorem:  $E$  is open  $\iff E^C$  is closed.
- (4) Theorem: arbitrary union of open sets is open, Finite intersection of open sets is open.
- (5) Theorem:  $\overline{E}$  is the smallest closed subset of  $X$  that contains  $E$ .

**Theorem.** Let  $(X, d)$  be a metric space and let  $p \in X$  and  $\varepsilon > 0$ . Every neighborhood is an open set; that is,  $N_\varepsilon(p)$  is an open set.

**Proof.** Our goal is to show that every point of  $N_\varepsilon(p)$  is an interior point of  $N_\varepsilon(p)$ . Let  $q \in N_\varepsilon(p)$ . We need to show that there exists  $\delta > 0$  such that  $N_\delta(q) \subseteq N_\varepsilon(p)$ . Let  $\delta = \frac{\varepsilon - d(p, q)}{2}$ . We claim that  $N_\delta(q)$  is a subset of  $N_\varepsilon(p)$ . Indeed, if  $x \in N_\delta(q)$ , then

$$d(q, x) < \delta \implies d(q, x) < \varepsilon - d(p, q)$$

and so

$$d(p, q) + d(q, x) < \varepsilon \iff d(p, x) < \varepsilon. \quad (\text{triangle inequality})$$

Thus,  $x \in N_\varepsilon(p)$ . ■

**Theorem.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . If  $p \in E'$ , then every neighborhood of  $p$  contains infinitely many points of  $E$ .

**Proof.** Suppose for sake of contradiction that there exists a neighborhood of  $p$  that contains finitely many points of  $E$ ; that is,

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \cap E \text{ is a finite set.}$$

Since  $N_\varepsilon(p) \cap (E \setminus \{p\}) \subseteq N_\varepsilon(p) \cap E$ , we can immediately conclude that  $N_\varepsilon(p) \cap (E \setminus \{p\})$  is finite also. Furthermore,  $N_\varepsilon(p) \cap (E \setminus \{p\})$  is nonempty since  $p \in E'$ . Let us denote the elements of  $N_\varepsilon(p) \cap (E \setminus \{p\})$  by  $x_1, \dots, x_n$ . Our goal is to find a  $\delta > 0$  such that the neighborhood with radius  $\delta > 0$ , we will not

contain any of the finite points we introduced. For each  $i \in \{1, \dots, n\}$ ,  $d(p, x_i) > 0$ ,

$$\delta = \min\{d(p, x_i) : i \in 1, \dots, n\} > 0.$$

Clearly,  $N_{\delta/2}(p) \cap (E \setminus \{p\}) = \emptyset$ . But this contradicts our assumption that  $p$  is NOT a limit point of  $E$ . ■

**Corollary.** A finite set has no limit points; that is, if  $E$  is finite, then  $E' = \emptyset$ .

**Proof.** This is just the contrapositive of the theorem above. ■

**Theorem.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We have that  $E$  is open if and only if  $E^c$  is closed.

**Proof.** ( $\implies$ ) Assume that  $E$  is open. We want to show that  $E^c$  is closed; that is, every limit point of  $E^c$  is contained in  $E^c$ . Let  $p$  be a limit point of  $(E^c)$ . Assume for sake of contradiction that  $p \notin E^c$ . Then  $p \in E$ . Since  $E$  is open,  $p$  is an interior point of  $E$ . Thus, there exists  $\delta > 0$  such that  $N_\delta(p) \subseteq E$ ; that is, there exists  $\delta > 0$  such that  $N_\delta(p) \cap E^c = \emptyset$ . Furthermore, we have that

$$N_\delta(p) \cap (E^c \setminus \{p\}) = \emptyset.$$

But this tells us that  $p$  is not a limit point which is a contradiction.

( $\impliedby$ ) Assume that  $E^c$  is closed. We want to show that  $E$  is open; that is, every  $x \in E$  is an interior point. Let  $p \in E$ . Assume for sake of contradiction that  $p \notin E^\circ$ . Then for all  $\delta > 0$ ,  $N_\delta(p) \not\subseteq E$ . Hence, for all  $N_\delta(p) \cap E^c \neq \emptyset$ . Therefore,

$$\forall \delta > 0 \ N_\delta(p) \cap (E^c \setminus \{p\}) \neq \emptyset.$$

That is,  $p$  is a limit point of  $E^c$ . But by assumption,  $E^c$  is closed. Thus,  $p \in E^c$ . But this contradicts the assumption that  $p \in E$ . ■

**Theorem.** Let  $(X, d)$  be a metric space. Let  $\{A_\alpha\}_{\alpha \in \Lambda}$  be a collection of open sets where  $\Lambda$  is an index set (can be finite or infinite). Then

$$\bigcup_{\alpha \in \Lambda} A_\alpha$$

is an open set.

**Proof.** Our goal is to show that every point of  $A = \bigcup_{\alpha \in \Lambda} A_\alpha$  is an interior point. Let  $p \in A$ . Thus, there exists  $\alpha \in \Lambda$  such that  $p \in A_\alpha$ . Since  $A_\alpha$  is open, so there exists  $\delta > 0$  such that  $N_\delta(p) \subseteq A_\alpha$ . But note that

$$A_{\alpha_0} \subseteq A.$$

Hence,

$$N_\delta(p) \subseteq A$$

and thus  $p$  is an interior point of  $A$ . ■

**Theorem.** Let  $A_1, \dots, A_n$  be open sets in the metric space  $(X, d)$ . Then

$$\bigcap_{k=1}^n A_k \text{ is open.}$$



**Proof.** Our goal is to show that every point of  $\bigcap_{k=1}^n A_k$  is an interior point of  $\bigcap_{k=1}^n A_k$ . Let  $p \in \bigcap_{k=1}^n A_k$ . Then for all  $1 \leq k \leq n$  such that  $p \in A_k$ . Since  $A_k$  is open for all  $k$ , we know that for all  $1 \leq k \leq n$ , there exists  $\delta_k > 0$  such that  $N_{\delta_k}(p) \subseteq A_k$ . Let  $\delta = \min\{\delta_i : 1 \leq i \leq n\}$ . Then we have that

$$N_\delta(p) \subseteq N_{\delta_k}(p) \subseteq A_k.$$

Consequently, we have

$$N_\delta(p) \subseteq \bigcap_{k=1}^n A_k.$$

Hence,  $p$  is an interior point of the intersection  $\bigcap_{k=1}^n A_k$ . ■

**Theorem.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ .

- (1)  $\overline{E}$  is a closed set.
- (2)  $E$  is closed if and only if  $E = \overline{E}$ .
- (3) If  $E \subseteq F$  and  $F$  is closed, then  $\overline{E} \subseteq F$ .

**Proof.** Our goal is to show that  $(\overline{E})^c$  is open. We need to show that every point of  $(\overline{E})^c$  is an interior point of  $(\overline{E})^c$ . Let  $p \in (\overline{E})^c$ . We have

$$\begin{aligned} p \in (\overline{E})^c &\implies p \notin \overline{E} \\ &\implies p \notin (E \cup E') \\ &\implies p \notin E \wedge p \notin E'. \end{aligned}$$

Note that

$$\begin{aligned} p \notin E' &\implies \exists \varepsilon > 0 \ N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset \\ &\implies \exists \varepsilon > 0 \ N_\varepsilon(p) \cap E = \emptyset. \end{aligned} \tag{1}$$

In what follows, we will show that  $N_\varepsilon(p) \cap E' = \emptyset$ . So, we have

$$\begin{aligned} N_\varepsilon(p) \cap (E \cup E') &= \emptyset \\ \implies N_\varepsilon(p) \cap \overline{E} &= \emptyset \\ \implies N_\varepsilon(p) &\subseteq (\overline{E})^c. \end{aligned}$$

Thus, we have that  $p$  is an interior point of  $(\overline{E})^c$ . It remains to show that  $N_\varepsilon(p) \cap E' = \emptyset$ . Assume for sake of contradiction that  $N_\varepsilon(p) \cap E' \neq \emptyset$ . Let  $q \in N_\varepsilon(p) \cap E'$ . Then we have  $q \in N_\varepsilon(p)$  and  $q \in E'$ . Hence, there exists  $\delta > 0$  such that  $N_\delta(q) \subseteq N_\varepsilon(p)$  and that  $N_\delta(q) \cap (E \setminus \{q\}) \neq \emptyset$ , respectively. But note that since  $N_q(p) \subseteq N_\varepsilon(p)$  and  $E \setminus \{q\} \subseteq E$  implies that

$$N_\varepsilon(p) \cap E \neq \emptyset$$

which contradicts (1). ■

**Definition (Metric Subspace).** Let  $(X, d)$  be a metric space and let  $Y$  be a nonempty subset of  $X$ . Then  $(Y, d|_{Y \times Y})$  is called a **metric subspace** of  $(X, d)$ .

**Remark.** Consider the metric subspace  $(Y, d|_{Y \times Y})$  of  $(X, d)$ . Let  $p \in Y$  and  $\varepsilon > 0$ . Then we define

$$\begin{aligned} N_\varepsilon^Y(p) &= \{y \in Y : d^Y(y, p) < \varepsilon\} = \{y \in Y : d(y, p) < \varepsilon\} \\ &= N_\varepsilon(p) \cap Y. \end{aligned}$$

# Chapter 6

## Week 6

### 6.1 Lecture 10

**Theorem** ( $E$  is open relative to  $Y$ ). Let  $(X, d)$  be a metric space and  $E \subseteq Y \subseteq X$  and  $Y \neq \emptyset$ .  $E$  is open relative to  $Y$  if and only if there exists an open set  $G_0 \subseteq X$  such that  $E = G \cap Y$ .

**Proof.** ( $\implies$ ) Assume that  $E$  is open relative to  $Y$ . Our goal is to show that there exists an open set  $G \subseteq X$  such that  $E = G \cap Y$ . Since  $E$  is open relative to  $Y$ , every  $a \in E$ , we have that  $a$  is an interior point of  $E$ ; that is, there exists  $\varepsilon_0 > 0$  such that  $N_{\varepsilon_0}^Y(a) \subseteq E$ . Hence, for all  $a \in E$ , there exists  $\varepsilon_a > 0$  such that  $N_{\varepsilon_a}(a) \cap Y \subseteq E$ . Let  $G = \bigcup_{a \in E} N_{\varepsilon_a}(a)$ . Clearly,  $G$  is open in  $X$  since

- (1) For all  $a \in E$ ,  $N_{\varepsilon_a}(a)$  is a neighborhood and so it is open in  $X$ .
- (2) A union of open sets is open in any metric space.

In what follows, we will prove that  $E = G \cap Y$ . Note that

$$G \cap Y = \left( \bigcup_{a \in E} N_{\varepsilon_a}(a) \right) \cap Y = \bigcup_{a \in E} (N_{\varepsilon_a} \cap Y) \subseteq \bigcup_{a \in E} E = E.$$

Suppose  $b \in E$ . We have  $b \in N_{\varepsilon_b}(b)$  and thus  $b \in G$ . Furthermore,  $b \in E$  implies  $b \in Y$  since  $E \subseteq Y$ . Thus, we see that  $E \subseteq G \cap Y$ .

( $\impliedby$ ) Assume that there exists  $G \subseteq X$  such that  $E = G \cap Y$ . We want to show that  $E$  is open relative to  $Y$ . Our goal is to show that for all  $a \in E$ , there exists  $\varepsilon > 0$  such that  $N_{\varepsilon}^Y(a) \subseteq E$ .

So, let  $a \in E$  be given. Our goal is to find  $\varepsilon > 0$  such that

$$N_{\varepsilon}(a) \cap Y \subseteq E.$$

By assumption, we have

$$\begin{aligned} a \in E = G \cap Y &\implies a \in G \\ &\implies_{G \text{ is open}} \exists \varepsilon > 0 \text{ such that } N_{\varepsilon}(a) \subseteq G. \end{aligned}$$

Hence, we see that

$$N_{\varepsilon}(a) \cap Y \subseteq G \cap Y = E.$$

■

**Theorem.** (i) If  $E \subseteq \mathbb{R}$  is bounded above, then  $\sup E \in \overline{E}$ .

(ii) If  $E \subseteq \mathbb{R}$  is bounded below, then  $\inf E \in \overline{E}$ .

**Proof.** Here we will prove (1) and the proof of (2) is completely analogous.

Since  $E$  is bounded above, we have that  $\sup E$  exists and is a real number by the least upper bound property of  $\mathbb{R}$ . Let  $\alpha = \sup E$ . Our goal is to show that  $\alpha \in \overline{E}$ ; that is, we want to show that for all  $\varepsilon > 0$ ,

$$N_\varepsilon(\alpha) \cap E \neq \emptyset. \quad (\text{Exercise 11 of HW4})$$

Let  $\varepsilon > 0$  be given. Since  $\alpha = \sup E$ , we know that there exists  $x \in E$  such that  $\alpha - \varepsilon < x$ . Hence, there exists  $x \in E$  such that

$$\alpha - \varepsilon < x \leq \alpha < \alpha + \varepsilon.$$

Hence,  $\alpha - \varepsilon < x < \alpha + \varepsilon$ ; that is,  $x \in N_\varepsilon(\alpha)$ . Therefore,  $N_\varepsilon(\alpha) \cap E \neq \emptyset$ . ■

**Definition (Open Covers).** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . A collection of sets  $\{O_\alpha\}_{\alpha \in \Lambda}$  is said to be an **open cover** of  $E$  if

(i) for every  $\alpha \in \Lambda$ ,  $O_\alpha$  is open in  $X$ .

(ii)  $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$ .

**Example.** Consider  $(\mathbb{R}, |\bullet|)$  and the subset  $E = [0, \infty)$ . The collection  $\{E_n\}_{n \in \mathbb{N}}$  defined by for all  $n \in \mathbb{N}$ , we have

$$E_n = \left(-\frac{1}{n}, n\right)$$

is an open cover of  $E$ . The reason is as follows:

(i) In homework 4, we proved that every open interval is an open set with the standard metric on  $\mathbb{R}$ .

(ii)  $E \subseteq \bigcup_{n=1}^{\infty} E_n$ . Indeed, let  $a \in E$ . If  $a = 0$ , then  $a$  belongs  $E_n$  for all  $n \in \mathbb{N}$ . So, we have

$$a \in \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, if  $a \neq 0$ , then  $\frac{1}{a} > 0$ . By the Archimedean Property of  $\mathbb{R}$ , there exists an  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \frac{1}{a}$ . So,  $a < n$ . Clearly,

$$-\frac{1}{n} < 0 < a < n.$$

Hence,  $a \in E_n = \left(-\frac{1}{n}, n\right)$ . Consequently,

$$a \in \bigcup_{m=1}^{\infty} E_m.$$

**Definition (Compactness).** Let  $(X, d)$  be a metric space and  $K \subseteq X$ . We say that  $K$  is **compact** if every open cover of  $K$  has a finite subcover; that is, if  $\{O_\alpha\}_{\alpha \in \Lambda}$  is an open cover of  $K$ , then there exists  $\alpha_1, \alpha_2, \dots, \alpha_n$ , we have

$$K \subseteq \bigcup_{i=1}^n O_{\alpha_i}.$$

**Example.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . If  $E$  is finite, then  $E$  is compact. The reason is as follows:

Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be any open cover. Our goal is to show that this open cover has a finite subcover; that is, we can choose  $\alpha_1, \dots, \alpha_n$  such that

$$E \subseteq \bigcup_{i=1}^n O_{\alpha_i}.$$

If  $E = \emptyset$ , there is nothing to prove. Otherwise,  $E \neq \emptyset$ , we can denote the elements of  $E$  by  $x_1, \dots, x_n$ . So,

$$E = \{x_1, \dots, x_n\}.$$

We have

$$x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

and so continue this process until you have reached the  $n$ th element of  $E$ . Hence, we have

$$x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}.$$

Thus, we have that

$$E \subseteq \bigcup_{i=1}^n O_{\alpha_i}$$

and so  $E$  is compact.

**Example.** Let  $(\mathbb{R}, |\cdot|)$  and

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Prove that  $E$  is compact (In general, if  $a_n \rightarrow a$  in  $\mathbb{R}$ , then the set  $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$  is compact).

Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be any open cover of  $E$ . Our goal is to show that this open cover has a finite subcover. Notice that  $0 \in E$  and  $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$  implies that

$$0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0}.$$

So, if  $0 \in O_{\alpha_0}$  and  $O_{\alpha_0}$  is open, we have

$$\exists \varepsilon > 0 \text{ such that } (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}.$$

By the Archimedean Property of  $\mathbb{R}$ , we see that there exists  $m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ . So, for all  $n \geq m$ , we have  $\frac{1}{n} < \varepsilon$ . Hence, for all  $n \geq m$ , we have  $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}$ . Note that  $1 \in E$  so there exists  $\alpha_1 \in \Lambda$  such that  $1 \in O_{\alpha_1}$ . Similarly,  $\frac{1}{2} \in E$  implies that there exists  $\alpha_2 \in \Lambda$  such that  $\frac{1}{2} \in O_{\alpha_2}$ . Continue this process until, we have

$$\frac{1}{m-1} \in E \implies \exists \alpha_m \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}}.$$

Thus, we conclude that

$$E \subseteq \bigcup_{n=0}^{\infty} O_{\alpha_n}$$

and so  $E$  is compact.

**Remark.** If  $X$  itself is compact, we say that  $(X, d)$  is a compact metric space; that is, if  $\{O_\alpha\}_{\alpha \in \Lambda}$  is any collection of open sets such that

$$X = \bigcup_{\alpha \in \Lambda} O_\alpha$$

then there exists  $\alpha_1, \dots, \alpha_n \in \Lambda$  such that

$$X = \bigcup_{i=1}^n O_{\alpha_i}.$$

**Theorem** (Compactness implies Closed). Let  $(X, d)$  be a metric space and  $K \subseteq X$  is compact. Then  $K$  is compact.

**Proof.** It is enough to show that  $K^c$  is open; that is, we need to find  $\varepsilon > 0$  such that  $N_\varepsilon(x) \subseteq K^c$  for every  $x \in K^c$ . That is, we want to show that

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \ d(x, a) > 0 \end{aligned}$$

For all  $x \in K$ , let  $\varepsilon_x = \frac{1}{4}d(x, a)$ . Clearly, we have

$$\forall x \in K \ N_{\varepsilon_x}(a) \cap N_{\varepsilon_x}(x) = \emptyset.$$

Note that  $\{N_{\varepsilon_x}(x)\}_{x \in K}$  is an open cover for  $K$ . Since  $K$  is compact, there is a finite subcover. That is, there exists  $x_1, \dots, x_n \in K$  such that

$$K \subseteq \bigcup_{i=1}^n N_{\varepsilon_{x_i}}(x_i)$$

and of course

$$N_{\varepsilon_{x_1}}(x_1) \cap N_{\varepsilon_{x_1}}(a) = \emptyset.$$

If we continue this process until the  $n$ th step, we have that

$$N_{\varepsilon_{x_n}}(x_n) \cap N_{\varepsilon_{x_n}}(a) = \emptyset.$$

Let  $\varepsilon = \min\{\varepsilon_{x_1}, \dots, \varepsilon_{x_n}\}$ . Clearly,  $N_\varepsilon(a) \subseteq N_{\varepsilon_{x_i}}(a) \forall 1 \leq i \leq n$ . Hence, we have

$$N_{\varepsilon_{x_1}} \cap N_\varepsilon(a) = \emptyset$$

and similarly for the  $n$ th step, we have

$$N_{\varepsilon_{x_n}}(x_n) \cap N_\varepsilon(a) = \emptyset.$$

Therefore, we have

$$N_\varepsilon(a) \cap [N_{\varepsilon_{x_1}} \cup \dots \cup N_{\varepsilon_{x_n}}] = \emptyset.$$

So,

$$N_\varepsilon(a) \cap K = \emptyset.$$

■

## 6.2 Lecture 11

### 6.2.1 Topics

- Metric subspace
- Theorem 2.35
- Theorem 2.33
- Theorem 2.37
- Theorem 2.36

**Theorem.** Closed subsets of compact sets are compact.

**Proof.**

■

**Corollary.** If  $F$  is closed and  $K$  is compact, then  $F \cap K$  is compact.

**Proof.**

■

**Theorem.** Suppose  $K \subseteq Y \subseteq X$ . We have  $E$  is compact if and only if  $K$  is compact relative to  $Y$ .

**Proof.**

■

**Theorem.** If  $E$  is an infinite subset of a compact set  $K$ , then  $E$  has a limit point in  $K$ .

**Proof.**

■

**Remark.** Let  $(X, d)$  be a metric space and  $K \subseteq X$ . The following statements are equivalent:

- (i)  $K$  is compact.
- (ii) Every infinite subset of  $K$  has a limit point in  $K$ .
- (iii) Every sequence in  $K$  has a subsequence that converges to a point in  $K$ .

**Corollary.** If  $K_1 \supseteq K_2 \supseteq \dots$  is a sequence of nonempty compact sets, then  $\bigcap_{n=1}^{\infty} K_n$  is nonempty.

**Proof.**

■

**Theorem.** Let  $\{K_\alpha\}$  a collection of compact subsets of  $X$ . If the intersection of any finite subcollection

is nonempty, then

$$\bigcap_{\alpha \in \Lambda} K_{\alpha} \neq \emptyset.$$

**Proof.**

■

**Theorem** (Nested Interval Property). If  $I_n = [a_n, b_n]$  is a sequence of closed intervals in  $\mathbb{R}$  such that  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ , then  $\bigcap_{n=1}^{\infty} I_n$  is nonempty.

**Proof.**

■