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# Linear Algebra Exercises

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# Chapter 1

## Vector Spaces

### 1.1 Vector Spaces

#### Exercise 1.2.1

Label the following statements as true or false.

- (a) Every vector space contains a zero vector.
- (b) A vector space may have more than one zero vector.
- (c) In any vector space,  $ax = bx$  implies that  $a = b$ .
- (d) In any vector space,  $ax = ay$  implies that  $x = y$ .

#### Exercise 1.2.7

Let  $S = \{0, 1\}$  and  $F = \mathbb{R}$ . In  $\mathcal{F}(S, \mathbb{R})$ , show that  $f = g$  and where  $f(t) = 2t + 1$ ,  $g(t) = 1 + 4t - 2t^2$ , and  $h(t) = 5^t + 1$ .

**Proof.** To show that  $f = g$ , we have to show that for each  $s \in S$  that  $f(s) = g(s)$ . Since  $S = \{0, 1\}$ , we can just evaluate both  $f$  and  $g$  for elements in  $S$ . Note that

$$f(0) = 2(0) + 1 = 1$$

and likewise,

$$g(0) = 1 + 4(0) - 2(0)^2.$$

Hence,  $f(0) = g(0)$ . Now let us evaluate both functions  $f$  and  $g$  at  $s = 1$ . Hence, we have

$$f(1) = 2(1) + 1 = 3$$

and

$$g(1) = 1 + 4(1) - 2(1)^2 = 3.$$

Thus, we must have  $f(s) = g(s)$  for all  $s \in S$ .

Now, we need to show that  $f + g = h$ . Like we did above, we have to show that this is the case for all  $s \in S$ . Note that

$$(f + g)(s) = f(s) + g(s).$$

Hence, we have

$$f(s) + g(s) = 2 + 6s - 2s^2.$$

Evaluating at  $s = 0$ , we have

$$f(0) + g(0) = 2.$$

and likewise,

$$h(0) = 5^0 + 1 = 2.$$

Hence,  $(f + g)(0) = h(0)$ . Now let us evaluate  $f + g$  at  $s = 1$

$$f(1) + g(1) = 2 + 6(1) - 2(1)^2 = 6$$

and likewise, we have

$$h(1) = 5^1 + 1 = 6.$$

Hence, we have  $(f + g)(1) = h(1)$ . Thus, we have that  $f + g = h$  for all  $s \in S$ . ■

### Exercise 1.2.8

In any vector space  $V$ , we have

$$(a + b)(x + y) = ax + ay + bx + by$$

for any  $x, y \in V$  and any  $a, b \in F$ .

**Proof.** Observe the following set equalities:

$$(a + b)(x + y) = a(x + y) + b(x + y) \quad (\text{VS 8})$$

$$= ax + ay + bx + by. \quad (\text{VS 7})$$

Hence, we have that

$$(a + b)(x + y) = ax + ay + bx + by$$

for any  $x, y \in V$  and any  $a, b \in F$ . ■

### Exercise 1.2.10

Let  $V$  denote the set of all differentiable real-valued functions defined on the real line. Prove that  $V$  is a vector space with the operations of addition and scalar multiplication defined in Example 3.

**Proof.** Let  $V$  denote the set of all differentiable real-valued functions defined on the real line. We need to show that  $V$  is a vector space over  $\mathbb{R}$  with addition and scalar multiplication defined in Example 3.

(VS 1) Let  $f, g \in V$ . We need to show that  $f + g = g + f$  for all  $x \in \mathbb{R}$ . Let  $x \in \mathbb{R}$ . Since  $f(x)$  and  $g(x)$  are also real numbers, we have

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

Hence, we have  $f + g = g + f$ .

(VS 2) Let  $f, g, h \in V$ . We need to show that  $f + (g + h) = (f + g) + h$ . Let  $x \in \mathbb{R}$ . Since  $f(x), g(x), h(x) \in \mathbb{R}$ , we can see that

$$\begin{aligned} (f + (g + h))(x) &= f(x) + (g + h)(x) \\ &= f(x) + g(x) + h(x) \\ &= (f + g)(x) + h(x) \\ &= ((f + g) + h)(x). \end{aligned}$$

Hence, we have  $f + (g + h) = (f + g) + h$ .

(VS 3) Observe that  $f_0(x) = 0$  for all  $x \in \mathbb{R}$  is also a real-valued function that is differentiable. We need to show that  $f + f_0 = f$ . Hence, we have

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x).$$

Hence,  $f + f_0 = f$  for all  $x \in \mathbb{R}$ .

(VS 4) Take  $c = -1$  and perform a scalar operation with a  $f \in V$ . Observe that  $(-1 \cdot f)(x) = (-1)f(x) = -f(x)$ . Denote  $g = -f$ . Since additive inverses exists in  $\mathbb{R}$ , we have

$$\begin{aligned}(f - g)(x) &= (f - f)(x) \\ &= f(x) - f(x) \\ &= 0 \\ &= f_0(x).\end{aligned}$$

Hence,  $f - g = f_0$  for all  $x \in \mathbb{R}$ .

(VS 5) Let  $f \in V$  and let  $x \in \mathbb{R}$ . We need to show that  $1 \cdot f = f$ . Since  $f(x) \in \mathbb{R}$ , we can see that multiplicative identities in  $\mathbb{R}$  are also preserved in  $V$ ; that is, we have

$$(1f)(x) = 1 \cdot f(x) = f(x).$$

Hence, we have  $1f = f$  for all  $x \in \mathbb{R}$ .

(VS 6) Let  $a, b \in \mathbb{R}$  and  $f \in V$ . Let  $x \in \mathbb{R}$  be arbitrary. Then observe that

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a(bf)(x).$$

Hence, we have  $(ab)f = a(bf)$  for all  $x \in \mathbb{R}$ .

(VS 7) Let  $a \in \mathbb{R}$  and let  $f, g \in V$ . We need to show that  $a(f + g) = af + ag$ . Let  $x \in \mathbb{R}$ . Observe that  $f(x), g(x) \in \mathbb{R}$  imply

$$\begin{aligned}(a(f + g))(x) &= a(f + g)(x) \\ &= a(f(x) + g(x)) \\ &= af(x) + ag(x) \\ &= (af)(x) + (ag)(x).\end{aligned}$$

Hence, we conclude that  $a(f + g) = af + ag$  for all  $x \in \mathbb{R}$ .

(VS 8) Let  $a, b \in \mathbb{R}$  and let  $f \in V$ . Let  $x \in \mathbb{R}$ . Then we have

$$\begin{aligned}((a + b)f)(x) &= (a + b)f(x) \\ &= af(x) + bf(x) \\ &= (af)(x) + (bf)(x).\end{aligned}$$

Hence, we have  $(a + b)f = af + bf$  for all  $x \in \mathbb{R}$ .

Since all the properties of a vector space have been satisfied, we conclude that  $V$  is a vector space. ■

### Exercise 1.2.11

Let  $V = \{0\}$  consist of a single vector  $O$  and define  $O + O = O$  and  $cO = O$  for each scalar  $c \in F$ . Prove that  $V$  is a vector space over  $F$ .

**Proof.** Let  $x, y \in O$ . Since  $V = \{0\}$ , we know that  $x$  and  $y$  are both the  $0$  vector. By using the addition defined on  $V$ , we can see that

$$x + y = O + O = y + x.$$

Note that  $x + y \in V$  implies that  $x + y = O + O$  and likewise  $y + z = O + O$  for every  $x, y, z \in V$ . Hence, we have that

$$x + (y + z) = O + (O + O) = (O + O) + O = (x + y) + z.$$

Since  $V$  consists of only the zero vector  $O$ , we know that (VS 3) and (VS 4) are satisfied. By

the same reasoning, (VS 5) is satisfied because

$$1 \cdot O = O.$$

Let  $a, b \in F$ . Then (VS 6) and (VS 7) are satisfied because

$$(ab)O = 0 = a \cdot 0 = a(bO) \text{ and } a(O + O) = 0 = O + O = aO + aO \text{ respectively.}$$

Let  $a, b \in F$  again. Then we have

$$\begin{aligned} (a + b)O &= 0 \\ &= O + O \\ &= aO + bO. \end{aligned}$$

Hence, (VS 8) is satisfied. ■

### Exercise 1.2.12

A real-valued function  $f$  defined on the real line is called an **even function** if  $f(-t) = f(t)$  for all  $t \in \mathbb{R}$ . Prove that the set of even functions defined on  $\mathbb{R}$  with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

**Proof.** First, let us show that for every  $t \in \mathbb{R}$  that  $(f + g)(t)$  is also an even function for every pair of even functions  $f, g$  and likewise  $(cf)(t)$  is an even function for every  $c \in \mathbb{R}$ . Observe that

$$\begin{aligned} (f + g)(-t) &= f(-t) + g(-t) \\ &= f(t) + g(t) \\ &= (f + g)(t). \end{aligned}$$

Hence, the function  $f + g$  is also even. Now observe that

$$\begin{aligned} (cf)(-t) &= cf(-t) \\ &= cf(t) \\ &= (cf)(t). \end{aligned}$$

Hence,  $cf$  is also even function.

Now we can show that the set of even functions is a vector space.

(VS 1) Let  $f, g$  be a pair of real-valued even functions. Let  $t \in \mathbb{R}$  be arbitrary. We need to show that  $f + g = g + f$ . Since  $f(t)$  and  $g(t)$  are real numbers, observe that

$$(f + g)(t) = f(t) + g(t) = g(t) + f(t) = (g + f)(t).$$

Hence, we have that  $f + g = g + f$ .

(VS 2) Let  $f, g, h$  be even functions and let  $t \in \mathbb{R}$  be arbitrary. We need to show that  $f + (g + h) = (f + g) + h$ . By the same reasoning we used to prove (VS 1), observe that

$$\begin{aligned} f(t) + ((g + h)(t)) &= f(t) + (g(t) + h(t)) \\ &= (f(t) + g(t)) + h(t) \\ &= ((f + g)(t)) + h(t) \end{aligned}$$

Hence, we have that  $f + (g + h) = (f + g) + h$ .

(VS 3) Let  $f$  be an even function. Note that the zero function is an even function. Denote the zero function as  $f_0$ . For every  $t \in \mathbb{R}$ , we have  $f_0(t) = O$ . We need to show that  $f + f_0 = f$ . Since  $f_0(t)$  is just a real number, we have

$$(f + f_0)(t) = f(t) + f_0(t) = f(t) + O = f(t).$$



(VS 4) Let  $f$  be an even function and let  $f_0$  be the zero function defined above. We need to show that there exists a even function  $g$  such that  $f + g = f_0$ . Let  $t \in \mathbb{R}$  be arbitrary. Note that

$$(f + g)(t) = f(t) + g(t)$$

Since  $f(t)$  and  $g(t)$  are real numbers and there exists an additive identity in the real numbers, we have that  $f(t) + g(t) = f_0(t)$ . Hence, (VS 5) is satisfied.

(VS 5) Let  $x \in \mathbb{R}$  and  $f$  an even function. Then we immediately have  $(1 \cdot f)(t) = 1f(t) = f(t)$ .

(VS 6) Let  $a, b \in \mathbb{R}$ . We need to show that  $(ab)f = a(bf)$ . Hence, let  $t \in \mathbb{R}$ . Then we have

$$(abf)(t) = a(bf)(t).$$

(VS 7) Let  $a \in \mathbb{R}$  and let  $f, g$  be even functions. Let  $t \in \mathbb{R}$ . We need to show that  $a(f + g) = af + ag$ . Then we have

$$\begin{aligned} a(f + g)(t) &= a[f(t) + g(t)] \\ &= af(t) + ag(t). \end{aligned}$$

Hence, (VS 7) is satisfied.

(VS 8) Now let  $a, b \in \mathbb{R}$  and let  $f$  be an even function. We need to show that  $(a + b)f = af + bf$ . Let  $t \in \mathbb{R}$  be arbitrary. Observe that

$$\begin{aligned} (a + b)f(t) &= af(t) + bf(t) \\ &= (af)(t) + (bf)(t) \end{aligned}$$

Hence, (VS 8) is satisfied. ■

### Exercise 1.2.13

Let  $V$  denote the set of ordered pairs of real numbers. If  $(a_1, a_2)$  and  $(b_1, b_2)$  are elements of  $V$  and  $c \in \mathbb{R}$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, a_2b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2).$$

Is  $V$  is a vector space over  $\mathbb{R}$  with these operations? Justify your answer.

**Proof.** We have that  $V$  is not a vector space of  $\mathbb{R}$ . To see why, let  $(2, 1), (4, 2) \in V$  where  $x = (2, 1)$  and  $y = (4, 2)$ . We will show that (VS 1) does not hold; that is,  $x + y \neq y + x$ . Hence, observe that

$$(2, 1) + (4, 2) = (2 + 1, 2) = (3, 2)$$

and

$$(4, 2) + (2, 1) = (4 + 2, 2) = (6, 2)$$

Hence, we have  $x + y \neq y + x$  and so  $V$  is **NOT** a vector space. ■

### Exercise 1.2.14

Let  $V = \{(a_1, a_2, \dots, a_n) : a_i \in C \text{ for } i = 1, 2, \dots, n\}$ ; so  $V$  is a vector space over  $C$  by Example 1. Is  $V$  is a vector space over the field of real numbers with the operations of coordinate-wise addition and multiplication?

**Proof.** Yes,  $V$  where

$$V = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n\}$$

is a vector space of  $\mathbb{R}$ .

- (VS 1) Let  $x, y \in V$  such that  $x = (a_1, a_2, \dots, a_n)$  and  $y = (b_1, b_2, \dots, b_n)$ . Since addition is entry-wise in  $V$  and each entry in both  $x$  and  $y$  are elements of  $\mathbb{R}$  (where  $\mathbb{R}$  is a field), we have  $a_i + b_i = b_i + a_i$  for all  $i = 1, 2, \dots, n$ . Hence,  $x + y = y + x$ .
- (VS 2) Let  $x, y, z \in V$  with  $x$  and  $y$  as defined as before where  $z$  contains entries  $c_i$  for all  $i = 1, 2, \dots, n$ . We can see that the entries of  $x, y, z$  are elements of  $\mathbb{R}$  so associativity is preserved; that is,  $a_i + (b_i + c_i) = (a_i + b_i) + c_i$  for all  $i = 1, 2, \dots, n$ . Hence, we have  $x + (y + z) = (x + y) + z$ .
- (VS 3) Since  $\mathbb{R}$  contains the zero element 0 and  $V$  is the set of  $n$ -tuples, there exists an element denoted by  $O$  such that this element consisting of entries that only have the zero element 0; that is,  $O = (0, 0, \dots, 0)$ . Take  $x \in V$ . Hence, we have  $a_i + 0 = a_i$  for every  $i = 1, 2, \dots, n$ . Thus, we must have  $x + O = x$ .
- (VS 4) Let  $x \in V$  be arbitrary as defined before. Since every entry in  $x$  is an element of  $\mathbb{R}$ ; that is, every  $a_i \in \mathbb{R}$  for all  $i = 1, 2, \dots, n$ , we know that every entry contains an element  $c_i$  such that  $a_i + c_i = 0$  for every  $i = 1, 2, \dots, n$ . Denote  $x' = (c_1, c_2, \dots, c_n)$ . Hence, we have  $x + x' = O$ .
- (VS 5) Let  $x \in V$ . Every entry  $a_i \in \mathbb{R}$  for all  $i = 1, 2, \dots, n$ , we have  $1 \cdot a_i = a_i$  which holds for all  $i$ . Denote this identity element as  $I$  with entries consisting only of 1. Hence, we have  $I \cdot x = x$ .
- (VS 6) Let  $e, r \in \mathbb{R}$  and let  $x \in V$ . For every entry  $a_i \in \mathbb{R}$  for all  $i = 1, 2, \dots, n$ , we have  $(er)a_i = e(ra_i)$  for all  $i = 1, 2, \dots, n$ . By using the operations of scalar multiplication for  $n$ -tuples, this tells us that  $(er)x = e(rx)$ . Hence, (VS 6) is satisfied.
- (VS 7) Let  $e \in \mathbb{R}$  and let  $x, y \in V$ . We need to show that  $e(x + y) = ex + ey$ . Note that  $a_i, b_i \in \mathbb{R}$  implies that  $e(a_i + b_i) = ea_i + eb_i$ . Hence, we have  $e(x + y) = ex + ey$ .
- (VS 8) Let  $e, r \in \mathbb{R}$  and  $x \in V$  as defined before. We need to show that  $(e + r)x = ex + rx$ . Since  $a_i \in \mathbb{R}$  with  $e, r \in \mathbb{R}$ , we are guaranteed to have  $(e + r)a_i = ea_i + ra_i$  for all  $i = 1, 2, \dots, n$ . Hence, we have  $(e + r)x = ex + rx$ .

■

### Exercise 1.2.16

Let  $V$  denote the set of all  $m \times n$  matrices with real entries; so  $V$  is a vector space over  $\mathbb{R}$  by Example 2. Let  $F$  be the field of rational numbers. Is  $V$  a vector space over  $F$  with the usual definitions of matrix addition and scalar multiplication?

**Proof**(VS 1) Let  $A, B \in V$ . Since  $A, B$  consist of elements  $A_{ij}, B_{ij} \in \mathbb{R}$ , we know that  $A_{ij} + B_{ij} = B_{ij} + A_{ij}$ . Hence,  $A + B = B + A$ .

(VS 2) Let  $A, B, C \in V$  with  $A, B$  defined as before and  $C$  containing real entries  $C_{ij}$ . With the same reasoning used to prove (VS 1), we know that  $A_{ij} + (B_{ij} + C_{ij}) = (A_{ij} + B_{ij}) + C_{ij}$ . Hence, we have  $A + (B + C) = (A + B) + C$ .

(VS 3) Let  $A \in V$  once again. Since the entries of  $A$  imply that there exists an element  $O$  such that  $A_{ij} + 0 = A_{ij}$ , we know that  $A + O = A$  where  $O$  is the **zero matrix** of  $V$ .

(VS 4) Since the real entries of  $A$  also consists of an element  $A'_{ij}$  such that  $A_{ij} + A'_{ij} = 0$ , this implies that  $A + A' = O$  where  $A'$  is the additive inverse matrix of  $V$ .

(VS 5) Let  $x \in V$  as defined as before. Every entry of  $A$ ,  $A_{ij} \in \mathbb{R}$ , has the following property:  $1 \cdot A_{ij} = A_{ij}$  for all for all  $1 \leq i \leq m$  and for all  $1 \leq j \leq n$ . The matrix whose entries consists of only one we can define as the **identity matrix** denoted by  $I$  where  $I_{ij} = 1$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Hence, we have  $A \cdot I = A$ .

(VS 6) Let  $r, t \in \mathbb{Q}$  and let  $A \in V$  as defined before. We need to show that  $(rt)A = r(tA)$ . Since  $A_{ij} \in \mathbb{R}$ , entry-wise scalar multiplication implies that  $(rt)A_{ij} = r(tA_{ij})$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Hence, we must have  $(rt)A = r(tA)$ .

(VS 7) Let  $r \in \mathbb{Q}$  and  $A, B \in V$  as defined before. Since  $A_{ij}, B_{ij} \in \mathbb{R}$ , we know that  $r(A_{ij} + B_{ij}) = rA_{ij} + rB_{ij}$ . Hence, we have  $r(A + B) = rA + rB$ .

(VS 8) Let  $r, t \in \mathbb{Q}$  and let  $A \in V$  as defined before. Since  $A_{ij} \in \mathbb{R}$  for all  $i, j$ , we must have  $(r + t)A_{ij} = rA_{ij} + tA_{ij}$ . Hence, we have  $(r + t)A = rA + tA$ .  
Hence,  $V$  is a vector space over  $\mathbb{Q}$ . ■

### Exercise 1.2.17

Let  $V = \{(a_1, a_2) : a_1, a_2 \in F\}$ , where  $F$  is a field. Define addition of elements of  $V$  coordinate-wise, and for  $c \in F$  and  $(a_1, a_2) \in V$ , define

$$c(a_1, a_2) = (ca_1, 0).$$

Is  $V$  a vector space over  $F$  with these operations? Justify your answer.

**Proof.** We claim that  $V$  is not a vector space over  $F$  because  $V$  fails to satisfy (VS 5). To see why, let  $(1, 2) \in V$ . Using (VS 5), we have

$$1 \cdot (1, 2) = (1, 0) \neq (1, 2).$$

Hence,  $V$  cannot be a vector space. ■

### Exercise 1.2.18

Let  $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ . For  $(a_1, a_2), (b_1, b_2) \in V$  and  $c \in \mathbb{R}$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is  $V$  a vector space over  $F = \mathbb{R}$  with these operations? Justify your answer?

**Proof.** We claim that  $V$  is not a vector space over  $\mathbb{R}$  and we will use (VS 1) to show this. Let  $x, y \in V$  be defined by  $x = (1, 2)$  and  $y = (3, 4)$ . Observe that

$$x + y = (1, 2) + (3, 4) = (7, 14)$$

and

$$y + x = (3, 4) + (1, 2) = (5, 10).$$

Clearly, we have  $x + y = (7, 14) \neq (5, 10) = y + x$  and so (VS 1) does not hold. ■

### Exercise 1.2.19

Let  $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ . Define addition of elements of  $V$  coordinate-wise, and for  $(a_1, a_2) \in V$  and  $c \in \mathbb{R}$ , define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is  $V$  a vector space over  $\mathbb{R}$  with these operations? Justify your answer.

**Proof.** We claim that  $V$  is not a vector space over  $\mathbb{R}$ . To see why, consider (VS 8). If we let  $(0, 1) \in V$  with  $c = 2 + 1 = 3$ . Observe that

$$(2 + 1)(0, 1) = \left(0, \frac{1}{2 + 1}\right) = \left(0, \frac{1}{3}\right).$$

Likewise, we have

$$2(0, 1) + 1(0, 1) = \left(0, \frac{1}{2}\right) + (0, 1) = \left(0, \frac{3}{2}\right).$$

Notice that  $(2+1)(0,1) \neq 2(0,1) + 1(0,1)$ . Hence,  $V$  cannot be a vector space over  $\mathbb{R}$ . ■

### Exercise 1.2.20

Let  $V$  denote the set of all real-valued functions  $f$  defined on the real line such that  $f(1) = 0$ . Prove that  $V$  is a vector space with the operations of addition and scalar multiplication defined in Example 3.

**Proof.** Define  $V$  as a vector space with the operations of addition and scalar multiplication defined in Example 3. We must show that  $V$  is a vector space.

(VS 1) Let  $f, g \in V$ . This means that  $f(1) = 0$  and  $g(1) = 0$ . We need to show that  $f + g = g + f$ . Since  $f(1), g(1) \in \mathbb{R}$  and commutativity holds in  $\mathbb{R}$ , we can write

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) \\ &= g(1) + f(1) \\ &= (g + f)(1)\end{aligned}$$

Hence, we have  $f + g = g + f$ .

(VS 2) Let  $f, g, h \in V$  then  $f(1) = g(1) = h(1) = 0$ . We need to show that  $f + (g + h) = (f + g) + h$ . Observe that

$$\begin{aligned}(f + (g + h))(1) &= f(1) + (g + h)(1) \\ &= f(1) + g(1) + h(1) \\ &= (f + g)(1) + h(1) \\ &= ((f + g) + h)(1).\end{aligned}$$

Hence, we have  $f + (g + h) = (f + g) + h$ .

(VS 3) Let  $f \in V$ . We need to show that  $f + f_0 = f$  for some  $f_0 \in V$ . Since  $V$  contains elements of  $f \in V$  such that  $f(1) = 0$ , we can choose  $f_0$  such that  $f_0(1) = 0$ . We can show that this is indeed the additive inverse of  $V$  by writing the following:

$$(f + f_0)(1) = f(1) + f_0(1) = f(1) + 0 = f(1).$$

Hence, we have  $f + f_0 = f$ .

(VS 4) Let  $f \in V$ . We need to find an element  $g \in V$  such that  $f + g = f_0$ . By definition of  $V$ ,  $f(1) = 0$ . We need to show that  $f + g = f_0$  with  $f_0$  defined as before. Choose  $g = -f$  as our additive inverse and observe that

$$\begin{aligned}(f + g)(1) &= (f - f)(1) \\ &= f(1) - f(1) \\ &= 0 - 0 \\ &= 0 \\ &= f_0(1).\end{aligned}$$

Hence,  $g = -f$  an element such that  $f + g = f_0$ .

(VS 5) Let  $f \in V$ . By definition of  $V$ , we have  $f(1) = 0$ . Since  $f(1) \in \mathbb{R}$ , we know that  $1 \cdot f(1) = f(1)$ . We need to show that  $1f = f$ . Observe that

$$(1f)(1) = 1 \cdot f(1) = f(1).$$

Hence, (VS 5) holds.

(VS 6) Let  $a, b \in \mathbb{R}$  and let  $x \in V$ . We need to show that  $(ab)f = a(bf)$ . By using scalar multiplication, we can see that

$$(ab \cdot f)(1) = (ab)f(1) = a(bf(1)) = a(b \cdot f)(1).$$

Hence, (VS 6) must hold.

(VS 7) Let  $a \in \mathbb{R}$  and let  $f, g \in V$ . We need to show that  $a(f + g) = af + ag$ . Observe that

$$\begin{aligned} a(f + g)(1) &= a(f(1) + g(1)) \\ &= af(1) + ag(1) \\ &= (af)(1) + (ag)(1). \end{aligned}$$

Hence,  $a(f + g) = af + ag$  and so (VS 7) is satisfied.

(VS 8) Let  $a, b \in \mathbb{R}$  and let  $f \in V$ . Observe that

$$\begin{aligned} (a + b)f(1) &= af(1) + bf(1) \\ &= (af)(1) + (bf)(1) \end{aligned}$$

Hence,  $(a + b)f = af + bf$ . Thus,  $V$  must be a vector space over  $\mathbb{R}$ . ■

### Exercise 1.2.21

Let  $V$  and  $W$  be vector spaces over a field  $F$ . Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Prove that  $Z$  is a vector space over  $F$  with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

**Proof.** Let  $V$  and  $W$  be vector spaces over a field  $F$ .

(VS 1) Let  $x, y \in Z$  where  $x = (v_1, w_1)$  and  $y = (v_2, w_2)$ . Since  $V$  and  $W$  are vector spaces, commutativity holds. Since the addition defined on both  $V$  and  $W$  is entry-wise, we can see that

$$\begin{aligned} x + y &= (v_1, w_1) + (v_2, w_2) \\ &= (v_1 + v_2, w_1 + w_2) \\ &= (v_2 + v_1, w_2 + w_1) \\ &= (v_2, w_2) + (v_1, w_1) \\ &= y + x. \end{aligned}$$

Hence, we can see that (VS 1) is satisfied.

(VS 2) Let  $x, y, z \in Z$  with  $x$  and  $y$  as defined before as well as  $z = (v_3, w_3)$ . Using the entry-wise addition defined for  $Z$ , we can see that

$$\begin{aligned} x + (y + z) &= (v_1, w_1) + ((v_2, w_2) + (v_3, w_3)) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (x + y) + z. \end{aligned}$$

(VS 3) Since  $V$  and  $W$  are vector spaces, we know that there exists an additive identity  $O_V$  and  $O_W$  respectively. Hence, we have  $(O_V, O_W) \in Z$  and denote  $O_Z = (O_V, O_W)$ . Now, let  $x \in Z$  as defined before. Observe that

$$\begin{aligned} x + O_Z &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + O_V, w_1 + O_W) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 4) Observe that  $V$  and  $W$  contain additive inverses for each  $v \in V$  and  $w \in W$  respectively. Since  $x = (v_1, w_1)$  with  $v_1 \in V$  and  $w_1 \in W$ , there exists an additive inverse  $v'_1 \in V$  and  $w'_1 \in W$  such that  $v_1 + v'_1 = O_V$  and  $w_1 + w'_1 = O_W$ . This implies that  $(v'_1, w'_1) \in Z$  which we will denote by  $x'$  such that

$$\begin{aligned} x + x' &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + v'_1, w_1 + w'_1) \\ &= (O_V, O_W) \\ &= O_Z. \end{aligned}$$

(VS 5) Let  $x \in Z$ . Since  $V$  and  $W$  are vector spaces (VS 5) implies that  $1 \cdot v_1 = v_1$  and  $1 \cdot w_1 = w_1$  respectively. Then observe that

$$\begin{aligned} 1 \cdot x &= 1 \cdot (v_1, w_1) \\ &= (1 \cdot v_1, 1 \cdot w_1) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 6) Let  $x \in Z$  and  $a, b \in F$ . Then

$$\begin{aligned} (ab)x &= (ab)(v_1, w_1) \\ &= ((ab)v_1, (ab)w_1) \\ &= (a(bv_1), a(bw_1)) && (V, W \text{ vector space}) \\ &= a(bv_1, bw_1) \\ &= a(bx) \end{aligned}$$

(VS 7) Let  $x, y \in Z$  as defined before. Let  $a \in F$ . Then

$$\begin{aligned} a(x + y) &= a((v_1, w_1) + (v_2, w_2)) \\ &= a((v_1 + v_2, w_1 + w_2)) \\ &= (a(v_1 + v_2), a(w_1 + w_2)) \\ &= (av_1 + av_2, aw_1 + aw_2) \\ &= (av_1, av_2) + (aw_1, aw_2) \\ &= a(v_1, v_2) + a(w_1, w_2) \\ &= ax + ay. \end{aligned}$$

(VS 8) Let  $a, b \in F$  and let  $x \in V$  as defined before. Since  $V$  and  $W$  are vector spaces, we know that

(VS 8) holds for bot entries  $v_1 \in V$  and  $w_1 \in W$ . Hence, observe that

$$\begin{aligned}
 (a+b)x &= (a+b)(v_1, w_1) \\
 &= ((a+b)v_1, (a+b)w_1) \\
 &= (av_1 + bv_1, aw_1 + bw_1) \\
 &= (av_1, aw_1) + (bv_1, bw_1) \\
 &= a(v_1, w_1) + b(v_1, w_1) \\
 &= ax + bx.
 \end{aligned}$$

Hence,  $Z$  is a vector space. ■

## 1.2 Subspaces

### Exercise 1.3.3

Prove that  $(aA + bB)^t = aA^t + bB^t$  for any  $A, B \in M_{m \times n}(F)$  and any  $a, b \in F$ .

**Proof.** Let  $A, B \in M_{m \times n}(F)$  and let  $a, b \in F$  be arbitrary. Using scalar multiplication defined on  $M_{m \times n}(F)$ , we have

$$\begin{aligned}
 (aA + bB)^t &= (aA)^t + (bB)^t \\
 &= aA^t + bB^t.
 \end{aligned}$$

Hence, we are done. ■

### Exercise 1.3.5

Prove that  $A + A^t$  is symmetric for any square matrix  $A$ .

**Proof.** Let  $A$  be an arbitrary square matrix. Since square matrices are symmetric, we have that  $A^t = A$ . We need to show that  $(A + A^t)^t$ . Observe that

$$\begin{aligned}
 (A + A^t)^t &= A^t + (A^t)^t \\
 &= A + A^t.
 \end{aligned}$$

Hence, we have  $A + A^t$  is symmetric. ■

### Exercise 1.3.

Prove that  $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$  for any  $A, B \in M_{n \times n}(F)$ .

**Proof.** Let  $A, B \in M_{n \times n}(F)$  and let  $a, b \in F$  be arbitrary. Now, let  $i = j$  and observe that

$$\begin{aligned}
 \operatorname{tr}(aA + bB) &= \sum_{i,j \in \mathbb{N}}^n (aA + bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n (aA)_{ij} + (bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n (aA)_{ij} + \sum_{i,j \in \mathbb{N}}^n (bB)_{ij} \\
 &= \sum_{i,j \in \mathbb{N}}^n aA_{ij} + \sum_{i,j \in \mathbb{N}}^n bB_{ij} \\
 &= a \sum_{i,j \in \mathbb{N}}^n A_{ij} + b \sum_{i,j \in \mathbb{N}}^n B_{ij} \\
 &= a \operatorname{atr}(A) + b \operatorname{atr}(B).
 \end{aligned}$$

Hence, we conclude

$$\operatorname{tr}(aA + bB) = a \operatorname{atr}(A) + b \operatorname{atr}(B)$$

for any  $A, B \in M_{n \times n}(F)$ . ■

### Exercise 1.3.7

Prove that diagonal matrices are symmetric matrices.

**Proof.** Let  $A \in M_{n \times n}(F)$  be diagonal. Let  $i \neq j$  where  $A_{ij} = 0$ . We need to show that  $A^t = A$ . If we apply a transpose on  $A$ , we get that  $A_{ji} = 0$  since  $A^t$  is also diagonal and square. Since  $A_{ij} = A_{ji} = 0$  for all  $1 \leq i \leq n$  and  $1 \leq j \leq n$ . Hence,  $A^t = A$ . ■

### Exercise 1.3.8

Determine whether the following sets are subspaces of  $\mathbb{R}^3$  under the operations of addition and scalar multiplication defined on  $\mathbb{R}^3$ . Justify your answers.

- (a)  $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

**Proof.** We claim that  $W_1$  is a subspace of  $\mathbb{R}^3$ .

- (a) Note that  $O_{\mathbb{R}^3} \in W_1$  where  $O_{\mathbb{R}^3} = (0, 0, 0)$  because  $0 = 3 \cdot 0$  and  $0 = -1 \cdot 0$ .
- (b) Let  $x, y \in W_1$  where  $x = (a_1, a_2, a_3)$  and  $y = (b_1, b_2, b_3)$ . We need to show that  $x + y \in W_1$ . Since  $a_1 = 3a_2$  and  $a_3 = -a_2$  as well as  $b_1 = 3b_2$  and  $b_3 = -b_2$ , we can write  $a_1 + b_1 = 3(a_2 + b_2)$  and  $a_3 + b_3 = -(a_2 + b_2)$ . Hence,  $x + y \in W_1$ .
- (c) Let  $c \in \mathbb{R}$  and  $x \in W_1$  with  $x$  defined as before. Then observe that  $ca_1 = c(3a_2) = 3(ca_2)$  and  $ca_3 = c(-a_2) = -(ca_2)$ . Hence,  $cx \in W_1$ .

Since all the properties of a Theorem 3 have been satisfied, we can conclude that  $W_1$  is a subspace of  $\mathbb{R}^3$ . ■

- (b)  $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
- (c)  $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
- (d)  $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - 3a_3 = 1\}$
- (e)  $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
- (f)  $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$



### Exercise 1.3.9

Let  $W_1, W_3, W_4$  be as in Exercise 8. Describe  $W_1 \cap W_3$ ,  $W_1 \cap W_4$ , and  $W_3 \cap W_4$  and observe that each is a subspace of  $\mathbb{R}^3$ .

**Proof.** ■

### Exercise 1.3.11

Prove that the set  $W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 0\}$  is a subspace of  $F^n$ , but  $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 1\}$  is not.

**Proof.** We need to show that  $W_1$  is a subspace of  $F^n$ . We proceed by satisfying the properties of Theorem 3 to do this.

- (a) Note that  $O_{F^n} \in W_1$  since  $0 + 0 + \dots + 0 = 0$   $n$  times.
- (b) Let  $x, y \in W_1$  with  $x = (a_1, a_2, \dots, a_n)$  and  $y = (b_1, b_2, \dots, b_n)$ . By definition of  $W_1$ , we can see that

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Hence,  $x + y \in W_1$  which tells us that  $W_1$  is closed under addition.

- (c) Let  $x \in W_1$  and  $c \in F$ . Then observe that

$$\sum_{i=1}^n (ca_i) = c \sum_{i=1}^n a_i = c \cdot 0 = 0.$$

Hence, we have  $cx \in W_1$ .

We claim that  $W_2$  is not a subspace because  $W_2$  is not closed under addition. Let  $(0, 1), (1, 0) \in F^2$ . Observe that  $0 + 1 = 1$  and  $1 + 0 = 1$ , but  $(0 + 1) + (1 + 0) = 1 + 1 = 2$ . Hence,  $(0, 1) + (1, 0) \notin W_2$ . ■

### Exercise 1.3.11

Is the set  $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$  a subspace of  $P(F)$  if  $n \geq 1$ ? Justify your answer.

### Exercise 1.3.12

Prove that the set of  $m \times n$  upper triangular matrices is a subspace of  $M_{m \times n}(F)$ .

**Proof.** Let  $V$  denote the set of  $m \times n$  upper triangular matrices. We will show that  $V$  is a subspace of  $M_{m \times n}(F)$  using Theorem 3.

- (a) The zero matrix  $O$  from  $M_{m \times n}(F)$  contains entries  $O_{ij} = 0$  whenever  $i > j$ . Hence,  $O \in V$ .
- (b) Let  $A, B \in V$ . By definition of  $V$ ,  $A$  and  $B$  are upper triangular where  $A_{ij} = 0$  and  $B_{ij} = 0$  whenever  $i > j$ . Observe that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$$

whenever  $i > j$ . Hence,  $A + B \in V$ .

- (c) Let  $c \in F$  and  $A \in V$  as defined before. Let  $i > j$  and observe that  $(cA)_{ij} = cA_{ij} = c \cdot 0 = 0$ . Hence,  $cA \in V$ .

Since all the properties of theorem 3 have been satisfied, we conclude that  $V$  is indeed a subspace of  $M_{m \times n}(F)$ . ■

### Exercise 1.3.13

Let  $S$  be nonempty set and  $F$  is a field. Prove that for any  $s_0 \in S$ , the set  $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ , is a subspace of  $\mathcal{F}(S, F)$ .

**Proof.** We will proceed to prove that  $S$  is a subspace of  $\mathcal{F}(S, F)$  over the field  $F$  by satisfying the properties of Theorem 3. Let  $V = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ .

- (a) Note that the zero function  $f_0 \in \mathcal{F}(S, F)$  where  $f_0(s_0) = 0$  for any  $s_0 \in S$  implies that  $f_0 \in V$ .

- (b) Let  $f, g \in V$ . By definition of  $V$ ,  $f(s_0) = 0$  and  $g(s_0) = 0$  for any  $s_0 \in S$ . We have  $f + g \in V$  since

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0.$$

Hence,  $V$  is closed under addition.

- (c) Let  $f \in V$  and  $c \in F$ . We have  $cf \in V$  since

$$(cf)(s_0) = cf(s_0) = c \cdot 0 = 0$$

for any  $s_0 \in S$ . Hence,  $V$  is closed under scalar multiplication. ■

### Exercise 1.3.14

Let  $S$  be a nonempty set and  $F$  a field. Let  $\mathcal{C}(S, F)$  denote the set of all functions  $f \in \mathcal{F}(S, F)$  such that  $f(s) = 0$  for all but a finite number of elements of  $S$ . Prove that  $\mathcal{C}(S, F)$  is a subspace of  $\mathcal{F}(S, F)$ .

**Proof.** We proceed by using Theorem 3 to prove that  $\mathcal{C}(S, F)$  is a subspace where  $S$  is a nonempty set and  $F$  is a field.

- (a) Note that the zero vector  $f_0 \in \mathcal{F}(S, F)$  is in  $\mathcal{C}(S, F)$  because  $f_0(x_n) = 0$  where  $x_n \in S$  for finitely many  $n$ .

- (b) Let  $f, g \in \mathcal{C}(S, F)$ . We need to show that  $f + g \in \mathcal{C}(S, F)$ . Let  $x_n \in S$  for finitely many  $n$ . Then using the addition defined on  $\mathcal{F}(S, F)$ , we can write

$$(f + g)(x_n) = f(x_n) + g(x_n) = 0 + 0 = 0.$$

- (c) Let  $f \in \mathcal{C}(S, F)$  and  $c \in F$ . We need to show that  $cf \in \mathcal{C}(S, F)$ . Let  $x_n \in S$  for finitely many  $n$ . Using the scalar operation defined on  $\mathcal{F}(S, F)$ , we can write

$$(cf)(x_n) = cf(x_n) = c \cdot 0 = 0.$$

Since all the properties of Theorem 3 have been satisfied, we conclude that  $\mathcal{C}(S, F)$  is indeed a subspace of  $\mathcal{F}(S, F)$ . ■

### Exercise 1.3.15

Is the set of all differentiable real-valued functions defined on  $\mathbb{R}$  a subspace of  $C(\mathbb{R})$ ?

**Proof.** We claim that the set of all differentiable real-valued functions defined on  $\mathbb{R}$  is a subspace of  $C(\mathbb{R})$ . Denote this set as  $V$ .

- (a) Note that the zero function  $f_0$  is differentiable for all  $x \in \mathbb{R}$  and continuous for all  $x \in \mathbb{R}$ .

Hence,  $f_0 \in V$ .

(b) Let  $f, g \in V$ . Using the addition operation defined on  $C(\mathbb{R})$ , we get that the sum  $(f+g)(x) = f(x) + g(x)$  is differentiable which implies that the sum of functions  $f, g$  is also continuous. Hence,  $f + g \in V$ .

(c) Let  $f \in V$  and let  $c \in \mathbb{R}$ . Then  $(cf)(x) = cf(x)$  is differentiable for all  $x \in \mathbb{R}$  which means that  $cf$  is also continuous. Hence,  $cf \in V$ .

Hence,  $V$  is a subspace of  $C(\mathbb{R})$ . ■

### Exercise 1.3.16

Let  $C^n(\mathbb{R})$  denote the set of all real-valued functions defined on the real line that have a continuous  $n$ th derivative. Prove that  $C^n(\mathbb{R})$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .

**Proof.** ■

### Exercise 1.3.17

Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $W \neq \emptyset$  and, whenever  $a \in F$  and  $x, y \in W$ , then  $ax \in W$  and  $x + y \in W$ .

**Proof.** For the forwards direction, let  $W \subseteq V$  where  $V$  is a vector space and  $W$  is a subspace of  $V$ . Let  $a \in F$  and  $x, y \in W$ . Since  $W$  is a subspace, we know that  $O_V \in W$ . So,  $W$  is nonempty. Since  $W$  is closed under addition and multiplication, we get that  $x + y \in W$  and  $ax \in W$  and we are done.

For the backwards direction, let  $W \neq \emptyset$  and  $W \subseteq V$ . Let  $a \in F$  and  $x, y \in W$  be arbitrary such that  $ax \in W$  and  $x + y \in W$ . We need to show that  $W$  is a subspace of  $V$ . We need only show that  $O_V \in W$  since  $W$  is closed under addition and scalar multiplication. Let  $x \in W$ . We can pick any  $c \in F$  such that  $c = 0$ . So, we have  $c \cdot x = 0 \cdot x = O_W$ . Since the zero vector  $O_W \in W$  is unique, we must have  $O_V = O_W$ . Hence,  $O_V \in W$  and we conclude that  $W$  is a subspace of  $V$ . ■

### Exercise 1.3.18

Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $O \in W$  and  $ax + y \in W$  whenever  $a \in F$  and  $x, y \in W$ .

**Proof.** ( $\Rightarrow$ ) Let  $W \subseteq V$  where  $W$  is a subspace of  $V$ . Since  $W$  is a subspace of  $V$ , we know that  $W$  is closed under addition and scalar multiplication. Let  $a \in F$  and  $x, y \in W$ . Using the third property of Theorem 3, we can see that  $ax \in W$ . Since  $W$  is closed under addition, we can take  $y \in W$  and  $ax \in W$  such that  $ax + y \in W$ . Since  $W$  is also a vector space by definition, we know that  $O_W \in W$ . But  $O_W = O_V$  so  $O_V \in W$ .

( $\Leftarrow$ ) Let  $a \in F$  and  $x, y \in W$ . We want to show that  $W \subseteq V$  is a subspace of  $V$ . We can do this by using Theorem 3.

(a) By assumption, the zero vector  $O_V \in W$ .

(b) Let  $x, y \in W$ . Choose  $a = 1$  such that  $ax + y = x + y$ . Since  $ax + y \in W$  and  $ax + y = x + y$ , we also have  $x + y \in W$ . Hence,  $W$  is closed under addition.

(c) Let  $x \in W$  and  $O_V \in W$ . Let  $a \in F$ . Then we have  $ax + O_V = ax \in W$ .

Hence,  $W$  is a subspace of  $V$  by Theorem 3. ■

### Exercise 1.3.19

Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $W_1 \cup W_2$  is a subspace of  $V$  if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

**Proof.** ( $\Rightarrow$ ) Let  $W_1 \cup W_2$  is a subspace of  $V$ . We need to show that  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ . We

proceed by showing the contrapositive. Assume  $W_2 \not\subseteq W_1$  and  $W_1 \not\subseteq W_2$ . We need to show that  $W_1 \cup W_2$  is **NOT** a subspace of  $V$ . By assumption,  $x \in W_1$  is not contained in  $W_2$  as well as  $y \in W_2$  is not contained in  $W_1$ . This implies that  $W_1 \cup W_2 \neq \emptyset$ . Since  $W_1 \cup W_2$  is empty where  $W_1 \cup W_2$  does not contain  $O_V$ , it cannot possibly be a subspace of  $V$ .

( $\Leftarrow$ ) Let  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ . We need to show that  $W_1 \cup W_2$  is a subspace of  $V$ . We proceed by using Theorem 3 to do this. Without loss of generality, assume  $W_1 \subseteq W_2$ . The proof will be the same if we use  $W_2 \subseteq W_1$ .

- (a) Since  $W_1$  is a subspace of  $V$ , we get that  $O_W \in W_1$ . Furthermore,  $W_1 \subseteq W_2$  implies that  $O_V \in W$ . Since  $O_V \in W_1$  and  $O_V \in W_2$ , we get that  $O_V \in W_1 \cup W_2$  by definition of union.
- (b) Let  $x, y \in W_1$ . Since  $W_1$  is a subspace, we get that  $x + y \in W_1$ . Since  $W_1 \subseteq W_2$ , we also get that  $x + y \in W_2$ . Since both  $x + y \in W_1$  and  $x + y \in W_2$ , we know that  $x + y \in W_1 \cup W_2$  by definition of the union.
- (c) Let  $x \in W_1$  and  $c \in F$ . Since  $W_1$  is closed under scalar multiplication, we have that  $cx \in W_1$ . But  $W_1 \subseteq W_2$  so  $W_2$  also contains  $cx \in W_1$ . So we must have  $cx \in W_1 \cup W_2$ .

Hence,  $W_1 \cup W_2$  is a subspace of a vector space  $V$ . ■

### Exercise 1.3.20

Prove that if  $W$  is a subspace of a vector space  $V$  and  $w_1, w_2, \dots, w_n$  are in  $W$ , then  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$ .

**Proof.** Let  $W$  be a subspace of a vector space  $V$ . Our goal is to show that the following statement:  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$  for all  $1 \leq i \leq n$ . We proceed by induction on  $i \geq 1$ . Let  $i = 1$ . Since  $W$  is closed under scalar multiplication, we know that  $a_1 \in F$  and  $w_1 \in W$  implies that  $a_1w_1 \in W$ . Now let  $i = 2$ , then  $w_1, w_2 \in W$  and  $a_1, a_2 \in F$  implies that  $a_1w_1 + a_2w_2 \in W$  since  $W$  is closed under scalar multiplication and addition. Now, assume that our result holds for all  $1 \leq i \leq n$ . We want to show that it also holds for  $i = n + 1$ . By our inductive hypothesis, we know that  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$ . Let  $a_{n+1} \in F$  and  $w_{n+1} \in W$ . Using vector addition and scalar multiplication, we know that

$$(a_1w_1 + a_2w_2 + \dots + a_nw_n) + a_{n+1}w_{n+1} \in W.$$

■

### Exercise 1.3.21

Let  $V$  denote the vector space of sequences in  $\mathbb{R}$ , as defined in Example 5 of section 1.2. Show that the set of convergent sequences  $(a_n)$  (that is, those for which  $\lim_{n \rightarrow \infty} a_n$  exists) is a subspace of  $V$ .

**Proof.** We will show that  $W$  (the set of convergent sequences in  $\mathbb{R}$ ) is a subspace of  $V$  by using Theorem 3.

- (a) Note that the zero sequence  $O_n$  is zero for all  $n = 1, 2, \dots$ . Hence, the limit of  $O_n$  converges to 0 and so we have  $O_n \in W$ .
- (b) Let  $(a_n)$  and  $(b_n)$  be two convergent sequences in  $\mathbb{R}$ . Then we get that  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  exists. Observe that

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

Since the  $a_n$  and  $b_n$  are both convergent sequences, we also get that the sum  $(a_n + b_n)$  also converges. Hence,  $(a_n + b_n) \in W$ .

- (c) Let  $t \in F$  and  $a_n \in W$  as before. Then we have

$$\lim_{n \rightarrow \infty} ta_n = t \lim_{n \rightarrow \infty} a_n.$$

Since any constant  $t \in F$  multiplied by a convergent sequence is convergent, we also get that the sequence  $(ta_n)$  is also convergent. Hence,  $ta_n \in W$ . ■

### Exercise 1.3.22

Let  $F_1$  and  $F_2$  be fields. A function  $g \in \mathcal{F}(F_1, F_2)$  is called an **even function** if  $g(-t) = g(t)$  for each  $t \in F_1$  and is called an **odd function** if  $g(-t) = -g(t)$  for each  $t \in F_1$ . Prove that the set of all even functions in  $\mathcal{F}(F_1, F_2)$  and the set of all odd functions in  $\mathcal{F}(F_1, F_2)$  are subspaces of  $\mathcal{F}(F_1, F_2)$ .

**Proof.** Let  $V$  be the set of all even functions and let  $W$  be the set of all odd functions. We will first prove that  $V$  is a subspace of  $\mathcal{F}(F_1, F_2)$  and the same with  $W$ .

- (a) Note that the zero function  $f_0$  from  $\mathcal{F}(F_1, F_2)$  is even since  $f_0(-t) = 0 = f_0(t)$  for all  $t \in F_1$ . Hence,  $f_0 \in V$ .
- (b) Let  $f, g \in V$ . We need to show that  $f + g \in V$ ; that is, we need to show that it is even. Let  $t \in F_1$ . Then observe that

$$(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(t)$$

Hence,  $(f + g)(-t) = (f + g)(t)$  for all  $t \in F_1$ .

- (c) Let  $f \in V$  and let  $c \in F_2$ . Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence,  $cf \in V$ .

Since all the properties of  $V$  are satisfied, we have that  $V$  is a subspace of  $\mathcal{F}(F_1, F_2)$ .

Now we will prove  $W$  is a subspace of  $\mathcal{F}(F_1, F_2)$ .

- (a) Note that the zero function  $f_0$  is in  $W$  because for any  $t \in F_1$  we have  $f_0(-t) = 0 = -1 \cdot 0 = -f_0(t)$ . Hence,  $f_0 \in W$ .
- (b) Let  $f, g \in W$  and Let  $t \in F_1$ . Observe that

$$(f + g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f + g)(t).$$

Hence,  $f + g \in W$ .

- (c) Let  $c \in F_2$  and  $f \in W$ . Let  $t \in F_1$  such that  $f(-t) = -f(t)$ . Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence,  $cf \in W$ .

Hence,  $W$  is a subspace of  $\mathcal{F}(F_1, F_2)$  ■

**Definition 1 (Sum of Two Sets).** If  $S_1$  and  $S_2$  are nonempty subsets of a vector spaces  $V$ , then the **sum**  $S_1$  and  $S_2$ , denoted  $S_1 + S_2$ , is the set  $\{x + y : x \in S_1 \text{ and } y \in S_2\}$ .

**Definition 2 (Direct Sum).** A vector space  $V$  is called the **direct sum** of  $W_1$  and  $W_2$  if  $W_1$  and  $W_2$  are subspaces of  $V$  such that  $W_1 \cap W_2 = \{0\}$  and  $W_1 + W_2 = V$ . We denote that  $V$  is the direct sum of  $W_1$  and  $W_2$  by writing  $V = W_1 \oplus W_2$ .

### Exercise 1.3.23

Let  $W_1 + W_2$  is a subspace of a vector space  $V$ .

- (a) Prove that  $W_1 + W_2$  is a subspace of  $V$  that contains both  $W_1$  and  $W_2$ .

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Define  $W_1 + W_2$  as the set

$$\{x_1 + y_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}.$$

- (a) Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that the zero vector  $O_V$  is contained in both  $W_1$  and  $W_2$ . Hence, the sum  $O_V = O_V + O_V \in W_1 + W_2$ .
- (b) Let  $u, v \in W_1 + W_2$  with  $u = x_1 + y_1$  and  $v = x_2 + y_2$ . Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that addition is closed in both subsets  $W_1$  and  $W_2$ . Hence,  $x_1 + x_2 \in W_1$  and  $y_1 + y_2 \in W_2$ . Observe that

$$\begin{aligned} u + v &= (x_1 + y_1) + (x_2 + y_2) \\ &= (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2. \end{aligned}$$

Hence, addition is closed in  $W_1 + W_2$ .

- (c) Let  $c \in F$  where  $F$  is a field and let  $u \in W_1 + W_2$  with  $u = x_1 + y_1$ . Since  $W_1$  and  $W_2$  are subspaces of  $V$ , we know that  $cx_1 \in W_1$  and  $cy_1 \in W_2$ . Observe that

$$\begin{aligned} cu &= c(x_1 + y_1) \\ &= cx_1 + cy_1 \in W_1 + W_2. \end{aligned}$$

Hence,  $W_1 + W_2$  is a subspace of  $V$ . ■

- (b) Prove that any subspace of  $V$  that contains both  $W_1$  and  $W_2$  must also contain  $W_1 + W_2$ .

**Proof.** Let  $X$  be a subspace of  $V$ . Suppose  $X$  contains both  $W_1$  and  $W_2$ ; that is,  $W_1 \subseteq X$  and  $W_2 \subseteq X$ . We must show that  $W_1 + W_2 \subseteq X$ . Let  $u \in W_1 + W_2$  with  $u = x_1 + y_1$ . By definition, we have  $x_1 \in W_1$  and  $y_1 \in W_2$ . Since  $W_1 \subseteq X$  and  $W_2 \subseteq X$ , we have  $x_1 \in X$  and  $y_1 \in X$ . Since  $X$  is a subspace of  $V$  and  $X$  is closed under addition, we have  $x_1 + y_1 \in X$ . Hence,  $u \in X$  and we conclude  $W_1 + W_2 \subseteq X$ . ■

### Exercise 1.3.24

Show that  $F^n$  is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \dots = a_{n-1} = 0\}.$$

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of  $F^n$ . Since  $W_1$  and  $W_2$  are subspaces, they both contain the zero vector  $O_{F^n}$ . Observe that for any element  $x \in W_1$ , we have the  $n$ th element  $a_n = 0$ . On the other hand,  $y \in W_2$  implies that  $a_1 = a_2 = \dots = a_{n-1} = 0$  except for the  $n$ th element. Thus, the only element that  $W_1$  and  $W_2$  have in common is the zero vector  $O_{F^n}$ . Hence, we have  $W_1 \cap W_2 = \{O_{F^n}\}$ .

Now, we show  $W_1 + W_2 = F^n$ . To do this, we need to show the following containments:

- (i)  $W_1 + W_2 \subseteq F^n$  and
- (ii)  $F^n \subseteq W_1 + W_2$ .

Let  $u \in W_1 + W_2$  with  $u = x_1 + x_2$  where  $x_1 \in W_1$  and  $y_1 \in W_2$ . Since  $W_1 \subseteq F^n$ , we know that  $x_1 \in F^n$ . Likewise,  $W_2 \subseteq F^n$  implies that  $y_1 \in F^n$ . Since  $F^n$  is a vector space where addition is closed, we have that  $u = x_1 + y_1 \in F^n$ . Hence,  $W_1 + W_2 \subseteq F^n$ .

Now, let  $u \in F^n$ . Since  $F^n$  is a vector space over  $F$ , each entry in  $u$  (where each entry is an element of a field  $F$ ) can be written and separated using the addition defined in  $F^n$  in the following

way:

$$\begin{aligned} u &= (a_1, a_2, \dots, a_n) \\ &= (a_1 + 0, a_2 + 0, \dots, 0 + a_n) \\ &= (a_1, a_2, \dots, 0) + (0, 0, \dots, a_n) \end{aligned}$$

where the first term is an element of  $W_1$  and the second term is an element of  $W_2$ . Hence,  $u \in W_1 + W_2$ .

Since both containments are satisfied, we have that  $W_1 + W_2 = F^n$ . Thus,  $W_1 \oplus W_2 = F^n$ . ■

### Exercise 1.3.25

Let  $W_1$  denote the set of all polynomials  $f(x)$  in  $P(F)$  such that in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

we have  $a_i = 0$  whenever  $i$  is even. Likewise, let  $W_2$  denote the set of all polynomials  $g(x)$  in  $P(F)$  such that in the representation

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0,$$

we have  $b_i = 0$  whenever  $i$  is odd. Prove that  $P(F) = W_1 \oplus W_2$ .

**Proof.** Let  $f(x) \in W_1$ . Then we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n$$

where  $a_i = 0$  where  $i$  is even. Likewise, let  $g(x) \in W_2$ . Hence,

$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_{n-1} x^{n-1} + b_n x^n$$

where  $b_i = 0$  for  $i$  odd. This tells us that the only representation that  $W_1$  and  $W_2$  have in common is the zero polynomial  $f(x) = 0$  where  $a_i = 0$  for all  $0 \leq i \leq n$ . Hence,  $W_1 \cap W_2 = \{0\}$ .

Now, we want to show that  $W_1 + W_2 = P(F)$ ; that is, we need to show  $P(F) \subseteq W_1 + W_2$  and  $W_1 + W_2 \subseteq P(F)$ . Starting with the former, let  $f \in P(F)$ . Observe that for  $a_i$  where  $1 \leq i \leq n$

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n \\ &= \underbrace{(a_1 x + a_3 x^3 + \dots + a_n x^n)}_{\in W_1} + \underbrace{(a_0 + a_2 x^2 + a_4 x^4 + \dots + a_n x^n)}_{\in W_2} \end{aligned}$$

The first term of the last equality contains coefficients  $a_i = 0$  for even  $i$  and the second term contains coefficients  $a_i = 0$  for odd  $i$ . This implies that  $f(x) \in W_1 + W_2$ . Hence,  $P(F) \subseteq W_1 + W_2$ .

Now, let  $u(x) \in W_1 + W_2$  with  $u(x) = f(x) + g(x)$  with  $f(x) \in W_1$  and  $g(x) \in W_2$ . Since  $W_1$  and  $W_2$  are subsets of  $P(F)$ , we have that  $f(x), g(x) \in P(F)$ . Since addition is closed in  $P(F)$ , we have that  $u(x) = f(x) + g(x) \in P(F)$ . Hence,  $W_1 + W_2 \subseteq P(F)$ . Since  $W_1 + W_2 = P(F)$  and  $W_1 \cap W_2 = \{0\}$ , we have  $W_1 \oplus W_2 = P(F)$ . ■

### Exercise 1.3.26

In  $M_{m \times n}(F)$  define  $W_1 = \{A \in M_{m \times n}(F) : A_{ij} = 0 \text{ whenever } i > j\}$  and  $W_2 = \{A \in M_{m \times n}(F) : A_{ij} \text{ whenever } i \leq j\}$ . Show that  $M_{m \times n}(F) = W_1 \oplus W_2$ . ( $W_1$  is the set of all triangular matrices as defined in the previous section.)

**Proof.** TO DO. ■

### Exercise 1.3.30

Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if each vector in  $V$  can be *uniquely* written as  $x_1 + x_2$ , where  $x_1 \in W_1$  and  $x_2 \in W_2$ .

---

**Proof.** Let  $W_1$  and  $W_2$  be subspaces of  $V$ . Let  $v \in V$ . Let  $x_1, x'_1 \in W_1$  and  $x_2, x'_2 \in W_2$  such that  $v = x_1 + x_2 = x'_1 + x'_2$ . Since  $W_1$  and  $W_2$  is closed under addition, we know that  $x_1 - x'_1 \in W_1$  and  $x_2 - x'_2 \in W_2$ . But observe that  $x_1 - x'_1 = x_2 - x'_2 \in W_1 \cap W_2$ . Since  $V$  is a direct sum of the two subspaces  $W_1$  and  $W_2$ , we know that  $W_1 \cap W_2 = \{O_V\}$  which implies that  $x_1 = x'_1$  and  $x_2 = x'_2$ . This tells us that every  $v$  can be expressed uniquely as  $x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ .

Conversely, suppose every vector  $v \in V$  can be *uniquely* written as  $x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ . We need to show that  $W_1 \oplus W_2 = V$ . Since  $v \in V$  is uniquely expressed in terms of  $x_1 + x_2$ , the only vector that the two subspaces  $W_1$  and  $W_2$  share is the zero vector. Hence,  $W_1 \cap W_2 = \{O_V\}$ . Now, we need to show that  $V = W_1 + W_2$ ; that is, we need to show  $V \subseteq W_1 + W_2$  and  $W_1 + W_2 \subseteq V$ . Suppose  $v \in V$ . Since  $x_1 \in W_1$  and  $x_2 \in W_2$ , we have that  $v \in W_1 + W_2$ . Hence,  $V \subseteq W_1 + W_2$ . Now, let  $v \in W_1 + W_2$ . Since  $v = x_1 + x_2$  where  $x_1 \in W_1$  and  $x_2 \in W_2$  and  $W_1 \subseteq V$  and  $W_2 \subseteq V$ , we know that  $x_1$  and  $x_2$  are contained within  $V$ . Since  $V$  is vector space, we know that sum of  $x_1$  and  $x_2$  are contained in  $V$ . Hence,  $v \in V$  and so  $V \subseteq W_1 + W_2$ . Thus,  $V = W_1 \oplus W_2$ . ■