Homework 8 Extra Credit

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Problem 1. Prove that

$$\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n) \leq \limsup a_n + \liminf b_n$$

provided that all expressions are meaningful.

Proof. Our goal is to show that

(1) $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n)$

(2) $\inf(a_n + b_n) \leq \limsup a_n + \liminf b_n$.

To show (1), we will assume that the left-hand side is NOT of the form $\infty + \infty$. Hence, there exists $n \geq n_0$ and $\ell \geq n$ such that

$$a_\ell \geq \inf\{a_k : k \geq n\},$$
$$b_\ell \geq \inf\{b_k : k \geq n\}.$$

Adding both inequalities above gives us

$$a_\ell + b_\ell \geq \inf\{a_k : k \geq n\} + \inf\{b_k : k \geq n\}.$$

Using the order limit theorem and algebraic limit theorem, we can write

$$\begin{aligned}\lim_{n\to\infty} \inf\{a_\ell + b_\ell : \ell \geq n\} &\geq \lim_{n\to\infty}[\inf\{a_k : k \geq n\} + \inf\{b_k : k \geq n\}] \\ &= \lim_{n\to\infty} \inf\{a_k : k \geq n\} + \lim_{n\to\infty} \inf\{b_k : k \geq n\}.\end{aligned}$$

Thus, we can conclude that

$$\liminf(a_n + b_n) \geq \liminf(a_n) + \liminf(b_n)$$

which establishes (1).

To show (2), we will consider the three cases; that is,

- $\liminf(a_n + b_n) \neq -\infty$
- $\limsup a_n = \infty$
- $\liminf(a_n + b_n) \neq -\infty$ and $\limsup a_n \neq \infty$.

Suppose $\liminf(a_n + b_n) \neq -\infty$. If this is the case, the right-hand side of (2) will always hold. Next, suppose $\limsup a_n = \infty$. Note that the right-hand side of (2) is NOT of the form $\infty + \infty$. So, we can conclude that $\liminf b_n \neq -\infty$ and so

$$\liminf b_n + \limsup a_n = \infty.$$

Now, suppose that $\liminf(a_n + b_n) \neq -\infty$ and $\limsup a_n \neq \infty$. Then using (1) and the algebraic limit theorem, we can write

$$\begin{aligned}\liminf b_n &= \liminf[(b_n + a_n) + (-a_n)] \\ &\geq \liminf(a_n + b_n) + \liminf(a_n) && \text{(ALT and (1))} \\ &= \liminf(a_n + b_n) - \limsup(a_n). && \text{(Exercise 9)}\end{aligned}$$

Then we have

$$\liminf(a_n + b_n) \leq \liminf b_n + \limsup a_n$$

which establishes (2). Hence, we now conclude that

$$\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n) \leq \limsup a_n + \liminf b_n$$

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Problem 2. Suppose X is a nonempty complete metric space, and (G_n) is a sequence of dense open subsets of X. Prove Baire's Theorem, namely, $\bigcap_{n=1}^{\infty} G_n$ is nonempty.

Proof. ■

Problem 3. Prove the following theorem:

Suppose (s_n) and (b_n) are two sequences of real numbers, (b_n) is a strictly increasing sequence that diverges to ∞, and $\lim\limits_{n\to\infty} \frac{s_{n+1}-s_n}{b_{n+1}-b_n} = L \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} \frac{s_n}{b_n} = L$.

Proof. Suppose (s_n) and (b_n) are two sequences of real numbers and (b_n) is a strictly increasing sequence that diverges to ∞, and that

$$\lim_{n\to\infty} \frac{s_{n+1} - s_n}{b_{n+1} - b_n} = L \in \mathbb{R}.$$

Our goal is to show that $\lim\limits_{n\to\infty} \frac{s_n}{b_n} = L$. In order to do this, we need to show that for any given $\varepsilon > 0$, we have

$$\limsup \frac{s_n}{b_n} \leq L + \varepsilon \tag{1}$$

$$\liminf \frac{s_n}{b_n} \geq L + \varepsilon \tag{2}$$

To this end, let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \frac{s_{n+1}-s_n}{b_{n+1}-b_n} = L$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, we have

$$\left|\frac{s_{n+1} - s_n}{b_{n+1} - b_n} - L\right| < \varepsilon$$

which can be written in the following way:

$$(L - \varepsilon)(b_{n+1} - b_n) < s_{n+1} - s_n < (L + \varepsilon)(b_{n+1} - b_n). \tag{*}$$

Now, fix an $\hat{N} > N$ denoted by $\hat{N} = N + 2$. Then observe that

$$s_n = [(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{\hat{N}} - s_{\hat{N}-1})] + s_{\hat{N}-1}.$$

Applying (*) to the equation above gives us

$$\begin{aligned}s_n &< (L + \varepsilon)[(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \cdots + (b_{\hat{N}} - b_{\hat{N}-1})] + s_{\hat{N}-1} \\ &= (L + \varepsilon)[b_n - b_{\hat{N}-1}] + s_{\hat{N}-1}\end{aligned}$$

Multiplying by b_n on both sides, we get

$$\begin{aligned}\frac{s_n}{b_n} &< (L+\varepsilon)\Big[\frac{b_n - b_{\hat{N}-1}}{b_n}\Big] + \frac{s_{\hat{N}-1}}{b_n} \\ &= (L+\varepsilon) + \frac{s_{\hat{N}-1} - b_{\hat{N}-1}(L+\varepsilon)}{b_n}\end{aligned}$$

Since we have (b_n) is a strictly increasing sequence that diverges to ∞, we must have $\frac{1}{b_n} \to 0$. Since

$$\alpha = s_{\hat{N}-1} - b_{\hat{N}-1}(L+\varepsilon)$$

is a fixed quantity (because $\hat{N}$ is fixed), we have that $\frac{\alpha}{b_n} \to 0$ by the algebraic limit theorem. By the order limit theorem, we can see that

$$\begin{aligned}\lim_{n\to\infty} \sup \frac{a_n}{b_n} &\leq \lim_{n\to\infty} \Big[(L+\varepsilon) + \frac{\alpha}{b_n}\Big] \\ &= (L+\varepsilon) + \lim_{n\to\infty} \frac{\alpha}{b_n} \qquad \text{(ALT)} \\ &= L + \varepsilon + 0 \\ &= L + \varepsilon.\end{aligned}$$

which establishes (1).

We can apply an analogous process to establish (2). By applying (*), we can see that

$$\frac{s_n}{b_n} > (L-\varepsilon) + \frac{s_{\hat{N}-1} - b_{\hat{N}-1}(L-\varepsilon)}{b_n}.$$

Now, set

$$\beta = s_{\hat{N}-1} - b_{\hat{N}-1}(L-\varepsilon).$$

By the order limit theorem, we see that

$$\begin{aligned}\lim_{n\to\infty} \inf \frac{s_n}{b_n} &\geq \lim_{n\to\infty} \Big[(L-\varepsilon) + \frac{\beta}{b_n}\Big] \\ &= (L-\varepsilon) + \lim_{n\to\infty} \frac{\beta}{b_n} \qquad \text{(ALT)} \\ &= L - \varepsilon + 0 \\ &= L - \varepsilon\end{aligned}$$

which establishes (2). Since $\varepsilon > 0$ is arbitrary, (1) and (2) imply that

$$\limsup \frac{s_n}{b_n} \leq L \quad \text{and} \quad \liminf \frac{s_n}{b_n} \geq L.$$

Thus, we can see that

$$\limsup \frac{s_n}{b_n} = \liminf \frac{s_n}{b_n} = L.$$

This tells us that

$$\lim_{n\to\infty} \frac{s_n}{b_n} = L.$$

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