# Math 230A: Homework 1

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September 4, 2024

1. Mark each statement True or False.

   1-1) If $x$ and $y$ are elements of an ordered field, then either $x \leq y$ or $y < x$. **True.**

   1-2) Every ordered field has the least upper bound property. **False.**

   1-3) If $E \subseteq \mathbb{R}$ is bounded above and $\alpha = \sup E$, then $\alpha \in E$. **False.**

   1-4) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, then $\alpha = \sup E$. **False.**

   1-5) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \in E$, then $\alpha = \sup E$. **True.**

   1-6) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \notin E$, then $\alpha \neq \sup(E)$. **False.**

2. Prove the following: Suppose $\alpha$ is an upper bound for $E \subset \mathbb{R}$. Then $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   **Proof.** Suppose $\alpha$ is an upper bound for $E \subseteq \mathbb{R}$. For the forwards direction, suppose $\alpha = \sup E$ and let $\varepsilon > 0$. Note that $\alpha - \varepsilon < \alpha$ implies that $\alpha - \varepsilon$ is NOT an upper bound of $E$. By definition, there must exist an element $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$. Thus, $\alpha < x_0 + \varepsilon$.

   For the backwards direction, let $\varepsilon > 0$ and let $\alpha$ be an upper bound of $E$. We need to show $\alpha = \sup E$; that is, we need to show that $\alpha$ is an upper bound of $E$ and that any $\gamma < \alpha$ implies $\gamma$ is NOT an upper bound of $E$. Notice that the first property is satisfied by our assumption. Thus, all that is left to show is the second property.

   Suppose $\gamma < \alpha$. Thus, we have $\alpha - \gamma > 0$, so pick $\varepsilon = \alpha - \gamma$. Our goal is to find some element $x$ in $E$ such that $x > \alpha - \varepsilon$. Choose $\varepsilon = \alpha - \gamma$. By assumption, there exists an element $x_0 \in E$ such that

   $$x_0 > \alpha - \varepsilon = \alpha - (\alpha - \gamma) = \gamma.$$

   Thus, $x_0 > \gamma$ for some $x_0 \in E$. Therefore, $\gamma$ is NOT an upper bound of $E$ and so we conclude that

   $$\alpha = \sup E.$$

   ■

3. Let $E = \{n/n + 1 : n \in \mathbb{N}\}$. Prove that $\sup E = 1$.

   **Proof.** To show that 1 is the supremum of $E$, we need to show that 1 is an upper bound for $E$ and that it satisfies the lemma found in Problem 2. First, we will show that 1 is an upper bound of $E$. Observe that for any $n \in \mathbb{N}$, we have

   $$\frac{n}{n+1} < \frac{n}{n} = 1.$$

   Thus, $E$ is bounded above by 1.

Let $\varepsilon > 0$. By the Archimedean Property, choose $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} < \varepsilon.$$

Then observe that

$$\frac{1}{n+1} = \frac{(n+1)-1}{n+1} = 1 - \frac{n}{n+1}.$$

Thus,

$$1 - \frac{n}{n+1} < \varepsilon \Leftrightarrow \frac{n}{n+1} > 1 - \varepsilon.$$

But note that $\frac{n}{n+1} \in E$. By Problem 2, we conclude that $\sup E = 1$.

■

4. State the analogue of the Very Useful Theorem for infimums.

**Solution.** Suppose $\beta$ is a lower bound for $E \subseteq \mathbb{R}$. Then $\beta = \inf E$ if and only if for all $\varepsilon > 0$, there exists $y_0 \in E$ such that $y_0 < \beta + \varepsilon$. ■

5. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Prove $a \leq b$.

**Proof.** Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Suppose for sake of contradiction that $a > b$. Note that $a - b > 0$. Pick $\varepsilon = \frac{a-b}{2}$. Then

$$\begin{aligned} b + \varepsilon = b + \frac{a-b}{2} = \frac{2b + a - b}{2} &= \frac{a+b}{2} \\ &< \frac{a+a}{2} \\ &= \frac{2a}{2} \\ &= a. \end{aligned}$$

But this means that $a < b+\varepsilon$ which is a contradiction of our assumption that $a \leq b+\varepsilon$. Thus, it must be the case that $a \leq b$. ■

6. Complete the following proof.

**Theorem (Greatest-lower-bound property of $\mathbb{R}$).** Every nonempty subset of $A$ of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf A$ exists and is a real number.

**Proof.** Continuation of the proof presented in homework. Set $\beta = -\alpha$ where $\alpha = \sup(-A)$. Thus, there exists $a \in A$ such that $-\gamma < -a$. Multiplying by a negative on both sides of this inequality, we get $\gamma > a$ for some $a \in A$. If $\gamma$ was a lower bound of $A$, then $\gamma \leq a$ for all $a \in A$. However, we have the negation of this, so we must have that $\gamma$ is NOT a lower bound of $A$, which satisfies part (ii). Since $\beta \leq \alpha$ for all $a \in A$ and $\beta < \gamma$ implies $\gamma$ is not a lower bound for $A$, we conclude that $\beta$ must be the infimum of $A$. ■

7. (i) Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many elements of $A$.

**Proof.** Let $\varepsilon > 0$. Our goal is to show that $(\sup A - \varepsilon, \sup A)$ has infinitely many points. Note that as a consequence of the result found in Problem 2, there exists

$x_0 \in A$ such that
$$\sup A - \varepsilon < x_0 \leq \sup A.$$
By assumption, we have that $\sup A \notin A$. Thus, we have
$$\sup A - \varepsilon < x_0 < \sup A$$
for some $x_0 \in A$. So, $(\sup A - \varepsilon, \sup A)$ is nonempty.

Now, we will show, through contradiction that there exists an $\varepsilon > 0$ such that $(\sup A - \varepsilon, \sup A)$ contains infinitely many points. Thus, suppose that $(\sup A - \varepsilon, \sup A) \cap A$ contains FINITELY many elements. Let us denote these elements by $a_i$ for $1 \leq i \leq n$ in
$$(\sup A - \varepsilon, \sup A) \cap A = \{a_1, \ldots, a_n : 1 \leq i \leq n\}.$$
For all $1 \leq k \leq n$, let the distance between each $a_k$ and $\sup A$ by
$$d_k = |\sup A - a_k|.$$
Since $a_k \in A$ and $\sup A \notin A$, we must have $|\sup A - a_k| > 0$. Let
$$d = \frac{\min\{d_1, \ldots, d_n\}}{2}.$$
But this implies that none of the $a_1, \ldots, a_n$ lies inside the interval $(\sup A - d, \sup A)$. That is,
$$A \cap (\sup A - d, \sup A) = \emptyset,$$
which is a contradiction. Thus, $(\sup A - \varepsilon, \sup A)$ must contain infinitely many points.

■

(ii) Let $B \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below. Show that if $\inf B \notin B$, then for all $\varepsilon > 0$ the open interval $(\inf B, \inf B + \varepsilon)$ contains infinitely many elements of $B$.

**Proof.** Let $\varepsilon > 0$. Our goal is to show that $(\inf B, \inf B + \varepsilon)$ contain infinitely many points of $B$. Note that as a consequence of Problem 3, there exists an $\beta \in B$ such that
$$\inf B \leq \beta < \inf B + \varepsilon.$$
Since $\inf B \notin B$, we must have
$$\inf B < \beta < \inf B + \varepsilon.$$
Thus, the open interval $(\inf B, \inf B + \varepsilon)$ is nonempty.

Now, we will show, through contradiction, that $(\inf B, \inf B + \varepsilon) \cap B$ contains infinitely many points. Suppose there exists an $\varepsilon > 0$ such that $(\inf B, \inf B + \varepsilon)$ FINITELY many points. Denote these points by $b_i$ for $1 \leq i \leq n$ in
$$(\inf B, \inf B + \varepsilon) \cap B = \{b_i : 1 \leq i \leq n\}$$
For all $1 \leq k \leq n$, let $d_k = |\inf B - b_k|$. Since $b_k \in B$ and $\inf B \notin B$, we must

have $|\inf B - b_k| > 0$. Let

$$d = \frac{\min\{d_1, d_2, \ldots, d_n\}}{2}.$$

But this implies that none of the $b_i$ for $1 \leq i \leq n$ lies within the open interval $(\inf B, \inf B + \varepsilon)$. That is, the set

$$(\inf B, \inf B + \varepsilon) \cap B = \emptyset$$

which is a contradiction. Thus, the open interval $(\inf B, \inf B + \varepsilon)$ must have infinitely many points. ■

8. Suppose the construction of $\mathbb{R}$ and the proof of the least-upper-bound property for $\mathbb{R}$ can be completed without directly using the well-ordering principle for $\mathbb{N}$. (Of course, statements equivalent to the well-ordering principle might have been used, but we are assuming that the nontrivial fact of their equivalence to the well-ordering principle was not utilized in the construction of $\mathbb{R}$ or in proving its Dedekind completeness.) Use the greatest-lower-bound property of $\mathbb{R}$ and the result of the previous exercises to prove the well-ordering principle for $\mathbb{N}$; that is, prove that every nonempty subset of $\mathbb{N}$ has a minimum.

**Proof.** Let $E$ be a nonempty subset of $\mathbb{N}$. First, we will show that $\inf E$ exists. Observe that $0$ is a lower bound of $E$. Thus, $E$ is bounded below. Thus, $\inf E$ must exists by the greatest-lower-bound property of $\mathbb{R}$.

Let $\varepsilon = \frac{1}{5}$. To show that $E$ contains a minimum, we suppose for contradiction that $\inf E \notin E$. By Part 2 of Problem 7, the open interval $(\inf E, \inf E + \frac{1}{5})$ intersects $E$ at infinitely many points (these points being natural numbers). Suppose we choose an $m \in \mathbb{N}$ such that $m \in (\inf E, \inf E + \frac{1}{5})$. Note that $m$ also lies interval $(m-1, m+1)$. Thus, $(\inf E, \inf E + \varepsilon) \subseteq (m-1, m+1)$. However, note that $(\inf E, \inf E + \frac{1}{5})$ contains infinitely many natural numbers which cannot happen within the open interval $(m-1, m+1)$ since this interval contains at most one element. Thus, this is a contradiction. Thus, it must be the case that $\inf E \in E$. ■

9. Let $A, B \subseteq \mathbb{R}$ be nonempty, bounded sets and let $c \in \mathbb{R}$. Define the following sets:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{ab : a \in A, b \in B\} \\ cA &= \{ca : a \in A\}. \end{aligned}$$

Prove that

9-1) $\inf(A + B) = \inf(A) + \inf(B)$.

**Proof.** In order to show that $\inf(A + B) = \inf A + \inf B$, we need to show that

(i) The set $A + B$ is bounded below by $\inf(A) + \inf(B)$.

(ii) For every $\varepsilon > 0$, there exists $\varphi \in A + B$ such that

$$\varphi < \inf(A) + \inf(B) + \varepsilon.$$

Note that $A + B \neq \emptyset$ since $a + b \in A + B$ and $A$ and $B$ are nonempty sets. Since $A$ and $B$ are also bounded below, we can see that $a \geq \inf A$ $a \in A$ and $b \geq \inf B$

for all $b \in B$ implies

$$a + b \geq \inf(A) + \inf(B) \quad \text{for all } a + b \in A + B,$$

which shows (i). Now, let $\varepsilon > 0$. Using the lemma found in Problem 3, there exists an $\alpha \in A$ and $\beta \in \beta$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{2}$$

and

$$\beta < \inf(B) + \frac{\varepsilon}{2},$$

respectively. Adding these two inequalities, we get

$$\varphi = \alpha + \beta < \inf(A) + \inf(B) + \varepsilon \quad \text{for some } \varphi \in A + B.$$

Thus, we conclude that $\inf(A + B) = \inf(A) + \inf(B)$. ■

9-2) $\sup(A + B) = \sup(A) + \sup(B)$

**Proof.** In order to show that $\sup(A + B) = \sup(A) + \sup(B)$, we need to show that

(i) $A + B$ is bounded above and
(ii) For every $\varepsilon > 0$ ,there exists a $\lambda > \sup(A) + \sup(B) - \varepsilon$.

Note that $A + B \neq \emptyset$ since $A$ and $B$ is nonempty.Since $A$ and $B$ are bounded above, we have that $a \leq \sup(A)$ for all $a \in A$ and $b \leq \sup(B)$ for all $b \in B$ implies that

$$a + b \leq \sup(A) + \sup(B)$$

for all $a + b \in A + B$, which proves (i). Let $\varepsilon > 0$. Since $\sup(A)$ and $\sup(B)$ exists, we see that there exists $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{2}$$

and

$$\beta > \sup(B) - \frac{\varepsilon}{2}$$

by the lemma found in Problem 2. Adding these two inequalities results in

$$\lambda = \alpha + \beta > \sup(A) + \sup(B) - \varepsilon.$$

Thus, $\sup(A + B) = \sup(A) + \sup(B)$ by the lemma found in Problem 2. ■

9-3) $\sup(-A) = -\inf(A)$

**Proof.** To show that $\sup(-A) = -\inf(A)$, we need to show that

(i) For all $-a \in -A$, we have $-a \leq -\inf(A)$ and
(ii) For all $\varepsilon > 0$, there exists $-\alpha \in -A$ such that

$$-\alpha < -\inf(A) - \varepsilon.$$

Since $A$ is bounded below and nonempty, we know that $a \geq \inf(A)$ for all $a \in A$.

Multiplying this inequality by a negative, we get that

$$-a \leq -\inf(A) \quad \text{for all } -a \in -A.$$

Thus, $-A$ is bounded above by $-\inf(A)$.
Let $\varepsilon > 0$. Since $\inf(A)$ exists, there exists $\gamma \in A$ such that

$$\gamma < \inf(A) + \varepsilon.$$

By multiplying by a negative on this inequality, we must have that

$$-\gamma > -\inf(A) - \varepsilon$$

for some $-\gamma \in -A$. Thus, we must have that $\sup(-A) = -\inf(A)$. ∎

9-4) $\inf(-A) = -\sup(A)$.

**Proof.** To show that $\inf(-A) = -\sup(A)$, we must show that

(i) For all $-a \in -A$, we have $-a \geq -\sup(A)$.
(ii) For all $\varepsilon > 0$, there exists $-\beta \in -A$ such that

$$-\beta < -\sup(A) + \varepsilon.$$

To show (i), let $a \in A$. Since $A$ is nonempty and bounded above, we have that

$$a \leq \sup(A).$$

Multiplying by a negative on this inequality, we get

$$-a \geq -\sup(A)$$

for all $-a \in -A$, which shows (i).
Now, let $\varepsilon > 0$. Since $\sup(A)$ exists, there exists $\gamma \in A$ such that

$$\gamma > \sup A - \varepsilon.$$

Multiplying by a negative, we get

$$-\gamma < -\sup A + \varepsilon$$

for some $-\gamma \in -A$, which shows (ii). By the lemma found in Problem 3, we conclude that $\inf(-A) = -\sup(A)$. ∎

9-5) $\sup(A - B) = \sup(A) - \inf(B)$.

**Proof.** Observe that

$$\begin{aligned} \sup(A - B) = \sup(A + (-B)) &= \sup(A) + \sup(-B) && \text{(9-2)} \\ &= \sup(A) - \inf(B) && \text{(9-3)} \end{aligned}$$

∎

9-6) $\inf(A - B) = \inf(A) - \sup(B)$.

**Proof.** Observe that

$$\begin{aligned}\inf(A-B) = \inf(A+(-B)) &= \inf(A)+\inf(-B) && \text{(9-1))}\\ &= \inf(A)-\sup(A). && \text{(9-4))}\end{aligned}$$

■

9-7) $\sup(cA) = c\sup(A)$ if $c > 0$.

**Proof.** Let $c > 0$. We will show that $\sup(cA) = c\sup(A)$. Note that $cA$ is nonempty because $ca \in cA$. First, we show that $cA$ is bounded above by $c\sup(A)$. Since $\sup(A)$ exists, we know that

$$a \leq \sup(A) \quad \text{for all } a \in A.$$

Multiplying by $c$, we must have

$$ca \leq c\sup(A) \quad \text{for all } ca \in cA.$$

Now, let $\varepsilon > 0$. Since $\sup(A)$ exists, there exists $\alpha \in A$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{c}.$$

Multiplying by $c > 0$, we must have

$$c\alpha > c\sup(A) - \varepsilon \quad \text{for some } c\alpha \in cA.$$

Thus, $c\sup(A) = \sup(cA)$ by the lemma found in Problem 2.

■

9-8) $\inf(cA) = c\inf(A)$ if $c > 0$.

**Proof.** Let $c > 0$. We will show that $\inf(cA) = c\inf(A)$. Note that $cA$ is nonempty because $ca \in cA$. First, we show that $cA$ is bounded below by $c\inf(A)$. Since $\inf(A)$ exists, we know that

$$a \geq \inf(A) \quad \text{for all } a \in A.$$

Multiplying by $c$, we must have

$$ca \geq c\inf(A) \quad \text{for all } ca \in cA.$$

Now, let $\varepsilon > 0$. Since $\inf(A)$ exists, there exists $\beta \in A$ such that

$$\beta < \inf(A) + \frac{\varepsilon}{c}.$$

Multiplying by $c > 0$, we must have

$$c\beta > c\inf(A) + \varepsilon \quad \text{for some } c\beta \in cA.$$

Thus, $c\inf(A) = \inf(cA)$ by the lemma found in Problem 2. ■

9-9) $\sup(cA) = c\inf(A)$ if $c < 0$.

**Proof.** Let $c < 0$. We will first show that $\sup(cA) = c\inf(A)$. We will first show that $cA$ is bounded above by $c\inf(A)$. Note that $\inf(A)$ exists. Thus, $a \geq \inf(A)$ for all $a \in A$. Since $c < 0$, $ca \leq c\inf(A)$ for all $ca \in cA$. Thus, $c\inf(A)$ is an upper bound of $cA$.

Now, let $\varepsilon > 0$. Since $\inf(A)$ exists, there must exists an $\alpha \in A$ such that

$$\alpha < \inf(A) - \frac{\varepsilon}{c}.$$

By multiplying through by $c < 0$, we have

$$c\alpha > c\inf(A) - \varepsilon$$

for some $c\alpha \in cA$. Thus, we conclude that $\sup(cA) = c\inf(A)$ by Problem 2. ■

9-10) $\inf(cA) = c\sup(A)$ if $c < 0$.

**Proof.** Let $c < 0$. We will show that $\inf(cA) = c\sup(A)$. We will first show that $cA$ is bounded below by $c\sup(A)$. Note that $\sup(A)$ exists. Thus, $a \leq \sup(A)$ for all $a \in A$. Since $c < 0$, $ca \geq c\sup(A)$ for all $ca \in cA$. Thus, $c\sup(A)$ is a lower bound of $cA$.

Now, let $\varepsilon > 0$. Since $\sup(A)$ exists, there must exists an $\omega \in A$ such that

$$\omega > \sup(A) + \frac{\varepsilon}{c}.$$

By multiplying through by $c < 0$, we have

$$c\omega < c\sup(A) + \varepsilon$$

for some $c\omega \in cA$. Thus, we conclude that $\inf(cA) = c\sup(A)$ by Problem 3. ■

9-11) Is it true that $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

**Proof.** This is false. Consider the sets $A = \{1, 2, 4, 8\}$ and $B = \{-5, -2, -3, -1\}$. Then $AB = \{-5, -4, -12, -8\}$. Observe that $\sup(A) \cdot \sup(B) = 8 \cdot -1 = -8$, but $\sup(AB) = -5 \neq -8 = \sup(A) \cdot \sup(B)$. ■

10. Recall that $\mathbb{Q}$ is a field; in particular, $\mathbb{Q}$ is closed under addition and multiplication.

10-1) Prove that if $p \in \mathbb{Q}$ and $t$ is an irrational number, then $p + t$ is an irrational number.

**Proof.** Suppose $p \in \mathbb{Q}$ and $t$ is an irrational number. Suppose for sake of contradiction that $p + t$ is an rational number. Then there exists $x, y \in \mathbb{Z}$ with $y \neq 0$ such that

$$p + t = \frac{x}{y}.$$

Subtracting $p$ on both sides, we get

$$t = \frac{x}{y} - p.$$

Since $\mathbb{Q}$ is a field and $\frac{x}{y}, p \in \mathbb{Q}$, we get that $\frac{x}{y} - p \in \mathbb{Q}$. But this tells us that $t$ is a rational number which is a contradiction. Thus, $t$ must be irrational. ■

10-2) Complete the following proof.

**Theorem.** Given any two real numbers $x < y$, there exists an irrational number $t$ satisfying $x < t < y$.

**Proof.** It follows from $x < y$ that $x - \sqrt{2} < y - \sqrt{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $p \in \mathbb{Q}$ such that $x - \sqrt{2} < p < y - \sqrt{2}$. Adding $\sqrt{2}$ on both sides gives us

$$x < p + \sqrt{2} < y.$$

Since $p \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, we get that $t = p + \sqrt{2}$ is irrational from 10-2). Thus, $x < t < y$ for some irrational number $t$.

■

11. Prove the following:

**Theorem (Nested Interval Property).** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each $I_n$ contains $I_{n+1}$. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, that is $\cap_n^\infty I_n \neq \emptyset$.

**Proof.** Let $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ because $a_1 \in A$. Since $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$ and that each $I_{n+1}$ is contained within each $I_n$ , we see that each $a_n \in A$ must be bounded above by $b_n$ for all $n \in \mathbb{N}$. Since $A \neq \emptyset$ and bounded above, there must exists a number $x$ such that $x = \sup(A)$. Since $x$ is an upper bound of $A$, we have $a_n \leq x \leq b_n$. Thus, $x \in I_n$ for all $n \in \mathbb{N}$ which means that

$$x \in \bigcap_{n=1}^{\infty} I_n,$$

proving that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

12. Complete the following proof:

Prove that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.

**Proof.** Suppose for sake of contradiction $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. So there exists $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. This implies that $x > 0$ and $x < 1/n$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < x.$$

But this contradicts our assumption that $x < 1/n$ holds for all $n \in \mathbb{N}$. Thus, the intersection

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$$

must be empty. ■