0.0.1 Topics

(1) Every neigborhood is an open set.

(2) If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

(3) Theorem: E is open $\iff E^C$ is closed.

(4) Theorem: arbitrary union of open sets is open, Finite intersection of open sets is open.

(5) Theorem: $\overline{E}$ is the smallest closed subset of X that contains E.

Theorem. Let (X, d) be a metric space and let $p \in X$ and $\varepsilon > 0$. Every neighborhood is an open set; that is, $N_\varepsilon(p)$ is an open set.

Proof. Our goal is to show that every point of $N_\varepsilon(p)$ is an interior point of $N_\varepsilon(p)$. Let $q \in N_\varepsilon(p)$. We need to show that there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$. Let $\delta = \frac{\varepsilon - d(p,q)}{2}$. We claim that $N_\delta(q)$ is a subset of $N_\varepsilon(p)$. Indeed, if $x \in N_\delta(q)$, then

$$d(q, x) < \delta \implies d(q, x) < \varepsilon - d(p, q)$$

and so

$$d(p, q) + d(q, x) < \varepsilon \iff d(p, x) < \varepsilon. \qquad \text{(triangle inequality)}$$

Thus, $x \in N_\varepsilon(p)$.

■

Theorem. Let (X, d) be a metric space and $E \subseteq X$. If $p \in E'$, then every neighborhood of p contains infinitely many points of E

Proof. Suppose for sake of contradiction that there exists a neighborhood of p that contains finitely many points of E; that is,

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \cap E \text{ is a finite set.}$$

Since $N_\varepsilon(p) \cap (E \setminus \{p\}) \subseteq N_\varepsilon(p) \cap E$, we can immediately conclude that $N_\varepsilon(p) \cap (E \setminus \{p\})$ is finite also. Furthermore, $N_\varepsilon(p) \cap (E \setminus \{p\})$ is nonempty since $p \in E'$. Let us denote the elements of $N_\varepsilon(p) \cap (E \setminus \{p\})$ by $x_1, \ldots, x_n$. Our goal is to find a $\delta > 0$ such that the neighborhood with radius $\delta > 0$, we will not contain any of the finite points we introduced. For each $i \in \{1, \ldots, n\}$, $d(p, x_i) > 0$,

$$\delta = \min\{d(p, x_i) : i \in 1, \ldots, n\} > 0.$$

Clearly, $N_{\delta/2}(p) \cap (E \setminus \{p\}) = \emptyset$. But this contradicts our assumption that p is NOT a limit point of E. ■

Corollary. A finite set has no limit points; that is, if E is finite, then $E' = \emptyset$.

Proof. This is just the contrapositive of the theorem above. ■

Theorem. Let (X, d) be a metric space and $E \subseteq X$. We have that E is open if and only if E^c closed.

Proof. ($\Longrightarrow$) Assume that E is open. We want to show that E^c is closed; that is, every limit point of E^c is contained in E^c. Let p be a limit point of (E^c). Assume for sake of contradiction that $p \notin E^c$. Then $p \in E$. Since E is open, p is an interior point of E. Thus, there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$; that is, there exists $\delta > 0$ such that $N_\delta(p) \cap E^c = \emptyset$. Furthermore, we have that

$$N_\delta(p) \cap (E^c \setminus \{p\}) = \emptyset.$$

But this tells us that p is not a limit point which is a contradiction.

($\Longleftarrow$) Assume that E^c is closed. We want to show that E is open; that is, every $x \in E$ is an interior point. Let $p \in E$. Assume for sake of contradiction that $p \notin E^\circ$. Then for all $\delta > 0$, $N_\delta(p) \nsubseteq E$. Hence, for all $N_\varepsilon(p) \cap E^c \neq \emptyset$. Therefore,

$$\forall \delta > 0 \; N_\delta(p) \cap (E^c \setminus \{p\}) \neq \emptyset.$$

That is, p is a limit point of E^c. But by assumption, E^c is closed. Thus, $p \in E^c$. But this contradicts the assumption that $p \in E$. ■

Theorem. Let (X, d) be a metric space. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets where Λ is an index set (can be finite or infinite). Then

$$\bigcup_{\alpha \in \Lambda} A_\alpha$$

is an open set.

Proof. Our goal is to show that every point of $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ is an interior point. Let $p \in A$. Thus, there exists $\alpha \in \Lambda$ such that $p \in A_\alpha$. Since A_α is open, so there exists $\delta > 0$ such that $N_\delta(p) \subseteq A_\alpha$. But note that

$$A_{\alpha_0} \subseteq A.$$

Hence,

$$N_\delta(p) \subseteq A$$

and thus p is an interior point of A.

■

Theorem. Let $A_1, \ldots, A_n$ be open sets in the metric space (X, d). Then

$$\bigcap_{k=1}^{n} A_k \text{ is open.}$$

Proof. Our goal is to show that every point of $\bigcap_{k=1}^n A_k$ is an interior point of $\bigcap_{k=1}^n A_k$. Let $p \in \bigcap_{k=1}^n A_k$. Then for all $1 \leq k \leq n$ such that $p \in A_k$. Since A_k is open for all k, we know that for all $1 \leq k \leq n$, there exists $\delta_k > 0$ such that $N_{\delta_k}(p) \subseteq A_k$. Let $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. Then we have that

$$N_\delta(p) \subseteq N_{\delta_k}(p) \subseteq A_k.$$

Consequently, we have

$$N_\delta(p) \subseteq \bigcap_{k=1}^{n} A_k.$$

Hence, p is an interior point of the intersection $\bigcap_{k=1}^n A_k$. ■

Theorem. Let (X, d) be a metric space and $E \subseteq X$.

(1) $\overline{E}$ is a closed set.

(2) E is closed if and only if $E = \overline{E}$.

(3) If $E \subseteq F$ and F is closed, then $\overline{E} \subseteq F$.

Proof. Our goal is to show that $(\overline{E})^c$ is open. We need to show that every point of $(\overline{E})^c$ is an interior point of $(\overline{E})^c$. Let $p \in (\overline{E})^c$. We have

$$\begin{aligned} p \in (\overline{E})^c &\implies p \notin \overline{E} \\ &\implies p \notin (E \cup E') \\ &\implies p \notin E \ \wedge \ p \notin E'. \end{aligned}$$

Note that

$$\begin{aligned} p \notin E' &\implies \exists \varepsilon > 0 \ N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset \\ &\implies \exists \varepsilon > 0 \ N_\varepsilon(p) \cap E = \emptyset. \end{aligned} \tag{1}$$

In what follows, we will show that $N_\varepsilon(p) \cap E' = \emptyset$. So, we have

$$\begin{aligned} &N_\varepsilon(p) \cap (E \cup E') = \emptyset \\ &\implies N_\varepsilon(p) \cap \overline{E} = \emptyset \\ &\implies N_\varepsilon(p) \subseteq (\overline{E})^c. \end{aligned}$$

Thus, we have that p is an interior point of $(\overline{E})^c$. It remains to show that $N_\varepsilon(p) \cap E' = \emptyset$. Assume for sake of contradiction that $N_\varepsilon(p) \cap E' \neq \emptyset$. Let $q \in N_\varepsilon(p) \cap E'$. Then we have $q \in N_\varepsilon(p)$ and $q \in E'$. Hence, there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$ and that $N_\delta(q) \cap (E \setminus \{q\}) \neq \emptyset$, respectively. But note that since $N_q(p) \subseteq N_\varepsilon(p)$ and $E \setminus \{q\} \subseteq E$ implies that

$$N_\varepsilon(p) \cap E \neq \emptyset$$

which contradicts (1). ■

Definition (Metric Subspace). Let (X, d) be a metric space and let Y be a nonempty subset of X. Then $(Y, d|_{Y \times Y})$ is called a **metric subspace** of (X, d).

Remark. Consider the metric subspace $(Y, d|_{Y \times Y}$ of (X, d). Let $p \in Y$ and $\varepsilon > 0$. Then we define

$$\begin{aligned} N_\varepsilon^Y(p) = \{y \in Y : d^Y(y, p) < \varepsilon\} &= \{y \in Y : d(y, p) < \varepsilon\} \\ &= N_\varepsilon(p) \cap Y. \end{aligned}$$