Understanding Analysis Notes

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Chapter 1

The Real Numbers

1.1 The Axiom of Completeness

Theorem 1. Every nonempty set of real numbers that is bounded above has a least upper bound.

Definition 1. We call a set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Otherwise, a set is *bounded below* if there exists a $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for every $a \in A$.

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it satisfies the following criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We denote the least upper bound of a set A by calling it the *supremum* of A i.e $\sup(A)$. Similarly, we denote the greatest upper bound of set A by calling it the *infimum* of A i.e $\inf(A)$.

Note that a set can have many upper/lower bounds. But there can only exist one supremum and one infimum. In other words, these bounds are unique. Furthemore, the infimum and supremum need not be in the set.

Consider the following set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

This set is bounded above and below. In addition, we can see that $\sup(A) = 1$ and $\inf(A) = 0$ (this is because each subsequent number in the sequence gets smaller and smaller).

Definition 2. We say that $a_0 \in \mathbb{R}$ is a *maximum* of the set A if $a_0 \in A$ and $a \leq a_0$ for all $a \in A$. Likewise, we say that $a_1 \in \mathbb{R}$ is a *minimum* of A if $a_1 \in A$ and $a \geq a_1$ for every $a \in A$.

If we have an open set $(0, 2)$ then the end points of this set are the infimum and supremum of the set respectively. Note that the maximum and the minimum do not exists because the infimum and the supremum are not in the set. If this set were to be closed, then the supremum and infimum would be in the set which implies that the max and min exists.

Now consider the Example

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

Notice that when we try and search for the supremum for this set, we cannot find one since we can always find a smaller number for an upper bound. One might say that $r = \sqrt{2}$ is the supremum of S but this is false since $r \notin \mathbb{Q}$ and is irrational.

Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and is bounded above. Let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Prove that $\sup(c + A) = c + \sup(A)$

Proof. We use defintion 0.2 to prove this proposition. First, we need to prove that this $\sup(c+A)$ is an upper bound. We have $\sup(A) = s$ for some $s \in A$ if $s \geq a$ for all $a \in A$. We find that adding $c \in \mathbb{R}$ gives us

$$c + s \geq c + a.$$

Hence, we have that $c+s$ is an upper bound for the set $c+A$.

Next, we prove that $\sup(c+A) = c+s$ is the *least upper bound*. We know that $c+s \geq c+a$ for all $a \in A$. Suppose we have another upper bound $b \in A$ such that $c+a \leq b$ for all $a \in A$. Another manipulation gives us $a \leq b-c$ for all $a \in A$. Since $\sup(A) = s$ is the least upper bound for A, it follows that $s \leq b-c$. Hence, we have

$$c + s \leq b \Rightarrow \sup(c+A) = c + \sup(A).$$

■

There is another way to restate part (ii) of defintion 0.2 i.e

Lemma 1. Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $s - \varepsilon < a$.

Proof. For the forward direction, suppose that $s = \sup A$ and consider $s - \varepsilon$. Since s is an upper bound, we have that $s - \varepsilon < s$. This means that $s - \varepsilon$ is not an upper bound. Hence, we can find an element $a \in A$ such that $s - \varepsilon < a$ because otherwise $s - \varepsilon$ would be an upper bound. This concludes the forward direction.

For the backwards direction, assume s is an upper bound. We must satisfy part (ii) of defintion 0.2. Let $\varepsilon > 0$, then $\varepsilon = s - b$. But since any number smaller than s is not an upper bound, we have that $s \leq b$ if b is any other upper bound for S. Hence, $s = \sup A$. ■

1.2 Consequences of Completeness

The first application of the Axiom of Completeness is a result that says that the real line contains no gaps.

Theorem 2. For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Our goal is to produce a real number x such that this element is in every closed interval I_n for every $n \in \mathbb{N}$. Using the Axiom of Completeness, we can denote the following sets

$$A = \{a_n : n \in \mathbb{N}\}$$
$$B = \{b_n : n \in \mathbb{N}\}$$

where A and B consists of the left-hand and right-hand endpoints respectively. Since every closed interval are nested, we know that every b_n serves as an upper bound for A. By the Axiom of completeness, we can say that a supremum exists for A and we can label this supremum as $x = \sup A$. By definition, this is an upper bound for A. Hence, we have that $a_n \leq x$. But since x is the least upper bound and every $b_n \in B$ is an upper bound for every $a_n \in A$, we have that $x \leq b_n$. Hence, we have that $a_n \leq x \leq b_n$ which means that $x \in I_n$ for all $n \in \mathbb{N}$. This precisely means that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. ■

1.2.1 The Density of the Rationals

Theorem 3.
- Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$
- Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$

Before we head on to the proof, it is important to notice that $\mathbb{N}$ is not bounded above and we shall not prove this fact since we are taking this property of the set to be a given just like all the properties that are contained in $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Proof. Assume for sake of contradiction that $\mathbb{N}$ is bounded above. Using the Axiom of Completeness, $\mathbb{N}$ contains a supremum, say, $\sup \mathbb{N} = \alpha$. Using lemma 1.3.8, we know that there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n. \qquad (\varepsilon = 1)$$

This impplies that

$$\alpha < n + 1$$

but this shows that $n+1 \in \mathbb{N}$ which is a contradiction because we assumed that $\alpha \geq n$ for all $n \in \mathbb{N}$ thereby rendering α to no longer be an upper bound for $\mathbb{N}$. Hence, we have that there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$. The second part of this theorem follows immediately by setting $x = 1/y$. ■

Theorem 4. For every two $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Our goal is to choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b \qquad (1)$$

The idea is to choose a denominator large enough so that when we increment by size $\frac{1}{n}$ that it will be too big to increment over the open interval (a, b). Using the (2) of the Archimedean Property, we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a. \qquad (2)$$

We now need to choose an $m \in \mathbb{Z}$ such that na is smaller than this chosen number. A diagram for choosing such a number is helpful. Hence,

Judging from our diagram, we can see that

$$m - 1 \leq na < m.$$

Focusing on the left part of the inequality, we can solve (2) for a and say that

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - 1/n) + 1 \\ &= nb \end{aligned}$$

This implies that $m < nb$ and consequently $na < m < nb$ which is equivalent to (1). ■

1.2.2 The Existence of Square Roots

Theorem 5. There exists $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

Proof. Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We need to show that $\alpha^2 = 2$. Hence, we need to show cases where $\alpha^2 < 2$ and $\alpha^2 > 2$. The idea behind these cases is to produce a contradiction that will show that having either one of these cases will violate the fact that $\alpha$is an upper bound for T and α is the least upper bound respectively.

Assume the first case, $\alpha^2 < 2$. We know that α is an upper bound for T. We need to construct an element that is larger than α. Hence, we construct

$$\alpha + \frac{1}{n} \in T \tag{1}$$

Squaring (1) we have that

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

We can use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}.$$

Rearranging we get that

$$\frac{2\alpha+1}{n_0} < 2 - \alpha^2$$

and consequently

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2-\alpha^2) = 2$$

But this means that $\alpha + 1/n_0 \in T$ showing that α is not an upper bound for T contradicting our assumption.

Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound.

■

1.3 Cardinality

1.3.1 Correspondence

Definition 3. A function $f : A \to B$ is *one-to-one* if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, there exists an element $a \in A$ for which $f(a) = b$.

An equivalent defintion for a function to be one-to-one is the following:

Definition 4. A function $f : A \to B$ is *one-to-one* if $f(a_1) \neq f(a_2)$ implies that $a_1 = a_2$.

A function that is both one-to-one and onto is said to be bijective. Meaning that we have a one-to-one correspodence between the sets A and B. Another way to explain a function being injective is to say that no two elements from A can map to the same element in B (think of the function x^2). And a function being onto can be explained as every element in A has to be mapped to an element in B.

From an algebraic perspective, we can denote a function being bijective to mean the same thing as two sets having the same cardinality i.e we can say that

Definition 5. Two sets A and B have the same cardinality if there exists $f : A \to B$ that is both one-to-one and onto. We can denote this symbolically as $A \sim B$

Some examples of bijective maps are

(i) Let the following map $f : \mathbb{N} \to \mathbf{E}$ be defined as $f(n) = 2n$. We can see that $\mathbb{N} \sim \mathbf{E}$. It's true that $\mathbf{E}$ is indeed a subset of $\mathbb{N}$, but do not conclude that it is a smaller set than $\mathbb{N}$ since they have the same cardinality or isomorphic to each other.

(ii) We can show this again. This time let us have a map $f : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ is odd.} \\ -n/2 \text{ if } n \text{ is even.} \end{cases}$$

We have that $\mathbb{N} \sim \mathbb{Z}$ indeed.

1.3.2 Countable Sets

Definition 6. A set A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an *uncountable set*.

Theorem 6. Let $\mathbb{Q}, \mathbb{R}$. Then

(i) The set $\mathbb{Q}$ is countable.

(ii) The set $\mathbb{R}$ is uncountable.

Proof.

1. Suppose we define A_n to be split into two sets. When $n = 1$, define A_n to be

$$A_1 = \{0\}$$

and define A_n when $n \geq 2$ as

$$A_n = \left\{ \pm \frac{p}{q} : \text{ where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

We can observe here that for every $n \in \mathbb{N}$ we can find every element of $\mathbb{Q}$ exactly once in the sets we have defined. So we can conclude that our map is onto. Since we designed our sets so that each rational numer appears once and the fact that for $n = 1$ and $n \geq 2$ produces two disjoint sets, we can see that our map is also one-to-one.

2. We can prove that second statement of theorem by contradiction. Assume for the sake of contradiction that there exists a *one-to-one* and *onto* function where $f : \mathbb{N} \to \mathbb{R}$. Letting $x_1 = f(1)$ and $x_2 = f(2)$ and so on, then we can enumerate each element of $\mathbb{R}$ i.e

$$\mathbb{R} = \{x_1, x_2, x_3, ...\}.$$

Using the Nested Interval Property, we will now produce a real number that is not in this set. Let I_n be a closed interval which does not contain x_n but contains x_{n+1}. Furthermore, I_{n+1} is contained within I_n. Note that within I_n there are two sets which are disjoint and x_{n+1} can be in either one of these sets. Now consider the following intersection $\cap_{n=1}^{\infty} I_n$. Using our construction that every $x_n \notin I_n$, then we can say that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

But this is a contradiction because the nested interval property asserts that this intersection is nonempty meaning that every $x \in \mathbb{R}$ is contained in the above set. Hence, we cannot emumerate every single element x_n of $\mathbb{R}$. Therefore, $\mathbb{R}$ is an *uncountable* set.

■

This gives us three insights:

1. The smallest type of infinite set is the countable set.
2. We can create another set by deleting or inserting elements into it.
3. Anything smaller than a countable set is either finite or countable.

We can create $\mathbb{R}$ by taking the union of $\mathbb{Q}$ and $\mathbb{I}$. Since $\mathbb{R}$ is not countable and $\mathbb{Q}$ is, this would mean that the set of irrational numbers $\mathbb{I}$ would be uncountable. This tells us that $\mathbb{I}$ is a bigger subset of $\mathbb{R}$ than $\mathbb{Q}$.

We can summarize these results in the follow two theorems:

Theorem 7. If $A \subseteq B$ and B is *countable*, then A is either countable or finite.

Theorem 8. (i) If $A_1, A_2, ...A_n$ are each countable sets, then the union of

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

(ii) If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

Chapter 2

Sequences and Series

2.1 The Limit of a Sequence

Understanding infinite series depends on understanding sequences that make up sequences of partial sums.

Definition 7. A sequence is a function whose domain is $\mathbb{N}$.

A way we describe sequences is to assign each $n \in \mathbb{N}$, use a mapping rule, and then have an output for the nth term. Mathematically we can describe it as a map $f : \mathbb{N} \to \mathbb{R}$.

Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$
2. $\{\frac{1+n}{n}\}_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ...)$
3. (a_n), where $a_n = 2^n$ for each $n \in \mathbb{N}$,
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

It should not be confused that in some instances, the index n will start at $n = 0$ or $n = n_0$ for some other $n_0 > 1$. It is important to keep in mind that sequences are just infinite lists of real numbers. The main point of our analysis deals with what happens at the "tail" end of a given sequence.

Definition 8 (Convergence of a Sequence)**.** A sequence (a_n) *converges* to a real number a if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \varepsilon$.

Furthermore, the convergence of a sequence (a_n) to a is denoted by

$$\lim_{n\to\infty} a_n = a.$$

To understand the last part of this definition, namely, $|a_n - a| < \varepsilon$, we can think of it as a neighborhood where a given value will be located in.

Definition 9. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

is called the *ε-neighborhood of a.*

We can think of $V_\varepsilon(a)$ as an interval where

$$a - \varepsilon < a < a + \varepsilon.$$

Another way is to think of it as a ball with radius $\varepsilon > 0$ centered at a. we can also think about the convergence of a sequence to a point with the following definition.

Definition 10. A sequence (a_n) converges to a if, given any $\varepsilon-$neighborhood $V_\varepsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\varepsilon(a)$. In other words, every $\varepsilon-$neighborhood contains all but a finite number of the terms of (a_n).

The main idea here is that for some $n \in \mathbb{N}$ along a sequence (a_n), all the points of the sequence converge to some point within a certain $\varepsilon-$neighborhood. Note that when increase the value of $n \in \mathbb{N}$, the smaller this $\varepsilon-$neighborhood has to be and vice versa.

Consider the sequence (a_n), where $a_n = \frac{1}{\sqrt{n}}$. From our regular understanding of calculus, one can see that the limit of this sequence goes to zero.

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{e^2}.$$

We now proceed by verifying that this choice $N \in \mathbb{N}$ has the desired property that $a_n \to 0$. Let $n \geq N$ such that $n > \frac{1}{\varepsilon^2}$. Hence, we have

$$\frac{1}{\sqrt{n}} < \varepsilon.$$

But this implies that $|a_n - 0| < \varepsilon$ and hence our sequence contains the desired property. ■

The main idea of these convergence proofs is to find an $N \in \mathbb{N}$ such that the value we want can be "hit" within some range that we specify with any number $\varepsilon > 0$.

Quantifiers

The phrase

"For all $\varepsilon > 0$", there exists $N \in \mathbb{N}$ such that ..."

means that for every positive integer I give you, there exists some index or natural number that contains some property that allows the sequence to converge to some value that we desire and as long as we satisfy this rule, then we can say that the sequence converges to our desired value. The template for our subsequent covergence proof will follow the steps below:

- "Let $\varepsilon > 0$" be arbitrary."
- Demonstrate that a specific choice of $N \in \mathbb{N}$ leads to the desired property. Note that finding this N often involves working backwards from $|a_n - a| < \varepsilon$.
- Show that this N actually works.
- Now assume $n \geq N$.
- With this choice of $\mathbb{N}$, you can work towards the property that $|a_n - a| < \varepsilon$

Show

$$\lim \left(\frac{n+1}{n}\right) = 1.$$

In other words, show that for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|a_n - 1| < \varepsilon$$

where

$$a_n = \frac{n+1}{n}.$$

To obtain our choice of $N \in \mathbb{N}$, we must work backwards from our conclusion. Hence, we have

$$\begin{aligned} a_n - 1 &< \varepsilon \\ \frac{n+1}{n} - \frac{n}{n} &< \varepsilon \\ \Leftrightarrow \frac{1}{n} &< \varepsilon \\ \Leftrightarrow \frac{1}{\varepsilon} &< n. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ is $N = 1/\varepsilon$.

Proof. Let $\varepsilon > 0$ be arbitrary. Choose $N = 1/\varepsilon$ such that

$$N > \frac{1}{\varepsilon}.$$

Let $n \geq N$. Then we proceed by showing that this choice of $N \in \mathbb{N}$ leads to the desired property. Hence,

$$\begin{aligned}
n &> \frac{1}{\varepsilon} \\
\varepsilon &> \frac{1}{n} \\
\varepsilon &> \frac{n+1}{n} - \frac{n}{n} \\
\varepsilon &> \frac{n+1}{n} - 1 \\
\varepsilon &> |a_n - 1|.
\end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ leads to $a_n \to 1$. We can now conclude that

$$\lim_{n\to\infty} a_n = 1.$$

■

Theorem 9 (Uniqueness of Limits)**.** The limit of a sequence, when it exists, must be unique.

Proof. Suppose we have $(a_n) \subseteq \mathbb{R}$. Suppose $a_n \to a$ and $a_n \to a'$. We want to show that

$$a = a'.$$

By definition, we have that

$$\begin{aligned}
|a_n - a| &< \varepsilon/2 \text{ for some } n_1 \in \mathbb{N} \\
|a_n - a'| &< \varepsilon/2 \text{ for some } n_2 \in \mathbb{N}.
\end{aligned}$$

We can show that $a = a'$ by showing that $|a - a'| < \varepsilon$. Hence, choose $N = \min\{n_1, n_2\}$ such that

$$\begin{aligned}
|a - a'| &< |a - a_n + a_n - a'| \\
&< |a - a_n| + |a_n - a'| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Hence, we have that $a = a'$ showing that our limit is unique. ■

2.1.1 Divergence

We can study the divergence of sequences by negating the definition we have above. Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5} \cdots\right)$$

We can prove that this sequence does not converge to zero. Why? When we choose an $\varepsilon = 1/10$, there is none of the term of the sequence converge within the neighborhood $(-1/10, 1/10)$ since the sequence oscillates between $-1/5$ and $1/5$. There is no $N \in \mathbb{N}$, that satisfies $a_n \to 0$. We can also give a counter-example in which we disprove the claim that (a_n) converges to $1/5$. Choose $\varepsilon = 1/10$. This produces the neighborhood $(1/10, 3/10)$. We can see that the sequence does in fact converge to $1/5$, but it does so in an oscillating fashion. Furthermore, the sequence does not stay within the neighbor we specified where we expect all the terms of the sequence to converge towards the value. Hence, there is no such $N \in \mathbb{N}$

where the property can be satisfied.

Definition 11. A sequence that does not converge is said to diverge.

2.2 The Algebraic and Order Limit Theorems

The goal of having a rigorous definition of convergence in Analysis is to prove statements about sequences in general like the notion of "boundedness" which we will define below.

Definition 12 (Bounded Sequences)**.** A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n). This naturally leads us to the point that all convergent sequences are bounded i.e

Theorem 10. Every convergent sequence is bounded.

Proof. Assume (x_n) converges to a limit ℓ. This means that given $\varepsilon = 1$, we can find an $N \in \mathbb{N}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}
&\Rightarrow |x_n - \ell| < 1 \\
&\Leftrightarrow -1 < x_n - \ell < 1 \\
&\Leftrightarrow \ell - 1 < x_n < \ell + 1.
\end{aligned}$$

Note the terms of the sequence (x_n) can be found in the open interval $(\ell - 1, \ell + 1)$. Since $\ell \in \mathbb{R}$ can either be positive or negative, we can conclude that

$$|x_n| < |\ell| + 1$$

for all $n \geq N$ where

$$M = \max\{|x_1|, |x_2|, ..., |\ell| + 1\}.$$

Hence, it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. ■

Theorem 11 (Algebraic Limit Theorem)**.** Let $\lim a_n = a$, and $\lim b_n = b$. Then,

(i) $\lim(ca_n) = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$ provided that $a \neq 0$.

Proof. We begin by proving part (i). Suppose $a_n \to a$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - a| < \varepsilon/|c|. \tag{1}$$

In order to show (i), we need to show that

$$|ca_n - ca| < \varepsilon.$$

Hence, observe that

$$\begin{aligned}|ca_n - ca| &= |c(a_n - a)| \\ &= |c||a_n - a| \\ &< |c|\frac{\varepsilon}{|c|} \\ &= \varepsilon.\end{aligned}$$

If $c = 0$, then our sequence (ca_n) reduces to the sequence $\{0, 0, 0, ..., 0\}$ which is clearly converging to $ca = 0$. Hence, we have attained our desired property that $\lim(ca_n) = ca$. The parts are left to you to prove. ■

Proof. To show part (ii), it suffices to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n + b_n - (a + b)| < \varepsilon.$$

Hence, we start with the left side of (ii). Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}|a_n + b_n - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, it follows that $\lim(a_n + b_n) = a + b$ as required.

■

Proof. To show part (iii), it suffices to show for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - ab| < \varepsilon.$$

Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \\ &< M\frac{\varepsilon}{2M} + |b|\frac{\varepsilon}{2|b|} && (a_n \text{ is bounded}) \\ &< \varepsilon\end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = ab$. ■

[]

Proof. To show part (iv), it suffices to show for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon.$$

Since $a_n \to a$ and $b_n \to b$ with $b \neq 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $n \geq N_1, N_2$, we can have

$$\begin{aligned}|a_n - a| &< M\varepsilon/2, \\ |b_n - b| &< \frac{|b|}{|a|} \cdot \frac{M\varepsilon}{2}.\end{aligned}$$

we can choose $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - ab + ab - b_n a}{b_n b}\right| \\
&= \left|\frac{b(a_n - a) + (b - b_n)a}{b_n b}\right| \\
&\leq \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b|} \cdot \frac{|b_n - b|}{|b_n|} \\
&< \frac{M\varepsilon}{2M} + \frac{|a|}{|b|} \cdot \frac{|b|M\varepsilon}{|a|2M} && (b_n \text{ bounded}) \\
&= \varepsilon.
\end{aligned}$$

Hence, it follows that $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ provided that $b \neq 0$.

■

Theorem 12 (Order Limit Theorem)**.** Assume $\lim a_n = a$ and $\lim b_n = b$.

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iv) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$, for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Proof. (i) We proceed by contradiction by assuming that $a < 0$. Suppose $a_n \geq 0$ and $a_n \to a$. Let $\varepsilon = |a|$ and suppose $n \geq N$. Then

$$|a_n - a| < |a| = -a.$$

But this means that $a_N < 0$ which is a contradiction since $a_N \geq 0$.

(ii) We can ensure that the sequence $b_n - a_n$ converges to $b - a$ by the Algebraic Limit Theorem. Since $b_n - a_n \geq 0$, we can use (i) to write $b - a \geq 0$. Hence, $a \leq b$.

(iii) Suppose there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$. Suppose $a_n = c$ then using (ii) yields $c \leq b$. Suppose $a_n \leq c$ for all $n \in \mathbb{N}$ then setting $b_n = c$ and using (ii) again yields $a \leq c$.

■

2.3 The Monotone Convergence Theorem

As we have seen in the last section, convergent sequences are bounded while the converse is not true. But if a sequence is monotone then surely it is convergent.

Definition 13 (Monotone Sequences)**.** A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 13 (Monotone Convergence Theorem)**.** If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be *monotone* and *bounded*. We need to show that (a_n) converges to some value s.

Let our set of points a_n be defined as

$$A = \{a_n : \text{ for all } n \in \mathbb{N}\}$$

and because we have a bounded sequence, we must have an upper bound s which can be defined as out supremum i.e

$$s = \sup\{a_n : \text{ for all } n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. We need to show that

$$|a_n - s| < \varepsilon$$

Since $s - \varepsilon$ is not an upper bound of A, there exists $N \in \mathbb{N}$ such that

$$s - \varepsilon < a_N.$$

Let's assume that (a_n) is an increasing sequence. By assuming $n \geq N$, we can say that $a_n \geq a_N$. Since $s + \varepsilon$ is an upper bound and s is the least upper bound, then we can say that

$$s - \varepsilon < a_N \leq a_n < s \leq s + \varepsilon$$

which imply that

$$s - \varepsilon < a_n < s + \varepsilon$$
$$\Rightarrow |a_n - s| < \varepsilon.$$

Hence, it follows that any *monotone* and *bounded* sequence converges. ∎

The key takeaway from this theorem is that we don't actually need to specify a value for a limit in order to show that it converges. As long as we have a monotone sequence and that we know it is bounded then we know for sure that the sequence converges.

Definition 14. Infinite Series Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots.$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \dots + b_m = \sum_{i=1}^{m} s_i,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges* to B if the sequence (s_m) converges to B. In this case, we write

$$\sum_{n=1}^{\infty} b_n = B.$$

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Because the terms in the sum are all positive, the sequence of partial sums are given by

$$s_m = \sum_{k=1}^{m} \frac{1}{k^2}$$

is increasing. Our goal is to show that this sequence is convergent so that the series converges. We proceed by using the Monotone Convergence Theorem to do this. Since we already have a monotone sequence of partial sums, only we need to do now find an upper bound for s_m. Observe that

$$\begin{aligned}
s_m &= 1 = \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \dots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \dots + \frac{1}{m(m-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{(m-1)} - \frac{1}{m}\right) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2.
\end{aligned}$$

The third second equality is found by taking the partial fractions of the line before it. Thus, we find that 2 is an upper bound for the sequence of partial sums, so we can conclude that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent. Let's consider the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums is defined as follows

$$s_m = \sum_{k=1}^{m} \frac{1}{k}.$$

Like our last example, we expect these sequence of terms to be bounded by 2 but upon further inspection, we have

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

which is not true. Similarly, we find that $s_8 > 2\frac{1}{2}$, and we can see that in general we have that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{k-1}} + \dots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^k} + \dots + \frac{2}{2^k}\right) \\
&= 1 + \dots + \left(2^{k-1}\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} \\
&= 1 + k\frac{1}{2}.
\end{aligned}$$

This shows that our sequence is unbounded because we found $M = 1 + k\left(\frac{1}{2}\right) > 0$ such that $s_k > M$. Despite how slow the sequence of partial of sums may be at reaching this point, it does end up surpassing every number on the postive real line. Since we have an unbounded sequence of partial sums, we conclude that the Harmonic series as divergent.

Theorem 14 (Cauchy Condensation Test)**.** Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=0}^{\infty} b_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges.

Proof. For the forwards direction, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. This means that the sequence of partial sums

$$t_k = b_1 + 2b_2 + \dots + 2^k b_{2k}$$

are bounded. Hence, there exists $M > 0$ such that $t_k \leq M$ for all $k \in \mathbb{N}$. Our goal is to show that the sequence of partial sums for the series

$$\sum_{n=0}^{\infty} b_n.$$

Since $b_n \geq 0$ and that for all $n \in \mathbb{N}$ b_n decreasing, we have that the partial sums t_k is monotone. Our goal is to show that

$$s_m = \sum_{k=0}^{m} b_k$$

is bounded. Hence, fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$ and hence $s_m \leq s_{2^{k+1}-1}$ which imply that

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + \ldots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \ldots + (b_{2k} + \ldots + b_{2k}) \\ &= b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2k} \\ &= t_k \end{aligned}$$

Hence, we have $s_m \leq s_{2^{k+1}-1} < t_k \leq M$ which means that (s_m) is bounded. By the Monotone Convergence Theorem, it follows that the series $\sum_{n=1}^{\infty} b_n$ converges. For the forwards direction, we proceed with contrapostive. Hence, assume for sake of contradiction that the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

is a divergent series. We want to show that the series

$$\sum_{n=0}^{\infty} b_n$$

is also a divergent series. ■

2.4 Subsequences and Bolzano-Weierstrass

In the last section, we observed that the convergence of partial sums of a particular series can be determined by the behavior of a subsequence of the partial sums.

Definition 15 (Subsequences)**.** Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

A few remarks about subsequences:

(a) The order of the subsequence is the same as in the original sequence. If we have the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

then the subsequences

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$$

and

$$\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots\right)$$

are permitted.

(b) Repetitions and swapping are not allowed. Like

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, ...\right)$$

and

$$\left(1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, ...\right)$$

Since subsequences have the same ordering as the original sequence, one can conjecture about them converging to the same limit.

Theorem 15. Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \to a$ and let (a_{n_k}) be a subsequence for (a_n). We want to show (a_{n_k}) converges to a as well. Since $(a_n) \to a$, there exists an N such that for any $n \geq N$, we have $|a_n - a| < \varepsilon$.

We claim that $n_k \geq k$ for any $k \in \mathbb{N}$. Let us proceed by inducting on k. Let the base case be $k = 1$. Since n_k is an *increasing* sequence of natural numbers, we see that $n_1 \geq 1$. Now let us assume $n_{k-1} \geq k - 1$. Since (a_{n_k}) in *increasing*, we have $a_k \geq a_{k-1} \geq k - 1$ which implies that $n_k \geq k$.

Since any choice of $n \geq N$, we can say that $n_k \geq k \geq N$. Hence, we have

$$|a_{n_k} - a| < \varepsilon$$

which is what we desired. ∎

Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > ... > 0,$$

the sequence (b^n) is *decreasing* and *bounded* below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some ℓ satisfying $0 \leq \ell < b$. To compute ℓ, notice that (b^{2n}) is a subsequence, so $b^{2n} \to \ell$ by Theorem 2.5.2. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $b^{2n} \to \ell \cdot \ell = \ell^2$. Because limits are unique (Theorem 2.2.7), $\ell^2 = \ell$, and thus $\ell = 0$.

Suppose we have an oscillating sequence of numbers

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\right)$$

Note that this sequence does not converge to any proposed limit yet if we take a subsequence of it, weget a sequence that converges! Observe, that the subsequence

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, ...\right)$$

and

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, ...\right)$$

converge to $1/5$ and $-1/5$ respectively. Since we have two subsequences that converge to two different limits, we immediately conclude that the original sequence diverges.

This leads us to our next theorem that states that

Theorem 16 (Bolzano-Weierstrass)**.** Every bounded sequence contains a convergent subsequence.

Proof. Let (a_n) be a *bounded* sequence. Then there exists $M > 0$ such that $a_n \in [-M, M]$. Suppose we divide this interval in half for k times: that is, let the length of the intervals be defined by the sequence $M(1/2)^{k-1}$. We claim that a subsequence (a_{n_k}) lies in either one of these intervals: that is, let $n_k > n_{k-1}$ for all $k \in \mathbb{N}$ such that $a_{n_k} \in I_k$.

Let us induct on k. Then let our base case be $k = 1$. Since we have an increasing sequence of natural numbers n_k, we have that $n_2 > n_1$ which means that $a_{n_2} \in I_2$ as well as $a_{n_1} \in I_1$. Now let us assume that this holds for all $k \leq \ell - 1$. We want to show that this holds for $k < \ell$. By the monotonicity of n_k, we have that $n_\ell > n_{\ell-1} > n_k > n_1$ which implies that $a_{n_\ell} \in I_\ell$ for all $\ell \in \mathbb{N}$. Furthermore, the sets

$$I_1 \subseteq I_2 \subseteq I_3 ...$$

form a nested sequence of closed intervals.

By the *Nested Interval Property*, we can conclude that there exists an $x \in I_k$ for all $k \in \mathbb{N}$ such that $\bigcup_{k=1}^{\infty} I_k \neq \emptyset$. Let $\varepsilon > 0$. Since $a_{n_k}, x \in I_k$ for all $k \in \mathbb{N}$ and $M(1/2)^{k-1} \to 0$ by the Algebraic Limit Theorem, we can choose an $N \in \mathbb{N}$ such that for any $k \geq n_k \geq N$, we have

$$|a_{n_k} - x| < \varepsilon.$$

Hence, $(a_{n_k}) \to x$. ■

2.5 The Cauchy Criterion

Definition 16 (Cauchy Sequence)**.** A sequence (a_n) is called a *Cauchy Sequence* if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|a_n - a_m| < \varepsilon.$$

In the regular convergence definition, we are given any $\varepsilon > 0$ where there is a point in the sequence $N \in \mathbb{N}$ such that past this point, all of our terms fall within an ε range around some limit point. In the Cauchy Criterion defintion, we begin with the same conditions but this time, all the terms of the sequence are all tightly packed together wthin the $\varepsilon > 0$ range we were given. It turns out, that these two definitions are equivalent: that is, *Cauchy sequences* are convergent sequences and convergent sequences are *Cauchy sequences*.

Theorem 17. Every convergent sequence is a Cauchy sequence.

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, there must existsa point $N \in \mathbb{N}$ after which we can conclude that

$$|x_n - x_m| < \varepsilon.$$

Let $\varepsilon > 0$. Since $(x_n) \to x$, we can choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|x_n - x| < \frac{\varepsilon}{2}, \tag{1}$$
$$|x_m - x| < \frac{\varepsilon}{2}. \tag{2}$$

Consider $|x_n - x_m|$. Then (1) and (2) imply that

$$\begin{aligned}
|x_n - x_m| &= |x_n - x + x - x_m| \\
&< |x_n - x| + |x - x_n| && \text{(Triangle Inequality)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence*. ■

We can prove the other direction, by using either the *Bolzano Weierstrass Theorem* or the *Monotone Convergence Theorem*. This is a little bit more difficult since we need to have a proposed limit for the sequence to converge to.

Lemma 2. Cauchy sequences are bounded.

Proof. Given $\varepsilon = 1$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < 1$ for amm $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$ (just substituted $m = N$ here). Hence, define

$$M = \max\{|x_1|, |x_1|, |x_1|, ..., |x_{N-1}|, |x_N| + 1\}.$$

Therefore, $|x_n| < M$ for all $n \in \mathbb{N}$ Hence, the *Cauchy sequence* (x_n) is *bounded*. ■

Theorem 18 (Cauchy Criterion in $\mathbb{R}$)**.** A sequence converges if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) This direction is just Theorem 2.6.2 which we have proved above.

($\Leftarrow$) Suppose (x_n) is a *Cauchy sequence*. Let $\varepsilon > 0$. Since (x_n) is a *bounded* sequence, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$ by the *Bolzano Weierstrass Theorem*. Let $\varepsilon > 0$. Then for some $N \in \mathbb{N}$, every $n_k \geq N$ has the property

$$|x_{n_k} - x| < \varepsilon.$$

Our goal now is to show that $(x_n) \to x$. Hence, consider $|x_n - x|$. Then for every $n, n_k \geq N$, we have

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &< |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $(x_n) \to x$. ∎

2.5.1 Completeness Revisited

We can summarize all of our results thus far in the following way

$$\text{AOC} \begin{cases} \text{NIP} \Rightarrow \text{BW} \Rightarrow \text{CC} \\ \text{MCT} \end{cases}$$

where AOC is our defining axiom to base all our reults on and giving us the notion that an ordered field contains no holes. We could also take the MCT to be our defining axiom and gives us the notion of least upper bounds by proving NIP. In addition, we could also take NIP to be our starting point but we need to have an extra hypothesis; that is, the Archimedean Property to prove all our results above (This is unavoidable).

It could be possible to assume the Arcimedean property holds, suppose one of the results we have proven is true, and derive the others yet this is sort of limited since $\mathbb{Q}$ contains a set that is not complete.

Below is the least of implications we can prove based on which theorem we would like to select asour defining axiom. Hence, we have

$$\text{NIP} + \text{Archimedean Property} \Rightarrow \text{AOC}$$

and

$$\text{BW} \Rightarrow \text{MCT} \Rightarrow \text{Archimedean Property}$$

2.6 Properties of Infinite Series

We have learned the convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n) where

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim s_n = A.$$

We called (s_n) the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$. Just like the *Algebraic Limit Theorem* for sequences, we can also do the same thing for series.

Theorem 19 (Properties of Infinite Series)**.** If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof. Suppose $\sum_{k=1}^{\infty} a_k = A$ and let $c \in \mathbb{R}$. Define the sequence of partial sums of $\sum_{k=1}^{\infty} ca_k$ as

$$t_k = cs_n = ca_1 + ca_2 + ca_3 + ... + ca_n.$$

By the *Algebraic Limit Theorem*, we know that $\lim cs_n = cA$. Hence,

$$\sum_{k=1}^{\infty} ca_k = cA.$$

To prove the addition rule, suppose $\sum_{k=1}^{\infty} b_k = B$. We want to show that

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B.$$

Define the sequence of partial sums for the two series as the following:

$$t_k = a_1 + a_2 + ... + a_n,$$
$$u_k = b_1 + b_2 + ... + b_n$$

Since $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, their sequence of partial sums also converges to the same value. Hence, let $\lim t_k = A$ and $\lim u_k = B$. By the *Algebraic Limit Theorem*, the sum of these two limits also converges i.e

$$\lim(t_k + u_k) = \lim t_k + \lim u_k = A + B.$$

Hence,

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B$$

■

We can summarize this theorem by keeping in mind that we can perform distribution over infinite addition and that we can add two infinite series together.

Theorem 20 (Cauchy Criterion for Series)**.** The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \varepsilon.$$

Suppose $\sum_{k=1}^{\infty} a_k$ converges. This is true if and only if the sequence of partial sums (t_k) converges. This is true if and only if (s_k) is *Cauchy* by the *Cauchy Criterion*. Hence, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$

$$|s_n - s_m| < \varepsilon.$$

Note that

$$\begin{aligned} |s_n - s_m| &= |\sum_{k=m+1}^{\infty} a_k - \sum_{k=m}^{m} a_k| \\ &= |\sum_{k=m+1}^{n} a_k| \\ &= |a_{m+1} + ... + a_n| < \varepsilon \end{aligned}$$

■

This gives us the opportunity to prove some basic facts about series.

Theorem 21. If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof. From the last theorem, we note that for every $\varepsilon > 0$ such that whenever $n \geq m \geq N$, we have

$$|s_n - s_m| = \Big| \sum_{k=m+1}^{\infty} a_k - 0 \Big| < \varepsilon$$

implies that $(a_n) \to 0$. ■

Keep in mind that the converse of this statement is not true! Just because (a_k) tends to 0 does not immediately imply that the series converges!

Theorem 22 (Comparison Test)**.** Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then we have

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let us show part (i). Suppose $\sum_{k=1}^{\infty} b_k$ converges. We want to show that $\sum_{k=1}^{\infty} a_k$ converges. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n > m \geq N$ and the fact that $a_k \leq b_k$ for all $k \in \mathbb{N}$

$$\begin{aligned} \Big| \sum_{k=m+1}^{n} a_k \Big| &\leq \Big| \sum_{k=m+1}^{n} b_k \Big| \\ &< \varepsilon. \end{aligned}$$

Hence, a_k converges as well.

Note that part (ii) is just the contrapositive of part (i) which is also true. ■

Note that the convergence of sequences and series are relatively immutable when it comes to changes in some finite number of initial terms: that is, the behavior of sequences and series can be found past some choice of $N \in \mathbb{N}$. In order for the above test to be of any use to us, it is important to have a few examples under our belt i.e any $p > 1$ implies that

$$\textstyle\sum_{n=1}^{\infty} 1/n^p \text{ converges if and only if } p > 1.$$

A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots .$$

If $r = 1$ and $a \neq 0$, the series diverges. We can use the following algebraic identity, for $r \neq 1$, to write the following:

$$(1 - r)(1 + r + r^2 \dots + r^{m-1}) = 1 - r^m$$

which allows us to rewrite the partial sum (s_m) of the above series to say that

$$s_m = a + ar + ar^2 + ar^3 + \dots + ar^{m-1} = \frac{a(1 - r^m)}{1 - r}$$

where $s_m = at_m$ where

$$t_m = 1 + r + r^2 + \dots + r^{m-1}$$

is a convergent sequence. Using the *Algebraic Limit Theorem*, therefore, allows us to say that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

if and only if $|r| < 1$.

The next theorem is a modification of the *Comparison Test* to handle series that contain negative terms.

Theorem 23 (Absolute Convergence Test)**.** If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof. Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. We want to show that $\sum_{n=1}^{\infty} a_n$ converges as well. Let $\varepsilon > 0$. By the *Cauchy Criterion* for series, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have

$$\begin{aligned} \left| \sum_{k=m+1}^{n} a_k \right| &\leq \sum_{k=m+1}^{n} |a_k| \\ &< \varepsilon. \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges. ■

Note that the converse of the above statement is false as taking the absolute value of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

produces the regular harmonic series which *diverges*.

Theorem 24 (Alternating Series Test)**.** Let (a_n) be a sequence satisfying,

(i) $a_1 \geq a_2 \geq a_3 \ldots \geq a_n \geq a_{n+1} \geq \ldots$ and

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. See exercise 2.7.1 for proof ■

Definition 17 (Absolute Convergence and Conditional Convergence)**.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

We can chart a few examples of some *conditionally convergent* series and *absolutely convergent* series.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \Rightarrow$ *conditionally convergent*
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \sum_{n=1}^{\infty} \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \Rightarrow$ *converges absolutely*

This tells us that any convergent series with positive terms must converge absolutely.

2.6.1 Rearrangements

We can obtain a rearrangement of an infinite series by permuting terms in the sum in some other order. In order for a sum to be a valid rearrangement, all the terms must appear and there should be no repeats.

Definition 18 (Rearrangements)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

We can now explain the weird behavior for why the *harmonic series* converges to a different limit when rearranging the terms; that is, it is because the *harmonic series* is a *conditionally convergent* series which leads us to the next theorem.

Theorem 25 (Rearrangement of Series)**.** If a series converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k=1}^{\infty} a_k$ *converges absolutely* to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Let us define the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ as

$$s_n = \sum_{k=1}^{n} a_k$$

and the sequence of partial sums for the rearranged series $\sum_{n=1}^{\infty} b_n$ as

$$t_m = \sum_{k=1}^{m} b_k.$$

Since $\sum_{n=1}^{\infty} a_n$ *converges absolutely*, let $\varepsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|s_n - A| < \frac{\varepsilon}{2}$$

as well some $N_2 \in \mathbb{N}$ such that whenever $n > m \geq N_2$, we have

$$\sum_{k=m+1}^{n} |a_k| < \frac{\varepsilon}{2}.$$

All that is left to do is to set a point in the sequence of the rearranged series where our ultimate goal is to have $|t_m - A| < \varepsilon$. Hence, define

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

Let $m \geq M$ such that, when using the *triangle inequality*, we get

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, we have that $\sum_{n=1}^{\infty} b_n$ converges to A. ■

2.7 Double Summations and Products

We discovered in an earlier section that given any doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbb{N}\}$, it can be an ambiguous task to define

$$\sum_{i,j=1}^{\infty} a_{ij}. \tag{1}$$

We also observed that performing *iterated summations*can lead to different summations. Of course, this can be avoided completely if we were to define the partial sum of (1) in the following way

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

for $m, n \in \mathbb{N}$. In order for the sum of (1) to converge we have to have the following hold:

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n \to \infty} s_{mn}$$

Exercise 2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n \to \infty} s_{mn}$. How does this value compare to the two iterated values for the sum already computed?

The double summation from section 2.1 is $a_{ij} = \frac{1}{2^{j-i}}$ where $\{a_{ij} : i, j \in \mathbb{N}\}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

Proof. To find $\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{mn}$, we first need to define the sequence of partial sums. We can fix j (the rows of the matrix) and define the sequence of partial sums for the series $\sum_{i,j=1}^{\infty} a_{ij}$ as

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}}\right) = -2 + \frac{1}{2^{n-1}}$$

which taking the limit leads to

$$\lim_{n\to\infty} \left(-2 + \frac{1}{2^{n-1}}\right) = -2.$$

■

The issue of rearrangements to an infinite series arises due to commutativity of addition in an infinite context. It was found that having an absolutely converging infinite series fixes this problem.

Exercise 2.8.2

Show that if the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some $b_i \in \mathbb{R}$, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

converges.

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \tag{1}$$

converges. This means that the (1) meets the *Cauchy Criterion*. Let $\varepsilon > 0$. This implies that there exists $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| < \varepsilon.$$

Consider $\left|\sum_{(i,j)\in A(m,n)} a_{ij}\right|$ where

$$A(m,n) = \{(i,j) : 1 \leq i \leq j \leq n\}.$$

Using the *Triangle Inequality*, we find that j

$$\begin{aligned} \left|s_{mm} - s_{nn}\right| &= \left|\sum_{(i,j)\in A(m,n)} a_{ij}\right| \\ &\leq \sum_{(i,j)\in A(m,n)} |a_{ij}| \qquad (2) \\ &< \varepsilon. \end{aligned}$$

Since (2) meets the *Cauchy Criterion* for series, we know that $\sum_{m,n}^{\infty} a_{ij}$ must be *Cauchy* and thus must converge as well. ■

Another proof using the Comparison Test goes something like this

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges. This means that for each $i \in \mathbb{N}$ the infinite series

$$\sum_{j=1}^{\infty} a_{ij} = r_i$$

for some $r_i \in \mathbb{R}$. Hence, we have the infinite series

$$\sum_{i=1}^{\infty} r_i. \tag{1}$$

Our goal is to show that (1) converges. Suppose we look at the terms

$$|r_i| = \Big| \sum_{j=1}^{\infty} a_{ij} \Big|.$$

Note by the *Triangle Inequality* that

$$\sum_{i=1}^{\infty} |r_i| \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|.$$

by assumption the infinite series to the right converges. Hence, the series to the left must also converge by the *Comparison Test*. Since $\sum |r_i|$ converges, then the series $\sum r_i$ converges by the *Absolute Convergence Test* . ■

Theorem 26. Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover, we have that

$$\lim_{n \to \infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1) , define

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$

■

Exercise 2.8.3

(a) Prove that (t_{nn}) converges.

Proof. From our definition of t_{nn} above we have

$$t_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

We want to show for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that

$|t_{nn} - L| < \varepsilon$. By assumption, we know that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \tag{1}$$

converges absolutely which implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges. Note that $t_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \to s_n = \sum_{i=1}^{\infty} r_i$ for some $r_i \in \mathbb{R}$. Furthermore, we have $s_n \to L$ since (1) converges. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have that

$$\begin{aligned}|t_{nn} - L| &= |t_{nn} - s_n + s_n - L| \\ &\leq |t_{nn} - s_n| + |s_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, the sequence of partial sums (t_{nn}) converges. ■

Another way we can prove this is to use the Monotone Convergence Theorem.

Proof. Our goal is to show that (t_{nn}) is converges to L. That is, our goal is to show that (t_{nn}) is bounded and monotone. We know that (t_{nn}) is monotone since all t_{nn} are non-negative terms and that $\sum_{n,m}^{\infty}|a_{ij}| = L$ where $L \geq 0$. To show that (t_{nn}) is bounded note that

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| \leq \sum_{i=1}^{m}\sum_{j=1}^{\infty}|a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Hence, (t_{nn}) is a bounded sequence. By the Monotone Convergence Theorem, (t_{nn}) converges. ■

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Proof. Suppose (t_{nn}) is a Cauchy Sequence. Then for some $N \in \mathbb{N}$ we have that for any $n \geq m > N$

$$|t_{nn} - t_{mm}| < \varepsilon.$$

We can rewrite this in the following way to say that

$$|\sum_{n,m} t_{ij}| < \varepsilon.$$

Our goal is to show that

$$|s_{nn} - s_{mm}| < \varepsilon.$$

Hence, for any $n \geq m > N$, we have that

$$\begin{aligned}|s_{nn} - s_{mm}| &\leq |t_{nn} - t_{mm}| \\ &= \Big|\sum_{n,m} t_{ij}\Big| \\ &< \varepsilon.\end{aligned}$$

Hence, (s_{nn}) converges.

■

Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Exercise 2.8.4

(a) Let $\varepsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\varepsilon}{2} < t_{mn} \leq B$.

Proof. Since (t_{mn}) bounded, we can say that $t_{mn} \leq B$. Since the set

$$\{t_{mn} : m, n \in \mathbb{N}\}$$

is bounded above and non-empty, we also have that $B = \sup\{t_{mn} : m, n \in \mathbb{N}\}$ exists. Hence, for any $\varepsilon > 0$, we have that $B - \frac{\varepsilon}{2}$ is not an upper bound. Hence, there exists some $t_{n_0 m_0}$ such that $B - \frac{\varepsilon}{2} < t_{m_0 n_0} \leq t_{mn}$. Furthermore, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq m > N_1$ since (t_{mn}) converges. Hence, we must have that $B - \frac{\varepsilon}{2} < t_{mn} \leq B$ ■

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \varepsilon$$

for all $m, n \geq N$.

Proof. Consider $|s_{mn} - S| < \varepsilon$. Since $(s_{nn}) \to S$, let $\varepsilon > 0$ such that for some $N_2 \in \mathbb{N}$ we have $n \geq m > N_2$, we have

$$|s_{nn} - S| < \frac{\varepsilon}{2}.$$

Since (s_{nn}) meets the Cauchy Criterion, we have that there exists $N_2 \in \mathbb{N}$ such that for any $n \geq m > N$, we have

$$|s_{nn} - s_{mn}| < \frac{\varepsilon}{2}.$$

Hence, observe that for any $n \geq m > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}|s_{mn} - S| &= |s_{mn} - s_{nn} + s_{nn} - S| \\ &\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, we have that $(s_{mn}) \to S$. ■

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 2.8.5

(a) Show that for all $m \geq N$
$$|(r_1 + r_2 + ... + r_m) - S| \leq \varepsilon.$$
Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

Proof. By exercise 2.8.4, we know that $s_{mn} \to S$. Note that
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} r_i \text{ for each } i.$$
Hence, we have
$$\lim_{m,n\to\infty} s_{mn} = \lim_{m\to\infty}\sum_{i=1}^{m} r_i = S$$
which is equivalent to saying that for all $m > N$ for some $N \in \mathbb{N}$ we have that
$$\left|\left(\sum_{i=1}^{m} r_i\right) - S\right| \leq \varepsilon.$$
■

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} a_{ij}$ converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Proof. Using the same process above for summing up the columns of $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ leads to
$$\left|\left(\sum_{j=1}^{n} c_j\right) - S\right| \leq \varepsilon.$$
Hence, we must have that
$$\lim_{n\to\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$
■

Another way of computing double sums is to sum along the diagonals of a rectangular matrix. Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array where
$$d_2 = a_{11}, d_3 = a_{12} + a_{21}, d_4 = a_{13} + a_{22} + a_{31}$$
and in general
$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$
Then, $\sum_{k=2}^{\infty} d_k$ represents another reasonable way of summing over every a_{ij} in the array.

Exercise 2.8.6

(a) Assuming the hypothesis and hence the conclusion of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

Proof. Our goal is to show that $\sum_{k=1}^{\infty} |d_k|$ converges. Since
$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}$$
we can define the sequence of partial sums for $\sum_{k=1}^{\infty} |d_k|$ as
$$\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\right|. \tag{1}$$

We know by Theorem 2.8.1 that the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ converges. Hence, we can write

$$\sum_{k=2}^{n}|d_k| = \sum_{i=1}^{n}\left|\sum_{j=1}^{n}a_{ij}\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$$

using the Triangle Inequality. By the Comparison Test, we must have that (1) converges as well. ■

2.7.1 Products of Series

We can take the product of two series by doing the following algebra below:

$$\begin{aligned}\left(\sum_{i=1}^{\infty}a_i\right)\left(\sum_{j=1}^{\infty}b_j\right) &= (a_1+a_2+a_3+\dots)(b_1+b_2+b_3\dots)\\ &= a_1b_1+(a_1b_2+a_2b_1)+(a_3b_1+a_2b_2+a_1b_3)+\dots\\ &= \sum_{k=2}^{\infty}d_k\end{aligned}$$

where

$$d_k = a_1b_{k-1}+a_2b_{k-2}+\cdots+a_{k-2}b_1.$$

Exercise 2.8.7

Assume that $\sum_{i=1}^{\infty}a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty}b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges so that we may apply Theorem 2.8.1.

Proof. Our goal is to show that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j$ converges absolutely. Observe that

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}a_ib_j\right| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j| \qquad (1)\\ &= \sum_{i=1}^{\infty}|a_i|\left(\sum_{j=1}^{\infty}|b_j|\right)\\ &= \sum_{i=1}^{\infty}|a_i|\cdot|B| \qquad (2)\end{aligned}$$

Since (2) converges absolutely by the Algebraic Series Theorem, we have that (1) converges. converges. ■

(b) Let $s_{nn}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j$, and prove that $\lim_{n\to\infty}s_{nn}=AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_ib_j = \sum_{k=2}^{\infty}d_k = AB,$$

where, as before, $d_k = a_1b_{k-1}+a_2b_{k-2}+\cdots+a_{k-1}b_1$.

Proof. We can show that (t_{nn}) converges via the Monotone Convergence Theorem where

$$t_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_jj|.$$

Observe that all the terms of (t_{nn}) are positive and increasing. Now all we need to show is that (t_{nn}) is bounded. Since $\sum_{i=1}^{n}|a_i|\leq M$ and $\sum_{j=1}^{\infty}|b_i|\leq L$ for some $M,L\in\mathbb{R}$, we have that

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_i b_j| \leq \sum_{i=1}^{\infty} |a_j| \sum_{j=1}^{\infty} |b_j|$$
$$\leq M \cdot L.$$

Hence, (t_{nn}) is a bounded sequence of partial sums. Now by theorem 2.8.1, we can say that

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB.$$

■

Chapter 3

Additional Topics

3.1 The Generalized Riemann Integral

3.1.1 The Riemann Integral as a Limit

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. A *tagged partition* is one where in addition to the partition P, we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. We can now define the *Riemann Sum* where given a function $f : [a, b] \to \mathbb{R}$, and a tagged partition $(P, \{c_k\}_{k=1}^n)$, the *Riemann sum* generated by this partition is defined by

$$R(f, P) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

By definition of the upper sum and the lower sum given in section 7.2, it follows immediately that

$$L(f, P) \leq R(f, P) \leq U(f, P)$$

for any bounded function f. From section 7.2.7, integrability is guaranteed when the supremum of the lower sums and infimum of the upper sums go to the same value. By the inequality above, it is quite clear that we expect $R(f, P)$ to also have the same value. We can characterize this by using an $\varepsilon - \delta$ definition applied to $R(f, P)$.

Definition 19 ($\delta-$fine Partions)**.** Let $\delta > 0$. A partition P is $\delta-$*fine* if every subinterval $[x_k, x_{k-1}]$ satisfies $x_k - x_{k-1} < \delta$. In other words, every subinterval has width less than δ.

Theorem 27. Limit Criterion for Riemann Integrability A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable with

$$\int_a^b f = A$$

if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any tagged partition $(P, \{c_k\})$ that is $\delta-$fine, it follow that

$$|R(f, P) - A| < \varepsilon.$$

The idea is that partitions become finer with the effect that the approximations get closer to the value of the integral. To rephrase the forwards direction of the theorem above, the integrability of some function implies that the approximations converge to the value of the integral independent of the tags chosen. In the backwards direction, the Riemann sum approximations accumulate around some value A which implies that a function is integrable and integrates to A.

Proof. ($\Rightarrow$) Let us assume that f is integrable on $[a, b]$. Given $\varepsilon > 0$, we must show that there exists a $\delta > 0$ such that if $(P, \{c_k\})$ is any tagged partition that is $\delta-$fine, then

$$\left| R(f, P) - \int_a^b f \right| < \varepsilon.$$

Since f is integrable, we can find a partition P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Let $M > 0$ be a bound on $|f|$, and let n be the number of subintervals of P_ε (this so P_ε) really consists of $n+1$ points in $[a, b]$. We claim that choosing

$$\delta = \frac{\varepsilon}{9nM}$$

has this desired property.

Let $(P, \{c_k\})$ be an arbitrary tagged partition of $[a, b]$ that is $\delta-$fine, and let $P' = P \cup P_\varepsilon$. The key is to establish the string of inequalities

$$L(f, P') - \frac{\varepsilon}{3} < L(f, P) \leq U(f, P) < U(f, P') + \frac{\varepsilon}{3}.$$

■

Exercise 8.1.1

(a) Explain why both the Riemann sum $R(f, P)$ and $\int_a^b f$ fall between $L(f, P)$ and $U(f, P)$.

Proof. Let $(P, \{c_k\})$ be a tagged partition of $[a, b]$. Then by definition of by the definition of M_k and m_k (the supremums and infimums of each subinterval $[x_{k-1}, x_k]$), we know that

$$m_k \leq f(c_k) \leq M_k$$

which imply that

$$L(f, P) \leq R(f, P) \leq U(f, P).$$

If f is integrable then $L(f) = U(f) = \int_a^b f$. This means

$$L(f, P) \leq \int_a^b f \leq U(f, P).$$

■

(b) Explain why $U(f, P') - L(f, P') < \varepsilon/3$.

Proof. Let $\varepsilon > 0$. By the properties of the supremum and infimum, we have

$$U(f, P') < U(f) + \frac{\varepsilon}{6}$$

and

$$L(f, P') > L(f) - \frac{\varepsilon}{6}.$$

Subtracting these two inequalities and assuming f is integrable ($U(f) = L(f)$), we end up with

$$U(f, P') - L(f, P') < (U(f) - L(f)) + \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

■

If we can show $U(f, P) < U(f, P') + \varepsilon/3$ (and similarly $L(f, P') - \varepsilon/3 > L(f, P)$), then it will follow that

$$\left| R(f, P) - \int_a^b f \right| < \varepsilon$$

and the proof will be done. To do this, we can try to estimate the distance between $U(f, P)$ and $U(f, P')$.

Exercise 8.1.2

Explain why $U(f, P) - U(f, P') \geq 0$.

Proof. If $P = P' \cup P_\varepsilon$, then it follows from lemma 7.2.4 that $U(f, P) \geq U(f, P')$ which implies that $U(f, P) - U(f, P') \geq 0$. ■

Observe that for any partition, the upper sum takes on the form

$$U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$$

which contains a good number of the M_k terms cancel out.

Exercise 8.1.3

(a) In terms of n, what is the largest number of terms of the form $M_k(x_k - x_{k-1})$ that could appear in one of $U(f, P)$ or $U(f, P')$ but not the other?

Proof. Since P_ε consists of $n-1$ points in $[a, b]$ and there are three points, that being the two endpoints and our sampling point c_k, we must have at most $3(n-1)$ points. ■

(b) Finish the proof in this direction by arguing that

$$U(f, P) - U(f, P') < \frac{\varepsilon}{3}.$$

Proof. Observe that for all $k \in \mathbb{N}$ that $M_k \leq 3(n-1)M$ for some $M > 0$ from part (a). Since P is $\delta-$fine, we must have $\Delta x_k < \varepsilon/9nM$. Hence, we must have

$$\begin{aligned}
U(f, P) - U(f, P') &= \sum_{k=1}^{n} M_k \Delta x_k \\
&\leq (3n-3)M \sum_{k=1}^{n} \Delta x_k \\
&< (3n-3)M \cdot \frac{\varepsilon}{9nM} \sum_{k=1}^{n} \\
&= (3n-3) \cdot \frac{\varepsilon}{9} \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

The same argument can be applied to the lower sums $L(f, P)$ and $L(f, P')$. Then observe that

$$L(f, P') - \frac{\varepsilon}{3} < L(f, P) \leq R(f, P) \leq U(f, P) < U(f, P') + \frac{\varepsilon}{3}$$

now holds which completes our proof that

$$\left| R(f, P) - \int_a^b f \right| < \varepsilon.$$

■

($\Leftarrow$) For the backwards direction, we can assume that $\varepsilon - \delta$criterion in Theorem 8.1.2 holds and show that f is integrable. To show this, we must have the upper sums are close to the lower sums. We now know that it is always the case that

$$L(f, P) \leq R(f, P) \leq U(f, P)$$

independent of the tags chosen to compute $R(f, P)$.

Exercise 8.1.4

(a) Show that if f is continuous, then it is possible to pick tags $\{c_k\}_{k=1}^{n}$ so that

$$R(f, P) = U(f, P).$$

Similarly, there are tags for which $R(f, P) = L(f, P)$ as well.

Proof. Let $\{c_k\}_{k=1}^n$ be an arbitrary tag on a partition P. Since f is continuous on the compact set $[a, b]$, we know that f must also be uniformly continuous. Denote the supremums of each subinterval $[x_{k-1}, x_k]$ by $M_k = f(z_k)$ for all k. Let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that whenever $|c_k - z_k| < \delta$, we have

$$f(c_k) - M_k < \frac{\varepsilon}{b-a}.$$

Then observe that for any partition P of $[a, b]$, we have

$$\begin{aligned} R(f, P) - U(f, P) &= \sum_{k=1}^n [f(c_k) - M_k]\Delta x_k \\ &< \frac{\varepsilon}{b-a} \sum_{k=1}^n \Delta x_k \\ &= \frac{\varepsilon}{b-a} \cdot b - a = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we must have $R(f, P) = U(f, P)$. A similar argument can be used to show $R(f, P) = L(f, P)$. ■

(b) If f is not continuous, it may not be possible to find tags for which $R(f, P) = U(f, P)$. Show, however, that given an arbitrary $\varepsilon > 0$, it is possible to pick tags for P so that

$$U(f, P) - R(f, P) < \varepsilon.$$

The analogous statement holds for lower sums.

Proof. Let $\varepsilon > 0$. Let $\{c_k\}_{k=1}^n$ be an arbitrary tag for P. Since $|f|$ is bounded by some $M > 0$, we know that the distance between the supremums of each subinterval M_k and each tag $f(c_k)$ can be bounded by M; that is, we have

$$M_k - f(c_k) \leq 2Mn.$$

Since the partition P is δ−fine, we know that we can choose $\delta = \frac{\varepsilon}{2Mn}$ such that every subinterval $[x_{k-1}, x_k]$ satisfies

$$\Delta x_k < \frac{\varepsilon}{2Mn}.$$

Then observe that

$$\begin{aligned} U(f, P) - R(f, P) &= \sum_{k=1}^n [M_k - f(c_k)]\Delta x_k \\ &\leq 2M \sum_{k=1}^n \Delta x_k \\ &< 2M \cdot \frac{\varepsilon}{2Mn} \sum_{k=1}^n \\ &= \varepsilon. \end{aligned}$$

The same argument can be applied to show

$$R(f, P) - L(f, P) < \varepsilon.$$

■

Exercise 8.1.5

Use the results of the previous exercise to finish the proof of Theorem 8.1.2.

Proof. Let $\varepsilon > 0$. Then let $(P, \{c_k\})$ be a tagged partition. Let $P = P_1 \cup P_2$ be a common refinement. By assumption, we can have

$$\begin{aligned} R(f, P_1) - R(f, P_2) &= \big[R(f, P_1) - A\big] + \big[A - R(f, P_2)\big] \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \end{aligned}$$

By using the results of part (a) and part(b), we have

$$\begin{aligned} U(f, P) - L(f, P) &= \big[U(f, P) - R(f, P_1)\big] + \big[R(f, P_1) - R(f, P_2\big] \\ &\quad + \big[R(f, P_2 - L(f, P)\big] \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Hence, f is integrable and $A = \int_a^b f$. ■

3.1.2 Gauges and $\delta(x)-$fine Partitions

The main component of the generalized Riemann Integral above is to have δ to be a function of x.

Definition 20 (Gauges)**.** A function $\delta : [a, b] \to \mathbb{R}$ is called a *gauge* on $[a, b]$ if $\delta(x) > 0$ for all $x \in [a, b]$.

Definition 21. Given a particular gauge $\delta(x)$, a tagged partition $(P, \{c_k\}_{k=1}^n)$ is $\delta(x)-$fine if every subinterval $[x_{k-1}, x_k]$ satisfies $x_{k-1} - x_k < \delta(c_k)$. In other words, each subinterval $[x_{k-1}, x_k]$ has width less than $\delta(c_k)$.

It's important to note that $\delta(x)$ is normally a constant function. The definition above is a more generalized version of what was stated earlier in definition 8.1.4.

Exercise 8.1.6

Consider the interval $[0, 1]$.

(a) If $\delta(x) = 1/9$, find a $\delta(x)-$fine tagged partition of $[0, 1]$. Does the choice of tags matter in this case?

Proof. Since $\delta(x)$ is just a constant, the choice of tags does not matter in this case. ■

(b) Let

$$\delta(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ x/3 & \text{if } 0 < x \leq 1. \end{cases}$$

Construct a $\delta(x)-$fine tagged partition of $[0, 1]$.

Proof. Let $P = \{([0, 1/7], 1/2), (\{1/2, 2/3\}, 0)\}, (\{2/3, 1\}, 1)$ is a $\delta(x)-$fine partition, then observe that

$$x_1 - x_0 < \delta(c_1) \Rightarrow \frac{1}{7} < \frac{1}{6}.$$

and

$$x_2 - x_1 < \delta(c_2) \Rightarrow \frac{11}{21} < \frac{1}{4}$$

and then finally,

$$x_3 - x_2 < \delta(c_3) \Rightarrow \frac{1}{3} < \frac{1}{2}.$$

■

Theorem 28. $\delta(x) - finePartitions$ Given a gauge $\delta(x)$ on an interval $[a, b]$, there exists a tagged partition $(P, \{c_k\}_{k=1}^n)$ that is $\delta(x)-$fine.

Proof. Let $I_0 = [a, b]$. It may be possible to find a tag that the trivial partition $P = \{a, b\}$ works. Specifically, if $b - a < \delta(x)$ for some $x \in [a, b]$, then we can set c_1 equal to such an x and notice that $(P, \{c_1\})$ is $\delta(x)-$fine. If no such x exists, then bisect $[a, b]$ into two equal halves. ■

Exercise 8.1.7

Finish the proof of Theorem 8.1.5.

Proof. Let each interval $I_k = [x_{k-1}, x_k]$ and define the gauge $\delta(c_k) = \varepsilon/2^{k-1}$ to be the length of each I_k. Then we for any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that for any $k \geq N$ the length

$$|x_k - x_{k-1}| < \varepsilon$$

since $\varepsilon/2^{k-1}$ converges to 0 as $k \to \infty$. Since this applies for every subinterval, the partition $(P, \{c_k\}_{k=1}^n)$ is $\delta(x)-$fine.

■

3.1.3 Generalized Riemann Integrability

Using gauges now, we can restate Riemann Integrability in a more generalized way.

Definition 22 (Generalized Riemann Integral)**.** A function f on $[a, b]$ has *generalized Riemann Integral* A if, for every $\varepsilon > 0$, there exists a gauge $\delta(x)$ on $[a, b]$ such that for each tagged partition $(P, \{c_k\}_{k=1}^n)$ that is $\delta(x)-$fine, it is true that

$$|R(f, P) - A| < \varepsilon.$$

In this case, we write $A = \int_a^b f$.

Theorem 29. If a function has a generalized Riemann integral, then the value of the integral is unique.

Proof. Assume that a function f has generalized Riemann integral A_1 and that it also has generalized Riemann integral A_2. We must prove $A_1 = A_2$. ■

Exercise 8.1.8

Finish the argument.

Proof. Let $\varepsilon > 0$. Since f has generalized Riemann integral A_1 and A_2, there exists a gauge

$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ such that for each tagged partition that is $\delta(x)$−fine, we must have

$$\begin{aligned}|A_1 - A_2| &= \Big|A_1 - R(f,P) + R(f,P) - A_2\Big| \\ &\le |A_1 - R(f,P)| + |R(f,P) - A_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the distance $|A_1 - A_2| < \varepsilon$ implies $A_1 = A_2$. ■

Exercise 8.1.9

Explain why every function that is Riemann-integrable with $\int_a^b f = A$ must also have generalized Riemann integral A.

Proof. If f is Riemann-integrable, we know that f must also be bounded by some $M > 0$ which is the same for all the subintervals $[x_{k-1}, x_k]$. This means every partition $(P, \{c_k\}_{k=1}^n)$ is $\delta(x)$−fine. Hence, f also contains a generalized Riemann integral A. ■

The converse statement of the above is not true since Dirichel's function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a non-Riemann-integrable function whose discontinuities are at every point of $\mathbb{R}$.

Theorem 30. Dirichlet's function $g(x)$ is generalized Riemann-integrable on $[0,1]$ with $\int_0^1 g = 0$.

Proof. Let $\varepsilon > 0$. The goal is to construct a gauge $\delta(x)$ on $[0,1]$ such that whenever $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)$−fine tagged partition, it follows that

$$0 \le \sum_{k=1}^n g(c_k)\Delta x_k < \varepsilon.$$

In this context, the gauge $\delta(x)$ represents the restriction on the size of $\Delta x_k = x_k - x_{k-1}$ where $\Delta x_k < \delta(c_k)$. Thus, the Riemann sums for the $g(x)$ consists of products of the form $g(c_k)\Delta x_k$. If we take irrational tags, then $g(c_k) = 0$ by definition of g. Hence, the only case we need to worry about is when we have rational tags.

Let $\{r_1, r_2, r_3, \dots\}$ be a countable set of rational numbers that are contained in $[0,1]$. Then for each r_k, define $\delta(r_k) = \varepsilon/2^{k+1}$. If $x \notin \mathbb{Q}$, then set $\delta(x) = 1$. ■

Exercise 8.1.10

Show that if $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)$−fine tagged partition, then $R(g,P) < \varepsilon$.

Proof. Please Check later. If $c_k \notin \mathbb{Q}$, it follows that $g(c_k) = 0$ for all k. Then it immediately follows that

$$R(g,P) < \varepsilon.$$

Otherwise, Let $\{r_1, r_2, r_3, \dots\}$ be a countable set of rational numbers that are contained in $[0,1]$. Since $r_k \in \mathbb{Q}$ for all k, we must have $g(r_k) = 1$. Using the definition of $\delta(r_k)$ and the fact that $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)$− fine tagged partition, we must have

$$0 \le R(g,P) = \sum_{k=1}^n g(r_k)\Delta x_k = \sum_{k=1}^n \Delta x_k < \sum_{k=1}^n \frac{\varepsilon}{2^{k-1}} < \varepsilon.$$

Hence, $R(g,P) < \varepsilon$. ■

(i) The failure of the Dirichlet's function to be Riemann-integrable is caused by the freedom to choose between $R(g, P) = 1$ and $R(g, P) = 0$ based on either rational or irrational tagged partitions.

(ii) Nonconstant gauges that depend on the value of x on some interval causes us to discriminate based on which tagged partitions qualify as $\delta(x)-$fine which makes it easier to achieve

$$|R(f, P) - A| < \varepsilon$$

for smaller and more deliberately selected set of tagged partitions.

3.1.4 The Fundamental Theorem of Calculus

(i) In the Theorem 7.5.1, we implicitly assumed the derivative of $F(x) = \int_a^b f$ was integrable. We do not need that assumption in the upcoming version of FTC.

(ii) While the MVT played a crucial role in the proof of Theorem 7.5.1, it turns out that we won't be needing that either.

Theorem 31. Assume $F : [a, b] \to \mathbb{R}$ is differentiable at each point in $[a, b]$ and set $f(x) = F'(x)$. Then, f has the generalized Riemann integral

$$\int_a^b f = F(b) - F(a).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Both this proof of Theorem 7.5.1 make use of the following fact.

Exercise 8.1.11

Show that

$$F(b) - F(a) = \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})].$$

Since $F : [a, b] \to \mathbb{R}$ is differentiable at each point in $[a, b]$, we must also have F continuous at each point in $[a, b]$. This means that the sum

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]$$

is telescoping and thus we must have

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] = F(b) - F(a).$$

If $\{c_k\}_{k=1}^n$ is a set of tags for P, then we can estimate the difference between the Riemann sum $R(f, P)$ and $F(b) - F(a)$ by

$$\begin{aligned} |F(b) - F(a) - R(f, P)| &= \left| \sum_{k=1}^{n} [F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})] \right| \\ &\leq \sum_{k=1}^{n} |F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})|. \end{aligned}$$

Let $\varepsilon > 0$. Our goal is to construct a gauge $\delta(c)$ such that

$$|F(b) - F(a) - R(f, P)| < \varepsilon$$

for all $(P, \{c_k\})$ that are $\delta(c)-$fine (Using the variable c in the gauge function is more convenient than x in this case.)

Exercise 8.1.12

For each $c \in [a, b]$, explain why there exists a $\delta(c) > 0$ (a $\delta > 0$ depending on c) such that

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \varepsilon$$

for all $0 < |x - c| < \delta(c)$. Since F is differentiable, then by we are guaranteed to find a $\delta(c) > 0$ such that whenever $0 < |x - c| < \delta(c)$ it follows that

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \varepsilon.$$

Exercise 8.1.13

(a) For a particular $c_k \in [x_{k-1}, x_k]$ of P, show that

$$|F(x_k) - F(c_k) - f(c_k)(x_k - x_{k-1})| < \varepsilon(x_k - c_k)$$

and

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

Note that the differentiability of F implies that the right hand limit and left hand limit are the same. Hence, we can state that

$$\lim_{x_k \to c_k} \frac{F(x_k) - F(c_k)}{x_k - c_k} = \lim_{x_{k-1} \to c_k} \frac{F(c_k) - F(x_{k-1})}{c_k - x_{k-1}}. \tag{1}$$

Then by definition of the derivative, the right hand side of (1) implies

$$\left|\frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c)\right| < \varepsilon$$

which leads to

$$|F(x_k) - F(c_k) - f(c_k)(x_k - x_{k-1})| < \varepsilon(x_k - c_k)$$

and likewise the left hand side of (1) implies

$$\left|\frac{F(c_k) - F(x_{k-1})}{c_k - x_{k-1}} - f(c)\right| < \varepsilon$$

which also implies

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

(b) Now, argue that

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

Let $\varepsilon > 0$. Using algebraic manipulations we can write, collecting terms, and using the results from part (a), we have

$$\begin{aligned} |F(x_k) - F(x_{k-1}) - F(c_k)(x_k - x_{k-1})| &\leq |F(x_k) - F(c_k) - f(c_k)(x_k - c_k)| \\ &\quad + |F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| \\ &< \varepsilon(x_k - c_k) + \varepsilon(c_k - x_{k-1}) \\ &= \varepsilon(x_k - x_{k-1}). \end{aligned}$$

Then

$$|F(b) - F(a) - R(f, P)| < \varepsilon \sum_{k=1}^{n} (x_k - x_{k-1})$$
$$= \varepsilon(b - a)$$

■

Consider the function

$$F(x) = \begin{cases} x^{3/2} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

that is differentiable everywhere, including $x = 0$, with

$$F'(x) = \begin{cases} (3/2)\sqrt{x} \sin(1/x) - (1/\sqrt{x}) \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

An interesting fact about the function above is that it is *unbounded* even though the ordinary Riemann integral only considers bounded functions on closed intervals. This shows that the Generalized Riemann integral does not restrict itself to a certain class of functions.

Theorem 32 (Change-of-variable Formula)**.** Let $g : [a, b] \to \mathbb{R}$ be differentiable at each point of $[a, b]$, and assume F is differentiable on the set $g([a, b])$. If $f(x) = F'(x)$ for all $x \in g[a, b]$, then

$$\int_a^b (f \circ g) \cdot g' = \int_{g(a)}^{g(b)} f.$$

Proof. The hypothesis of the theorem guarantees that the function $(F \circ g)(x)$ is differentiable for all $x \in [a, b]$. ■

Exercise 8.1.14

(a) Why are we sure that f and $(F \circ g)'$ have generalized Riemann integrals?

Proof. Since F is differentiable and satisfies $F'(x) = f(x)$ for all $x \in g[a, b]$, we know that f must have a Generalized Riemann Integral. Likewise, the differentiability as well as the continuity of F and g guarantee Riemann integrability of their composition which also implies that the $(F \circ g)$ to have a Generalized Riemann integral. ■

(b) Use Theorem 8.1.9 to finish the proof.

Proof. Assume g is differentiable and F differentiable with $F'(x) = f(x)$ for all $x \in g[a, b]$. By part (a), we must have the following

$$\int_a^b (f \circ g) \cdot g' = (F \circ g)(b) - (F \circ g)(a)$$
$$= F(g(b)) - F(g(a))$$
$$= \int_{g(a)}^{g(b)} f.$$

Hence, we conclude that

$$\int_a^b (f \circ g) \circ g' = \int_{g(a)}^{g(b)} f.$$

■

3.2 Metric Spaces

In this section, we aim to give a more generalized view of what it means to have a "distance" over sets other than $\mathbb{R}$. Do the theorems and properties we have proved about sequences, series, and functions carry over to sets like $\mathbb{R}^2$ or even in higher dimensions like $\mathbb{R}^n$? We will be mainly testing our notions that we have developed throughout the book on sets such as $\mathbb{R}^2$ and $C[0,1]$, the space of continuous functions on $[0,1]$.

Definition 23 (Metric Spaces). Given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* on X if for all $x, y \in X$:

(i) $d(x,y) \geq 0$ with $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$, and

(iii) for all $z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$.

A *metric space* is a set X together with a metric d.

- Property (iii) in the definition above is just the triangle inequality.
- The set X can have different metrics on it.
- Whenever a metric space is mentioned, we usually specify what metric are using.

Exercise 8.2.1

Decide which of the following are metrics on $X = \mathbb{R}^2$. For each, we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the plane.

(a) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(b) $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(c) $d(x,y) = |x_1 x_2 + y_1 y_2|$.

Proof. (a) We claim that $d(x,y)$ is a metric on $X = \mathbb{R}^2$. Let $x', y' \in \mathbb{R}^2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For part (i), suppose $x' \neq y'$. Then observe that by property of the square root, we know that $d(x,y) > 0$. Otherwise, $d(x,y) = 0$.

For part (ii), observe that

$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \\ &= d(y,x). \end{aligned}$$

For part(iii), let $x, y, z \in \mathbb{R}^2$. Then observe that

$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (z_1 - y_1)^2 + (z_2 - y_2)^2} \\ &\leq \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} + \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2} \\ &= d(x,z) + d(z,y). \end{aligned}$$

Hence, we conclude that $d(x,y)$ is a metric on $\mathbb{R}^2$.

(b) We have $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is a metric on $\mathbb{R}^2$. For property (i), observe that $d(x,y) > 0$ if either $x \geq y$ or $x < y$. This holds because $|\cdot| > 0$. If $x = y$, then it follows immediately that $d(x,y) = 0$. To show the triangle inequality, we will use the formula

$$\max\{a,b\} = a + b + ||a| - |b||.$$

Then observe that for any $x, y, z \in \mathbb{R}^2$, we have

$$\begin{aligned}
d(x,y) &= \max\{|x_1 - y_1|, |x_2 - y_2|\} \\
&= \frac{1}{2}\Big[|x_1 - y_1| + |x_2 - y_2| + ||x_1 - x_1| + |x_2 - y_2||\Big] \\
&\leq \frac{1}{2}\Big[|x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\
&+ \Big||x_1 - z_1| + |z_1 - y_1| - |x_2 - z_2| + |z_2 - y_2|\Big|\Big] \\
&= \frac{1}{2}\Big[|x_1 - z_1| + |x_2 - z_2| + \Big||x_1 - z_1| - |x_2 - z_2|\Big|\Big] \\
&+ \frac{1}{2}\Big[|z_1 - y_1| + |z_2 - y_2| + \Big||z_1 - y_1| - |z_2 - y_2|\Big|\Big] \\
&= \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\} \\
&= d(x,z) + d(z,y).
\end{aligned}$$

(c) $d(x,y) = |x_1x_2 + y_1y_2|$ cannot be a metric since $d(x,y) \neq 0$ for all $x, y \in \mathbb{R}^2$.

■

- The metric in part (a) is the Euclidean distance between two points in a plane.
- $d(x,y) = |x - y|$ is a metric over $\mathbb{R}$ (the main metric we have been working with throughout the book).

Exercise 8.2.2

Let $C[0,1]$ be the collection of continuous functions on the closed interval $[0,1]$. Decide which of the following are metrics on $C[0,1]$.

(a) $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$.

(b) $d(f,g) = |f(1) - g(1)|$.

(c) $d(f,g) = \int_0^1 |f - g|$.

Proof. (a) Observe that for any two functions $f, g \in C[0,1]$ that are distinct, we know that

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \geq |f(x) - f(x)| > 0.$$

If $f = g$, then it immediately follows that $d(f,g) = 0$. Hence, property (i) is satisfied.

Observe that part (ii) is satisfied by taking

$$\begin{aligned}
d(f,g) &= \sup\{|f(x) - g(x)| : x \in [0,1]\} \\
&= \sup\{|g(x) - f(x)| : x \in [0,1]\} \\
&= d(g,f).
\end{aligned}$$

For part (iii), let $f, g, h \in C[0,1]$, then we must have

$$\begin{aligned}
d(f,g) &= \sup |f(x) - g(x)| \\
&= \sup |f(x) - h(x) + h(x) - g(x)| \\
&\leq \sup |f(x) - h(x)| + \sup |h(x) - g(x)| \\
&= d(f,h) + d(h,g).
\end{aligned}$$

(b) The first property fails (take $f(1) = 1$ and $g(x) = x$).

(c) We claim that $d(f,g) = \int_0^1 |f - g|$ is a metric on $\mathbb{R}^2$. Note that for any two distinct functions $f, g \in C[0,1]$, we must have $|f - g| > 0$. By exercise 7.4.4, we must have $\int_0^1 |f - g| > 0$. Otherwise, $f = g$ implies $\int_0^1 |f - g| = 0$. If $\int_0^1 |f - g| = 0$, then we must have $|f - g| = 0$ and hence, $f = g$. To show the triangle inequality, let $f, g, h \in C[0,1]$ be integrable (since they are part of a set of continuous functions that are bounded). Hence, observe that

$$\begin{aligned} d(f,g) &= \int_0^1 |f-g| \\ &\leq \int_0^1 |f-h| + |h-g| \\ &= \int_0^1 |f-h| + \int_0^1 |h-g| \\ &= d(f,h) + d(h,g). \end{aligned}$$

■

Define the *discrete metric* on any set X where for any $x, y \in X$, let

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Exercise 8.2.3

Verify that the discrete metric is actually a metric.

Proof. Observe that if $x \neq y$, then by definition we must have $\rho(x,y) > 0$. Otherwise, $\rho(x,y) = 0$ by definition. It is clear that $\rho(x,y) = \rho(y,x)$. To show the triangle inequality, let $x, y, z \in X$, then we must have

$$\begin{aligned} \rho(x,y) &= 1 + 0 \\ &\leq 1 + 1 \\ &= \rho(x,z) + \rho(z,y). \end{aligned}$$

Hence, $\rho(x,y)$ is a metric on any arbitrary set X. ■

3.2.1 Basic Definitions

Definition 24 (Convergence In A General Metric Space). Let (X,d) be a metric space. A sequence $(x_n) \subseteq X$ *converges* to an element $x \in X$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

Definition 25 (Cauchy Sequences). A sequence (x_n) in a metric space is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$.

Exercise 8.2.4

Show that a convergent sequence is Cauchy.

Proof. Since $(x_n) \subseteq$ is a Cauchy sequence, we can pick an $N \in \mathbb{N}$ such that for any $n, m \geq N$, we must have

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \text{and} \quad d(x, x_m) < \frac{\varepsilon}{2}.$$

Using the same choice of $N \in \mathbb{N}$ so that $n, m \geq N$, we must have that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, (x_n) is Cauchy. ■

- Notice that this is only the forwards direction of the Cauchy Criterion we studied under $\mathbb{R}$.
- For metric spaces other than $\mathbb{R}$, the converse of the Cauchy Criterion does not necessarily hold.
- We need to develop an ordering of our space similar to how the Axiom of Completeness is used in $\mathbb{R}$ (This is called *completeness*).
- The convergence of Cauchy sequences is taken to be the definition of completeness.

Definition 26 (Complete Metric Spaces). A metric space (X, d) is *complete* if every Cauchy sequence in X converges to an element of X.

Exercise 8.2.5

(a) Consider $\mathbb{R}^2$ with the discrete metric $\rho(x, y)$ examined in Exercise 8.2.3. What do Cauchy sequences look like in this pace? Is $\mathbb{R}^2$ complete with respect to this metric?

Proof. Cauchy sequences under the discrete metric under $\mathbb{R}^2$ would have $x_n = (x_{n_1}, x_{n_2})$ and $x_m = (x_{m_1}, x_{m_2})$ such that

$$\rho(x_n, x_m) = \begin{cases} 0 & \text{if } x_n = x_m \\ 1 & \text{if } x_{n_i} \neq x_{m_i} \text{ where } 1 \leq i \leq 2. \end{cases}$$

Yes, $\rho(x, y)$ is complete under $\mathbb{R}^2$. ■

(b) Show that $C[0, 1]$ is complete with respect to the metric in Exercise 8.2.2(a).

Proof. The metric from Exercise 8.2.2 (a) is

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Let $f_n, f_m \in C[0, 1]$. The Cauchy sequence under sup norm metric will be

$$d(f_n, f_m) = \sup_{x \in [0,1]} |f_n(x) - f_m(x)|$$

We want to show that the Cauchy sequence of functions (f_n) converges under $C[0, 1]$. Since (f_n) is a Cauchy sequence under $\mathbb{R}$, we know that is satisfies the Cauchy Criterion. Hence, (f_n) must converge uniformly. By choosing $N \in \mathbb{N}$, we can let $m, n \geq N$ and $x \in [0, 1]$ such that

$$\begin{aligned} d(f_n, f) &= \sup_{x \in [0,1]} |f_n(x) - f(x)| \\ &\leq \sup_{x \in [0,1]} |f_n(x) - f_m(x)| + \sup_{x \in [0,1]} |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, the sup norm metric is complete under $C[0, 1]$. ■

(c) Define $C^1[0, 1]$ to be the collection of differentiable functions on $[0, 1]$ whose derivatives are also continuous. Is $C^1[0, 1]$ complete with respect to the metric defined in Exercise 8.2.2(a)?

Proof. No, $C^1[0, 1]$ is not complete under metric defined in Exercise 8.2.2 (a). Define

$$h'_n = \frac{x}{\sqrt{x^2 + 1/n}}.$$

Note that the convergence of $h'n \to h$ where $h(x) = x/|x|$ is not uniform. Hence, we cannot have completeness on $C^1[0,1]$ when we have pointwise convergence instead of uniform convergence. ■

The sup metric is usually written as

$$\|f-g\|_\infty = d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$$

and setting $g=0$ gives us the "sup norm"

$$\|f\|_\infty = d(f,0) = \sup\{|f(x)| : x \in [0,1]\}.$$

From now on, we will assume that the space $C[0,1]$ is paired with the metric above unless otherwise specified.

Definition 27 (Continuity in General Metric Spaces)**.** Let (X,d_1) and (Y,d_2) be metric spaces. A function $f : X \to Y$ is *continuous* at $x \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x),f(y)) < \varepsilon$ whenever $d_1(x,y) < \delta$.

Exercise 8.2.6

Which of these functions from $C[0,1]$ to $\mathbb{R}$ (with the usual metric) are continuous?

(a) $g(f) = \int_0^1 fk$ where k is some fixed function in $C[0,1]$.

Proof. We claim that $g(f) = \int_0^1 fk$ where k is some fixed function in $C[0,1]$. Let $\varepsilon > 0$. Under the usual metric under $\mathbb{R}$, suppose there exists a $\delta > 0$ such that $|x-c| < \delta$. Since $f \in C[0,1]$, f is also continuous. Hence, we can use the same δ such that

$$|f(x)-f(c)| < \varepsilon.$$

Since k is a fixed function in $C[0,1]$, we must have

$$\begin{aligned} |g(f(x)) - g(f(c))| &= \left|k \int_0^1 (f(x)-f(c))\ dx\right| \\ &\le M \int_0^1 |f(x)-f(c)|\ dx && (k \text{ is bounded}) \\ &< M \int_0^1 \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Hence, $g(f)$ is continuous in $C[0,1]$. ■

(b) $g(f) = f(1/2)$.

Proof. Let $\varepsilon > 0$. Since $f \in C[0,1]$, we know that f must be continuous. Hence, we can choose a $\delta > 0$ such that for any $|x-c| < \delta$, we have

$$|g(f(x)) - g(f(c))| = |f(1/2) - f(1/2)| = 0 < \varepsilon.$$

Hence, g is continuous on $C[0,1]$. ■

(c) $g(f) = f(1/2)$, but this time with respect to the metric on $C[0,1]$ from Exercise 8.2.2 (c).

Proof. Not continuous. Let $f = 0$ and let $\delta > 0$. Now define

$$h_\delta(x) = \begin{cases} 1/2 & x \in V_\delta(1/2) \\ 0 & \text{otherwise} . \end{cases}$$

Observe that for any $\delta > 0$, we have that $d(h_\delta, f) = \delta$. Using the metric from part (c), we will end up with $d(h_\delta, h) = (h - f)(1/2) = 1/2$. Thus, we can't satisfy $\varepsilon < 1/2$. ■

3.2.2 Topology on Metric Spaces

Definition 28 ($\varepsilon-$neighborhoods)**.** Given $\varepsilon > 0$ and an element x in the metric space (X, d), the *$\varepsilon-$neighborhood of x* is the set

$$V_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

Exercise 8.2.7

Describe the $\varepsilon-$neighborhoods in $\mathbb{R}^2$ for each of the different metrics described in Exercise 8.2.1. How about the discrete metric?

Proof. ■

Now we are able to define *open sets, limit points,* and *closed sets* like we did before on $\mathbb{R}$ but this time with more general spaces and different metrics. Reframing our definitions of these concepts in $\mathbb{R}$ in terms of a general space X, we

- call a set $O \subseteq X$ *open* if for every $x \in O$ we can find a neighborhood $V_\varepsilon(x) \subseteq O$.
- A point x is a *limit point* of a set A if every $V_\varepsilon(x)$ intersects A in some point other than x. A set C is *closed* if it contains its limit points.

Exercise 8.2.8

Let (X, d) be a metric space.

(a) Verify that a typical $\varepsilon-$neighborhood $V_\varepsilon(x)$ is an open set. Is the set

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$$

a closed set?

Proof. Observe that $V_\varepsilon(x) \subseteq V_\varepsilon(x)$. Hence, $V_\varepsilon(x)$ is an open set. Yes, we have

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$$

is a closed set. To see why, let x be a limit point of $C_\varepsilon(x)$. By Theorem 3.2.2, there exists a sequence $x_n \subseteq C_\varepsilon(x)$ such that $\lim x_n = x$ with $x_n \neq x$ for all $n \in \mathbb{N}$. This means that we can choose an $N \in \mathbb{N}$ such that for all $n \geq N$, we must have $d(x_n, x) \leq \varepsilon$. Hence, $x \in C_\varepsilon(x)$ and so $C_\varepsilon(x)$ must be a closed set. ■

(b) Show that a set $E \subseteq X$ is open if and only if its complement is closed.

Proof. ($\Rightarrow$) Let x be a limit point of E^c. Then for all $\varepsilon-$neighborhoods, the intersection

$$V_\varepsilon(x) \cap E^c$$

is nonempty. Since E is open, we know that $V_\varepsilon(x) \subseteq E$. But we have $E \cap E^c$ is empty so we must have $x \in E^c$ because otherwise, there exists a $V_\varepsilon(x)$ such that $V_\varepsilon(x) \subseteq E$. Hence, E^c must be closed.

($\Leftarrow$) Let E^c be a closed set. Let $x \in E$. Since x is not a limit point of E^c, x must not be a limit point of E^c. This means there exists an intersection $V_\varepsilon(x) \cap E^c$ that is empty. Hence, $V_\varepsilon(x) \subseteq E$ implying that E is open. ■

Exercise 8.2.9

(a) Show that the set $Y = \{f \in C[0, 1] : \|f\|_\infty \leq 1\}$ is closed in $C[0, 1]$.

Proof. Suppose f is a limit point of Y. Let (f_n) be a Cauchy sequence that converges uniformly to f. Choose $\varepsilon = 1$. Then observe that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we must have

$$\|f_n - f\| \leq 1.$$

Then observe that

$$\begin{aligned} |f(x)| &\leq |f_n(x) - f(x) + f(x)| \\ &\leq |f_n(x) - f(x)| + |f(x)| \\ &\leq \|f_n - f\|_\infty \\ &\leq 1. \end{aligned}$$

Since $|f(x)| \leq \|f(x)\|$, we must also have $\|f(x)\| \leq 1$. Hence, f is contained in Y and thus we must have Y closed. ∎

(b) Is the set $T = \{f \in C[0,1] : f(0) = 0\}$ open, closed, or neither in $C[0,1]$?

Proof. T is closed. Let $\varepsilon > 0$. Let $f \in C[0,1]$. Then there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\|f\| < \varepsilon.$$

Hence, $f(0) = 0$ which implies $f \in T$. ∎

Definition 29 (Compact Metric Spaces)**.** A subset K of a metric space (X,d) is *compact* if every sequence in K has a convergent subsequence that converges to a limit in K.

In $\mathbb{R}$, we came across a proposition that a set is compact if and only if it is closed and bounded. For more general metric spaces, however, this proposition only holds true in the forwards direction.

Definition 30 (Boundedness)**.** A subset K of a metric space (X,d) is *bounded* if there exists an $R > 0$ such that for all $x, y \in X$, we have $d(x,y) < R$.

Exercise 8.2.10

(a) Show that if K is compact subset of the metric space (X,d), then K is closed and bounded.

Proof. Since K is a compact subset of the metric space (X,d), every sequence $(x_n) \subseteq K$ contains a subsequence (x_{n_k}) that converges to a limit x that is contained in K. Let (x_n) be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that for any $n > n_k \geq N$, we must have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $x \in K$ and $(x_n) \to x$, K must be closed. Since every sequence (x_n) converges, we know that every (x_n) is bounded by some $M > 0$. Hence, K must also be bounded. ∎

(b) Show that $Y \subseteq C[0,1]$ from Exercise 8.2.9 (a) is closed and bounded but not compact.

Proof. The results from part (a) of Exercise 8.2.9 imply that Y is bounded and closed. To see why Y is not compact, suppose we have a sequence of continuous functions (f_n) defined by $f_n = x^n$. Since $Y \subseteq C[0,1]$, we know that the $(f_n) \to f$ uniformly. But the pointwise limit of $\lim f_n(x)$ is *not continuous* and every subsequence of (f_n) will necessarily converge pointwise to $f \notin C[0,1]$. Hence, Y cannot be compact in $C[0,1]$. ∎

- The concept of *equicontinuity* of functions is key to the solution of part (c) above.
- Look back to the Arzeli-Ascoli Theorem in chapter 6 before solving the exercise above.

- The result found in part (b) can only be made possible if, in addition to our assumptions, Y contained a collection of functions that are equicontinuous.

Definition 31 (Closure). Given a subset E of a metric space (X, d), the *closure* $\overline{E}$ is the union of E together with its limit points. The *interior* of E is denoted by E° and is defined as

$$E^\circ = \{x \in E : \text{ there exists } V_\varepsilon(x) \subseteq E\}.$$

Exercise 8.2.11

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

Proof. See the solution in Exercise 3.2.14. ■

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ c) = \overline{E^c}$.

Proof. See the solution in Exercise 3.2.14. ■

Exercise 8.2.12

(a) show

$$\overline{V_\varepsilon(x)} \subseteq \{y \in X : d(x, y) \leq \varepsilon\},$$

in an arbitrary metric space (X, d).

Proof. Let x be a limit point of $\overline{V_\varepsilon(x)}$. By definition of $\overline{V_\varepsilon(x)}$, we know that $x \in \overline{V_\varepsilon(x)}$. Then, there exist exists a sequence (x_n) such that $x_n \to x$ with $x_n \neq x$ for all $n \in \mathbb{N}$. Hence, for some $N \in \mathbb{N}$ we know that for any $n \geq N$, we have $d(x_n, x) \leq \varepsilon$. But this is the definition of $C_\varepsilon(x)$. Hence, $x \in C_\varepsilon(x)$. ■

(b) To keep things from sounding too familiar, find an example of a specific metric space where

$$\overline{V_\varepsilon(x)} \neq \{y \in X : d(x, y) \leq \varepsilon\}.$$

Proof. Take $(\mathbb{R}, |\cdot|)$. Then observe that $\overline{V_\varepsilon(x)} \neq \{n \in \mathbb{N} : |1/n| \leq \varepsilon\}$ where $V_\varepsilon(x) = \{0\}$. ■

Definition 32 (Dense Sets). A set $A \subseteq X$ is *dense* in the metric space (X, d) if $\overline{A} = X$. A subset E of a metric space (X, d) is *nowhere-dense* in X if $\overline{E}^\circ$ is empty.

Exercise 8.2.13

If E is a subset of a metric space (X, d), show that E is nowhere-dense in X if and only if $\overline{E}^c$ is dense in X.

Proof. ($\Rightarrow$) Suppose E is nowhere-dense in X. Then $\overline{E}^\circ$ is empty. Let $\varepsilon > 0$. Then for any $x \in \overline{E}^\circ$ is contained in $(\overline{E}^\circ)^c$. By exercise 8.2.11, we know that $(\overline{E}^\circ)^c = \overline{(\overline{E}^\circ)^c}$. But notice that we must have $\overline{(\overline{E}^\circ)^c} = X$. Hence, $\overline{E}^c$ must be dense in X.

($\Leftarrow$) Suppose $\overline{E}^c$ is dense in X. Let $x \in \overline{E}^\circ$. Let $\varepsilon > 0$. Since x is neither an element of nor a limit point of $\overline{E}^c$, we know that for every $V_\varepsilon(x)$, we have $V_\varepsilon(x) \cap \overline{E}$ is empty. This tells us that $\overline{E}^\circ$ is empty. Hence, E is nowhere dense in X. ■

3.3 Euler's Sum

Recall Euler's famous series derivation

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

which used the Taylor series representation

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{1}$$

There is also the infinite product representation

$$\sin(x) = x\Big(1 - \frac{x}{\pi}\Big)\Big(1 + \frac{x}{\pi}\Big)\Big(1 - \frac{x}{2\pi}\Big)\Big(1 + \frac{x}{2\pi}\Big)\cdots . \tag{2}$$

We have developed the sufficient theory to show why (1) is true, but not (2). There have been many derivations for (2) using multi-variable calculus, Fourier series, and even complex integration. However, we will try to show (2) by using the properties of uniformly convergent series and Taylor series expansions.

3.3.1 Walli's Product

We currently don't have enough machinery at our disposal to be able to prove the infinite product representation of $\sin(x)$ in (2), but we can prove the special case when

$$\frac{\pi}{2} = \lim_{n\to\infty} \prod_{n=1}^{n} \Big(\frac{2n \cdot 2n}{(2n-1)(2n+1)}\Big) \tag{3}$$

where (3) is the partial products of (2) but with $x = \pi/2$.

Exercise 8.3.1

Supply the details to show (3) above.

Proof. Plugging in $x = \pi/2$ into (2), we get that

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \Big(1 - \frac{1}{2n}\Big)\Big(1 + \frac{1}{2n}\Big) = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2}.$$

Taking the reciprocal of the infinite product above, we end up with

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{\pi}{2}.$$

■

Now we will prove why (3) holds. Set

$$b_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\ dx, \text{ for } n = 0, 1, 2, \ \ldots$$

If we look at the $n = 0$ and $n = 1$ case, we can easily obtain the following equations

$$b_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \text{ and } b_1 = \int_0^{\frac{\pi}{2}} \sin(x)\ dx = 1.$$

Exercise 8.3.2

Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$, and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\ dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\ dt.$$

Proof. Refer to the solution in part (a) of Exercise 7.5.6. ■

Exercise 8.3.3

(a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation
$$b_n = \frac{n-1}{n} b_{n-2} \text{ for all } n \geq 2.$$

Proof. Let $h(x) = \sin^n(x)$ and $k'(x) = \sin(x)$. Let $n \geq 2$. Then by the integration-by-parts formula and using the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, we must have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx &= \int_0^{\frac{\pi}{2}} \sin^n(x) \cdot \sin(x)\ dx \\
&= \Big[-\sin^{n-1}(x) \cdot \cos(x) \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot \cos^2(x)\ dx \\
&= \Big[-\sin^{n-1}(x) \cdot \cos(x) \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot [1 - \sin^2(x)]\ dx
\end{aligned}$$

The first term on the last equality cancels out and the second term can be expanded into

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot [1 - \sin^2(x)]\ dx = &\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx \\
&+ \int_0^{\frac{\pi}{2}} (n-1)\sin^{n}(x)\ dx.
\end{aligned}$$

Hence, we end up with

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx = \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n}(x)\ dx. \tag{1}$$

Finally, subtracting the second term on the right side of (1), simplifying, and dividing by n on both sides gives us our desired result

$$\begin{aligned}
b_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\ dx \\
&= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\ dx \\
&= \frac{n-1}{n} b_{n-2}.
\end{aligned}$$

■

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).

Proof. The first three even terms are

$$\begin{aligned}
b_2 &= \frac{1}{2} b_0 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}, \\
b_4 &= \frac{3}{4} b_2 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}, \\
b_6 &= \frac{5}{6} b_4 = \frac{5}{6} \cdot \frac{3\pi}{16} = \frac{5\pi}{32}.
\end{aligned}$$

The first odd terms are

$$\begin{aligned} b_3 &= \frac{2}{3}b_1 = \frac{2}{3}\cdot 1 = \frac{2}{3} \\ b_5 &= \frac{4}{5}b_3 = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15} \\ b_7 &= \frac{6}{7}b_5 = \frac{6}{7}\cdot\frac{8}{15} = \frac{16}{35}. \end{aligned}$$

■

(c) Write a general expression for b_{2n} and b_{2n+1}.

Proof. Using the formula we derived in part (a), plugging in the desired cases gives us

$$b_{2n} = \frac{2n-1}{2n}b_{2(n-1)} \quad \text{and} \quad b_{2n+1} = \frac{2n}{2n+1}b_{2n-1}.$$

■

For the $(n+1)$th term, we have the following bound $0 \leq \sin^{n+1}(x) \leq \sin^n(x)$ on $[0, \pi/2]$. But this tells us that (b_n) is a decreasing sequence of functions. Since (b_n) is bounded and decreasing, we know that it must converge. It turns out that $(b_n) \to 0$ but this isn't the limit that we want to concern ourselves at the moment.

Exercise 8.3.4

Show

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Walli's product formula in (3).

Proof. For $k \geq 1$, observe that

$$\begin{aligned} \frac{b_{2n}}{b_{2n+1}} &= \frac{(2n-1)(2n+1)}{(2n)(2n)} \cdot \frac{b_{2n-2}}{b_{2n-1}} \\ &= \frac{(2n-1)(2n+1)}{(2n)(2n)} \cdot \frac{(2n-3)(2n-1)}{(2n-2)(2n-2)} \cdot \frac{b_{2n-4}}{b_{2n-3}}. \end{aligned}$$

Notice when expanding the terms on the numerator and the denominator of b_{2n}/b_{2n+1}, we will always have the same coefficient. Hence, the limit of b_{2n}/b_{2n+1} gives us our result that

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1.$$

■

Some techniques to dealing with the notation in (3) is to use the following equations

$$2\cdot 4\cdot 6\cdots(2n) = 2^n n!$$

and

$$1\cdot 3\cdot 5\cdot\cdots(2n+1) = \frac{(2n+1)!}{2\cdot 4\cdot 6\cdots(2n)} = \frac{(2n+1)!}{2^n n!}.$$

Exercise 8.3.5

Derive the following alternative form of Walli's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Proof. ■

3.3.2 Taylor Series

To prove (2), we need to somehow generate the Taylor series for $\arcsin(x)$. This can't be done directly from Taylor's Formula for the coefficients. We need to first find the expansion for $1/\sqrt{1-x}$ by dealing with

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

first.

Exercise 8.3.6

Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Proof. Let $f(x) = 1/\sqrt{1-x}$. Using Taylor's coefficient formula, we have the first three derivatives of f

$$f^{(1)}(x) = \frac{1}{2} \cdot (1-x)^{-3/2},$$
$$f^{(2)}(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot (1-x)^{-5/2},$$
$$f^{(3)}(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot (1-x)^{-7/2}.$$

For $n \geq 1$, we can use induction to show

$$f^{(n)}(x) = \left[\prod_{k=1}^{n} \frac{2k-1}{2k}\right](1-x)^{-(2n+1)/2}.$$

Plugging in $x = 0$ and using the techniques given to us above, we now have the desired formula

$$c_n = \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n)!}{2^{2n}(n!)^2}$$

where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n.$$

■

Observe that the coefficients above should look familiar to the formulas produced from Walli's product.

Exercise 8.3.7

Show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Proof. The first statement is shown in Exercise 2.7.10. Observe that

$$c_n \leq \frac{1}{2^{2n}} \leq \frac{1}{n}.$$

Since $\sum 1/n$ diverges, we must also have $\sum c_n$ diverge by the Comparison test. ■

Now our goal is to establish at which particular points in the domain of f where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

is valid. This can be done by using Lagrange's Remainder Theorem.

To properly show that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

holds for all $x \in (-1, 1)$, we need to show that the error function

$$E_N(x) = \frac{1}{\sqrt{1-x}} - \sum_{n=0}^{N} c_n x^n$$

approaches zero as $N \to \infty$. This can be done using Lagrange's Remainder Theorem (Theorem 6.6.3).

Exercise 8.3.8

Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem, show that equation (4) is valid for all $|x| < 1/2$. What goes wrong when we try try to use this method to prove (4) for $x \in (1/2, 1)$?

Proof. Since f is $N+1$ times differentiable on $(-1/2, 1/2)$, there exists a c such that $|c| < |x|$ where the error function $E_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)x^n}{(N+1)!}$$

by Lagrange's Remainder Theorem. Observe that

$$f^{(N+1)}(c) = \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big](1-c)^{-(2N+3)/2} < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\Big(\frac{2}{3}\Big)^{(2N+3)/2}.$$

Since $|x| < 1/2$ and $|c| < |x|$, we can now write

$$E_N(x) < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big] \frac{2^{3/2}}{3^{(2N+3)/2}(N+1)!} \xrightarrow{N\to\infty} 0.$$

Hence, (4) holds for all $x \in (-1, 1)$. If we try to prove $E_N \to 0$ on $(-1/2, 1)$, then we produce a sequence that diverges. ■

3.3.3 The Integral Form of the Remainder

The goal of the previous exercise is to recognize a different method is needed to estimate the error function $E_N(x)$. The following theorem is one such way to do this.

Theorem 33. Integral Remainder Theorem Let f be differentiable $N+1$ times on $(-R, R)$ and assume $f^{(N+1)}$ is continuous. Define $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, \dots, N$, and let

$$S_N(x) = \sum_{k=0}^{N} a_k x^k.$$

For all $x \in (-R, R)$, the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N \, dt.$$

Proof. The case $x = 0$ is easy to check, so let's take $x \neq 0$ in $(-R, R)$ and keep in mind that x is a fixed constant in what follows. To avoid a few technical distractions, let's just consider the case $x > 0$.

Exercise 8.3.9

(a) Show
$$f(x) = f(0) + \int_0^x f'(t)\ dt.$$
Since f is continuous differentiable for all $t \in (0, x)$, we can use part (i) of FTC to write
$$\int_0^x f'(t)\ dt = f(x) - f(0).$$
Solving for $f(x)$ gives us our desired result
$$f(x) = f(0) + \int_0^x f'(t)\ dt.$$

(b) Now use a previous result from this section to show
$$f(x) = f(0) + f'(0)x + \int_0^x f"(t)(x-t)\ dt.$$
Taking advantage of f being continuously differentiable $N+1$ times for all $t \in (0, x)$ and using the integration-by-parts formula found in Exercise 8.3.2, we have
$$\begin{aligned}\int_0^x f"(t)(x-t)\ dt &= \Big[f'(t)(x-t)\Big]_0^x + \int_0^x f'(t)\ dt \\ &= -xf'(0) + [f(x) - f(0)].\end{aligned}$$
Solving for $f(x)$ once again, we get our desired result
$$f(x) = f(0) + f'(0)x + \int_0^x f"(t)(x-t)\ dt.$$

(b) Continue in this fashion to complete the proof of the theorem.

Continuing the process in parts (a) and (b) and using the fact that f is $N+1$ times differentiable for all $x \in (-R, R)$, we have that
$$\begin{aligned}f(x) &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt + \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x_k \\ &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt + S_N(x).\end{aligned}$$
Subtracting $S_N(x)$ from both sides above and using the fact that $E_N(x) = f(x) - S_N(x)$ gives us our desired result
$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt.$$
■

We will use this fact to now show that (4) holds.

Exercise 8.3.10

(a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1, 1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

Proof. ■

(b) For a general x satisfying $|x| < 1$, show
$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\ dt.$$

Proof. Let $|x| < 1$. Let $N = 2$. Observe that

$$f^{(3)}(t) = \frac{15}{8}(1-t)^{-7/2}.$$

Using the Integral Remainder Theorem, we have that

$$\begin{aligned} E_2(x) &= \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2 \ dt \\ &= \frac{15}{16}\int_0^x (1-t)^{-7/2}(x-t)^2 \ dt \\ &= \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \ dt. \end{aligned}$$

■

(c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \leq |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking that this overestimate is in fact larger than $|E_2(x)|$ at three computed values from part (a).

Proof. The inequality above is valid since

$$\begin{aligned} \left|\frac{x-t}{1-t}\right| &= \sqrt{\left(\frac{x-t}{1-t}\right)^2} \\ &\leq \sqrt{(x-t)^2} \\ &= |x-t| \\ &\leq |x| \end{aligned}$$

which holds for t. ■

(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1, 1)$.

Proof. Let f be differentiable $N+1$ times. Using the inequality found in part (c) and $|x| < 1$,we can write

$$\begin{aligned} |E_N(x)| &= \frac{1}{N!}\left[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\right]\left|\int_0^x \left(\frac{x-t}{1-t}\right)^N \cdot \frac{1}{(1-t)^{3/2}} \ dt\right| \\ &\leq \frac{c_{N+1}}{N!}\int_0^x \left|\frac{x-t}{1-t}\right|^N \cdot \left|\frac{1}{(1-t)^{3/2}}\right| \ dt \\ &\leq \frac{c_{N+1}}{N!}\int_0^x \frac{|x|^N}{(1-t)^{3/2}} \ dt \\ &< \frac{c_{N+1}}{N!}\int_0^x \frac{1}{(1-t)^{3/2}} \ dt \\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}\int_0^x \ dt \\ &= \frac{c_{N+1}x}{2\sqrt{2}\cdot N!} \\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}. \end{aligned}$$

Since $c_{N+1} = \prod_{k=1}^{N+1} \frac{2k-2}{2k} \to 0$ from Exercise 8.2.7, we can write

$$|E_N(x)| < \frac{c_{N+1}}{2\sqrt{2}\cdot N!} \to 0$$

as $N \to \infty$. Hence, $|E_N(x)| \to 0$ which tells us that $E_N \to f(x)$ uniformly. ■

Now that we have established that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

holds for all $x \in (-1, 1)$, we are now in the position to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{2n+1}$$

for all $|x| < 1$ using term-by-term anti-differentiation of (4).

Exercise 8.3.11

Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1. \tag{5}$$

Proof. From our result in part (d) of Exercise 8.3.10, we know that substituting $x = x^2$ into

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

give us

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} c_n x^{2n}$$

which holds for all $x \in (-1, 1)$. By assumption, we know that the derivative of $\arcsin(x)$ is $1/\sqrt{1-x}$. Using Term-by-term Antidifferentiation, we get that

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1.$$

■

Exercise 8.3.12

Our work thus far shows that the Taylor series in (5) is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \leq 1$. Carefully, explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1, 1]$.

Proof. Since (5) is valid for all $|x| < 1$, it suffices to show that (5) holds for $x = 1$ and likewise $x = -1$ so that we may show that (5) holds for $|x| \leq 1$ using Theorem 6.5.2. Plugging in $x = 1$ gives us

$$\sum_{n=0}^{\infty} \frac{c_n}{2n+1}.$$

Let's define

$$\gamma_n = \frac{c_n}{2n+1}.$$

Using the Cauchy Condensation Test, we can prove that $\sum 2^n \gamma_{2^n}$ so that $\sum \gamma_n$ converges. Observe that

$$\sum_{n=0}^{\infty} \frac{c_{2^n}}{2^{2n+1}+1} \leq \frac{1}{2} \sum_{n=0}^{\infty} c_{2^n}.$$

Now our goal is to use the Comparison Test to show that the right side of the inequality above converges which will immediately imply that the left side converges. Hence, observe that we have the following bound

$$\begin{aligned}\frac{1}{2}c_{2^n} &= \frac{1}{2}\cdot\frac{(2^{n+1})!}{(2^n!)^2\cdot 2^{2^{2n+1}}}\\ &\le \frac{2^{n+1}-1}{2\cdot(2^n)!}\\ &\le \frac{2^{n+1}}{2\cdot n!}.\end{aligned}$$

Observe that the last inequality forms a series that converges via the ratio test. Hence, we must have $\frac{1}{2}\sum c_{2^n}$ converge via the Comparison test. Hence, we must have $\sum c_n/(2n+1)$ converges via the Cauchy Condensation Test. Hence, by Theorem 6.5.2 we must have (5) converge at $x = |1|$ for all $x \in [-1, 1]$. ∎

3.3.4 Summing $\sum_{n=1}^{\infty} 1/n^2$

Suppose we let $x = \sin(\theta)$ in (5) where we restrict our domain to $-\pi/2 \le \theta \le \pi/2$. Then we have

$$\theta = \arcsin(\sin(\theta)) = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)$$

which converges uniformly on $[-\pi/2, \pi/2]$.

Exercise 8.3.13

(a) Show

$$\int_0^{\pi/2}\theta\ d\theta = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Walli's product.

Proof. Observe that the series

$$\theta = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)$$

converges uniformly to θ for all $-\pi/2 \le \theta \le \pi/2$. Hence, we are able to move integration from outside the summation to inside the summation. Using this fact, we write

$$\begin{aligned}\int_0^{\pi/2}\theta\ d\theta &= \int_0^{\pi/2}\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)\ d\theta\\ &= \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\Big[\int_0^{\pi/2}\sin^{2n+1}(\theta)\ d\theta\Big]\\ &= \sum_{n=0}^{\infty}\frac{c_n}{2n+1}b_{2n+1}. && \text{(Walli's Formula)}\end{aligned}$$

∎

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Proof. Looking at the left side of part (a), we can integrate to get

$$\int_0^{\pi/2}\theta\ d\theta = \Big[\frac{1}{2}\theta^2\Big]_0^{\pi/2} = \frac{\pi^2}{8}.$$

Focusing our attention to the right side of (a), we see that b_{2n+1} can be expanded to

$$b_{2n+1} = \frac{2n}{(2n+1)} \cdot \frac{(2n-2)}{(2n-1)} \cdot \frac{(2n-4)}{(2n-3)} \cdot \frac{(2n-6)}{(2n-5)} \cdot \dots .$$

Likewise, c_n can be expanded into

$$\begin{aligned} c_n &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &= \frac{(2n) \cdot (2n-1) \cdot (2n-2) \cdot (2n-3) \cdot (2n-4) \cdot \dots}{2^{2n}(n!)^2} \end{aligned}$$

Notice that when we multiply b_{2n+1} and c_n together as seen in part (a), we see that the $(2n-k)$ terms for k odd cancel, leaving the $(2n-\ell)$ terms for ℓ even on the top. Hence, we have

$$\begin{aligned} c_n b_{2n+1} &= \frac{(2n)^2 \cdot (2n-2)^2 \cdot (2n-4)^2 \cdot (2n-4)^2 \cdot \dots}{4^n(n!)^2(2n+1)} \\ &= \frac{(2n)^2 \cdot (2(n-1))^2 \cdot (2(n-2))^2 \cdot \dots}{4^n(n!)^2(2n+1)} \\ &= \frac{4^n \cdot n^2 \cdot (n-1)^2 \cdot (n-2)^2 \cdot \dots}{4^n(n!)^2(2n+1)} \\ &= \frac{4^n(n!)^2}{4^n(n!)^2} \cdot \frac{1}{2n+1}. \\ &= \frac{1}{2n+1}. \end{aligned}$$

Hence, part (a) leads to

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Since the infinite sum in part (a) converges uniformly for all $-\pi/2 \leq \theta \leq \pi/2$, we are free to rearrange the sum however we like. Observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\ &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{4n^2} \\ &= \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{1}{4n^2}. \end{aligned}$$

Subtracting the second term on the right hand side to both sides above and collecting terms and dividing, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

■

3.3.5 Riemann-Zeta Function

The general formula that developed by Euler for the result we arrived at is written as a function of s where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for all } s > 1.$$

It is said that Euler was able to work out the sum for even s. There are a lot of deep properties about the function above, but among them, the most prominent would be about how $\zeta(s)$ is connected to the prime numbers given in the following formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\left(\frac{1}{1-7^{-s}}\right)\cdots \tag{6}$$

where the product is taken over all the primes. It is not surprising that delving deep into investigation of such properties will require more sophisticated machinery. However, the formula above is quite accessible. We see that expanding the product on the right hand side of (6) and using the fact that every natural number n contains a unique prime factorization, leading to the following formula

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \cdots .$$

3.4 Inventing the Factorial Function

The goal of this section is construct a function $f(x)$, defined on all of $\mathbb{R}$ with the property that $f(n) = n!$ for all $n \in \mathbb{N}$. This can be done easily by defining a piecewise function such that

$$f(x) = \begin{cases} n! & \text{if } n \le x < n+1, n \in \mathbb{N} \\ 1 & \text{if } x < 1. \end{cases}$$

Some questions we can explore is its continuity, differentiability (if differentiable then how many times?). Our goal now is to define a function that extends the definition of the factorial $n!$ in a meaningful way to non-natural n.

Exercise 8.4.1

For each $n \in \mathbb{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

Proof. ■

(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$, and revisit part (a).

Proof. The statement above is clearly true for $n = 1$. Now assume $n\# = \frac{1}{2}n(n+1)$ holds for $1 \le n \le k-1$. We want to show that $n\#$ holds for the kth case. By using the definition of $n\#$, we can write

$$\begin{aligned} k\# &= k + (k-1)\# \\ &= k + \frac{1}{2}k(k-1) \\ &= \frac{1}{2}(k^2 + k) \\ &= \frac{1}{2}k(k+1). \end{aligned}$$

Since $n\# = \frac{1}{2}n(n+1)$ holds for the kth case, we know that it holds for any $n \in \mathbb{N}$. ■

We can replace the discrete variable $n \in \mathbb{N}$ for values of $x \in \mathbb{R}$ and the resulting formula

$$x\# = \frac{1}{2}x(x+1)$$

will still make sense.

3.4.1 The Exponential Function

How is the exponential function like 2^x defined on $\mathbb{R}$? Typically, 2^x is defined through a series of domain expansions. Starting with the function defined on $\mathbb{N}$, we can expand its domain by using reciprocals, then to $\mathbb{Q}$ using roots, and then $\mathbb{R}$ using continuity. Our goal in this section is to expand the domain of 2^x using a different method.

Our first step is to properly define the natural exponential function e^x. Recall in chapter 6, we constructed a series expansion for e^x. This time, we do the opposite direction; that is, create a proper definition of e^x. We can do this by using the results we have found in our studies of power series expansions.

Define

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Exercise 8.4.2

Verify that the series converges absolutely for all $x \in \mathbb{R}$, that $E(x)$ is differentiable on $\mathbb{R}$, and $E'(x) = E(x)$.

Proof. First we prove that the series above converges absolutely for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Observe that

$$\Big|\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big| \leq \sum_{n=0}^{\infty} \Big|\frac{x^n}{n!}\Big|.$$

By using the ratio test for power series found in section 6.5, we have

$$\begin{aligned}\Big|\frac{a_{n+1}}{a_n}\Big| &= \Big|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\Big| \\ &= \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Since the limit above is 0, we know that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

converges absolutely for all $x \in \mathbb{R}$. Given any compact set in $\mathbb{R}$, we know that the convergence of (1) to $E(x)$ is uniform. Hence, it must be continuous on any $A \subseteq \mathbb{R}$ and differentiable n times. Differentiating

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and reordering indices we find that $E'(x) = E(x)$. ■

Exercise 8.4.3

(a) Use the results of Exercise 2.8.7 and the binomial formula to show that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$. By definition of $E(x+y)$, using the binomial formula, we can write

$$\begin{aligned}
E(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k \cdot x^{n-k}}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k}{k!} \cdot \frac{x^{n-k}}{(n-k)!} \\
&= \Big[\sum_{m=0}^{\infty} \frac{x^m}{m!}\Big]\Big[\sum_{k=0}^{\infty} \frac{y^k}{k!}\Big]. && (n-k=m)
\end{aligned}$$

Since

$$\begin{aligned}
E(x) &= \sum_{m=0}^{\infty} \frac{x^m}{m!}, \\
E(y) &= \sum_{k=0}^{\infty} \frac{y^k}{k!}
\end{aligned}$$

both converge absolutely (by Exercise 2.8.7), we can write

$$E(x+y) = E(x)E(y).$$

∎

(b) Show that $E(0) = 1, E(-x) = 1/E(x)$, and $E(x) > 0$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. The first fact immediately follows when $x = 0$. Now let us show the second fact. Using the first fact and part(a), we can write

$$1 = E(0) = E(x-x) = E(x)E(-x) \Leftrightarrow E(x)E(-x) = 1.$$

Dividing through by $E(x)$ on both sides leads us to our result

$$E(-x) = \frac{1}{E(x)}.$$

For the last fact, observe that $E(x) > 0$ follows immediately when we consider any $x \geq 0$. Suppose we let x be negative, then using the fact that

$$E(-x) = \frac{1}{E(x)}$$

where $E(x) > 0$ for any $x > 0$ implies that $E(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} > 0$. ∎

The takeaway here is that the power series $E(x)$ contains all the "normal" properties that is associated with the exponential function e^x.

Exercise 8.4.4

Define $e = E(1)$. Show $E(n) = e^n$ and $E(m/n) = (\sqrt[n]{e})^m$ for all $m, n \in \mathbb{Z}$.

Proof. Let $P(n)$ be the statement that $E(n) = e^n$ for all $n \in \mathbb{Z}$. Let our base case be $n = 1$. Then by definition, we must have $E(1) = e$. Now assume $E(n) = e^n$ holds for all $n \in \mathbb{Z}^+$. We want to show that $E(n+1) = e^{n+1}$ holds. Observe that by part (a) of Exercise 4.4.3, we have

$$\begin{aligned} E(n+1) &= E(n) \cdot E(1) \\ &= e^n \cdot e \\ &= e^{n+1}. \end{aligned}$$

Hence, $E(n) = e^n$ for all $n \in \mathbb{Z}^+$. To show that the statement also holds for all $n \in \mathbb{Z}^-$, we can just multiply n by a negative to get

$$\begin{aligned} E(-n) &= \frac{1}{E(n)} \\ &= \frac{1}{e^n} \\ &= e^{-n}. \end{aligned}$$

Lastly, we show $E(m/n) = (\sqrt[n]{e})^m$ for all $n, m \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$. Observe that

$$E(1) = E\left(\frac{n}{n}\right) = (\sqrt[n]{e})^n \Rightarrow E(1/n) = \sqrt[n]{e}.$$

Furthermore, we can rewrite m/n in the following way where

$$\frac{m}{n} = \sum_{i=0}^{m} \frac{1}{n}.$$

Then we see that

$$\begin{aligned} E\left(\frac{m}{n}\right) &= E\left(\sum_{i=0}^{m} \frac{1}{n}\right) \\ &= E\left(\frac{1}{n}\right) \cdot E\left(\frac{1}{n}\right) \cdot E\left(\frac{1}{n}\right) \cdot \ \cdots \ m \text{ times} \\ &= \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \ \cdots \ m \text{ times} \\ &= (\sqrt[n]{e})^m. \end{aligned}$$

■

To complete our list of properties of e^x, all we need is its behavior as $x \to \pm\infty$.

Definition 33. Given $f : [a, \infty) \to \mathbb{R}$, we say that $\lim_{x\to\infty} f(x) = L$ if, for all $\varepsilon > 0$, there exists $M > a$ such that whenever $x \geq M$ it follows that $|f(x) - L| < \varepsilon$.

Exercise 8.4.5

Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \ldots$. To get started notice that when $x \geq 0$, all the terms in (1) are positive.

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Choose $M = 1/\varepsilon > a$. Then observe that for any $x \geq M$, we have

$$\left|\frac{x^n}{e^x} - 0\right| = \frac{x^n}{e^x} < \frac{x^n}{x^{n+1}} = \frac{1}{x} < \varepsilon.$$

Hence, $\lim_{x\to\infty} x^n e^{-x} = 0$. ■

3.4.2 Other Bases

Having established a rigorous foundation for e^x, we can now do the same for t^x for any real number $t > 0$.

Exercise 8.4.6

(a) Explain why we know e^x has an inverse function; that is, let's call it $\log(x)$ defined for any real $x > 0$ and satisfying

(i) $\log(e^y) = y$ for all $y \in \mathbb{R}$ and

(ii) $e^{\log(x)} = x$, for all $x > 0$.

Proof. If we are considering $f(x) = e^x$ defined on $(0, \infty)$, then we get that $f(x)$ is a bijective function for all $x \in (0, \infty)$. To see why, suppose we let $x, y \in (0, \infty)$. Since $\log(x)$ is defined for all $x \in (0, \infty)$, we can say that

$$\begin{aligned} E(x) &= E(y) \\ e^x &= e^y \\ \log(e^x) &= \log(e^y) \\ x &= y. \end{aligned}$$

Hence, $E(x) = e^x$ is an injective function. Now let's show surjectivity. Then letting $x = \log(y)$, observe that

$$E(x) = e^x = e^{\log(y)} = y.$$

Hence, $E(x)$ is a surjective function. Since $E(x)$ is both injective and surjective, we know that $E(x)$ must be bijective and thus must have an inverse function. ■

(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)

Proof. Let $y = f(x) = e^x$. Using the result from Exercise 5.2.12, the fact that $f'(x) = e^x$, and $e^{\log(x)}$, we get that

$$\begin{aligned} (\log x)' &= \frac{1}{f'(x)} \\ &= \frac{1}{e^{\log(x)}} \\ &= \frac{1}{x}. \end{aligned}$$

■

(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log(x) + \log(y) \text{ for all } x, y > 0.$$

Proof. Let $x, y \in (0, \infty)$ with $x = e^y$ and $y = e^x$. Our logarithm properties, we then have $\log(x) = y$ and $\log(y) = x$. Then by using the properties of e^x and $\log(x)$, observe that

$$\begin{aligned} \log(xy) &= \log(e^y \cdot e^x) \\ &= \log(e^{y+x}) \\ &= y + x \\ &= \log(x) + \log(y). \end{aligned}$$

Hence, we have

$$\log(xy) = \log(x) + \log(y).$$

■

(d) For $t > 0$ and $n \in \mathbb{N}$, t^n has the usual interpretation as $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \text{ for all } n \in \mathbb{N}.$$

Proof. Let $t > 0$ and $n \in \mathbb{N}$. Observe that $t = e^{\log(t)}$ and then

$$t^n = \left(e^{\log(t)}\right)^n = e^{n\log(t)}.$$

■

Definition 34. Given $t > 0$, define the exponential function t^x to be

$$t^x = e^{x\log t} \text{ for all } x \in \mathbb{R}.$$

Exercise 8.4.7

(a) Show $t^{m/n} = (\sqrt[n]{t})^m$ for all $m, n \in \mathbb{N}$.

Proof. Let $m, n \in \mathbb{N}$. Then

$$t^{m/n} = (t^{1/n})^m = (\sqrt[n]{t})^m.$$

■

(b) Show $\log(t^x) = x\log t$, for all $t > 0$ and $x \in \mathbb{R}$.

Proof. Let $t > 0$ and $x \in \mathbb{R}$. Then observe that

$$\begin{aligned} t^x = e^{x\log t} &\Rightarrow \log(t^x) = \log(e^{x\log t}) \\ &\Rightarrow \log(t^x) = x\log t. \end{aligned}$$

■

(c) Show t^x is differentiable on $\mathbb{R}$ and find the derivative.

Proof. Let $x, t \in \mathbb{R}$. To show that $f(x) = t^x$ is differentiable, we can use the definition of differentiability. Using the fact that $t^x = e^{x\log t}$, we have

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{t^x - t^c}{x - c} \\ &= \lim_{x\to c} \frac{e^{x\log t} - e^{c\log t}}{x - c}. \end{aligned}$$

Observe that $g(x) = e^{x\log t}$ is differentiable. Hence, the limit in the last equality exists and therefore $f'(c)$ exists. Using the Chain Rule, we get

$$f'(x) = (t^x)' = (e^{x\log t})' = \log(t)e^{x\log t} = \log(t)t^x.$$

■

The strategy we have been partaking in so far is a similar to how we would define what $n!$ would mean if it was replaced by $x \in \mathbb{R}$ instead of $n \in \mathbb{N}$.

3.4.3 The Functional Equation

Our goal now is to somehow extend the domain of the factorial from the set of natural number; that is,

$$n! = n(n-1)! \text{ for all } n \in \mathbb{N}$$

all the way to the set of real numbers with

$$x! = x(x-1)! \text{ for all } x \in \mathbb{R}.$$

Of course, we cannot forget about $n = 1$ implying that $0! = 1$.

Exercise 8.4.8

Inspired by the fact that $0! = 1$ and $1! = 1$, let $h(x)$ satisfy

(i) $h(x) = 1$ for all $0 \leq x \leq 1$, and

(ii) $h(x) = xh(x-1)$ for all $x \in \mathbb{R}$.

(a) Find a formula for $h(x)$ on $[1,2]$, $[2,3]$, and $[n, n+1]$ for arbitrary $n \in \mathbb{N}$.

Proof. On $[1,2]$, observe that

$$h(2) = 2 \cdot h(1) = 2 \cdot 1 \cdot h(0) = 2$$

and likewise

$$h(1) = 1.$$

This tells us that $h(x)$ on $[1,2]$ must be defined as $h(x) = x$. Whereas on $[2,3]$, we have

$$h(3) = 3 \cdot h(2) = 3 \cdot 2 \cdot h(1) = 3! = 3$$

and

$$h(2) = 2 \cdot h(1) = 2 \cdot 1 \cdot h(0) = 2! = 2.$$

which tells us that $h(x)$ on $[2,3]$ must be defined as $h(x) = x(x-1)$. On $[n, n+1]$, observe that

$$h(n) = n \cdot h(n-1) = n \cdot (n-1) \cdot h(n-2) = n!$$

and

$$h(n+1) = (n+1) \cdot n \cdot (n-1) \cdot (n-2) = (n+1)!.$$

This tells us that $h(x)$ on $[n, n+1]$ will be defined as

$$h(x) = \prod_{i=1}^{n-1} x - i$$

which can be proven using induction. ■

(b) Now do the same for $[-1,0], [-2,-1]$, and $[-n, -n+1]$.

Proof. ■

(c) Sketch h over the domain $[-4,4]$.

Proof. To do. ■

Our function above $h(x)$ satisfies $h(n) = n!$ and it is at least continuous for $x \geq 0$. However, we still run into the problem where our piecewise function contains non-differentiable corners. We conclude that from the exercise above that $x!$ will have the same asymptotic behavior as h at negative integers x. Hence, it won't be defined on $x \in \mathbb{Z}^-$.

3.4.4 Improper Riemann Integrals

Our goal in this section is to provide a rigorous foundation for the formula

$$\int_0^\infty e^{-t}\, dt.$$

This is know in our regular Calculus classes as the *improper Riemann integral* which is defined by taking the limit of "proper" integrals over unbounded regions such as $[0, \infty)$.

Definition 35. Assume f is defined on $[a, \infty)$ and integrable on every interval of the form $[a,b]$. Then define $\int_a^\infty f$ to be

$$\lim_{b \to \infty} \int_a^b f,$$

provided the limit exists. In this case, we say the improper integral $\int_a^\infty f$ *converges*.

Exercise 8.4.9

(a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\varepsilon > 0$, there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \varepsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)

Proof. For the forwards direction, suppose that the improper integral $\int_a^\infty f$ converges. Let $\varepsilon > 0$. By assumption, we can find an $M > a$ such that whenever $d > c \geq M$, it follows that

$$\left| \int_a^d f - L \right| < \frac{\varepsilon}{2} \quad \text{whenever } d \geq M > a,$$
$$\left| \int_a^c f - L \right| < \frac{\varepsilon}{2} \quad \text{whenever } c \geq M > a.$$

Observe that

$$\int_c^d f = \int_c^a f + \int_a^d f = \int_a^d f - \int_a^c f.$$

Then we have

$$\begin{aligned}\left| \int_c^d f \right| &= \left| \int_a^d f - \int_a^c f \right| \\ &\leq \left| \int_a^d f - L \right| + \left| L - \int_a^c f \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Now assume the converse. Let $\varepsilon > 0$. We want to show that

$$\lim_{n\to\infty} \int_a^{a+n} f = L.$$

By assumption, there exists a natural number $N > a$ such that whenever $a + n > n \geq N$, we have

$$\begin{aligned}\left| \int_a^{a+n} f - L \right| &= \left| \left(\int_a^n f + \int_n^{a+n} f \right) - L \right| \\ &= \left| \left(\int_a^n f - L \right) + \int_n^{a+n} f \right| \\ &\leq \left| \int_a^n f - L \right| + \left| \int_n^{a+n} f \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

■

(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.

Proof. Let $\varepsilon > 0$. Our goal is to show that there exists an $M > a$ such that whenever $d > c \geq M$, we have

$$\left| \int_c^d f \right| < \varepsilon.$$

Since $0 \leq f \leq g$ and $\int_0^\infty g$ converges, there exists an $M > a$ such that whenever $d > d \geq M$, we have that

$$\Big| \int_c^d f \Big| \leq \Big| \int_c^d g \Big| < \varepsilon.$$

Hence, we must have that $\int_0^\infty f$ converges as well. ■

(c) Part (a) is a Cauchy criterion, and part (b) is a comparison test. State and prove an absolute convergence test for improper integrals.

Definition 36. Absolute Convergence Test For Improper Integrals If $\int_0^\infty |f|$ converges, then $\int_0^\infty f$ converges as well.

Proof. Suppose $\int_0^\infty |f|$ converges. Then observe that

$$\Big| \int_0^\infty f \Big| \leq \int_0^\infty |f|.$$

Since the right-hand side converges, we know that the left-hand side of the inequality also converges via the Comparison test. Hence, $\int_0^\infty f$ converges. ■

Exercise 8.4.10

(a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t}\,dt = 1.$$

Proof. Using part (i) of FTC implies

$$\begin{aligned}
\int_0^\infty e^{-t}\,dt &= \lim_{b\to\infty} \int_0^b e^{-t}\,dt \\
&= \lim_{b\to\infty} \Big[-e^{-b} + e^0 \Big] \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

■

(b) Show

$$\int_0^\infty e^{-t}\,dt, \text{ for all } \alpha > 0. \tag{3}$$

Proof. Let $\alpha > 0$. Using part (i) of FTC, we have

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\,dt &= \lim_{b\to\infty} \int_0^b e^{-\alpha t}\,dt \\
&= \lim_{b\to\infty} \Big[\frac{-e^{-bt}}{b} + \frac{e^0}{\alpha} \Big] \\
&= 0 + \frac{1}{\alpha} \\
&= \frac{1}{\alpha}.
\end{aligned}$$

■

Let us now consider the left side of (3). Differentiating the left hand side, we certainly get the following

$$\left[\frac{1}{\alpha}\right]' = \frac{-1}{\alpha^2}.$$

On the right hand side of (3), however, it is not so obvious whether or not we can "distribute" differentiation inside the integral of (3). Let us pretend that we can so we have

$$[e^{-\alpha t}]' = e^{-\alpha t} \cdot (-t).$$

Now let us actually find out if our conjecture that

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\, dt.$$

Exercise 8.4.11

(a) Evaluate $\int_0^b te^{-\alpha t}\, dt$ using the integration-by-parts formula from Exercise 7.5.6. The result will be an expression in α and b.

Proof. Using the integration-by-parts formula, we get that

$$\begin{aligned}
\int_0^b te^{-\alpha t}\, dt &= \left[\frac{-t}{\alpha}e^{-\alpha t}\right]_0^b + \frac{1}{\alpha}\int_0^b e^{-\alpha t}\, dt \\
&= \left[\frac{-be^{-\alpha b}}{\alpha}\right] + \frac{1}{\alpha}\left[\frac{-1}{\alpha}e^{-\alpha t}\right]_0^b \\
&= \left[\frac{-be^{-\alpha b}}{\alpha}\right] + \frac{1}{\alpha}\left[\frac{-1}{\alpha}e^{-\alpha b} + \frac{1}{\alpha}\right] \\
&= \frac{-be^{-\alpha b}}{\alpha} - \frac{e^{-\alpha b}}{\alpha^2} + \frac{1}{\alpha^2}
\end{aligned}$$

■

(b) Now compute $\int_0^\infty te^{-\alpha t}\, dt$ and verify equation (4).

Proof. Letting $b \to \infty$ in the result in part (a), gives us

$$\int_0^\infty te^{-\alpha t}\, dt = \lim_{b\to\infty}\int_0^b te^{-\alpha t}\, dt = \frac{1}{\alpha^2}.$$

■

Since the above (4) ended up working out, we have to now create a rigorous foundation for why this works.

3.4.5 Differentiating Under the Integral

Suppose we have a function of two variables $f(x,t)$ that is defined for all $x \in [a,b]$ and $t \in [c,d]$. The domain for f can be called the *rectangle* D in $\mathbb{R}^2$.

Let's say that we have f continuous at some point (x_0, t_0) in D? To have this make more sense, observe that we have a different metric under $\mathbb{R}^2$ which contains the Euclidean distance formula

$$\|(x,t) - (x_0,t_0)\| = \sqrt{(x-x_0)^2 + (t-t_0)^2}.$$

Definition 37. A function $f : D \to \mathbb{R}$ is continuous at (x_0, t_0) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\|(x,t) - (x_0,t_0)\| < \delta$, it follows that

$$|f(x,t) - f(x_0,t_0)| < \varepsilon.$$

Exercise 8.4.12

Assume the function $f(x,t)$ is continuous on the rectangle $D = \{(x,t) : a \le x \le b, c \le t \le d\}$. Explain why the function

$$F(x) = \int_c^d f(x,t)\ dt$$

is properly defined for all $x \in [a,b]$.

Proof. All we need to do is show that F is integrable on D. To do this, we need to show that F is continuous. Let $\varepsilon > 0$ and let $x \in [a,b]$. Since $f : D \to \mathbb{R}$ is continuous at (x_0, t_0), there exists a $\delta > 0$ such that whenever $\|(x,t) - (x_0,t_0)\| < \delta$, it follows that

$$|f(x,t) - f(x_0,t_0)| < \frac{\varepsilon}{d-c}.$$

Then observe that

$$\begin{aligned}
|F(x) - F(u)| &= \left| \int_c^d f(x,t)\ dt - \int_c^d f(u,v)\ dt \right| \\
&= \left| \int_c^d f(x,t) - f(u,v)\ dt \right| \\
&\le \int_c^d |f(x,t) - f(u,v)|\ dt \\
&< \int_c^d \frac{\varepsilon}{d-c}\ dt = \varepsilon.
\end{aligned}$$

Hence, we have F must be continuous on D and therefore it must be integrable on D. Not only is F continuous but it is uniformly continuous since D is a compact set. ■

There is a direct analogue of continuous functions on compact sets in the $\mathbb{R}^2$ setting.

Theorem 34. If $f(x,t)$ is continuous on D, then $F(x) = \int_c^d f(x,t)\ dt$ is uniformly continuous on $[a,b]$.

Exercise 8.4.13

Prove Theorem 8.4.5.

Proof. Apply the same argument as in exercise 8.4.12. ■

Now let us add the assumption that for every fixed value $t \in [c,d]$, we end up with a differentiable function $f(x,t)$ such that the limit

$$f_x(x,t) = \lim_{z \to x} \frac{f(z,t) - f(x,t)}{z-x}$$

exists for all $(x,t) \in D$. Furthermore, suppose $f_x(x,t)$ is continuous.

Theorem 35. If $f(x,t)$ and $f_x(x,t)$ are continuous on D, then the function $F(x) = \int_c^d f(x,t)\ dt$ is differentiable and

$$F'(x) = \int_c^d f_x(x,t)\ dt.$$

Proof. Fix $x \in [a,b]$ and let $\varepsilon > 0$ be arbitrary. Our goal is to find a $\delta > 0$ such that

$$\left| \frac{F(z) - F(x)}{z-x} - \int_c^d f_x(x,t)\ dt \right| < \varepsilon. \tag{5}$$

whenever $0 < |z-x| < \delta$. ■

Exercise 8.4.14

Finish the proof of Theorem 8.4.6.

Proof. Suppose $f(x,t)$ and $f_x(x,t)$ are continuous on D. Let $\varepsilon > 0$. Observe that

$$\begin{aligned}\frac{F(z)-F(x)}{z-x} &= \frac{1}{z-x}\Big[\int_c^d f(z,t)\ dt - \int_c^d f(x,t)\ dt\Big]\\ &= \int_c^d \frac{f(z,t)-f(x,t)}{z-x}\ dt.\end{aligned}$$

Since $f(x,t)$ is differentiable, there exists $\alpha \in (a,b)$ such that

$$f_x(\alpha,t) = \frac{f(z,t)-f(x,t)}{z-x}$$

by MVT. Since f_x is continuous on D and D is a compact set, we know that f_x is uniformly continuous. Hence, there exists $\delta_1 > 0$ such that whenever $\|(x,t)-(z,t)\| < \delta_1$ where

$$|f_x(\alpha,t) - f_x(x,t)| < \frac{\varepsilon}{d-c}.$$

By assumption, there exists a $\delta_2 > 0$ and then assume $0 < |z-x| < \delta_2$. Then choose $\delta = \min\{\delta_1,\delta_2\}$ such that whenever $0 < |z-x| < \delta$, we have that

$$\begin{aligned}\Big|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)\ dt\Big| &= \Big|\int_c^d \frac{f(z,t)-f(x,t)}{z-x}\ dt - \int_c^d f_x(x,t)\ dt\Big|\\ &= \Big|\int_c^d f_x(\alpha,t)\ dt - \int_c^d f_x(x,t)\ dt\Big|\\ &= \Big|\int_c^d f_x(\alpha,t)\ - f_x(x,t)\ dt\Big|\\ &\le \int_c^d |f_x(\alpha,t) - f_x(x,t)|\ dt\\ &< \frac{\varepsilon}{d-c}\int_c^d\ dt = \varepsilon.\end{aligned}$$

We conclude that $F(x)$ is differentiable and that

$$F'(x) = \int_c^d f_x(x,t)\ dt.$$

■

3.4.6 Improper Integrals, Revisited

We see that Theorem 8.4.2 forms a rigorous foundation for what it means to differentiate under the integral sign. However, we are partly there since we are still trying to form rigorous theory for (3) where the integral is improper. Observe that (3) is a function $f(x,t)$ where t is defined on an unbounded interval where $t \in [c,\infty)$.

Suppose we fix $x \in A$ a subset of $\mathbb{R}$. Then for this choice of x, we have that

$$F(x) = \int_c^\infty f(x,t)\ dt = \lim_{n\to\infty}\int_c^d f(x,t)\ dt, \tag{6}$$

provided that the limit exists.

Note that (6) is a *pointwise* statement; that is, given an arbitrary $x \in A$ and $\varepsilon > 0$, there exists an $M(x)$ such that

$$\Big|F(x) - \int_c^d f(x,t)\ dt\Big| < \varepsilon$$

whenever $d \ge M$.

Definition 38. Given $f(x,t)$ defined on $D = \{(x,t) : x \in A, c \le t\}$, assume $F(x) = \int_c^\infty f(x,t)\ dt$ exists for all $x \in A$. We say the improper integral *converges uniformly* to $F(x)$ on A if for all $\varepsilon > 0$, there exists $M > c$ such that
$$\left|F(x) - \int_c^d f(x,t)\ dt\right| < \varepsilon$$

Exercise 8.4.15

(a) Show that the improper integral $\int_0^\infty e^{-xt}\ dt$ converges uniformly to $1/x$ on the set $[1/2, \infty)$.

Proof. Let $x \in [1/2, \infty)$. Let $\varepsilon > 0$. Taking the integral of $\int_{1/2}^d e^{-xt}\ dt$ leads us to
$$\int_{1/2}^d e^{-xt}\ dt = \frac{1}{x}(e^{-dx} - e^{(-1/2)x}).$$
Since $\lim\limits_{x\to\infty} 1/x = 0$, we can find a $M > 1/2$ such that
$$\left|\frac{1}{x} - 0\right| < \frac{\varepsilon}{2}$$
whenever $x \ge M$. Furthermore, we know that $f(x,t) = e^{-xt}$ is a uniformly continuous function on $[1/2, \infty)$. Hence, there we know that there exists a $\delta > 0$ such that whenever $\|-1/2 - d\| < \delta$, we have
$$|f(-1/2, t) - f(d,t)| < \frac{\varepsilon}{4}.$$
Using the same choice of $M > 1/2$, we have that
$$\begin{aligned}
\left|\frac{1}{x} - \int_{1/2}^d e^{-xt}\ dt\right| &= \left|\frac{1}{x} - \frac{1}{x}\left(e^{-dx} - e^{(-1/2)x}\right)\right| \\
&= \left|\frac{1}{x} + \frac{1}{x}\left(e^{(-1/2)x} - e^{-dx}\right)\right| \\
&\le \left|\frac{1}{x}\right| + \frac{1}{x}|e^{(-1/2)x} - e^{-dx}| \\
&\le \left|\frac{1}{x} - 0\right| + 2|e^{(-1/2)x} - e^{-dx}|. \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, we have that
$$\int_{1/2}^d e^{-xt}\ dt \Rightarrow \frac{1}{x}$$
uniformly. ■

(b) Is the convergence uniform on $(0, \infty)$?

Proof. No, the convergence is not uniform. To see why, observe that
$$\left|\frac{1}{x} - \int_0^d e^{-xt}\ dt\right| = \left|\frac{1}{x} - \frac{1}{x}(e^{-dx} - 1)\right|.$$

We can see that the convergence of $1/x \to 0$ depends on our choice of x because clearly we cannot bound $1/x$ by 0. Hence, the convergence of $\int_0^\infty e^{-xt}\ dt$ on $(0, \infty)$ is pointwise. ■

Exercise 8.4.16

Prove the following analogue of the Weierstrass M-Test for improper integrals: If $f(x,t)$ satisfies $|f(x,t)| \leq g(t)$ and $\int_a^\infty g(t)\ dt$ converges, then $\int_a^\infty f(x,t)\ dt$ converges uniformly on A.

Proof. Our goal is to use the Absolute Comparison test for improper integrals and the comparison test for improper integrals found in Exercise 8.4.9 (b) to show that $\int_0^\infty f(x,t)\ dt$ converges uniformly. Observe that

$$\left|\int_0^\infty f(x,t)\ dt\right| \leq \int_0^\infty |f(x,t)|\ dt \leq \int_0^\infty g(t)\ dt.$$

Since $\int_0^\infty g(t)\ dt$ converges (which means that our choice of M is independent of $x \in [0,\infty)$), we know that $\left|\int_0^\infty f(x,t)\ dt\right|$ must converge absolutely. Hence, we must have $\int_0^\infty f(x,t)\ dt$ converge uniformly by the comparison test. ■

An immediate consequence of Definition 8.4.7 is that the uniform convergence of an improper integral implies that the sequence of functions defined by

$$F_n(x) = \int_c^{c+n} f(x,t)\ dt$$

converges uniformly to $F(x)$ on the closed interval $[a,b]$.

Theorem 36. If $f(x,t)$ continuous on $D = \{(x,t) : a \leq x \leq b, c \leq t\}$, then

$$F(x) = \int_c^\infty f(x,t)\ dt$$

is uniformly continuous on $[a,b]$, provided the integral converges uniformly.

Exercise 8.4.17

Prove Theorem 8.4.8.

Proof. Let $\varepsilon > 0$ and let $x, y \in (a,b)$. Since $f(x,t)$ is continuous on D and D is a compact set, we know that $f(x,t)$ must be uniformly continuous on D. Hence, there exists a $\delta > 0$ such that whenever $\|(x,t) - (y,t)\| < \delta$, we must have

$$|f(x,t) - f(y,t)| < \frac{\varepsilon}{d-c}.$$

Then using the same choice of $\delta > 0$, we can say

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_c^d f(x,t)\ dt - \int_c^d f(y,t)\ dt\right| \\
&= \left|\int_c^d f(x,t) - f(y,t)\ dt\right| \\
&\leq \int_c^d |f(x,t) - f(y,t)|\ dt \\
&< \frac{\varepsilon}{d-c}\int_c^d\ dt = \varepsilon.
\end{aligned}$$

Hence, F is a uniformly continuous function. ■

Theorem 37. Assume the function $f(x,t)$ is continuous on $D = \{(x,t) : a \leq x \leq b, c \leq t\}$ and $F(x) = \int_c^\infty f(x,t)\ dt$ exists for each $x \in [a,b]$. If the derivative function $f_x(x,t)$ exists and is continuous, then

$$F'(x) = \int_c^\infty f_x(x,t)\ dt,$$

provided the integral in (7) converges uniformly.

Exercise 8.4.18

Prove Theorem 8.4.9.

Proof. Let $\varepsilon > 0$. We can use MVT (since F is differentiable) to write

$$\begin{aligned} F'(x) &= \frac{F(z) - f(y)}{z - y} \\ &= \frac{1}{z-y}\int_c^d f(z,t) - f(y,t)\ dt \\ &= \int_c^d \frac{f(z,t) - f(y,t)}{z-y}\ dt \end{aligned}$$

for some $x \in (z, y)$. Since the derivative $f_x(x,t)$ also exists, we can use MVT again to find an $\alpha \in (z, y)$ such that

$$f_x(\alpha, t) = \frac{f(z,t) - f(y,t)}{z-y}.$$

We know that f_x is continuous. Hence, there exists a $\delta > 0$ such that whenever $\|(\alpha, t) - (x,t)\| < \delta$, we have

$$|f_x(\alpha,t) - f_x(x,t)| < \frac{\varepsilon}{d-c}.$$

Since the improper integral $\int_c^d f_x(x,t)\ dt$ converges uniformly, we can find an $M > c$,

$$\begin{aligned} \left|F'(x) - \int_c^d f_x(x,t)\ dt\right| &= \left|\int_c^d f_x(\alpha,t)\ dt - \int_c^d f_x(x,t)\ dt\right| \\ &= \left|\int_c^d f_x(\alpha,t) - f_x(x,t)\ dt\right| \\ &\leq \int_c^d |f_x(\alpha,t) - f_x(x,t)|\ dt \\ &< \frac{\varepsilon}{d-c}\int_c^d dt = \varepsilon. \end{aligned}$$

Hence, we conclude that

$$F'(x) = \int_c^\infty f_x(x,t)\ dt.$$

■

3.4.7 The Factorial Function

We can now give a more rigorous justification for

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\ dt, \text{ for all } \alpha > 0. \tag{8}$$

Exercise 8.4.19

(a) Although we verified it directly, show how to use the theorems in this section to give a second justification for the formula

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\ dt, \text{ for all } \alpha > 0$$

Proof. Note that $f(\alpha, t) = e^{-\alpha t}$ is continuous on D implies that

$$F(\alpha) = \frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\ dt \tag{1}$$

exists and is also continuous on D. If we differentiate with respect to α, we get that $f_\alpha(\alpha, t)$ is also continuous on D. Thus, we can say that (1) is differentiable and that

$$\frac{1}{\alpha^2} = F'(\alpha) = \int_c^d f_\alpha(\alpha, t)\ dt$$

is defined by Theorem 8.4.9. ■

(b) Now derive the formula

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\ dt, \text{ for all } \alpha > 0.$$

If we set $\alpha = 1$ in equation (8) we get

$$n! = \int_0^\infty t^n e^{-t}\ dt.$$

Proof. We proceed using induction. Observe that for the $n = 1$ case, we know that

$$\int_0^\infty te^{-\alpha t}\ dt = \frac{1}{\alpha^2}$$

by Exercise 8.4.11. Now suppose

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\ dt$$

for $n \geq 1$. We want to show that this holds for the $n+1$ case; that is, we want to show

$$\frac{(n+1)!}{\alpha^{n+2}} = \int_0^\infty t^{n+1} e^{-\alpha t}\ dt.$$

Observe by the integration-by-parts formula that

$$\begin{aligned}\int_0^d t^{n+1} e^{-\alpha t}\ dt &= \left[\frac{-t^{n+1}}{\alpha} e^{-\alpha t}\right]_0^d + \frac{(n+1)}{\alpha}\int_0^d t^n e^{-\alpha t}\ dt \\ &= \frac{-d^{n+1}}{\alpha} e^{-\alpha d} + \frac{(n+1)}{\alpha}\int_0^d t^n e^{-\alpha t}\ dt.\end{aligned}$$

We see that as we take the limit as $d \to \infty$, we have

$$\lim_{d\to\infty} \frac{-d^{n+1}}{\alpha} e^{-\alpha d} = 0.$$

Now using our induction hypothesis, we have

$$\begin{aligned}\lim_{d\to\infty}\int_0^d t^{n+1} e^{-\alpha t}\ dt &= \frac{(n+1)}{\alpha}\lim_{d\to\infty}\int_0^d t^n e^{-\alpha t}\ dt. \\ &= \frac{(n+1)}{\alpha}\int_0^\infty t^n e^{-\alpha t}\ dt \\ &= \frac{(n+1)}{\alpha}\cdot\frac{n!}{\alpha^{n+1}} \\ &= \frac{(n+1)!}{\alpha^{n+2}}.\end{aligned}$$

Hence, we conclude that

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\ dt.$$

■

Now we are well on our way to extending the $n!$ on the right-hand and left-hand side of the above equation to $x \in \mathbb{R}$.

Definition 39 (Factorial Function)**.** For $x \geq 0$, define the *factorial function*

$$x! = \int_0^\infty t^x e^{-t}\, dt.$$

Exercise 8.4.20

(a) Show that $x!$ is an infinitely differentiable function on $(0, \infty)$ and produce a formula for the n^{th} derivative. In particular show that $(x!)" > 0$.

Proof. Note that $f(x,t) = t^x e^{-t}$ is continuous and so is $f_x(x,t) = \log(t)t^x e^{-t}$. Observe that we can differentiate t^x as many times as we want. Hence, we have that

$$x! = \int_0^\infty t^x e^{-t}\, dt \tag{1}$$

is infinitely differentiable by Theorem 8.4.6. Differentiating (1), we have that

$$\begin{aligned} \frac{d}{dx}[x!] &= \frac{d}{dx}\Big[\int_0^\infty t^x e^{-t}\, dt\Big] \\ &= \int_0^\infty \frac{\partial}{\partial x}[t^x e^{-t}]\, dt \\ &= \int_0^\infty \log(t) t^x e^{-t}\, dt \end{aligned}$$

It can be shown, using induction, that for every $n \geq 1$, that

$$\frac{d^n}{dx^n}[x!] = \int_0^\infty \log^n(t) t^x e^{-t}\, dt.$$

■

(b) Use the integration-by-parts formula employed earlier to show that $x!$ satisfies the functional equation

$$(x+1)! = (x+1)x!.$$

Proof. Using the integration-by-parts formula and the definition of $x!$, we can write

$$\begin{aligned} (x+1)! &= \int_0^\infty t^{x+1} e^{-t}\, dt \\ &= \lim_{d\to\infty}\Big[-t^{x+1}e^{-t}\Big]_0^d + (x+1)\int_0^\infty t^x e^{-t}\, dt \\ &= 0 + (x+1)x! \\ &= (x+1)x!. \end{aligned}$$

Hence, we have

$$(x+1)! = (x+1)x!.$$

■

Theorem 38 (Bohr-Mollerup Theorem)**.** There is a unique positive function f defined on $x \geq 0$ satisfying

(i) $f(0) = 1$.

(ii) $f(x+1) = (x+1)f(x)$, and

(iii) $\log(f(x))$ is convex.

Since $x!$ satisfies properties (i), (ii), and (iii), it follows that $f(x) = x!$.

Proof. Geometrically, if $[a, b]$ and $[a', b']$ are two intervals in the domain of a convex function ϕ, and $a \leq a'$ and $b \leq b'$, then the slopes the function ϕ would satisfy the following inequality

$$\frac{\phi(b) - \phi(a)}{b - a} \leq \frac{\phi(b') - \phi(a')}{b' - a'}.$$

Since f satisfies properties (i) and (ii) we know $f(n) = n!$ for all $n \in \mathbb{N}$. Now let us fix $n \in \mathbb{N}$ and $x \in (0, 1]$. ∎

Exercise 8.4.21

(a) Use the convexity of $\log(f(x))$ and the three intervals $[n-1, n]$, $[n, n+x]$, and $[n, n+1]$ to show

$$x\log(n) \leq \log(f(n+x)) - \log(n!) \leq x\log(n+1).$$

Proof. On the interval $[n-1, n]$ and the fact that $f(n) = n!$, we have

$$\begin{aligned}\log(f(n)) - \log(f(n-1)) &= \log(n!) - \log(n-1!)\\ &= \log(n(n-1)!) - \log(n-1)!\\ &= \log(n) + \log(n-1)! - \log(n-1)!\\ &= \log(n).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\log(f(n+1)) - \log(f(n)) &= \log(n+1)! - \log(n)!\\ &= \log((n+1)!n) - \log(n)\\ &= \log(n+1) + \log(n) - \log(n)\\ &= \log(n+1).\end{aligned}$$

Then using the convexity of $\log(f(x))$, we have

$$\log(n) \leq \frac{\log(f(n+x)) - \log(n!)}{x} \leq \log(n+1)$$

which implies

$$x\log(n) \leq \log(f(n+x)) - \log(n!) \leq x\log(n+1).$$

∎

(b) Show $\log(f(n+x)) = \log(f(x)) + \log((x+1)(x+2)\dots(x+n))$.

Proof. We can use induction to show the statement above. Let our base case be $n = 1$. Then observe that

$$f(x+1) = f(x)(x+1)$$

which implies that

$$\begin{aligned}\log(f(x+1)) &= \log(f(x)(x+1))\\ &= \log(f(x)) + \log((x+1)).\end{aligned}$$

Now assume the statement holds for $n \geq 1$. For the $n+1$ case, observe that

$$\begin{aligned}\log(f(x+(n+1))) &= \log(f((x+n)+1))\\ &= \log(f(x+n)((x+n)+1))\\ &= \log(f(x)) + \log((x+1)(x+2)\dots(x+n)(x+(n+1))).\end{aligned}$$

■

(c) Now establish that

$$0 \leq \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \leq x\log\left(1+\frac{1}{n}\right).$$

Proof. Using our result from part (a), we can subtract $x\log(n)$ on both sides to get

$$0 \leq \log(f(n+x)) - \log(n!) - x\log(n) \leq x\log(n+1) - x\log(n). \tag{1}$$

Now observe the middle of the inequality above. Using our logarithm properties that we derived from earlier sections and using the result from part (b), we get that

$$\begin{aligned}
\log(f(n+x)) - \log(n!) - x\log(n) &= \log(f(x)) + \log((x+1)(x+2)\dots(x+n)) \\
&\quad - \log(n^x n!) \\
&= \log(f(x)) - \Big(\log(n^x n!) \\
&\quad - \log((x+1)(x+2)\dots(x+n))\Big) \\
&= \log(f(x)) \\
&\quad - \log\left(\frac{n^x n!}{(x+1)(x+2)\dots(x+n)}\right).
\end{aligned}$$

Now, focusing on the right side of the inequality in (1), we have

$$\begin{aligned}
x\log(n+1) - x\log(n) &= x\Big(\log(n+1) - \log(n)\Big) \\
&= x\log\left(\frac{n+1}{n}\right) \\
&= x\log\left(1+\frac{1}{n}\right).
\end{aligned}$$

Hence, we have reached our desired result that

$$0 \leq \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \leq x\log\left(1+\frac{1}{n}\right).$$

■

(d) Conclude that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\dots(x+n)}, \text{ for all } x \in (0,1].$$

Proof. Observe that the right hand side of the inequality found in part (c) converges; that is,

$$\lim_{n\to\infty} x\log\left(1+\frac{1}{n}\right) = 0.$$

Now let $\varepsilon > 0$. Since the limit above converges, we know that we can find an $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in (0,1]$, we have that

$$\left|\log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\dots(x+n)}\right)\right| < \varepsilon.$$

This implies that

$$\log(f(x)) = \lim_{n\to\infty} \log\left(\frac{n^x n!}{(x+1)(x+2)\dots(x+n)}\right).$$

However, we know that the limit above holds whenever

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\dots(x+n)}.$$

■

(e) Finally, show that the conclusion in (d) holds for all $x \geq 0$.

Proof. First, we show that the limit in part (d) converges when $x = 0$ into the limit found in part (d), we get

$$\lim_{n\to\infty} \frac{n!}{1\cdot 2\cdots\ n} = \lim_{n\to\infty} \frac{n!}{n!} = 1! = f(0).$$

Now suppose $x > 0$. By parts (a) and (c), we know that

$$\frac{n^x n!}{(n+x)!}$$

is bounded as well as decreasing for all $n \geq 1$. By MCT, we must have

$$\lim_{n\to\infty} \frac{n^x n!}{(n+x)!} = f(x).$$

■

3.4.8 The Gamma Function

The function we have been rigorously developing is the gamma function which is denoted by the following notation

$$\Gamma(x) = (x-1)! = \int_0^\infty t^{x-1} e^{-t}\ dt.$$

Just as we have seen in the prior exercises, we have $\Gamma(n+1) = n!$ and $x\Gamma(x) = x(x+1)!$. This is the convention in many fields such as Number Theory, Probability Theory, Geometry and so on. If we try to extend the Gamma function to all of $\mathbb{R}$ using the functional equation $x! = x(x-1)!$, we will get asymptotes at each $x \in \mathbb{Z}^-$. In this case, we can consider the reciprocal function $1/x!$ which we can define as 0 for every $x \in \mathbb{Z}^-$.

Exercise 8.4.22

(a) Where does $g(x) = \frac{x}{x!(-x)!}$ equal zero? What other familiar function has the same set of roots?

Proof. We have $g(x) = \frac{x}{x!(-x)!}$ equals zero at every $x \in \mathbb{Z}$. A similar function that has the same roots as $g(x)$ is $f(x) = \sin(\pi x)$. ■

(b) The function e^{-x^2} provides the raw material for the all-important Gaussian bell curve from probability, where it is known that $\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$ Use this fact (and some standard integration techniques) to evaluate $(1/2)!$.

Proof. Using the definition of $x!$, we have

$$\begin{aligned}
(1/2)! &= \int_0^\infty t^{1/2}e^{-t}\,dt \\
&= \lim_{d\to\infty}\int_0^d t^{1/2}e^{-t}\,dt \\
&= \lim_{d\to\infty}\left[t^{1/2}e^{-t}\right]_0^d + \frac{1}{2}\int_0^\infty t^{-1/2}e^{-t}\,dt \\
&= \int_0^\infty \frac{1}{2}\cdot t^{-1/2}e^{-(\sqrt{t})^2}\,dt && (\text{let } u=\sqrt{t}) \\
&= \int_0^\infty e^{-u^2}\,du && (du = 1/2\cdot t^{-1/2}\,dt)
\end{aligned}$$

Hence, we have that

$$(1/2)! = \frac{\sqrt{\pi}}{2}.$$

■

(c) Now use (a) and (b) to conjecture a striking relationship between the factorial function and a well-known function from trigonometry.

Proof. The factorial function (most notably the gamma function) can be used to find values of the sine function. It can also be applied to finding values of the Riemann Zeta function found in the last section. ■

Exercise 8.4.23

As a parting shot, use the value for $(1/2)!$ and the Gauss product formula in equation (9) to derive the famous product formula for π discovered by John Wallis in the 1650's:

$$\frac{\pi}{2} = \lim_{n\to\infty}\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

Proof. In the last section, we defined

$$\prod_{i=1}^n \frac{2i}{2i-1} = \frac{2^{2n}(n!)^2}{(2n)!}$$

as well as

$$\prod_{i=1}^n \frac{2i}{2i+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Using our result from part (b) of Exercise 8.4.22 and the Gauss product formula found in equation (9), we can write

$$\begin{aligned}
\frac{\sqrt{\pi}}{2} = (1/2)! &= \int_0^\infty t^{1/2}e^{-t}\,dt \\
&= \lim_{n\to\infty}\frac{n^{1/2}n!}{(\frac{1}{2}+1)(\frac{1}{2}+2)\dots(\frac{1}{2}+n)} \\
&= \lim_{n\to\infty}\frac{n^{1/2}2^n n!}{(2n+1)!}.
\end{aligned}$$

By exercise 8.3.5, we have that

$$\sqrt{\pi} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Using this fact and the others stated at the beginning of this proof, we have that

$$\begin{aligned}
\lim_{n\to\infty}\prod_{i=1}^{n}\frac{2i\cdot 2i}{(2i-1)(2i+1)} &= \Big(\lim_{n\to\infty}\prod_{i=1}^{n}\frac{2i}{2i-1}\Big)\cdot\Big(\lim_{n\to\infty}\prod_{i=1}^{n}\frac{2i}{2i+1}\Big)\\
&= \Big(\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{n^{1/2}(2n)!}\Big)\cdot\Big(\lim_{n\to\infty}\frac{2^{2n}(n!)^2 n^{1/2}}{(2n+1)!}\Big)\\
&= \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2}\\
&= \frac{\pi}{2}.
\end{aligned}$$

■

3.5 Fourier Series

Fourier's development of a special type of series involving trigonometric functions caused a rework into what it really means to have a "limit" or to "converge" towards a specific

3.5.1 Trigonometric Series

The basic use of series representations is to express a given function $f(x)$ in terms of the sum of simpler and "nicer" functions such as polynomials. In the case of power series representations, we have the following set of component functions $\{1, x, x^2, x^3, \dots\}$ such that the following series takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

A *trigonometric series* on the other hand is an infinite series composed of component functions that are trigonometric; that is, we have

$$\{1, \cos(x), \sin(x), \sin(2x), \cos(3x), \sin(3x), \dots\}.$$

Thus, we can write a trigonometric series in the following way

$$\begin{aligned}
f(x) &= a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + a_3\cos(3x) + \cdots\\
&= a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).
\end{aligned}$$

Now we will see how this formulation can be used to solve partial differential equations. Consider the following problem proposed by d'Alembert

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

which describes the motion of a vibrating string. Suppose the solution to the partial differential equation above is $u(x,t)$ which models that displacement of the string at time $t \geq 0$. Suppose the string is attached at each end of this interval. Then the boundary conditions apply

$$u(0,t) = 0 \text{ and } u(\pi,t) = 0 \tag{2}$$

for all values of $t \geq 0$. If we let $t = 0$ (at the instant the string is released), then we can assume that

$$\frac{\partial u}{\partial t}(x,0) = 0;$$

that is, the initial velocity of the string is defined to be 0.

Exercise 8.5.1

(a) Verify that

$$u(x,t) = b_n \sin(nx)\cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbb{N}$ and $b_n \in \mathbb{R}$. What goes wrong if $n \notin \mathbb{N}$.

Proof. We first compute the partial derivatives of $u(x,t)$. Observe the first partial derivative with respect to x of $u(x,t)$ is

$$\frac{\partial u}{\partial x} = nb_n \cos(nx)\cos(nt).$$

Then differentiating with respect to the same variable leads to

$$\frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx)\cos(nt).$$

Likewise, differentiating with respect to t leads to

$$\frac{\partial u}{\partial t} = -nb_n \sin(nx)\sin(nt).$$
$$\frac{\partial^2 u}{\partial t^2} = -n^2 b_n \sin(nx)\cos(nt).$$

We can see that (1) is satisfied using our given solution $u(x,t)$. Now, using the boundary conditions found in (2) and the fact that the sine function is 0 with $n\pi$ for all $n \in \mathbb{N}$, we can see that

$$u(0,t) = b_n \sin(0)\cos(nt) = 0$$

and

$$u(\pi,t) = b_n \sin(n\pi)\cos(nt) = 0.$$

Hence, (2) is satisfied. By the same reasoning, we can see that (3) is satisfied.

If $n \notin \mathbb{N}$, then $u(x,t)$ would result in nonzero sine values rendering equations (1), (2), (3) invalid. ■

(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Proof. Since $\sin(x)$ and $\cos(x)$ are both continuous differentiable up to n times, we can use term-by-term differentiation of the finite sum of these trigonometric functions to show that, indeed, equations (1), (2), and (3) are all satisfied. ■

Observe that

$$u(x,t) = \sum_{n=1}^{N} b_n \sin(nx)\cos(nt) \tag{4}$$

solves the PDE in (1) which is called d'Alembert's *wave equation*. The solutions to (1), however, depend on how the string is originally "plucked". Suppose at $t = 0$, the string is given some initial displacement

$$f(x) = u(x,0).$$

Letting $t = 0$ in our general solution in (4), we find that

$$f(x) = \sum_{n=1}^{N} b_n \sin(nx). \tag{5}$$

Given there exists coefficients b_n for all $1 \le n \le N$ such that our solution $f(x)$ can be written in terms of the finite sum of sine functions found in (5), we can be sure that the PDE in (1) can be solved by

$u(x,t)$ given in (4). We can then ask, more generally, how can we construct solutions that are linear combinations of functions found in the set

$$\{\sin(x), \sin(2x), \sin(3x), \dots\}.$$

It turns out that we can take an *infinite* sum of (5) to model the position of $f(x)$ for any $x \in [0, \pi]$. The function $f(x)$ can be thought of as the initial temperature applied to some boundary of a heat-conducting material.

3.5.2 Periodic Functions

A more general formulation of our problem is to find suitable coefficients (a_n) and (b_n) to express a function $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \tag{6}$$

It is important to note that every component that makes up (6) is 2π periodic. That is, any function that is represented by trigonometric functions is necessarily periodic. Turning our attention on the interval $(-\pi, \pi]$, we can take a function such as $f(x) = x^2$, restrict its domain to $(-\pi, \pi]$, and then extend f periodically to all of $\mathbb{R}$ using the rule $f(x) = f(x + 2k\pi)$ for all $k \in \mathbb{Z}$

3.5.3 Types of Convergence

The way we express functions in terms of a series of trigonometric functions depends on the type of convergence. As we have been discussing all throughout this book, we have to look at the partial sums. Hence, we have

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx). \tag{7}$$

The idea of expressing $f(x)$ in terms of a trigonometric series involves finding the coefficients $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that

$$f(x) = \lim_{N \to \infty} S_N(x). \tag{8}$$

Showing that

$$\int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 \, dx \to 0$$

is a natural way to understand (8). This convergence is called the L^2 *convergence* (won't be discussed here). Another type of convergence that we will look at in this section is the *Cesaro mean convergence* which involves taking the *averages* of the partial sums and showing that they converge uniformly to $f(x)$.

3.5.4 Fourier Coefficients

Before discussing the Cesaro mean convergence, we will look at a few basic calculus facts.

Exercise 8.5.2

Using trigonometric identities when necessary, verify the following integrals.

(a) For all $n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos(nx) \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx) \, dx = 0$$

> **Proof.** We will begin by showing the first integral. Hence, using the fact that $\sin(x)$ is odd

and the fact that $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$, we can say

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(nx)\ dx &= \left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{n}[\sin(n\pi) - \sin(-n\pi)] \\
&= \frac{2}{n}\sin(n\pi) \\
&= 0.
\end{aligned}$$

Now we show the second integral. Using the fact that $\cos(x)$ is an even function, we can write

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(nx)\ dx &= \left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{n}[\cos(-n\pi) - \cos(n\pi)] \\
&= \frac{1}{n}[\cos(n\pi) - \cos(n\pi)] \\
&= 0.
\end{aligned}$$

■

(b) For all $n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\ dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\ dx = \pi.$$

Proof. Using the trigonometric identity $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$, we can write

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2(2xn)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos(2xn)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(2xn)\ dx
\end{aligned}$$

The first integral evaluates to

$$\frac{1}{2}\int_{-\pi}^{\pi} dx = \frac{1}{2}[x]_{-\pi}^{\pi} = \pi$$

Then, using the fact that sine is an odd function and the fact that $\sin(2\pi n) = 0$ for all $n \in \mathbb{N}$, the second integral evaluates to

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(2xn)\ dx &= \frac{1}{2n}[\sin(2xn)]_{-\pi}^{\pi} \\
&= \frac{1}{2n}[\sin(2\pi n) - \sin(-2\pi n)] \\
&= \frac{1}{2n}[\sin(2\pi)] && (\sin(-2\pi n) = -\sin(2\pi n)) \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos^2(x)\ dx = \pi.$$

Similarly, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin^2(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [1-\cos(2xn)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(2xn)\ dx \\
&= \frac{1}{2}\cdot 2\pi - 0 \\
&= \pi.
\end{aligned}$$

■

(c) For all $m, n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx = 0.$$

Proof. Let $m, n \in \mathbb{N}$. Suppose $m = n$. Then using a change of variable (letting $u = \sin(mx)$), we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\sin(mx)\ dx &= \frac{1}{m}\int_{-\pi}^{\pi} u\ du \\
&= \frac{1}{2m}[\sin^2(\pi x)]_{-\pi}^{\pi} \\
&= \frac{1}{2m}\sin^2(\pi m) + \frac{1}{2m}\sin^2(\pi m) \\
&= \frac{1}{m}\sin^2(\pi m) \\
&= 0.
\end{aligned}$$

If $m \neq n$, then using the trigonometric identity

$$\cos(mx)\sin(nx) = \frac{1}{2}[\sin((m+n)x) - \sin((m-n)x)]$$

we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)x) - \sin((m-n)x)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((m-n)x)\ dx \\
&= \left.\frac{\cos((m-n)x)}{2(m-n)}\right|_{-\pi}^{\pi} + \left.\frac{\cos((m+n)x)}{2(m+n)}\right|_{-\pi}^{\pi}
\end{aligned}$$

Using the fact that cosine is an even function we can evaluate the first term which leads to

$$\begin{aligned}
\left.\frac{\cos((m-n)x)}{2(m-n)}\right|_{-\pi}^{\pi} &= \frac{1}{2(m-n)}[\cos((m-n)\pi) - \cos(-(m-n)\pi)] \\
&= \frac{1}{(m-n)}[\cos((m-n)\pi) - \cos((m-n)\pi)] \\
&= 0.
\end{aligned}$$

Using the same reasoning, the second term can be written as

$$\begin{aligned}\frac{\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos(-(m+n)\pi)] \\ &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos((m+n)\pi)] \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx = 0.$$

Our goal now is to show

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx = 0$$

for all $m \neq n$. Let $m \neq n$. Using the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)],$$

the first integral evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos((m+n)x) + \cos((m-n)x)]\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\ dx \\ &= \frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi}.\end{aligned}$$

Using the fact that the sine function is odd and the fact that $\sin((m+n)\pi) = 0$ for all $m, n \in \mathbb{N}$, we can write

$$\begin{aligned}\frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)} \cdot 2\sin((m+n)\pi) \\ &= \frac{\sin((m+n)\pi)}{(m+n)} \\ &= 0\end{aligned}$$

Likewise, the second term evaluates to

$$\begin{aligned}\frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)} \cdot 2\sin((m-n)\pi) \\ &= \frac{\sin((m-n)\pi)}{(m-n)} \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0.$$

Now we show the second integral. Similarly, we use the trigonometric identity

$$\sin(a)\sin(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)].$$

Then observe that

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)x) + \sin((m-n)x)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((m-n)x)\ dx \\
&= \frac{-\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi}.
\end{aligned}$$

Starting with the first term, we can utilize the fact that $\cos(x)$ is an even function to get

$$\begin{aligned}
\frac{-\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos((m+n)\pi)] \\
&= 0.
\end{aligned}$$

Likewise, the second integral evaluates to

$$\begin{aligned}
\frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)}[\cos((m-n)\pi) - \cos((m-n)\pi)] \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \sin(mx)\sin(mx)\ dx = 0.$$

■

We can interpret the integrals in exercise 8.5.2 as dot products allowing us to see that all the functions from the set

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \dots\}$$

are all *orthogonal* to each other. This, in turn, forms a *basis* for a large class of functions.

Our ultimate goal is to find coefficients (a_n) and (b_n) in equation (6). We can assume that we are in possession of some expression described in (6) that will allow us to find formulas for both coefficients. This is similar to how we found the coefficients of a taylor series back in chapter 6, but instead of differentiating repeatedly, we aim to integrate.

In order to compute a_0, we need to integrate each side of (6) on the closed interval $[-\pi, \pi]$ to get

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\ dx &= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\right]\ dx \\
&= \int_{-\pi}^{\pi} a_0\ dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx) + b_n \sin(mx)]\ dx \\
&= a_0 2\pi + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\ dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(mx)\ dx \\
&= a_0 2\pi + 0 + 0 \\
&= a_0 2\pi.
\end{aligned}$$

Solving for a_0 gives us

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx. \tag{9}$$

One might look at the interchanging of the integral and summation used in the derivation with suspicion. The point, as of now, is not to justify each step of the derivation, but rather work backwards so that we may find a representation to prove. The rigor will follow later.

Now suppose we fix $m \geq 1$. To compute a_m, we first multiply (6) by $\cos(mx)$ on both sides and integrate over the closed interval $[-\pi, \pi]$.

Exercise 8.5.3

Derive the formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx \text{ and } b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx$$

for all $m \geq 1$.

Proof. Let $m \geq 1$. Multiplying by $\cos(mx)$ on both sides of (6) gives us

$$f(x)\cos(mx) = a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx).$$

Now, taking the integral on the closed interval $[-\pi, \pi]$ on (6) gives us

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx = a_0 &\int_{-\pi}^{\pi}\cos(mx)\ dx \\ &+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}[a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx)]\ dx.\end{aligned}$$

Using exercise 8.5.2, we know that all the integrals in the above equation evaluate to 0 for all $n \in \mathbb{N}$, except when $n = m$ for which the right side of the equation evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx &= \int_{-\pi}^{\pi} a_m\cos^2(mx)\ dx \\ &= a_m \cdot \pi\end{aligned}$$

Solving for a_m gives us our result

$$a_m = \frac{1}{m}\int_{-\pi}^{\pi}\cos^2(mx)\ dx.$$

Using the same reasoning to show the first integral, we have that

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx = 0$$

for all $n \in \mathbb{N}$ except when $n = m$ for which the integral above evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx = a_0 &\int_{-\pi}^{\pi}\sin(mx)\ dx \\ &+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}[a_n\cos(nx)\sin(mx) + b_n\sin(nx)\sin(mx)]\ dx \\ &= b_m\int_{-\pi}^{\pi}\sin^2(mx)\ dx \\ &= b_m \cdot \pi.\end{aligned}$$

Solving for b_m gives us our desired result

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx.$$

■

Let's see how these coefficients are computed with simple functions.

Let

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < \pi \\ 0 \text{ if } x = 0 \text{ or } x = \pi \\ -1 \text{ if } -\pi < x < 0. \end{cases}$$

Notice that f is an odd function which means that we can evaluate our integrals easily using the symmetry argument; that is, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = 0 \text{ and } \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx = 0$$

for all $n \geq 1$. Using the same argument, we can simplify b_n to get

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\ dx \\ &= \frac{2}{\pi}\Big(\frac{-1}{n}\cos(nx)\Big|_0^{\pi}\Big) \\ &= \frac{2}{\pi}\Big(\frac{-1}{n}\cos(n\pi) + \frac{1}{n}\Big) \\ &= \begin{cases} 4/n\pi \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases} \end{aligned}$$

Plugging in our results into (6) then gives us the following representation of $f(x)$

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)x).$$

Exercise 8.5.4

(a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is *not* uniform on any interval containing 0.

Proof. At the endpoints, $f(x)$ takes on a different value making our convergence pointwise instead of being uniform. ■

(b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?

Proof. Our goal is to produce coefficients a_0, a_m, and b_m so that we have the following representation of $f(x)$; that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

Using the formula for a_0 derived earlier and using the fact that $|x|$ is an even function, we produce

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\ dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x\ dx \\ &= \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big|_0^{\pi} \\ &= \frac{\pi}{2}. \end{aligned}$$

Using the symmetry from the absolute value function once again, we get that computing a_m

leads to

$$\begin{aligned}
a_m &= \frac{2}{\pi}\int_0^\pi x\cos(mx)\ dx \\
&= \frac{2}{\pi}\left[\frac{1}{m}x\sin(mx)\right]_0^\pi - \frac{2}{m\pi}\int_0^\pi \sin(mx)\ dx \\
&= \frac{2}{m^2\pi}\cos(mx)\Big|_0^\pi \\
&= \frac{2}{m^2\pi}[\cos(m\pi)-1] \\
&= \begin{cases} 0 & \text{if } m \text{ even} \\ -4/m^2\pi & \text{if } m \text{ odd.}\end{cases}
\end{aligned}$$

Since $\sin(mx)$ is an odd function and $|x|$ is an even function, we find that their product $|x|\sin(mx)$ is also odd. Hence, the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx$$

evaluates to zero. Hence, $b_m = 0$. Therefore, $f(x) = |x|$ can be represented by the following Fourier series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2}.$$

By looking at the coefficients, we can see that the convergence of (a_n) goes to zero and does not depend on our choice of $x \in [-\pi,\pi]$. ■

3.5.5 The Riemann-Lebesgue Lemma

Observe that by looking at the Fourier coefficients of (a_n) and (b_n) tend to zero as we take the limit as n approaches ∞. We will see why this works in our upcoming convergence proof.

If we take a look at any function $h(x)$ and $\sin(nx)$, we will notice that when we take their product $h(x)\sin(nx)$, the oscillatory nature of $\sin(nx)$ do not affect the value of h too much. This means that the value from the integral

$$\int_{-\pi}^{\pi} h(x)\sin(nx)\ dx$$

leads to a small value (because of the positive and negative oscillations of $h(x)\sin(nx)$ leads values cancelling out).

Theorem 39 (Riemann-Lebesgue Lemma). Assume $h(x)$ is continuous on $(-\pi,\pi]$. Then,

$$\int_{-\pi}^{\pi} h(x)\sin(nx)\ dx \to 0 \quad\text{and}\quad \int_{-\pi}^{\pi} h(x)\cos(nx)\ dx \to 0$$

as $n\to\infty$.

Proof. Using our assumption that h is continuous on the interval $(-\pi,\pi]$, we can periodically extend h to be continuous on all of $\mathbb{R}$. This implies that we have to assume that $\lim\limits_{n\to-\pi^+} h(x) = h(\pi)$ holds. ■

Exercise 8.5.5

Explain why h is uniformly continuous on $\mathbb{R}$.

Proof. We know that h is uniformly continuous on $\mathbb{R}$ since we can extend the continuity of h to any interval $(-n\pi, n\pi]$ for all $n\in\mathbb{Z}^+$ as well as having the implication that $\lim\limits_{x\to-n\pi^+} h(x) = h(n\pi)$. ■

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \frac{\varepsilon}{2}$. Notice that the period of $\sin(nx)$ is $2\pi/n$. Hence, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Now consider a particular interval $[a, b]$ of length $2\pi/n$ over which $\sin(nx)$ moves through one complete oscillation.

Exercise 8.5.6

Show that $\left|\int_a^b h(x)\sin(nx)\ dx\right| < \varepsilon/n$, and use this fact to complete the proof.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{\frac{1}{2n}, \frac{\varepsilon}{4mN}\} > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon/2$. Using the period of $\sin(nx)$ which is $2\pi/n$, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Then using the length of $[a, b]$ given to us above and the fact that $|\sin(nx)| \leq 1$, we can write

$$\int_a^b |h(x)||\sin(nx)|\ dx \leq \int_a^b |h(x)|\ dx.$$

Since h is integrable, we know that h must also be bounded. Hence, there exists an $M > 0$ such that $|h(y)| \leq M$ for all $y \in [a, b]$. Now observe that

$$\begin{aligned}
\left|\int_a^b h(x)\sin(nx)\ dx\right| &\leq \int_a^b |h(x)||\sin(nx)|\ dx \\
&= \int_a^b |h(x)|\ dx \\
&< \left(\frac{\varepsilon}{2} + |h(y)|\right)\int_a^b\ dx \\
&= \left(\frac{\varepsilon}{2} + |h(y)|\right)(b - a) \\
&= \frac{\varepsilon}{2}(b - a) + |h(y)|(b - a) \\
&\leq \frac{\varepsilon}{2}(b - a) + M(b - a) \\
&< \frac{\varepsilon}{2}\cdot\frac{2\pi}{n} + M\cdot\frac{2\pi}{n} \\
&< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} \\
&= \frac{\varepsilon}{n}.
\end{aligned}$$

Hence, we have that

$$\left|\int_a^b h(x)\sin(nx)\ dx\right| < \frac{\varepsilon}{n}.$$

We can use the same reasoning to show that if $|\cos(nx)| \leq 1$, then

$$\left|\int_a^b h(x)\cos(nx)\ dx\right| < \frac{\varepsilon}{n}.$$

Letting $\varepsilon > 0$, we know that $\varepsilon/n \to 0$ as $n \to \infty$. This tells us that we can find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\left|\int_a^b h(x)\sin(nx)\ dx\right| < \frac{\varepsilon}{n} \to 0$$

and

$$\left|\int_a^b h(x)\cos(nx)\ dx\right| < \frac{\varepsilon}{n} \to 0.$$

■

In attempt to make the proof above shorter, we can just say that since h is uniformly continuous on $\mathbb{R}$, we know that h must be bounded on any compact interval $[a, b] \subseteq \mathbb{R}$. Hence, there must exists an $M > 0$ such that $|h(x)| \leq M$ for any $x \in [a, b]$. Using the same set of assumptions above and letting

$\delta = \varepsilon/4Mn$, we can write

$$\begin{aligned}
\left| \int_a^b h(x)\sin(nx)\ dx \right| &\le \int_a^b |h(x)||\sin(nx)|\ dx \\
&\le M \int_a^b |\sin(nx)|\ dx \\
&\le M \int_a^b \ dx \\
&= M(b-a) \\
&< M \cdot \frac{2\pi}{n} \\
&< 2M \cdot \frac{\varepsilon}{2Mn} = \frac{\varepsilon}{n}.
\end{aligned}$$

Note that the lemma above still holds even if we take out the continuity assumption. This makes the use of Fourier series much more versatile in terms of its application to a wider variety of functions and is capable of more interesting behavior than our power series counterpart.

3.5.6 A Pointwise Convergence Proof

The formulas for the Fourier coefficients that we have produced so far require the implicit assumption that our function needs to be Riemann Integrable. This tells us that it is advantageous to have our class of integrable functions to be as large as possible in order to properly represent $f(x)$ in terms of (6). The goal now is to determine how many more assumptions do we need in order to have Fourier series converge back to $f(x)$. This involves determining which type of convergence we hope to establish.

Consider

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

once again. On the left-hand side, we would like to have $f(x)$ be

(i) bounded,

(ii) integrable,

(iii) continuous,

(iv) differentiable

(vi) f' continuous.

Suppose on the right-hand side, we list the types of convergences needed to properly represent $f(x)$ using (6). These types of convergences include

(i) pointwise convergence

(ii) uniform convergence

(iii) L^2 convergence, and

(iv) Cesaro mean convergence.

Theorem 40 (Pointwise Convergence of Fourier Series)**.** Let $f(x)$ be continuous on $(-\pi, \pi]$, and let $S_N(x)$ be the Nth partial sum of the Fourier series described in equation (7), where the coefficients (a_n) and (b_n) are given by equations (9) and (10). It follows that

$$\lim_{N\to\infty} S_N(x) = f(x)$$

pointwise at any $x \in (-\pi, \pi]$ where $f'(x)$ exists.

Before we embark on the proof, let us get a few known facts out of the way.

(i) $\cos(\alpha-\theta)=\cos(\alpha)\cos(\theta)+\sin(\alpha)\sin(\theta)$.

(ii) $\sin(\alpha+\theta)=\sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)$.

(iii) For any $\theta\neq 2n\pi$, we have

$$\frac{1}{2}+\cos(\theta)+\cos(2\theta)+\cos(3\theta)+\cdots+\cos(N\theta)=\frac{\sin((N+1/2)\theta)}{2\sin(\theta/2)}.$$

This function is called the *Dirichlet Kernel*. The proof of this fact will be omitted because it involves complex analysis; that is, it results from taking the real part of a geometric sum of complex exponentials.

(iv) Letting

$$D_N(\theta)=\begin{cases}\frac{\sin((N+1/2)\theta)}{2\sin(\theta/2)}, & \text{if } \theta\neq 2n\pi\\ 1/2+N, & \text{if } \theta=2n\pi\end{cases}$$

from (iii), we can see that

$$\int_{-\pi}^{\pi} D_N(\theta)\ d\theta=\pi.$$

(v) The Riemann-Lebesgue Lemma.

Let us fix a point $x\in(-\pi,\pi]$. Our first goal is to simplify $S_N(x)$. Since x is a fixed constant, we can write equations (9) and (10) using t as the "dummy" variable for integration. Using facts (i) and (iii), we can see that

$$\begin{aligned}
S_N(x)&=a_0+\sum_{n=1}^{N}a_n\cos(nx)+b_n\sin(nx)\\
&=\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\ dt\Big]+\sum_{n=1}^{N}\Big[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)\ dt\Big]\cos(nx)\\
&+\sum_{n=1}^{N}\Big[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)\ dx\Big]\sin(nx)\\
\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\Big[\frac{1}{2}+\sum_{n=1}^{N}\cos(nt)\cos(nx)+\sin(nt)\sin(nx)\Big]\ dt\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\Big[\frac{1}{2}+\sum_{n=1}^{N}\cos(n(t-x))\Big]\ dt\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_N(t-x)\ dx.
\end{aligned}$$

Letting $u=t-x$ and using the fact that f is extended to be $2\pi-$periodic, we can express $S_N(x)$ in terms of $D_N(u)$. By fact (iv), it follows that

$$S_N(x)=\frac{1}{\pi}\int_{-\pi-x}^{\pi-x}f(u+x)D_N(u)\ du=\frac{1}{\pi}\int_{-\pi}^{\pi}f(u+x)D_N(u)\ du.$$

Since D_N is periodic, we know that our computation of the integral is independent of the choice of interval granted that we cover one full period. Now showing that $S_N(x)\to f(x)$ is a matter of showing $|S_N(x)-f(x)|$ gets arbitrarily small when we let $N\to\infty$. Using (iv), we can express $f(x)$ as we have done with $S_N(x)$ in terms of $D_N(u)$. Hence, we have

$$f(x)=\frac{f(x)}{\pi}\int_{-\pi}^{\pi}D_N(u)\ du=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)D_N(u)\ du$$

and thus we have

$$S_N(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}(f(u+x)-f(x))D_N(u)\ du. \tag{11}$$

We can use (ii), to rewrite the Dirichlet kernel $D_N(u)$ as

$$\begin{aligned} D_N(u) &= \frac{\sin((N+1/2)u)}{2\sin(u/2)} \\ &= \frac{\sin(Nu)\cos(u/2)+\cos(Nu)\sin(u/2)}{2\sin(u/2)} \\ &= \frac{1}{2}\Big[\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)}+\cos(Nu)\Big]. \end{aligned}$$

Now observe that

$$\begin{aligned} S_N(x)-f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(u+x)-f(x))D_N(u)\ du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(u+x)-f(x))\Big[\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)}+\cos(Nu)\Big]\ du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(u+x)-f(x))\Big(\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)}\Big) \\ &+(f(u+x)-f(x))\cos(Nu)\ du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}p_x(u)\sin(Nu)\ du+\frac{1}{2\pi}\int_{-\pi}^{\pi}q_x(u)\cos(Nu)\ du, \end{aligned}$$

where in the last equality, we set

$$p_x(u)=\frac{(f(u+x)-f(x))\cos(u/2)}{\sin(u/2)} \quad \text{and} \quad q_x(u)=f(u+x)-f(x).$$

Exercise 8.5.7

(a) First, argue why the integral involving $q_x(u)$ tends to zero as $N\to\infty$.

Proof. Observe that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}q_x(u)\cos(Nu)\ du &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(h(u+x)-h(x))\cos(Nu)\ du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}h(u+x)\cos(Nu)\ du \\ &- \frac{1}{2\pi}\int_{-\pi}^{\pi}h(x)\cos(Nu)\ du. \end{aligned}$$

Since $h(x)$ continuous on $(-\pi,\pi]$, we know that both

$$\int_{-\pi}^{\pi}h(u+x)\cos(Nu)\ du\to 0$$

and

$$\int_{-\pi}^{\pi}h(x)\cos(Nu)\ du\to 0$$

as $N\to\infty$. Hence, we must also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}q_x(u)\cos(Nu)\ du\to 0$$

as $N\to\infty$

■

(b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Proof. Our goal is to show that $\frac{1}{2}p_x(u)$ is a continuous function so that we may use the Riemann-Lebesgue formula to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\ du = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}p_x(u)\sin(Nu)\ du = 0.$$

We notice that $\frac{1}{2}p_x(u)$ is clearly continuous. Let us show that it is continuous at $x = 0$. Taking advantage of the famous trigonometric limit

$$\lim_{u\to 0}\frac{u/2}{\sin(u/2)} = 1.$$

and the fact that f is differentiable at x, we can write the expression $p_x(u)$ and add a limit as $u \to 0$; that is,

$$\begin{aligned}
\frac{1}{2}\lim_{u\to 0} p_x(u) &= \frac{1}{2}\lim_{u\to 0}\frac{(f(u+x)-f(x))\cos(u/2)}{u\sin(u/2)} \\
&= \lim_{u\to 0}\frac{f(u+x)-f(u)}{2\sin(u/2)} \\
&= \lim_{u\to 0}\frac{(f(u+x)-f(x))}{2\sin(u/2)}\cdot\frac{u}{u} \\
&= \lim_{u\to 0}\frac{(f(u+x)-f(x))\cos(u/2)}{u} \\
&\cdot \lim_{u\to 0}\frac{u/2}{\sin(u/2)}\cdot\lim_{u\to 0}\cos(u/2) \\
&= f'(x)\cdot 1\cdot 1.
\end{aligned}$$

Hence, we have that $\frac{1}{2}p_x(u)$ is a continuous function. Now, using the Riemann-Lebesgue theorem, we can say that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\lim_{u\to 0} p_x(u)\right)\sin(Nu)\ du &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\lim_{u\to 0} p_x(u)\right)\sin(Nu)\ du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(Nu)\ du \\
&= 0.
\end{aligned}$$

Hence, we also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\ du = 0$$

■

This completes the proof that $S_N(x) \to f(x)$ for all x where f is differentiable. If f' exists everywhere, then we get that $S_N \to f$ pointwise. It is not too hard to see that if f' is continuous, then the convergence will be uniform. Furthermore, the smoothness of f contributes to the rate at which the terms of the Fourier series converges. Even more interesting, the more differentiable f is, the faster the rate of convergence of the partial sums S_N to f.

3.5.7 Cesaro Mean Convergence

Now, we will look at a special type of convergence called the Cesaro mean convergence.

Exercise 8.5.8

Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converges to the same limit. Give an example to show that it is possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Proof. Let $\varepsilon > 0$. Suppose $(x_n) \to x$. We can choose $N \in \mathbb{N}$ such that for any $N \in \mathbb{N}$, we have

$$|x_n - x| < \varepsilon.$$

Then using the same choice of $N \in \mathbb{N}$, we can write

$$\begin{aligned}
|y_n - x| &= \Big|\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} x\Big| \\
&= \frac{1}{n}\Big|\sum_{i=1}^{n}(x_i - x)\Big| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}|x_i - x| \\
&< \frac{\varepsilon}{n}\sum_{i=1}^{n} \\
&= \frac{\varepsilon}{n}\cdot n = \varepsilon.
\end{aligned}$$

■

Theorem 41 (Fejér's Theorem)**.** Let $S_n(x)$ be the nth partial sum of the Fourier series for a function f on $(-\pi, \pi]$. Define

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x).$$

If f is continuous on $(-\pi, \pi]$, then $\sigma_N(x) \to f(x)$ uniformly.

Proof. We can fashion an argument just like the proof in Theorem 8.5.3 but this time, it will be much simpler. In addition to the facts mentioned before the proof of Theorem 8.5.3, we are going to use

$$\sin(\theta) + \sin(2\theta) + \sin(3\theta) + \cdots + \sin(N\theta) = \frac{\sin(\frac{N\theta}{2})\sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}.$$

■

Exercise 8.5.9

Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)}\left[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}\right]^2.$$

Proof. Let $\theta \neq 2\pi n$ so that we may have

$$D_n(\theta) = \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$$

be defined. Before we begin the derivation, we need a few more identities to make the computations simpler; that is, let us use $\sin(a)\cos(a) = 1/2\sin(2a)$ and $1/2(1+\cos(2a)) = \cos^2(a/2)$. Notice that we may write

$$D_N(\theta) = \frac{1}{2} + \sum_{n=1}^{N}\cos(n\theta) = \frac{1}{2}\Big(\cos(N\theta) + \frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big).$$

Using the identities given to us, let's manipulate the left-hand side of the equation to get

$$
\begin{aligned}
\frac{1}{N+1}\Big[\frac{1}{2}+\sum_{n=1}^{N}D_n(\theta)\Big] &= \frac{1}{N+1}\Big[\frac{1}{2}+\frac{1}{2\sin(\theta/2)}\sum_{n=1}^{N}\sin((n+1/2)\theta)\Big] \\
&= \frac{1}{N+1}\Big[\frac{1}{2}+\frac{1}{2\sin(\theta/2)}\Big(\cos(\theta/2)\sum_{n=1}^{N}\sin(n\theta) \\
&+\sin(\theta/2)\sum_{n=1}^{N}\cos(n\theta)\Big)\Big] \\
&= \frac{1}{2(N+1)}\Big[1+\frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta)+\sum_{n=1}^{N}\cos(n\theta)\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2}+\sum_{n=1}^{N}\cos(n\theta)+\frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta)+\frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[D_N(\theta)+\frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta)+\frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2}\Big(\cos(N\theta)+\frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big) \\
&+\frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta)+\frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2}\Big(\cos(N\theta)+\frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big) \\
&+\frac{\cos(\theta/2)}{\sin(\theta/2)}\Big(\frac{\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2})}{\sin(\theta/2)}\Big)+\frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)\sin^2(\theta/2)}[\mathbf{A}]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{A} &= \frac{1}{2}(\cos(N\theta)\sin^2(\theta/2)+\sin(\theta/2)\sin(N\theta)\cos(\theta/2)) \\
&+\cos(\theta/2)\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2})+\frac{1}{2}\sin^2(\theta/2). \qquad (1)
\end{aligned}
$$

Working backwards and using the identities we mentioned at the beginning of this proof, we can manipulate $\sin^2((N+1)\frac{\theta}{2})$ to get

$$
\begin{aligned}
\sin^2((N+1)\theta/2) &= [\sin(N\theta/2)\cos(\theta/2)+\cos(N\theta/2)\sin(\theta/2)]^2 \\
&= (\sin^2(N\theta/2)\cos^2(\theta/2) \\
&+ 2\sin(N\theta/2)\cos(N\theta/2)\sin(\theta/2)\cos(\theta/2) \\
&+ \cos^2(N\theta/2)\sin^2(\theta/2)) \\
&= \sin^2(N\theta/2)\cos^2(\theta/2)+\frac{1}{2}\sin(\theta)\sin(N\theta) \\
&+ \cos^2(N\theta/2)\sin^2(\theta/2).
\end{aligned}
$$

Hence, all we need to show is

$$
\sin^2((N+1)\theta/2) = \sin^2(N\theta/2)\cos^2(\theta/2)+\frac{1}{2}\sin(\theta)\sin(N\theta)+\cos^2(N\theta/2)\sin^2(\theta/2)
$$

and we're done. Manipulating (1), we write

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2}(\cos(N\theta)\sin^2(\theta/2) + \sin(\theta/2)\sin(N\theta)\cos(\theta/2)) \\
&+ \cos(\theta/2)\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2}) + \frac{1}{2}\sin^2(\theta/2) \\
&= \sin^2(\theta)\frac{1}{2}(1+\cos(N\theta)) + \frac{1}{2}\sin(\theta/2)\cos(\theta/2)\sin(N\theta) \\
&+ \cos(\theta/2)\sin(N\theta/2)\sin((N+1)\theta/2) \\
&= \sin^2(\theta)\cos^2(N\theta/2) + \frac{1}{4}\sin(\theta)\sin(N\theta) + \frac{1}{4}\sin(\theta)\sin(N\theta) \\
&+ \sin^2(N\theta/2)\cos^2(\theta).
\end{aligned}$$

Hence, we have that

$$\mathbf{A} = \sin^2((N+1)\theta/2)$$

and thus we can conclude that

$$\frac{1}{N+1}\Big[\frac{1}{2} + \sum_{n=1}^{N} D_n(\theta)\Big] = \frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\theta/2)}\Big]^2$$

for $\theta \neq 2\pi n$. ■

The result above is called the *Fejér's kernel* and will be denoted by $F_N(\theta)$ (similar to the Dirichlet kernel $D_N(\theta)$) derived earlier).

Exercise 8.5.10

(a) Show that

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ du.$$

Proof. Let $\theta \neq 2\pi n$ and $u = t - x$. Using our results from Exercise 8.5.9, we can write

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} S_n(x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\Big[\frac{1}{N+1}\sum_{n=0}^{N} D_n(u)\Big]\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{N+1}\Big[\frac{1}{2} + \sum_{n=1}^{N} D_n(u)\Big]\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\Big]^2\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ dx.
\end{aligned}$$

Thus, we have

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ dx.$$

■

(b) Graph the function $F_N(u)$ for several values of N. Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \to 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi, \pi]$).

Proof. We can leave the graphing to your favorite Ti-84 calculator. We shall prove that $F_N \to 0$ uniformly. Let $u \in \{u : |u| \geq \delta\}$ where $\delta > 0$. Let $\varepsilon > 0$. Since $1/2(N+1) \to 0$, there exists an $M \in \mathbb{N}$ such that for any $N \geq M$, we have

$$\left|\frac{1}{2(N+1)}\right| < \varepsilon.$$

Furthermore, the sine function is bounded by 1. Hence, we can write the following

$$\begin{aligned}
\left|\frac{1}{2(N+1)}\left[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\right]^2\right| &\leq \left|\frac{1}{2(N+1)}\right| \cdot \left|\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\right|^2 \\
&\leq \frac{1}{2(N+1)} \\
&< \varepsilon.
\end{aligned}$$

Hence, we must also have

$$\frac{1}{2(N+1)}\left[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\right]^2 \to 0$$

uniformly. ∎

(c) Prove that $\int_{-\pi}^{\pi} F_N(u)\, du = \pi$.

Proof. We will use the fact that $\int_{-\pi}^{\pi} D_N(u)\, du = \pi$ to show the result. Using the definition of $F_N(u)$ and reordering indices, we can (specifically we will be using the left-hand side of the equation) write

$$\begin{aligned}
\int_{-\pi}^{\pi} F_N(u)\, du &= \int_{-\pi}^{\pi} \left[\frac{1}{N+1}\Big(\frac{1}{2} + \sum_{n=1}^{N} D_n(u)\Big)\right] du \\
&= \frac{1}{N+1}\int_{-\pi}^{\pi} \Big(D_0(u) + \sum_{n=1}^{N} D_n(u)\Big)\, du. \\
&= \frac{1}{N+1}\int_{-\pi}^{\pi} \Big[\sum_{n=1}^{N+1} D_{n-1}(u)\Big]\, du \\
&= \frac{1}{N+1}\sum_{n=1}^{N+1} \Big[\int_{-\pi}^{\pi} D_{n-1}(u)\, du\Big] \\
&= \frac{\pi}{N+1}\sum_{n=0}^{N+1} \\
&= \frac{\pi}{N+1} \cdot N+1 \\
&= \pi.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} F_N(u)\, du = \pi.$$

∎

(d) To finish the proof of Fejér's Theorem, first choose $\delta > 0$ so that

$$|u| < \delta \text{ implies } |f(x+u) - f(x)| < \varepsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x.

Proof. Let $\varepsilon > 0$. First, we show that $\sigma_n(x) \to f(x)$ on $|u| \le \delta$. Using the fact that f is uniformly continuous on $(-\pi, \pi]$, we can write

$$\begin{aligned}
|\sigma_n(x) - f(x)| &= \Big|\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ du - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F_N(u)\ du\Big| \\
&= \frac{1}{\pi}\Big|\int_{-\pi}^{\pi} (f(u+x) - f(x))F_N(u)\ du\Big| \\
&\le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(u+x) - f(x)|F_N(u)\ du \\
&< \frac{\varepsilon}{\pi}\int_{-\pi}^{\pi} F_N(u)\ du \\
&= \frac{\varepsilon}{\pi}\cdot\pi = \varepsilon.
\end{aligned}$$

Now suppose $|u| \ge \delta$. Since f is uniformly continuous, we can bound f by some $M > 0$. Hence, we must have

$$|f(u+x) - f(x)| \le 2M.$$

Then we can use the fact that $F_N \to 0$ on sets such as $|u| \ge \delta$ to write

$$\begin{aligned}
|\sigma_n(x) - f(x)| &\le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(u+x) - f(x)||F_N(u)|\ du \\
&\le \frac{2M}{\pi}\int_{-\pi}^{\pi} |F_N(u)|\ du \\
&< \frac{2M}{\pi}\cdot\frac{\varepsilon}{4M}\int_{-\pi}^{\pi} du \\
&= \frac{\varepsilon}{2\pi}\cdot 2\pi \\
&= \varepsilon.
\end{aligned}$$

■

3.5.8 Weierstrass Approximation Theorem

Fejér's Theorem gives us a nice and concise argument for the Weierstrass Approximation Theorem found in section 6.7. We can restate here as a reminder.

Theorem 42 (Weierstrass Approximation Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there exists a polynomial $p(x)$ satisfying

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$.

We can use the fact that $\sin(x)$ and $\cos(x)$ is bounded on a compact set to show theorem above.

Exercise 8.5.11

(a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a, b]$ is $[0, \pi]$.

Proof. Let $\varepsilon > 0$. Since $f(x)$ is continuous on $[0, \pi]$, we can pick a $\delta > 0$ such that whenever $|u| < \delta$, we have

$$|f(u+x) - f(x)| < \varepsilon.$$

Observe that

$$\int_0^{\pi} F_N(u)\ du = \frac{1}{2}\int_{-\pi}^{\pi} F_N(u)\ du = \frac{\pi}{2}.$$

We can use σ_N defined in Fejér's Theorem to write

$$\begin{aligned}
|\sigma_N(x) - f(x)| &= \Big|\frac{2}{\pi}\int_0^{\pi} f(u+x)F_N(u)\ du - \frac{2}{\pi}\int_0^{\pi} f(x)F_N(u)\ du\Big| \\
&= \frac{2}{\pi}\Big|\int_0^{\pi} (f(u+x) - f(x))F_N(u)\ du\Big| \\
&\leq \frac{2}{\pi}\int_0^{\pi} |f(u+x) - f(x)|F_N(u)\ du \\
&< \frac{2\varepsilon}{\pi}\int_0^{\pi} F_N(u)\ du \\
&= \frac{2\varepsilon}{\pi}\cdot\frac{\pi}{2} = \varepsilon.
\end{aligned}$$

Note that the polynomial we found was just $\sigma_N(x) = p(x)$. Hence, we have that

$$|f(x) - p(x)| < \varepsilon.$$

■

(b) Show how the case for an arbitrary interval $[a, b]$ follows from this one.

Proof. Taking advantage of f's uniformly continuity on $(-\pi, \pi]$, we can extend uniform continuity of f to all of $\mathbb{R}$. Using Fejér's Theorem, we can find a polynomial $p(x)$ (in this case, $p(x) = \sigma_N(x)$) such that

$$|f(x) - p(x)| < \varepsilon$$

for any interval $[a, b]$. ■

3.6 A Construction of R from Q

In this section, we will focus on constructing a proof for the existence of the Real Numbers.

Theorem 43 (Existence of Real Numbers)**.** There exists an ordered field in which every nonempty set that is bounded above has a least upper bound. In addition, this field contains $\mathbb{Q}$ as a subfield.

A review of chapter 1 tells us that

(i) We can't do analysis on the set of rational numbers because we don't have the square root defined (nor does it exists) which does not allow us to define the convergence of Cauchy sequences.

A way that we solved this problem is to create the following axiom:

Theorem 44 (Axiom of Completeness)**.** Every nonempty set of real numbers that is bounded above has a least upper bound.

Note that we actually need to demonstrate that we can actually extend the rational numbers to contain nonempty sets that have least upper bounds, thereby making the axiom into a theorem worth proving.

3.6.1 Dedekind Cuts

Let's begin by assuming that all the properties that come with the rational numbers are available to us. Let us also assume that we don't have real numbers at this point.

Definition 40 (Cuts)**.** A subset A of the rational numbers is called a *cut* if it possesses the following three properties:

(c1) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(c2) If $r \in A$, then A also contains every rational $q < r$.

(c3) A does not have a maximum; that is, if $r \in A$, then there exists $s \in A$ with $r < s$.

Exercise 8.6.1

(a) Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{t \in \mathbb{Q} : t < r\}$ is a cut.

Proof. For the first property (C1), we show that $C_r \neq \emptyset$ and $C_r \neq \mathbb{Q}$. Suppose $r \in \mathbb{Q}$ is fixed. Now define $t = r - 1$. Then clearly, we have $t < r$. But this means $t \in C_r$. Hence, $C_r \neq \emptyset$. Now, we show $C_r \neq \mathbb{Q}$. Define $s = r + 1 \in \mathbb{Q}$. Then clearly, $s > r$. Hence, $s \notin C_r$ and thus $C_r \neq \mathbb{Q}$.

For the second property (C2), we need to show that if $r \in C_r$, then C_r also contains every rational $q < r$. Suppose we let $t, q \in \mathbb{Q}$ be arbitrary with $t > q$ with $t \in C_r$. Then by definition of C_r, we must have $t < r$. But we have $q < t < r$. Hence, we have $q < r$ for any $q \in \mathbb{Q}$.

Lastly, we need to show that C_r does not have a maximum; that is, for any $x \in C_r$, there exists an $s \in C_r$ with $x < s$. Let $x \in C_r$. Then by definition of C_r, we must have $x < r$. Suppose we take the midpoint between x and r. Then we must have $x < \frac{x+r}{2} < r$. Setting $s = \frac{x+r}{2}$ gives us our desired result. ■

Avoid thinking of cuts as only having this form. Which of the following subsets of $\mathbb{Q}$ are cuts?

(b) $S = \{t \in \mathbb{Q} : t \leq 2\}$

Proof. This is not a cut because S contains a maximum. In this case, the maximum is 2. ■

(c) $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$

Proof. The set T is a cut. First, we show (C1). Observe that $0 \in T$. Hence, $T \neq \emptyset$. Then observe that $2 \notin T$. Hence, $T \neq \mathbb{Q}$.

Next, we show (C2). Let $r \in T$. Then by definition of T, we must have either $r^2 < 2$ or $r < 0$. If $r < 0$, then we can easily pick $q < r$ for any $q < 0$. Otherwise, we have $r^2 < 2$ such that if we let $q \in \mathbb{Q}$ be arbitrary with $q^2 < r^2$. Then we easily have $q < r$.

Finally, we show (C3). Let $r \in T$. Then either we have $r^2 < 2$ or $r < 0$. Consider

$$\begin{aligned}\left(r + \frac{1}{n}\right)^2 &= r^2 + \frac{2r}{n} + \frac{1}{n} \\ &= r^2 + \frac{2r+1}{n}.\end{aligned}$$

Let us set $s = r + \frac{1}{n_0}$. If $r > 0$, then we can pick an n_0 large enough such that

$$\frac{1}{n_0} < \frac{4 - r^2}{2r + 1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < r^2 + \frac{2r+1}{n_0} \cdot \frac{4 - r^2}{2r+1} < 4.$$

Hence, we have $r < s < 2$. Otherwise, we choose n_0 large enough so that

$$\frac{1}{n_0} < \frac{-r^2}{2r + 1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < 0.$$

This implies that $r < s < 0$. ■

(d) $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$.

Proof. Is a cut. Same reasoning can be used to prove that U is a cut, but we can assume $a \geq 0$ this time. ■

Exercise 8.6.2

Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Proof. Suppose for sake of contradiction that $r \geq s$. Since $r \in A$ and $s \in \mathbb{Q}$, we know that $s < r$. But this means that $s \in A$ which leads to a contradiction. ■

Definition 41. Define the *real numbers* $\mathbb{R}$ to be the set of all cuts in $\mathbb{Q}$.

Here we have defined the set $\mathbb{R}$ whose elements are subsets of $\mathbb{Q}$ which is awkward because we think of numbers as elements of a set rather than the sets being elements themselves. Our goal now is to create an algebraic structure on $\mathbb{R}$. This involves answering the following question

"What do we mean by an ordered field?".

3.6.2 Field and Order Properties

Suppose we were given a set F and two elements $x, y \in F$. The *operation* that we impose on F takes x and y as an ordered pair (x, y) and maps it to a third element $z \in F$. Here we are trying to emulate our basic notions of adding and multiplying two numbers.

Definition 42 (Field)**.** A set F is a *field* if there exists two operations — addition $(x + y)$ and multiplication (xy) — that satisfy the following list of conditions:

(f1) (commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

(f2) (associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

(f3) (identities exist) There exist two special elements 0 and 1 with $0 \neq 1$ such that $x + 0 = x$ and $1 \cdot x = x$ for all $x \in F$.

(f4) (inverses exist) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$. If $x \neq 0$, there exists an element x^{-1} such that $xx^{-1} = 1$.

(f5) (distributive property) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Exercise 8.6.3

Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$, respectively.

Proof. ($\mathbb{N}$) This is not a field since an additive inverse does not exist; that is, $-1 \notin \mathbb{N}$.

($\mathbb{Z}$) The set of integers is not a field because every element in $x \in \mathbb{Z}$ does not contain an inverse except for 1.

($\mathbb{Q}$) Field.

■

Just an aside, we can actually use the field properties to show, for example, that for any $x, y, z \in \mathbb{Q}$, we have that $x + y = x + z$ implies $y = z$.

Definition 43. Ordering An *ordering* on a set F is a relation, represented by $\leq$, with the following three properties:
For arbitrary $x, y \in F$, at least one of the statements $x \leq y$ or $y \leq x$ is true.

(o2) If $x \leq y$ and $y \leq x$, then $x = y$.

(o3) If $x \le y$ and $y \le z$, then $x \le z$. Note that writing $y \ge x$ is equivalent to $x \le y$. The strict inequality $x < y$ is used to mean $x \le y$ but $x \ne y$.

(o4) If $y \le z$, then $x + y \le x + z$.

(o5) If $x \ge 0$ and $y \ge 0$, then $xy \ge 0$.

We have thus far defined $\mathbb{R}$ to be the collection of $\mathbb{Q}$ cuts. Now we want to invent the ordering and basic operations outlined in the last two definitions. The easiest out of these is the ordering. Let A and B be two arbitrary elements of $\mathbb{R}$. Define $A \le B$ to mean $A \subseteq B$.

Exercise 8.6.4

Show that this defines an ordering on $\mathbb{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Proof. Observe that the first property (o1) follows immediately. For the second property (o2), suppose $A \le B$ and $B \le A$. Then we have $A \subseteq B$ and $B \subseteq A$. This implies that $A = B$. Finally, for the third property (o3), let $A, B, C \in \mathbb{R}$. If $A \le B$ and $B \le C$, then we have $A \subseteq B$ and $B \subseteq C$. Then we have $A \subseteq B \subseteq C$. Hence, we have $A \subseteq C$. Hence, we have an ordering on $\mathbb{R}$. ∎

3.6.3 Algebra in R

Given A and B in $\mathbb{R}$, define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Before trying to verify if $A + B$ defines an operation, we have first ask if $A + B$ defines a cut. Suppose we verify (c2) of our definition of a cut. Suppose A, B are cuts. Let $a + b \in A + B$ be arbitrary and let $s \in \mathbb{Q}$ satisfy $s < a + b$. Then, $s - b < a$, which implies that $s - b \in A$ because A is a cut. But then

$$s = (s - b) + b \in A + B,$$

and (c2) is proved.

Exercise 8.6.5

(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

Proof. For property (c1), let $a \in A$. Then $a = (a - b) + b$ where $a - b \in A$ and $b \in B$ implies $a \in A + B$. Hence, we have that $A + B \ne \emptyset$. Let $c \in C$ where $C \subseteq \mathbb{Q}$, then we find that $A + C \not\subseteq A + B$. Hence, the sum $a + c \notin A + B$. Hence, $A + B \ne \mathbb{Q}$. To show property (c3), Let $a + b \in A + B$ be arbitrary. Since A is a cut, we can find an $\alpha \in A$ such that $a < \alpha$. Likewise, B being a cut implies that there exists $\beta \in B$ such that $b < \beta$. Adding the two inequalities together, we get that $a + b < \alpha + \beta$. Denote the set $S = \{\alpha + \beta : \alpha \in A \text{ and } \beta \in B\}$. Then we have $A + B \subseteq S$ which proves property (c3). Since properties (c1), (c2), and (c3) have been satisfied, we can now conclude that $A + B$ is a cut. ∎

(b) Check that addition in $\mathbb{R}$ is commutative (f1) and associative (f2).

Proof. Since $A + B$ is a cut and its elements $a + b$, where $a \in A$ and $b \in B$ are both cuts, subsets of $\mathbb{Q}$ where $\mathbb{Q}$ is an ordered field, we know that addition must be commutative and associative. This means that

$$A + B = B + A.$$

If we have an extra set C that is also a cut, then we would have $(a + b) + c = a + (b + c)$ implying that

$$(A + B) + C = A + (B + C).$$

Hence, we have (f1) and (f2) are satisfied. ∎

(c) Show that property (o4) holds.

Proof. Suppose we have $A \subseteq C$ where A and C are both cuts. Let B be a cut. Then $A \subseteq C$

implies $a \leq c$. If we have $b \in B$ be arbitrary, then we can use the ordering of $\mathbb{Q}$ to write $b + a \leq b + c$. But this means that $B + A \subseteq B + C$ and hence property (o4) is satisfied. ■

(d) Show that the cut
$$O = \{p \in \mathbb{Q} : p < 0\}$$
successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions : $A + O \subseteq A$ and $A \subseteq A + O$.)

Proof. First, we show $A + O \subseteq A$. Let $a + p \in A + O$ be arbitrary. Since $p < 0$, we must have $a + p \leq a$. But this means that $A + O \subseteq A$. For the second inclusion $A \subseteq A + O$, let $a \in A$. Then observe that $a = (a - p) + p$. Since $p < 0$, we can write $a = (a - p) + p$. But since $a - p \in A$ and $p \in O$, we know that $a \in A + O$. Hence, we have $A \subseteq A + O$. ■

Suppose we try and prove additive inverses? Given an $A \in \mathbb{R}$, we need to construct a cut $-A$ such that $A + (-A) = O$. Without the help of the supremum, which we can't even use because it is strictly defined on $\mathbb{R}$ without inherently assuming that it exists, how do we go about defining this set?

Suppose we are given $A \in \mathbb{R}$, then define
$$-A = \{r \in \mathbb{Q} : \text{ there exists } t \notin A \text{ with } t < -r\}.$$

Exercise 8.6.6

(a) Prove that $-A$ defines a cut.

Proof. To show the (c1), let $r \in \mathbb{Q}$. Since $\mathbb{Q}$ is a field, we can rewrite r in the following way
$$r = r + (t - t) = (r + t) - t = (t + r) - t < -t$$
with $t \notin A$. Multiplying through the inequality by a negative leads us to $-r > t$. This tells us that $r \in -A$ and we must have $-A \neq \emptyset$. However, this also implies that $-r \notin -A$ and hence, $-A \neq \mathbb{Q}$. To show (c2), let $r \in -A$. Then by definition of $-A$, there exists a $t \notin A$ such that $-r > t$. Suppose we let $-q \in \mathbb{Q}$ be arbitrary with $-q > -r$. Using the fact that $\mathbb{Q}$ is a field, we can multiply the inequality by a negative to get $r > q$ our desired result. Hence, (c2) is satisfied.

Finally, we show (c3). Suppose $r \in -A$. Then there exists $t \notin A$ such that $-r > t$. Multiplying through the inequality by a negative gives us $r < -t$ which is our desired result. Hence, $-A$ defines a cut. ■

(b) What goes wrong if we set $-A = \{r \in \mathbb{Q} : -r \in A\}$?

Proof. If we have $-A$ defined as the set above, then it would fail (c3). We can see this when we define $A = \{r \in \mathbb{Q} : r < 0\}$ and let $-A = \{r \in \mathbb{Q} : r \leq 0\}$ which tells us that $-A$ contains a maximum.

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in the definition of a field.

Proof. Let $a \in A$ and $r \in -A$. Let $a + r \in O$. Since $r \in -A$, we know that there exists $t \notin A$ with $t < -r$. Now, we can write $a + r < a - t$. But $-t < 0$ so this tells us that $a + r < 0$. Hence, $a + r \in O$ and we have $A + (-A) \subseteq O$. Now we need to show $O \subseteq A + (-A)$. Let us fix $o \in O$ and finding $a \in A$ such that $a + b = o$. Let $\varepsilon = |o|/2 = -o/2$. Taking advantage of properties (c1) and (c2), we can choose a rational $t \notin A$ such that $t - \varepsilon \in A$. If no such t existed then we would either have $A = \mathbb{Q}$ or $A = \emptyset$. Now, $t \notin A$ implies $-(t + \varepsilon) \in -A$. Then
$$o = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A),$$
and hence we conclude $O \subseteq A + (-A)$. Hence, (f4) is proven. ■

Remark. Another possible way you could show the reverse inclusion in part (c) is to rewrite $o \in O$ such that $o = (o + r) - r$ and show that $0 + r \in -A$ for which, in addition to $-r \in A$, leads to $O \subseteq A + (-A)$. Since there exists $t \notin A$ such that $-r > t$ and $o < 0$, we have

$$o + r < r \Rightarrow o + r < r < -t.$$

Multiplying the inequality by a negative gives us $-(o+r) < t$, which tells us that $o+r \in -A$. Since we also have $-r \in A$, we conclude that $o \in A + (-A)$ and hence $O \subseteq A + (-A)$.

Suppose we try to create the structure for multiplication using cuts. This can be quite difficult because of the fact that the product of two negative numbers is positive. A way to do this is to define multiplication on non-negative cuts.

Given $A \geq O$ and $B \geq O$ in $\mathbb{R}$, define the product

$$AB = \{ab : a \in A, b \in B \text{ with } a, b \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

Exercise 8.6.7

(a) Show that AB is a cut and that property (o5) holds.

Proof. To show (c1), let $a, b \in \mathbb{Q}$ with the property that $a, b \geq 0$. Since $\mathbb{Q}$ is a field, we can take the product of a and b and get $a \cdot b \geq 0$ implying that $AB \neq \emptyset$. Likewise, if we take any $p \in \mathbb{Q}$ with $p < 0$ and multiply it by any strictly non-negative $a \in \mathbb{Q}$, then $p \cdot a < 0$ implying that $AB \neq \emptyset$ once again. We know that $AB \neq \emptyset$ since $q \leq 0$ is not in AB. Hence, (c1) is satisfied.

To show (c2), we need to show that for all $q \in \mathbb{Q}$ that $r \in AB$ implies $q < r$. Suppose $r = a \cdot b$ with $a \in A$ and $b \in B$. Since A and B are cuts, we can use (c2) to state that for all $\alpha \in A$ and for all $\beta \in B$, we have such $\alpha < a$ and $\beta < b$ respectively. Then taking the product of these inequalities, we get our desired result that $q = \alpha \cdot \beta < a \cdot b = r$. Hence, (c2) is satisfied.

Finally, we show (c3). Let $r \in AB$ where $r = a \cdot b$ where $a \in A$ and $b \in B$. Since A and B are cuts, we can find $w \in A$ and $z \in B$ such that $r = a \cdot b < z \cdot w = \omega$. Hence, (c3) is satisfied.

Now we need to show that property (o5) holds for AB. Assume $A \geq O$ and $B \geq O$. We know that $O \subseteq A$ and $O \subseteq B$. This means that if we let $p, w \in O$, then $p, w \in A$ and $p, w \in B$. Note that $p \cdot w < 0$. Since $ab \geq 0$, we know that $pw \leq ab$ which implies that $pw \in AB$ and hence, $O \subseteq AB$. ■

(b) Propose a good candidate for the multiplicative identity on $\mathbb{R}$ and show that this works for all cuts $A \geq O$.

Proof. A good candidate for the multiplicative identity is I. Let us define it as follows:

$$I = \{p \in \mathbb{Q} : p < 1\}.$$

We want to show that $AI = A$; that is, we need to show the following two inclusions:

$$AI \subseteq A \text{ and } A \subseteq AI.$$

Let $x \in AI$. Then we have $xp < x \cdot 1 = x$ which tells us $xp \in A$ and we have $AI \subseteq A$. Now if $x \in A$, then we have $x \cdot 1 > xp$ and hence $x \in AI$. Thus, we have $A \subseteq AI$. Hence, $AI = A$. ■

(c) Show the distributive property (f5) holds for non-negative cuts. Before moving on to the proof, define $A(B + C)$ as

$$A(B + C) = \{a(b + c) : a \in A, b + c \in B + C \text{ with } a, b + c \geq 0\} \cup \{p \in \mathbb{Q} : p < 0\}.$$

Then let

$$P = \{a(b + c) : a \in A, b + c \in B + C \text{ with } a, b + c \geq 0\}.$$

Proof. The main goal is to show that $A(B+C) = AB + AC$; that is, we need to show the following two inclusions:

$$A(B+C) \subseteq AB + AC \text{ and } AB + AC \subseteq A(B+C).$$

Since $x \in A(B+C)$, then either $x \in P$ or $x \in O$. Suppose we have $x \in P$. Then we have $x = a(b+c)$. Since $\mathbb{Q}$ is an ordered field, we can use the distributive property to write $a(b+c) = ab + ac \in AB + BC$. Hence, $a(b+c) \in AB + BC$ and $A(B+C) \subseteq AB + BC$. Now suppose $x \in O$, then $x < 0$. Since the products AB and BC are non-negative, we know that we must have $ab \geq 0$ and $bc \geq 0$. This tells us that their sum $ab + ac \geq 0$ and hence, we have $x < 0 \leq ab + ac$. Since $x \leq ab + ac$, we can write $O \subseteq AB + AC$.

Next, we show $AB + BC \subseteq A(B+C)$. Let $x \in AB + BC$. Then $x = r_1 + r_2$ where $r_1 \in AB$ and $r_2 \in BC$. If $r_1 \in AB$, then either $r_1 = ab \geq 0$ or $r_1 < 0$. Likewise, $r_2 \in BC$ implies that either $r_2 = ac \geq 0$ or $r_2 < 0$. Suppose $r_1 = ab \geq 0$ and $r_2 = ac \geq 0$. Then we can simply use the distributive property to write $ab + ac = a(b+c) \in A(B+C)$. Hence, $ab + ac \in A(B+C)$. Now suppose $r_1 \geq 0$ and $r_2 < 0$. Then $x = r_1 + r_2 < r_1 + 0$. But note that $AB + O \subseteq O$ which tells us that $x \in A(B+C)$. The case for when $r_2 \geq 0$ and $r_1 < 0$ is similar. If both $r_1 < 0$ and $r_2 < 0$, then $x = r_1 + r_2 < 0 \leq a(b+c)$ and hence, $x \in A(B+C)$. Thus, we can conclude that $AB + BC \subseteq A(B+C)$. ■

We can define products of two positive cuts by observing that $-A \geq O$ whenever $O \leq A$. In addition, given any $A \leq O$, we can use property (o4) to write $A + (-A) \leq O + (-A)$ which results in $O \leq -A$.
Now let $A, B \in \mathbb{R}$ and define the following possible cases for which we can multiple A and B:

$$AB = \begin{cases} \text{as given} & \text{if } A \geq O \text{ and } B \geq O \\ -[A(-B)] & \text{if } A \geq O \text{ and } B \geq O \\ -[(-A)B] & \text{if } A < O \text{ and } B \geq O \\ (-A)(-B) & \text{if } A < O \text{ and } B < O. \end{cases}$$

We leave these as exercises for you to do. The proofs follow the same pattern as the ones we have done so far with cases usually falling into when terms are either positive or negative.

3.6.4 Least Upper Bounds

We can now use the fact that $\mathbb{R}$ is an ordered field to show that the field is complete. Recall in Chapter 1 that we define "complete" in terms of least upper bounds. Below we restate the definitions seen in Chapter 1.

Definition 44. A set $\mathcal{A} \subseteq \mathbb{R}$ is *bounded above* if there exists a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in \mathcal{A}$. The number B is called an *upper bound* for $\mathcal{A}$.

A real number $S \in \mathbb{R}$ is the *least upper bound* for a set $\mathcal{A} \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) S is an upper bound for $\mathcal{A}$ and

(ii) if B is any upper bound for $\mathcal{A}$, then $S \leq B$.

Exercise 8.6.8

Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

(a) First, prove that $S \in \mathbb{R}$ by showing that it is a cut.

Proof. First, we show (c1). Since S is defined as the union of nonempty sets A, we also have that $S \neq \emptyset$. Since all the sets within S are just a union of cuts, we know that these cuts also have the property that $A \neq \mathbb{Q}$ and hence $S \neq \mathbb{Q}$.

Next, we show property (c2). Let $x \in S$. Since S is defined as the union of all cuts $A \in \mathcal{A}$, we know that $x \in S$ implies that there exists an A' such that $x \in A'$. Let $q \in \mathbb{Q}$ But since A' is a cut, we know that we can use (c2) to say that $q < x$. Hence, (c2) is satisfied.

Lastly, we show property (c3). Let $x \in S$. Using the same reasoning to show (c2), we know that $x \in A'$ implies that there exists $\alpha \in Q$ such that $x < \alpha$.

Since all the properties of a cut are satisfied, we can now conclude that $S \in \mathbb{R}$ is also a cut. ■

(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Proof. In order to show that S is the least upper bound of $\mathcal{A}$, we need to show that S is an upper bound and that S is the least upper bound.

First, we show S is an upper bound; that is, $S \geq A$ for all $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be arbitrary. Observe that since $S = \bigcup A$, we know that $A \subseteq \bigcup A = S$. But this tells us that $A \leq S$. Since A is arbitrary, we know that S must be an upper bound.

Lastly, we show that S is the *least upper bound*. Let $B \in \mathcal{A}$ be any upper bound. Since each set A in the union $S = \bigcup A$ is bounded by B (that is, $A \subseteq B$), it follows that $S \subseteq B$. But this tells us that $S \leq B$ and hence, S must be the least upper bound of $\mathcal{A}$. ■

This finishes the proof that $\mathbb{R}$ is complete. However, we need to solve the problem of the slight mistake in saying that $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. We can solve this issue by saying that $\mathbb{R}$ contains a subfield that looks and acts exactly like $\mathbb{Q}$.

Exercise 8.6.9

Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbb{Q} : t < r\}$$

where $r \in \mathbb{Q}$. (See Exercise 8.6.1)

(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbb{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

Proof. Let $r, s \in \mathbb{Q}$. First, we show $C_r + C_s = C_{r+s}$; that is, we need to show that following two inclusions:

$$C_r + C_s \subseteq C_{r+s} \text{ and } C_{r+s} \subseteq C_r + C_s.$$

Let $z \in C_r + C_s$. Then we have $z = x + y$ where $x \in C_r$ and $y \in C_s$. By definition of C_r and C_s, we know that $x < r$ and $y < s$ respectively. Hence, adding both of these inequalities produces $z = x + y < r + s$ which tells us that $z \in C_{r+s}$. Thus, we have $C_r + C_s \subseteq C_{r+s}$.

Now, we show the reverse inclusion. Let $z \in C_{r+s}$. Then $z < r + s$ by definition of C_{r+s}. Note that subtract s from both sides of the inequality to get $z - s < r$. This tells us that $z - s \in C_r$. Likewise, subtract r from both sides of the inequality to get $z - r < s$ which implies that $z - r \in C_s$. We can add these two distinct elements to get

$$(z - r) + (z - s) = 2z - (r + s) < 2(r + s).$$

Dividing by 2 then gives us

$$z - \frac{r + s}{2} < r + s.$$

Hence, $z \in C_r + C_s$. and we conclude that $C_{r+s} \subseteq C_r + C_s$.

Let $r, s \geq 0$ in $\mathbb{Q}$. We want to show $C_r C_s = C_{rs}$. To do this, we need to show the following two inclusions; that is,

$$C_r C_s \subseteq C_{rs} \text{ and } C_{rs} \subseteq C_r C_s.$$

First, let $x \in C_r C_s$. Then either $x = \alpha\beta$ where $\alpha \cdot \beta \geq 0$ where $\alpha \in C_r$ and $\beta \in C_s$ or $x < 0$. If we assume the former, we can take $\alpha \in C_r$ imply $\alpha < r$ and $\beta \in C_s$ imply $\beta < s$. Multiplying these two inequalities together, we must have $\alpha\beta < rs$ which implies $x \in C_{rs}$ and hence $C_r C_s \subseteq C_{rs}$. Suppose $x < 0$. Since $r, s \geq 0$ implies $rs \geq 0$, we know that $x < 0 < rs$. Hence, we must have $x \in C_{rs}$.

Now, we show the reverse inclusion. Suppose $x \in C_{rs}$, then $x < rs$ for $r, s \geq 0$ in $\mathbb{Q}$. Suppose $r, s > 0$ (if $r, s = 0$ then the result follows immediately), then dividing by r on both sides gives us $x/r < s$ which tells us that $x/r \in C_s$. Similarly, we can divide by s to get $x/s < r$ which implies $x/s \in C_r$. Hence, taking the product of these two elements leads to

$$\frac{x}{s} \cdot \frac{x}{r} < (rs)^2 \Leftrightarrow \frac{x}{s^2} \cdot \frac{x}{r^2} < rs.$$

Hence, we must have $x \in C_r C_s$ and thus we have $C_{rs} \subseteq C_r C_s$. ■

(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbb{Q}$.

Proof. Suppose $r \leq s$ in $\mathbb{Q}$. Let $z \in C_r$. Using the fact that $r \leq s$ and using definition of C_r, we have $z < r \leq s$. This tells us that $z \leq s$ which subsequently tells us that $z \in C_s$. Hence, $C_r \subseteq C_s$ which is equivalent to $C_r \leq C_s$. Now, let $C_r \leq C_s$. Suppose for sake of contradiction that $r > s$. Suppose we take the midpoint of r and s. Then we have $a = \frac{r+s}{2}$ which implies $a < r$. This tells us that $a \in C_r$, but $a \notin C_s$ since $a > s$. Hence, we have a contradiction. Hence, we must have $r \leq s$ in $\mathbb{Q}$. ■

3.6.5 Cantor's Approach

A weirdly different approach used by Cantor to characterizing completeness is to define a real number as the entirety of a Cauchy sequence. But we run into some issues when we characterize it this way because we can have two different Cauchy sequences converge to the same limit. Does this mean two different numbers converge to the same number? A quite confusing problem to wrestle with indeed. A way to fix this issue is to create *equivalency classes* of Cauchy sequences where we have two sequences say (x_n) and (y_n) and say that they're in the same equivalence class if and only if $(x_n - y_n) \to 0$.