1 Handout-13

1.1 Topics

- Discuss more consequences of Cauchy-Riemann equations.
- Introduce holomorphic functions

1.2 Recap

Let $D \subseteq \mathbb{C}$, let $f : D \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$. Assume that the partial derivatives of u and v are continuous. Then the following statements are equivalent.

(i) f is complex differentiable on D.

(ii) The Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on } D.$$

1.3 Laplace's Equation and Harmonic Functions

Definition. Let $D \subseteq \mathbb{C}$ be open and let $f : D \to \mathbb{C}$. If f is complex differentiable on D, then we say f is holomorphic on D. Let $a \in D$. We say f is holomorphic at a if we can find an open set $D' \subseteq D$ such that $a \in D'$ and f is holomorphic on D'.

Lemma. Let $D \subseteq \mathbb{C}$ be open, let $f : D \to \mathbb{C}$. Then the following statements are equivalent

(i) f is holomorphic on D

(ii) f is holomorphic at $a \in D$ for all $a \in D$.

Let $D \subseteq \mathbb{C}$, let D be an open set, and let $f : D \to \mathbb{C}$ be holomorphic. Let $f = u + iv$. In addition, assume that u and v have second order continuous partial derivatives. By Cauchy-Riemann equation, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on } D.$$

Therefore, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Since we assumed, the second partial derivatives are continuous, we have

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus, we have proved the following proposition.

Proposition (Laplace's Equation)**.** Let $D \subseteq \mathbb{C}$ be an open set, let $f : D \to \mathbb{C}$ be a holomorphic function and let $f = u + iv$. In addition, assume that u and v have second order continuous partial derivatives. Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Definition (Harmonic Function)**.** Let $D \subseteq \mathbb{R}^2$ be open. A function $u : D \to \mathbb{R}$ is called **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on D.

Thus, we learned that if $f : D \to \mathbb{C}$ with D being an open set, f being holomorphic, $\Re(f)$ and $\Im(f)$ have continuous partial order partial derivatives, then $\Re(f)$ and $\Im(f)$ are harmonic functions on D. From the Cauchy-Riemann equations, we see that u and v are heavily dependent on each other. This begs the following two questions:

(1) Can we determine v from u?

(2) Suppose that $u : D \to \mathbb{R}$ that is harmonic. Is it possible to find a holomorphic f such that $u = \Re(f)$?

It turns out that the answers to these questions depends on the topology of D. But first we recall some basic topological facts in order to answer these questions.

1.4 Basic Topological Facts

Definition. Let $D \subseteq \mathbb{C}$ be an open set. Let $z, w \in D$. A path in D joining z to w is a continuous map $\gamma : [a, b] \to D$ such that $\gamma(a) = z$ and $\gamma(b) = w$, where $[a, b]$ is a closed interval in $\mathbb{R}$.

- Let $D \subseteq \mathbb{C}$ be an open set. Let $z, w \in D$. Then z and w can be joined by a line segment if $\gamma : [0, 1] \to D$ is given by $\gamma(t) = (1 - t)z + tw$.
- An **open set** $D \subseteq \mathbb{C}$ is connected if any two points $z, w \in D$ can be joined by a sequence of line segments; that is, we can find points $z_1, \ldots, z_k$ such that z and z_1 can be joined by a line segment, z_i and z_{i+1} can be joined by a line segment for $i = 1, 2, \ldots, k - 1$ and z_k and w can be joined by a line segment.

Example.

- $B(a, R)$ is connected.
- The annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ where $r_1, r_2 > 0$ such that $r_1 < r_2$ is connected.
- $D = B(0, 1) \cup B(5, 2)$ is not connected as they are disjoint.

Remark. In topology, one uses a more general version of connectedness. Our definition of connectedness is specific to open subsets of $\mathbb{C}$ (or for any set in $\mathbb{R}^2$ rather).

One consequence of connectedness is outlined in the proposition below:

Proposition. Let $D \subseteq \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be a holomorphic function. Suppose that f is locally constant on D. If D is connected, then f is constant.

Theorem. Let $D \subseteq \mathbb{C}$ be an open rectangle, whose sides parallel to the real and imaginary axes. Let $u : D \to \mathbb{R}$ be a harmonic function. Then, we can find $v : D \to \mathbb{R}$ such that $f : u + iv$ is holomorphic on D.