Contents

Linear Algebra Exercises

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November 29, 2023

Chapter 1

Vector Spaces

1.1 Linear Dependence and Linear Independence

Exercise 1.5.1

Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vector in S.

Proof. **True** ■

(b) Any set containing the zero vector is linearly dependent.

Proof. **True** ■

(c) The empty set is linearly dependent.

Proof. **False**. It is linearly independent. ■

(d) Subsets of linearly dependent sets are linearly dependent. **True** by Theorem 6.

(e) Subsets of linearly independent sets are linearly independent.

Proof. **True** by corollary to Theorem 6. ■

(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

Proof. **True** this is by definition. ■

Exercise 1.5.4

In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Proof. Choose a finite amount of scalars $a_1, a_2, \ldots, a_n \in F$ to create the following linear combination:

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = (0, 0, \ldots, 0). \tag{1}$$

To show that the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, we need to show that the scalars $a_1, a_2, \ldots, a_n \in F$ have the trivial representation; that is, $a_1 = a_2 = \cdots = a_n = 0$. Since the jth coordinate of e_j is 1 but 0 in all the other entries, we have that

$$\begin{aligned}
&a_1(1, 0, \ldots, 0) + a_2(0, 1, \ldots, 0) + \cdots + a_n(0, 0, \ldots, 1)\\
&= (a_1, 0, \ldots, 0) + (0, a_2, \ldots, 0) + \cdots + (0, 0, \ldots, a_n)\\
&= (a_1, a_2, \ldots, a_n).
\end{aligned}$$

Hence, we have

$$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

Equating each entry of the left side of the equation above to 0, we find that $a_i = 0$ for all $1 \leq j \leq n$. Hence, the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent. ■

Exercise 1.5.4

Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

Proof. Just like the prior exercise, we need to show that we can find scalars $a_0, a_1, \ldots, a_n \in F$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where $a_i = 0$ for all $0 \leq i \leq n$. Note that the 0 polynomial is just

$$0 + 0x + 0x^2 + \cdots + 0x^n = 0.$$

Hence, equating coefficients we immediately get that $a_i = 0$ for all $0 \leq i \leq n$. Thus, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent. ■

Exercise 1.5.6

In $M_{m \times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Proof. First, we create a linear combination of a finite amount vectors in $E = \{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ with scalars δ_k for $1 \leq k \leq N$ with $N = mn$ as the number of total entries in each matrix in $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Note that after doing our scalar multiplication and summing up each term, we find that each $\delta_k E_{ij} = \delta_k$ in our linear combination can equated with a corresponding i and j entry in the zero matrix such that $\delta_k = 0$ for all $1 \leq k \leq N$. Hence, E is a linearly independent set. ■

Exercise 1.5.7

Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

Proof. Define W as the linearly independent spanning set of the set of diagonal matrices in $M_{2\times 2}$ where

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

To see why W is a linearly independent set, choose scalars $\delta_1, \delta_2 \in F$ such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing scalar multiplication and vector addition gives us the following equation

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the zero matrix is a diagonal matrix, we know that equation entries where $i = j$yields $\delta_1 = \delta_2 = 0$. Hence, W is a linearly independent set that generates the set of diagonal matrices of $M_{2\times 2}(F)$. ■

Exercise 1.5.8

Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = \mathbb{R}$, then S is linearly independent.

Proof. ■

(b) Prove that if F has characteristic two, then S is linearly dependent.

Proof. ■

Exercise 1.5.9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other. I have written two proofs for this:

Proof. Let u and v be distinct vectors in a vector space V.

($\Rightarrow$) Since $\{u, v\}$ is a linearly dependent set, we can find scalars $a_1, a_2 \in F$ such that

$$a_1 u + a_2 v = 0 \tag{1}$$

Suppose v is not a multiple of u and choose $a_1 \neq 0$ since $\{u, v\}$ is linearly dependent. We need to show that u is a multiple of v. Solving for u, we get that

$$u = -\frac{a_2}{a_1} v.$$

Hence, u is a multiple of v.

($\Leftarrow$) Suppose u or v is a scalar multiple of the other. Assume u is the scalar multiple of v. Then for some $c \neq 0 \in F$, we have $u = cv$. Hence, we have $u - cv = 1u - cv = 0$. This tells us that $\{u, v\}$ is linearly dependent. ■

Exercise 1.5.12

Prove Theorem 1.6 and its corollary.

Proof. See proof in notes. ■

Exercise 1.5.13

Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Proof. Let u and v be distinct vectors in V.

For the forwards direction, assume $\{u, v\}$ is a linearly independent set. We need to show that $\{u + v, u - v\}$ is linearly independent. Hence, we need to find $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0. \tag{1}$$

Note that (1) leads to

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu. \end{aligned}$$

Since $\{u, v\}$ is a linearly independent set, we have that

$$au - bv = 0$$

and

$$av + bu = 0$$

for $a = b = 0$. Hence,

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ and so $\{u - v, u + v\}$ is a linearly independent set.

For the backwards direction, suppose $\{u+v, u-v\}$ is linearly independent. We need to show that $\{u, v\}$ is linearly independent. Note that $a, b \in F$ such that

$$a(u+v) + b(u-v) = 0$$

for $a = b = 0$ since $\{u-v, u+v\}$ is linearly independent. Note that

$$\begin{aligned} a(u+v) + b(u-v) &= au + av + bu - bv \\ &= au - bv + av + bu \\ &= 0 + av + bu \\ &= 0. \end{aligned}$$

Thus, $av + bu = 0$ where a, b both zero. Thus, the set $\{u, v\}$ is linearly independent. ■

(b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

Proof. For the forwards direction, suppose $\{u, v, w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that

$$a_1u + a_2v + a_3w = 0$$

with $a_1 = a_2 = a_3 = 0$. We need to show that $\{u+v, u+w, v+w\}$ is linearly independent; that is, we need to show that we can find scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u+v) + a_2(u+w) + a_3(v+w) = 0 \tag{1}$$

for $a_1 = a_2 = a_3 = 0$. Observe that (1) can be written in the following way

$$(a_1u + a_3v + a_2w) + (a_1v + a_2u + a_3w) = 0 \tag{2}$$

Since $\{u, v, w\}$ is linearly independent, we know that $a_1 = a_2 = a_3 = 0$. But this also has to mean that $\{u+v, u+w, v+w\}$ is a linearly independent set.

For the backwards direction, suppose $\{u+v, u+w, v+w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u+v) + a_2(u+w) + a_3(v+w) = 0. \tag{1}$$

We need to show that $\{u, v, w\}$ is linearly independent. Observe that (1) can be re-written as

$$\begin{aligned} &(a_1u + a_3v + a_2w) + (a_1v + a_2u + a_3w) = 0 \\ &\Rightarrow 0 + (a_1v + a_2u + a_3w) = 0 \\ &\Rightarrow a_1v + a_2u + a_3w = 0 \end{aligned}$$

where $a_1 = a_2 = a_3 = 0$. Hence, $\{u, v, w\}$ is linearly independent. ■

Exercise 1.5.14

Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

Proof. For the forwards direction, Let S be a linearly dependent. Then we need to show that either $S = \{0\}$ or S contains distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \text{span}(S)$. Suppose there does not exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \text{span}(S)$. This tells us that S only contains the singleton $S = \{v\}$. Furthermore, we must require $v = 0$ since S is linearly dependent. Otherwise, $v \neq 0$ would imply that S is linearly independent. Hence, we have $S = \{0\}$. Now suppose $S \neq \{0\}$. Since S is linearly dependent, there exists scalars $a_1, a_2, \ldots, a_{n+1}$ and vectors $v, u_1, u_2, \ldots, u_n$ such that

$$a_1v + a_2u_1 + a_3u_2 + \cdots + a_{n+1}u_n = 0$$

with all $a_1, a_2, \ldots, a_n, a_{n+1}$ not all equal to zero. Solving for v, we get that

$$v = -(a_1^{-1}a_2)u_1 - (a_1^{-1}a_3)u_2 - \cdots - (a_1^{-1}a_{n+1})u_n.$$

Since v is a linear combination of $u_1, u_2, \ldots, u_n$, we have that $v \in \operatorname{span}(S)$.

Conversely, suppose that either $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$. Assume $S = \{0\}$. Then S is linearly dependent because the singleton is the zero vector. Now suppose v is a linear combination of vectors $u_1, u_2, \ldots, u_n$. Then there exists scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $u_1, u_2, \ldots, u_n \in S$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Subtracting v from both sides of this equation yields the following equation

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - 1v = 0.$$

Since not all scalars in the equation above are zero and $v, u_1, u_2, \ldots, u_n \in S$, we must have that S is a linearly dependent set. ■

Exercise 1.5.15

Prove that a set $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for some k where $1 \leq k < n$.

Proof. Suppose $u_{k+1} \notin \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for all $1 \leq k < n$. Since S is linearly dependent, we know that the zero vector is contained in S. Choose $k = 1$ such that $u_1 = 0$ and we are done. On the other hand, suppose $u_1 \neq 0$. We need to show that $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$. Choose $k = n - 1$. Then clearly $n = k + 1$. Since S is linear independent, choose scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0. \tag{1}$$

where $a_1, a_2, \ldots, a_k, a_{k+1}$ not all zero. Solving for u_{k+1} by subtracting $a_{k+1}u_{k+1}$ on both sides of (1) and multiplying $-a_{k+1}^{-1}$ on both sides of (1), we end up with the following equation:

$$u_{k+1} = -(a_{k+1}^{-1}a_1)u_1 - (a_{k+1}^{-1}a_2)u_2 - \cdots - (a_{k+1}^{-1}a_k)u_k.$$

This tells us that u_{k+1} can be written as a linear combination of vectors $u_1, u_2, \ldots, u_k$. Hence, $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots u_k\})$.

Conversely, either $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(u_1, u_2, \ldots, u_k)$. Suppose $u_1 = 0$. Then S contains the zero vector so S must be linearly dependent. On the other hand, choose $k = n - 1$ where $1 \leq k < n$ such that $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ implies that there exists scalars $a_1, a_2, \ldots, a_k$ such that

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k \\ \Rightarrow u_n &= a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}. \end{aligned} \tag{1}$$

Subtracting u_n on both sides of (1) implies that

$$(a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}) - 1u_n = 0.$$

Since not all scalars in the linear combination above are zero, we know that S must be linearly dependent.

■

Exercise 1.5.16

Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Proof. ($\Rightarrow$) Suppose S is a linearly independent set. Let S' be any finite subset of S. By corollary to Theorem 6, we can see that $S' \subseteq S$ implies that S' is also linearly independent. ($\Leftarrow$) We will proceed by proving the contrapositive. Let $S' \subseteq S$ be a finite subset that is linearly dependent set.

We will prove that S is a linearly dependent set. Since $S' \subseteq S$, we have that S must be a linearly dependent set by Theorem 6. ■

Exercise 1.5.17

Let M be a square upper triangular matrix (as defined in Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

Proof. Let M be a square upper triangular matrix. Note that M have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & a_{n-1n} \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

We can define the following columns as vectors $v_1, v_2, \ldots v_n$ where

$$v_1 = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ 0 \end{pmatrix}, \ldots, v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Choose scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that

$$\delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_n v_n = 0$$

where the zero vector 0 is denoted as

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using scalar multiplication and addition, we arrive at the following system of equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} + \cdots + \delta_{n-1} a_{1n-1} + \delta_n a_{1n} &= 0 \\ \delta_2 a_{22} + \delta_3 a_{23} + \cdots + \delta_n a_{2n} &= 0 \\ &\vdots \\ \delta_{n-1} a_{n-1n-1} + \delta_n a_{n-1n} &= 0 \\ \delta_n a_{nn} &= 0. \end{aligned}$$

Since all the diagonal entries of M are non-zero, we can see from the equation above that $\delta_n = 0$ which subsequently tells us that $\delta_{n-1} = 0$. We claim that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ for all $n \geq 1$. We can prove this via induction. Let our base case be $n = 2$. Then observe that we have an upper triangular 2×2 matrix such that the linear combination of the columns lead to the following system of linear equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} &= 0 \\ 0 + \delta_2 a_{22} &= 0. \end{aligned}$$

Observe that $\delta_2 = 0$ which also implies that $\delta_1 = 0$. Now suppose our claim that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ holds for all $n \geq 1$. We want to show that our claim still holds for the $n+1$ case. Observe that $\delta_{n+1} = 0$ derived from an $n+1 \times n+1$ matrix. Using the same process that proved the base case, we find that $\delta_n = \delta_{n+1} = 0$. By our inductive hypothesis, we know that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ for all $n \geq 1$. This tells us that $\delta_1 = \cdots = \delta_{n+1}$ for all $n \geq 1$. Hence, the columns of M are linearly independent.

■

Exercise 1.5.18

Let S be a set of nonzero polynomials in $P(F)$ such that no two have the same degree. Prove that S is linearly independent.

Proof. TO DO. ■

Exercise 1.5.19

Prove that if $\{A_1, A_3, \ldots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, A_3^t, \ldots, A_k^t\}$ is also linearly independent.

Proof. TO DO. ■

Exercise 1.5.20

Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where f and g are defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$ with $r \neq s$. Suppose for sake of contradiction that the set $\{f, g\}$ is a linearly dependent set of vectors. By Exercise 1.5.9, we know that either f is a multiple of g or g is a multiple of f. Assume f is a multiple of g. Hence, there exists a $c \in R$ such that $f(t) = cg(t)$. In other words,

$$f(t) = cg(t) \Rightarrow e^{rt} = ce^{st}.$$

This equality is valid only when $r = s$ because otherwise f would not be a scalar multiple of g. But note that $r \neq s$ by assumption. Hence, we have a contradiction and thus $\{f, g\}$ must be linearly independent. ■