0.1 Lecture 4

Definition (Sequence)**.** We call a **sequence**, we mean a function f on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence f by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in A.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

Theorem. Every infinite subset of a countable set is countable.

Proof. Let A be a countable set. Let $E \subseteq A$ and E is infinite. Our goal is to show that E is countable. Since A is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let n_1 be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let n_2

■

Remark.

Corollary.

Example 0.1.1 ($\mathbb{N} \times \mathbb{N}$ is countable)**.**

Example 0.1.2 ($\mathbb{Q}$ is countable)**.**

Theorem. Countable union of at most countable sets is at most countable.

Corollary.

Corollary.

Theorem. Finite product of countable sets is countable.

Example 0.1.3 ($\mathbb{Q}$ is countable)**.**

0.2 Lecture 5

0.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks
- Inequalities

0.2.2 Summary of Last Lecture

1. How to prove A is at most countable:
   - Show that $A \subseteq S$ where S is countable.
   - Find an injective function $f : A \to S$ where S is countable.
2. How to prove A is infinite:
   - Prove that A has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subseteq A$ such that $A \sim B$.
3. Find a proper subset $B \subseteq A$ such that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

Theorem (D). Let A be the set of all sequences whose terms are the digits 0 and 1 (that is A is the collection of all binary sequences). This set A is uncountable.

Proof. First, notice that A is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

$$\text{for all } h(n) = \text{The binary sequence whose } n\text{th term is the digit 1 and all other terms are zero.}$$

Clearly, h is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, A is infinite

Suppose for sake of contradiction that A is NOT uncountable. Since A is infinite and not uncountable, this assumption tells us that A must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned} f(1) &= (a_1^1, a_2^1, \dots) \\ f(2) &= (a_1^2, a_2^2, \dots) \\ f(3) &= (a_1^3, a_2^3, \dots) \\ &\vdots \\ f(n) &= (a_1^n, a_2^n, \dots) \\ &\vdots \end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that A contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N}\ b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence (b_n) is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

0.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of a is defined to be $|a|$. But note that the **absolute value of** a is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers a and b in $\mathbb{R}$, the **standard distance** between a and b is

$$\operatorname{dist}(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as h gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

Theorem (Basic Properties of Standard Size in $\mathbb{R}$)**.** Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Remark. Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

Definition. Let V be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on V if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space V equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

Theorem (Basic Properties of Standard Distance in $\mathbb{R}$)**.** For all $a, b \in \mathbb{R}$.

(i) $\text{dist}(a, b) \geq 0$

(ii) $\text{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\text{dist}(a, b) = \text{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

0.2.4 Metric Spaces

Definition (Metric Spaces)**.** Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X equipped with a metric d is called a **metric space**. (Sometimes we write (X, d) is a metric space)

Remark. X does **NOT** need to be a vector space. Furthermore, d is a more general notion of size than $\|\cdot\|$.

Example 0.2.1. $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)

0.2.5 Inequalities