Homework 5

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Problem 1. Let $(V, \|\cdot\|)$ be a normed space and Y be a vector subspace of V. Last time, we saw that $V/Y = \{v + Y : v \in V\}$ is also a vector space. Now, assume that Y is closed in $(V, \|\cdot\|)$.

(i) Let v and v' such that $v - v' \in Y$. Show that $\inf_{y\in Y} \|v + Y\| = \inf_{y \in Y} \|v' + y\|$.

Proof. From problem 2(i) of Homework 4, $v - v' \in Y$ implies that $v + Y = v' + Y$. Hence, we have

$$\begin{aligned} v + Y = v' + Y &\implies \|v + y\| = \|v' + y\| \ \forall y \in Y \\ &\implies \inf_{y\in Y} \|v + y\| = \inf_{y\in Y} \|v' + y\|. \end{aligned}$$

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(ii) For $[v] = v + Y \in V/Y$, define

$$\|[v]\|_0 = \inf_{y\in Y} \|v + y\|.$$

Show that $\|\cdot\|_0$ defines a norm on V/W.

Proof. Clearly, we have $\|[v]\|_0 \geq 0$ since $\|\cdot\|$ satisfies property (I).

(I) Suppose $v + Y = 0_{V/W}$ where $[0] = 0_{V/W} = 0_V + Y$. Then by definition of $\|\cdot\|_0$, we have $\|[0]\|_0 = 0$. From part (a), we have

$$\begin{aligned} \|[v]\|_0 = \|[0]\|_0 &\iff \inf_{y\in Y} \|v + y\| = 0 \\ &\iff \|[v]\|_0 = 0. \end{aligned}$$

Hence, the property (I) is satisfied.

(II) Let $\alpha \in F$ where F is a field. Then we have

$$\begin{aligned} \|[\alpha v]\|_0 &= \|\alpha v + Y\|_0 \\ &= \inf_{y\in Y} \|\alpha v + Y\| \\ &= \inf_{y\in Y} \|\alpha(v + Y)\| \\ &= \inf_{y\in Y} |\alpha|\|v + Y\| \\ &= |\alpha| \inf_{y\in Y} \|v + Y\| && (\|\cdot\| \text{ is a norm}) \\ &= |\alpha|\|[v]\|_0. \end{aligned}$$

(III) Let $v_1, v_2 \in V/W$. Then since $\|\cdot\|$ is a norm, we have that

$$\begin{aligned}
\|[v_1+v_2]\|_0 &= \|(v_1+v_2)+Y\|_0 \\
&= \|(v_1+Y)+(v_2+Y)\|_0 \\
&= \inf_{y\in Y} \|(v_1+y_1)+(v_2+y_2)\| \\
&\leq \inf_{y\in Y} \Big[\|v_1+y_1\|+\|v_2+y_2\|\Big] \\
&= \inf_{y\in Y}\|v_1+y_1\| + \inf_{y\in Y}\|v_2+y_2\| \\
&= \|[v_1]\|_0 + \|[v_2]\|_0.
\end{aligned}$$

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(iii) For any $v \in V$, show that $\|[v]\|_0 \leq \|u\|$.

Proof. By the triangle inequality, we have

$$\|v\| = \|v\| + \|0_Y\| \geq \|v+0_Y\| \geq \inf_{y\in Y}\|v+Y\| = \|[v]\|_0.$$

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(iv) We have a Canonical map $\pi : V \to V \setminus Y$, $\pi(u) = [u]$. Show that π is linear and continuous. Here continuity means that if $\|v_n - v\| \to 0$ in V, then $\|[v_n]-[v]\|_0 \to 0$ in V/W.

Proof. First, we show that π is linear. For any $u_1, u_2 \in V$, we have

$$\begin{aligned}
\pi(u_1+u_1) &= [u_1+u_2] \\
&= (u_1+u_2)+Y \\
&= (u_1+Y)+(u_2+Y) \\
&= [u_1]+[u_2] \\
&= \pi(u_1)+\pi(u_2).
\end{aligned}$$

Let $\alpha \in F$ where F is a field and let $u \in V$. Then we have

$$\pi(\alpha u) = [\alpha u] = (\alpha u)+Y = \alpha(u+Y) = \alpha[u] = \alpha\pi(u).$$

Hence, we conclude that π is a linear map from $V \to V/Y$. Now, we want to show that π is, indeed, continuous for any $v \in V$. Let v_n be a sequence in V such that $v_n \to v$; that is, $\|v_n - v\| \to 0$. Our goal is to show that $\|[v_n]-[v]\| \to 0$. By part (b), we can see that

$$0 \leq \|[v_n]-[v]\|_0 \leq \|v_n - v\| \to 0.$$

Hence, we have $\|[v_n] \to [v]\|_0 \to 0$ by Squeeze Theorem and so we conclude that $\pi : V \to V/Y$ is a continuous function on V. ■

Problem 2. Consider the normed space $(\ell^\infty, \|\cdot\|_\infty)$. Define a sequence $e^{(n)}$ in ℓ^∞ by

$$x^{(n)} = (\delta_j^{(n)}), \quad \delta_j^{(n)} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

(i) Compute $\|x^{(n)} - x^{(n')}\|_\infty$ for $n \neq n'$.

Solution. Let $n > m$. Then we see that

$$\delta_j^{(m)} = (0, 0, \ldots, \underbrace{1}_{j=m}, 0, \ldots)$$

and

$$\delta_j^{(n)} = (0, 0, \dots, \underbrace{1}_{j=n}, 0, \dots)$$

Then we have

$$\delta_j^{(n)} - \delta_j^{(m)} = \begin{cases} 1 & \text{if } j = n \\ -1 & \text{if } j = m \\ 0 & \text{if } j \neq n, m \end{cases}$$

Clearly, we can see that $\|x^{(n)} - x^{(m)}\|_\infty = 1$. ■

(ii) Does $(x^{(n)})$ have a convergent subsequence?

Proof. We claim that $x^{(n)}$ does not have a convergent subsequence. Indeed, from part (i), we can see that

$$\|x^{(n)} - x^{(m)}\|_\infty \geq \frac{1}{2} \ \forall n \neq m$$

and

$$\|x^{(n)}\|_\infty = 1$$

for all $n \in \mathbb{N}$. By a theorem proven in class, we can see that $x^{(n)}$ does not have a convergent subsequence. ■

(iii) Prove that $\mathcal{S}(\ell^\infty) = \{x = (x_j) \in \ell^\infty : \|x\|_\infty = 1\}$ is closed and bounded but not compact.

Proof. It follows immediately that $S(\ell^\infty)$ is bounded by construction. We will show that $S(\ell^\infty)$ is closed. Let $x \in \overline{S(\ell^\infty)}$. Then there exists a sequence $x^{(n)}$ in $S(\ell^\infty)$ such that $x^{(n)} \to x$ for some x. This implies that $x^{(n)}$ is a Cauchy sequence in ℓ^∞. Since ℓ^∞ is a Banach space, it follows that $x^{(n)} \to y$ for some $y \in \ell^\infty$. Our goal is to show that $y \in s(\ell^\infty)$; that is, we want to show that $\|y\|_\infty = 1$. Using the triangle inequality , we can see that

$$\begin{aligned} \|y\|_\infty &\leq \|y - x^{(n)}\|_\infty + \|x^{(n)}\|_\infty \\ &= \|y - x^{(n)}\|_\infty + 1. \end{aligned}$$

Hence, we have

$$\|y\|_\infty - 1 \leq \|y - x^{(n)}\|_\infty. \tag{1}$$

Similarly, we have

$$\|x^{(n)}\|_\infty \leq \|x^{(n)} - y\|_\infty + \|y\|_\infty$$

and so,

$$1 - \|y\|_\infty \leq \|x^{(n)} - y\|_\infty. \tag{2}$$

Now, (1) and (2) imply that

$$0 \leq |\|y\|_\infty - 1| \leq \|x^{(n)} - y\|_\infty \to 0.$$

Since $|\cdot|$ and $\|\cdot\|_\infty$ are continuous functions, we have

$$\lim_{n\to\infty} |\|y\|_\infty - 1| = \left| \lim_{n\to\infty} (\|y\|_\infty - 1) \right| = 0$$

and so we conclude that $\|y\|_\infty = 1$ which proves that $S(\ell^\infty)$ closed. But note that by part (ii), $x^{(n)}$ does not have convergent subsequence. By Sequential Compactness, it follows that $S(\ell^\infty)$ is not a compact set. ■

Problem 3. Let $(V, \|\cdot\|)$ be a normed space and Y be a subspace of V such that $Y \neq V$. Let $v \in V \setminus Y$. Define $d(v, Y) = \inf_{y \in Y} \|v - y\|$.

(i) Show that if $d(v, Y) = 0$, then $v \in \overline{Y}$.

Proof. Suppose that $d(v, Y) = 0$. Our goal is to show that $v \in \overline{Y}$; that is, we want to show that for any $\varepsilon > 0$, $B(y, \varepsilon) \cap Y \neq \emptyset$. Let $\varepsilon > 0$ be given. By a characterization of the infimum, we know there exists $\hat{y} \in Y$ such that

$$\|v - \hat{y}\| < \inf_{y \in Y} \|v - y\| + \varepsilon.$$

By assumption, $d(v, Y) = \inf_{y \in Y} \|v - Y\| = 0$ and so we have, from the above inequality that

$$\|v - \hat{y}\| < \varepsilon.$$

Hence, $\hat{y} \in B(v, \varepsilon)$. Since $\hat{y} \in Y$, we can conclude that

$$B(v, \varepsilon) \cap Y \neq \emptyset$$

and so $\hat{y} \in \overline{Y}$. ■

(ii) Assume that Y is closed. Prove that $d > 0$.

Proof. Suppose that Y is closed. Our goal is to show that $d(v, Y) > 0$ for all $v \in V \setminus Y$. To this end, let $v \in V \setminus Y$. Suppose for sake of contradiction that $d(v, Y) \leq 0$. If $d(v, Y) < 0$, then we have $\|v - y\| < 0$ which is absurd. If $f(v, Y) = 0$, then from part (a) we have that $v \in \overline{Y}$. But Y is closed and so $Y = \overline{Y}$. This tell us that $v \in Y$ which contradicts our assumption that $v \in V \setminus Y$. ■

Problem 4. Read section 2.6 of Kryszig and write down statements of key theorems, lemmas, and propositions.

Theorem (Range and Null Space)**.** Let T be a linear operator. Then:

(a) The range $R(T)$ is a vector space.

(b) If $\dim(T) = n < \infty$, then $\dim(R(T)) \leq n$.

(c) The null space $N(T)$ is a vector space.

Theorem (Inverse Operator)**.** Let X, Y be vector spaces, both real or both complex. Let $T : D(T) \to Y$ be a linear operator with domain $D(T) \subseteq X$ and range $R(T) \subseteq Y$. Then:

(a) The inverse $T^{-1} : R(T) \to D(T)$ exists if and only if

$$T(x) = 0 \Longrightarrow x = 0.$$

(b) If T^{-1} exists, it is a linear operator.

(c) If $\dim(D(T)) = n < \infty$ and T^{-1} exists, then $R(T) = \dim(D(T))$.

Lemma (Inverse of Product)**.** Let $T : X \to Y$ and $S : Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the inverse $(ST)^{-1} : Z \to X$ of the product (the comoposite) ST exists, and

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Problem 5 (i)**.** Let $T : D(T) \to W$ be a linear operator. Assume that $T^{-1} : R(T) \to D(T)$ exists. Show that if $\{v_1, \ldots, v_n\}$ is linearly dependent on $D(T)$, then $\{T(v_1), \ldots, T(v_n)\}$ is linearly dependent on W.

Proof. We will show the claim through contrapositive. Suppose $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent; that is, the equation

$$\sum_{i=1}^{n} c_i T(v_i) = 0 \tag{*}$$

has the trivial solution $c_i = 0$ for all $1 \leq i \leq n$. Since T is linear, (*) implies that

$$T\Big(\sum_{i=1}^{n} c_i v_i\Big) = 0.$$

Note that T^{-1} exists and so T must be injective (and surjective). Hence, we have $N(T) = \{0\}$. Thus, we have

$$\sum_{i=1}^{n} c_i v_i \in N(T) \Longrightarrow \sum_{i=1}^{n} c_i v_i = 0.$$

But then $c_i = 0$ for all $1 \leq i \leq n$. Hence, we see that $\{v_1, \ldots, v_n\}$ is a linearly independent set on V. ■

Problem 6 (ii). Let V and W be two vector spaces and $T : V \to W$ be a linear operator. Assume that $T : V \to W$ be a linear operator. Assume that V and W are finite dimensional and $\dim(V) = \dim(W)$. Prove that $R(T) = W$ if and only if T^{-1} exists.

Proof. Assume that V and W are finite dimensional and $\dim(V)$ and $\dim(W)$.

($\Longrightarrow$) Suppose $R(T) = W$. Let $\dim(V) = \dim(W) = n$. Our goal is to show that T^{-1} exists. It suffices to show that T is both surjective and injective. Note that, by assumption, T is immediately surjective. So, it suffices to show that T is injective. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\omega = \{w_1, w_2, \ldots, w_n\}$. Since T is surjective, we get $T(v_i) = w_i$ for $1 \leq i \leq n$. Let $x, y \in V$. Then since β is a basis, we have

$$\begin{aligned} x &= \sum_{i=1}^{n} c_i v_i, \\ y &= \sum_{i=1}^{n} b_i v_i \end{aligned}$$

Suppose $T(x) = T(y)$. Then by the linearity and surjective of T, we see that

$$\begin{aligned} T(x) = T(y) &\Longrightarrow T\Big(\sum_{i=1}^{n} c_i v_i\Big) = T\Big(\sum_{i=1}^{n} b_i v_i\Big) \\ &\Longrightarrow \sum_{i=1}^{n} c_i T(v_i) = \sum_{i=1}^{n} b_i T(v_i) \\ &\Longrightarrow \sum_{i=1}^{n} (c_i - b_i) T(v_i) = 0 \\ &\Longrightarrow \sum_{i=1}^{n} (c_i - b_i) w_i = 0. \end{aligned}$$

Since ω is basis for W, w_i for all $1 \leq i \leq n$ are linearly independent. Hence, $c_i - b_i = 0$ for all $1 \leq i \leq n$. Hence, $c_i = b_i$ for all $1 \leq i \leq n$. This tells us that $x = y$. Thus, T must be injective. Thus, T^{-1} must exists.

($\Longleftarrow$) If T^{-1} exists, then T must be a bijective map between V and W. Hence, we immediately have that $R(T) = W$.

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