Math 234A: Homework 1

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Problem 1

(i) **(Parallelogram identity)** Let $z, w \in \mathbb{C}$. Show that

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

> **Proof.** Let $z, w \in \mathbb{C}$ with $z = x + iy$ and $w = u + iv$ with $x, y \in \mathbb{R}$ and $u, v \in \mathbb{R}$. Our goal is to show that
> $$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$
> Consider $|z - w|^2$ and notice that
> $$z - w = (x - u) + i(y - v).$$
> By definition of the modulus, we have
> $$\begin{aligned}
> |z - w|^2 &= (z - w)\overline{z - w} \\
> &= ((x - u) + i(y - v))((x - u) - i(y - v)) \\
> &= (x - u)^2 + (y - v)^2 \\
> &= x^2 - 2xu + u^2 + y^2 - 2yv + u^2 \\
> &= (x^2 + y^2) - 2(xu + yv) + (u^2 + v^2) \\
> &= |z|^2 - 2(xu + yv) + |w|^2.
> \end{aligned}$$
> Note that
> $$z + w = (x + u) + i(y + v).$$
>
> $$\begin{aligned}
> |z + w|^2 &= (z + w)\overline{(z + w)} \\
> &= ((x + u) + i(y + v))((x + u) - i(y + v)) \\
> &= (x + u)^2 + (y + v)^2 \\
> &= x^2 + 2xu + u^2 + y^2 + 2yv + v^2 \\
> &= |z|^2 + 2(xu + yv) + |w|^2.
> \end{aligned}$$
> Adding these two moduli together gives us
> $$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2 = 2(|z|^2 + |w|^2)$$
> which is our desired result. ∎

(ii) **(Binomial Expansion)**: Let $z, w \in \mathbb{C}$ and n be a positive integer. Show that

$$(z + w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. Let $z, w \in \mathbb{C}$. We proceed via induction on $n \in \mathbb{Z}^+$ to show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Let $n = 1$ be our base case. Then we have

$$\sum_{k=0}^{1} \binom{n}{k} z^k w^{n-k} = \binom{1}{0} z^0 w + \binom{1}{1} z^1 w^0$$
$$= (z+w)^1,$$

which tells us that the result holds in our base case. Now, suppose the result holds for nth case. We will show the result holds for the $n+1$ case. By our induction hypothesis, we see that

$$(z+w)^{n+1} = (z+w)(z+w)^n$$
$$= (z+w) \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1}.$$

Reordering indices in the first summation by setting $m = k+1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1} = \sum_{m=1}^{n+1} \binom{n}{m-1} z^m w^{(n+1)-m}$$
$$+ \sum_{k=0}^{n} \binom{n}{k} z^k w^{(n-k)+1}.$$

Then separating the first and last term of each summation, respectively, we have

$$(z+w)^{n+1} = \binom{n}{n} z^n w + \sum_{m=1}^{n} \binom{n}{m-1} z^m w^{(n-k)+1} + \sum_{k=1}^{n} \binom{n}{k} z^k w^{n-k+1} + \binom{n}{0} w^{n+1}$$
$$= \binom{n}{n} z^n w + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] z^k w^{(n+1)-k} + \binom{n}{0} w^{n+1}.$$

Using the fact that **(Need to show this!)**

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

and collecting the first and last terms of the summation, we see that

$$(z+w)^{n+1} = \binom{n}{0} z^0 w^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} z^k w^{(n+1)-k} + \binom{n}{n} z^{n+1} w^0$$
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} z^k w^{(n+1)-k}$$

which completes our induction argument. ∎

Problem 2

For $z, w \in \mathbb{C}$. Define $\langle z, w \rangle = \Re(z\overline{w})$. (If we think of $\mathbb{C}$ as two dimensional real vector space, then $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{C}$).

(i) Cauchy Schwarz Inequality: Show that $|\langle z, w \rangle|^2 \le |z|^2|w|^2$ for all $z, w \in \mathbb{C}$.

Proof. First, we would like to show that for any $z, w \in \mathbb{C}$

$$\langle z, w \rangle^2 + \langle iz, w \rangle^2 = |z|^2|w|^2.$$

Let $z, w \in \mathbb{C}$. By definition of $\langle z, w \rangle$, we see that

$$\langle z, w \rangle^2 + \langle iz, w \rangle^2 = (xu + yv)^2 + (xv - uy)^2. \tag{1}$$

Observe that

$$(xu + yv)^2 = (xu)^2 + 2xuyv + (yv)^2 \tag{2}$$

and

$$(xv - uy)^2 = (xv)^2 - 2xuyv + (uy)^2. \tag{3}$$

Then adding (2) and (3) results in

$$\begin{aligned}
\langle z, w \rangle^2 + \langle iz, w \rangle^2 &= (xu)^2 + (yv)^2 + (xv)^2 + (uy)^2 \\
&= x^2u^2 + y^2v^2 + x^2v^2 + u^2y^2 \\
&= u^2(x^2 + y^2) + v^2(x^2 + y^2) \\
&= (x^2 + y^2)(u^2 + v^2) \\
&= |z|^2|w|^2.
\end{aligned}$$

Now, we need to show that $|\langle z, w \rangle|^2 \le |z|^2|w|^2$. By using the result above, we can write

$$|\langle z, w \rangle|^2 \le |\langle z, w \rangle|^2 + |\langle iz, w \rangle|^2 = |z|^2|w|^2$$

and we are done. ∎

(ii) Triangle Inequalities: Show

$$|z + w| \le |z| + |w|$$

and

$$||z| - |w|| \le |z - w|$$

for all $z, w \in \mathbb{C}$.

Proof. Let $z, w \in \mathbb{C}$. We will first show that $|z + w| \le |z| + |w|$. First, we will show the following results:

$$|z + w|^2 = |z|^2 + 2|\langle z, w \rangle| + |w|^2 \tag{1}$$

and

$$|z - w|^2 = |z|^2 - 2|\langle z, w \rangle| + |w|^2. \tag{2}$$

Let $z = x + iy$ and $w = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that

$$z + w = (x + u) + i(y + v)$$

and

$$z - w = (x - u) + i(y - v).$$

Using the definition of the modulus, we see that

$$|z + w|^2 = (z + w)\overline{(z + w)} = ((x + u) + i(y + v))((x + u) - i(y + v))$$
$$= (x + u)^2 + (y + v)^2$$
$$= x^2 2xu + u^2 + y^2 + 2yv + v^2$$
$$= (x^2 + y^2) + 2(xu + yv) + (y^2 + v^2) \quad (\text{since } \Re(z\overline{w}) = xu + yv)$$
$$= |z|^2 + 2\langle z, w \rangle + |w|^2. \quad\quad (\text{since } \langle z, w \rangle = \Re(z\overline{w}))$$

Similarly, we have

$$|z - w|^2 = (z - w)\overline{(z - w)} = ((x - u) + i(y - v))((x - u) - i(y - v))$$
$$= (x - u)^2 + (y - v)^2$$
$$= x^2 - 2xu + u^2 + y^2 - 2yv + v^2$$
$$= x^2 + y^2 - 2(xu + yv) + v^2$$
$$= (x^2 + y^2) - 2\langle z, w \rangle + (v^2 + u^2)$$
$$= |z|^2 - 2\langle z, w \rangle + |w|^2.$$

Now, let us prove that $|z + w| \leq |z| + |w|$. Consider $|z + w|^2$. By part (a), we see that

$$|z + w|^2 = |z|^2 + 2\langle z, w \rangle + |w|^2$$
$$\leq |z|^2 + 2zw + |w|^2$$
$$\leq |z|^2 + 2|z||w| + |w|^2$$
$$= (|z| + |w|)^2.$$

By taking the square root of both sides, we see that

$$|z + w| \leq |z| + |w|.$$

To show the second inequality, consider $|z - w|^2$. Then using part (a) again, we have

$$|z - w|^2 = |z|^2 - 2\langle z, w \rangle + |w|^2$$
$$\geq |z|^2 - 2|z||w| + |w|^2$$
$$= (|z| - |w|)^2.$$

By taking the square root of both sides, we see that

$$|z - w| \geq ||z| - |w||.$$

∎

Problem 3

(Lagrange Identity) Let $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Show that

$$\left| \sum_{k=1}^{n} z_k w_k \right|^2 = \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2 - \sum_{1 \leq i < j \leq n} |z_i \overline{w_j} - z_j \overline{w_i}|^2.$$

Use this to deduce that

$$\left| \sum_{k=1}^{n} z_k w_k \right|^2 \leq \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2.$$

Proof. We will show that

$$\sum_{1 \le i < j \le n} |z_i \overline{w_j} - z_j \overline{w_i}|^2 = \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2 - \left| \sum_{k=1}^n z_k w_k \right|^2.$$

Observe that

$$\sum_{1 \le i < j \le n} |z_i \overline{w_j} - z_j \overline{w_i}|^2 = \sum_{i=1}^n \sum_{j=1}^n |z_i \overline{w_j} - z_j \overline{w_i}|^2$$

$$= \sum_{i=1}^n \sum_{j=1}^n (z_i \overline{w_j} - z_j \overline{w_i})(\overline{z_i} w_j - \overline{z_j} w_i)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \left[|z_i|^2 |w_j|^2 - 2 z_j \overline{z_i w_i} w_j + |z_j|^2 |w_i|^2 \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n |z_i|^2 |w_j|^2 - 2 \sum_{i=1}^n \sum_{j=1}^n z_j \overline{z_i w_i} w_j + \sum_{i=1}^n \sum_{j=1}^n |z_j|^2 |w_i|^2$$

$$= \sum_{i=1}^n |z_i|^2 \sum_{j=1}^n |w_i|^2 + \sum_{i=1}^n |w_i|^2 \sum_{j=1}^n |z_j|^2 - 2 \sum_{i=1}^n \sum_{j=1}^n z_j \overline{z_i w_i} w_j$$

$$= \sum_{i=1}^n |z_i| \sum_{i=1}^n |w_i|^2 - \sum_{i=1}^n \overline{z_i w_i} \sum_{i=1}^n z_i w_i \qquad \text{(for } i = j\text{)}$$

$$= \sum_{i=1}^n |z_i|^2 \sum_{i=1}^n |w_i|^2 - \left| \sum_{i=1}^n z_i w_i \right|^2.$$

Thus, we conclude that

$$\left| \sum_{k=1}^n z_k w_k \right|^2 = \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2 - \sum_{1 \le i < j \le n} |z_i \overline{w_j} - z_j \overline{w_i}|^2$$

$$\le \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2.$$

■

Problem 4

Express the following complex number in the form $\alpha + i\beta$:

(i) $(1+i)^{-1}$

Solution. Observe that
$$(1+i)^{-1} = \frac{1}{1+i}$$
and that
$$\frac{1}{1+i} \cdot \frac{(1-i)}{(1-i)} = \frac{(1-i)}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i.$$

■

(ii) $(1+i)/2i$

Solution. Observe that

$$\frac{(1+i)}{2i} = \frac{1}{2i}(1+i) = \frac{1}{2i} + \frac{1}{2} = \frac{1}{2} - \frac{1}{2}i.$$

■

(iii) $(5 + 5i)^{10}$

> **Solution.** Let $z = 1 + i$. Observe that we can write
>
> $$(5 + 5i)^{10} = 5^{10}(1 + i)^{10}.$$
>
> Note that
> $$r = \sqrt{1^2 + 1^2} = \sqrt{2}.$$
>
> Furthermore, we have
> $$\tan^{-1}(1/1) = \tan^{-1}(1) = \frac{\pi}{4}.$$
>
> Using De Moivre's formula, we can write
> $$z^{10} = (\sqrt{2})^{10}(\cos(10\theta) + i\sin(10\theta)$$
> $$= (\sqrt{2})^{10}(\cos(5\pi/2) + i\sin(5\pi/2)) \qquad = (\sqrt{2})^{10}i.$$
>
> Then we have
> $$(5 + 5i)^{10} = 5^{10}(\sqrt{2})^{10}i = 312500000i.$$
>
> ■

(iv) $\left(\frac{2+i}{3-2i}\right)^2$

> **Solution.** Our first step is to get $\frac{2+i}{3-2i}$ in terms of $\alpha + i\beta$. Thus, observe that
>
> $$\frac{2+i}{3-2i} = \frac{2+i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{7i+4}{13} = \frac{4}{13} + i\frac{7}{13}.$$
>
> Furthermore, we have
> $$\left(\frac{4}{13} + i\frac{7}{13}\right)^2 = \frac{1}{169}(4 + 7i)^2 = \frac{1}{169}(16 + 46i - 49) = \frac{1}{169}(-33 + 46i).$$
>
> Thus, we have that
> $$\left(\frac{2+i}{3-2i}\right)^2 = \frac{-33}{169} + \frac{46}{169}i$$
>
> ■

(v) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3.$

> **Solution.** Denote $z = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$. Then observe that
>
> $$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}.$$
>
> Furthermore, we have
> $$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$
>
> Using De Moivre's formula, we have that
> $$z^3 = 1^3 \cdot \left(\cos\left(3 \cdot \frac{2\pi}{3}\right) + i\sin\left(3 \cdot \frac{2\pi}{3}\right)\right) = \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1.$$
>
> ■

Problem 5

Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos \varphi + i \sin \varphi)$ where $n \in \mathbb{Z}^+$ and

$$w = \gamma^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i \sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]$$

where $k \in \mathbb{Z}$. Show that $w^n = z$.

Proof. Note that for any $n \in \mathbb{N}$ that

$$z^n = (\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi. \tag{1}$$

Using (1), we can see that

$$
\begin{aligned}
w^n &= \left(\gamma^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i \sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]\right)^n \\
&= (\gamma^{1/n})^n\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i \sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]^n \\
&= \gamma\left[\cos\left(n \cdot \frac{\varphi + 2\pi k}{n}\right) + i \sin\left(n \cdot \frac{\varphi + 2\pi k}{n}\right)\right] \\
&= \gamma(\cos \varphi + i \sin \varphi) \\
&= z
\end{aligned}
$$

which ends our proof. $\blacksquare$

Problem 6

(**Computing fourth roots**): Find your distinct complex numbers w such that $w^4 = z$ for

(i) $z = i$.

Solution. Note that $z = i = 0 + 1 \cdot i$. For $0 \leq k \leq 3$, we see that the arguments are

$$\frac{\varphi + 2\pi k}{4} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}.$$

For $k = 0$, we see that

$$\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 + \sqrt{2}/2}{2}} + i\sqrt{\frac{1 + \sqrt{2}/2}{2}}.$$

For $k = 1$, we see that

$$
\begin{aligned}
\cos\left(\frac{5\pi}{8}\right) + i \sin\left(\frac{5\pi}{8}\right) &= -\sqrt{\frac{1 + \cos(5\pi/4)}{2}} + i\sqrt{\frac{1 - \cos(5\pi/4)}{2}} \\
&= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}.
\end{aligned}
$$

For $k = 2$, we see that

$$\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right) = \cos\left(\frac{1}{2} \cdot \frac{9\pi}{4}\right) + i\sin\left(\frac{1}{2} \cdot \frac{9\pi}{4}\right)$$

$$= \sqrt{\frac{1 + \cos(9\pi/4)}{2}} + i\sqrt{\frac{1 - \cos(9\pi/4)}{2}}$$

$$= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}.$$

Lastly, for $k = 3$, we have

$$\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right) = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}.$$

$\blacksquare$

(ii) $z = -i$.

Solution. Note that $z = -i = 0 - 1 \cdot i$. For $0 \leq k \leq 3$, the arguments are

$$\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}.$$

Let $k = 0$. Then using the result found in problem 5 and the half-angle identity, we have

$$\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right) = -\sqrt{\frac{1 + \cos(3\pi/4)}{2}} + i\sqrt{\frac{1 - \cos(3\pi/4)}{2}}$$

$$= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$

Let $k = 1$. We get that

$$\cos\left(\frac{\frac{3\pi}{2} + 2\pi}{4}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2\pi}{4}\right) = \cos\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{8} + \frac{\pi}{2}\right).$$

Using the sum formula, we see that

$$\cos\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = \left[\cos\left(\frac{3\pi}{8}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{3\pi}{8}\sin\left(\frac{\pi}{2}\right)\right)\right]$$

$$+ i\left[\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{8}\sin\left(\frac{\pi}{2}\right)\right)\right]$$

$$= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}.$$

For $k = 2$, we employ the same process. Thus, we have

$$\cos\left(\frac{3\pi}{8} + \pi\right) + i\sin\left(\frac{3\pi}{8} + \pi\right) = \left[\cos\left(\frac{3\pi}{8}\right)\cos(\pi) - \sin\left(\frac{3\pi}{8}\right)\sin(\pi)\right]$$

$$+ i\left[\sin\left(\frac{3\pi}{8}\right)\cos(\pi) + \cos\left(\frac{3\pi}{8}\right)\sin(\pi)\right]$$

$$= -\cos\left(\frac{3\pi}{8}\right) - i\sin\left(\frac{3\pi}{8}\right)$$

$$= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}.$$

Lastly, for $k = 3$, we have

$$
\begin{aligned}
\cos\left(\frac{15\pi}{8}\right) + i\sin\left(\frac{15\pi}{8}\right) &= \cos\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) \\
&= \left[\cos\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{8}\sin\left(\frac{3\pi}{2}\right)\right)\right] \\
&\quad + i\left[\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{2}\right) + \cos\left(\frac{3\pi}{8}\right)\sin\left(\frac{3\pi}{2}\right)\right] \\
&= \sin\left(\frac{3\pi}{8}\right) - i\cos\left(\frac{3\pi}{8}\right) \\
&= \sqrt{\frac{1 - \frac{2}{2}}{2}} + i\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}.
\end{aligned}
$$

$\blacksquare$

(iii) $z = 1$.

Solution. Note that $z = 1 = 1 + i \cdot 0$. For $0 \leq k \leq 3$, we have the arguments

$$
0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}
$$

with each argument corresponding to $k \in [0, 3]$. For $k = 0$, observe that

$$
\cos(0) + i\sin(0) = 1.
$$

For $k = 1$, we have

$$
\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i = i.
$$

Similarly, if $k = 2$ we get

$$
\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.
$$

$\blacksquare$

(iv) $z = -1$.

Solution. Let $0 \leq k \leq 3$ and $n = 4$. We will compute all the solution along the interval $0 \leq k \leq 3$. Note that $z = i = 0 + i \cdot 1$ and that $\varphi = \arg z = \pi/2$ and that for $0 \leq k \leq 3$, we have

$$
\frac{\varphi + 2\pi k}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \pi.
$$

Using the result in problem 5, observe that for $k = 0$, we have

$$
\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.
$$

Similarly, for $k = 1$, we have

$$
\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.
$$

For $k = 2$,

$$
\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.
$$

For $k = 3$, we have
$$\cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1.$$

Thus, we have
$$z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -1.$$

∎

Problem 8

Let $z, a \in \mathbb{C}$.

(i) Show that $|1 - z\bar{a}|^2 - |z - a|^2 = (1 - |z|)^2(1 - |a|^2)$.

Proof. Note that $|a|^2 = |\bar{a}|^2$ and that
$$\langle z, a \rangle = \Re(z\bar{a}) = xu + yv = \Re(\bar{z}a) = \langle 1, z\bar{a} \rangle.$$

if $z = z + iy$ and $a = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that
$$\begin{aligned}
|1 - z\bar{a}|^2 - |z - a|^2 &= [1 - 2\langle 1, z\bar{a} \rangle + |z\bar{a}|^2] - [|z|^2 - 2\langle z, a \rangle + |a|^2] \\
&= |z|^2|a|^2 - |z|^2 - |a|^2 - 1 \\
&= (1 - |z|^2)(1 - |a|^2).
\end{aligned}$$

Hence, we see that
$$|1 - z\bar{a}|^2 - |z - a|^2 = (1 - |z|^2)(1 - |a|^2).$$

∎

(ii) Assume that $|a| < 1$. Show that
$$|z| < 1 \Leftrightarrow \left| \frac{z - a}{1 - \bar{a}z} \right| < 1 \tag{1}$$

and
$$|z| = 1 \Leftrightarrow \left| \frac{z - a}{1 - \bar{a}z} \right| = 1. \tag{2}$$

Proof. Suppose $|z| < 1$ and $|a| < 1$. We will first show that the forwards direction of (1).
$$\left| \frac{z - a}{1 - \bar{a}z} \right| < 1.$$

It suffices to show that
$$|z - a|^2 < |1 - \bar{a}z|^2.$$

By part (i), we see that
$$|z - a|^2 = |1 - z\bar{a}|^2 - (1 - |z|^2)(1 - |a|^2). \tag{3}$$

Since $|z| < 1$ and $|a| < 1$, we see that
$$1 - |z|^2 > 0 \text{ and } 1 - |a|^2 > 0.$$

Thus, we see that
$$-(1 - |z|^2)(1 - |a|^2) < 0.$$

Therefore, (3) implies that
$$|z - a|^2 < |1 - \bar{a}z|^2$$

which further implies that
$$\left| \frac{z - a}{1 - \bar{a}z} \right| < 1.$$

Now, suppose for the backwards direction that

$$\left| \frac{z-a}{1-\overline{a}z} \right| < 1.$$

Note that this is equivalent to saying

$$|z-a|^2 < |1-\overline{a}z|^2$$

By using the fact that

$$|z-a|^2 = |z|^2 - 2\langle z, a \rangle + |w|^2$$

and

$$|1-\overline{a}z|^2 = 1 - 2\langle 1, \overline{a}z \rangle + |\overline{a}z|^2.$$

Then observe that

$$|z-a|^2 < |1-\overline{a}z|^2 \Rightarrow |z|^2 - 2\langle z, a \rangle + |w|^2 < 1 - 2\langle 1, \overline{a}z \rangle + |\overline{a}|^2 |z|^2.$$

So, we must have

$$|z|^2 - 2\langle z, a \rangle + |a|^2 < 1 - 2\langle 1, \overline{a}z \rangle + |\overline{a}|^2 |z|^2$$
$$|z|^2 + |a|^2 < 1 + |a|^2 |z|^2$$

which imply that

$$|z|^2 < (1 - |a|^2) + |a|^2 |z|^2 \Rightarrow |z|^2 (1 - |a|^2) < 1 - |a|^2$$
$$\Rightarrow |z|^2 < \frac{1 - |a|^2}{1 - |a|^2} = 1.$$

Thus, we see that $|z| < 1$. To prove (2), we can replace the inequalities above with an equal sign and the proof will be the same. $\blacksquare$