1 Handout-11

In this handout, we will relate complex differentiability with differentiability in multivariable calculus.

Definition. Let $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ be a function. We say that f is complex differentiable on D if f is differentiable at each $a \in D$.

If f is differentiable on D, then we can define a function $f' : D \to \mathbb{C}$ by $z \mapsto f'(z)$. The function f' is called the complex derivative of f.

Remark. Assume that $D = [a, b] \subseteq \mathbb{R}$. By abusing notation, let us write $f(x)$ with $x \in [a, b]$. Let $f(x) = u(x) + iv(x)$. Then, one can show that f is differentiable if and only if u and v are differentiable. If f is differentiable, then $f'(x) = u'(x) + iv'(x)$.

Example (Computing Complex Derivatives). (i) Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$, $n \in \mathbb{Z}^+$. Then $f'(z) = nz^{n-1}$. The result follows from

$$z^n - w_n = (z - w)(z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}).$$

(ii) Let $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ for $z \in \mathbb{C}$ and $a_0, \dots, a_n \in \mathbb{C}$ are constants and $n \in \mathbb{Z}^+$. Then

$$P'(z) = \sum_{k=1}^{n} k a_k z^{k-1}.$$

(iii) Power series: Let $c_0, c_1, c_2, \dots$ be a sequence of complex numbers and $a \in \mathbb{C}$. A series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ is called a power series centered at a with coefficients $\{c_n\}_{n=0}^{\infty}$. Assume that the series converges on an open ball $B(a, R)$ for some R; that is, for each $z \in B(a, R)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ exists. Then we can define $f : B(a, R) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n.$$

(iv) Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \overline{z}$. We claim that f is not complex differentiable at $z = 0$. We have $f(z) - f(0) = \overline{z}$ and thus,

$$\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{\overline{z}}{z} = \begin{cases} 1 & \text{if } z \to 0 \text{ along the real axis} \\ -1 & \text{if } z \to 0 \text{ along the imaginary axis} \end{cases}$$

Consider the function $\operatorname{Log} : \mathbb{C}^\bullet \to \mathbb{C}$. We say that it is NOT continuous along the negative real axis. Let

$$D = \mathbb{C}_- = \mathbb{C}^\bullet \setminus \{z \in \mathbb{C} | z < 0\}.$$

Then, $\operatorname{Log} : D \to \mathbb{C}$ is complex differentiable and

$$(\operatorname{Log} z)' = \frac{1}{z}$$

on D. Assuming complex differentiability, we can use $e^{\operatorname{Log} z} = z$ and chain rule:

$$\begin{aligned} &\implies e^{\operatorname{Log} z} \cdot (\operatorname{Log} z)' = 1 \\ &\implies (\operatorname{Log} z)' = \frac{1}{e^{\operatorname{Log} z}} = \frac{1}{z}. \end{aligned}$$

(v)(vi) Show that $f : \mathbb{C} \to \mathbb{C}$, $f(z) = |z|^2 = z\overline{z}$ not complex differentiable except for $z = 0$. Also, show that $f'(0) = 0$.

Remark. When $D \subseteq \mathbb{C}$ is open, the notion of complex differentiability becomes interesting and has very different behavior from that of $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}$ open, f differentiable. We will expand on this later.

Let us start with recalling some multivariable calculus concepts. Let $D \subseteq \mathbb{R}^p$ be open and $f : D \to \mathbb{R}^q$ be a function. We say that f is differentiable at $\vec{a} \in D$ if we can find a linear function $A : \mathbb{R}^p \to \mathbb{R}^q$ such that

(i) $f(\vec{x}) = f(\vec{a}) + A(\vec{x} - \vec{a}) + \gamma(\vec{x})$

(ii) $\lim\limits_{\vec{x}\to\vec{a}} \frac{\gamma(\vec{x})}{\|\vec{x}-\vec{a}\|=0}$.

The linear map is called the Jacobian of f at a and it is written as $J(f; a)$. The following proposition gives us a connection between complex differentiability and differentiability in the multivariable sense when $D \subseteq \mathbb{C}$ is open.

Proposition. Let $D \subseteq \mathbb{C}$ be an open set and $f : D \to \mathbb{C}$. Let $a \in D$ and $\ell \in \mathbb{C}$. Then the following statements are equivalent.

(i) f is complex differentiable at a and $f'(a) = \ell$.

(ii) f is differentiable at $a \in D$ in the sense of multivariable calculus (here we think of $D \subseteq \mathbb{R}^2$ open and $\mathbb{C}$ is identified with $\mathbb{R}^2$) and

$$J(f, a) = \begin{pmatrix} \Re(\ell) & -\Im(\ell) \\ \Im(\ell) & \Re(\ell) \end{pmatrix}.$$

In fact, if we indetify $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x + iy$, then $J(f, a)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Re(\ell)x - \Im(\ell)y \\ \Re(\ell)y + \Im(\ell)x \end{pmatrix}$; that is, $J(f, a)z = \ell \cdot z$.