0.1 Outer Measure on $\mathbb{R}$

0.1.1 Motivation and Definition of Outer Measure

- This section focuses on developing the notion of assigning a "size" to subintervals when it comes to more complicated unions of open intervals.
- The hope is to create an integration theory that will cover a broader class of functions than covered by Riemann integration.
- We start off by giving a definition of the length of an open interval.

Definition 0.1.1 (Length of Open Interval; $\ell(I)$). The **length** $\ell(I)$ of an open interval I is defined bt

$$\ell(I) = \begin{cases} b-a & \text{if } I = (a,b) \text{ for some } a,b \in \mathbb{R} \text{ with } a < b, \\ 0 & \text{if } I = \emptyset, \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R}, \\ \infty & \text{if } I = (-\infty, \infty). \end{cases}$$

Suppose we have a subset $A \subset \mathbb{R}$. Then the size of A should be the sum of the lengths of a sequence of open intervals whose union contains A. The infimum of all possible sums gives the definition of the size of A, denoted $|A|$ which we call the **outer measure** of A.

Definition 0.1.2 (Outer Measure; $|A|$). The **outer measure** $|A|$ of a set $A \subset \mathbb{R}$ is defined by

$$|A| = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

- Notice that the definition of outer measure involves an infinite sum.
- Recall that the infinite sum converges if the sequence of partial sums converges.
- If one of the elements in the sequence is ∞ (that is, $t_k = \infty$) for some k, then the infinite sum is ∞.
- Otherwise, the infinite sum $\sum_{k=1}^{\infty t_k}$ is defined to be the limit of the increasing sequence of partial sums where
$$\sum_{k=1}^{\infty} t_k = \lim_{n \to \infty} \sum_{k=1}^{n} t_k.$$

0.1.2 Properties of Outer Measure

Proposition 0.1.1 (Countable sets have outer measure 0). Every countable subset of $\mathbb{R}$ has outer measure 0.

Proof. Suppose $A = \{a_1, a_2, \ldots\}$ is a countable subset of $\mathbb{R}$. Let $\varepsilon > 0$. For $k \in \mathbb{Z}^+$, let

$$I_k = \left(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\right).$$

Then $I_1, I_2, \dots$ is a sequence of open intervals whose union contains A. Since

$$\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon,$$

we have $|A| \leq 2\varepsilon$. Since ε is an arbitrary positive number, we must have $|A| = 0$. ■

- Note that $\mathbb{Q}$ is a countable set, and thus contains a measure of 0.

Proposition 0.1.2 (Outer Measure Preserves Order)**.** Suppose A and B are subsets of $\mathbb{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Proof. Suppose $I_1, I_2, \dots$ is a sequence of open intervals such that

$$B \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Since $A \subseteq B$, we can see that A is also contained within

$$\bigcup_{k=1}^{\infty} I_k.$$

Thus, we have

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k).$$

If we take the infimum over all the sequences of open intervals whose union contains B, we see that $|A| \leq |B|$. ■

Definition 0.1.3 (Translation; $t + A$)**.** If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then the translation $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

- If $t > 0$, then translating A by adding t moves A by $|t|$ units to the right.
- If $t < 0$, then translating A by adding t moves A to the left by $|t|$ units.
- In a similar manner to how functions do not change their shape when translated either horizontally or vertically, intervals do not change their shape/length when moved either left or right on the number line.
- By standard convention, we take $t + (-\infty) = -\infty$ and $t + \infty = \infty$.

Proposition 0.1.3 (Outer Measure is Translation Invariant)**.** Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$. Then $|t + A| = |A|$.

Proof. Let $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. Suppose $I_1, I_2, \dots$ is a sequence of open intervals such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Then $t+I_1, t+I_2, \ldots$ is a sequence of open intervals such that

$$t+A \subseteq \bigcup_{k=1}^{\infty} t+I_k.$$

Since translation by t of intervals I_k of A does not change, we have that

$$\ell(t+I_k) = \ell(I_k).$$

So, we see that

$$|t+A| \leq \sum_{k=1}^{\infty} \ell(t+I_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

But this tells us that the union of the sequence of intervals I_k of A contains $t+A$. So, taking the infimum over all such sequences, we have $|t+A| \leq |A|$.

Now, observe that $A = -t+(t+A)$ for $t \in \mathbb{R}$. Then we see that

$$|A| = |-t+(t+A)| \leq |t+A|.$$

Thus, $|A| = |t+A|$. ■

Proposition 0.1.4 (Countable Subadditivity of Outer Measure). Suppose $A_1, A_2, \ldots, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|.$$

Proof. If $|a_k| = \infty$ for some $k \in \mathbb{Z}^+$, then the inequality above holds. Thus, assume $|A_k| < \infty$ for all $k \in \mathbb{Z}^+$.

Let $\varepsilon > 0$. Using a lemma found in introductory real analysis textbooks, we can write that for each $k \in \mathbb{Z}^+$, let $I_{1,k}, I_{2,k}, \ldots$ be a sequence of open intervals whose union contains A_k such that

$$\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq \frac{\varepsilon}{2^k} + |A_k|.$$

Taking the infinite sum of both sides of this inequality, we get

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq \varepsilon + \sum_{k=1}^{\infty} |A_k|.$$

Note that

$$\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

by using the geometric sum formula. Consider the doubly indexed collection of open intervals $\{I_{j,k} : j,k \in \mathbb{Z}^+\}$ into a sequence of open intervals such that

$$\bigcup_{k=1}^{\infty} \ell(I_{j,k}) \subseteq \bigcup \{I_{j,k} : j,k \in \mathbb{Z}^+\}.$$

Suppose in step k, starting with $k=2$, we adjoin the $k-1$ intervals whose indices add up to k; that is,

$$\underbrace{I_{1,1}}_{2},\underbrace{I_{1,2},I_{2,1}}_{3},\underbrace{I_{1,3},I_{2,2}}_{4},\underbrace{I_{1,5},1_{2,4},I_{3,3},I_{4,2},I_{5,1}}_{6}\dots$$

Thus, we must have

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \le \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\ell(I_{j,k}) \le \varepsilon + \sum_{k=1}^{\infty}|A_k|.$$

Since $\varepsilon > 0$ is arbitrary, we must have that

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \le \sum_{k=1}^{\infty}|A_k|.$$

■

0.1.3 Outer Measure of Closed Bounded Interval

The next property we will prove is that the length of each closed interval $[a,b]$ with $a<b$ and $a,b\in\mathbb{R}$ is equal to $[a,b]=b-a$. If $\varepsilon>0$, we see that $(a-\varepsilon,b+\varepsilon),\emptyset,\emptyset,\dots$ is a sequence of open intervals whose union contains $[a,b]$. Then $|[a,b]|\le b-a+2\varepsilon$. Since this inequaltiy holds for all $\varepsilon>0$, we must have that

$$|[a,b]|\le b-a.$$

Definition 0.1.4 (Open Cover). Suppose $A\subseteq\mathbb{R}$.

- A collection $\mathcal{C}$ of open subsets of $\mathbb{R}$ is called an **open cover** of A if A is contained in the union of all the sets in $\mathbb{C}$.
- An open cover $\mathcal{C}$ of A is said to have a **finite subcover** if A is contained in the union of some finite list of sets in $\mathcal{C}$

Proof. ■

Proposition 0.1.5 (Heine-Borel Theorem). Every open cover of a closed bounded subset of $\mathbb{R}$ has a finite subcover.

Proof. Suppose F is a closed bounded subset of $\mathbb{R}$ and $\mathcal{C}$ is an open cover of F (Note that $\mathcal{C}$ is the collection of open subsets in $\mathbb{R}$).

First, suppose $F=[a,b]$ for some $a,b\in\mathbb{R}$ with $a<b$. Thus, $\mathcal{C}$ is an open cover of $[a,b]$. Let

$$D=\{d\in[a,b]:[a,d] \text{ has a finite subcover from } \mathcal{C}\}.$$

Let's show that this set is nonempty first. Note that $a\in D$ since $a\in G$ for some $G\in\mathcal{C}$. Thus, $D\neq\emptyset$ and so we can say that a supremum exists for D (clearly, D is bounded above). Let

$$s=\sup D.$$

So, $s\in[a,b]$. Hence, there exists an open set $G\in\mathcal{C}$ such that $s\in G$. Let $\delta>0$ be such that $(s-\delta,s+\delta)\subset G$. Since $s=\sup D$, there exists $d\in(s-\delta,s]$ and $n\in\mathbb{Z}^+$ and $G_1,G_2,\dots,G_n\in\mathcal{C}$ such that

$$[a,d]\subset\bigcup_{k=1}^{n} G_k.$$

Now, let $d' \in [s, s+\delta)$ be arbitrary. Then

$$[a, d'] \subset G \cup \bigcup_{k=1}^{n} G_k.$$

Since $[a, d']$ contains a finite subcover from $\mathcal{C}$, we get that $d' \in D$ for all $d' \in [s, s+\delta) \cap [a, b]$. Furthermore, this implies that $b = s$. So, with $d' = b$, $[a, b]$ must contain a finite subcover from $\mathcal{C}$, completing the proof in the case that $F = [a, b]$.

Now, suppose F is an arbitrary closed bounded subset of $\mathbb{R}$ and that $\mathcal{C}$ is an open cover of F. Let $a, b \in \mathbb{R}$ be such that $F \subseteq [a, b]$. Now $\mathcal{C} \cup \{\mathbb{R} \setminus F\}$ is an open cover of $\mathbb{R}$ and hence it is an open cover of $[a, b]$. By our first case, there exists $G_1, \ldots, G_n \in \mathcal{C}$ such that

$$[a, b] \subset \left(\bigcup_{k=1}^{n} G_k \right) \cup (\mathbb{R} \setminus F).$$

Thus,

$$F \subset \bigcup_{k=1}^{n} G_k,$$

completing the proof. ■

Proposition 0.1.6 (Outer Measure of a Closed Interval). Suppose $a, b \in \mathbb{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proof. The proof of $|[a, b]| \leq b - a$ can be found here.

Now, we want to show that $|[a, b]| \geq b - a$. Suppose $I_1, I_2, \ldots$ is a sequence of open intervals such that $[a, b] \subset \bigcup_{k=1}^{\infty} I_k$. Using the Heine-Borel Theorem, there exists $n \in \mathbb{Z}^+$ such that

$$[a, b] \subset \bigcup_{k=1}^{n} I_k.$$

We will now show, by induction on n, that the inclusion above implies that

$$\sum_{k=1}^{n} \ell(I_k) \geq b - a. \tag{1}$$

Proving this will show that

$$\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) \geq b - a, \tag{2}$$

completing the proof that $|[a, b]| \geq b - a$. If $n = 1$, we have that (1) implies (2). Now, suppose $n > 1$. Then (1) implies (2) for all choices of $a, b \in \mathbb{R}$ with $a < b$. Suppose $I_1, \ldots, I_n, I_{n+1}$ are open intervals such that

$$[a, b] \subset \bigcup_{k=1}^{n+1} I_k.$$

Thus, b must be in at least one of these intervals. By relabeling, we can assume that $b \in I_{n+1}$. Suppose $I_{n+1} = (c, d)$. If $c \leq a$, then $\ell(I_{n+1}) \geq b - a$ and there is nothing further to prove; thus, assume that $a < c < b < d$. Thus, we can assume that

$$[a, c] \subset \bigcup_{k=1}^{n} I_k.$$

Using our induction hypothesis, we have

$$\sum_{k=1}^{n} \ell(I_k) \geq c - a.$$

Since $d > b$ and $c \in [a, b]$, we can see that $\ell(I_{n+1}) = d - c$. Thus, we have that

$$\begin{aligned} \sum_{k=1}^{n+1} \ell(I_k) &= \ell(I_{n+1}) + \sum_{k=1}^{n} \ell(I_k) \\ &\geq (c - a) + (d - c) \\ &= d - a \\ &\geq b - a, \end{aligned}$$

completing the proof. ∎

Proposition 0.1.7 (Nontrivial Intervals Are Uncountable). Every interval in $\mathbb{R}$ that contains at least two distinct elements is uncountable.

Proof. ∎

0.1.4 Outer Measure is Not Additive

Let us state a result that will be used in the upcoming proof of this result.

The result states that a nonempty intersection of two equivalence classes implies that the two classes are equal to each other.

Proposition 0.1.8. If $a, b \in [-1, 1]$ and $\tilde{a} \cap \tilde{b} \neq \emptyset$ where

$$\tilde{a} = \{c \in [-1, 1] : a - c \in \mathbb{Q}\},$$

($\tilde{b}$ is defined similarly) then $\tilde{a} = \tilde{b}$.

Proof. Let $d \in \tilde{a} \cap \tilde{b}$. Then $d \in \tilde{a}$ and $d \in \tilde{b}$. By definition of $\tilde{a}$ and $\tilde{b}$, we have $a - d \in \mathbb{Q}$ and $b - d \in \mathbb{Q}$, respectively. Since $\mathbb{Q}$ is a field, we can use subtraction to get

$$a - d - (b - d) = a - b.$$

This implies that $a - b \in \mathbb{Q}$ since $a - d \in \mathbb{Q}$ and $b - d \in \mathbb{Q}$. Thus, we have

$$a - c = (a - b) + (b - c)$$

is a rational number if and only $b - c \in \mathbb{Q}$. But, $\tilde{a} \cap \tilde{b} \neq \emptyset$. So, we must have

$$a - c = b - c \Rightarrow \tilde{a} = \tilde{b}.$$

∎

Proposition 0.1.9 (Nonadditivity of Outer Measure)**.** There exists disjoint subsets of A and B of $\mathbb{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

Proof. Continuing from the proof before this result, we can see that $a \in \tilde{a}$ for each $a \in [-1, 1]$. Thus, we have

$$[-1, 1] = \bigcup_{a \in [-1,1]} \tilde{a}.$$

■

0.1.5 Exercise 2A

1. Prove that if A and B are subsets of $\mathbb{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

   Proof. Let A and B be subsets of $\mathbb{R}$ and $|B| = 0$. Hence, by finite subadditivity, we must have

   $$|A \cup B| \leq |A| + |B| = |A|.$$

   Since $A \subseteq A \cup B$, we can use result 2.5 to conclude that

   $$|A| \leq |A \cup B|.$$

   Thus, we can conclude that $|A \cup B| = |A|$. ■

2. Suppose $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t||A|$.

   Proof. Let $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. We proceed by showing the following two inequalities:

   $$|tA| \leq |t||A| \text{ and } |tA| \geq |t||A|.$$

   Let $I_1, I_2, \ldots$ be a sequence of open intervals such that

   $$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

   Thus, we have

   $$|A| \leq \sum_{k=1}^{\infty} \ell(I_k) \tag{1}$$

   Now, let $tI_1, tI_2, \ldots$ be a sequence of open interval such that

   $$tA \subseteq \bigcup_{k=1}^{\infty} tI_k.$$

   Thus, we have

   $$|tA| \leq \sum_{k=1}^{\infty} \ell(tI_k). \tag{2}$$

   Noting that each the length of each subinterval of tA is just the length of each subinterval of A but scaled by $|t|$ where $t > 0$, we must have that $\ell(tI_k) = |t|\ell(I_k)$. To show the first inequality, we can multiply (1) by $|t|$ to get

$$|t||A| \leq |t| \sum_{k=1}^{\infty} |t|\ell(I_k) = \sum_{k=1}^{\infty} \ell(tI_k).$$

Then taking the infimum over all sequences of open intervals containing tA, we have that $|t||A| \leq |tA|$.

Now, let $\varepsilon > 0$. Then we have

$$\sum_{k=1}^{\infty} \ell(tI_k) \leq |tA| + \varepsilon.$$

By (1), we get that

$$\sum_{k=1}^{\infty} \ell(tI_k) = |t| \sum_{k=1}^{\infty} \ell(I_k) \geq |t||A|.$$

Since $\varepsilon > 0$ is arbitrary, we must have $|t||A| \leq |tA|$. Thus, we conclude that $|tA| = |t||A|$.

■

3. Prove that if $A, B \subset \mathbb{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

Proof. Let $A, B \subset \mathbb{R}$ and $|A| < \infty$. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be a sequence of open intervals such that

$$B \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}_k.$$

Thus, we see that

$$|B| \leq \sum_{k=1}^{\infty} \ell(\mathcal{I}_k).$$

Similarly, let $I_1, I_2, \ldots$ and $\mathcal{J}_1, \mathcal{J}_2, \ldots$ be sequences of open intervals for $B \setminus A$ and A, respectively. Then we have

$$B \setminus A \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and } A \subseteq \bigcup_{k=1}^{\infty} \mathcal{J}_k.$$

Observe that

$$\sum_{k=1}^{\infty} \ell(I_k) + \sum_{k=1}^{\infty} \ell(\mathcal{J}_k) \geq \sum_{k=1}^{\infty} \ell(\mathcal{I}_k) \geq |B|.$$

Now, taking the infimum over all sequences I_k and $\mathcal{J}_k$ for $k \in \mathbb{N}$, we must have

$$|B| \leq |B \setminus A| + |A|$$

which can be rewritten to get our result that

$$|B \setminus A| \geq |B| - |A|.$$

■