Linear Algebra Notes

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Contents

Chapter 1

Vector Spaces

1.1 Vector Spaces

1.1.1 Basics

> **Definition 1** (Vector Spaces). A **vector space** (or **linear space**) over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in V there is a unique element ax in V, such that the following conditions hold:
>
> (VS 1) For all $x, y \in V$, $x + y = y + x$ (commutativity of addition).
>
> (VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (associativity of addition).
>
> (VS 3) There exists an element in V denoted by O such that $x + O = x$ for each $x \in V$
>
> (VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = O$.
>
> (VS 5) For each element $x \in V$, we have $1x = x$.
>
> (VS 6) For each $a, b \in F$ and each element $x \in V$, then $(ab)x = a(bx)$.
>
> (VS 7) For each element $a \in F$ and each pair $x, y \in V$, we have $a(x + y) = ax + ay$.
>
> (VS 8) For each pair $a, b \in F$ and each $x \in V$, we have $(a + b)x = ax + bx$.
>
> The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.

- The elements of a field F are called **scalars** and the elements of a vector space V are called **vectors** (these should not be confused!).

- Every vector space will always be defined over a given field, mostly defined over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ unless otherwise noted.

- Every vector space should specify the operations of addition and scalar multiplication.

> **Definition 2** (n-tuples). An object of the form $(a_1, a_2, \ldots, a_n)$, where the entries $a_1, a_2, \ldots, a_n$ are elements of a field F, is called an **n-tuple** with entries from F. The elements $a_1, a_2, \ldots, a_n$ are called **entries** or **components** of the n-tuple.

> **Definition 3.** We say that two n-tuples, $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, are **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

> **Example.** The set of all n-tuples with entries from a field F denoted by F_n is a vector space. To see why, suppose $u, v \in F_n$ where $u = (a_1, a_2, \ldots a_n)$ and $v = (b_1, b_2, \ldots, b_n)$. If we take term-by-term addition of the

entries in both u and v, then we end up with

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and likewise,

$$cu = (ca_1, ca_2, \ldots, ca_n).$$

These same set of operations define $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ and likewise, $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.

- Note that vectors in F^n can be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

  rather than **row vectors** $(a_1, a_2, \ldots, a_n)$.

- 1-tuples are are just scalars or an just an element from F.

Definition 4. An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry a_{ij} with ($1 \le i \le m, 1 \le j \le n$) is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \ldots a_{in}$ compose the i**th row** of the matrix, and the entries $a_{1j}, a_{2j}, \ldots a_{mj}$ compose the jth column of the matrix.

- The rows make a vector space which we denote F^n.

- Likewise, the columns make a vector space we denote F^m

Definition 5 (Zero Matrix)**.** The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

Definition 6 (Square Matrix)**.** A matrix is **square** if the number of rows and columns of a matrix are equal.

- Just like our tuple example, the set of all $m \times n$ matrices with entries from a field F form a vector space. Denote this vector space as $M_{m \times n}(F)$ endowed with two operations; that is, **matrix addition** and **scalar multiplication**. Suppose for $A, B \in M_{m \times n}(F)$ and $c \in F$, we have

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

  and

$$(cA)_{ij} = cA_{ij}$$

  for $1 \le i \le m$ and $1 \le j \le n$. In other words, the two operations can be performed entry-wise.

- The operations from our tuple case extends very naturally to $M_{m \times n}(F)$. In other words, if we add two matrices A_{ij} and B_{ij}, then we would expect to that $A_{ij} + B_{ij} \in M_{m \times n}(F)$ as well and likewise for the scalar multiplication case.

> **Definition 7** (Set of All Functions)**.** Let S be any nonempty set and F be any field, and let $\mathscr{F}(S, F)$ denote the set of all functions from S to F.

> **Definition 8.** Two functions $f, g \in \mathscr{F}(S, F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$.

The set of functions above forms a vector space with the operations of addition and scalar multiplication defined for every $f, g \in \mathscr{F}(S, F)$ and $c \in F$ with

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$

> **Definition 9** (Set of All Polynomials)**.** A **polynomial** with coefficients from a field F is an expression of the form
> $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$
> where n is a nonnegative integer and each $a_k \in F$ is called the **coefficient** of x_k.

> **Definition 10** (Zero Polynomial)**.** We call $f(x) = 0$ the **zero polynomial** if $a_n = a_{n-1} = \cdots = a_0 = 0$.

> **Definition 11** (Degree)**.** The **degree** of a given polynomial f is defined to be the largest exponent of x that appears in the representation
> $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

> **Definition 12** (Equality of Polynomials)**.** We call two polynomials f, g, where
> $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
> and
> $$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$
> are **equal** if $m = n$ and $a_i = b_i$ for all $i = 0, 1, \ldots, n$.

Suppose we have $c \in F$ and say we evaluated the polynomial $f \in F$ at c. Then we would have the following

$$f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

where $f(c) \in F$.

> **Definition 13** (Basic Operations of Polynomials)**.** Define polynomial **addition** $f + g$ as the following:
> $$f(x) + g(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0).$$
> Let $c \in F$. Let scalar **multiplication** be defined by
> $$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0.$$

The operations above form a vector space for $P(F)$ (the set of all polynomials).

> **Definition 14** (Sequences)**.** A **sequence** in F is a function $\sigma : \mathbb{Z}^+ \to F$. A given sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \ldots$ is denoted (a_n).

Let V be the set of all sequences $\sigma(n) \in F$. For every $(a_n), (b_n) \in V$ with $t \in F$, we have the following operations

$$(a_n) + (b_n) = (a_n + b_n) \text{ and } t(a_n) = (ta_n).$$

1.1.2 Non-examples

Example. Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$ where R is a field. For every $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Note that S is **NOT** a vector space since (VS 1), (VS 2), and (VS 8) fail.

Example. Let S be the same set as in the last example. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$$

and

$$c(a_1, a_2) = (ca_1, 0).$$

Note that S is **NOT** a vector space with these operations since (VS 3), (VS 4), and (VS 5) fail.

1.1.3 Basic Extensions from Definition

Theorem 1 (Cancellation Law for Vector Addtion)**.** If $x, y, z \in V$ such that $x + z = y + z$, then $x = y$.

Proof. There exists a vector $v \in V$ such that $z + v = O$ (VS 4). Thus, we have

$$\begin{aligned}
x &= x + O \\
&= x + (z + v) \\
&= (x + z) + v \\
&= (y + z) + v \\
&= y + (z + v) \\
&= y + O \\
&= y.
\end{aligned}$$

Hence, we have $x = y$. ∎

Corollary. The vector described O described in (VS 3) is unique.

Proof. Let $x \in V$. Suppose there exists two elements $O, O' \in V$ such that

$$x + O = x \tag{1}$$

and

$$x + O' = x. \tag{2}$$

Our objective is to show that $O = O'$. Equating (1) and (2), we find that

$$x + O = x + O' \tag{3}$$

Using the Cancellation Law for Vector Addition, (3) implies that $O = O'$. Hence, O is a unique vector in V. ∎

Corollary. The vector y described in (VS 4) is unique.

Proof. Let $x \in V$. Suppose there exists $y, y' \in V$ such that

$$x + y = O \text{ and } x + y' = O.$$

Equating the two equations above, we get that

$$x + y = x + y'.$$

Applying the Cancellation Law for Vector Addition once again, we get that $y = y'$. Hence, y is a unique

vector in V. ∎

Theorem 2. In any vector space V, the following statements are true:

(a) $0x = O$ for any $x \in V$.

(b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.

(c) $a0 = 0$ for any $a \in F$.

Proof. (a) Using (VS 8), (VS 3), and (VS 1), it follows that

$$
\begin{aligned}
0x + 0x &= (0 + 0)x \\
&= 0x \\
&= 0x + O \\
&= O + 0x.
\end{aligned}
$$

Hence, $0x = O$ by Theorem 1.1.

(b) The vector $-(ax)$ is the unique element of V such that $ax + [-(ax)] = O$. If we know that $ax + (-a)x = O$, we can use Corollary 2 to Theorem 1.1 to state that $(-a)x = -(ax)$. But by (VS 8), we have

$$ax + (-ax) = [a + (-a)]x = 0x = O$$

by (a). Hence, we have that $(-a)x = -(ax)$. In particular, we have $(-1)x = -x$. Thus, (VS 6) implies that

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

(c) The proof of (c) is similar to the proof of (a). ∎