

Understanding Analysis Notes

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Chapter 1

Additional Topics

1.1 Fourier Series

Fourier's development of a special type of series involving trigonometric functions caused a rework into what it really means to have a "limit" or to "converge" towards a specific

1.1.1 Trigonometric Series

The basic use of series representations is to express a given function $f(x)$ in terms of the sum of simpler and "nicer" functions such as polynomials. In the case of power series representations, we have the following set of component functions $\{1, x, x^2, x^3, \dots\}$ such that the following series takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

A *trigonometric series* on the other hand is an infinite series composed of component functions that are trigonometric; that is, we have

$$\{1, \cos(x), \sin(x), \sin(2x), \cos(3x), \sin(3x), \dots\}.$$

Thus, we can write a trigonometric series in the following way

$$\begin{aligned} f(x) &= a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + a_3 \cos(3x) + \dots \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \end{aligned}$$

Now we will see how this formulation can be used to solve partial differential equations. Consider the following problem proposed by d'Alembert

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

which describes the motion of a vibrating string. Suppose the solution to the partial differential equation above is $u(x, t)$ which models that displacement of the string at time $t \geq 0$. Suppose the string is attached at each end of this interval. Then the boundary conditions apply

$$u(0, t) = 0 \text{ and } u(\pi, t) = 0 \tag{2}$$

for all values of $t \geq 0$. If we let $t = 0$ (at the instant the string is released), then we can assume that

$$\frac{\partial u}{\partial t}(x, 0) = 0;$$

that is, the initial velocity of the string is defined to be 0.

Exercise 8.5.1

(a) Verify that

$$u(x, t) = b_n \sin(nx) \cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbb{N}$ and $b_n \in \mathbb{R}$. What goes wrong if $n \notin \mathbb{N}$.

Proof. We first compute the partial derivatives of $u(x, t)$. Observe the first partial derivative with respect to x of $u(x, t)$ is

$$\frac{\partial u}{\partial x} = nb_n \cos(nx) \cos(nt).$$

Then differentiating with respect to the same variable leads to

$$\frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx) \cos(nt).$$

Likewise, differentiating with respect to t leads to

$$\begin{aligned} \frac{\partial u}{\partial t} &= -nb_n \sin(nx) \sin(nt). \\ \frac{\partial^2 u}{\partial t^2} &= -n^2 b_n \sin(nx) \cos(nt). \end{aligned}$$

We can see that (1) is satisfied using our given solution $u(x, t)$. Now, using the boundary conditions found in (2) and the fact that the sine function is 0 with $n\pi$ for all $n \in \mathbb{N}$, we can see that

$$u(0, t) = b_n \sin(0) \cos(nt) = 0$$

and

$$u(\pi, t) = b_n \sin(n\pi) \cos(nt) = 0.$$

Hence, (2) is satisfied. By the same reasoning, we can see that (3) is satisfied.

If $n \notin \mathbb{N}$, then $u(x, t)$ would result in nonzero sine values rendering equations (1), (2), (3) invalid. ■

(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Proof. Since $\sin(x)$ and $\cos(x)$ are both continuous differentiable up to n times, we can use term-by-term differentiation of the finite sum of these trigonometric functions to show that, indeed, equations (1), (2), and (3) are all satisfied. ■

Observe that

$$u(x, t) = \sum_{n=1}^N b_n \sin(nx) \cos(nt) \tag{4}$$

solves the PDE in (1) which is called d'Alembert's *wave equation*. The solutions to (1), however, depend on how the string is originally "plucked". Suppose at $t = 0$, the string is given some initial displacement

$$f(x) = u(x, 0).$$

Letting $t = 0$ in our general solution in (4), we find that

$$f(x) = \sum_{n=1}^N b_n \sin(nx). \tag{5}$$

Given there exists coefficients b_n for all $1 \leq n \leq N$ such that our solution $f(x)$ can be written in terms of the finite sum of sine functions found in (5), we can be sure that the PDE in (1) can be solved by

$u(x, t)$ given in (4). We can then ask, more generally, how can we construct solutions that are linear combinations of functions found in the set

$$\{\sin(x), \sin(2x), \sin(3x), \dots\}.$$

It turns out that we can take an *infinite* sum of (5) to model the position of $f(x)$ for any $x \in [0, \pi]$. The function $f(x)$ can be thought of as the initial temperature applied to some boundary of a heat-conducting material.

1.1.2 Periodic Functions

A more general formulation of our problem is to find suitable coefficients (a_n) and (b_n) to express a function $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \quad (6)$$

It is important to note that every component that makes up (6) is 2π periodic. That is, any function that is represented by trigonometric functions is necessarily periodic. Turning our attention on the interval $(-\pi, \pi]$, we can take a function such as $f(x) = x^2$, restrict its domain to $(-\pi, \pi]$, and then extend f periodically to all of \mathbb{R} using the rule $f(x) = f(x + 2k\pi)$ for all $k \in \mathbb{Z}$.

1.1.3 Types of Convergence

The way we express functions in terms of a series of trigonometric functions depends on the type of convergence. As we have been discussing all throughout this book, we have to look at the partial sums. Hence, we have

$$S_N(x) = a_0 + \sum_{n=1}^N a_n \cos(nx) + b_n \sin(nx). \quad (7)$$

The idea of expressing $f(x)$ in terms of a trigonometric series involves finding the coefficients $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that

$$f(x) = \lim_{N \rightarrow \infty} S_N(x). \quad (8)$$

Showing that

$$\int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 dx \rightarrow 0$$

is a natural way to understand (8). This convergence is called the L^2 convergence (won't be discussed here). Another type of convergence that we will look at in this section is the *Cesaro mean convergence* which involves taking the *averages* of the partial sums and showing that they converge uniformly to $f(x)$.

1.1.4 Fourier Coefficients

Before discussing the Cesaro mean convergence, we will look at a few basic calculus facts.

Exercise 8.5.2

Using trigonometric identities when necessary, verify the following integrals.

- (a) For all $n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos(nx) dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx) dx = 0$$

Proof. We will begin by showing the first integral. Hence, using the fact that $\sin(x)$ is odd

and the fact that $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$, we can say

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx) \, dx &= \left[\frac{1}{n} \sin(nx) \right]_{-\pi}^{\pi} \\ &= \frac{1}{n} [\sin(n\pi) - \sin(-n\pi)] \\ &= \frac{2}{n} \sin(n\pi) \\ &= 0.\end{aligned}$$

Now we show the second integral. Using the fact that $\cos(x)$ is an even function, we can write

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx) \, dx &= \left[-\frac{1}{n} \cos(nx) \right]_{-\pi}^{\pi} \\ &= \frac{1}{n} [\cos(-n\pi) - \cos(n\pi)] \\ &= \frac{1}{n} [\cos(n\pi) - \cos(n\pi)] \\ &= 0.\end{aligned}$$

■

(b) For all $n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos^2(nx) \, dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx) \, dx = \pi.$$

Proof. Using the trigonometric identity $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$, we can write

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2(2xn) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [1 + \cos(2xn)] \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(2xn) \, dx\end{aligned}$$

The first integral evaluates to

$$\frac{1}{2} \int_{-\pi}^{\pi} dx = \frac{1}{2} [x]_{-\pi}^{\pi} = \pi$$

Then, using the fact that sine is an odd function and the fact that $\sin(2\pi n) = 0$ for all $n \in \mathbb{N}$, the second integral evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(2xn) \, dx &= \frac{1}{2n} [\sin(2xn)]_{-\pi}^{\pi} \\ &= \frac{1}{2n} [\sin(2\pi n) - \sin(-2\pi n)] \\ &= \frac{1}{2n} [\sin(2\pi)] \quad (\sin(-2\pi n) = -\sin(2\pi n)) \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos^2(x) \, dx = \pi.$$

Similarly, we have

$$\begin{aligned}
 \int_{-\pi}^{\pi} \sin^2(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [1 - \cos(2xn)] \, dx \\
 &= \frac{1}{2} \int_{-\pi}^{\pi} dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(2xn) \, dx \\
 &= \frac{1}{2} \cdot 2\pi - 0 \\
 &= \pi.
 \end{aligned}$$

(c) For all $m, n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = 0.$$

Proof. Let $m, n \in \mathbb{N}$. Suppose $m = n$. Then using a change of variable (letting $u = \sin(mx)$), we get

$$\begin{aligned}
 \int_{-\pi}^{\pi} \cos(mx) \sin(mx) \, dx &= \frac{1}{m} \int_{-\pi}^{\pi} u \, du \\
 &= \frac{1}{2m} [\sin^2(\pi x)]_{-\pi}^{\pi} \\
 &= \frac{1}{2m} \sin^2(\pi m) + \frac{1}{2m} \sin^2(\pi m) \\
 &= \frac{1}{m} \sin^2(\pi m) \\
 &= 0.
 \end{aligned}$$

If $m \neq n$, then using the trigonometric identity

$$\cos(mx) \sin(nx) = \frac{1}{2} [\sin((m+n)x) - \sin((m-n)x)]$$

we obtain

$$\begin{aligned}
 \int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin((m+n)x) - \sin((m-n)x)] \, dx \\
 &= \frac{1}{2} \int_{-\pi}^{\pi} \sin((m+n)x) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin((m-n)x) \, dx \\
 &= \frac{\cos((m-n)x)}{2(m-n)} \Big|_{-\pi}^{\pi} + \frac{\cos((m+n)x)}{2(m+n)} \Big|_{-\pi}^{\pi}
 \end{aligned}$$

Using the fact that cosine is an even function we can evaluate the first term which leads to

$$\begin{aligned}
 \frac{\cos((m-n)x)}{2(m-n)} \Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)} [\cos((m-n)\pi) - \cos(-(m-n)\pi)] \\
 &= \frac{1}{(m-n)} [\cos((m-n)\pi) - \cos((m-n)\pi)] \\
 &= 0.
 \end{aligned}$$

Using the same reasoning, the second term can be written as

$$\begin{aligned}\frac{\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos(-(m+n)\pi)] \\ &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos((m+n)\pi)] \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx = 0.$$

Our goal now is to show

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = 0$$

for all $m \neq n$. Let $m \neq n$. Using the trigonometric identity

$$\cos(a) \cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)],$$

the first integral evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos((m+n)x) + \cos((m-n)x)] \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos((m+n)x) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos((m-n)x) \, dx \\ &= \frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi}.\end{aligned}$$

Using the fact that the sine function is odd and the fact that $\sin((m+n)\pi) = 0$ for all $m, n \in \mathbb{N}$, we can write

$$\begin{aligned}\frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)} \cdot 2 \sin((m+n)\pi) \\ &= \frac{\sin((m+n)\pi)}{(m+n)} \\ &= 0\end{aligned}$$

Likewise, the second term evaluates to

$$\begin{aligned}\frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)} \cdot 2 \sin((m-n)\pi) \\ &= \frac{\sin((m-n)\pi)}{(m-n)} \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0.$$

Now we show the second integral. Similarly, we use the trigonometric identity

$$\sin(a) \sin(b) = \frac{1}{2}[\sin(a+b) - \sin(a-b)].$$

Then observe that

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin((m+n)x) + \sin((m-n)x)] \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \sin((m+n)x) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin((m-n)x) \, dx \\ &= \frac{-\cos((m+n)x)}{2(m+n)} \Big|_{-\pi}^{\pi} + \frac{\cos((m-n)x)}{2(m-n)} \Big|_{-\pi}^{\pi}.\end{aligned}$$

Starting with the first term, we can utilize the fact that $\cos(x)$ is an even function to get

$$\begin{aligned}\frac{-\cos((m+n)x)}{2(m+n)} \Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)} [\cos((m+n)\pi) - \cos((m+n)\pi)] \\ &= 0.\end{aligned}$$

Likewise, the second integral evaluates to

$$\begin{aligned}\frac{\cos((m-n)x)}{2(m-n)} \Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)} [\cos((m-n)\pi) - \cos((m-n)\pi)] \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \sin(mx) \sin(mx) \, dx = 0.$$

■

We can interpret the integrals in exercise 8.5.2 as dot products allowing us to see that all the functions from the set

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \dots\}$$

are all *orthogonal* to each other. This, in turn, forms a *basis* for a large class of functions.

Our ultimate goal is to find coefficients (a_n) and (b_n) in equation (6). We can assume that we are in possession of some expression described in (6) that will allow us to find formulas for both coefficients. This is similar to how we found the coefficients of a Taylor series back in chapter 6, but instead of differentiating repeatedly, we aim to integrate.

In order to compute a_0 , we need to integrate each side of (6) on the closed interval $[-\pi, \pi]$ to get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \, dx &= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right] \, dx \\ &= \int_{-\pi}^{\pi} a_0 \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx) + b_n \sin(mx)] \, dx \\ &= a_0 2\pi + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx) \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(mx) \, dx \\ &= a_0 2\pi + 0 + 0 \\ &= a_0 2\pi.\end{aligned}$$

Solving for a_0 gives us

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx. \quad (9)$$

One might look at the interchanging of the integral and summation used in the derivation with suspicion. The point, as of now, is not to justify each step of the derivation, but rather work backwards so that we may find a representation to prove. The rigor will follow later.

Now suppose we fix $m \geq 1$. To compute a_m , we first multiply (6) by $\cos(mx)$ on both sides and integrate over the closed interval $[-\pi, \pi]$.

Exercise 8.5.3

Derive the formulas

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx \text{ and } b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx$$

for all $m \geq 1$.

Proof. Let $m \geq 1$. Multiplying by $\cos(mx)$ on both sides of (6) gives us

$$f(x) \cos(mx) = a_0 \cos(mx) + \sum_{n=1}^{\infty} a_n \cos(mx) \cos(nx) + b_n \cos(mx) \sin(nx).$$

Now, taking the integral on the closed interval $[-\pi, \pi]$ on (6) gives us

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx &= a_0 \int_{-\pi}^{\pi} \cos(mx) \, dx \\ &\quad + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(mx) \cos(nx) + b_n \cos(mx) \sin(nx)] \, dx. \end{aligned}$$

Using exercise 8.5.2, we know that all the integrals in the above equation evaluate to 0 for all $n \in \mathbb{N}$, except when $n = m$ for which the right side of the equation evaluates to

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx &= \int_{-\pi}^{\pi} a_m \cos^2(mx) \, dx \\ &= a_m \cdot \pi \end{aligned}$$

Solving for a_m gives us our result

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(mx) \, dx.$$

Using the same reasoning to show the first integral, we have that

$$\int_{-\pi}^{\pi} f(x) \sin(mx) \, dx = 0$$

for all $n \in \mathbb{N}$ except when $n = m$ for which the integral above evaluates to

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx &= a_0 \int_{-\pi}^{\pi} \sin(mx) \, dx \\ &\quad + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx) \sin(mx) + b_n \sin(nx) \sin(mx)] \, dx \\ &= b_m \int_{-\pi}^{\pi} \sin^2(mx) \, dx \\ &= b_m \cdot \pi. \end{aligned}$$

Solving for b_m gives us our desired result

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx.$$

■

Let's see how these coefficients are computed with simple functions.

Let

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ 0 & \text{if } x = 0 \text{ or } x = \pi \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$

Notice that f is an odd function which means that we can evaluate our integrals easily using the symmetry argument; that is, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0 \text{ and } \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0$$

for all $n \geq 1$. Using the same argument, we can simplify b_n to get

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx \\ &= \frac{2}{\pi} \left(\frac{-1}{n} \cos(nx) \Big|_0^{\pi} \right) \\ &= \frac{2}{\pi} \left(\frac{-1}{n} \cos(n\pi) + \frac{1}{n} \right) \\ &= \begin{cases} 4/n\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Plugging in our results into (6) then gives us the following representation of $f(x)$

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)x).$$

Exercise 8.5.4

- (a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is *not* uniform on any interval containing 0.

Proof. At the endpoints, $f(x)$ takes on a different value making our convergence pointwise instead of being uniform. ■

- (b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?

Proof. Our goal is to produce coefficients a_0, a_m , and b_m so that we have the following representation of $f(x)$; that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Using the formula for a_0 derived earlier and using the fact that $|x|$ is an even function, we produce

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx \\ &= \frac{1}{\pi} \int_0^{\pi} x dx \\ &= \frac{1}{\pi} \cdot \frac{1}{2} x^2 \Big|_0^{\pi} \\ &= \frac{\pi}{2}. \end{aligned}$$

Using the symmetry from the absolute value function once again, we get that computing a_m

leads to

$$\begin{aligned}
 a_m &= \frac{2}{\pi} \int_0^\pi x \cos(mx) \, dx \\
 &= \frac{2}{\pi} \left[\frac{1}{m} x \sin(mx) \right]_0^\pi - \frac{2}{m\pi} \int_0^\pi \sin(mx) \, dx \\
 &= \frac{2}{m^2\pi} \cos(mx) \Big|_0^\pi \\
 &= \frac{2}{m^2\pi} [\cos(m\pi) - 1] \\
 &= \begin{cases} 0 & \text{if } m \text{ even} \\ -4/m^2\pi & \text{if } m \text{ odd.} \end{cases}
 \end{aligned}$$

Since $\sin(mx)$ is an odd function and $|x|$ is an even function, we find that their product $|x| \sin(mx)$ is also odd. Hence, the integral

$$\frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin(mx) \, dx$$

evaluates to zero. Hence, $b_m = 0$. Therefore, $f(x) = |x|$ can be represented by the following Fourier series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

By looking at the coefficients, we can see that the convergence of (a_n) goes to zero and does not depend on our choice of $x \in [-\pi, \pi]$. ■

1.1.5 The Riemann-Lebesgue Lemma

Observe that by looking at the Fourier coefficients of (a_n) and (b_n) tend to zero as we take the limit as n approaches ∞ . We will see why this works in our upcoming convergence proof.

If we take a look at any function $h(x)$ and $\sin(nx)$, we will notice that when we take their product $h(x) \sin(nx)$, the oscillatory nature of $\sin(nx)$ do not affect the value of h too much. This means that the value from the integral

$$\int_{-\pi}^\pi h(x) \sin(nx) \, dx$$

leads to a small value (because of the positive and negative oscillations of $h(x) \sin(nx)$ leads values cancelling out).

Theorem 1 (Riemann-Lebesgue Lemma). Assume $h(x)$ is continuous on $(-\pi, \pi]$. Then,

$$\int_{-\pi}^\pi h(x) \sin(nx) \, dx \rightarrow 0 \quad \text{and} \quad \int_{-\pi}^\pi h(x) \cos(nx) \, dx \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. Using our assumption that h is continuous on the interval $(-\pi, \pi]$, we can periodically extend h to be continuous on all of \mathbb{R} . This implies that we have to assume that $\lim_{n \rightarrow -\pi^+} h(x) = h(\pi)$ holds. ■

Exercise 8.5.5

Explain why h is uniformly continuous on \mathbb{R} .

Proof. We know that h is uniformly continuous on \mathbb{R} since we can extend the continuity of h to any interval $(-n\pi, n\pi]$ for all $n \in \mathbb{Z}^+$ as well as having the implication that $\lim_{x \rightarrow -n\pi^+} h(x) = h(n\pi)$. ■

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \frac{\varepsilon}{2}$. Notice that the period of $\sin(nx)$ is $2\pi/n$. Hence, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Now consider a particular interval $[a, b]$ of length $2\pi/n$ over which $\sin(nx)$ moves through one complete oscillation.

Exercise 8.5.6

Show that $\left| \int_a^b h(x) \sin(nx) \, dx \right| < \varepsilon/n$, and use this fact to complete the proof.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{\frac{1}{2n}, \frac{\varepsilon}{4mN}\} > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon/2$. Using the period of $\sin(nx)$ which is $2\pi/n$, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Then using the length of $[a, b]$ given to us above and the fact that $|\sin(nx)| \leq 1$, we can write

$$\int_a^b |h(x)| |\sin(nx)| \, dx \leq \int_a^b |h(x)| \, dx.$$

Since h is integrable, we know that h must also be bounded. Hence, there exists an $M > 0$ such that $|h(y)| \leq M$ for all $y \in [a, b]$. Now observe that

$$\begin{aligned} \left| \int_a^b h(x) \sin(nx) \, dx \right| &\leq \int_a^b |h(x)| |\sin(nx)| \, dx \\ &= \int_a^b |h(x)| \, dx \\ &< \left(\frac{\varepsilon}{2} + |h(y)| \right) \int_a^b dx \\ &= \left(\frac{\varepsilon}{2} + |h(y)| \right) (b - a) \\ &= \frac{\varepsilon}{2} (b - a) + |h(y)| (b - a) \\ &\leq \frac{\varepsilon}{2} (b - a) + M(b - a) \\ &< \frac{\varepsilon}{2} \cdot \frac{2\pi}{n} + M \cdot \frac{2\pi}{n} \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} \\ &= \frac{\varepsilon}{n}. \end{aligned}$$

Hence, we have that

$$\left| \int_a^b h(x) \sin(nx) \, dx \right| < \frac{\varepsilon}{n}.$$

We can use the same reasoning to show that if $|\cos(nx)| \leq 1$, then

$$\left| \int_a^b h(x) \cos(nx) \, dx \right| < \frac{\varepsilon}{n}.$$

Letting $\varepsilon > 0$, we know that $\varepsilon/n \rightarrow 0$ as $n \rightarrow \infty$. This tells us that we can find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\left| \int_a^b h(x) \sin(nx) \, dx \right| < \frac{\varepsilon}{n} \rightarrow 0$$

and

$$\left| \int_a^b h(x) \cos(nx) \, dx \right| < \frac{\varepsilon}{n} \rightarrow 0.$$

■

In attempt to make the proof above shorter, we can just say that since h is uniformly continuous on \mathbb{R} , we know that h must be bounded on any compact interval $[a, b] \subseteq \mathbb{R}$. Hence, there must exist an $M > 0$ such that $|h(x)| \leq M$ for any $x \in [a, b]$. Using the same set of assumptions above and letting

$\delta = \varepsilon/4Mn$, we can write

$$\begin{aligned}
\left| \int_a^b h(x) \sin(nx) \, dx \right| &\leq \int_a^b |h(x)| |\sin(nx)| \, dx \\
&\leq M \int_a^b |\sin(nx)| \, dx \\
&\leq M \int_a^b 1 \, dx \\
&= M(b-a) \\
&< M \cdot \frac{2\pi}{n} \\
&< 2M \cdot \frac{\varepsilon}{2Mn} = \frac{\varepsilon}{n}.
\end{aligned}$$

Note that the lemma above still holds even if we take out the continuity assumption. This makes the use of Fourier series much more versatile in terms of its application to a wider variety of functions and is capable of more interesting behavior than our power series counterpart.

1.1.6 A Pointwise Convergence Proof

The formulas for the Fourier coefficients that we have produced so far require the implicit assumption that our function needs to be Riemann Integrable. This tells us that it is advantageous to have our class of integrable functions to be as large as possible in order to properly represent $f(x)$ in terms of (6). The goal now is to determine how many more assumptions do we need in order to have Fourier series converge back to $f(x)$. This involves determining which type of convergence we hope to establish.

Consider

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

once again. On the left-hand side, we would like to have $f(x)$ be

- (i) bounded,
- (ii) integrable,
- (iii) continuous,
- (iv) differentiable
- (vi) f' continuous.

Suppose on the right-hand side, we list the types of convergences needed to properly represent $f(x)$ using (6). These types of convergences include

- (i) pointwise convergence
- (ii) uniform convergence
- (iii) L^2 convergence, and
- (iv) Cesaro mean convergence.

Theorem 2 (Pointwise Convergence of Fourier Series). Let $f(x)$ be continuous on $(-\pi, \pi]$, and let $S_N(x)$ be the N th partial sum of the Fourier series described in equation (7), where the coefficients (a_n) and (b_n) are given by equations (9) and (10). It follows that

$$\lim_{N \rightarrow \infty} S_N(x) = f(x)$$

pointwise at any $x \in (-\pi, \pi]$ where $f'(x)$ exists.

Before we embark on the proof, let us get a few known facts out of the way.

(i) $\cos(\alpha - \theta) = \cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta)$.

(ii) $\sin(\alpha + \theta) = \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)$.

(iii) For any $\theta \neq 2n\pi$, we have

$$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots + \cos(N\theta) = \frac{\sin((N+1/2)\theta)}{2\sin(\theta/2)}.$$

This function is called the *Dirichlet Kernel*. The proof of this fact will be omitted because it involves complex analysis; that is, it results from taking the real part of a geometric sum of complex exponentials.

(iv) Letting

$$D_N(\theta) = \begin{cases} \frac{\sin((N+1/2)\theta)}{2\sin(\theta/2)}, & \text{if } \theta \neq 2n\pi \\ 1/2 + N, & \text{if } \theta = 2n\pi \end{cases}$$

from (iii), we can see that

$$\int_{-\pi}^{\pi} D_N(\theta) d\theta = \pi.$$

(v) The Riemann-Lebesgue Lemma.

Let us fix a point $x \in (-\pi, \pi]$. Our first goal is to simplify $S_N(x)$. Since x is a fixed constant, we can write equations (9) and (10) using t as the "dummy" variable for integration. Using facts (i) and (iii), we can see that

$$\begin{aligned} S_N(x) &= a_0 + \sum_{n=1}^N a_n \cos(nx) + b_n \sin(nx) \\ &= \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right] + \sum_{n=1}^N \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \right] \cos(nx) \\ &\quad + \sum_{n=1}^N \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \right] \sin(nx) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^N \cos(nt) \cos(nx) + \sin(nt) \sin(nx) \right] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^N \cos(n(t-x)) \right] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_N(t-x) dx. \end{aligned}$$

Letting $u = t - x$ and using the fact that f is extended to be 2π -periodic, we can express $S_N(x)$ in terms of $D_N(u)$. By fact (iv), it follows that

$$S_N(x) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(u+x) D_N(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) D_N(u) du.$$

Since D_N is periodic, we know that our computation of the integral is independent of the choice of interval granted that we cover one full period. Now showing that $S_N(x) \rightarrow f(x)$ is a matter of showing $|S_N(x) - f(x)|$ gets arbitrarily small when we let $N \rightarrow \infty$. Using (iv), we can express $f(x)$ as we have done with $S_N(x)$ in terms of $D_N(u)$. Hence, we have

$$f(x) = \frac{f(x)}{\pi} \int_{-\pi}^{\pi} D_N(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) D_N(u) du$$

and thus we have

$$S_N(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(u+x) - f(x)) D_N(u) du. \quad (11)$$

We can use (ii), to rewrite the Dirichlet kernel $D_N(u)$ as

$$\begin{aligned} D_N(u) &= \frac{\sin((N+1/2)u)}{2\sin(u/2)} \\ &= \frac{\sin(Nu)\cos(u/2) + \cos(Nu)\sin(u/2)}{2\sin(u/2)} \\ &= \frac{1}{2} \left[\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)} + \cos(Nu) \right]. \end{aligned}$$

Now observe that

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u+x) - f(x)) D_N(u) \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u+x) - f(x)) \left[\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)} + \cos(Nu) \right] \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(u+x) - f(x)) \left(\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)} \right) \\ &\quad + (f(u+x) - f(x)) \cos(Nu) \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} p_x(u) \sin(Nu) \, du + \frac{1}{2\pi} \int_{-\pi}^{\pi} q_x(u) \cos(Nu) \, du, \end{aligned}$$

where in the last equality, we set

$$p_x(u) = \frac{(f(u+x) - f(x)) \cos(u/2)}{\sin(u/2)} \quad \text{and} \quad q_x(u) = f(u+x) - f(x).$$

Exercise 8.5.7

- (a) First, argue why the integral involving $q_x(u)$ tends to zero as $N \rightarrow \infty$.

Proof. Observe that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} q_x(u) \cos(Nu) \, du &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (h(u+x) - h(x)) \cos(Nu) \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(u+x) \cos(Nu) \, du \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) \cos(Nu) \, du. \end{aligned}$$

Since $h(x)$ continuous on $(-\pi, \pi]$, we know that both

$$\int_{-\pi}^{\pi} h(u+x) \cos(Nu) \, du \rightarrow 0$$

and

$$\int_{-\pi}^{\pi} h(x) \cos(Nu) \, du \rightarrow 0$$

as $N \rightarrow \infty$. Hence, we must also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} q_x(u) \cos(Nu) \, du \rightarrow 0$$

as $N \rightarrow \infty$

■

- (b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Proof. Our goal is to show that $\frac{1}{2}p_x(u)$ is a continuous function so that we may use the Riemann-Lebesgue formula to show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p_x(u) \sin(Nu) \, du = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} p_x(u) \sin(Nu) \, du = 0.$$

We notice that $\frac{1}{2}p_x(u)$ is clearly continuous. Let us show that it is continuous at $x = 0$. Taking advantage of the famous trigonometric limit

$$\lim_{u \rightarrow 0} \frac{u/2}{\sin(u/2)} = 1.$$

and the fact that f is differentiable at x , we can write the expression $p_x(u)$ and add a limit as $u \rightarrow 0$; that is,

$$\begin{aligned} \frac{1}{2} \lim_{u \rightarrow 0} p_x(u) &= \frac{1}{2} \lim_{u \rightarrow 0} \frac{(f(u+x) - f(x)) \cos(u/2)}{u \sin(u/2)} \\ &= \lim_{u \rightarrow 0} \frac{f(u+x) - f(x)}{2 \sin(u/2)} \\ &= \lim_{u \rightarrow 0} \frac{(f(u+x) - f(x))}{2 \sin(u/2)} \cdot \frac{u}{u} \\ &= \lim_{u \rightarrow 0} \frac{(f(u+x) - f(x)) \cos(u/2)}{u} \\ &\quad \cdot \lim_{u \rightarrow 0} \frac{u/2}{\sin(u/2)} \cdot \lim_{u \rightarrow 0} \cos(u/2) \\ &= f'(x) \cdot 1 \cdot 1. \end{aligned}$$

Hence, we have that $\frac{1}{2}p_x(u)$ is a continuous function. Now, using the Riemann-Lebesgue theorem, we can say that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\lim_{u \rightarrow 0} p_x(u) \right) \sin(Nu) \, du &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} \lim_{u \rightarrow 0} p_x(u) \right) \sin(Nu) \, du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin(Nu) \, du \\ &= 0. \end{aligned}$$

Hence, we also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p_x(u) \sin(Nu) \, du = 0$$

■

This completes the proof that $S_N(x) \rightarrow f(x)$ for all x where f is differentiable. If f' exists everywhere, then we get that $S_N \rightarrow f$ pointwise. It is not too hard to see that if f' is continuous, then the convergence will be uniform. Furthermore, the smoothness of f contributes to the rate at which the terms of the Fourier series converges. Even more interesting, the more differentiable f is, the faster the rate of convergence of the partial sums S_N to f .

1.1.7 Cesaro Mean Convergence

Now, we will look at a special type of convergence called the Cesaro mean convergence.

Exercise 8.5.8

Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converges to the same limit. Give an example to show that it is possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Proof. Let $\varepsilon > 0$. Suppose $(x_n) \rightarrow x$. We can choose $N \in \mathbb{N}$ such that for any $N \in \mathbb{N}$, we have

$$|x_n - x| < \varepsilon.$$

Then using the same choice of $N \in \mathbb{N}$, we can write

$$\begin{aligned} |y_n - x| &= \left| \frac{1}{n} \sum_{i=1}^n x_i - \frac{1}{n} \sum_{i=1}^n x \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^n (x_i - x) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n |x_i - x| \\ &< \frac{\varepsilon}{n} \sum_{i=1}^n 1 \\ &= \frac{\varepsilon}{n} \cdot n = \varepsilon. \end{aligned}$$

■

Theorem 3 (Fejér's Theorem). Let $S_n(x)$ be the n th partial sum of the Fourier series for a function f on $(-\pi, \pi]$. Define

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^N S_n(x).$$

If f is continuous on $(-\pi, \pi]$, then $\sigma_N(x) \rightarrow f(x)$ uniformly.

Proof. We can fashion an argument just like the proof in Theorem 8.5.3 but this time, it will be much simpler. In addition to the facts mentioned before the proof of Theorem 8.5.3, we are going to use

$$\sin(\theta) + \sin(2\theta) + \sin(3\theta) + \cdots + \sin(N\theta) = \frac{\sin(\frac{N\theta}{2}) \sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}.$$

■

Exercise 8.5.9

Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})} \right]^2.$$

Proof. Let $\theta \neq 2\pi n$ so that we may have

$$D_n(\theta) = \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$$

be defined. Before we begin the derivation, we need a few more identities to make the computations simpler; that is, let us use $\sin(a) \cos(a) = 1/2 \sin(2a)$ and $1/2(1 + \cos(2a)) = \cos^2(a/2)$. Notice that we may write

$$D_N(\theta) = \frac{1}{2} + \sum_{n=1}^N \cos(n\theta) = \frac{1}{2} \left(\cos(N\theta) + \frac{\sin(N\theta) \cos(\theta/2)}{\sin(\theta/2)} \right).$$

Using the identities given to us, let's manipulate the left-hand side of the equation to get

$$\begin{aligned}
\frac{1}{N+1} \left[\frac{1}{2} + \sum_{n=1}^N D_n(\theta) \right] &= \frac{1}{N+1} \left[\frac{1}{2} + \frac{1}{2 \sin(\theta/2)} \sum_{n=1}^N \sin((n+1/2)\theta) \right] \\
&= \frac{1}{N+1} \left[\frac{1}{2} + \frac{1}{2 \sin(\theta/2)} \left(\cos(\theta/2) \sum_{n=1}^N \sin(n\theta) \right. \right. \\
&\quad \left. \left. + \sin(\theta/2) \sum_{n=1}^N \cos(n\theta) \right) \right] \\
&= \frac{1}{2(N+1)} \left[1 + \frac{\cos(\theta/2)}{\sin(\theta/2)} \sum_{n=1}^N \sin(n\theta) + \sum_{n=1}^N \cos(n\theta) \right] \\
&= \frac{1}{2(N+1)} \left[\frac{1}{2} + \sum_{n=1}^N \cos(n\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)} \sum_{n=1}^N \sin(n\theta) + \frac{1}{2} \right] \\
&= \frac{1}{2(N+1)} \left[D_N(\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)} \sum_{n=1}^N \sin(n\theta) + \frac{1}{2} \right] \\
&= \frac{1}{2(N+1)} \left[\frac{1}{2} \left(\cos(N\theta) + \frac{\sin(N\theta) \cos(\theta/2)}{\sin(\theta/2)} \right) \right. \\
&\quad \left. + \frac{\cos(\theta/2)}{\sin(\theta/2)} \sum_{n=1}^N \sin(n\theta) + \frac{1}{2} \right] \\
&= \frac{1}{2(N+1)} \left[\frac{1}{2} \left(\cos(N\theta) + \frac{\sin(N\theta) \cos(\theta/2)}{\sin(\theta/2)} \right) \right. \\
&\quad \left. + \frac{\cos(\theta/2)}{\sin(\theta/2)} \left(\frac{\sin((N+1)\frac{\theta}{2}) \sin(\frac{N\theta}{2})}{\sin(\theta/2)} \right) + \frac{1}{2} \right] \\
&= \frac{1}{2(N+1) \sin^2(\theta/2)} [\mathbf{A}]
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2} (\cos(N\theta) \sin^2(\theta/2) + \sin(\theta/2) \sin(N\theta) \cos(\theta/2)) \\
&\quad + \cos(\theta/2) \sin((N+1)\frac{\theta}{2}) \sin(\frac{N\theta}{2}) + \frac{1}{2} \sin^2(\theta/2).
\end{aligned} \tag{1}$$

Working backwards and using the identities we mentioned at the beginning of this proof, we can manipulate $\sin^2((N+1)\frac{\theta}{2})$ to get

$$\begin{aligned}
\sin^2((N+1)\theta/2) &= [\sin(N\theta/2) \cos(\theta/2) + \cos(N\theta/2) \sin(\theta/2)]^2 \\
&= (\sin^2(N\theta/2) \cos^2(\theta/2) \\
&\quad + 2 \sin(N\theta/2) \cos(N\theta/2) \sin(\theta/2) \cos(\theta/2) \\
&\quad + \cos^2(N\theta/2) \sin^2(\theta/2)) \\
&= \sin^2(N\theta/2) \cos^2(\theta/2) + \frac{1}{2} \sin(\theta) \sin(N\theta) \\
&\quad + \cos^2(N\theta/2) \sin^2(\theta/2).
\end{aligned}$$

Hence, all we need to show is

$$\sin^2((N+1)\theta/2) = \sin^2(N\theta/2) \cos^2(\theta/2) + \frac{1}{2} \sin(\theta) \sin(N\theta) + \cos^2(N\theta/2) \sin^2(\theta/2)$$

and we're done. Manipulating (1), we write

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2}(\cos(N\theta) \sin^2(\theta/2) + \sin(\theta/2) \sin(N\theta) \cos(\theta/2)) \\
&\quad + \cos(\theta/2) \sin((N+1)\frac{\theta}{2}) \sin(\frac{N\theta}{2}) + \frac{1}{2} \sin^2(\theta/2) \\
&= \sin^2(\theta) \frac{1}{2}(1 + \cos(N\theta)) + \frac{1}{2} \sin(\theta/2) \cos(\theta/2) \sin(N\theta) \\
&\quad + \cos(\theta/2) \sin(N\theta/2) \sin((N+1)\theta/2) \\
&= \sin^2(\theta) \cos^2(N\theta/2) + \frac{1}{4} \sin(\theta) \sin(N\theta) + \frac{1}{4} \sin(\theta) \sin(N\theta) \\
&\quad + \sin^2(N\theta/2) \cos^2(\theta).
\end{aligned}$$

Hence, we have that

$$\mathbf{A} = \sin^2((N+1)\theta/2)$$

and thus we can conclude that

$$\frac{1}{N+1} \left[\frac{1}{2} + \sum_{n=1}^N D_n(\theta) \right] = \frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\theta/2)} \right]^2$$

for $\theta \neq 2\pi n$. ■

The result above is called the *Fejér's kernel* and will be denoted by $F_N(\theta)$ (similar to the Dirichlet kernel $D_N(\theta)$ derived earlier).

Exercise 8.5.10

(a) Show that

$$\sigma_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) F_N(u) du.$$

Proof. Let $\theta \neq 2\pi n$ and $u = t - x$. Using our results from Exercise 8.5.9, we can write

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1} \sum_{n=0}^N S_n(x) \\
&= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t-x) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \left[\frac{1}{N+1} \sum_{n=0}^N D_n(u) \right] dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \frac{1}{N+1} \left[\frac{1}{2} + \sum_{n=1}^N D_n(u) \right] dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)} \right]^2 dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) F_N(u) dx.
\end{aligned}$$

Thus, we have

$$\sigma_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) F_N(u) dx. \quad \blacksquare$$

(b) Graph the function $F_N(u)$ for several values of N . Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \rightarrow 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi, \pi)$).

Proof. We can leave the graphing to your favorite Ti-84 calculator. We shall prove that $F_N \rightarrow 0$ uniformly. Let $u \in \{u : |u| \geq \delta\}$ where $\delta > 0$. Let $\varepsilon > 0$. Since $1/2(N+1) \rightarrow 0$, there exists an $M \in \mathbb{N}$ such that for any $N \geq M$, we have

$$\left| \frac{1}{2(N+1)} \right| < \varepsilon.$$

Furthermore, the sine function is bounded by 1. Hence, we can write the following

$$\begin{aligned} \left| \frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)} \right]^2 \right| &\leq \left| \frac{1}{2(N+1)} \right| \cdot \left| \frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)} \right|^2 \\ &\leq \frac{1}{2(N+1)} \\ &< \varepsilon. \end{aligned}$$

Hence, we must also have

$$\frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)} \right]^2 \rightarrow 0$$

uniformly. ■

(c) Prove that $\int_{-\pi}^{\pi} F_N(u) \, du = \pi$.

Proof. We will use the fact that $\int_{-\pi}^{\pi} D_N(u) \, du = \pi$ to show the result. Using the definition of $F_N(u)$ and reordering indices, we can (specifically we will be using the left-hand side of the equation) write

$$\begin{aligned} \int_{-\pi}^{\pi} F_N(u) \, du &= \int_{-\pi}^{\pi} \left[\frac{1}{N+1} \left(\frac{1}{2} + \sum_{n=1}^N D_n(u) \right) \right] \, du \\ &= \frac{1}{N+1} \int_{-\pi}^{\pi} \left(D_0(u) + \sum_{n=1}^N D_n(u) \right) \, du. \\ &= \frac{1}{N+1} \int_{-\pi}^{\pi} \left[\sum_{n=1}^{N+1} D_{n-1}(u) \right] \, du \\ &= \frac{1}{N+1} \sum_{n=1}^{N+1} \left[\int_{-\pi}^{\pi} D_{n-1}(u) \, du \right] \\ &= \frac{\pi}{N+1} \sum_{n=0}^{N+1} 1 \\ &= \frac{\pi}{N+1} \cdot (N+1) \\ &= \pi. \end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} F_N(u) \, du = \pi. \quad \blacksquare$$

(d) To finish the proof of Fejér's Theorem, first choose $\delta > 0$ so that

$$|u| < \delta \text{ implies } |f(x+u) - f(x)| < \varepsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x .

Proof. Let $\varepsilon > 0$. First, we show that $\sigma_n(x) \rightarrow f(x)$ on $|u| \leq \delta$. Using the fact that f is uniformly continuous on $(-\pi, \pi]$, we can write

$$\begin{aligned} |\sigma_n(x) - f(x)| &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) F_N(u) \, du - \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) F_N(u) \, du \right| \\ &= \frac{1}{\pi} \left| \int_{-\pi}^{\pi} (f(u+x) - f(x)) F_N(u) \, du \right| \\ &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(u+x) - f(x)| F_N(u) \, du \\ &< \frac{\varepsilon}{\pi} \int_{-\pi}^{\pi} F_N(u) \, du \\ &= \frac{\varepsilon}{\pi} \cdot \pi = \varepsilon. \end{aligned}$$

Now suppose $|u| \geq \delta$. Since f is uniformly continuous, we can bound f by some $M > 0$. Hence, we must have

$$|f(u+x) - f(x)| \leq 2M.$$

Then we can use the fact that $F_N \rightarrow 0$ on sets such as $|u| \geq \delta$ to write

$$\begin{aligned} |\sigma_n(x) - f(x)| &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(u+x) - f(x)| F_N(u) \, du \\ &\leq \frac{2M}{\pi} \int_{-\pi}^{\pi} F_N(u) \, du \\ &< \frac{2M}{\pi} \cdot \frac{\varepsilon}{4M} \int_{-\pi}^{\pi} du \\ &= \frac{\varepsilon}{2\pi} \cdot 2\pi \\ &= \varepsilon. \end{aligned}$$

■

1.1.8 Weierstrass Approximation Theorem

Fejér's Theorem gives us a nice and concise argument for the Weierstrass Approximation Theorem found in section 6.7. We can restate here as a reminder.

Theorem 4 (Weierstrass Approximation Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there exists a polynomial $p(x)$ satisfying

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$.

We can use the fact that $\sin(x)$ and $\cos(x)$ is bounded on a compact set to show theorem above.

Exercise 8.5.11

- (a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a, b]$ is $[0, \pi]$.

Proof. Let $\varepsilon > 0$. Since $f(x)$ is continuous on $[0, \pi]$, we can pick a $\delta > 0$ such that whenever $|u| < \delta$, we have

$$|f(u+x) - f(x)| < \varepsilon.$$

Observe that

$$\int_0^\pi F_N(u) \, du = \frac{1}{2} \int_{-\pi}^\pi F_N(u) \, du = \frac{\pi}{2}.$$

We can use σ_N defined in Fejér's Theorem to write

$$\begin{aligned} |\sigma_N(x) - f(x)| &= \left| \frac{2}{\pi} \int_0^\pi f(u+x) F_N(u) \, du - \frac{2}{\pi} \int_0^\pi f(x) F_N(u) \, du \right| \\ &= \frac{2}{\pi} \left| \int_0^\pi (f(u+x) - f(x)) F_N(u) \, du \right| \\ &\leq \frac{2}{\pi} \int_0^\pi |f(u+x) - f(x)| F_N(u) \, du \\ &< \frac{2\varepsilon}{\pi} \int_0^\pi F_N(u) \, du \\ &= \frac{2\varepsilon}{\pi} \cdot \frac{\pi}{2} = \varepsilon. \end{aligned}$$

Note that the polynomial we found was just $\sigma_N(x) = p(x)$. Hence, we have that

$$|f(x) - p(x)| < \varepsilon.$$

■

- (b) Show how the case for an arbitrary interval $[a, b]$ follows from this one.

Proof. Taking advantage of f 's uniform continuity on $(-\pi, \pi]$, we can extend uniform continuity of f to all of \mathbb{R} . Using Fejér's Theorem, we can find a polynomial $p(x)$ (in this case, $p(x) = \sigma_N(x)$) such that

$$|f(x) - p(x)| < \varepsilon$$

for any interval $[a, b]$.

■

1.2 A Construction of \mathbb{R} from \mathbb{Q}

In this section, we will focus on constructing a proof for the existence of the Real Numbers.

Theorem 5 (Existence of Real Numbers). There exists an ordered field in which every nonempty set that is bounded above has a least upper bound. In addition, this field contains \mathbb{Q} as a subfield.

A review of chapter 1 tells us that

- (i) We can't do analysis on the set of rational numbers because we don't have the square root defined (nor does it exist) which does not allow us to define the convergence of Cauchy sequences.

A way that we solved this problem is to create the following axiom:

Theorem 6 (Axiom of Completeness). Every nonempty set of real numbers that is bounded above has a least upper bound.

Note that we actually need to demonstrate that we can actually extend the rational numbers to contain nonempty sets that have least upper bounds, thereby making the axiom into a theorem worth proving.

1.2.1 Dedekind Cuts

Let's begin by assuming that all the properties that come with the rational numbers are available to us. Let us also assume that we don't have real numbers at this point.

Definition 1 (Cuts). A subset A of the rational numbers is called a *cut* if it possesses the following three properties:

- (c1) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
- (c2) If $r \in A$, then A also contains every rational $q < r$.
- (c3) A does not have a maximum; that is, if $r \in A$, then there exists $s \in A$ with $r < s$.

Exercise 8.6.1

- (a) Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{t \in \mathbb{Q} : t < r\}$ is a cut.

Proof. For the first property (C1), we show that $C_r \neq \emptyset$ and $C_r \neq \mathbb{Q}$. Suppose $r \in \mathbb{Q}$ is fixed. Now define $t = r - 1$. Then clearly, we have $t < r$. But this means $t \in C_r$. Hence, $C_r \neq \emptyset$. Now, we show $C_r \neq \mathbb{Q}$. Define $s = r + 1 \in \mathbb{Q}$. Then clearly, $s > r$. Hence, $s \notin C_r$ and thus $C_r \neq \mathbb{Q}$.

For the second property (C2), we need to show that if $r \in C_r$, then C_r also contains every rational $q < r$. Suppose we let $t, q \in \mathbb{Q}$ be arbitrary with $t > q$ with $t \in C_r$. Then by definition of C_r , we must have $t < r$. But we have $q < t < r$. Hence, we have $q < r$ for any $q \in \mathbb{Q}$.

Lastly, we need to show that C_r does not have a maximum; that is, for any $x \in C_r$, there exists an $s \in C_r$ with $x < s$. Let $x \in C_r$. Then by definition of C_r , we must have $x < r$. Suppose we take the midpoint between x and r . Then we must have $x < \frac{x+r}{2} < r$. Setting $s = \frac{x+r}{2}$ gives us our desired result. ■

Avoid thinking of cuts as only having this form. Which of the following subsets of \mathbb{Q} are cuts?

- (b) $S = \{t \in \mathbb{Q} : t \leq 2\}$

Proof. This is not a cut because S contains a maximum. In this case, the maximum is 2. ■

- (c) $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$

Proof. The set T is a cut. First, we show (C1). Observe that $0 \in T$. Hence, $T \neq \emptyset$. Then observe that $2 \notin T$. Hence, $T \neq \mathbb{Q}$.

Next, we show (C2). Let $r \in T$. Then by definition of T , we must have either $r^2 < 2$ or $r < 0$. If $r < 0$, then we can easily pick $q < r$ for any $q < 0$. Otherwise, we have $r^2 < 2$ such that if we let $q \in \mathbb{Q}$ be arbitrary with $q^2 < r^2$. Then we easily have $q < r$.

Finally, we show (C3). Let $r \in T$. Then either we have $r^2 < 2$ or $r < 0$. Consider

$$\begin{aligned} \left(r + \frac{1}{n}\right)^2 &= r^2 + \frac{2r}{n} + \frac{1}{n} \\ &= r^2 + \frac{2r+1}{n}. \end{aligned}$$

Let us set $s = r + \frac{1}{n_0}$. If $r > 0$, then we can pick an n_0 large enough such that

$$\frac{1}{n_0} < \frac{4 - r^2}{2r + 1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < r^2 + \frac{2r+1}{n_0} \cdot \frac{4 - r^2}{2r + 1} < 4.$$

Hence, we have $r < s < 2$. Otherwise, we choose n_0 large enough so that

$$\frac{1}{n_0} < \frac{-r^2}{2r + 1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < 0.$$

This implies that $r < s < 0$. ■

(d) $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$.

Proof. ■

Exercise 8.6.2

Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Proof. Suppose for sake of contradiction that $r \geq s$. Since $r \in A$ and $s \in \mathbb{Q}$, we know that $s < r$. But this means that $s \in A$ which leads to a contradiction. ■

Definition 2. Define the *real numbers* \mathbb{R} to be the set of all cuts in \mathbb{Q} .

Here we have defined the set \mathbb{R} whose elements are subsets of \mathbb{Q} which is awkward because we think of numbers as elements of a set rather than the sets being elements themselves. Our goal now is to create an algebraic structure on \mathbb{R} . This involves answering the following question

"What do we mean by an ordered field?".

1.2.2 Field and Order Properties

Suppose we were given a set F and two elements $x, y \in F$. The *operation* that we impose on F takes x and y as an ordered pair (x, y) and maps it to a third element $z \in F$. Here we are trying to emulate our basic notions of adding and multiplying two numbers.

Definition 3 (Field). A set F is a *field* if there exists two operations — addition $(x + y)$ and multiplication (xy) — that satisfy the following list of conditions:

- (f1) (commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.
- (f2) (associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (f3) (identities exist) There exist two special elements 0 and 1 with $0 \neq 1$ such that $x + 0 = x$ and $1 \cdot x = x$ for all $x \in F$.
- (f4) (inverses exist) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$. If $x \neq 0$, there exists an element x^{-1} such that $xx^{-1} = 1$.
- (f5) (distributive property) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Exercise 8.6.3

Using the usual definitions of addition and multiplication, determine which of these properties are possessed by \mathbb{N} , \mathbb{Z} , and \mathbb{Q} , respectively.

Proof. (\mathbb{N}) This is not a field since an additive inverse does not exist; that is, $-1 \notin \mathbb{N}$.

(\mathbb{Z}) The set of integers is not a field because every element in $x \in \mathbb{Z}$ does not contain an inverse except for 1.

(\mathbb{Q}) Field. ■

Just an aside, we can actually use the field properties to show, for example, that for any $x, y, z \in \mathbb{Q}$, we have that $x + y = x + z$ implies $y = z$.

Definition 4. Ordering An *ordering* on a set F is a relation, represented by \leq , with the following three properties:

For arbitrary $x, y \in F$, at least one of the statements $x \leq y$ or $y \leq x$ is true.

(o1) If $x \leq y$ and $y \leq x$, then $x = y$.

(o3) If $x \leq y$ and $y \leq z$, then $x \leq z$. Note that writing $y \geq x$ is equivalent to $x \leq y$. The strict

inequality $x < y$ is used to mean $x \leq y$ but $x \neq y$.

(o4) If $y \leq z$, then $x + y \leq x + z$.

(o5) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

We have thus far defined \mathbb{R} to be the collection of \mathbb{Q} cuts. Now we want to invent the ordering and basic operations outlined in the last two definitions. The easiest out of these is the ordering. Let A and B be two arbitrary elements of \mathbb{R} . Define $A \leq B$ to mean $A \subseteq B$.

Exercise 8.6.4

Show that this defines an ordering on \mathbb{R} by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Proof. Observe that the first property (o1) follows immediately. For the second property (o2), suppose $A \leq B$ and $B \leq A$. Then we have $A \subseteq B$ and $B \subseteq A$. This implies that $A = B$. Finally, for the third property (o3), let $A, B, C \in \mathbb{R}$. If $A \leq B$ and $B \leq C$, then we have $A \subseteq B$ and $B \subseteq C$. Then we have $A \subseteq B \subseteq C$. Hence, we have $A \subseteq C$. Hence, we have an ordering on \mathbb{R} . ■

1.2.3 Algebra in \mathbb{R}

Given A and B in \mathbb{R} , define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Before trying to verify if $A + B$ defines an operation, we have first ask if $A + B$ defines a cut. Suppose we verify (c2) of our definition of a cut. Suppose A, B are cuts. Let $a + b \in A + B$ be arbitrary and let $s \in \mathbb{Q}$ satisfy $s < a + b$. Then, $s - b < a$, which implies that $s - b \in A$ because A is a cut. But then

$$s = (s - b) + b \in A + B,$$

and (c2) is proved.

Exercise 8.6.5

(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

Proof. For property (c1), let $a \in A$. Then $a = (a - b) + b$ where $a - b \in A$ and $b \in B$ implies $a \in A + B$. Hence, we have that $A + B \neq \emptyset$. Let $c \in \mathbb{Q}$ where $C \subseteq \mathbb{Q}$, then we find that $A + C \not\subseteq A + B$. Hence, the sum $a + c \notin A + B$. Hence, $A + B \neq \mathbb{Q}$. To show property (c3), Let $a + b \in A + B$ be arbitrary. Since A is a cut, we can find an $\alpha \in A$ such that $a < \alpha$. Likewise, B being a cut implies that there exists $\beta \in B$ such that $b < \beta$. Adding the two inequalities together, we get that $a + b < \alpha + \beta$. Denote the set $S = \{\alpha + \beta : \alpha \in A \text{ and } \beta \in B\}$. Then we have $A + B \subseteq S$ which proves property (c3). Since properties (c1), (c2), and (c3) have been satisfied, we can now conclude that $A + B$ is a cut. ■

(b) Check that addition in \mathbb{R} is commutative (f1) and associative (f2).

Proof. Since $A + B$ is a cut and its elements $a + b$, where $a \in A$ and $b \in B$ are both cuts, subsets of \mathbb{Q} where \mathbb{Q} is an ordered field, we know that addition must be commutative and associative. This means that

$$A + B = B + A.$$

If we have an extra set C that is also a cut, then we would have $(a + b) + c = a + (b + c)$ implying that

$$(A + B) + C = A + (B + C).$$

Hence, we have (f1) and (f2) are satisfied. ■

(c) Show that property (o4) holds.

Proof. Suppose we have $A \subseteq C$ where A and C are both cuts. Let B be a cut. Then $A \subseteq C$ implies $a \leq c$. If we have $b \in B$ be arbitrary, then we can use the ordering of \mathbb{Q} to write $b + a \leq b + c$. But this means that $B + A \subseteq B + C$ and hence property (o4) is satisfied. ■

(d) Show that the cut

$$O = \{p \in \mathbb{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions : $A + O \subseteq A$ and $A \subseteq A + O$.)

Proof. First, we show $A + O \subseteq A$. Let $a + p \in A + O$ be arbitrary. Since $p < 0$, we must have $a + p \leq a$. But this means that $a + p \in A$. For the second inclusion $A \subseteq A + O$, let $a \in A$. Then observe that $a = (a - p) + p$. Since $p < 0$, we can write $a = (a - p) + p$. But since $a - p \in A$ and $p \in O$, we know that $a \in A + O$. Hence, we have $A \subseteq A + O$. ■

Suppose we try and prove additive inverses? Given an $A \in \mathbb{R}$, we need to construct a cut $-A$ such that $A + (-A) = O$. Without the help of the supremum, which we can't even use because it is strictly defined on \mathbb{R} without inherently assuming that it exists, how do we go about defining this set?

Suppose we are given $A \in \mathbb{R}$, then define

$$-A = \{r \in \mathbb{Q} : \text{there exists } t \notin A \text{ with } t < -r\}.$$

Exercise 8.6.6

(a) Prove that $-A$ defines a cut.

Proof. To show the (c1), let $r \in \mathbb{Q}$. Since \mathbb{Q} is a field, we can rewrite r in the following way

$$r = r + (t - t) = (r + t) - t = (t + r) - t < -t$$

with $t \notin A$. Multiplying through the inequality by a negative leads us to $-r > t$. This tells us that $r \in -A$ and we must have $-A \neq \emptyset$. However, this also implies that $-r \notin -A$ and hence, $-A \neq \mathbb{Q}$. To show (c2), let $r \in -A$. Then by definition of $-A$, there exists a $t \notin A$ such that $-r > t$. Suppose we let $-q \in \mathbb{Q}$ be arbitrary with $-q > -r$. Using the fact that \mathbb{Q} is a field, we can multiply the inequality by a negative to get $r > q$ our desired result. Hence, (c2) is satisfied.

Finally, we show (c3). Suppose $r \in -A$. Then there exists $t \notin A$ such that $-r > t$. Multiplying through the inequality by a negative gives us $r < -t$ which is our desired result. Hence, $-A$ defines a cut. ■

(b) What goes wrong if we set $-A = \{r \in \mathbb{Q} : -r \in A\}$?

Proof. If we have $-A$ defined as the set above, then it would fail (c3). We can see this when we define $A = \{r \in \mathbb{Q} : r < 0\}$ and let $-A = \{r \in \mathbb{Q} : r \leq 0\}$ which tells us that $-A$ contains a maximum.

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in the definition of a field.

Proof. Let $a \in A$ and $r \in -A$. Let $a + r \in O$. Since $r \in -A$, we know that there exists $t \notin A$ with $t < -r$. Now, we can write $a + r < a - t$. But $-t < 0$ so this tells us that $a + r < 0$. Hence, $a + r \in O$ and we have $A + (-A) \subseteq O$. Now we need to show $O \subseteq A + (-A)$. Let us fix $o \in O$ and find $a \in A$ such that $a + b = o$. Let $\varepsilon = |o|/2 = -o/2$. Taking advantage of properties (c1) and (c2), we can choose a rational $t \notin A$ such that $t - \varepsilon \in A$. If no such t existed then we would either have $A = \mathbb{Q}$ or $A = \emptyset$. Now, $t \notin A$ implies $-(t + \varepsilon) \in -A$. Then

$$o = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A),$$

and hence we conclude $O \subseteq A + (-A)$. Hence, (f4) is proven. ■

Remark. Another possible way you could show the reverse inclusion in part (c) is to rewrite $o \in O$ such that $o = (o + r) - r$ and show that $0 + r \in -A$ for which, in addition to $-r \in A$, leads to

$O \subseteq A + (-A)$. Since there exists $t \notin A$ such that $-r > t$ and $o < 0$, we have

$$o + r < r \Rightarrow o + r < r < -t.$$

Multiplying the inequality by a negative gives us $-(o+r) < t$, which tells us that $o+r \in -A$. Since we also have $-r \in A$, we conclude that $o \in A + (-A)$ and hence $O \subseteq A + (-A)$.

Suppose we try to create the structure for multiplication using cuts. This can be quite difficult because of the fact that the product of two negative numbers is positive. A way to do this is to define multiplication on non-negative cuts.

Given $A \geq O$ and $B \geq O$ in \mathbb{R} , define the product

$$AB = \{ab : a \in A, b \in B \text{ with } a, b \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

Exercise 8.6.7

- (a) Show that AB is a cut and that property (o5) holds.

Proof. To show (c1), let $a, b \in \mathbb{Q}$ with the property that $a, b \geq 0$. Since \mathbb{Q} is a field, we can take the product of a and b and get $a \cdot b \geq 0$ implying that $AB \neq \emptyset$. Likewise, if we take any $p \in \mathbb{Q}$ with $p < 0$ and multiply it by any strictly non-negative $a \in \mathbb{Q}$, then $p \cdot a < 0$ implying that $AB \neq \emptyset$ once again. We know that $AB \neq \emptyset$ since $q \leq 0$ is not in AB . Hence, (c1) is satisfied.

To show (c2), we need to show that for all $q \in \mathbb{Q}$ that $r \in AB$ implies $q < r$. Suppose $r = a \cdot b$ with $a \in A$ and $b \in B$. Since A and B are cuts, we can use (c2) to state that for all $\alpha \in A$ and for all $\beta \in B$, we have such $\alpha < a$ and $\beta < b$ respectively. Then taking the product of these inequalities, we get our desired result that $q = \alpha \cdot \beta < a \cdot b = r$. Hence, (c2) is satisfied.

Finally, we show (c3). Let $r \in AB$ where $r = a \cdot b$ where $a \in A$ and $b \in B$. Since A and B are cuts, we can find $w \in A$ and $z \in B$ such that $r = a \cdot b < z \cdot w = \omega$. Hence, (c3) is satisfied.

Now we need to show that property (o5) holds for AB . Assume $A \geq O$ and $B \geq O$. We know that $O \subseteq A$ and $O \subseteq B$. This means that if we let $p, w \in O$, then $p, w \in A$ and $p, w \in B$. Note that $p \cdot w < 0$. Since $ab \geq 0$, we know that $pw \leq ab$ which implies that $pw \in AB$ and hence, $O \subseteq AB$. ■

- (b) Propose a good candidate for the multiplicative identity on \mathbb{R} and show that this works for all cuts $A \geq O$.

Proof. We propose the following candidate for the multiplicative identity on \mathbb{R} : For $A \geq O$, we have

$$A^{-1} = \{a \in \mathbb{Q} : \exists p \in \mathbb{Q} \Rightarrow a \cdot p^{-1} < 1 \text{ and } p > 0 \text{ with } p^{-1} \notin A\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

■

- (c) Show the distributive property (f5) holds for non-negative cuts. Before moving on to the proof, define $A(B+C)$ as

$$A(B+C) = \{a(b+c) : a \in A, b+c \in B+C \text{ with } a, b+c \geq 0\} \cup \{p \in \mathbb{Q} : p < 0\}.$$

Then let

$$P = \{a(b+c) : a \in A, b+c \in B+C \text{ with } a, b+c \geq 0\}.$$

Proof. The main goal is to show that $A(B+C) = AB + AC$; that is, we need to show the following two inclusions:

$$A(B+C) \subseteq AB + AC \text{ and } AB + AC \subseteq A(B+C).$$

Since $x \in A(B+C)$, then either $x \in P$ or $x \in O$. Suppose we have $x \in P$. Then we have $x = a(b+c)$. Since \mathbb{Q} is an ordered field, we can use the distributive property to write $a(b+c) = ab+ac \in AB+BC$. Hence, $a(b+c) \in AB+BC$ and $A(B+C) \subseteq AB+BC$. Now suppose $x \in O$, then $x < 0$. Since the products AB and BC are non-negative, we know that

we must have $ab \geq 0$ and $bc \geq 0$. This tells us that their sum $ab + ac \geq 0$ and hence, we have $x < 0 \leq ab + ac$. Since $x \leq ab + ac$, we can write $O \subseteq AB + AC$.

Next, we show $AB + BC \subseteq A(B + C)$. Let $x \in AB + BC$. Then $x = r_1 + r_2$ where $r_1 \in AB$ and $r_2 \in BC$. If $r_1 \in AB$, then either $r_1 = ab \geq 0$ or $r_1 < 0$. Likewise, $r_2 \in BC$ implies that either $r_2 = ac \geq 0$ or $r_2 < 0$. Suppose $r_1 = ab \geq 0$ and $r_2 = ac \geq 0$. Then we can simply use the distributive property to write $ab + ac = a(b + c) \in A(B + C)$. Hence, $ab + ac \in A(B + C)$. Now suppose $r_1 \geq 0$ and $r_2 < 0$. Then $x = r_1 + r_2 < r_1 + 0$. But note that $AB + O \subseteq O$ which tells us that $x \in A(B + C)$. The case for when $r_2 \geq 0$ and $r_1 < 0$ is similar. If both $r_1 < 0$ and $r_2 < 0$, then $x = r_1 + r_2 < 0 \leq a(b + c)$ and hence, $x \in A(B + C)$. Thus, we can conclude that $AB + BC \subseteq A(B + C)$. ■

We can define products of two positive cuts by observing that $-A \geq O$ whenever $O \leq A$. In addition, given any $A \leq O$, we can use property (o4) to write $A + (-A) \leq O + (-A)$ which results in $O \leq -A$.

Now let $A, B \in \mathbb{R}$ and define the following possible cases for which we can multiply A and B :

$$AB = \begin{cases} \text{as given} & \text{if } A \geq O \text{ and } B \geq O \\ -[A(-B)] & \text{if } A \geq O \text{ and } B \leq O \\ -[(-A)B] & \text{if } A \leq O \text{ and } B \geq O \\ (-A)(-B) & \text{if } A \leq O \text{ and } B \leq O. \end{cases}$$

We leave these as exercises for you to do. The proofs follow the same pattern as the ones we have done so far with cases usually falling into when terms are either positive or negative.

1.2.4 Least Upper Bounds

We can now use the fact that \mathbb{R} is an ordered field to show that the field is complete. Recall in Chapter 1 that we define "complete" in terms of least upper bounds. Below we restate the definitions seen in Chapter 1.

Definition 5. A set $\mathcal{A} \subseteq \mathbb{R}$ is *bounded above* if there exists a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in \mathcal{A}$. The number B is called an *upper bound* for \mathcal{A} .

A real number $S \in \mathbb{R}$ is the *least upper bound* for a set $\mathcal{A} \subseteq \mathbb{R}$ if it meets the following two criteria:

- (i) S is an upper bound for \mathcal{A} and
- (ii) if B is any upper bound for \mathcal{A} , then $S \leq B$.

Exercise 8.6.8

Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

- (a) First, prove that $S \in \mathbb{R}$ by showing that it is a cut.

Proof. First, we show (c1). Since S is defined as the union of nonempty sets A , we also have that $S \neq \emptyset$. Since all the sets within S are just a union of cuts, we know that these cuts also have the property that $A \neq \mathbb{Q}$ and hence $S \neq \mathbb{Q}$.

Next, we show property (c2). Let $x \in S$. Since S is defined as the union of all cuts $A \in \mathcal{A}$, we know that $x \in S$ implies that there exists an A' such that $x \in A'$. Let $q \in \mathbb{Q}$. But since A' is a cut, we know that we can use (c2) to say that $q < x$. Hence, (c2) is satisfied.

Lastly, we show property (c3). Let $x \in S$. Using the same reasoning to show (c2), we know that $x \in A'$ implies that there exists $\alpha \in \mathbb{Q}$ such that $x < \alpha$.

Since all the properties of a cut are satisfied, we can now conclude that $S \in \mathbb{R}$ is also a cut. ■

- (b) Now, show that S is the least upper bound for \mathcal{A} .

Proof. In order to show that S is the least upper bound of \mathcal{A} , we need to show that S is an upper bound and that S is the least upper bound.

First, we show S is an upper bound; that is, $S \geq A$ for all $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be arbitrary.

Observe that since $S = \bigcup A$, we know that $A \subseteq \bigcup A = S$. But this tells us that $A \leq S$. Since A is arbitrary, we know that S must be an upper bound.

Lastly, we show that S is the *least upper bound*. Let $B \in \mathcal{A}$ be any upper bound. Since each set A in the union $S = \bigcup A$ is bounded by B (that is, $A \subseteq B$), it follows that $S \subseteq B$. But this tells us that $S \leq B$ and hence, S must be the least upper bound of \mathcal{A} . ■

This finishes the proof that \mathbb{R} is complete. However, we need to solve the problem of the slight mistake in saying that \mathbb{R} contains \mathbb{Q} as a subfield. We can solve this issue by saying that \mathbb{R} contains a subfield that looks and acts exactly like \mathbb{Q} .

Exercise 8.6.9

Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbb{Q} : t < r\}$$

where $r \in \mathbb{Q}$. (See Exercise 8.6.1)

- (a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbb{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

Proof. Let $r, s \in \mathbb{Q}$. First, we show $C_r + C_s = C_{r+s}$; that is, we need to show that following two inclusions:

$$C_r + C_s \subseteq C_{r+s} \text{ and } C_{r+s} \subseteq C_r + C_s.$$

Let $z \in C_r + C_s$. ■

- (b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in \mathbb{Q} .

Proof. ■