Linear Algebra Notes

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October 24, 2023

Contents

Chapter 1

Vector Spaces

1.1 Vector Spaces

1.1.1 Basics

Definition 1 (Vector Spaces). A **vector space** (or **linear space**) over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in V there is a unique element ax in V, such that the following conditions hold:

(VS 1) For all $x, y \in V$, $x + y = y + x$ (commutativity of addition).

(VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in V denoted by O such that $x + O = x$ for each $x \in V$

(VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = O$.

(VS 5) For each element $x \in V$, we have $1x = x$.

(VS 6) For each $a, b \in F$ and each element $x \in V$, then $(ab)x = a(bx)$.

(VS 7) For each element $a \in F$ and each pair $x, y \in V$, we have $a(x + y) = ax + ay$.

(VS 8) For each pair $a, b \in F$ and each $x \in V$, we have $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.

- The elements of a field F are called **scalars** and the elements of a vector space V are called **vectors** (these should not be confused!).
- Every vector space will always be defined over a given field, mostly defined over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ unless otherwise noted.
- Every vector space should specify the operations of addition and scalar multiplication.

Definition 2 (n-tuples). An object of the form $(a_1, a_2, \dots, a_n)$, where the entries $a_1, a_2, \dots, a_n$ are elements of a field F, is called an **n-tuple** with entries from F. The elements $a_1, a_2, \dots, a_n$ are called **entries** or **components** of the n-tuple.

Definition 3. We say that two n-tuples, $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$, are **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

Example. The set of all n-tuples with entries from a field F denoted by F_n is a vector space. To see why, suppose $u, v \in F_n$ where $u = (a_1, a_2, \dots a_n)$ and $v = (b_1, b_2, \dots, b_n)$. If we take term-by-term addition of the entries in both u and v, then we end up with

$$u+v=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)$$

and likewise,

$$cu=(ca_1,ca_2,\ldots,ca_n).$$

These same set of operations define $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ and likewise, $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.

- Note that vectors in F^n can be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

  rather than **row vectors** $(a_1,a_2,\ldots,a_n)$.

- 1-tuples are are just scalars or an just an element from F.

Definition 4. An $m\times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry a_{ij} with $(1\le i\le m, 1\le j\le n)$ is an element of F. We call the entries a_{ij} with $i=j$ the **diagonal entries** of the matrix. The entries $a_{i1},a_{i2},\ldots a_{in}$ compose the i**th row**of the matrix, and the entries $a_{1j},a_{2j},\ldots a_{mj}$ compose the jth column of the matrix.

- The rows make a vector space which we denote F^n.
- Likewise, the columns make a vector space we denote F^m

Definition 5 (Zero Matrix)**.** The $m\times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

Definition 6 (Square Matrix)**.** A matrix is **square** if the number of rows and columns of a matrix are equal.

- Just like our tuple example, the set of all $m\times n$ matrices with entries from a field F form a vector space. Denote this vector space as $M_{m\times n}(F)$ endowed with two operations; that is, **matrix addition** and **scalar multiplication**. Suppose for $A,B\in M_{m\times n}(F)$ and $c\in F$, we have

$$(A+B)_{ij}=A_{ij}+B_{ij}$$

  and

$$(cA)_{ij}=cA_{ij}$$

  for $1\le i\le m$ and $1\le j\le n$. In other words, the two operations can be performed entry-wise.

- The operations from our tuple case extends very naturally to $M_{m\times n}(F)$. In other words, if we add two matrices A_{ij} and B_{ij}, then we would expect to that $A_{ij}+B_{ij}\in M_{m\times n}(F)$ as well and likewise for the scalar multiplication case.

Definition 7 (Set of All Functions). Let S be any nonempty set and F be any field, and let $\mathscr{F}(S,F)$ denote the set of all functions from S to F.

Definition 8. Two functions $f, g \in \mathscr{F}(S,F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$.

The set of functions above forms a vector space with the operations of addition and scalar multiplication defined for every $f, g \in \mathscr{F}(S,F)$ and $c \in F$ with

$$(f+g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$

Definition 9 (Set of All Polynomials). A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each $a_k \in F$ is called the **coefficient** of x_k.

Definition 10 (Zero Polynomial). We call $f(x) = 0$ the **zero polynomial** if $a_n = a_{n-1} = \cdots = a_0 = 0$.

Definition 11 (Degree). The **degree** of a given polynomial f is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Definition 12 (Equality of Polynomials). We call two polynomials f, g, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

, **equal** if $m = n$ and $a_i = b_i$ for all $i = 0, 1, \ldots, n$.

Suppose we have $c \in F$ and say we evaluated the polynomial $f \in F$ at c. Then we would have the following

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$$

where $f(c) \in F$.

Definition 13 (Basic Operations of Polynomials). Define polynomial **addition** $f + g$ as the following:

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Let $c \in F$. Let scalar **multiplication** be defined by

$$cf(x) = ca_n x^n + ca_{n-1}x^{n-1} + \cdots + ca_1 x + ca_0.$$

The operations above form a vector space for $P(F)$ (the set of all polynomials).

Definition 14 (Sequences). A **sequence** in F is a function $\sigma : \mathbb{Z}^+ \to F$. A given sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \ldots$ is denoted (a_n).

Let V be the set of all sequences $\sigma(n) \in F$. For every $(a_n), (b_n) \in V$ with $t \in F$, we have the following operations

$$(a_n) + (b_n) = (a_n + b_n) \text{ and } t(a_n) = (ta_n).$$

1.1.2 Non-examples

Example. Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$ where R is a field. For every $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Note that S is **NOT** a vector space since (VS 1), (VS 2), and (VS 8) fail.

Theorem 1 (Cancellation Law for Vector Addition)**.** Given vectors $x, y, z \in V$ such that $x + z = y + z$, then $x = y$.

Proof. Note that we can use (VS 4) to state that there exists

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