## 0.1 Lecture 3

### 0.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

### 0.1.2 Review, Existence of Roots

**Proposition.** There is no rational number whose square is 2.

**Theorem.** There is a unique positive real number $\alpha$ satisfying $\alpha^2 = 2$.

**Proof.** (i) **Uniqueness:** Suppose there are two of them $\alpha_1$ and $\alpha_2$. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that $A$ is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

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**Remark.** A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number $\alpha$ such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

### 0.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

**Definition** (Usual Way of Defining Functions)**.** Let $A$ and $B$ be two sets. A **function** from $A$ to $B$ denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

**Definition** (The Correct Way of Defining Functions)**.** Let $A$ and $B$ be two sets. A function from $A$ to $B$ is a triple $(f, A, B)$ where $f$ is a **relation** form $A$ to $B$ satisfying

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, $A$ is called the **domain** of $f$ and $B$ is called the **codomain** of $f$)

**Example.** Let $A = \emptyset$ and $B$ be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to $B$ is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

**Definition** (Image, Range, Onto (Surjective))**.** Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of $f$ as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of $f$ as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say $f$ is **Onto (Surjective)**.

**Definition** (Preimage). Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of $D$ under $f$ is denoted by

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

**Definition** (One-to-One (Injective)). Consider a function $f : A \to B$. We call $f$ one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of $A$.

### 0.1.4 Equivalent Sets

**Definition.** Let $A$ and $B$ be two sets. We say that $A$ and $B$ have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

**Remark.**

- An injective and surjective mapping is a bijective mapping.
- $A$ and $B$ have the same cardinal number

$$\begin{aligned} &= A \text{ and } B \text{ have the same cardinality} \\ &= A \text{ and } B \text{ can be put in the } \textbf{one-to-one correspondence} \\ &= \operatorname{card} A = \operatorname{card} B \\ &= A \text{ and } B \text{ are equivalent} \\ &= A \text{ and } B \text{ are equipotent} \end{aligned}$$

**Example.** Consider $\{1, 2, 3\} \sim \{a, b, c\}$. Indeed, the function $f : \{1, 2, 3\} \to \{a, b, c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

**Example.** $\mathbb{N} \sim \{2, 4, 6, \dots\}$. Indeed, the function $f : \mathbb{N} \to \{2, 4, 6, \dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

**Example.** $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

**Example.** $(-\infty, \infty) \sim (0, \infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty, \infty)$ and $(0, \infty)$.

**Example.** $(0, \infty) \sim (0, 1)$. Indeed, the function $f : (0, \infty) \to (0, 1)$ defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

**Example.** $[0, 1) \sim (0, 1)$. Indeed, the function $f : [0, 1) \to (0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

**Definition** ($\sim$ is an equivalence relation)**.** Let $A$ and $B$ be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}$$
$$\mathbb{N} = \{1, 2, 3, \dots\}$$

### 0.1.5 Finite, Infinite, Countable, At most countable

**Definition** (Finite, Infinite, Countable, At most countable)**.** Let $A$ be any set.

(a) We say that $A$ is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number $n$.

  (*) When $A \sim \mathbb{N}_n$, we say $A$ has $n$ elements and we write $\text{card}(A) = n$.

  (*) Also, we set $\text{card}(\emptyset) = 0$.

(b) The set $A$ is said to be **infinite** if it is not finite.

(c) The set $A$ is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set $A$ is said to be **uncountable** if it is neither countable or finite.

(e) The set $A$ is said to be **at most countable** if it is either finite or countable.

**Remark.** Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$\begin{gathered} A \text{ is finite} \iff B \text{ is finite} \\ A \text{ is countable} \iff B \text{ is countable} \\ A \text{ is uncountable} \iff B \text{ is uncountable} \end{gathered}$$

(iii) The union of two finite sets is finite. If $A$ is infinite and $B$ is infinite, then $A \setminus B$ is infinite.

(iv) If $A$ **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.