Analysis Rudin Notes

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List of Theorems

Chapter 1

Basic Topology

1.1 Finite, Countable, and Uncountable Sets

1.1.1 Function Concepts

Definition (Functions, Domains, Values, and Range). Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element of x of A there is a associated, in some manner, an element of B, which we denote by $f(x)$.

- The function f is said to be a *function* from A into B;
- The set A is called the *domain* of f;
- The elements $f(x)$ are called the *values* of f;
- The set of values of f is called the *range* of f

Definition (Images, Onto). Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, then $f(E)$ is called the set of all elements $f(x)$, for $x \in E$.

- We call $f(E)$, the *image* of E under f.
- We call $f(A)$ the *range* of f. Clearly, we have $f(A) \subset B$.
- If $f(A) = B$, we say that f maps *onto* B.

Definition (Inverse Images, One-to-one).

- If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f.
- If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B.
- Another way to state this is to say that for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ is called a 1-1 function.
- Alternatively, if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ is also called a 1-1 function (this is just contrapositive of the last statement).

Definition (Correspondence). If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinality*, or, that A and B are *equivalent*. For this, we write $A \sim B$.

This relation contains the following properties:

- **Reflexive**: $A \sim A$.
- **Symmetric**: If $A \sim B$, then $B \sim A$.
- **Transitive:** If $A \sim B$ and $B \sim C$, then $A \sim C$.

Definition (Finite, Infinite, Countable, Uncountable, At most countable). For any positive integer n, let $\mathbb{N}_n$ be the set whose elelemts are the integers $1, 2, \dots, n$; let $\mathbb{N}$ be the set consisting of all positive integers. For any set A, we say:

(a) A is *finite* if $A \sim \mathbb{N}_n$ for some n (the empty set is also considered to be finite).

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim \mathbb{N}$.

(d) A is *uncountable* if A is neither finite n or countable.

(e) A is *at most countable* if A is finite or countable.

Proposition. Every infinite subset of a countable set A is countable.

Proof. Suppose $E \subset A$, and E is infinite. Arrange the elements of x of A as a sequence (x_n) of distinct elements. Construct a sequence (n_k) as follows: Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \dots n_{k-1}$ with $(k = 2, 3, 4, \dots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$ with $(k = 1, 2, 3, \dots)$, we obtain a 1-1 correspondence between E and $\mathbb{N}$. Thus, by definition, we see that E is an infinite subset of A that is countable. ■

Definition. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

- We can have sets whose elements are also sets.
- To make this easier to understand, we usually denote these kinds of sets as a collection of sets.

Definition (Union). The *union* of the sets E_α is defined to be the set S such that $x \in S$ if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If our collection of sets A is finite, then we can use the notation

$$S = \bigcup_{m=1}^{n} E_m$$

for $E_1, E_2, \ldots, E_n \in A$

On the other hand, when the collection of sets A contains a countable number of elements, then we can use the notation

$$S = \bigcup_{m=1}^{\infty} E_m.$$

The ∞ on the top of the union symbol should not be confused with $+\infty$ and $-\infty$.

Definition (Intersection). The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if $x \in E_\alpha$ for every $\alpha \in A$, we have

$$P = \bigcap_{\alpha \in A} E_\alpha.$$

Like the union, A can either have a finite collection of sets or a countable collection of sets. Thus, we have

$$P = \bigcap_{m=1}^{n} E_m \text{ and } P = \bigcap_{m=1}^{\infty} E_m,$$

respectively.

Definition (Nonempty Intersections and Disjoint Sets). If we have $A \cap B \neq \emptyset$, then we say that A and B *intersect*. Otherwise, we say that they are *disjoint*.

Here are some list of algebraic properties of sets:

- **Commutativity**: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- **Associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Distributivity:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \subset A \cup B$.
- $A \cap B \subset A$.
- $A \cup \emptyset$ and $A \cap \emptyset = \emptyset$.
- If $A \subset B$, then
$$A \cup B = B, \quad A \cap B = A$$

Theorem. Let $\{E_n\}$ with $n \in \mathbb{N}$ be a countable collection of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n,$$

Then S is countable.

Proof. Let every set E_n be arranged in a sequence (x_{n_k}) with $k = 1, 2, 3, \ldots,$. We can consider an infinite array such that, in each row, we have all the elements of each E_n. If we take the diagonal entries, starting from left to right, we can rearrange these entries into a sequence

$$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; \ldots ,$$

starting from $n = 2$. Notice how the sum of each index in the sequence adds up to the index of the sequence above. Thus, there exists a subset of T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable, using our result about infinite subsets of countable sets. Since each $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus S is countable. ■

Corollary. Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Then

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is at most countable, for T is equivalent to a subset of

$$\bigcup_{n=1}^{\infty} B_n,$$

where $B_n \in A$.

Theorem. Let A be a countable set, and let B_n be the set of all $n-$tuples $(a_1, \ldots, a_n)$, where $a_k \in A (k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_m is countable.

1.2 Metric Spaces

Definition (Metric Space). A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$ called the *distance* from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

- Suppose a subset Y of X is a metric space, with the same distance function.
- Y must also be a metric space with the same distance function as X (metric).

Definition.

- By the *segment* (a, b) we mean the set of all $x \in \mathbb{R}$ such that $a < x < b$.
- We call an *interval* $[a, b]$ to mean the set of all $x \in \mathbb{R}$ such that $a \leq x \leq b$ for $a, b \in \mathbb{R}$ with $a < b$.
- We call a *half-open interval* either $[a, b)$ or $(a, b]$ to mean $a \leq x < b$ and $a < x \leq b$,

respectively.

- We call a k-*cell* to mean that if $a_i < b_i$, for $i = 1, \dots, k$, the set of all points $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy $a_i \leq x_i \leq b_i (1 \leq i \leq k)$.
- If $x \in \mathbb{R}^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at x and radius r is defined to be the set
$$B(x, r) = \{y \in \mathbb{R}^k : |y - x| < r\}$$
or alternatively, $|y - x| \leq r$.
- We call a set $E \subset \mathbb{R}^k$ *convex* if
$$\lambda x + (1 - \lambda) y \in E$$
whenever $x, y \in E$, and $0 < \lambda < 1$.

Definition. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.

(d) E is closed if every limit point of E is a point of E.

(e) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \neq E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E.

(i) E is *bounded* if there exists a $M \in \mathbb{R}$ and $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(j) E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

Remark. In $\mathbb{R}^1$, neighborhoods are segments and in $\mathbb{R}^2$ neighborhoods are interiors of circles.

Theorem (Neighborhoods are Open)**.** Every neighborhood is an open set.

Proof. Let $p \in X$. Consider the neighborhood $N_r(p)$ for some $r > 0$. Let $y \in X$. Similarly, we can construct a neighborhood $N_h(y)$ for some $h > 0$. Observe that the distance between p and y is
$$d(p, y) = r - h.$$
Our goal is to show that $N_h(y) \subset N_r(p)$ in order for $N_r(p)$ to be open. Let $x \in N_h(y)$.

Using the triangle inequality, we can see that

$$\begin{aligned} d(p,x) &\le d(p,y) + d(y,x) \\ &< (r-h) + h \\ &= r. \end{aligned}$$

This tells us that $x \in N_r(p)$, proving that $N_r(p)$ is an open set. ■

Theorem. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof. Suppose for sake of contradiction that there exists a neighborhood N of p which contains only a finite number of points of E. Let $q_1, q_2, \ldots, q_n$ be the points of $N \cap E$ such that $q_m \neq p$ for all m. Observe that

$$r = \min_{1 \le m \le n} d(p, q_m) > 0$$

since each $d(p, q_m) > 0$. Since each $d(p, q_m) < \delta_m$ and not $d(p, q_m) = \delta_m$, we have that none of the $q_m \in N_r(p)$ where $q_m \neq p$. So, p must not be a limit point of E which is a contradiction. Thus, every neighborhood of p must contain infinitely many points of E. ■

Corollary. A finite point set has no limit points.

Example 1.2.1 (Examples of Closed, Open, Perfect, Bounded Sets).

(a) The set of all $z \in \mathbb{C}$ such that $|z| < 1$. **Open and Bounded**

(b) The set of all $z \in \mathbb{C}$ such that $|z| \le 1$. **Closed, Perfect, Bounded**

(c) A nonempty finite set. **(Closed, Bounded)**

(d) The set of all integers. **(Closed)**

(e) The set

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Note that no point of E is a limit point of E; that is, there are no limit points contained in E. **Bounded**

(f) The set of all complex numbers (that is, $\mathbb{R}^2$).

(g) The segment (a, b). **(Bounded)**

Note that (g) is not open in $\mathbb{R}^1$ but open in $\mathbb{R}^2$.

Theorem. Let $\{E_\alpha\}$ be a (either finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha \right)^c = \bigcap_\alpha (E_\alpha^c).$$

Theorem. A set E is open if and only if its complement is closed.

Proof. For the forwards direction, let x be a limit point of E^c. Thus, for all neighborhoods $N(x)$, $N(x)$ contains a point $q \neq x$ such that $q \in E^c$. Since E is open, $N(e) \subset E$ for every $e \in E$. Since $q \notin E^c$, q must NOT be an interior point of E. Thus, $x \in E^c$, proving that E^c is closed.

For the backwards direction, suppose E^c is closed. Let $x \in E$. Since E^c is closed, all the limit points of E^c must be contained in E^c. But $x \notin E^c$, so x must not be a limit point of E^c. Thus, there exists a neighborhood $N(x)$ such that $N(x) \cap E^c$ is empty. Thus, x must be an interior point of E; that is, $N(x) \subset E$. Hence, E must be open. ■

Corollary. A set F is closed if and only if its complement is open.

Theorem. (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

c For any finite collection $G_1, G_2, \ldots, G_n$ of open sets $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{i=1}^n F_i$ is closed.

Proof. ■

Example 1.2.2.

- The finiteness found in part (c) and (d) of the preceding theorem is essential.
- Suppose $G_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$. We can see that G_n is an open subset of $\mathbb{R}$. But observe that
$$G = \bigcap_{n=1}^{\infty} (-1/n, 1/n)$$
only contains one point, namely 0, which implies that G is not an open subset of $\mathbb{R}$.

- The intersection of an infinite collection of open sets **need not** be open.
- The union of an infinite collection of closed sets need not be closed.

Definition (Closure). If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\overline{E} = E \cup E'$.

Theorem. If X is a metric space and $E \subset X$, then

(a) $\overline{E}$ is closed,

(b) $E = \overline{E}$ if and only if E is closed.

(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Proof. ■

Definition (Open Relative to Y). Suppose $E \subset Y \subset X$, where X is a metric space. To say that the set X is *open relative* to Y is to say that for each $p \in E$, there is an associated

$r > 0$ such that $q \in E$ whenever

$$d(p, q) < r \text{ and } q \in Y.$$

Theorem. Suppose $Y \subset X$. A subset of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof. For the forwards direction, suppose E is open relative to Y. For each $p \in E$, there exists a positive number r_p such that $q \in E$ whenever

$$d(p, q) < r_p \text{ and } q \in Y.$$

Let V_p be the set

$$\{q \in Y : d(p, q) < r_p\}$$

by definition. Since each V_p is just a neighborhood, we know that each V_p has to be an open subset of X. Thus, we can set

$$G = \bigcup_{p \in E} V_p$$

which implies G is open by part (a) of Theorem 2.24. Since $p \in V_p$ for all $p \in E$ and $p \in Y$ (since $E \subset Y$), we have $E \subset Y \cap G$. (**This is using the result from set theory that states $E \subset Y$ and $E \subset V_p$ implies $E \subset Y \cap V_p$.**) With our choice of V_p, we can see that $Y \cap V_p \subset E$. Since each $V_p \subset G$, we see that $Y \cap G \subset E$, which shows that $E = Y \cap G$.

For the backwards direction, suppose $E = Y \cap G$ for some open subset of G of X. Since G is open, for every $p \in E$, we can construct a neighborhood V_p such that $V_p \subset G$. So, $V_p \cap Y \subset E$. Thus, E is open relative to Y. ■

1.3 Compact Sets

Definition (Open Cover). By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition (Finite Subcover). A subset K of a metrix space X is said to be *compact* if every open cover of K contains a *finite* subcover. That is, if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subset \bigcup_{\alpha_i}^{n} G_{\alpha_i}.$$

Theorem. Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.

Proof. ■

Theorem. Compact subsets of metric spaces are closed.

Proof. ■

Theorem. Closed subsets of compact sets are compact.

Corollary. If F is closed and K is compact, then $F \cap K$ is compact.

Theorem. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Proof. ■

Corollary. If $\{K_n\}$ is a countable collection of nonempty compact sets such that $K_n \supset K_{n+1} (n \in \mathbb{N})$, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Theorem. If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$, such that $I_n \supset I_{n+1} (n \in \mathbb{N})$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. ■

Theorem. Let k be a positive integer. If $\{I_n\}$ is a sequence of $k-$cells such that $I_n \supset I_{n+1} (n \in \mathbb{N})$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. ■

Theorem. Every $k-$cell is compact.

Proof. ■

Theorem. If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof. ■

1.4 Connected Sets

Definition (Separated).

- Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. That is, if no point of A lies in the closure of B and no point of B lies in the closure of A.
- A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

Remark. Separated sets are disjoint, but disjoint sets are not always separated. For example, suppose we have two sets $[0, 1]$ and $(1, 2)$. They are both disjoint since 1 is not an element of both sets. But if we take the closure of $(1, 2)$, then the intersection with $[0, 1]$ is nonempty.

Theorem. A subset of E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property:

$$\text{If } x \in E, y \in E, \text{ and } x < z < y, \text{ then } z \in E.$$

Chapter 2

Numerical Sequences and Series

2.1 Convergent Sequences

Definition (Convergence of Sequences in Metric Spaces). A sequence (P_n) in a metric space X is said to **converge** if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$ where d denotes the distance in X.

- Another way we say the statement above is to say that $(p_n) \to p$, or that p is the limit of (p_n), or that
$$\lim_{n\to\infty} p_n = p.$$
- If (p_n) does not converge, then it is said to **diverge**.
- We can be more specific about the convergence of a sequence by specifying the convergence of the sequence with respect to the metric space.

Theorem. Let (p_n) be a sequence in a metric space X.

(a) (p_n) converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.

(b) If $p \in X$, $p' \in X$, and if (p_n) converges to p and to p', then $p' = p$.

(c) If (p_n) converges, then (p_n) is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence (p_n) in E such that $p = \lim_{n\to\infty} p_n$.

Theorem. Suppose $(s_n), (t_n)$ are complex sequences and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$ for any number c;

(c) $\lim_{n\to\infty} s_n t_n = st$;

(d) $\lim_{n\to\infty}(1/s_n) = 1/s$, provided $s_n \neq 0 (n = 1, 2, 3, \dots)$ and $s \neq 0$.

Theorem. (a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, 3 \dots)$ and

$$x_n = (\alpha_{1,n}, \dots, \alpha_{k,n}).$$

Then $(x_n) \to x$ with $x = (\alpha_1, \alpha_2, \dots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k).$$

(b) Suppose $(x_n), (y_n)$ are sequences in $\mathbb{R}^k$, (β_n) is a sequence of real numbers, and $x_n \to x$, $y_n \to y$, and $\beta_n \to \beta$. Then

$$\lim_{n\to\infty}(x_n + y_n) = x + y, \quad \lim_{n\to\infty} x_n \cdot y_n = x \cdot y, \quad \lim_{n\to\infty} \beta_n x_n = \beta x.$$

2.2 Subsequences

Definition. Given a sequence (p_n), consider a sequences (n_k) of positive integers such that $n_1 < n_2 < \dots$. Then the sequence (p_{n_i}) is called a **subsequence** of (p_n). If (p_{n_i}) converges, its limit is called the **subsequential limit** of (p_n).

Recall that if (p_n) converges to p iff every subsequence of (p_n) converges to p.

Theorem. (a) If (p_n) is a sequence in a compact metric space X, then some subsequence of (p_n) converges to a point of x.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Chapter 3

Differentiation

3.1 The Derivative of a Real Function

Definition (5.1). Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$, form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b,\ t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t).$$

Theorem (5.2). Let f be defined on $[a, b]$. If f differentiable at a point $x \in [a, b]$, then f is continuous at x.