0.1 Lecture 3

0.1.1 Topics

- Polar Representation of Complex Numbers
- Convergence of Sequences in $\mathbb{C}$

0.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^{\bullet} = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos \varphi, \sin \varphi)$$

with $\tan \varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos \psi, \sin \psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^{\cdot}$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos \varphi + i \sin \varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^{\cdot}$, then

$$z = \gamma(\cos \varphi + i \sin \varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument). Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos \varphi + i \sin \varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\text{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos \varphi + i \sin \varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi'))$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi \varphi' [\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$
\begin{aligned}
zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\
&= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\
&= \gamma\gamma'(\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')).
\end{aligned}
$$

∎

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^\cdot$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$
\begin{aligned}
z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\
&= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi).
\end{aligned}
$$

Corollary (De Moivre's Theorem). Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$
z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).
$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example (An example of De Moivre's Theorem). Suppose we had the complex number

$$
z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.
$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$
\varphi = \arg(z) = \frac{\pi}{3}.
$$

Using De Moivre's Theorem, we can write

$$
\begin{aligned}
z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i\sin\left(10 \cdot \frac{\pi}{3}\right) \\
&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}.
\end{aligned}
$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^\cdot = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$
(r, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)
$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

0.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$). Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_{\varepsilon} \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_{\varepsilon}$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences). Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$
\begin{aligned}
|\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\
&\leq \alpha|z_n - z| + \beta|w_n - w| \\
&< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$
\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\
&\leq |w_n||z_n - z| + |z||w_n - w| \\
&\leq M|z_n - z| + |z||w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, we conclude that

$$
z_n w_n \to zw.
$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$
|z_n^{-1} - z^{-1}| < \varepsilon.
$$

Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$
|z_n - z| < \frac{\varepsilon |z|^2}{2}.
$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$
|z_n| > \frac{|z|}{2}.
$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$
\begin{aligned}
|z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\
&< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, we conclude that $z_n^{-1} \to z^{-1}$.

(iv) ($\Longrightarrow$) Let $\varepsilon > 0$. Since $(z_n) \to z$, we can choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$
\begin{aligned}
|\Re(z_n) - \Re(z)| &= |\Re(z_n - z)| \leq |z_n - z| < \varepsilon \\
|\Im(z_n) - \Im(z)| &= |\Im(z_n - z)| \leq |z_n - z| < \varepsilon.
\end{aligned}
$$

Hence, the real and imaginary part of (z_n) converge.

($\Longleftarrow$) Letting $\varepsilon > 1$ again. Our goal is to find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$
|z_n - z| < \varepsilon.
$$

Since the real and imaginary part of z_n converge, we know that there exists $N_1, N_2 \in \mathbb{N}$ such that, we have

$$
|\Re(z_n) - \Re(z)| < \frac{\varepsilon}{2} \tag{1}
$$

$$
|\Im(z_n) - \Im(z)| < \frac{\varepsilon}{2} \tag{2}
$$

whenever $n \geq N_1$ and $n \geq N_2$, respectively. Now, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$
\begin{aligned}
|z_n - z| &= |(\Re(z_n) - \Re(z)) + i(\Im(z_n) - \Im(z))| \\
&\leq |\Re(z_n) - \Re(z)| + |\Im(z_n) - \Im(z)| \qquad (|i| = 1) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, we see that $(z_n) \to z$.

$\blacksquare$