Understanding Analysis Notes

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Chapter 1

Additional Topics

1.1 Euler's Sum

Recall Euler's famous series derivation

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

which used the Taylor series representation

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{1}$$

There is also the infinite product representation

$$\sin(x) = x\Big(1 - \frac{x}{\pi}\Big)\Big(1 + \frac{x}{\pi}\Big)\Big(1 - \frac{x}{2\pi}\Big)\Big(1 + \frac{x}{2\pi}\Big)\cdots . \tag{2}$$

We have developed the sufficient theory to show why (1) is true, but not (2). There have been many derivations for (2) using multi-variable calculus, Fourier series, and even complex integration. However, we will try to show (2) by using the properties of uniformly convergent series and Taylor series expansions.

1.1.1 Walli's Product

We currently don't have enough machinery at our disposal to be able to prove the infinite product representation of $\sin(x)$ in (2), but we can prove the special case when

$$\frac{\pi}{2} = \lim_{n\to\infty} \prod_{n=1}^{n} \Big(\frac{2n \cdot 2n}{(2n-1)(2n+1)}\Big) \tag{3}$$

where (3) is the partial products of (2) but with $x = \pi/2$.

Exercise 8.3.1

Supply the details to show (3) above.

Proof. Plugging in $x = \pi/2$ into (2), we get that

$$1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{2n}\right)\left(1 + \frac{1}{2n}\right) = \frac{\pi}{2}\prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)^2}.$$

Taking the reciprocal of the infinite product above, we end up with

$$\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)} = \frac{\pi}{2}.$$

■

Now we will prove why (3) holds. Set

$$b_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\ dx, \text{ for } n = 0, 1, 2, \ \ldots$$

If we look at the $n = 0$ and $n = 1$ case, we can easily obtain the following equations

$$b_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \text{ and } b_1 = \int_0^{\frac{\pi}{2}} \sin(x)\ dx = 1.$$

Exercise 8.3.2

Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$, and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\ dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\ dt.$$

Solution. Refer to the solution in part (a) of Exercise 7.5.6. ■

Exercise 8.3.3

(a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation

$$b_n = \frac{n-1}{n}b_{n-2} \text{ for all } n \geq 2.$$

Proof. Let $h(x) = \sin^n(x)$ and $k'(x) = \sin(x)$. Let $n \geq 2$. Then by the integration-by-parts formula and using the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, we must have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx &= \int_0^{\frac{\pi}{2}} \sin^n(x) \cdot \sin(x)\ dx \\
&= \Big[-\sin^{n-1}(x) \cdot \cos(x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot \cos^2(x)\ dx \\
&= \Big[-\sin^{n-1}(x) \cdot \cos(x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot [1 - \sin^2(x)]\ dx
\end{aligned}$$

The first term on the last equality cancels out and the second term can be expanded into

$$\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\cdot[1-\sin^2(x)]\ dx = \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n}(x)\ dx.$$

Hence, we end up with

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx = \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n}(x)\ dx. \tag{1}$$

Finally, subtracting the second term on the right side of (1), simplifying, and dividing by n on both sides gives us our desired result

$$\begin{aligned} b_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\ dx \\ &= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\ dx \\ &= \frac{n-1}{n} b_{n-2}. \end{aligned}$$

■

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).

Solution. The first three even terms are

$$\begin{aligned} b_2 &= \frac{1}{2}b_0 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}, \\ b_4 &= \frac{3}{4}b_2 = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16}, \\ b_6 &= \frac{5}{6}b_4 = \frac{5}{6}\cdot\frac{3\pi}{16} = \frac{5\pi}{32}. \end{aligned}$$

The first odd terms are

$$\begin{aligned} b_3 &= \frac{2}{3}b_1 = \frac{2}{3}\cdot 1 = \frac{2}{3} \\ b_5 &= \frac{4}{5}b_3 = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15} \\ b_7 &= \frac{6}{7}b_5 = \frac{6}{7}\cdot\frac{8}{15} = \frac{16}{35}. \end{aligned}$$

■

(c) Write a general expression for b_{2n} and b_{2n+1}.

Solution. Using the formula we derived in part (a), plugging in the desired cases gives us

$$b_{2n} = \frac{2n-1}{2n} b_{2(n-1)} \quad \text{and} \quad b_{2n+1} = \frac{2n}{2n+1} b_{2n-1}.$$

■

For the $(n+1)$th term, we have the following bound $0 \leq \sin^{n+1}(x) \leq \sin^n(x)$ on $[0, \pi/2]$. But this tells us that (b_n) is a decreasing sequence of functions. Since (b_n) is bounded and decreasing, we know that it must converge. It turns out that $(b_n) \to 0$ but this isn't the limit that we want to concern ourselves at the moment.

Exercise 8.3.4

Show

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Walli's product formula in (3).

Proof. For $k \geq 1$, observe that

$$\begin{aligned} \frac{b_{2n}}{b_{2n+1}} &= \frac{(2n-1)(2n+1)}{(2n)(2n)} \cdot \frac{b_{2n-2}}{b_{2n-1}} \\ &= \frac{(2n-1)(2n+1)}{(2n)(2n)} \cdot \frac{(2n-3)(2n-1)}{(2n-2)(2n-2)} \cdot \frac{b_{2n-4}}{b_{2n-3}}. \end{aligned}$$

Notice when expanding the terms on the numerator and the denominator of b_{2n}/b_{2n+1}, we will always have the same coefficient. Hence, the limit of b_{2n}/b_{2n+1} gives us our result that

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1.$$

■

Some techniques to dealing with the notation in (3) is to use the following equations

$$2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!$$

and

$$1 \cdot 3 \cdot 5 \cdot \cdots (2n+1) = \frac{(2n+1)!}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n+1)!}{2^n n!}.$$

Exercise 8.3.5

Derive the following alternative form of Walli's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Proof. ■

1.1.2 Taylor Series

To prove (2), we need to somehow generate the Taylor series for $\arcsin(x)$. This can't be done directly from Taylor's Formula for the coefficients. We need to first find the expansion for $1/\sqrt{1-x}$ by dealing with

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

first.

Exercise 8.3.6

Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Proof. Let $f(x) = 1/\sqrt{1-x}$. Using Taylor's coefficient formula, we have the first three derivatives of f

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{2} \cdot (1-x)^{-3/2}, \\ f^{(2)}(x) &= \frac{1}{2} \cdot \frac{3}{2} \cdot (1-x)^{-5/2}, \\ f^{(3)}(x) &= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot (1-x)^{-7/2}. \end{aligned}$$

For $n \geq 1$, we can use induction to show

$$f^{(n)}(x) = \Big[\prod_{k=1}^{n} \frac{2k-1}{2k}\Big](1-x)^{-(2n+1)/2}.$$

Plugging in $x = 0$ and using the techniques given to us above, we now have the desired formula

$$c_n = \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n)!}{2^{2n}(n!)^2}$$

where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n.$$

■

Observe that the coefficients above should look familiar to the formulas produced from Walli's product.

Exercise 8.3.7

Show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Proof. The first statement is shown in Exercise 2.7.10. Observe that

$$c_n \leq \frac{1}{2^{2n}} \leq \frac{1}{n}.$$

Since $\sum 1/n$ diverges, we must also have $\sum c_n$ diverge by the Comparison test. ■

Now our goal is to establish at which particular points in the domain of f where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

is valid. This can be done by using Lagrange's Remainder Theorem.

To properly show that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

holds for all $x \in (-1, 1)$, we need to show that the error function

$$E_N(x) = \frac{1}{\sqrt{1-x}} - \sum_{n=0}^{N} c_n x^n$$

approaches zero as $N \to \infty$. This can be done using Lagrange's Remainder Theorem (Theorem 6.6.3).

Exercise 8.3.8

Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem, show that equation (4) is valid for all $|x| < 1/2$. What goes wrong when we try try to use this method to prove (4) for $x \in (1/2, 1)$?

Proof. Since f is $N+1$ times differentiable on $(-1/2, 1/2)$, there exists a c such that $|c| < |x|$ where the error function $E_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)x^n}{(N+1)!}$$

by Lagrange's Remainder Theorem. Observe that

$$f^{(N+1)}(c) = \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big](1-c)^{-(2N+3)/2} < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\Big(\frac{2}{3}\Big)^{(2N+3)/2}.$$

Since $|x| < 1/2$ and $|c| < |x|$, we can now write

$$E_N(x) < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\frac{2^{3/2}}{3^{(2N+3)/2}(N+1)!} \xrightarrow{N\to\infty} 0.$$

Hence, (4) holds for all $x \in (-1, 1)$. If we try to prove $E_N \to 0$ on $(-1/2, 1)$, then we produce a sequence that diverges. ■

1.1.3 The Integral Form of the Remainder

The goal of the previous exercise is to recognize a different method is needed to estimate the error function $E_N(x)$. The following theorem is one such way to do this.

Theorem 1.1.1: Integral Remainder Theorem

Let f be differentiable $N+1$ times on $(-R, R)$ and assume $f^{(N+1)}$ is continuous. Define $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, \ldots, N$, and let

$$S_N(x) = \sum_{k=0}^{N} a_k x^k.$$

For all $x \in (-R, R)$, the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{1}{N!} \int_0^x f^{(N+1)}(t)(x-t)^N \, dt.$$

Proof. The case $x = 0$ is easy to check, so let's take $x \neq 0$ in $(-R, R)$ and keep in mind that x is a fixed constant in what follows. To avoid a few technical distractions, let's just consider the case $x > 0$.

Exercise 8.3.9

(a) Show

$$f(x) = f(0) + \int_0^x f'(t) \, dt.$$

Since f is continuous differentiable for all $t \in (0, x)$, we can use part (i) of FTC to write

$$\int_0^x f'(t) \, dt = f(x) - f(0).$$

Solving for $f(x)$ gives us our desired result

$$f(x) = f(0) + \int_0^x f'(t) \, dt.$$

(b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t) \, dt.$$

Taking advantage of f being continuously differentiable $N+1$ times for all $t \in (0, x)$ and using the integration-by-parts formula found in Exercise 8.3.2, we have

$$\begin{aligned}\int_0^x f''(t)(x-t) \, dt &= \Big[f'(t)(x-t)\Big]_0^x + \int_0^x f'(t) \, dt \\ &= -xf'(0) + [f(x) - f(0)].\end{aligned}$$

Solving for $f(x)$ once again, we get our desired result

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t) \, dt.$$

(b) Continue in this fashion to complete the proof of the theorem.

Continuing the process in parts (a) and (b) and using the fact that f is $N+1$ times differentiable for all $x \in (-R, R)$, we have that

$$\begin{aligned} f(x) &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N \ dt + \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x_k \\ &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N \ dt + S_N(x). \end{aligned}$$

Subtracting $S_N(x)$ from both sides above and using the fact that $E_N(x) = f(x) - S_N(x)$ gives us our desired result

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N \ dt.$$

■

We will use this fact to now show that (4) holds.

Exercise 8.3.10

(a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1, 1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

Solution. ■

(b) For a general x satisfying $|x| < 1$, show

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \ dt.$$

Solution. Let $|x| < 1$. Let $N = 2$. Observe that

$$f^{(3)}(t) = \frac{15}{8}(1-t)^{-7/2}.$$

Using the Integral Remainder Theorem, we have that

$$\begin{aligned} E_2(x) &= \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2 \ dt \\ &= \frac{15}{16}\int_0^x (1-t)^{-7/2}(x-t)^2 \ dt \\ &= \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} \ dt. \end{aligned}$$

■

(c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \le |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking that this overestimate is in fact larger than $|E_2(x)|$ at three computed values from part (a).

Solution. The inequality above is valid since

$$\begin{aligned}\left|\frac{x-t}{1-t}\right| &= \sqrt{\left(\frac{x-t}{1-t}\right)^2} \\ &\leq \sqrt{(x-t)^2} \\ &= |x-t| \\ &\leq |x|\end{aligned}$$

which holds for t. ■

(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1, 1)$.

Proof. Let f be differentiable $N+1$ times. Using the inequality found in part (c) and $|x| < 1$,we can write

$$\begin{aligned}|E_N(x)| &= \frac{1}{N!}\Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\Big|\int_0^x \left(\frac{x-t}{1-t}\right)^N \cdot \frac{1}{(1-t)^{3/2}}\,dt\Big| \\ &\leq \frac{c_{N+1}}{N!}\int_0^x \left|\frac{x-t}{1-t}\right|^N \cdot \left|\frac{1}{(1-t)^{3/2}}\right| dt \\ &\leq \frac{c_{N+1}}{N!}\int_0^x \frac{|x|^N}{(1-t)^{3/2}}\,dt \\ &< \frac{c_{N+1}}{N!}\int_0^x \frac{1}{(1-t)^{3/2}}\,dt \\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}\int_0^x dt \\ &= \frac{c_{N+1}x}{2\sqrt{2}\cdot N!} \\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}.\end{aligned}$$

Since $c_{N+1} = \prod_{k=1}^{N+1} \frac{2k-2}{2k} \to 0$ from Exercise 8.2.7, we can write

$$|E_N(x)| < \frac{c_{N+1}}{2\sqrt{2}\cdot N!} \to 0$$

as $N \to \infty$. Hence, $|E_N(x)| \to 0$ which tells us that $E_N \to f(x)$ uniformly. ■

Now that we have established that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

holds for all $x \in (-1, 1)$, we are now in the position to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{2n+1}$$

for all $|x| < 1$ using term-by-term anti-differentiation of (4).

Exercise 8.3.11

Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1. \tag{5}$$

Proof. From our result in part (d) of Exercise 8.3.10, we know that substituting $x = x^2$ into

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

give us

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} c_n x^{2n}$$

which holds for all $x \in (-1, 1)$. By assumption, we know that the derivative of $\arcsin(x)$ is $1/\sqrt{1-x}$. Using Term-by-term Antidifferentiation, we get that

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1.$$

■

Exercise 8.3.12

Our work thus far shows that the Taylor series in (5) is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \leq 1$. Carefully, explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1, 1]$.

Proof. Since (5) is valid for all $|x| < 1$, it suffices to show that (5) holds for $x = 1$ and likewise $x = -1$ so that we may show that (5) holds for $|x| \leq 1$ using Theorem 6.5.2. Plugging in $x = 1$ gives us

$$\sum_{n=0}^{\infty} \frac{c_n}{2n+1}.$$

Let's define

$$\gamma_n = \frac{c_n}{2n+1}.$$

Using the Cauchy Condensation Test, we can prove that $\sum 2^n \gamma_{2^n}$ so that $\sum \gamma_n$ converges. Observe that

$$\sum_{n=0}^{\infty} \frac{c_{2^n}}{2^{2n+1}+1} \leq \frac{1}{2} \sum_{n=0}^{\infty} c_{2^n}.$$

Now our goal is to use the Comparison Test to show that the right side of the inequality above converges which will immediately imply that the left side converges. Hence, observe that we have the following bound

$$\begin{aligned}\frac{1}{2}c_{2^n} &= \frac{1}{2}\cdot\frac{(2^{n+1})!}{(2^n!)^2\cdot 2^{2^{2n+1}}}\\ &\leq \frac{2^{n+1}-1}{2\cdot(2^n)!}\\ &\leq \frac{2^{n+1}}{2\cdot n!}.\end{aligned}$$

Observe that the last inequality forms a series that converges via the ratio test. Hence, we must have $\frac{1}{2}\sum c_{2^n}$ converge via the Comparison test. Hence, we must have $\sum c_n/(2n+1)$ converges via the Cauchy Condensation Test. Hence, by Theorem 6.5.2 we must have (5) converge at $x = |1|$ for all $x \in [-1, 1]$. ∎

1.1.4 Summing $\sum_{n=1}^{\infty} 1/n^2$

Suppose we let $x = \sin(\theta)$ in (5) where we restrict our domain to $-\pi/2 \leq \theta \leq \pi/2$. Then we have

$$\theta = \arcsin(\sin(\theta)) = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)$$

which converges uniformly on $[-\pi/2, \pi/2]$.

Exercise 8.3.13

(a) Show

$$\int_0^{\pi/2}\theta\ d\theta = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Walli's product.

Proof. Observe that the series

$$\theta = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)$$

converges uniformly to θ for all $-\pi/2 \leq \theta \leq \pi/2$. Hence, we are able to move integration from outside the summation to inside the summation. Using this fact, we write

$$\begin{aligned}\int_0^{\pi/2}\theta\ d\theta &= \int_0^{\pi/2}\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\sin^{2n+1}(\theta)\ d\theta\\ &= \sum_{n=0}^{\infty}\frac{c_n}{2n+1}\left[\int_0^{\pi/2}\sin^{2n+1}(\theta)\ d\theta\right]\\ &= \sum_{n=0}^{\infty}\frac{c_n}{2n+1}b_{2n+1}. && \text{(Walli's Formula)}\end{aligned}$$

∎

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Proof. Looking at the left side of part (a), we can integrate to get

$$\int_0^{\pi/2} \theta \, d\theta = \left[\frac{1}{2}\theta^2\right]_0^{\pi/2} = \frac{\pi^2}{8}.$$

Focusing our attention to the right side of (a), we see that b_{2n+1} can be expanded to

$$b_{2n+1} = \frac{2n}{(2n+1)} \cdot \frac{(2n-2)}{(2n-1)} \cdot \frac{(2n-4)}{(2n-3)} \cdot \frac{(2n-6)}{(2n-5)} \cdot \ldots .$$

Likewise, c_n can be expanded into

$$\begin{aligned} c_n &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &= \frac{(2n)\cdot(2n-1)\cdot(2n-2)\cdot(2n-3)\cdot(2n-4)\cdot\ldots}{2^{2n}(n!)^2} \end{aligned}$$

Notice that when we multiply b_{2n+1} and c_n together as seen in part (a), we see that the $(2n-k)$ terms for k odd cancel, leaving the $(2n-\ell)$ terms for ℓ even on the top. Hence, we have

$$\begin{aligned} c_n b_{2n+1} &= \frac{(2n)^2\cdot(2n-2)^2\cdot(2n-4)^2\cdot(2n-4)^2\cdot\ldots}{4^n(n!)^2(2n+1)} \\ &= \frac{(2n)^2\cdot(2(n-1))^2\cdot(2(n-2))^2\cdot\ldots}{4^n(n!)^2(2n+1)} \\ &= \frac{4^n\cdot n^2\cdot(n-1)^2\cdot(n-2)^2\cdot\ldots}{4^n(n!)^2(2n+1)} \\ &= \frac{4^n(n!)^2}{4^n(n!)^2}\cdot\frac{1}{2n+1}. \\ &= \frac{1}{2n+1}. \end{aligned}$$

Hence, part (a) leads to

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Since the infinite sum in part (a) converges uniformly for all $-\pi/2 \leq \theta \leq \pi/2$, we are free to rearrange the sum however we like. Observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\ &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{4n^2} \\ &= \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{1}{4n^2}.\end{aligned}$$

Subtracting the second term on the right hand side to both sides above and collecting terms and dividing, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

■

1.1.5 Riemann-Zeta Function

The general formula that developed by Euler for the result we arrived at is written as a function of s where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for all } s > 1.$$

It is said that Euler was able to work out the sum for even s. There are a lot of deep properties about the function above, but among them, the most prominent would be about how $\zeta(s)$ is connected to the prime numbers given in the following formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\left(\frac{1}{1-7^{-s}}\right)\cdots \tag{6}$$

where the product is taken over all the primes. It is not surprising that delving deep into investigation of such properties will require more sophisticated machinery. However, the formula above is quite accessible. We see that expanding the product on the right hand side of (6) and using the fact that every natural number n contains a unique prime factorization, leading to the following formula

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \cdots.$$

1.2 Inventing the Factorial Function

The goal of this section is construct a function $f(x)$, defined on all of $\mathbb{R}$ with the property that $f(n) = n!$ for all $n \in \mathbb{N}$. This can be done easily by defining a piecewise function such that

$$f(x) = \begin{cases} n! & \text{if } n \le x < n+1, n \in \mathbb{N} \\ 1 & \text{if } x < 1. \end{cases}$$

Some questions we can explore is its continuity, differentiability (if differentiable then how many times?). Our goal now is to define a function that extends the definition of the factorial $n!$ in a meaningful way to non-natural n.

Exercise 8.4.1

For each $n \in \mathbb{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

Solution. ■

(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$, and revisit part (a).

Proof. The statement above is clearly true for $n = 1$. Now assume $n\# = \frac{1}{2}n(n+1)$ holds for $1 \leq n \leq k-1$. We want to show that $n\#$ holds for the kth case. By using the definition of $n\#$, we can write

$$\begin{aligned} k\# &= k + (k-1)\# \\ &= k + \frac{1}{2}k(k-1) \\ &= \frac{1}{2}(k^2 + k) \\ &= \frac{1}{2}k(k+1). \end{aligned}$$

Since $n\# = \frac{1}{2}n(n+1)$ holds for the kth case, we know that it holds for any $n \in \mathbb{N}$. ■

We can replace the discrete variable $n \in \mathbb{N}$ for values of $x \in \mathbb{R}$ and the resulting formula

$$x\# = \frac{1}{2}x(x+1)$$

will still make sense.

1.2.1 The Exponential Function

How is the exponential function like 2^x defined on $\mathbb{R}$? Typically, 2^x is defined through a series of domain expansions. Starting with the function defined on $\mathbb{N}$, we can expand its domain by using reciprocals, then to $\mathbb{Q}$ using roots, and then $\mathbb{R}$ using continuity. Our goal in this section is to expand the domain of 2^x using a different method.

Our first step is to properly define the natural exponential function e^x. Recall in chapter 6, we constructed a series expansion for e^x. This time, we do the opposite direction; that is, create a proper definition of e^x. We can do this by using the results we have found in our studies of power series expansions.

Define

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Exercise 8.4.2

Verify that the series converges absolutely for all $x \in \mathbb{R}$, that $E(x)$ is differentiable on $\mathbb{R}$, and $E'(x) = E(x)$.

Solution. First we prove that the series above converges absolutely for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Observe that

$$\Big|\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big| \le \sum_{n=0}^{\infty} \Big|\frac{x^n}{n!}\Big|.$$

By using the ratio test for power series found in section 6.5, we have

$$\begin{aligned}\Big|\frac{a_{n+1}}{a_n}\Big| &= \Big|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\Big| \\ &= \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Since the limit above is 0, we know that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

converges absolutely for all $x \in \mathbb{R}$. Given any compact set in $\mathbb{R}$, we know that the convergence of (1) to $E(x)$ is uniform. Hence, it must be continuous on any $A \subseteq \mathbb{R}$ and differentiable n times. Differentiating

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and reordering indices we find that $E'(x) = E(x)$. ■

Exercise 8.4.3

(a) Use the results of Exercise 2.8.7 and the binomial formula to show that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$. By definition of $E(x+y)$, using the binomial formula, we can write

$$\begin{aligned}E(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{y^k \cdot x^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{y^k}{k!} \cdot \frac{x^{n-k}}{(n-k)!} \\ &= \Big[\sum_{m=0}^{\infty} \frac{x^m}{m!}\Big]\Big[\sum_{k=0}^{\infty} \frac{y^k}{k!}\Big]. && (n-k=m)\end{aligned}$$

Since

$$E(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!},$$
$$E(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

both converge absolutely (by Exercise 2.8.7), we can write

$$E(x+y) = E(x)E(y).$$

■

(b) Show that $E(0) = 1, E(-x) = 1/E(x)$, and $E(x) > 0$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. The first fact immediately follows when $x = 0$. Now let us show the second fact. Using the first fact and part(a), we can write

$$1 = E(0) = E(x - x) = E(x)E(-x) \iff E(x)E(-x) = 1.$$

Dividing through by $E(x)$ on both sides leads us to our result

$$E(-x) = \frac{1}{E(x)}.$$

For the last fact, observe that $E(x) > 0$ follows immediately when we consider any $x \geq 0$. Suppose we let x be negative, then using the fact that

$$E(-x) = \frac{1}{E(x)}$$

where $E(x) > 0$ for any $x > 0$ implies that $E(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} > 0$. ■

The takeaway here is that the power series $E(x)$ contains all the "normal" properties that is associated with the exponential function e^x.

Exercise 8.4.4

Define $e = E(1)$. Show $E(n) = e^n$ and $E(m/n) = (\sqrt[n]{e})^m$ for all $m, n \in \mathbb{Z}$.

Proof. Let $P(n)$ be the statement that $E(n) = e^n$ for all $n \in \mathbb{Z}$. Let our base case be $n = 1$. Then by definition, we must have $E(1) = e$. Now assume $E(n) = e^n$ holds for all $n \in \mathbb{Z}^+$. We want to show that $E(n+1) = e^{n+1}$ holds. Observe that by part (a) of Exercise 4.4.3, we have

$$\begin{aligned} E(n+1) &= E(n) \cdot E(1) \\ &= e^n \cdot e \\ &= e^{n+1}. \end{aligned}$$

Hence, $E(n) = e^n$ for all $n \in \mathbb{Z}^+$. To show that the statement also holds for all $n \in \mathbb{Z}^-$, we can just multiply n by a negative to get

$$\begin{aligned} E(-n) &= \frac{1}{E(n)} \\ &= \frac{1}{e^n} \\ &= e^{-n}. \end{aligned}$$

Lastly, we show $E(m/n) = (\sqrt[n]{e})^m$ for all $n, m \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$. Observe that

$$E(1) = E\left(\frac{n}{n}\right) = (\sqrt[n]{e})^n \implies E(1/n) = \sqrt[n]{e}.$$

Furthermore, we can rewrite m/n in the following way where

$$\frac{m}{n} = \sum_{i=0}^{m} \frac{1}{n}.$$

Then we see that

$$\begin{aligned} E\left(\frac{m}{n}\right) &= E\left(\sum_{i=0}^{m} \frac{1}{n}\right) \\ &= E\left(\frac{1}{n}\right) \cdot E\left(\frac{1}{n}\right) \cdot E\left(\frac{1}{n}\right) \cdot \ \cdots \ m \text{ times} \\ &= \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \ \cdots \ m \text{ times} \\ &= (\sqrt[n]{e})^m. \end{aligned}$$

■

To complete our list of properties of e^x, all we need is its behavior as $x \to \pm\infty$.

Definition 1.2.1

Given $f : [a, \infty] \to \mathbb{R}$, we say that $\lim_{x\to\infty} f(x) = L$ if, for all $\epsilon > 0$, there exists $M > a$ such that whenever $x \geq M$ it follows that $|f(x) - L| < \epsilon$.

Exercise 8.4.5

Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \ldots$. To get started notice that when $x \geq 0$, all the terms in (1) are positive.

Proof. Let $\epsilon > 0$. Choose $M = 1/\sqrt{\epsilon}$. Assume for any $x \geq 1/\sqrt{\epsilon}$, we have

$$\begin{aligned} \left|\frac{x^n}{e^x} - 0\right| &= \frac{x^n}{e^x} \\ &< \frac{x^n}{x^{2n}} \\ &= \frac{1}{x^n} \end{aligned}$$

■

1.2.2 Other Bases

Having established a rigorous foundation for e^x, we can now do the same for t^x for any real number $t > 0$.

Exercise 8.4.6

(a) Explain why we know e^x has an inverse function; that is, let's call it $\log(x)$ defined for any real $x > 0$ and satisfying

(i) $\log(e^y) = y$ for all $y \in \mathbb{R}$ and

(ii) $e^{\log(x)} = x$, for all $x > 0$.

Solution. If we are considering $f(x) = e^x$ defined on $(0, \infty)$, then we get that $f(x)$ is a bijective function for all $x \in (0, \infty)$. To see why, suppose we let $x, y \in (0, \infty)$. Since $\log(x)$ is defined for all $x \in (0, \infty)$, we can say that

$$\begin{aligned} E(x) &= E(y) \\ e^x &= e^y \\ \log(e^x) &= \log(e^y) \\ x &= y. \end{aligned}$$

Hence, $E(x) = e^x$ is an injective function. Now let's show surjectivity. Then letting $x = \log(y)$, observe that

$$E(x) = e^x = e^{\log(y)} = y.$$

Hence, $E(x)$ is a surjective function. Since $E(x)$ is both injective and surjective, we know that $E(x)$ must be bijective and thus must have an inverse function. ■

(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)

Proof. Let $y = f(x) = e^x$. Using the result from Exercise 5.2.12, the fact that $f'(x) =$

e^x, and $e^{\log(x)}$, we get that

$$\begin{aligned}(\log x)' &= \frac{1}{f'(x)}\\ &= \frac{1}{e^{\log(x)}}\\ &= \frac{1}{x}.\end{aligned}$$

■

(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log(x) + \log(y) \text{ for all } x, y > 0.$$

Proof. Let $x, y \in (0, \infty)$ with $x = e^y$ and $y = e^x$. Our logarithm properties, we then have $\log(x) = y$ and $\log(y) = x$. Then by using the properties of e^x and $\log(x)$, observe that

$$\begin{aligned}\log(xy) &= \log(e^y \cdot e^x)\\ &= \log(e^{y+x})\\ &= y + x\\ &= \log(x) + \log(y).\end{aligned}$$

Hence, we have

$$\log(xy) = \log(x) + \log(y).$$

■

(d) For $t > 0$ and $n \in \mathbb{N}$, t^n has the usual interpretation as $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \text{ for all } n \in \mathbb{N}.$$

Proof. Let $t > 0$ and $n \in \mathbb{N}$. Observe that $t = e^{\log(t)}$ and then

$$t^n = \left(e^{\log(t)}\right)^n = e^{n\log(t)}.$$

■

Definition 1.2.2

Given $t > 0$, define the exponential function t^x to be

$$t^x = e^{x \log t} \text{ for all } x \in \mathbb{R}.$$

Exercise 8.4.7

(a) Show $t^{m/n} = (\sqrt[n]{t})^m$ for all $m, n \in \mathbb{N}$.

Proof. ■

(b) Show $\log(t^x) = x \log t$, for all $t > 0$ and $x \in \mathbb{R}$.

Proof. ■

(c) Show t^x is differentiable on $\mathbb{R}$ and find the derivative.

Proof. ■

The strategy we have been partaking in so far is a similar to how we would define what $n!$ would mean if it was replaced by $x \in \mathbb{R}$ instead of $n \in \mathbb{N}$.

1.2.3 The Functional Equation