Math 234A Lecture Notes

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Contents

Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial x^2+1 does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\, dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system $\mathbb{R}$.

(i) $(\mathbb{R}, +, \cdot)$ is a field.

(ii) We have an order relation on $\mathbb{R}$.

(iii) For all $x \in \mathbb{R}$, $x^2+1>0$. Hence, the polynomial equation $x^2+1=0$ does not have any solutions in $\mathbb{R}$.

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2+1=0$. Then we define the following set
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \operatorname{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that $\mathbb{C}$ forms a field.

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane $\mathbb{C}$ as a working model for complex numbers.
- Discuss the Euclidean Topology on $\mathbb{C}$.
- Discuss polar representation of a complex number.

1.3 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

Lemma. Let F be a field containing $\mathbb{R}$ and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then $\mathbb{C}$ is a **subfield** of F.

(ii) Let F be another field containing $\mathbb{R}$ and containing a solution i' of $x^2 + 1 = 0$ and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then $\mathbb{C}$ and $\mathbb{C}'$ are isomorphic as fields.

The second part of this lemma is simply saying that $\mathbb{C}$ is a unique subfield of F.

1.3.1 A model for $\mathbb{C}$

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication $\cdot$ as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers $\mathbb{C}$ have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \to \mathbb{C}$ by $x \to (x, 0)$. Then $\mathcal{J}$ is a subfield and forms an isomorphism between $\mathbb{R}$ and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

(i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.

(ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homorphism.

(iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that $\mathcal{J}$ is a bijective map.

■

Definition (Constructing a solution for $x^2 + 1 = 0$)**.** Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number)**.** Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate)**.** Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\overline{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the $x-$axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(iii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iv) $\Re(z) = \frac{1}{2}(z + \overline{z})$.

(v) $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $z\overline{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number)**.** Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

Proposition (More Properties of Complex Numbers)**.** Given $z, w \in \mathbb{C}$, we have the following properties:

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|zw| = |z||w|$.

(iii) $|\Re(z)| \le |z|$.

(iv) $|\Im(z)| \le |z|$.

(v) $|z + w| \le |z| + |w|$.

(vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

1.3.2 Viewing $\mathbb{C}$ as a Metric Space

In $\mathbb{C}$, the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space)**.** For $z, w \in \mathbb{C}$, we call $\mathbb{C}$ a metric space if it satisfies the following properties:

(i) $d(z, w) = d(w, z)$.

(ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have $\mathbb{C}$ is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of Complex Numbers
- Convergence of Sequences in $\mathbb{C}$

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^{\bullet} = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_{+} = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos\varphi, \sin\varphi)$$

with $\tan\varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos\psi, \sin\psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_{+} \times \mathbb{R} \to \mathbb{C}^{\cdot}$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^{\cdot}$, then

$$z = \gamma(\cos\varphi + i\sin\varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument)**.** Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\operatorname{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\ &= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')). \end{aligned}$$

■

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^{\cdot}$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi). \end{aligned}$$

Corollary (De Moivre's Theorem)**.** Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example (An example of De Moivre's Theorem)**.** Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i\sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i \sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\
&\leq |w_n||z_n - z| + |z||w_n - w| \\
&\leq M|z_n - z| + |z||w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we conclude that

$$z_n w_n \to zw.$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon|z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned}
|z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\
&< \frac{\varepsilon|z|^2}{2} \cdot \frac{2}{|z|^2} \\
&= \varepsilon.
\end{aligned}$$

Thus, we conclude that $z_n^{-1} \to z^{-1}$.

(iv) ($\Longrightarrow$) Let $\varepsilon > 0$. Since $(z_n) \to z$, we can choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|\Re(z_n) - \Re(z)| = |\Re(z_n - z)| \leq |z_n - z| < \varepsilon \\
|\Im(z_n) - \Im(z)| = |\Im(z_n - z)| \leq |z_n - z| < \varepsilon.
\end{aligned}$$

Hence, the real and imaginary part of (z_n) converge.

($\Longleftarrow$) Letting $\varepsilon > 1$ again. Our goal is to find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n - z| < \varepsilon.$$

Since the real and imaginary part of z_n converge, we know that there exists $N_1, N_2 \in \mathbb{N}$ such that, we have

$$|\Re(z_n) - \Re(z)| < \frac{\varepsilon}{2} \tag{1}$$
$$|\Im(z_n) - \Im(z)| < \frac{\varepsilon}{2} \tag{2}$$

whenever $n \geq N_1$ and $n \geq N_2$, respectively. Now, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned} |z_n - z| &= |(\Re(z_n) - \Re(z)) + i(\Im(z_n) - \Im(z))| \\ &\leq |\Re(z_n) - \Re(z)| + |\Im(z_n) - \Im(z)| && (|i| = 1) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we see that $(z_n) \to z$.

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Chapter 3

Week 3

3.1 Lecture 4

3.1.1 Topics

(i) Convergence of Series

(ii) Exponential, Sine, and Cosine functions

3.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

Definition (Convergence of Series)**.** If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i}\Big[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\Big]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$
$$\frac{1}{2}is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider

$$\frac{1}{2}i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \implies s_n = \frac{1}{i}\left[1 - \frac{1}{2^n}\right].$$

Thus, taking the limit as $n \to \infty$ gives us

$$\lim_{n\to\infty} s_n = \frac{1}{i}.$$

Example. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since $|z| < 1$, we know that

$$\sum_{n-1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n\to\infty} \sum_{k=1}^{n} z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as $n \to \infty$, we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series)**.** Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in $\mathbb{C}$.

3.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function)**.** For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series)**.** We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define
$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z + w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \textbf{ Show this!}.$$

Chapter 4

Week 4

4.1 Lecture 4

4.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

4.1.2 Class Exercises

(i) Show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z.

(ii) Show that $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$ converges for all z.

(iii) Show that $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all z.

Definition (Exponential, Cosine, and Sine)**.** We define

$$
\begin{aligned}
\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\end{aligned}
$$

Our main goal for this lecture is to show that $\exp(z+w) = \exp(z)\exp(w)$.

4.1.3 Cauchy Multiplication Theorem

Theorem (Cauchy Multiplication Theorem)**.** Assume that $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converges absolutely. Then

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} z_k w_{n-k}\Big) = \Big(\sum_{n=0}^{\infty} z_n\Big)\Big(\sum_{n=0}^{\infty} w_n\Big) \tag{1}$$

where the series on the left-hand side of (1) converges absolutely.

Proof. We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is bounded. Observe that

$$\begin{aligned}\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big| &\le \sum_{n=0}^{N}\sum_{i+j=n}|z_i w_j| \\ &= \sum_{0\le i+j\le N}|z_i w_j| \\ &\le \sum_{0\le i,j\le N}|z_i w_j| \\ &= \sum_{i=0}^{N}|z_i|\sum_{j=0}^{N}|w_j|.\end{aligned}$$

By our assumption, we can see that the sequence of partial sums of $\sum_{n=0}^{\infty}$ and $\sum_{n=0}^{\infty} w_n$ are bounded, and thus the left side of the equation above is bounded. Thus, we see that

$$\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \Big|\sum_{n=0}^{2N}\sum_{i+j=n} z_i w_j - \sum_{i=0}^{N} z_i \sum_{j=0} w_j\Big| \to 0$$

as $N\to\infty$. Let us define the following sets

$$\begin{aligned}T_N &= \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : i\ge 0, j\ge 0, 0\le i+j\le N\}\\ t_n &= \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : 0\le i\le N, 0\le j\le N\}.\end{aligned}$$

Observe that $T_N\subseteq t_n$ and $t_N\subseteq T_{2N}\subseteq t_{2N}$. Thus,

$$\begin{aligned}\alpha_N &= \Big|\sum_{(i,j)\in T_{2N}\setminus t_N} z_i w_j\Big| \le \sum_{(i,j)\in T_{2N}\setminus t_N}|z_i w_j| \\ &\le \sum_{(i,j)\in t_{2N}\setminus t_N}|z_i w_j| \\ &= \sum_{i=0}^{2N}\sum_{j=0}^{2N}|z_i w_j| - \sum_{i=0}^{N}\sum_{j=0}^{N}|z_i w_j| \\ &= \sum_{i=0}^{2N}|z_i|\sum_{j=0}^{2N}|w_j| - \sum_{i=0}^{N}|z_i|\sum_{j=0}^{N}|w_j| \to 0 \text{ as } N\to\infty.\end{aligned}$$

■

Corollary. For any $z, w\in\mathbb{C}$, we have $\exp(z)\cdot\exp(w) = \exp(z+w)$.

Proof. Let

$$C_n = \sum_{k=0}^{n}\frac{z^k}{k!}\cdot\frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by $n!$ on both sides of the equation above, we see that

$$n!C_n = \sum_{k=0}^{n}\binom{n}{k} z^k w^{n-k} = (z+w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^{N} C_n = \sum_{n=0}^{N} \frac{n! C_n}{n!} = \sum_{n=0}^{N} \frac{(z+w)^n}{n!}.$$

Let $n \to \infty$. Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w).$$

■

4.2 Lecture 5

4.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

4.2.2 Sine and Cosine Functions

Recall that we defined

$$\begin{aligned} \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C} \\ \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}. \end{aligned}$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2.$

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z + i \sin z. \end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x+iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.
- $\Im(\exp(z)) = e^x \sin y, \ |\exp(z)| = e^x$.
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x+iy$ and $w = u+iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x+iy$. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ■

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\bullet}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\bullet} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Problem 1. Let $z \in \mathbb{C}^{\bullet}$. Find all possible solutions of $\exp(w) = z$.

Solution. Taking the log of both sides of $\exp(w) = z$, we see that

$$w = \log(z) = \ln|z| + i\Big(\operatorname{Arg}(z) + 2\pi k\Big),\ k \in \mathbb{Z}.$$

■

Lemma. For $z \in \mathbb{C}^\bullet$, we have

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

4.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \operatorname{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \operatorname{Log} z + 2\pi ik$$

is a **multivalued function** and $\log z = \operatorname{Log} z$ if we want the solution to be in S.

Chapter 5

Week 5

5.1 Lecture 6

5.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on $\mathbb{C}$:

$$d(z, w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ using the map $\mathbb{C} \to \mathbb{R}^2$ with $\alpha + i\beta \to (\alpha, \beta)$. Thus, we can visualize $\mathbb{C}$ using $\mathbb{R}^2$.

Recall that definition of continuity for functions in $\mathbb{R}$.

Definition (continuity in $\mathbb{R}$). Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **continuous** for each $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say $|x - a| < \delta$, we can also say that $x \in (a - \delta, a + \delta)$. Similarly, we have $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

5.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x, a) < \delta\}$$

which is a more general statement than the one found in the definition above.

Definition (Open Ball). Consider the usual metric $d(z, w) = |z - w|$ in $\mathbb{C}$. Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a, and radius $\varepsilon > 0$, we mean the set

$$B(a, \varepsilon) = \{z \in \mathbb{C} : d(z, a) < \varepsilon\}.$$

Definition (Open Set)**.** Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Example. (i) $\mathbb{C}$ is open.

(ii) Let $\varepsilon > 0$ and $a \in \mathbb{C}$. Then $B(a, \varepsilon)$ is also open; that is, open balls are open.

(iii) Let D_1 and D_2 be open sets. Show that $D_1 \cap D_2$ is open. Let $x \in D_1 \cap D_2$ To show that $D_1 \cap D_2$ is open, we need to find $\delta > 0$ such that $B(x, \delta) \subseteq D_1 \cap D_2$. Since $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$. Since D_1 and D_2 are open sets, we can find $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$B(x, \varepsilon) \subseteq D_1 \text{ and } B(x, \varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that $d(x,p) < \varepsilon$ for all $p \in D_1$ and $d(x,q) < \varepsilon'$. Then pick $\delta = \min\{d(x,p), d(x,q)\}$. Since x is in both D_1 and D_2, we have that

$$B(x, \delta) \subseteq D_1 \cap D_2.$$

Hence, $D_1 \cap D_2$ is open.

1. The empty set is open.
2. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a collection of open ets, where Λ is an indexing set. Define the set

$$\bigcup_{\lambda \in \Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that $\bigcup_{\lambda \in \Lambda} D_\lambda$ is also open.

5.1.3 Interior Points

Definition (Interior Point)**.** Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Proposition. D is open if and only if each $a \in D$ is an interior point of D.

Example. Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton $\{0\}$. Show that $\{0\}$ is not open. Note that the intersection of arbitrary many open sets may not be open.

5.1.4 Closed sets

Definition (Closed Set)**.** We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

Example. Define a closed ball with center at $a \in \mathbb{C}$ and radius $\varepsilon > 0$ by

$$\overline{B}(a, \varepsilon) = \{z \in \mathbb{C} : |z - a| \leq \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that $\overline{B}(a, \varepsilon)$ is closed.

Example.
- For any $a \in \mathbb{C}$, we say that the singleton $\{a\}$ is closed.
- If $A_1, A_2, \dots, A_n$ are closed sets. Then
$$\bigcup_{i=1}^{n} A_i \text{ is also closed.}$$
We can just show that the complement of this set is open.
- Let $\{A_\Lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Define
$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\Lambda \ \forall \lambda \in \Lambda\},$$
then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.

Definition (Boundary Points)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a, \varepsilon) \cap D \neq \emptyset$ and $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a, \varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.

Example. Any point on the circle $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$ is a boundary point of $B(a, \varepsilon)$, where $\varepsilon > 0$.

Example (Boundary Points of Singletons)**.**
- Note that the boundary point of the singleton is just the singleton itself.
- Let $D = \mathbb{C} \setminus \{a\}$. Just like the first item, the boundary points of this set is just $\{a\}$.

Definition (Accumulation Point)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,
$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

Example.
(i) Let $D = \mathbb{C}$. Then $\{0\}$ is an accumulation point of D.
(ii) Let a such that $|a| = 1$. Then a is an accumulation point of $D = \{z : |z| < 1\}$.
(iii) Let $D = \{1, i\}$. Then D has no accumulation points.
(iv) If $D = \{w\}$ for some nonzero $w \in \mathbb{C}$. Thus, $D' = \emptyset$. That is, every singleton set contains no accumulation points. Furthermore, any finite set does not have any accumulation points.
(v) Let $D = \mathbb{R}$. Then the accumulation points of D is just $\mathbb{R}$ itself. That is, any point of $\mathbb{R}$ is an accumulation point.
(vi) Let $D = B(0, 1)$. The accumulation points are contained in the following set:
$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Another way to define the set of accumulation points of D is the following:
$$\{a \in \mathbb{C} : \forall \varepsilon > 0, B(a, \varepsilon) \cap (D \setminus \{a\})\}.$$

Definition (Closure)**.** Let $D \subseteq \mathbb{C}$. We denote the **closure of** D by $\mathbf{D}$ and define $\overline{D}$ as
$$\overline{D} = D \cup \{x \in D : \forall \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

The closure is the smallest closed set that contains D!

Lemma. (i) $\overline{D}$ is closed in $\mathbb{C}$.

(ii) The following statements are equivalent:

(a) $a \in \mathbb{C}$ is an accumulation point of D

(b) There is a sequence (z_n) with $z_n \in D$, $z_n \neq a$ for all $n \in \mathbb{Z}_+$ and $z_n \to a$.

Proof. Proof is in the homework! ■

Definition (Bounded)**.** We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$; that is, $D \subseteq \overline{B}(0, R)$.

Chapter 6

Week 6

6.1 Lecture 7

6.1.1 Topics

Our goal for this lecture is to discuss complex powers of a non-zero complex number.

6.1.2 Complex Powers

Let $z \in \mathbb{C}^\bullet$. We want to define z^w for $w \in \mathbb{C}$. Recall from real number system that, for $x \in \mathbb{R}_+$, $x^a = e^{a \ln x}$. We want our complex powers to extend this idea.

Definition. We define $z^w = \exp(w \log z)$. Note that

$$\begin{aligned} \log z &= \operatorname{Log} z + 2\pi i k \\ &= \ln|z| + i(\operatorname{Arg}(z) + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$

is a multi-valued function.

For this reason, the complex power of a non-zero complex number will also be multi-valued. Hence, by choosing a fixed k, we can find a specific value of z^w.

Example. (i) Compute i^i. Recall that $\log i = i\left(\frac{\pi}{2} + 2\pi k\right)$. Hence, we have

$$i^i = \exp(i \log i) = \{e^{\frac{-\pi}{2} + 2\pi k} : k \in \mathbb{Z}\}.$$

If we are just referring to the principal branch, then

$$i^i = e^{\frac{-\pi}{2}}.$$

(ii) Compute: $(-1)^{1+i}$. Note that

$$\log(-1) = i\pi + 2\pi i k = i(2k+1)\pi.$$

Then we see that for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} (-1)^{1+i} &= \exp((2k+1)\pi(i-1)) \\ &= e^{-(2k+1)\pi} \exp(e^{(2k+1)\pi i}) \\ &= e^{-(2k+1)\pi} \Big[\cos(2k+1)\pi + i \sin(2k+1)\pi \Big] \\ &= -e^{-(2k+1)\pi}. \end{aligned}$$

Let us summarize some facts about $\exp(z)$.

- For all $x \in \mathbb{R}$, we have $\exp(z) = e^x$. Thus, from now on, we write e^z in place of $\exp(z)$.
- For $z \in \mathbb{C}^\bullet$, we have $\log z$ is a multi-valued function. That is, we have

$$\log z = \ln|z| + i(\operatorname{Arg} z + 2\pi k), \;\; k \in \mathbb{Z}.$$

- For $z \in \mathbb{C}^\bullet$, we have $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ (this is the principal branch log).
- For $z \in \mathbb{C}^\bullet$, we have $z^w = e^{w \log z}$ is also a multi-valued function.

Example.
- Find all solutions of $\sin z = 1$.
- Find all possible solutions of $\cos z = i$.

Problem 2. Find all solutions of $e^z = i$.

Solution. ■

6.2 Lecture 8

- Limits of a function
- Continuity
- Derivative

6.2.1 Functional Limit

Definition (Functional Limit)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. Let a be an accumulation point of D. Let $\ell \in \mathbb{C}$. We say that ℓ is **the limit of** $f(z)$ as z approaches to a if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $z \in D$ with $0 < |z - a| < \delta$, we have

$$|f(z) - \ell| < \varepsilon;$$

that is, $z \in D \cap (B(a, \delta) \setminus \{a\})$ implies $f(z) \in B(\ell, \varepsilon)$.

Example. Let $f : B(0, 1) \to \mathbb{C}$ defined by $f(z) = \frac{iz}{2}$ then $\lim\limits_{z \to i} f(z) = -\frac{1}{2}$.

Proof. Observe that

$$|f(z) - \ell| = \left|\frac{i}{2}z - \left(-\frac{1}{2}\right)\right| = \left|\frac{i}{2}\right||z - i| = \frac{1}{2}|z - i|.$$

So, given any $\varepsilon > 0$, we can take any $0 < \delta < 2\varepsilon$ such that

$$0 < |z - i| < \delta$$

implies

$$|f(z) - \ell| < \varepsilon.$$

■

Some facts about limits are:

Theorem. Let $D \subseteq \mathbb{C}$ with $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ be two functions let us write $f = u + iv$.

(i) Let $a = \alpha + i\beta$. Then $\lim\limits_{z \to a} f(x) = \ell$ if and only if

$$\lim_{(x,y)\to(\alpha,\beta)} u(x,y) = \Re(\ell) \text{ and } \lim_{(x,y)\to(\alpha,\beta)} v(x,y) = \Im(\ell); \text{ that is,}$$

$$\lim_{z\to a} f(z) = \lim_{z\to a} u(z) + i \lim_{z\to a} v(z).$$

(ii) Let $\lim_{z\to a} f(z) = \ell$ and $\lim_{z\to a} g(z) = \ell'$ and $c, d \in \mathbb{C}$. Then

$$\lim_{z\to a}(cf(z) + dg(z)) = c\ell + d\ell'.$$

(iii) $\lim_{z\to a}(f(z)g(z)) = \lim_{z\to a} f(z) \cdot \lim_{z\to a} g(z)$.

(iv) We have

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$$

whenever $\lim_{z\to a} g(z) \neq 0$.

Problem 3. Let $\lim_{z\to i} f(z) = 2 + i$ and $\lim_{z\to i} g(z) = 1 - i$. Compute the following:

$$\lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2}\right].$$

Solution. Observe that

$$\begin{aligned} \lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2}\right] &= \lim_{z\to i}(f(z))^3 + \lim_{z\to i} \frac{(1+i)g(z)}{z^2} \\ &= (2+i)^3 + \frac{(1+i)(1-i)}{i^2} \\ &= (2+i)^3 - 2. \end{aligned}$$

■

Definition (Limits at Infinity). (i) Let $f : D \to \mathbb{C}$ be a function and $a \in \mathbb{C}$ be an accumulation point of D. Then we say $\lim_{z\to a} f(z) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that

$$z \in D \cap (B(a,\delta) \setminus \{a\})$$

implies $|f(z)| \geq M$; that is, f is unbounded as z approaches to a.

(ii) Let f be a complex function defined on the complement of a ball in $\mathbb{C}$. We say $\lim_{z\to\infty} f(z) = \ell$ if for all $\varepsilon > 0$, there exists $R > 0$ such that $|z| > R$ implies

$$|f(z) - \ell| < \varepsilon.$$

Example. (i) Consider $\lim_{z\to 0} \frac{1}{z} = \infty$. Let $M > 0$. Let $\delta = \frac{1}{M}$. Then $0 < |z| < \delta$ implies that

$$\left|\frac{1}{z}\right| > \frac{1}{\delta} = M.$$

(ii) Consider $\lim_{z\to\infty} \frac{1}{z} = 0$. Let $\varepsilon > 0$. Choose $R = \frac{1}{\varepsilon} > 0$. Then $|z| > R$ implies

$$\left|\frac{1}{z}\right| < \frac{1}{R} = \varepsilon.$$

Problem 4. (i) Show that $\lim_{z\to\infty} f(z) = \ell$ if and only if

$$\lim_{z\to\infty} f\left(\frac{1}{z}\right) = \ell.$$

(ii) $\lim_{z\to a} f(z) = \infty$ if and only if $\lim_{z\to a} \frac{1}{f(z)} = 0$.

(iii) Give a definition of $\lim_{z\to\infty} f(z) = \infty$. Show that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \infty$.

(iv) Compute $\lim_{z\to\infty} \frac{z-1}{z+i}$.

(v) Compute $\lim_{z\to\infty} \frac{gz+i}{z^2+z+1}$.

(vi) Can you compute $\lim_{z\to\infty} e^{-z}$?

6.2.2 Continuity

Definition (Continuity)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. We say f **is continuous at** $a \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(a,\delta) \cap D$, we have $f(z) \in B(f(a), \varepsilon)$.

Example. Compute $\lim_{z\to\infty} \frac{z-i}{z+i}$. Observe that

$$\lim_{w\to 0} \frac{\frac{1}{w} - i}{\frac{1}{w} + i} = \lim_{w\to 0} \frac{1-wi}{1+wi} = 1.$$

Lemma. Let $f : D \to \mathbb{C}$ be a function. Then the following statements are equivalent.

(i) If f is continuous at $a \in D$.

(ii) For any sequence (a_n) and $a_n \in D$ such that $a_n \to a$, we have

$$f(a_n) \to f(a).$$

We say that $f : D \to \mathbb{C}$ is continuous if f is continuous if f is continuous for every $a \in D$.

Some facts that we can prove as an exercise:

Lemma. Let $f : D \to \mathbb{C}$ with $D \subseteq \mathbb{C}$. Then we have

(1) f is continuous if and only if $\Re(f)$ and $\Im(f)$ are continuous (Note that $\Re(f)$ and $\Im(f)$ are functions of two variables in $\mathbb{R}$; that is, $\Re(f) : D \to \mathbb{R}$ an $\Im(f) : D \to \mathbb{R}$).

(2) Linear combinations and products of continuous functions are continuous.

(3) Composition of functions are continuous.

(4) If f is continuous on D and $f(z) \neq 0$ for all $z \in D$, then $\frac{1}{f}$ is continuous on D.

(5) If f is continuous, then $|f|$ is also continuous (note that $|f| = \sqrt{(\Re(f))^2 + (\Im(f))^2}$).

Proof. Left as an exercise. ■

Example. (1) $\exp : \mathbb{C} \to \mathbb{C}^\bullet$ is continuous.

(2) $\sin, \cos, \sinh, \cosh$ are all continuous.

(3) $\mathrm{Arg} : \mathbb{C}^\bullet \to (-\pi, \pi]$ is continuous on $\mathbb{C}_- = \mathbb{C}^\bullet \setminus \{\text{negative real axis}\}$ **(this will be on the**

next homework).

(4) Log : $\mathbb{C}^\bullet \to \{x + iy : -\pi < y \leq \pi\}$ is continuous on $\mathbb{C}_-$ (**this will be on the next homework**)

(5) Arg and Log are both not continuous at any $z < 0$ (that is, points on negative axis).

(6) Fix $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Then

$$f(z) = a_0 + a_1 z + \cdots + a_n z_n$$

is continuous on $\mathbb{C}$; that is, any polynomial with complex coefficients will be continuous.

Note that $\text{Arg} : \mathbb{C}^\bullet \to (-\pi, \pi] \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a discontinuous.

6.2.3 Complex Derivative

Lemma. Let $f : D \to \mathbb{C}$ be a function and a is an accumulation point of D. Then the following statements are equivalent.

(1) $\lim\limits_{z \to a} f(z) = \ell$

(2) Define $\tilde{f} : D \cup \{a\} \to \mathbb{C}$ by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ \ell & \text{if } z = a. \end{cases}$$

Then $\tilde{f}$ is continuous at a.

The reason we define (2) the way it is is because we may have a removable discontinuity.

We will us the lemma above to define complex differentiability.

Definition (Differentiability). Let $f : D \to \mathbb{C}$ be a function and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. We say that f is differentiable at a if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists.}$$

If f is differentiable at a, we denote this limit by $f'(a)$. We say f is differentiable on D if f is differentiable at each $a \in D$. In this case, we can define a new function $f' : D \to \mathbb{C}$ where $z \mapsto f'(z)$.