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# Linear Algebra Exercises

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# Chapter 1

## Vector Spaces

### 1.1 Subspaces

#### Exercise 1.3.3

Prove that  $(aA + bB)^t = aA^t + bB^t$  for any  $A, B \in M_{m \times n}(F)$  and any  $a, b \in F$ .

**Proof.** Let  $A, B \in M_{m \times n}(F)$  and let  $a, b \in F$  be arbitrary. Using scalar multiplication defined on  $M_{m \times n}(F)$ , we have

$$\begin{aligned}(aA + bB)^t &= (aA)^t + (bB)^t \\ &= aA^t + bB^t.\end{aligned}$$

Hence, we are done. ■

#### Exercise 1.3.4

Prove that  $(A^t)^t = A$  for each  $A \in M_{n \times n}(F)$ .

**Proof.** Let  $A \in M_{n \times n}(F)$ . By definition of transpose, we have

$$\left((A^t)^t\right)_{ij} = (A^t)_{ji} = A_{ij}$$

for all  $1 \leq i, j \leq n$ . Hence,  $(A^t)^t = A$ . ■

#### Exercise 1.3.5

Prove that  $A + A^t$  is symmetric for any square matrix  $A$ .

**Proof.** Let  $A$  be an arbitrary square matrix. Since square matrices are symmetric, we have that  $A^t = A$ . We need to show that  $(A + A^t)^t$ . Observe that

$$\begin{aligned}(A + A^t)^t &= A^t + (A^t)^t \\ &= A + A^t.\end{aligned}$$

Hence, we have  $A + A^t$  is symmetric. ■

#### Exercise 1.3.

Prove that  $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$  for any  $A, B \in M_{n \times n}(F)$ .

**Proof.** Let  $A, B \in M_{n \times n}(F)$  and let  $a, b \in F$  be arbitrary. Now, let  $i = j$  and observe that

$$\begin{aligned}
 \operatorname{tr}(aA + bB) &= \sum_{i,j \in \mathbb{N}}^n (aA + bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n (aA)_{ij} + (bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n (aA)_{ij} + \sum_{i,j \in \mathbb{N}}^n (bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n aA_{ij} + \sum_{i,j \in \mathbb{N}}^n bB_{ij} \\
 &= a \sum_{i,j \in \mathbb{N}}^n A_{ij} + b \sum_{i,j \in \mathbb{N}}^n B_{ij} \\
 &= a \operatorname{atr}(A) + b \operatorname{atr}(B).
 \end{aligned}$$

Hence, we conclude

$$\operatorname{tr}(aA + bB) = a \operatorname{atr}(A) + b \operatorname{atr}(B)$$

for any  $A, B \in M_{n \times n}(F)$ . ■

### Exercise 1.3.7

Prove that diagonal matrices are symmetric matrices.

**Proof.** Let  $A \in M_{n \times n}(F)$  be diagonal. Let  $i \neq j$  where  $A_{ij} = 0$ . We need to show that  $A^t = A$ . If we apply a transpose on  $A$ , we get that  $A_{ji} = 0$  since  $A^t$  is also diagonal and square. Since  $A_{ij} = A_{ji} = 0$  for all  $1 \leq i \leq n$  and  $1 \leq j \leq n$ . Hence,  $A^t = A$ . ■

### Exercise 1.3.8

Determine whether the following sets are subspaces of  $\mathbb{R}^3$  under the operations of addition and scalar multiplication defined on  $\mathbb{R}^3$ . Justify your answers.

- (a)  $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

**Proof.** We claim that  $W_1$  is a subspace of  $\mathbb{R}^3$ .

- (a) Note that  $O_{\mathbb{R}^3} \in W_1$  where  $O_{\mathbb{R}^3} = (0, 0, 0)$  because  $0 = 3 \cdot 0$  and  $0 = -1 \cdot 0$ .
- (b) Let  $x, y \in W_1$  where  $x = (a_1, a_2, a_3)$  and  $y = (b_1, b_2, b_3)$ . We need to show that  $x + y \in W_1$ . Since  $a_1 = 3a_2$  and  $a_3 = -a_2$  as well as  $b_1 = 3b_2$  and  $b_3 = -b_2$ , we can write  $a_1 + b_1 = 3(a_2 + b_2)$  and  $a_3 + b_3 = -(a_2 + b_2)$ . Hence,  $x + y \in W_1$ .
- (c) Let  $c \in \mathbb{R}$  and  $x \in W_1$  with  $x$  defined as before. Then observe that  $ca_1 = c(3a_2) = 3(ca_2)$  and  $ca_3 = c(-a_2) = -(ca_2)$ . Hence,  $cx \in W_1$ .

Since all the properties of a Theorem 3 have been satisfied, we can conclude that  $W_1$  is a subspace of  $\mathbb{R}^3$ . ■

- (b)  $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
- (c)  $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
- (d)  $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - 3a_3 = 1\}$
- (e)  $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
- (f)  $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

### Exercise 1.3.9

Let  $W_1, W_3, W_4$  be as in Exercise 8. Describe  $W_1 \cap W_3$ ,  $W_1 \cap W_4$ , and  $W_3 \cap W_4$  and observe that each is a subspace of  $\mathbb{R}^3$ .

**Proof.** ■

### Exercise 1.3.11

Prove that the set  $W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 0\}$  is a subspace of  $F^n$ , but  $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 1\}$  is not.

**Proof.** We need to show that  $W_1$  is a subspace of  $F^n$ . We proceed by satisfying the properties of Theorem 3 to do this.

- (a) Note that  $O_{F^n} \in W_1$  since  $0 + 0 + \dots + 0 = 0$   $n$  times.
- (b) Let  $x, y \in W_1$  with  $x = (a_1, a_2, \dots, a_n)$  and  $y = (b_1, b_2, \dots, b_n)$ . By definition of  $W_1$ , we can see that

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Hence,  $x + y \in W_1$  which tells us that  $W_1$  is closed under addition.

- (c) Let  $x \in W_1$  and  $c \in F$ . Then observe that

$$\sum_{i=1}^n (ca_i) = c \sum_{i=1}^n a_i = c \cdot 0 = 0.$$

Hence, we have  $cx \in W_1$ .

We claim that  $W_2$  is not a subspace because  $W_2$  is not closed under addition. Let  $(0, 1), (1, 0) \in F^2$ . Observe that  $0 + 1 = 1$  and  $1 + 0 = 1$ , but  $(0 + 1) + (1 + 0) = 1 + 1 = 2$ . Hence,  $(0, 1) + (1, 0) \notin W_2$ . ■

### Exercise 1.3.11

Is the set  $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$  a subspace of  $P(F)$  if  $n \geq 1$ ? Justify your answer.

### Exercise 1.3.12

Prove that the set of  $m \times n$  upper triangular matrices is a subspace of  $M_{m \times n}(F)$ .

**Proof.** Let  $V$  denote the set of  $m \times n$  upper triangular matrices. We will show that  $V$  is a subspace of  $M_{m \times n}(F)$  using Theorem 3.

- (a) The zero matrix  $O$  from  $M_{m \times n}(F)$  contains entries  $O_{ij} = 0$  whenever  $i > j$ . Hence,  $O \in V$ .
- (b) Let  $A, B \in V$ . By definition of  $V$ ,  $A$  and  $B$  are upper triangular where  $A_{ij} = 0$  and  $B_{ij} = 0$  whenever  $i > j$ . Observe that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$$

whenever  $i > j$ . Hence,  $A + B \in V$ .

- (c) Let  $c \in F$  and  $A \in V$  as defined before. Let  $i > j$  and observe that  $(cA)_{ij} = cA_{ij} = c \cdot 0 = 0$ . Hence,  $cA \in V$ .

Since all the properties of theorem 3 have been satisfied, we conclude that  $V$  is indeed a subspace of  $M_{m \times n}(F)$ . ■

### Exercise 1.3.13

Let  $S$  be nonempty set and  $F$  is a field. Prove that for any  $s_0 \in S$ , the set  $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ , is a subspace of  $\mathcal{F}(S, F)$ .

**Proof.** We will proceed to prove that  $S$  is a subspace of  $\mathcal{F}(S, F)$  over the field  $F$  by satisfying the properties of Theorem 3. Let  $V = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ .

- (a) Note that the zero function  $f_0 \in \mathcal{F}(S, F)$  where  $f_0(s_0) = 0$  for any  $s_0 \in S$  implies that  $f_0 \in V$ .

- (b) Let  $f, g \in V$ . By definition of  $V$ ,  $f(s_0) = 0$  and  $g(s_0) = 0$  for any  $s_0 \in S$ . We have  $f + g \in V$  since

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0.$$

Hence,  $V$  is closed under addition.

- (c) Let  $f \in V$  and  $c \in F$ . We have  $cf \in V$  since

$$(cf)(s_0) = cf(s_0) = c \cdot 0 = 0$$

for any  $s_0 \in S$ . Hence,  $V$  is closed under scalar multiplication. ■

### Exercise 1.3.14

Let  $S$  be a nonempty set and  $F$  a field. Let  $\mathcal{C}(S, F)$  denote the set of all functions  $f \in \mathcal{F}(S, F)$  such that  $f(s) = 0$  for all but a finite number of elements of  $S$ . Prove that  $\mathcal{C}(S, F)$  is a subspace of  $\mathcal{F}(S, F)$ .

**Proof.** We proceed by using Theorem 3 to prove that  $\mathcal{C}(S, F)$  is a subspace where  $S$  is a nonempty set and  $F$  is a field.

- (a) Note that the zero vector  $f_0 \in \mathcal{F}(S, F)$  is in  $\mathcal{C}(S, F)$  because  $f_0(x_n) = 0$  where  $x_n \in S$  for finitely many  $n$ .

- (b) Let  $f, g \in \mathcal{C}(S, F)$ . We need to show that  $f + g \in \mathcal{C}(S, F)$ . Let  $x_n \in S$  for finitely many  $n$ . Then using the addition defined on  $\mathcal{F}(S, F)$ , we can write

$$(f + g)(x_n) = f(x_n) + g(x_n) = 0 + 0 = 0.$$

- (c) Let  $f \in \mathcal{C}(S, F)$  and  $c \in F$ . We need to show that  $cf \in \mathcal{C}(S, F)$ . Let  $x_n \in S$  for finitely many  $n$ . Using the scalar operation defined on  $\mathcal{F}(S, F)$ , we can write

$$(cf)(x_n) = cf(x_n) = c \cdot 0 = 0.$$

Since all the properties of Theorem 3 have been satisfied, we conclude that  $\mathcal{C}(S, F)$  is indeed a subspace of  $\mathcal{F}(S, F)$ . ■

### Exercise 1.3.15

Is the set of all differentiable real-valued functions defined on  $\mathbb{R}$  a subspace of  $C(\mathbb{R})$ ?

**Proof.** We claim that the set of all differentiable real-valued functions defined on  $\mathbb{R}$  is a subspace of  $C(\mathbb{R})$ . Denote this set as  $V$ .

- (a) Note that the zero function  $f_0$  is differentiable for all  $x \in \mathbb{R}$  and continuous for all  $x \in \mathbb{R}$ .



Hence,  $f_0 \in V$ .

(b) Let  $f, g \in V$ . Using the addition operation defined on  $C(\mathbb{R})$ , we get that the sum  $(f+g)(x) = f(x) + g(x)$  is differentiable which implies that the sum of functions  $f, g$  is also continuous. Hence,  $f+g \in V$ .

(c) Let  $f \in V$  and let  $c \in \mathbb{R}$ . Then  $(cf)(x) = cf(x)$  is differentiable for all  $x \in \mathbb{R}$  which means that  $cf$  is also continuous. Hence,  $cf \in V$ .

Hence,  $V$  is a subspace of  $C(\mathbb{R})$ . ■

### Exercise 1.3.16

Let  $C^n(\mathbb{R})$  denote the set of all real-valued functions defined on the real line that have a continuous  $n$ th derivative. Prove that  $C^n(\mathbb{R})$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .

**Proof.** ■

### Exercise 1.3.17

Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $W \neq \emptyset$  and, whenever  $a \in F$  and  $x, y \in W$ , then  $ax \in W$  and  $x+y \in W$ .

**Proof.** For the forwards direction, let  $W \subseteq V$  where  $V$  is a vector space and  $W$  is a subspace of  $V$ . Let  $a \in F$  and  $x, y \in W$ . Since  $W$  is a subspace, we know that  $O_V \in W$ . So,  $W$  is nonempty. Since  $W$  is closed under addition and multiplication, we get that  $x+y \in W$  and  $ax \in W$  and we are done.

For the backwards direction, let  $W \neq \emptyset$  and  $W \subseteq V$ . Let  $a \in F$  and  $x, y \in W$  be arbitrary such that  $ax \in W$  and  $x+y \in W$ . We need to show that  $W$  is a subspace of  $V$ . We need only show that  $O_V \in W$  since  $W$  is closed under addition and scalar multiplication. Let  $x \in W$ . We can pick any  $c \in F$  such that  $c = 0$ . So, we have  $c \cdot x = 0 \cdot x = O_W$ . Since the zero vector  $O_W \in W$  is unique, we must have  $O_V = O_W$ . Hence,  $O_V \in W$  and we conclude that  $W$  is a subspace of  $V$ . ■

### Exercise 1.3.18

Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $O \in W$  and  $ax+y \in W$  whenever  $a \in F$  and  $x, y \in W$ .

**Proof.** ( $\Rightarrow$ ) Let  $W \subseteq V$  where  $W$  is a subspace of  $V$ . Since  $W$  is a subspace of  $V$ , we know that  $W$  is closed under addition and scalar multiplication. Let  $a \in F$  and  $x, y \in W$ . Using the third property of Theorem 3, we can see that  $ax \in W$ . Since  $W$  is closed under addition, we can take  $y \in W$  and  $ax \in W$  such that  $ax+y \in W$ . Since  $W$  is also a vector space by definition, we know that  $O_W \in W$ . But  $O_W = O_V$  so  $O_V \in W$ .

( $\Leftarrow$ ) Let  $a \in F$  and  $x, y \in W$ . We want to show that  $W \subseteq V$  is a subspace of  $V$ . We can do this by using Theorem 3.

(a) By assumption, the zero vector  $O_V \in W$ .

(b) Let  $x, y \in W$ . Choose  $a = 1$  such that  $ax+y = x+y$ . Since  $ax+y \in W$  and  $ax+y = x+y$ , we also have  $x+y \in W$ . Hence,  $W$  is closed under addition.

(c) Let  $x \in W$  and  $O_V \in W$ . Let  $a \in F$ . Then we have  $ax+O_V = ax \in W$ .

Hence,  $W$  is a subspace of  $V$  by Theorem 3. ■

### Exercise 1.3.19

Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $W_1 \cup W_2$  is a subspace of  $V$  if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

**Proof.** ( $\Rightarrow$ ) Let  $W_1 \cup W_2$  is a subspace of  $V$ . We need to show that  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ . We

proceed by showing the contrapositive. Assume  $W_2 \not\subseteq W_1$  and  $W_1 \not\subseteq W_2$ . We need to show that  $W_1 \cup W_2$  is **NOT** a subspace of  $V$ . By assumption,  $x \in W_1$  is not contained in  $W_2$  as well as  $y \in W_2$  is not contained in  $W_1$ . This implies that  $W_1 \cup W_2 \neq \emptyset$ . Since  $W_1 \cup W_2$  is empty where  $W_1 \cup W_2$  does not contain  $O_V$ , it cannot possibly be a subspace of  $V$ .

( $\Leftarrow$ ) Let  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ . We need to show that  $W_1 \cup W_2$  is a subspace of  $V$ . We proceed by using Theorem 3 to do this. Without loss of generality, assume  $W_1 \subseteq W_2$ . The proof will be the same if we use  $W_2 \subseteq W_1$ .

- (a) Since  $W_1$  is a subspace of  $V$ , we get that  $O_V \in W_1$ . Furthermore,  $W_1 \subseteq W_2$  implies that  $O_V \in W$ . Since  $O_V \in W_1$  and  $O_V \in W_2$ , we get that  $O_V \in W_1 \cup W_2$  by definition of union.
- (b) Let  $x, y \in W_1$ . Since  $W_1$  is a subspace, we get that  $x + y \in W_1$ . Since  $W_1 \subseteq W_2$ , we also get that  $x + y \in W_2$ . Since both  $x + y \in W_1$  and  $x + y \in W_2$ , we know that  $x + y \in W_1 \cup W_2$  by definition of the union.
- (c) Let  $x \in W_1$  and  $c \in F$ . Since  $W_1$  is closed under scalar multiplication, we have that  $cx \in W_1$ . But  $W_1 \subseteq W_2$  so  $W_2$  also contains  $cx \in W_1$ . So we must have  $cx \in W_1 \cup W_2$ .

Hence,  $W_1 \cup W_2$  is a subspace of a vector space  $V$ . ■

### Exercise 1.3.20

Prove that if  $W$  is a subspace of a vector space  $V$  and  $w_1, w_2, \dots, w_n$  are in  $W$ , then  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$ .

**Proof.** Let  $W$  be a subspace of a vector space  $V$ . Our goal is to show that the following statement:  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$  for all  $1 \leq i \leq n$ . We proceed by induction on  $i \geq 1$ . Let  $i = 1$ . Since  $W$  is closed under scalar multiplication, we know that  $a_1 \in F$  and  $w_1 \in W$  implies that  $a_1w_1 \in W$ . Now let  $i = 2$ , then  $w_1, w_2 \in W$  and  $a_1, a_2 \in F$  implies that  $a_1w_1 + a_2w_2 \in W$  since  $W$  is closed under scalar multiplication and addition. Now, assume that our result holds for all  $1 \leq i \leq n$ . We want to show that it also holds for  $i = n + 1$ . By our inductive hypothesis, we know that  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$ . Let  $a_{n+1} \in F$  and  $w_{n+1} \in W$ . Using vector addition and scalar multiplication, we know that

$$(a_1w_1 + a_2w_2 + \dots + a_nw_n) + a_{n+1}w_{n+1} \in W.$$

■

### Exercise 1.3.21

Let  $V$  denote the vector space of sequences in  $\mathbb{R}$ , as defined in Example 5 of section 1.2. Show that the set of convergent sequences  $(a_n)$  (that is, those for which  $\lim_{n \rightarrow \infty} a_n$  exists) is a subspace of  $V$ .

**Proof.** We will show that  $W$  (the set of convergent sequences in  $\mathbb{R}$ ) is a subspace of  $V$  by using Theorem 3.

- (a) Note that the zero sequence  $O_n$  is zero for all  $n = 1, 2, \dots$ . Hence, the limit of  $O_n$  converges to 0 and so we have  $O_n \in W$ .
- (b) Let  $(a_n)$  and  $(b_n)$  be two convergent sequences in  $\mathbb{R}$ . Then we get that  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  exists. Observe that

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

Since the  $a_n$  and  $b_n$  are both convergent sequences, we also get that the sum  $(a_n + b_n)$  also converges. Hence,  $(a_n + b_n) \in W$ .

- (c) Let  $t \in F$  and  $a_n \in W$  as before. Then we have

$$\lim_{n \rightarrow \infty} ta_n = t \lim_{n \rightarrow \infty} a_n.$$

Since any constant  $t \in F$  multiplied by a convergent sequence is convergent, we also get that the sequence  $(ta_n)$  is also convergent. Hence,  $ta_n \in W$ . ■

### Exercise 1.3.22

Let  $F_1$  and  $F_2$  be fields. A function  $g \in \mathcal{F}(F_1, F_2)$  is called an **even function** if  $g(-t) = g(t)$  for each  $t \in F_1$  and is called an **odd function** if  $g(-t) = -g(t)$  for each  $t \in F_1$ . Prove that the set of all even functions in  $\mathcal{F}(F_1, F_2)$  and the set of all odd functions in  $\mathcal{F}(F_1, F_2)$  are subspaces of  $\mathcal{F}(F_1, F_2)$ .

**Proof.** Let  $V$  be the set of all even functions and let  $W$  be the set of all odd functions. We will first prove that  $V$  is a subspace of  $\mathcal{F}(F_1, F_2)$  and the same with  $W$ .

- (a) Note that the zero function  $f_0$  from  $\mathcal{F}(F_1, F_2)$  is even since  $f_0(-t) = 0 = f_0(t)$  for all  $t \in F_1$ . Hence,  $f_0 \in V$ .
- (b) Let  $f, g \in V$ . We need to show that  $f + g \in V$ ; that is, we need to show that it is even. Let  $t \in F_1$ . Then observe that

$$(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(t)$$

Hence,  $(f + g)(-t) = (f + g)(t)$  for all  $t \in F_1$ .

- (c) Let  $f \in V$  and let  $c \in F_2$ . Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence,  $cf \in V$ .

Since all the properties of  $V$  are satisfied, we have that  $V$  is a subspace of  $\mathcal{F}(F_1, F_2)$ .

Now we will prove  $W$  is a subspace of  $\mathcal{F}(F_1, F_2)$ .

- (a) Note that the zero function  $f_0$  is in  $W$  because for any  $t \in F_1$  we have  $f_0(-t) = 0 = -1 \cdot 0 = -f_0(t)$ . Hence,  $f_0 \in W$ .
- (b) Let  $f, g \in W$  and Let  $t \in F_1$ . Observe that

$$(f + g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f + g)(t).$$

Hence,  $f + g \in W$ .

- (c) Let  $c \in F_2$  and  $f \in W$ . Let  $t \in F_1$  such that  $f(-t) = -f(t)$ . Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence,  $cf \in W$ .

Hence,  $W$  is a subspace of  $\mathcal{F}(F_1, F_2)$ . ■

**Definition 1 (Sum of Two Sets).** If  $S_1$  and  $S_2$  are nonempty subsets of a vector spaces  $V$ , then the **sum**  $S_1$  and  $S_2$ , denoted  $S_1 + S_2$ , is the set  $\{x + y : x \in S_1 \text{ and } y \in S_2\}$ .

**Definition 2 (Direct Sum).** A vector space  $V$  is called the **direct sum** of  $W_1$  and  $W_2$  if  $W_1$  and  $W_2$  are subspaces of  $V$  such that  $W_1 \cap W_2 = \{0\}$  and  $W_1 + W_2 = V$ . We denote that  $V$  is the direct sum of  $W_1$  and  $W_2$  by writing  $V = W_1 \oplus W_2$ .

### Exercise 1.3.23

Let  $W_1 + W_2$  is a subspace of a vector space  $V$ .

- (a) Prove that  $W_1 + W_2$  is a subspace of  $V$  that contains both  $W_1$  and  $W_2$ .

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Define  $W_1 + W_2$  as the set

$$\{x_1 + y_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}.$$

- (a) Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that the zero vector  $O_V$  is contained in both  $W_1$  and  $W_2$ . Hence, the sum  $O_V = O_V + O_V \in W_1 + W_2$ .
- (b) Let  $u, v \in W_1 + W_2$  with  $u = x_1 + y_1$  and  $v = x_2 + y_2$ . Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that addition is closed in both subsets  $W_1$  and  $W_2$ . Hence,  $x_1 + x_2 \in W_1$  and  $y_1 + y_2 \in W_2$ . Observe that

$$\begin{aligned} u + v &= (x_1 + y_1) + (x_2 + y_2) \\ &= (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2. \end{aligned}$$

Hence, addition is closed in  $W_1 + W_2$ .

- (c) Let  $c \in F$  where  $F$  is a field and let  $u \in W_1 + W_2$  with  $u = x_1 + y_1$ . Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that  $cx_1 \in W_1$  and  $cy_1 \in W_2$ . Observe that

$$\begin{aligned} cu &= c(x_1 + y_1) \\ &= cx_1 + cy_1 \in W_1 + W_2. \end{aligned}$$

Hence,  $W_1 + W_2$  is a subspace of  $V$ . ■

- (b) Prove that any subspace of  $V$  that contains both  $W_1$  and  $W_2$  must also contain  $W_1 + W_2$ .

**Proof.** Let  $X$  be a subspace of  $V$ . Suppose  $X$  contains both  $W_1$  and  $W_2$ ; that is,  $W_1 \subseteq X$  and  $W_2 \subseteq X$ . We must show that  $W_1 + W_2 \subseteq X$ . Let  $u \in W_1 + W_2$  with  $u = x_1 + y_1$ . By definition, we have  $x_1 \in W_1$  and  $y_1 \in W_2$ . Since  $W_1 \subseteq X$  and  $W_2 \subseteq X$ , we have  $x_1 \in X$  and  $y_1 \in X$ . Since  $X$  is a subspace of  $V$  and  $X$  is closed under addition, we have  $x_1 + y_1 \in X$ . Hence,  $u \in X$  and we conclude  $W_1 + W_2 \subseteq X$ . ■

### Exercise 1.3.24

Show that  $F^n$  is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \dots = a_{n-1} = 0\}.$$

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of  $F^n$ . Since  $W_1$  and  $W_2$  are subspaces, they both contain the zero vector  $O_{F^n}$ . Observe that for any element  $x \in W_1$ , we have the  $n$ th element  $a_n = 0$ . On the other hand,  $y \in W_2$  implies that  $a_1 = a_2 = \dots = a_{n-1} = 0$  except for the  $n$ th element. Thus, the only element that  $W_1$  and  $W_2$  have in common is the zero vector  $O_{F^n}$ . Hence, we have  $W_1 \cap W_2 = \{O_{F^n}\}$ .

Now, we show  $W_1 + W_2 = F^n$ . To do this, we need to show the following containments:

- (i)  $W_1 + W_2 \subseteq F^n$  and
- (ii)  $F^n \subseteq W_1 + W_2$ .

Let  $u \in W_1 + W_2$  with  $u = x_1 + x_2$  where  $x_1 \in W_1$  and  $y_1 \in W_2$ . Since  $W_1 \subseteq F^n$ , we know that  $x_1 \in F^n$ . Likewise,  $W_2 \subseteq F^n$  implies that  $y_1 \in F^n$ . Since  $F^n$  is a vector space where addition is closed, we have that  $u = x_1 + y_1 \in F^n$ . Hence,  $W_1 + W_2 \subseteq F^n$ .

Now, let  $u \in F^n$ . Since  $F^n$  is a vector space over  $F$ , each entry in  $u$  (where each entry is an element of a field  $F$ ) can be written and separated using the addition defined in  $F^n$  in the following

way:

$$\begin{aligned} u &= (a_1, a_2, \dots, a_n) \\ &= (a_1 + 0, a_2 + 0, \dots, 0 + a_n) \\ &= (a_1, a_2, \dots, 0) + (0, 0, \dots, a_n) \end{aligned}$$

where the first term is an element of  $W_1$  and the second term is an element of  $W_2$ . Hence,  $u \in W_1 + W_2$ .

Since both containments are satisfied, we have that  $W_1 + W_2 = F^n$ . Thus,  $W_1 \oplus W_2 = F^n$ . ■

### Exercise 1.3.25

Let  $W_1$  denote the set of all polynomials  $f(x)$  in  $P(F)$  such that in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

we have  $a_i = 0$  whenever  $i$  is even. Likewise, let  $W_2$  denote the set of all polynomials  $g(x)$  in  $P(F)$  such that in the representation

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0,$$

we have  $b_i = 0$  whenever  $i$  is odd. Prove that  $P(F) = W_1 \oplus W_2$ .

**Proof.** Let  $f(x) \in W_1$ . Then we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n$$

where  $a_i = 0$  where  $i$  is even. Likewise, let  $g(x) \in W_2$ . Hence,

$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_{n-1} x^{n-1} + b_n x^n$$

where  $b_i = 0$  for  $i$  odd. This tells us that the only representation that  $W_1$  and  $W_2$  have in common is the zero polynomial  $f(x) = 0$  where  $a_i = 0$  for all  $0 \leq i \leq n$ . Hence,  $W_1 \cap W_2 = \{0\}$ .

Now, we want to show that  $W_1 + W_2 = P(F)$ ; that is, we need to show  $P(F) \subseteq W_1 + W_2$  and  $W_1 + W_2 \subseteq P(F)$ . Starting with the former, let  $f \in P(F)$ . Observe that for  $a_i$  where  $1 \leq i \leq n$

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n \\ &= \underbrace{(a_1 x + a_3 x^3 + \dots + a_n x^n)}_{\in W_1} + \underbrace{(a_0 + a_2 x^2 + a_4 x^4 + \dots + a_n x^n)}_{\in W_2} \end{aligned}$$

The first term of the last equality contains coefficients  $a_i = 0$  for even  $i$  and the second term contains coefficients  $a_i = 0$  for odd  $i$ . This implies that  $f(x) \in W_1 + W_2$ . Hence,  $P(F) \subseteq W_1 + W_2$ .

Now, let  $u(x) \in W_1 + W_2$  with  $u(x) = f(x) + g(x)$  with  $f(x) \in W_1$  and  $g(x) \in W_2$ . Since  $W_1$  and  $W_2$  are subsets of  $P(F)$ , we have that  $f(x), g(x) \in P(F)$ . Since addition is closed in  $P(F)$ , we have that  $u(x) = f(x) + g(x) \in P(F)$ . Hence,  $W_1 + W_2 \subseteq P(F)$ . Since  $W_1 + W_2 = P(F)$  and  $W_1 \cap W_2 = \{0\}$ , we have  $W_1 \oplus W_2 = P(F)$ . ■

### Exercise 1.3.26

In  $M_{m \times n}(F)$  define  $W_1 = \{A \in M_{m \times n}(F) : A_{ij} = 0 \text{ whenever } i > j\}$  and  $W_2 = \{A \in M_{m \times n}(F) : A_{ij} \text{ whenever } i \leq j\}$ . Show that  $M_{m \times n}(F) = W_1 \oplus W_2$ . ( $W_1$  is the set of all triangular matrices as defined in the previous section.)

**Proof.** TO DO. ■

### Exercise 1.3.30

Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if each vector in  $V$  can be *uniquely* written as  $x_1 + x_2$ , where  $x_1 \in W_1$  and  $x_2 \in W_2$ .

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of  $V$ . Let  $v \in V$ . Let  $x_1, x'_1 \in W_1$  and  $x_2, x'_2 \in W_2$  such that  $v = x_1 + x_2 = x'_1 + x'_2$ . Since  $W_1$  and  $W_2$  is closed under addition, we know that  $x_1 - x'_1 \in W_1$  and  $x_2 - x'_2 \in W_2$ . But observe that  $x_1 - x'_1 = x_2 - x'_2 \in W_1 \cap W_2$ . Since  $V$  is a direct sum of the two subspaces  $W_1$  and  $W_2$ , we know that  $W_1 \cap W_2 = \{O_V\}$  which implies that  $x_1 = x'_1$  and  $x_2 = x'_2$ . This tells us that every  $v$  can be expressed uniquely as  $x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ .

Conversely, suppose every vector  $v \in V$  can be *uniquely* written as  $x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ . We need to show that  $W_1 \oplus W_2 = V$ . Since  $v \in V$  is uniquely expressed in terms of  $x_1 + x_2$ , the only vector that the two subspaces  $W_1$  and  $W_2$  share is the zero vector. Hence,  $W_1 \cap W_2 = \{O_V\}$ . Now, we need to show that  $V = W_1 + W_2$ ; that is, we need to show  $V \subseteq W_1 + W_2$  and  $W_1 + W_2 \subseteq V$ . Suppose  $v \in V$ . Since  $x_1 \in W_1$  and  $x_2 \in W_2$ , we have that  $v \in W_1 + W_2$ . Hence,  $V \subseteq W_1 + W_2$ . Now, let  $v \in W_1 + W_2$ . Since  $v = x_1 + x_2$  where  $x_1 \in W_1$  and  $x_2 \in W_2$  and  $W_1 \subseteq V$  and  $W_2 \subseteq V$ , we know that  $x_1$  and  $x_2$  are contained within  $V$ . Since  $V$  is vector space, we know that sum of  $x_1$  and  $x_2$  are contained in  $V$ . Hence,  $v \in V$  and so  $V \subseteq W_1 + W_2$ . Thus,  $V = W_1 \oplus W_2$ . ■

## 1.2 Linear Combinations

### Exercise 1.4.7

In  $F^n$ , let  $e_j$  denote the vector whose  $j$ th coordinate is 1 and whose other coordinates are 0. Prove that  $\{e_1, e_2, \dots, e_n\}$  generates  $F^n$ .

**Proof.** Denote the set  $V = \{e_1, e_2, \dots, e_n\}$ . Our goal is to find scalars  $\delta_1, \delta_2, \dots, \delta_n \in F$  such that for all  $e_j \in V$  for  $1 \leq j \leq n$ ,

$$\delta_1 e_1 + \delta_2 e_2 + \dots + \delta_n e_n = (a_1, a_2, \dots, a_n). \quad (1)$$

Since  $e_j = 1$  for the  $j$ th coordinate and the rest of the entries are zeros, we have that

$$\delta_j e_j = \delta_j (0, 0, \underbrace{1}_{j\text{th entry}}, \dots, 0) = (0, 0, \delta_j, \dots, 0)$$

for all  $1 \leq j \leq n$ . Summing up each term on the left side of (1) and equating each coordinate to the right side of (1), we get that  $a_j = \delta_j$  for each  $1 \leq j \leq n$ . Hence, the set  $V$  can span the vector space  $F^n$ . ■

### Exercise 1.4.8

Show that  $P_n(F)$  is generated by  $\{1, x, \dots, x^n\}$ .

**Proof.** Let  $V = \{1, x, \dots, x^n\}$ . We need to find scalars  $\delta_1, \delta_2, \dots, \delta_n \in F$  such that

$$\delta_0 + \delta_1 x + \delta_2 x^2 + \dots + \delta_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n. \quad (1)$$

Immediately, we have that equating coefficients in (1) gives us  $a_i = \delta_i$  for all  $1 \leq i \leq n$ . Hence,  $V$  generates  $P_n(F)$ . ■

### Exercise 1.4.9

Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate  $M_{2 \times 2}(F)$ .

**Proof.** Define  $V$  with the given  $2 \times 2$  matrices above. We need to find scalars  $\delta_i \in F$  for all  $1 \leq i \leq 4$

such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \delta_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \delta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (1)$$

Distributing each scalar  $\delta_i$  for all  $1 \leq i \leq 4$  for each term in (1), summing up each matrix in (1), and equating each term entry-wise, we have that  $a_{11} = \delta_1$ ,  $a_{12} = \delta_2$ ,  $a_{21} = \delta_3$ , and  $a_{22} = \delta_4$ . Hence,  $V$  spans  $M_{2 \times 2}(F)$ . ■

#### Exercise 1.4.10

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of  $\{M_1, M_2, M_3\}$  is the set of all symmetric  $2 \times 2$  matrices.

**Proof.** We need to show that the set  $\{M_1, M_2, M_3\}$  spans the set of all symmetric  $2 \times 2$  matrices. We need to find scalars  $a_1, a_2, a_3$  such that

$$a_1 M_1 + a_2 M_2 + a_3 M_3 = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = A \quad (1)$$

where  $A$  is any  $2 \times 2$  symmetric matrix. Observe that for  $i = j$ , we have  $\delta_{12} = \delta_{21}$ . Performing scalar multiplication on matrices  $M_1, M_2$ , and  $M_3$ , addition of all three terms on (1), and equating entry-wise, we get that  $\delta_{12} = \delta_{21} = \delta_3$  and  $\delta_{11} = \delta_1$  and  $\delta_2 = a_{22}$ . Hence, we have

$$\begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^t = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}.$$

Thus, the set  $\{M_1, M_2, M_3\}$  spans the set of all symmetric  $2 \times 2$  matrices. ■

#### Exercise 1.4.11

Prove that  $\text{span}(\{x\}) = \{ax : a \in F\}$  for any vector  $x$  in a vector space  $V$ . Interpret this result geometrically in  $\mathbb{R}^3$ .

**Proof.** We need to show that  $\text{span}(\{x\}) = \{ax : a \in F\}$ , we need to show two containments; that is,  $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$  and  $\{ax : a \in F\} \subseteq \text{span}(\{x\})$ . Let  $v \in \text{span}(\{x\})$ . Then observe that we can find  $\delta \in F$  such that multiplying by  $x \in V$  leads  $v = \delta x$ . But this means that  $v \in \{ax : a \in F\}$  by definition. Hence,  $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$ . Let  $v \in \{ax : a \in F\}$ . Then  $v = ax$  for some  $a \in F$ . But this is a linear combination of  $x$  that makes  $v$ . So  $v \in \text{span}(\{x\})$  and hence,  $\{ax : a \in F\} \subseteq \text{span}(\{x\})$ . This result can be viewed as the scaling of vectors in  $\mathbb{R}^3$ . ■

#### Exercise 1.4.12

Show that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $\text{span}(W) = W$ .

**Proof.** ( $\Rightarrow$ ) Let  $W$  be a subspace of  $V$ . To show that  $\text{span}(W) = W$ , we need to show two containments; that is,  $\text{span}(W) \subseteq W$  and  $W \subseteq \text{span}(W)$ . Clearly,  $W$  contains itself. Hence,  $\text{span}(W) \subseteq W$  Theorem 5. Let  $v \in W$ . Since  $W$  is a subspace, we can find scalars  $a_1, a_2, \dots, a_n \in F$  and vectors  $w_1, w_2, \dots, w_n \in W$  such that

$$v = a_1 w_1 + a_2 w_2 + \dots + a_n w_n \in W$$

using the result in Exercise 1.3.20. But this tells us that  $v \in \text{span}(W)$ . Hence,  $W \subseteq \text{span}(W)$ .

( $\Leftarrow$ ) Since the span of any subset of  $W$  is a subspace and  $W = \text{span}(W)$ , we have that  $W$  is a subspace as well by Theorem 5. ■

### Exercise 1.4.13

Show that if  $S_1$  and  $S_2$  are subsets of a vector space  $V$  such that  $S_1 \subseteq S_2$ , then  $\text{span}(S_1) \subseteq \text{span}(S_2)$ . In particular, if  $S_1 \subseteq S_2$  and  $\text{span}(S_1) = V$ , deduce that  $\text{span}(S_2) = V$ .

**Proof.** Let  $S_1$  and  $S_2$  be subsets of a vector space  $V$ . Let  $v \in \text{span}(S_1)$ . We can find scalars  $\delta_1, \delta_2, \dots, \delta_n \in F$  and  $x_1, x_2, \dots, x_n \in S_1$  such that

$$v = \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n.$$

Since  $S_1 \subseteq S_2$ , we know that  $x_1, x_2, \dots, x_n \in S_2$  so we must have  $v \in \text{span}(S_2)$ . Hence,  $\text{span}(S_1) \subseteq \text{span}(S_2)$ .

Now, let  $\text{span}(S_1) = V$ . We need to show that  $\text{span}(S_2) = V$ ; that is, we need to show  $\text{span}(S_2) \subseteq V$  and  $V \subseteq \text{span}(S_2)$ . By assumption,  $S_2 \subseteq V$  and  $\text{span}(S_2)$  is a subspace. Clearly,  $\text{span}(S_2) \subseteq V$ . Since  $\text{span}(S_1) = V$  and  $\text{span}(S_1) \subseteq \text{span}(S_2)$ , we have  $V \subseteq \text{span}(S_2)$ . Hence,  $\text{span}(S_2) = V$ . ■

### Exercise 1.4.14

Show that if  $S_1$  and  $S_2$  are arbitrary subsets of a vector space  $V$ , then  $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$ . (The sum of two subsets is defined in the exercises of Section 1.3.)

**Proof.** Let  $S_1$  and  $S_2$  be subsets of a vector space  $V$ . We need to show  $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$ ; that is,  $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$  and  $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$ . Let  $v \in \text{span}(S_1 \cup S_2)$ . We can find  $\delta_1, \delta_2, \dots, \delta_n \in F$  such that  $x_1, x_2, \dots, x_n \in S_1 \cup S_2$  implies that

$$v = \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n.$$

Hence, either  $x_1, x_2, \dots, x_n \in S_1$  or  $x_1, x_2, \dots, x_n \in S_2$ . If  $x_1, x_2, \dots, x_n \in S_1$ , then  $v \in \text{span}(S_1)$ . Since  $\text{span}(S_2)$  is a subspace, we know that  $0_V \in \text{span}(S_2)$ . Hence,  $0_V \in \text{span}(S_1)$  and  $v \in \text{span}(S_1)$  imply that  $v + 0_V = v \in \text{span}(S_1) + \text{span}(S_2)$ . The other case follows a similar process. Hence,  $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$ .

Let  $s \in \text{span}(S_1) + \text{span}(S_2)$ . Hence,  $s = u + v$  where  $u \in \text{span}(S_1)$  and  $v \in \text{span}(S_2)$ . The former implies that we can find scalars  $a_1, a_2, \dots, a_n \in F$  such that  $x_1, x_2, \dots, x_n \in S_1$  where

$$u = \sum_{i=1}^n a_i x_i$$

and the latter implies that there exists scalars  $b_1, b_2, \dots, b_n \in F$  such that  $y_1, y_2, \dots, y_n \in S_2$  where

$$v = \sum_{i=1}^n b_i y_i.$$

Since both  $x_i \in S_1$  and  $y_i \in S_2$  for all  $1 \leq i \leq n$ , we have  $x_i, y_i \in S_1 \cup S_2$  for all  $1 \leq i \leq n$ . So we must have  $s \in \text{span}(S_1 \cup S_2)$ . Hence,  $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$ . ■

### Exercise 1.4.15

Let  $S_1$  and  $S_2$  be subsets of a vector space  $V$ . Prove that  $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$ . Give an example in which  $\text{span}(S_2 \cap S_2)$  and  $\text{span}(S_1) \cap \text{span}(S_2)$  are equal and one in which they are not unequal.

**Proof.** Let  $S_1$  and  $S_2$  be subsets of a vector space  $V$ . Let  $v \in \text{span}(S_1 \cap S_2)$ . Then we can find scalars  $a_i \in F$  and vectors  $x_i \in S_1 \cap S_2$  for all  $1 \leq i \leq n$  such that

$$v = \sum_{i=1}^n a_i x_i.$$

If  $x_i \in S_1 \cap S_2$  for all  $1 \leq i \leq n$ , then  $x_i \in S_1$  and  $x_i \in S_2$  for all  $1 \leq i \leq n$ . This implies



that  $v \in \text{span}(S_1)$  and  $v \in \text{span}(S_2)$ . Hence,  $v \in \text{span}(S_1) \cap \text{span}(S_2)$ . Thus, we conclude that  $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$ . ■

**Example.** Define  $S_1$  as the set

$$\{(1, 1, 0) \in \mathbb{R}^3\}$$

and  $S_2$  as the set

$$\{(1, 1, 0), (1, 0, 1), (0, 1, 1) \in \mathbb{R}^3\}.$$

Observe that  $S_1 \cap S_2 = \{(1, 1, 0)\}$  and thus the of this set yields  $\{a(1, 1, 0) : a \in F \text{ and } (1, 1, 0) \in \mathbb{R}^3\}$ . The span of  $S_1$  yields the following set  $\{a(1, 1, 0) : a \in F \text{ and } (1, 1, 0) \in \mathbb{R}^3\}$  and the span of  $S_2$  yields the following set

$$\{a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) : a, b, c \in F \text{ and } (1, 1, 0), (1, 0, 1), (0, 1, 1) \in \mathbb{R}^3\}.$$

Note that  $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$  since  $\text{span}(S_1) \cap \text{span}(S_2) \neq \emptyset$ .

Now define  $S_1 = \{O_V\} = S_2$ . Clearly,  $S_1 \cap S_2 = \{O_V\}$ ,  $\text{span}(S_1) \cap \text{span}(S_2) = \{O_V\}$ , and that  $\text{span}(S_1 \cap S_2)$  is also equal to this set.  $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

### Exercise 1.4.16

Let  $V$  be a vector space and  $S$  a subset of  $V$  with the property that whenever  $v_1, v_2, \dots, v_n \in S$  and  $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ , then  $a_1 = a_2 = \dots = a_n = 0$ . Prove that every vector in the span of  $S$  can be *uniquely* written as a linear combination of vectors of  $S$ .

**Proof.** Let  $x \in \text{span}(S)$ . Suppose there exists two sets of scalars  $a_1, a_2, \dots, a_n \in F$  and  $b_1, b_2, \dots, b_n \in F$  such that whenever  $v_1, v_2, \dots, v_n \in S$  such that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = x \tag{1}$$

and

$$b_1v_1 + b_2v_2 + \dots + b_nv_n = x \tag{2}$$

implies that  $a_i$ . Setting (1) and (2) together, we have

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = b_1v_1 + b_2v_2 + \dots + b_nv_n \tag{3}$$

which implies that

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \dots + (a_n - b_n)v_n = 0.$$

Since  $v_1, v_2, \dots, v_n \in S$ , we have that  $a_i - b_i = 0$  and thus  $a_i = b_i$  for all  $1 \leq i \leq n$ . Hence, every vector in the span of  $S$  can be *uniquely* written as a linear combination of vectors of  $S$ . ■

## 1.3 Linear Dependence and Linear Independence

### Exercise 1.5.1

Label the following statements as true or false.

- (a) If  $S$  is a linearly dependent set, then each vector in  $S$  is a linear combination of other vector in  $S$ .

**Proof. True** ■

- (b) Any set containing the zero vector is linearly dependent.

**Proof. True** ■

- (c) The empty set is linearly dependent.

**Proof. False.** It is linearly independent. ■

- (d) Subsets of linearly dependent sets are linearly dependent.

**Proof.** This is **False**. We can have a linearly independent subset of a set that is linearly dependent. ■

(e) Subsets of linearly independent sets are linearly independent.

**Proof.** **True** by corollary to Theorem 6. ■

(f) If  $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$  and  $x_1, x_2, \dots, x_n$  are linearly independent, then all the scalars  $a_i$  are zero.

**Proof.** **True** this is by definition. ■

#### Exercise 1.5.4

In  $F^n$ , let  $e_j$  denote the vector whose  $j$ th coordinate is 1 and whose other coordinates are 0. Prove that  $\{e_1, e_2, \dots, e_n\}$  is linearly independent.

**Proof.** Choose a finite amount of scalars  $a_1, a_2, \dots, a_n \in F$  to create the following linear combination:

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = (0, 0, \dots, 0). \quad (1)$$

To show that the set  $\{e_1, e_2, \dots, e_n\}$  is linearly independent, we need to show that the scalars  $a_1, a_2, \dots, a_n \in F$  have the trivial representation; that is,  $a_1 = a_2 = \cdots = a_n = 0$ . Since the  $j$ th coordinate of  $e_j$  is 1 but 0 in all the other entries, we have that

$$\begin{aligned} & a_1(1, 0, \dots, 0) + a_2(0, 1, \dots, 0) + \cdots + a_n(0, 0, \dots, 1) \\ &= (a_1, 0, \dots, 0) + (0, a_2, \dots, 0) + \cdots + (0, 0, \dots, a_n) \\ &= (a_1, a_2, \dots, a_n). \end{aligned}$$

Hence, we have

$$(a_1, a_2, \dots, a_n) = (0, 0, \dots, 0).$$

Equating each entry of the left side of the equation above to 0, we find that  $a_i = 0$  for all  $1 \leq i \leq n$ . Hence, the set  $\{e_1, e_2, \dots, e_n\}$  is linearly independent. ■

#### Exercise 1.5.4

Show that the set  $\{1, x, x^2, \dots, x^n\}$  is linearly independent in  $P_n(F)$ .

**Proof.** Just like the prior exercise, we need to show that we can find scalars  $a_0, a_1, \dots, a_n \in F$  such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

where  $a_i = 0$  for all  $0 \leq i \leq n$ . Note that the 0 polynomial is just

$$0 + 0x + 0x^2 + \cdots + 0x^n = 0.$$

Hence, equating coefficients we immediately get that  $a_i = 0$  for all  $0 \leq i \leq n$ . Thus, the set  $\{1, x, x^2, \dots, x^n\}$  is linearly independent. ■

#### Exercise 1.5.6

In  $M_{m \times n}(F)$ , let  $E^{ij}$  denote the matrix whose only nonzero entry is 1 in the  $i$ th row and  $j$ th column. Prove that  $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$  is linearly independent.

**Proof.** First, we create a linear combination of a finite amount vectors in  $E = \{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$  with scalars  $\delta_k$  for  $1 \leq k \leq N$  with  $N = mn$  as the number of total entries in each matrix in  $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ . Note that after doing our scalar multiplication and summing up each term, we find that each  $\delta_k E_{ij} = \delta_k$  in our linear combination can equated with a corresponding  $i$  and  $j$  entry in the zero matrix such that  $\delta_k = 0$  for all  $1 \leq k \leq N$ . Hence,  $E$  is a linearly independent set. ■

### Exercise 1.5.7

Recall from Example 3 in Section 1.3 that the set of diagonal matrices in  $M_{2 \times 2}(F)$  is a subspace. Find a linearly independent set that generates this subspace.

**Proof.** Define  $W$  as the linearly independent spanning set of the set of diagonal matrices in  $M_{2 \times 2}$  where

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

To see why  $W$  is a linearly independent set, choose scalars  $\delta_1, \delta_2 \in F$  such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing scalar multiplication and vector addition gives us the following equation

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the zero matrix is a diagonal matrix, we know that equation entries where  $i = j$  yields  $\delta_1 = \delta_2 = 0$ . Hence,  $W$  is a linearly independent set that generates the set of diagonal matrices of  $M_{2 \times 2}(F)$ . ■

### Exercise 1.5.8

Let  $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  be a subset of the vector space  $F^3$ .

- (a) Prove that if  $F = \mathbb{R}$ , then  $S$  is linearly independent.

**Proof.** ■

- (b) Prove that if  $F$  has characteristic two, then  $S$  is linearly dependent.

**Proof.** ■

### Exercise 1.5.9

Let  $u$  and  $v$  be distinct vectors in a vector space  $V$ . Show that  $\{u, v\}$  is linearly dependent if and only if  $u$  or  $v$  is a multiple of the other. I have written two proofs for this:

**Proof.** Let  $u$  and  $v$  be distinct vectors in a vector space  $V$ .

( $\Rightarrow$ ) Since  $\{u, v\}$  is a linearly dependent set, we can find scalars  $a_1, a_2 \in F$  such that

$$a_1 u + a_2 v = 0 \tag{1}$$

Suppose  $v$  is not a multiple of  $u$  and choose  $a_1 \neq 0$  since  $\{u, v\}$  is linearly dependent. We need to show that  $u$  is a multiple of  $v$ . Solving for  $u$ , we get that

$$u = -\frac{a_2}{a_1} v.$$

Hence,  $u$  is a multiple of  $v$ .

( $\Leftarrow$ ) Suppose  $u$  or  $v$  is a scalar multiple of the other. Assume  $u$  is the scalar multiple of  $v$ . Then for some  $c \neq 0 \in F$ , we have  $u = cv$ . Hence, we have  $u - cv = 1u - cv = 0$ . This tells us that  $\{u, v\}$  is linearly dependent. ■

### Exercise 1.5.12

Prove Theorem 1.6 and its corollary.

**Proof.** See proof in notes. ■

### Exercise 1.5.13

Let  $V$  be a vector space over a field of characteristic not equal to two.

- (a) Let  $u$  and  $v$  be distinct vectors in  $V$ . Prove that  $\{u, v\}$  is linearly independent if and only if  $\{u + v, u - v\}$  is linearly independent.

**Proof.** Let  $u$  and  $v$  be distinct vectors in  $V$ .

For the forwards direction, assume  $\{u, v\}$  is a linearly independent set. We need to show that  $\{u + v, u - v\}$  is linearly independent. Hence, we need to find  $a, b \in F$  such that

$$a(u + v) + b(u - v) = 0. \quad (1)$$

Note that (1) leads to

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu. \end{aligned}$$

Since  $\{u, v\}$  is a linearly independent set, we have that

$$au - bv = 0$$

and

$$av + bu = 0$$

for  $a = b = 0$ . Hence,

$$a(u + v) + b(u - v) = 0$$

for  $a = b = 0$  and so  $\{u - v, u + v\}$  is a linearly independent set.

For the backwards direction, suppose  $\{u + v, u - v\}$  is linearly independent. We need to show that  $\{u, v\}$  is linearly independent. Note that  $a, b \in F$  such that

$$a(u + v) + b(u - v) = 0$$

for  $a = b = 0$  since  $\{u - v, u + v\}$  is linearly independent. Note that

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu \\ &= 0 + av + bu \\ &= 0. \end{aligned}$$

Thus,  $av + bu = 0$  where  $a, b$  both zero. Thus, the set  $\{u, v\}$  is linearly independent. ■

- (b) Let  $u, v$ , and  $w$  be distinct vectors in  $V$ . Prove that  $\{u, v, w\}$  is linearly independent if and only if  $\{u + v, u + w, v + w\}$  is linearly independent.

**Proof.** For the forwards direction, suppose  $\{u, v, w\}$  is linearly independent. Then choose scalars  $a_1, a_2, a_3 \in F$  such that

$$a_1u + a_2v + a_3w = 0$$

with  $a_1 = a_2 = a_3 = 0$ . We need to show that  $\{u + v, u + w, v + w\}$  is linearly independent; that is, we need to show that we can find scalars  $a_1, a_2, a_3 \in F$  such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0 \quad (1)$$

for  $a_1 = a_2 = a_3 = 0$ . Observe that (1) can be written in the following way

$$(a_1u + a_3v + a_2w) + (a_1v + a_2u + a_3w) = 0 \quad (2)$$

Since  $\{u, v, w\}$  is linearly independent, we know that  $a_1 = a_2 = a_3 = 0$ . But this also has to mean that  $\{u + v, u + w, v + w\}$  is a linearly independent set.

For the backwards direction, suppose  $\{u + v, u + w, v + w\}$  is linearly independent. Then choose scalars  $a_1, a_2, a_3 \in F$  such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0. \quad (1)$$

We need to show that  $\{u, v, w\}$  is linearly independent. Observe that (1) can be re-written as

$$\begin{aligned} (a_1u + a_3v + a_2w) + (a_1v + a_2u + a_3w) &= 0 \\ \Rightarrow 0 + (a_1v + a_2u + a_3w) &= 0 \\ \Rightarrow a_1v + a_2u + a_3w &= 0 \end{aligned}$$

where  $a_1 = a_2 = a_3 = 0$ . Hence,  $\{u, v, w\}$  is linearly independent. ■

### Exercise 1.5.14

Prove that a set  $S$  is linearly dependent if and only if  $S = \{0\}$  or there exists distinct vectors  $v, u_1, u_2, \dots, u_n \in S$  such that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$ .

**Proof.** For the forwards direction, Let  $S$  be a linearly dependent. Then we need to show that either  $S = \{0\}$  or  $S$  contains distinct vectors  $v, u_1, u_2, \dots, u_n \in S$  such that  $v \in \text{span}(S)$ . Suppose there does not exist distinct vectors  $v, u_1, u_2, \dots, u_n \in S$  such that  $v \in \text{span}(S)$ . This tells us that  $S$  only contains the singleton  $S = \{v\}$ . Furthermore, we must require  $v = 0$  since  $S$  is linearly dependent. Otherwise,  $v \neq 0$  would imply that  $S$  is linearly independent. Hence, we have  $S = \{0\}$ . Now suppose  $S \neq \{0\}$ . Since  $S$  is linearly dependent, there exists scalars  $a_1, a_2, \dots, a_{n+1}$  and vectors  $v, u_1, u_2, \dots, u_n$  such that

$$a_1v + a_2u_1 + a_3u_2 + \dots + a_{n+1}u_n = 0$$

with all  $a_1, a_2, \dots, a_n, a_{n+1}$  not all equal to zero. Solving for  $v$ , we get that

$$v = -(a_1^{-1}a_2)u_1 - (a_1^{-1}a_3)u_2 - \dots - (a_1^{-1}a_{n+1})u_n.$$

Since  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$ , we have that  $v \in \text{span}(S)$ .

Conversely, suppose that either  $S = \{0\}$  or there exists distinct vectors  $v, u_1, u_2, \dots, u_n \in S$  such that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$ . Assume  $S = \{0\}$ . Then  $S$  is linearly dependent because the singleton is the zero vector. Now suppose  $v$  is a linear combination of vectors  $u_1, u_2, \dots, u_n$ . Then there exists scalars  $a_1, a_2, \dots, a_n \in F$  and distinct vectors  $u_1, u_2, \dots, u_n \in S$  such that

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n.$$

Subtracting  $v$  from both sides of this equation yields the following equation

$$a_1u_1 + a_2u_2 + \dots + a_nu_n - 1v = 0.$$

Since not all scalars in the equation above are zero and  $v, u_1, u_2, \dots, u_n \in S$ , we must have that  $S$  is a linearly dependent set. ■

### Exercise 1.5.15

Prove that a set  $S = \{u_1, u_2, \dots, u_n\}$  be a finite set of vectors. Prove that  $S$  is linearly dependent if and only if  $u_1 = 0$  or  $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$  for some  $k$  where  $1 \leq k < n$ .

**Proof.** Suppose  $u_{k+1} \notin \text{span}(\{u_1, u_2, \dots, u_k\})$  for all  $1 \leq k < n$ . Since  $S$  is linearly dependent, we know that the zero vector is contained in  $S$ . Choose  $k = 1$  such that  $u_1 = 0$  and we are done. On the other hand, suppose  $u_1 \neq 0$ . We need to show that  $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$ . Choose  $k = n - 1$ . Then clearly  $n = k + 1$ . Since  $S$  is linearly dependent, choose scalars  $a_1, a_2, \dots, a_n \in F$  such that

$$a_1u_1 + a_2u_2 + \dots + a_ku_k + a_{k+1}u_{k+1} = 0. \quad (1)$$

where  $a_1, a_2, \dots, a_k, a_{k+1}$  not all zero. Solving for  $u_{k+1}$  by subtracting  $a_{k+1}u_{k+1}$  on both sides of (1) and multiplying  $-a_{k+1}^{-1}$  on both sides of (1), we end up with the following equation:

$$u_{k+1} = -(a_{k+1}^{-1}a_1)u_1 - (a_{k+1}^{-1}a_2)u_2 - \dots - (a_{k+1}^{-1}a_k)u_k.$$

This tells us that  $u_{k+1}$  can be written as a linear combination of vectors  $u_1, u_2, \dots, u_k$ . Hence,  $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$ .

Conversely, either  $u_1 = 0$  or  $u_{k+1} \in \text{span}(u_1, u_2, \dots, u_k)$ . Suppose  $u_1 = 0$ . Then  $S$  contains the zero vector so  $S$  must be linearly dependent. On the other hand, choose  $k = n - 1$  where  $1 \leq k < n$  such that  $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$  implies that there exists scalars  $a_1, a_2, \dots, a_k$  such that

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \dots + a_ku_k \\ \Rightarrow u_n &= a_1u_1 + a_2u_2 + \dots + a_{n-1}u_{n-1}. \end{aligned} \tag{1}$$

Subtracting  $u_n$  on both sides of (1) implies that

$$(a_1u_1 + a_2u_2 + \dots + a_{n-1}u_{n-1}) - 1u_n = 0.$$

Since not all scalars in the linear combination above are zero, we know that  $S$  must be linearly dependent. ■

#### Exercise 1.5.16

Prove that a set  $S$  of vectors is linearly independent if and only if each finite subset of  $S$  is linearly independent.

**Proof.** ( $\Rightarrow$ ) Suppose  $S$  is a linearly independent set. Let  $S'$  be any finite subset of  $S$ . By corollary to Theorem 6, we can see that  $S' \subseteq S$  implies that  $S'$  is also linearly independent. ( $\Leftarrow$ ) We will proceed by proving the contrapositive. Let  $S' \subseteq S$  be a finite subset that is linearly dependent set. We will prove that  $S$  is a linearly dependent set. Since  $S' \subseteq S$ , we have that  $S$  must be a linearly dependent set by Theorem 6. ■

#### Exercise 1.5.17

Let  $M$  be a square upper triangular matrix (as defined in Section 1.3) with nonzero diagonal entries. Prove that the columns of  $M$  are linearly independent.

**Proof.** Let  $M$  be a square upper triangular matrix. Note that  $M$  have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & a_{n-1n} \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

We can define the following columns as vectors  $v_1, v_2, \dots, v_n$  where

$$v_1 = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ 0 \end{pmatrix}, \dots, v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Choose scalars  $\delta_1, \delta_2, \dots, \delta_n \in F$  such that

$$\delta_1v_1 + \delta_2v_2 + \dots + \delta_nv_n = 0$$

where the zero vector  $0$  is denoted as

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using scalar multiplication and addition, we arrive at the following system of equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} + \cdots + \delta_{n-1} a_{1n-1} + \delta_n a_{1n} &= 0 \\ \delta_2 a_{22} + \delta_3 a_{23} + \cdots + \delta_n a_{2n} &= 0 \\ &\vdots \\ \delta_{n-1} a_{n-1n-1} + \delta_n a_{n-1n} &= 0 \\ \delta_n a_{nn} &= 0. \end{aligned}$$

Since all the diagonal entries of  $M$  are non-zero, we can see from the equation above that  $\delta_n = 0$  which subsequently tells us that  $\delta_{n-1} = 0$ . We claim that  $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$  for all  $n \geq 1$ . We can prove this via induction. Let our base case be  $n = 1$ . Then we have  $\delta_1 a_{11} = 0$  with  $a_{11} \neq 0$  implies  $\delta_1 = 0$ . Now let  $n = 2$ . Then observe that we have an upper triangular  $2 \times 2$  matrix such that the linear combination of the columns lead to the following system of linear equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} &= 0 \\ 0 + \delta_2 a_{22} &= 0. \end{aligned}$$

Observe that  $\delta_2 = 0$  which also implies that  $\delta_1 = 0$ . Now suppose our claim that  $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$  holds for all  $n \geq 1$ . We want to show that our claim still holds for the  $n + 1$  case. Observe that  $\delta_{n+1} = 0$  derived from an  $(n+1) \times (n+1)$  matrix. Using the same process that proved the base case, we find that  $\delta_n = \delta_{n+1} = 0$ . By our inductive hypothesis, we know that  $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$  for all  $n \geq 1$ . This tells us that  $\delta_1 = \cdots = \delta_{n+1}$  for all  $n \geq 1$ . Hence, the columns of  $M$  are linearly independent. ■

### Exercise 1.5.18

Let  $S$  be a set of nonzero polynomials in  $P(F)$  such that no two have the same degree. Prove that  $S$  is linearly independent.

**Proof.** Let  $0 \leq n \leq k$  such that  $S = \{p_0(x), p_1(x), p_2(x), \dots, p_k(x)\}$  where every  $p_n(x)$  non-zero such that no two polynomials in this set have the same degree. Define  $p_n(x) = x^n + x^{n+1} + \cdots + x^k$ . Choose scalars  $a_1, a_2, \dots, a_k \in F$  such that

$$a_1 p_1(x) + a_2 p_2(x) + \cdots + a_k p_k(x) = 0 \quad (1)$$

We need to show that  $\delta_1 = \delta_2 = \cdots = \delta_k = 0$ . Observe that (1) implies that

$$a_0 + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_k)x^k = 0. \quad (2)$$

Setting each  $x^n$  to both sides of the equation above leads to the following system of linear equations:

$$\begin{aligned} a_0 &= 0 \\ a_0 + a_1 &= 0 \\ a_0 + a_1 + a_2 &= 0 \\ &\vdots \\ a_0 + a_1 + a_2 + \cdots + a_k &= 0. \end{aligned}$$

It can be proved via induction that  $a_1 = a_2 = \cdots = a_k = 0$  for all  $1 \leq n \leq k$ . Hence,  $S$  is a linearly independent set. ■

### Exercise 1.5.19

Prove that if  $\{A_1, A_2, \dots, A_k\}$  is a linearly independent subset of  $M_{n \times n}(F)$ , then  $\{A_1^t, A_2^t, A_3^t, \dots, A_k^t\}$  is also linearly independent.

**Proof.** Suppose  $\{A_1, A_2, \dots, A_k\}$  is a linearly independent set. Choose a finite set of scalars  $\delta_1, \delta_2, \dots, \delta_k \in F$  such that

$$\delta_1 A_1 + \delta_2 A_2 + \dots + \delta_k A_k = 0$$

with  $\delta_1 = \delta_2 = \dots = \delta_k = 0$ . Apply the transpose to both sides, we know that  $0^t = 0$  and  $(\delta_i A_i)^t = \delta_i (A_i)^t$  for all  $1 \leq i \leq k$ . Hence, we have

$$\begin{aligned} (\delta_1 A_1 + \delta_2 A_2 + \dots + \delta_k A_k)^t &= 0^t \\ \Rightarrow (\delta_1 A_1)^t + (\delta_2 A_2)^t + \dots + (\delta_k A_k)^t &= 0 \\ \Rightarrow \delta_1 A_1^t + \delta_2 A_2^t + \dots + \delta_k A_k^t &= 0. \end{aligned}$$

Hence, the set  $\{A_1^t, A_2^t, \dots, A_k^t\}$  is also linearly independent. ■

### Exercise 1.5.20

Let  $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$  be the functions defined by  $f(t) = e^{rt}$  and  $g(t) = e^{st}$ , where  $r \neq s$ . Prove that  $f$  and  $g$  are linearly independent in  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .

**Proof.** Let  $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$  where  $f$  and  $g$  are defined by  $f(t) = e^{rt}$  and  $g(t) = e^{st}$  with  $r \neq s$ . Suppose for sake of contradiction that the set  $\{f, g\}$  is a linearly dependent set of vectors. By Exercise 1.5.9, we know that either  $f$  is a multiple of  $g$  or  $g$  is a multiple of  $f$ . Assume  $f$  is a multiple of  $g$ . Hence, there exists a  $c \in \mathbb{R}$  such that  $f(t) = cg(t)$ . In other words,

$$f(t) = cg(t) \Rightarrow e^{rt} = ce^{st}.$$

To solve for  $c$ , let  $t = 0$ . Then we have

$$f(0) = cg(0) \Rightarrow 1 = c.$$

Hence, we have

$$e^{rt} = e^{st}.$$

This equality is valid only when  $r = s$  because otherwise  $f$  would not be a scalar multiple of  $g$ . But note that  $r \neq s$  by assumption. Hence, we have a contradiction and thus  $\{f, g\}$  must be linearly independent. ■

### Exercise 1.5.21

Let  $S_1$  and  $S_2$  be disjoint linearly independent subsets of  $V$ . Prove that  $S_1 \cup S_2$  is linearly dependent if and only if  $\text{span}(S_1) \cap \text{span}(S_2) \neq \{0\}$ .

**Proof.** ( $\Rightarrow$ ) We will proceed via contrapositive. Since  $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$ , let  $v \in \text{span}(S_1)$  such that we can find a  $w \in \text{span}(S_2)$  such that both  $v = w$  where both  $v = 0$  and  $w = 0$ . Since  $v \in \text{span}(S_1)$ , we can find a finite set of vectors  $x_1, x_2, \dots, x_n \in S_1$  and scalars  $a_1, a_2, \dots, a_n \in F$  such that

$$v = \sum_{i=1}^n a_i x_i = 0.$$

Likewise,  $w \in \text{span}(S_2)$  implies that we can find  $y_1, y_2, \dots, y_n \in S_2$  and scalars  $b_1, b_2, \dots, b_n \in F$  such that

$$w = \sum_{j=1}^n b_j y_j = 0.$$



Observe that

$$v = w \Rightarrow \sum_{i=1}^n a_i x_i = \sum_{j=1}^n b_j y_j.$$

Then we have

$$\sum_{i=1}^n a_i x_i - \sum_{j=1}^n b_j y_j = 0.$$

Since  $S_1$  and  $S_2$  are disjoint linearly independent sets, we know that  $x_i \notin S_2$  and  $y_i \notin S_1$  and that  $a_i = 0$  and  $b_j = 0$  for all  $1 \leq i \leq n$  and  $1 \leq j \leq n$  respectively. Hence,  $S_1 \cup S_2$  is a linearly independent set.

( $\Leftarrow$ ) We will proceed via contrapositive for this direction as well. We need to show that  $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$ . Since  $S_1 \cup S_2$  is linearly independent, we can choose scalars  $a_1, a_2, \dots, a_n \in F$  and vectors  $x_1 - y_1, x_2 - y_2, \dots, x_n - y_n \in S_1 \cup S_2$  such that

$$\sum_{i=1}^n a_i (x_i - y_i) = 0 \tag{1}$$

with  $a_i = 0$  for all  $1 \leq i \leq n$ . Observe that (1) implies that

$$\underbrace{\sum_{i=1}^n a_i x_i}_{\in \text{span}(S_1)} = \underbrace{\sum_{i=1}^n a_i y_i}_{\in \text{span}(S_2)}.$$

Since  $S_1 \cup S_2$  is linearly independent,  $a_i = 0$  for all  $1 \leq i \leq n$  which implies that

$$\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i y_i = 0$$

and thus  $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$ . ■