# Extra Credit Problems

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April 29, 2025

**Problem 1.** Assume $\alpha : [a,b] \to \mathbb{R}$ is increasing and $G : [a,b] \to \mathbb{R}$ be differentiable with continuous derivative. Prove that

$$\int_a^b G\ d\alpha = \Big[G(x)\alpha(x)\Big]_a^b - \int_a^b \alpha\ dG. \tag{*}$$

**Proof.** First, observe that since $\alpha : [a,b] \to \mathbb{R}$ is increasing and $x$ is an increasing and continuous function, we have that $\alpha \in R[a,b]$ by Theorem 6.9 (in Rudin). Since $G : [a,b] \to \mathbb{R}$ is differentiable and its derivative $G'$ is continuous, it follows that $G' \in R[a,b]$ by Theorem 6.8 (in Rudin). As a consequence, both $G'$ and $\alpha$ are bounded on $[a,b]$. Hence, it follows that from the product of two integrable functions that $\alpha G' \in R[a,b]$, and so

$$\int_a^b \alpha(x)G'(x)\ dx = \int_a^b \alpha\ dG \text{ exists.} \tag{1}$$

Also, since $G$ is differentiable, we have that $G$ is continuous on $[a,b]$. This means that, with respect to the integrator $\alpha$ on $[a,b]$, we have $G \in R_\alpha[a,b]$. Hence, we have

$$\int_a^b G\ d\alpha \text{ exists.} \tag{2}$$

From (1), we know that for any $r > 0$, we can find a partition $P_1 = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that

$$U(\alpha G', P) - L(\alpha G', P) < r.$$

Using this partition and using the fact that $G$ is differentiable on $[a,b]$, it follows from the Mean Value Theorem that for all $1 \leq i \leq n$, there exists a $t_i \in (x_{i-1}, x_i)$ such that

$$G'(t_i) = \frac{G(x_i) - G(x_{i-1})}{\Delta x_i} \Longrightarrow G'(t_i)\Delta x_i = G(x_i) - G(x_{i-1}). \tag{3}$$

As a step towards showing (*), we first need to show that

$$\sum_{i=1}^n \alpha(x_i)G'(t_i)\Delta x_i + \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i = G(b)\alpha(b) - G(a)\alpha(a). \tag{**}$$

Indeed, note that

$$\sum_{i=1}^n [G\alpha(x_i) - G\alpha(x_{i-1})] = G(b)\alpha(b) - G(a)\alpha(a).$$

and so from (3), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}[G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1})] \\
&= \sum_{i=1}^{n}[G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1}) + G(x_{i-1})\alpha(x_i) - G(x_{i-1})\alpha(x_i)] \\
&= \sum_{i=1}^{n}\big[\alpha(x_i)[G(x_i) - G(x_{i-1})] + G(x_{i-1})[\alpha(x_i) - \alpha(x_{i-1})]\big] \\
&= \sum_{i=1}^{n}\alpha(x_i)[G(x_i) - G(x_{i-1})] + \sum_{i=1}^{n}G(x_{i-1})[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i + \sum_{i=1}^{n}G(x_{i-1})\Delta\alpha_i
\end{aligned}$$

Hence, we see that (**) is satisfied. Next, we will show that the following equations hold:

$$\sum_{i=1}^{n}G(x_{i-1})\Delta\alpha_i = \int_a^b G\ d\alpha. \tag{4}$$

and

$$\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i = \int_a^b \alpha(x)G'(x)\ dx \tag{5}$$

Note that the first equation follows from the fact that

$$L(G,\alpha,Q) \le \sum_{i=1}^{n}G(x_{i-1})\Delta\alpha_i \le U(G,\alpha,Q)$$

for any $Q \in \Pi[a,b]$. Taking the supremum and infimum of the respective sides of the inequality above over all partitions in $[a,b]$, we have

$$L(G,\alpha) \le \sum_{i=1}^{n}G(x_{i-1})\Delta\alpha_i \le U(G,\alpha).$$

Since $G \in R_\alpha[a,b]$, we obtain

$$\sum_{i=1}^{n}G(x_{i-1})\Delta\alpha_i = \int_a^b G\ d\alpha.$$

Now, our goal is to show that

$$\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i = \int_a^b \alpha(x)G'(x)\ dx.$$

It suffices to show that for any $\varepsilon > 0$

$$\Big|\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i - \int_a^b \alpha(x)G'(x)\ dx\Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Note that $\alpha \in R[a,b]$ implies that there exists a $P_2 \in \Pi[a,b]$ such that

$$U(\alpha,P_2) - L(\alpha,P_2) < \frac{\varepsilon}{2M}.$$

Choose $\hat{P} = P_1 \cup P_2$. Also, since $t_i \in [x_{i-1},x_i]$ for all $1 \le i \le n$, we have (by Theorem 6.7)

$$\Big|\sum_{i=1}^{n}\alpha(t_i)G'(t_i)\Delta x_i - \int_a^b \alpha(x)G'(x)\ dx\Big| < \frac{\varepsilon}{2}.$$

Now, all that is left to show is that

$$\Big|\sum_{i=1}^{n}\alpha(t_i)G'(t_i)\Delta x_i-\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Big|<\frac{\varepsilon}{2}.$$

Since $G'$ is bounded on $[a,b]$, there exists an $M>0$ such that $|G'(x)|\leq M$ for all $x\in[a,b]$. Then using $\hat{P}$ as our partition, we write

$$\begin{aligned}\Big|\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i-\sum_{i=1}^{n}\alpha(t_i)G'(t_i)\Delta x_i\Big| &= \Big|\sum_{i=1}^{n}G'(t_i)[\alpha(x_i)-\alpha(t_i)]\Delta x_i\Big|\\ &\leq \sum_{i=1}^{n}|G'(t_i)||\alpha(x_i)-\alpha(t_i)|\Delta x_i\\ &\leq M\sum_{i=1}^{n}|\alpha(x_i)-\alpha(t_i)|\Delta x_i\\ &\leq M\sum_{i=1}^{n}\Big[\sup_{I_i}\alpha-\inf_{I_i}\alpha\Big]\Delta x_i\\ &= M(U(\alpha,\hat{P})-L(\alpha,\hat{P}))\\ &< M\cdot\frac{\varepsilon}{2M}=\frac{\varepsilon}{2}.\end{aligned}$$

where $I_i=[x_{i-1},x_i]$ and so we conclude that

$$\Big|\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i-\sum_{i=1}^{n}\alpha(t_i)G'(t_i)\Delta x_i\Big|<\frac{\varepsilon}{2}.$$

Using the triangle inequality, we have

$$\begin{aligned}\Big|\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i-\int_a^b\alpha(x)G'(x)\ dx\Big| &\leq \Big|\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i-\sum_{i=1}^{n}\alpha(t_i)G'(t_i)\Delta_i\Big|\\ &\quad+\Big|\sum_{i=1}^{n}\alpha(t_i)G'(t_i)-\int_a^b\alpha(x)G'(x)\ dx\Big|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{n}\alpha(x_i)G'(t_i)\Delta x_i=\int_a^b\alpha(x)G'(x)\ dx$$

and so we can conclude that

$$\begin{aligned}&\int_a^b\alpha(x)G'(x)\ dx+\int_a^b G\ d\alpha=\Big[G(x)\alpha(x)\Big]_a^b\\ &\implies\int_a^b G\ d\alpha=\Big[G(x)\alpha(x)\Big]_a^b-\int_a^b\alpha(x)G'(x)\ dx\\ &\implies\int_a^b G\ d\alpha=\Big[G(x)\alpha(x)\Big]_a^b-\int_a^b\alpha\ dG\end{aligned}$$

which is our desired result. ■

**Problem 2.** Define $f(x)=\int_x^{x+1}\sin t^2\ dt$.

(2-1) Prove that $|f(x)|<\frac{1}{x}$ if $x>0$.

(2-2) Prove that

$$2xf(x)=\cos(x^2)-\cos((x+1)^2)+r(x)$$

where $|r(x)| < \frac{c}{x}$ and $c$ is constant.

(2-3) Does $\displaystyle\int_0^\infty \sin(t^2)\, dt$ converge?

(2-1)