Math 241A Final

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May 16, 2025

Problem 1. (a) Let (X, d) be a metric space (assume that $X \neq \emptyset$) and let $\alpha > 0$. Define $d_\alpha : X \times X \to \mathbb{R}$ by

$$d_\alpha(x, x') = \frac{d(x, x')}{1 + \alpha d(x, x')}.$$

Prove that d_α defines a metric on X.

(b) Let A be a non-empty set. Let $X = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = \{(a_1, a_2, \dots, a_n) : a_i \in A,\ 1 \leq i \leq n\}$. Does d define a metric on X. If it does, provide a detailed proof. If it does not, justify in detail why it does not. Here, $\#A$ denotes the number of elements in the set A.

Proof. (a) Clearly, we see that $d_\alpha(x, y) \geq 0$ since $d(x, y) \geq 0$ for all $x, y \in X$ and $\alpha > 0$.

First, we will show that $d_\alpha(x, y) = 0$ if and only if $x = y$. Hence,

$$\begin{aligned} d_\alpha(x, y) = 0 &\iff \frac{d(x, y)}{1 + \alpha d(x, y)} = 0 \\ &\iff d(x, y) = 0 \\ &\iff x = y && (d \text{ is a metric on } X) \end{aligned}$$

Hence, we see that property (1) of metrics is satisfied.

Second, we will show that $d_\alpha(x, y) = d_\alpha(y, x)$ for all $x, y \in X$. We have

$$d_\alpha(x, y) = \frac{d(x, y)}{1 + \alpha d(x, y)} = \frac{d(y, x)}{1 + \alpha d(y, x)} = d_\alpha(y, x).$$

Hence, the symmetric property of metrics is satisfied.

Lastly, we show that d satisfied the triangle inequality. Indeed, we will first show that the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{t}{1 + \alpha t}$$

is an increasing function. Suppose $s \leq t$ with $s, t \in [0, \infty)$. Our goal is to show that $f(s) \leq f(t)$. Indeed, we have

$$\begin{aligned} s + \alpha st \leq t + \alpha st &\iff s(1 + \alpha t) \leq t(1 + \alpha s) \\ &\iff \frac{s}{1 + \alpha s} \leq \frac{t}{1 + \alpha t} \\ &\iff f(s) \leq f(t). \end{aligned}$$

Hence, f is an increasing function. We shall use this property to prove that d_α satisfies the triangle inequality. Indeed, let $x, y, z \in X$. We will consider three cases:

(1) $d(x, y) \leq d(x, z)$

(2) $d(x, y) \leq d(z, y)$

(3) $d(x, y) > d(x, z)$ and $d(x, z) > d(z, y)$

Starting with case (1), we can use the monotonicity of f to get

$$\begin{aligned} d_\alpha(x,y) = \frac{d(x,y)}{1+\alpha d(x,y)} \leq \frac{d(x,z)}{1+\alpha d(x,z)} &= d_\alpha(x,z) \\ &\leq d_\alpha(x,z) + d_\alpha(z,y) \end{aligned}$$

Similarly, if $d(x,y) \leq d(z,y)$, then

$$\begin{aligned} d_\alpha(x,y) = \frac{d(x,y)}{1+\alpha d(x,y)} \leq \frac{d(z,y)}{1+\alpha d(z,y)} &= d_\alpha(z,y) \\ &\leq d_\alpha(x,z) + d_\alpha(z,y). \end{aligned}$$

Finally, assume $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$. Indeed, we have

$$\begin{aligned} d_\alpha(x,y) = \frac{d(x,y)}{1+\alpha d(x,y)} &\leq \frac{d(x,z)+d(z,y)}{1+\alpha d(x,y)} \\ &= \frac{d(x,z)}{1+\alpha d(x,y)} + \frac{d(z,y)}{1+\alpha d(x,y)} \\ &\leq \frac{d(x,z)}{1+\alpha d(x,z)} + \frac{d(z,y)}{1+\alpha d(z,y)} \\ &= d_\alpha(x,z) + d_\alpha(z,y). \end{aligned}$$

Hence, we conclude that d_α satisfies the triangle inequality.

(b) We claim that d defines a metric on X. Clearly, we can see that for any $\mathbf{a}$, $\mathbf{b} \in X$, we have $d(\mathbf{a},\mathbf{b}) \geq 0$.

Starting with the first property, we can see that $\mathbf{a} = \mathbf{b}$ holds true if and only if $a_i = b_i$ for all $1 \leq i \leq n$. This holds true if and only if

$$\#\{1 \leq i \leq n : a_i \neq b_i\} = 0 \Longleftrightarrow d(\mathbf{a},\mathbf{b}) = 0.$$

Next, we will show that d satisfies the symmetric property. Clearly, we have

$$d(\mathbf{a},\mathbf{b}) = \#\{1 \leq i \leq n : a_i \neq b_i\} = \#\{1 \leq i \leq n : b_i \neq a_i\} = d(\mathbf{b},\mathbf{a}).$$

Finally, we show that the d satisfies the triangle inequality. Denote the following sets

$$\begin{aligned} A &= \{1 \leq i \leq n : a_i \neq b_i\} \\ B &= \{1 \leq i \leq n : a_i \neq c_i\} \\ C &= \{1 \leq i \leq n : c_i \neq b_i\} \end{aligned}$$

Note that if $a_i \neq b_i$, then either $a_i \neq c_i$ or $c_i \neq b_i$. This tells us that $A \subseteq B \cup C$. Hence, we have

$$\#A \leq \#(B \cup C) = \#B + \#C \Longrightarrow \#A \leq \#B + \#C$$

and so

$$d(\mathbf{a},\mathbf{b}) \leq d(\mathbf{a},\mathbf{c}) + d(\mathbf{c},\mathbf{b}).$$

Hence, we conclude that d defines a metric on X.

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Problem 2. Let (X,d) be a metric space (assume that $X \neq \emptyset$). Let $\alpha > 0$ and let d_α be the metric on X defined in problem 1(a).

(a) Let (x_n) be a sequence in X and $x \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$. Justify whether the following holds: $\lim_{n\to\infty} d_\alpha(x_n,x) = 0$.

Proof. We claim that $\lim_{n\to\infty} d_\alpha(x_n,x)=0$ holds given $\lim_{n\to\infty} d(x_n,x)=0$. Indeed, we have

$$\begin{aligned}\lim_{n\to\infty} d_\alpha(x_n,x) &= \lim_{n\to\infty}\frac{d(x_n,x)}{1+\alpha d(x_n,x)}\\ &= \frac{\lim_{n\to\infty} d(x_n,x)}{1+\alpha\lim_{n\to\infty} d(x_n,x)}\\ &= \frac{0}{1+\alpha\cdot 0}\\ &= 0.\end{aligned}$$

■

(b) Let (x_n) be a sequence in X and $x\in X$ such that $\lim_{n\to\infty} d_\alpha(x_n,x)=0$. Justify whether the following holds: $\lim_{n\to\infty} d(x_n,x)=0$.

Solution. The limit $\lim_{n\to\infty} d(x_n,x)=0$. Define the function $f:[0,1)\to\infty$ by

$$f(x)=\frac{x}{1-\alpha x}.$$

Our first goal is to show that $f(x)\le 2x$ for all $x\in[0,1/2]$. Let $x\in[0,1/2]$. Then we have

$$\begin{aligned}0\le x\le\frac{1}{2} &\implies -\frac{1}{2}\le -x\le 0\\ &\implies -\frac{\alpha}{2}\le -\alpha x\le 0\\ &\implies \frac{2-\alpha}{2}\le 1-\alpha x\le 1\\ &\implies 1-\alpha x\ge\frac{2-\alpha}{2}\\ &\implies \frac{1}{1-\alpha x}\le\frac{2}{2-\alpha}\le 2\\ &\implies \frac{x}{1-\alpha x}\le 2x.\end{aligned}$$

One can prove easily that

$$d(x,y)=\frac{d_\alpha(x,y)}{1-\alpha d_\alpha(x,y)}.$$

Indeed, we have

$$\begin{aligned}d_\alpha(x,y)=\frac{d(x,y)}{1+\alpha d(x,y)} &\implies d_\alpha(x,y)+\alpha d(x,y)d_{\alpha(x,y}=d(x,y)\\ &\implies d_\alpha(x,y)=d(x,y)[1-\alpha d_\alpha(x,y)]\\ &\implies d(x,y)=\frac{d_\alpha(x,y)}{1-\alpha d_\alpha(x,y)}.\end{aligned}$$

We can see from our derivation above that $d_\alpha(x,y)\le\frac{1}{2}$. Hence, we have

$$0\le d(x_n,x)\le 2d_\alpha(x_n,x).$$

Now, assuming that $d_\alpha(x_n,x)\to 0$, we can apply the squeeze theorem on the inequality above to get

$$d(x_n,x)\to 0.$$

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(c) Let (x_n) be a sequence in X such that $\lim_{m,n\to\infty} d(x_n,x_m)=0$. Justify whether the following holds:

$\lim_{m,n\to\infty} d_\alpha(x_n,x_m)=0$.

Proof. We claim that $\lim_{m,n\to\infty} d_\alpha(x_n, x_m) = 0$ holds given $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. Indeed, using the Algebraic Limit Theorem, we have

$$\begin{aligned}
\lim_{m,n\to\infty} d_\alpha(x_n, x_m) &= \lim_{m,n\to\infty}\Big[\frac{d(x_n,x_m)}{1+\alpha d(x_n,x_m)}\Big] \\
&= \frac{\lim_{m,n\to\to\infty} d(x_n,x_m)}{1+\alpha \lim_{m,n\to\infty} d(x_n,x_m)} \\
&= \frac{0}{1+\alpha\cdot 0} \\
&= 0.
\end{aligned}$$

■

(d) Let (x_n) be a sequence in X such that $\lim_{m,n\to\infty} d_\alpha(x_n, x_m) = 0$.

Solution. Using the fact that $d(x,x') \leq 2d_\alpha(x,x')$ from part (b), we can see that if $d_\alpha(x_n,x_m) \to 0$ as $m,n\to\infty$, we have

$$d(x_n,x_m) \to 0 \text{ as } m,n\to\infty.$$

■

Problem 3. Let $X = C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

(a) Define $d : X\times X \to \mathbb{R}$ by

$$d(f,g) = \Big(\int_0^1 (f(t)-g(t))^2\, dt\Big)^{\frac{1}{2}}.$$

Prove that d defines a metric on X. **Hint:** Think about whhat does $\int_0^1 f(t)g(t)\, dt$ represent.

Proof. Note that it is clear that $d(f,g) \geq 0$ for any $f,g\in X$.

Suppose $d(f,g) = 0$. Then we have

$$\begin{aligned}
&\Big(\int_0^1 [f(t)-g(t)]^2\, dt\Big)^{1/2} = 0 \\
&\implies \int_0^1 [f(t)-g(t)]^2\, dt = 0.
\end{aligned}$$

Note that $f,g\in C[0,1]$ implies $f-g\in C[0,1]$ and that $(f-g)^2\in C[0,1]$. Since $(f-g)^2\geq 0$ on $[0,1]$, $(f-g)^2\in C[0,1]$, and $\int_0^1 [f(t)-g(t)]^2\, dt = 0$, we have

$$\begin{aligned}
&[f(t)-g(t)]^2 = 0 \\
&\implies f(t) = g(t) \quad \forall t\in[0,1].
\end{aligned}$$

Conversely, if $f(t) = g(t)$, then

$$\Big(\int_a^b [f(t)-g(t)]^2\, dt\Big)^{\frac{1}{2}} = 0 \implies d(f,g) = 0.$$

Next, we show symmetry of d. Indeed, we have

$$\begin{aligned}
d(f,g) &= \Big(\int_0^1 (f(t)-g(t))^2\, dt\Big)^{1/2} \\
&= \Big(\int_0^1 (g(t)-f(t))^2\, dt\Big)^{1/2} \\
&= d(g,f) \quad \forall f,g\in C[0,1].
\end{aligned}$$

Thus, we have $d(f,g) = d(g,f)$.

Finally, we show that d satisfies the triangle inequality. Let $f, g, h \in C[0,1]$. We have

$$\begin{aligned}
d^2(f,g) &= \int_0^1 (f(t)-g(t))^2 \, dt \\
&= \int_0^1 |f(t)-g(t)|^2 \, dt \\
&= \int_0^1 |f(t)-h(t)+h(t)-g(t)|^2 \, dt \\
&\leq \int_0^1 (|f(t)-h(t)|+|h(t)-g(t)|)^2 \, dt \\
&= \int_0^1 |f(t)-h(t)|^2 \, dt + 2\int_0^1 |f(t)-h(t)||h(t)-g(t)| \, dt + \int_0^1 |h(t)-g(t)|^2 \, dt.
\end{aligned}$$

Now, consider the middle term and notice, by applying the Cauchy-Schwarz Inequality, we have

$$2\int_0^1 |f(t)-h(t)||h(t)-g(t)| \, dt \leq 2\Big(\int_0^1 |f(t)-h(t)|^2 \, dt\Big)^{1/2}\Big(\int_0^1 |h(t)-g(t)|^2 \, dt\Big)^{1/2}.$$

Hence, we have

$$\begin{aligned}
d^2(f,g) &\leq \int_0^1 |f(t)-h(t)|^2 \, dt + 2\Big(\int_0^1 |f(t)-h(t)|^2 \, dt\Big)^{1/2}\Big(\int_0^1 |h(t)-g(t)|^2 \, dt\Big)^{1/2} \\
&\quad + \int_0^1 |h(t)-g(t)|^2 \, dt \\
&= \bigg(\Big(\int_0^1 |f(t)-h(t)|^2 \, dt\Big)^{1/2} + \Big(\int_0^1 |h(t)-g(t)|^2 \, dt\Big)^{1/2}\bigg)^2 \\
&= \bigg(\Big(\int_0^1 (f(t)-h(t))^2 \, dt\Big)^{1/2} + \Big(\int_0^1 (h(t)-g(t))^2 \, dt\Big)^{1/2}\bigg)^2 \\
&= \Big(d(f,h)+d(h,g)\Big)^2.
\end{aligned}$$

Now, we obtain

$$d(f,g) \leq d(f,h) + d(h,g).$$

We conclude that d is a metric on X. ■

(b) Consider the sequence (f_n) in X defined by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1-1/n \\ \sqrt{n(t-(1-\frac{1}{n}))} & \text{if } 1-1/n \leq t \leq 1. \end{cases}$$

Prove that (f_n) is a Cauchy sequence in X.

Proof. Our goal is to show that $d(f_n, f_m) \to 0$ as $m, n \to \infty$. From the hint, we have

$$\begin{aligned}
&(f_n - f_m)^2 = f_n^2 - 2f_n f_m + f_m^2 \leq f_n^2 + f_m^2 \\
&\implies \int_0^1 (f_n - f_m)^2 \, dt \leq \int_0^1 f_n^2 \, dt + \int_0^1 f_m^2 \, dt \\
&\implies d^2(f_n, f_m) \leq \int_0^1 f_n^2 \, dt + \int_0^1 f_m^2 \, dt.
\end{aligned}$$

Next, we compute the first term on the right-hand side of the inequality above. Computing the second term will follow analogously. Note that for any $t \in [0, 1-1/n]$, we have $f_n(t) = 0$. Hence,

$$\int_0^{1-1/n} f_n^2(t) \, dt = 0.$$

Now, we have

$$\begin{aligned}\int_0^1 f_n\,dt &= \int_0^{1-1/n} f_n^2\,dt + \int_{1-1/n}^1 f_n^2\,dt \\ &= 0 + \int_{1-1/n}^1 f_n^2\,dt \\ &= \int_{1-1/n}^1 n\left(t - \left(1-\frac{1}{n}\right)\right)dt \\ &= \int_{1-1/n}^1 [nt - n + 1]\,dt \\ &= \frac{n}{2}t^2 - nt + t\Big]_{1-1/n}^1 \\ &= \frac{n}{2} - n + 1 - \left(\frac{n}{2}(1-1/n)^2 - (n-1)(1-1/n)\right) \\ &= \frac{1}{n} - \frac{1}{2n} \\ &= \frac{1}{2n}.\end{aligned}$$

Similarly, we have

$$\int_0^1 f_m\,dt = \frac{1}{2m}.$$

So, we see that

$$0 \leq d^2(f_n, f_m) \leq \int_0^1 f_n^2\,dt + \int_0^1 f_m^2\,dt = \frac{1}{2n} + \frac{1}{2m} \to 0 \text{ as } n, m \to \infty.$$

Via the squeeze theorem, we have $d^2(f_n, f_m) \to 0$ which implies $d(f_n, f_m) \to 0$ (since $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is continuous). Hence, we conclude that (f_n) is a Cauchy sequence in X. ■

(c) Prove that the sequence (f_n) does not converge in X by explicitly identifying the potential limit function. Note that (f_n) does converge in a space that is bigger than $C[0,1]$.

Proof. The pointwise limit of the sequence (f_n) is

$$f(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t = 1 \end{cases}.$$

Indeed, if $t = 1$, then

$$f_n(1) = 1$$

and for any $t \neq 1$ in $[0,1]$, we have $f_n(t) = 0$. This tells us that (f_n) is a sequence of continuous functions but the pointwise limit $f(t)$ is clearly not continuous. ■

Problem 4. (a) Let $(V, \|\cdot\|)$ be a normed space. Consider the closed unit ball

$$\overline{B}(0,1) = \{v \in V : \|v\| \leq 1\}.$$

Prove that $\overline{B}(0,1)$ is convex; that is, $v, w \in \overline{B}(0,1)$ implies $(1-t)v + tw \in \overline{B}(0,1)$ for all $0 \leq t \leq 1$.

(b) Let $V = \mathbb{R}^n$. Define $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ by $\|\vec{x}\| = \left(\sum_{j=1}^n |x_j|^{1/2}\right)^2$. Does $\|\cdot\|$ define a norm on $\mathbb{R}^n$?

(c) Let $V = \mathbb{R}^n$ and $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ by $\|\vec{x}\| = \#\{1 \leq i \leq n : x_i \neq 0\}$. Here, $\#A$ denotes the number of elements in the set A. Prove that $\|\cdot\|$ satisfies N1, N2, and N4 but it does not satisfy N3. Compare this with problem 1-c and state your observation.

Proof. (a) Our goal is to show that for all $v, w \in \overline{B}(0,1)$, we have

$$(1-t)v + tw \in \overline{B}(0,1)$$

for all $0 \leq t \leq 1$; that is, we need to show that

$$\|(1-t)v + tw\| \leq 1.$$

Let $v, w \in \overline{B}(0,1)$. Then we have $\|v\| \leq 1$ and $\|w\| \leq 1$. Thus, we have

$$\begin{aligned}\|(1-t)v + tw\| &\leq \|(1-t)v\| + \|tw\| \\ &= |1-t|\|v\| + |t|\|w\| \\ &\leq |1-t| + |t| \\ &= (1-t) + tt[0,1] \\ &= 1.\end{aligned}$$

Indeed, we have $(1-t)v + tw \in \overline{B}(0,1)$ and so $\overline{B}(0,1)$ is convex.

(b) We claim that $\|\cdot\|$ does not define a metric on $\mathbb{R}^n$. Let $n = 2$ and let $(1,0)$ and $(0,1)$ be two vectors in $\mathbb{R}^2$. Then with respect to the norm given, we have

$$\|(1,0) + (0,1)\| = \|(1,1)\| = (1^{1/2} + 1^{1/2})^2 = 4.$$

However, $\|(1,0)\| = (1^{1/2} + 0^{1/2})^2 = 1$ and similarly, $\|(0,1)\| = 1$. This implies that

$$\|(1,0) + (0,1)\| \geq \|(1,0)\| + \|(0,1)\|$$

which violates the triangle inequality for norms.

(c) (N1) It is clear that for any $x \in \mathbb{R}^n$, we have $\|x\| \geq 0$.

(N2) Note that

$$\begin{aligned}\vec{x} = 0 &\iff x_i = 0 \ \ \forall 1 \leq i \leq n \\ &\iff \#\{1 \leq i \leq n : x_i \neq 0\} = 0 \\ &\iff \|\vec{x}\| = 0.\end{aligned}$$

(N4) Let $\vec{x}, \vec{y}, \in \mathbb{R}^n$. Denote the following sets

$$\begin{aligned}A &= \{1 \leq i \leq n : x_i + y_i \neq 0\} \\ B &= \{1 \leq i \leq n : x_i \neq 0\} \\ C &= \{1 \leq i \leq n : y_i \neq 0\}.\end{aligned}$$

We see that if $x_i + y_i \neq 0$, then $x_i \neq -y_i$. From here, it follows that either $x_i \neq 0$ or $y_i \neq 0$. Hence, $A \subseteq B \cup C$ and so we have $\#A \leq \#(B \cup C) = \#B + \#C$. Thus, $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

(N3) By definition, we can see that scaling the components of any vector $\vec{x}$ in $\mathbb{R}^n$ should not change the number of nonzero components; that is, $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$ cannot be possible. Indeed, as a counterexample, let $V = \mathbb{R}^2$ and consider $(1,0)$ in $\mathbb{R}^2$. Clearly, $\|(1,0)\| = 1$. But note that if we multiply by any nonzero scalar α, then immediately $\alpha\|1,0\| = \alpha \cdot 1 = \alpha > 1 = \|(1,0)\|$.

From observation, the difference between this problem and problem 1-c) is that here, we are measuring the number of non-zero components in a vector in $\mathbb{R}^n$, whereas problem 1-c) compares elements of a direct products based on the positions at which they differ. Furthermore, the fact that $\|\cdot\|$ fails the scaling property tells us that d from 1-c) cannot be induced by a norm. In particular, d fails the following property

$$d(\alpha x, \alpha y) = |\alpha| d(x,y).$$

■

Problem 5. Let $(V, \|\cdot\|)$ be a normed space (real or complex).

(a) Let $v, w \in V$ be two non-zero elements of V. Show that

$$\|v\|\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \|v - w\| + \Big|\|v\| - \|w\|\Big|$$

and use this to prove that

$$\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \frac{4\|v-w\|}{\|v\| + \|w\|}.$$

Proof. Observe that

$$\begin{aligned}
\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| &= \frac{1}{\|v\|\|w\|}\|\|v\|w\| - w\|v\|\| \\
&= \frac{1}{\|v\|\|w\|}\|\|v\|w\| - \|w\|w + \|w\|w - w\|v\|\| \\
\\
&\leq \frac{1}{\|v\|\|w\|}\|\|w\|(v-w) + w(\|w\| - \|v\|)\| \\
&\leq \frac{1}{\|v\|\|w\|}\Big(\|w\|\|v-w\| + \Big|\|v\| - \|w\|\Big|\|w\|\Big) \\
&= \frac{1}{\|v\|}\Big(\|v-w\| + \Big|\|v\| - \|w\|\Big|\Big).
\end{aligned}$$

Since $v \neq 0$, we can divide by $\|v\|$ on both sides of the inequality above to obtain

$$\|v\|\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \|v - w\| + \Big|\|v\| - \|w\|\Big|. \tag{1}$$

Similarly, we can show that (by adding and subtracting $\|v\|v$ to the quantity above, applying the triangle inequality, and collecting terms)

$$\|w\|\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \|v - w\| + \Big|\|v\| - \|w\|\Big|. \tag{2}$$

Using (1) and (2), we get that

$$(\|v\| + \|w\|)\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq 2\|v - w\| + 2\Big|\|v\| - \|w\|\Big|.$$

Using the reverse triangle inequality on the second term on the right hand side of the inequality above, we have

$$(\|v\| + \|w\|)\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq 4\|v - w\|.$$

Thus, we have our desired result

$$\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \frac{4\|v-w\|}{\|v\| + \|w\|}.$$

■

(b) Assume that $\langle,\rangle$ is an inner product on V such that $\|v\| = \sqrt{\langle v, v\rangle}$. Let $v, w \in V$ be two non-zero elements of V. Prove that

$$\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\| \leq \frac{2\|v-w\|}{\|v\| + \|w\|}.$$

Proof. Our goal is to show that

$$\frac{1}{4}(\|v\| + \|w\|)^2\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\|^2 \leq \|v - w\|^2; \tag{*}$$

that is,

$$\frac{1}{4}(\|v\| + \|w\|)^2\left|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right|^2 \leq \|v\|^2 - 2\Re\langle v, w\rangle + \|w\|^2$$

Starting with the left-hand side, we have

$$\frac{1}{4}(\|v\|+\|w\|)^2\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\|^2=\frac{1}{4\|v\|^2\|w\|^2}\left[\|v\|^2+2\|v\|\|w\|+\|w\|^2\right]\Big\|v\|w\|-w\|v\|\Big\|^2. \tag{1}$$

Set $\tilde{v}=v\|w\|$ and $\tilde{w}=w\|v\|$. Then we have

$$\begin{aligned}\|\tilde{v}-\tilde{w}\|^2&=\|\tilde{v}\|^2-2\Re\langle\tilde{v},\tilde{w}\rangle+\|\tilde{w}\|^2\\&=\|w\|^2\|v|^2-2\Re\langle\|w\|v,\|v\|w\rangle+\|v\|^2\|w\|^2\\&=2\|w\|^2\|v\|^2-2\|w\|\|v\|\Re\langle v,w\rangle.\end{aligned}$$

From (1), we have

$$\begin{aligned}\frac{1}{4}(\|v\|+\|w\|)^2\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\|^2&=\left[\|v\|^2+2\|v\|\|w\|+\|w\|^2\right]\left(\frac{1}{2}-\frac{1}{2\|w\|\|v\|}\Re\langle v,w\rangle\right)\\&=\frac{\|v\|^2}{2}-\frac{\|v\|}{2\|w\|}\Re\langle v,w\rangle+\|v\|\|w\|-\Re\langle v,w\rangle\\&+\frac{\|w\|^2}{2}-\frac{\|w\|}{2\|v\|}\Re\langle v,w\rangle\\&\leq\frac{1}{2}\left[\|v\|^2-2\Re\langle v,w\rangle+\|w\|^2\right]-\left[\frac{\|v\|}{\|w\|}\Re\langle v,w\rangle+\frac{\|w\|}{\|v\|}\Re\langle v,w\rangle\right]\\&\leq\frac{1}{2}\left[\|v\|^2-2\Re\langle v,w\rangle+\|w\|^2\right]\\&\leq\|v\|^2-2\Re\langle v,w\rangle+\|w\|^2\\&=\|v-w\|^2.\end{aligned}$$

Hence, (*) is obtained by rearranging terms and taking the square root of both sides of the above inequality. ∎

(c) In homework, we proved that the following inequalities for non-zero elements $v,w\in V$:

$$\|v+w\|\leq\|v\|+\|w\|-\left(2-\left\|\frac{v}{\|w\|}+\frac{w}{\|w\|}\right\|\right)\min\{\|v\|,\|w\|\}$$

and

$$\|v+w\|\geq\|v\|+\|w\|-\left(2-\left\|\frac{v}{\|w\|}+\frac{w}{\|w\|}\right\|\right)\max\{\|v\|,\|w\|\}.$$

Prove that equality holds in any of these inequalities if either $\|v\|=\|w\|$ or $v=cw$ for some $c>0$.

Proof. Suppose $\|v\|=\|w\|$. Then we have $\max\{\|v\|,\|w\|\}=\min\{\|v\|,\|w\|\}$. It follows immediately that

$$\|v+w\|=\|v\|+\|w\|-\left(2-\left\|\frac{v}{\|v\|}+\frac{w}{\|w\|}\right\|\right)\max\{\|v\|,\|w\|\}.$$

Suppose $v=cw$ for some $c>0$. Then observe that

$$\begin{aligned}\|v+w\|&\leq\|cw\|+\|w\|-\left(2-\left\|2\cdot\frac{w}{\|w\|}\right\|\right)\min\{\|v\|,\|w\|\}\\&=\|cw\|+\|w\|-\left(2-2\cdot\frac{\|w\|}{\|w\|}\right)\min\{\|v\|,\|w\|\}\\&=\|cw\|+\|w\|\\&=(|c|+1)\|w\|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|v+w\|&\geq\|cw\|+\|w\|-\left(2-\left\|2\cdot\frac{w}{\|w\|}\right\|\right)\min\{\|v\|,\|w\|\}\\&=\|cw\|+\|w\|-\left(2-2\cdot\frac{\|w\|}{\|w\|}\right)\max\{\|v\|,\|w\|\}\\&=\|cw\|+\|w\|\\&=(|c|+1)\|w\|.\end{aligned}$$

Hence, equality holds in this case too. ∎

(d) Using (c), for any non-zero $v, w \in V$, prove that

$$\frac{\|v-w\| - |\|v\| - \|w\||}{\min\{\|v\|, \|w\|\}} \leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\max\{\|v\|, \|w\|\}}.$$

Proof. First, we prove the left-hand side of the inequality above. Let $v, w \in V \setminus \{0\}$. Using the triangle inequality, we have

$$\begin{aligned}\left\| \frac{v}{\|v\|} + \frac{w}{\|w\|} \right\| &= \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} + \frac{w}{\|w\|} + \frac{w}{\|w\|} \right\| \\ &\leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| + 2 \cdot \frac{\|w\|}{\|w\|} \\ &= \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| + 2.\end{aligned}$$

Hence, we have

$$\left\| \frac{v}{\|v\|} + \frac{w}{\|w\|} \right\| - 2 \leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\|$$

that is,

$$-\left(2 - \left\| \frac{v}{\|v\|} + \frac{w}{\|w\|} \right\|\right) \leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\|. \tag{*}$$

Using the first inequality in the previous part and (*), we have

$$\begin{aligned}\frac{\|v-w\| - |\|v\| - \|w\||}{\min\{\|v\|, \|w\|\}} &\leq \frac{\|v+w\| - (\|v\| + \|w\|)}{\min\{\|v\|, \|w\|\}} \\ &\leq \frac{\|v+w\| - \|v\| - \|w\|}{\min\{\|v\|, \|w\|\}} \\ &\leq -\left(2 - \left\| \frac{v}{\|v\|} + \frac{w}{\|w\|} \right\|\right) \\ &\leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\|.\end{aligned}$$

This gives us the left-hand side of our desired result.

For the right-hand side of the desired inequality, we assume (without loss of generality) that $\|v\| > \|w\|$. From the first inequality in part (a), we have

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\|v\|} = \frac{\|v-w\| + |\|v\| - \|w\||}{\max\{\|v\|, \|w\|\}}.$$

All together, we have

$$\frac{\|v-w\| - |\|v\| - \|w\||}{\min\{\|v\|, \|w\|\}} \leq \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\max\{\|v\|, \|w\|\}}.$$

■

(e) Using (d), give a different proof of (a).

Proof. Suppose without loss of generality that $\|v\| > \|w\|$. Hence, we have

$$\|v\| = \max\{\|v\|, \|w\|\} \geq \|w\|.$$

Using the right-hand side of what was proved in part (d), we have

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\|v\|} \implies \|v\| \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \|v-w\| + |\|v\| - \|w\||. \tag{1}$$

Also, Since $\max\{\|v\|, \|w\|\} \geq \|w\|$, we have

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\max\{\|v\|, \|w\|\}} \leq \frac{\|v-w\| + |\|v\| - \|w\||}{\|w\|}.$$

This implies that

$$\|w\|\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\| \leq \|v-w\|+|\|v\|-\|w\||. \tag{2}$$

Adding (1) and (2), we get

$$(\|v\|+\|w\|)\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\| \leq 2(\|v-w\|+|\|v\|-\|w\||).$$

Using the reverse triangle inequality on $|\|v\|-\|w\||$, we have

$$(\|v\|+\|w\|)\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\| \leq 4\|v-w\|$$

and so

$$\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\| \leq \frac{4\|v-w\|}{\|v\|+\|w\|}.$$

■

Problem 6. Let $V=\mathbb{R}^n$, $\vec{b}\in V$, and $A=(a_{ij})_{1\leq i,j\leq n}$ be an $n\times n$ matrix. Define $T: V\to V$ by $T\vec{x}=A\vec{x}+\vec{b}$.

(a) Suppose that $\max_{1\leq k\leq n}\sum_{j=1}^n |a_{jk}|<1$. Prove that T has a unique fixed point.

Proof. Since $\mathbb{R}^n$ is a complete metric space with respect to the d_∞ metric, it suffices to show that T is a contraction by the Banach Fixed Point Theorem. It is enough to show that there exists $0<\alpha<1$ such that for any $\vec{x},\vec{y}\in\mathbb{R}^n$

$$\|T\vec{x}-T\vec{y}\|_\infty \leq \alpha\|\vec{x}-\vec{y}\|_\infty.$$

Let $\vec{x},\vec{y}\in\mathbb{R}^n$. Hence, we have

$$\begin{aligned}
\|T\vec{x}-T\vec{y}\|_\infty &= \|(A\vec{x}+\vec{b})-(A\vec{y}+\vec{b})\|_\infty \\
&= \|A\vec{x}-A\vec{y}\|_\infty \\
&= \|A(\vec{x}-\vec{y})\|_\infty \\
&= \max_{1\leq k\leq n}\Big|\sum_{j=1}^n a_{jk}(x_k-y_k)\Big| \\
&\leq \|\vec{x}-\vec{y}\|_\infty \max_{1\leq k\leq n}\sum_{j=1}^n |a_{jk}|.
\end{aligned}$$

Now, now that

$$0<\max_{1\leq k\leq n}\sum_{j=1}^n |a_{jk}|<1$$

and so set $\alpha=\max_{1\leq k\leq n}\sum_{j=1}^n |a_{jk}|$. Hence,

$$\|T\vec{x}-T\vec{y}\|_\infty \leq \alpha\|\vec{x}-\vec{y}\|_\infty.$$

Thus, T is a contraction as desired. Hence, T contains a unique fixed point. ■

(b) Suppose that $\sum_{k=1}^n\sum_{j=1}^n a_{jk}^2<1$. Prove that T has a unique fixed point.

Proof. Just as in part (a), it suffices to show that T is a contraction by the Banach Fixed Point Theorem. Let $\vec{x},\vec{y}\in\mathbb{R}^n$. Our approach is to use the 2-norm to show the result. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|T\vec{x}-T\vec{y}\|^2 = \|T(\vec{x}-\vec{y})\|^2 &= \sum_{j=1}^n\Big[\sum_{k=1}^n a_{jk}(x_k-y_k)\Big]^2 \\
&\leq \sum_{j=1}^n\Big[\Big(\sum_{k=1}^n a_{jk}^2\Big)^{1/2}\Big(\sum_{\ell=1}^n (x_\ell-y_\ell)^2\Big)^{1/2}\Big]^2 \\
&= \|\vec{x}-\vec{y}\|^2\sum_{j=1}^n\sum_{k=1}^n a_{jk}^2.
\end{aligned}$$

Note that
$$0 < \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2 < 1$$
so set
$$\alpha^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2 \implies \alpha = \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2}.$$
Taking the square root, we have
$$\|T\vec{x} = T\vec{y}\| \leq \alpha\|\vec{x}-\vec{y}\|.$$
Hence, T is a contraction. Thus, T contains a unique fixed point. ■

Problem 7. (a) Let $a, b \in \mathbb{R}$ such that $a < b$. Let $\mathbf{J} = [a,b]^n = \underbrace{[a,b]\times[a,b]\times\cdots\times[a,b]}_{n \text{ times}} \subseteq \mathbb{R}^n$. Assume that U is an open set in $\mathbb{R}^n$ containing $\mathbf{J}$, $f : U \to \mathbb{R}^n$ is continuous, and component functions $f_1, \dots, f_n$ of f have continuous partial derivative. In addition, assume that

(i) $f(\mathbf{J}) \subseteq J$.

(ii) There is $0 < \alpha < 1$ such that on $\mathbf{J}$ the following holds:
$$\left|\frac{\partial f_i}{\partial x_j}\right| < \frac{\alpha}{n} \quad \forall\, 1 \leq j \leq n \;\; 1 \leq i \leq n.$$

Prove that f has a fixed point in $\mathbf{J}$.

(b) Use (a) to prove that the system of equations
$$x^2 - 20x + y^2 + 10 = 0$$
$$xy^2 + x - 20y + 10 = 0$$
has a solution in $[0,2]\times[0,2]$.

Proof. (a) It suffices to show that f is a contraction; that is, there exists an $0 < \alpha < 1$ such that for all $\vec{x}, \vec{y} \in U$, we have
$$\|f(\vec{x}) - f(\vec{y})\|_\infty \leq \alpha\|\vec{x}-\vec{y}\|_\infty.$$
Let $\vec{x}, \vec{y} \in U$ with
$$\vec{x} = (x_1, x_2, \dots, x_n)$$
and
$$\vec{y} = (y_1, y_2, \dots, y_n).$$
Note that since U is an open set in $\mathbb{R}^n$, we know that U is a convex set. Hence, U is a connected in $\mathbb{R}^n$. As a consequence, let L be a line segment connecting $\vec{x}$ and $\vec{y}$. Since each component f_i of f is differentiable and contains partial derivatives that are continuously differentiable, we can use the Mean Value Theorem to find a $\vec{c} \in L$ such that
$$f(\vec{x}) - f(\vec{y}) = \nabla f(\vec{c}) \cdot (\vec{x}-\vec{y}).$$
By definition of the standard dot product on $\mathbb{R}^n$, we have
$$\nabla f \cdot (\vec{x}-\vec{y}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial f_j}(\vec{c}) \cdot (\vec{x}-\vec{y}).$$
From here, we claim that $\alpha \in (0,1)$ is the desired constant we were looking for. Indeed, using our assumption that
$$\left|\frac{\partial f_i}{\partial x_j}\right| < \frac{\alpha}{n},$$

we have

$$\begin{aligned}
|f(\vec{x}) - f(\vec{y})| &= \Big| \sum_{i=1}^{n} \frac{\partial f_i}{\partial f_j}(\vec{c}) \cdot (\vec{x} - \vec{y}) \Big| \\
&\leq \sum_{i=1}^{n} \Big| \frac{\partial f_i}{\partial f_j}(\vec{c}) \cdot (\vec{x} - \vec{y}) \Big| \\
&\leq \|\vec{x} - \vec{y}\|_\infty \sum_{i=1}^{n} \Big| \frac{\partial f_i}{\partial x_j}(\vec{c}) \Big| \\
&< \|\vec{x} - \vec{y}\|_\infty \sum_{i=1}^{n} \frac{\alpha}{n} \\
&= \|\vec{x} - \vec{y}\|_\infty \frac{\alpha}{n} \sum_{i=1}^{n} \\
&= \alpha \|\vec{x} - \vec{y}\|_\infty.
\end{aligned}$$

Taking the maximum of the left-hand side of the inequality above, we have

$$\|f(\vec{x} - f(\vec{y}))\|_\infty \leq \alpha \|\vec{x} - \vec{y}\|_\infty.$$

Hence, we conclude that f is a contraction and so f contains a fixed point.

(b) Let $\vec{x} = (x, y) \in \mathbf{J} = [0,2] \times [0,2]$ and let $x_1 = x$ and $x_2 = y$. Denote $f_1(\vec{x}) = x^2 - 20x + y^2 + 10$ and $f_2(x) = xy^2 + x - 20y + 10$. It is clear that f_1 and f_2 are differentiable and contain partial derivatives that are continuous. Since f_1 and f_2 are continuous, it also follows that $f = (f_1, f_2)$ is continuous and $f(\mathbf{J}) \subseteq J$. All that is left to show is that there exists some $0 < \alpha < 1$ such that for all $i = 1,2$ and $j = 1,2$, we have

$$\Big| \frac{\partial f_i}{\partial x_j} \Big| < \frac{\alpha}{n}.$$

First, we compute the partial derivatives of each component function. Indeed, we have

$$\begin{aligned}
\frac{\partial f_1}{\partial x} &= 2x - 20 & \frac{\partial f_1}{\partial y} &= 2y \\
\frac{\partial f_2}{\partial x} &= 2xy^2 + 1 & \frac{\partial f_2}{\partial y} &= 2x^2 y - 20.
\end{aligned}$$

Since $\frac{\partial f_1}{\partial x}$ is continuous at 0, for any $\varepsilon > 0$, we can find an $0 < \alpha_1 < 1$ such that if $|x| < \alpha_1$, then

$$\Big| \frac{\partial f_1}{\partial x} \Big| < \frac{\varepsilon}{2}.$$

In particular, for $\varepsilon = \alpha_1$, then

$$\Big| \frac{\partial f_1}{\partial x} \Big| < \frac{\alpha_1}{2} \tag{1}$$

if $|x| < \alpha_1$. Similarly, since $\frac{\partial f_1}{\partial y}$ is continuous at 0, we know that for any $\varepsilon > 0$, there exists an $0 < \alpha_2 < 1$ such that if $|y| < \alpha_2$, we have

$$\Big| \frac{\partial f_1}{\partial y} \Big| < \frac{\varepsilon}{2}.$$

In particular, for $\varepsilon = \alpha_2$, we have

$$\Big| \frac{\partial f_1}{\partial y} \Big| < \frac{\alpha_2}{2}. \tag{2}$$

Again, since $\frac{\partial f_2}{\partial x}$ is continuous at 0, for any $\varepsilon > 0$, we can find an $0 < \alpha_3 < 1$ such that if $|x| < \alpha_3$, we have

$$\Big| \frac{\partial f_2}{\partial x} \Big| < \frac{\varepsilon}{2}.$$

In particular, for $\varepsilon = \alpha_3$, we have

$$\Big| \frac{\partial f_2}{\partial x} \Big| < \frac{\alpha_3}{2}. \tag{3}$$

Lastly, since $\frac{\partial f_2}{\partial y}$ is continuous at 0, we know that for any $\varepsilon > 0$, we can find an $0 < \alpha_4 < 1$ such that if $|y| < \alpha_4$, then

$$\left|\frac{\partial f_2}{\partial y}\right| < \frac{\varepsilon}{2}.$$

In particular, for $\varepsilon = \alpha_4$, we have

$$\left|\frac{\partial f_2}{\partial y}\right| < \frac{\alpha_4}{2} \tag{4}$$

if $|y| < \alpha_4$. Let $\alpha = \frac{1}{2}\min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (clearly, this is between 0 and 1). Then for all $i = 1,2$ and $j = 1,2$, we have

$$\left|\frac{\partial f_i}{\partial x_j}\right| < \frac{\alpha}{2}.$$

Using part (a), we can now conclude that f contains a fixed point in $[0,2] \times [0,2]$.

■

Problem 8. Let $(V, \|\cdot\|)$ be a finite dimensional normed space (real or complex). Let W be a **non-zero proper subspace** of V and let $v_0 \in V$ such that $v_0 \notin W$. Let $\delta = \min\{\|v_0 - w\| : w \in W\}$.

(a) Prove that $\delta > 0$.

Proof. Since $v_0 \notin W$, we have $0 < \|v_0 - w\|$. Hence, 0 is a lower bound for the set

$$\{\|v_0 - w\| : w \in W\}.$$

Taking the infimum of the right-hand side, we see that $\delta > 0$. ■

(b) Prove that there is $f \in V'$ such that $\|f\| = 1$, $f(v_0) = \delta$, and $f(w) = 0$ for all $w \in W$. Here, V' is the dual of V.

Proof. First note that since V is finite-dimensional, any given linear functional on V is automatically contained in V'; that is, every linear functional in a finite-dimensional normed space is bounded where $V^* = V'$.

First, we construct the desired linear functional and show that it contains all the desired properties. Define the set $W_1 = \text{span}\{v_0, w\}$ for $w \in W$. Observe that for each $w_1 \in W_1$, w_1 can be uniquely expressed as

$$w_1 = \alpha v_0 + w$$

where $w \in W$ and $\alpha \in \mathbb{F}$. Define $\tilde{f}(w_1) = \alpha\delta$. We will first show that $\tilde{f}$ is linear. Let $w_1, w_2 \in W_1$. Then

$$\begin{aligned} w_1 &= \varphi_1 + \alpha_1 v_0 \\ w_2 &= \varphi_2 + \alpha_2 v_0 \end{aligned}$$

are unique representations of w_1 and w_2, respectively. Let $\alpha, \beta \in \mathbb{F}$. Then we can see that

$$\alpha w_1 + \beta w_2 = \alpha\varphi_1 + \beta\varphi_2 + (\alpha\alpha_1 + \beta\alpha_2)v_0.$$

Hence, we obtain

$$\begin{aligned} \tilde{f}(\alpha w_1 + \beta w_2) &= (\alpha\alpha_1 + \beta\alpha_2)\delta \\ &= \alpha(\alpha_1\delta) + \beta(\alpha_2\delta) \\ &= \alpha\tilde{f}(w_1) + \beta\tilde{f}(w_2). \end{aligned}$$

Hence, we can see that $\tilde{f}$ is linear in V (and bounded since V is finite-dimensional). Note that $w \in W$ can expressed as $w = 0 \cdot v_0 + w$ and so $\tilde{f}(w) = 0$. Similarly, $v_0 \in V$ implies $v_0 = 1 \cdot v_0 + 0$. Hence, $\tilde{f}(v_0) = \delta$.

Next, we will show that $\|\tilde{f}\| = 1$. Our goal is to show that $\|\tilde{f}\| \leq 1$ and $\|\tilde{f}\| \geq 1$. Note that for every $w_1 \in W_1 \setminus \{0\}$, we have

$$|\tilde{f}(w_1)| \leq \|w_1\|.$$

Assuming that $w_1 \in W_1 \setminus \{0\}$, we have
$$\frac{|\tilde{f}(w_1)|}{\|w_1\|} \leq 1.$$
Taking the supremum over all $w_1 \in W \setminus \{0\}$, we have
$$\sup_{w_1 \in W_1 \setminus \{0\}} \frac{|\tilde{f}(w_1)|}{\|w_1\|} \leq 1 \Longrightarrow \|\tilde{f}\| \leq 1.$$

Now, we show that $\|\tilde{f}\| \geq 1$. Let $\varepsilon > 0$ be given. By definition of δ, there exists $w' \in W$ such that
$$\begin{aligned} \|v_0 - w'\| < \delta + \varepsilon \Longrightarrow \frac{1}{\delta+\varepsilon} &< \frac{1}{\|v_0 - w'\|} \\ \Longrightarrow \frac{|\tilde{f}(v_0 - w')|}{\delta + \varepsilon} &< \frac{|\tilde{f}(v_0 - w')|}{\|v_0 - w'\|} \leq \|\tilde{f}\|. \end{aligned}$$
Note that $\tilde{f}(v_0 - w') = \delta$ and so we have
$$\frac{\delta}{\delta + \varepsilon} \leq \|\tilde{f}\| \underbrace{\Longrightarrow}_{\varepsilon \to 0} \frac{\delta}{\delta} \leq \|\tilde{f}\| \Longrightarrow 1 \leq \|\tilde{f}\|.$$
Hence, we conclude that $\|\tilde{f}\| = 1$. Since V is finite-dimensional, we can extend W_1 to V inductively. ■

Problem 9. Let $V = C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

(a) Prove that $(V, \langle,\rangle)$ is an inner product space where
$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

Proof.(IP1) Let $f, g, h \in V$. Then by the linearity of the integral, we have
$$\begin{aligned} \langle f + g, h \rangle &= \int_0^1 (f+g)(t)h(t)\, dt \\ &= \int_0^1 [f(t) + g(t)]h(t)\, dt \\ &= \int_0^1 [f(t)h(t) + g(t)h(t)]\, dt \\ &= \int_0^1 f(t)h(t)\, dt + \int_0^1 g(t)h(t)\, dt \\ &= \langle f, h \rangle + \langle g, h \rangle. \end{aligned}$$
Hence, (IP1) is satisfied.

(IP2) Let $\alpha \in \mathbb{F}$ and $f, g \in V$. Then using the linearity of the integral again, we have
$$\begin{aligned} \langle \alpha f, g \rangle &= \int_0^1 (\alpha f)(t)g(t)\, dt \\ &= \int_0^1 \alpha f(t)g(t)\, dt \\ &= \alpha \int_0^1 f(t)g(t)\, dt \\ &= \alpha \langle f, g \rangle. \end{aligned}$$

■

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

Hence, (IP2) is satisfied.

(IP3) Let $f \in V$. Since f is continuous on $[0,1]$, it follows that f^2 is also continuous on $[0,1]$ (since $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x^2$ is continuous and the composition of φ and f is a continuous function). Also, $f^2 \in R[0,1]$ and $f^2 \geq 0$ on $[0,1]$. Hence,

$$\langle f, f \rangle = \int_0^1 f^2(t)\, dt \geq 0.$$

(IP4) Let $f \in V$ be such that $\langle f, f \rangle = 0$. Since $f^2 \geq 0$, f^2 is continuous on $[0,1]$, $\int_0^1 f^2(t)\, dt = 0$, we have $f^2 = 0$ on $[0,1]$ and $f = 0$ on $[0,1]$. Conversely, if $f = 0$, then $f^2 = 0$ on $[0,1]$. Thus,

$$\int_0^1 f^2\, dt = 0 \Longrightarrow \langle f, f \rangle = 0.$$

Hence, we see from the properties above that $(V, \langle , \rangle)$ is an inner product space.

(b) Prove that $(V, \langle , \rangle)$ is **NOT** a Hilbert space.

Proof. We can induce a norm $\| \cdot \| : V \to \mathbb{R}$ out of the inner product $\langle , \rangle$ by defining

$$\|f\| = \sqrt{\langle f, f \rangle} = \Big(\int_0^1 f^2(t)\, dt \Big)^{1/2}.$$

From this norm, we can rewrite the metric defined in problem 3 can be written in the following way:

$$d(f,g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle} = \Big(\int_0^1 (f-g)^2(t)\, dt \Big)^{1/2}.$$

Using the sequence of functions (f_n) defined in part (b) of problem 3, we know that (f_n) is Cauchy with respect to this metric d, but (f_n) does not converge to a continuous function. Hence, $(V, \langle , \rangle)$ cannot be complete in V. ■