Homework 4

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Problem 1. Prove the following theorem.

Theorem (Generalize Mean Value Theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $g : [a,b] \to \mathbb{R}$ is integrable and either $g \geq 0$ on $[a,b]$ or $g \leq 0$ on $[a,b]$. Then there exists a $c \in [a,b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

Proof. Our goal is to find a $c \in [a,b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function and $[a,b]$ is a compact set, f attains its maximum and minimum on $[a,b]$. Then

$$m = \sup_{x \in [a,b]} f(x) \text{ and } m = \inf_{x \in [a,b]} f(x)$$

and hence,

$$m \leq f(x) \leq M \implies mg(x) \leq f(x)g(x) \leq Mg(x) \ \forall x \in [a,b]. \tag{$\dagger$}$$

Now, note that the result follows immediately if $g(x) = 0$ for all $x \in [a,b]$. Hence, suppose that $g(x) \neq 0$ on $[a,b]$; that is, we either have $g(x) > 0$ or $g(x) < 0$. Suppose $g(x) > 0$ on $[a,b]$ (The proof to the other case is completely analogous). By the Order Theorem for Integrals, ($\dagger$) implies

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g. \tag{1}$$

Furthermore, $g(x) > 0$ on $[a,b]$ implies that

$$\int_a^b g > 0.$$

Hence, we can divide by $\int_a^b g$ on (1). So, we have

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

But note that f is continuous on $[a,b]$ and so, by the Intermediate Value Theorem, there exists a $c \in [a,b]$ such that

$$f(c) = \frac{\int_a^b fg}{\int_a^b g} \implies \int_a^b fg = f(c) \int_a^b g$$

which is our desired result. ■

Problem 2. Prove the following theorem.

Theorem (Rudin, Theorem 6.15)**.** If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f \; d\alpha = f(s).$$

Proof. Our goal is to show that $\int_a^b f \; d\alpha = f(s)$. Define the partition

$$P_n = \left\{a, s - \frac{1}{n}, s + \frac{1}{n}, b\right\}.$$

Since $f \in R_\alpha[a, b]$, we have

$$\int_a^b f \; d\alpha = \lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n). \tag{*}$$

So, it suffices to show that

$$\lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n) = f(s).$$

Note that

$$\alpha(x) = I(x - s) = \begin{cases} 1 & \text{if } x > s \\ 0 & \text{if } x \leq s \end{cases}$$

By definition of α, we have

$$\begin{aligned} k = 1; \Delta\alpha_1 &= \alpha(s - \delta) - \alpha(a) = 0 - 0 = 0 \\ k = 2; \Delta\alpha_2 &= \alpha(s + \delta) - \alpha(s - \delta) = 1 - 0 = 1 \\ k = 3; \Delta\alpha_3 &= \alpha(b) - \alpha(s + \delta) = 1 - 1 = 0. \end{aligned}$$

Hence, we have

$$U(f, \alpha, P_n) = \sum_{k=1}^{3} M_k \Delta\alpha_k = M_2 = \sup_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x)$$

and similarly,

$$L(f, \alpha, P_n) = \sum_{k=1}^{3} m_k \Delta\alpha_k = m_2 = \inf_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x).$$

Because f is continuous at s, we know by exercise 4 of homework 3 that

$$\lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} \sup_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x) = f(s)$$

and

$$\lim_{n\to\infty} L(f, \alpha, P_n) = \lim_{n\to\infty} \inf_{x\in[s-\frac{1}{n}, s+\frac{1}{n}]} f(x) = f(s)$$

which is our desired result. ■

Problem 3. Prove the following theorem.

Theorem (Rudin, Theorem 6.16)**.** (a) Let $N \in \mathbb{N}$. Let $c_1, \dots, c_N$ be nonnegative numbers. Suppose $s_1, \dots, s_N$ are distinct points in (a, b), and let $\alpha(x) = \sum_{n=1}^{N} c_n I(x - s_n)$. Let f be continuous at s. Then

$$\int_a^b f\ d\alpha = \sum_{n=1}^{N} c_n f(s_n).$$

(b) Suppose $c_n \geq 0$ for $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} c_n$ converges, (s_n) is a sequence of distinct points in (a, b), and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous at s. Then

$$\int_a^b f\ d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof. (i) Denote $\alpha_n = I(x - s_n)$ for $1 \leq n \leq N$. Applying the previous problem N number f times, we can see that

$$\int_a^b f\ d\alpha_n = f(s_n).$$

Then using problem 5 from homework 3, we have

$$\begin{aligned}\int_a^b f\ d\alpha &= \int_a^b\ d\Big(\sum_{n=1}^{N} c_n\alpha_n\Big)\\ &= \sum_{n=1}^{N}\int_a^b f\ d(c_n\alpha_n)\\ &= \sum_{n=1}^{N} c_n \int_a^b f\ d\alpha_n\\ &= \sum_{n=1}^{N} c_n f(s_n).\end{aligned}$$

(ii) Note that $\sum_{n=1}^{\infty} c_n f(s_n) = \lim\limits_{N\to\infty} \sum_{n=1}^{N} c_n f(s_n)$. Using part (ii) and applying the limit as $N \to \infty$, we have

$$\begin{aligned}\lim_{N\to\infty}\int_a^b f\ d\alpha &= \lim_{N\to\infty}\sum_{n=1}^{N} c_n f(s_n)\\ \implies \int_a^b f\ d\alpha &= \sum_{n=1}^{\infty} c_n f(s_n).\end{aligned}$$

Note that we can apply the limit above because $\int_a^b f\ d\alpha$ is just a real number and applying the limit to it will keep it the same.

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Problem 4. Let $p, q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

4-1) Prove that if $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p\ d\alpha = 1 = \int_a^b g^q\ d\alpha,$$

then $\displaystyle\int_a^b fg\ d\alpha \leq 1$.

4-2) Prove that if $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, then

$$\int_a^b |fg|\ d\alpha \leq \Big[\int_a^b |f|^p\ d\alpha\Big]^{\frac{1}{p}}\Big[\int_a^b |g|^q\ d\alpha\Big]^{\frac{1}{q}}.$$

Proof. 4-1) Since $f \geq$, $g \geq 0$, and $p, q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}.$$

Note that if $f, g = 0$ on $[a, b]$, then the result immediately holds. So, suppose $f, g > 0$ on $[a, b]$. Since $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, we can use the Order Theorem for Integrals and the Algebraic Theorem for integrals to write

$$\begin{aligned}
\int_a^b |fg| \, d\alpha &= \int_a^b fg \, d\alpha && (fg > 0) \\
&\leq \int_a^b \left[\frac{f^p}{p} + \frac{g^q}{q}\right] \, d\alpha \\
&= \int_a^b \frac{f^p}{p} \, d\alpha + \int_a^b \frac{g^q}{q} \, d\alpha \\
&= \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Hence, we have

$$\int_a^b |fg| \, d\alpha \leq 1.$$

4-2) Using the fact in 4-1), we have that

$$\begin{aligned}
\int_a^b |fg| \, d\alpha &= \int_a^b fg \, d\alpha \\
&\leq 1 \\
&= 1 \cdot 1 \\
&= 1^{1/p} \cdot 1^{1/q} \\
&= \left(\int_a^b f^p d\alpha\right)^{1/p} \cdot \left(\int_a^b g^q \, d\alpha\right)^{1/q}
\end{aligned}$$

which is our desired result.

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Problem 5. (a) Let f be continuous on $[a, b]$ such that for any subinterval $[c, d] \subseteq [a, b]$

$$\int_c^d f(t) \, dt = 0.$$

Prove that $f(x) = 0$ on $[a, b]$.

(b) Let f be continuous on $[a, b]$ such that

$$\int_a^x f(t) \, dt = \int_x^b f(t) \, dt, \ \forall x \in [a, b].$$

Proof. (a) Let f be continuous on $[a, b]$ such that for any subinterval $[c, d] \leq [a, b]$

$$\int_c^d f(t)\ dt = 0.$$

Suppose for sake of contradiction that $f(c) \neq 0$ for some $c \in [a, b]$. Since f is continuous on $[a, b]$, it follows that f is continuous at c. We have two cases to consider; that is, either

(1) $f(c) > 0$

(2) $f(c) < 0$

We will prove the case where $f(c) > 0$. The other case will follow analogously. Indeed, since f is continuous at c, we have

$$f(c) > 0 \Longrightarrow \exists \delta > 0 \text{ such that } \forall x \in (c - \delta, c + \delta)\ f(x) > 0.$$

By the Order Theorem for Integrals, we have

$$f(x) > 0 \Longrightarrow \int_{c-\delta}^{c+\delta} f(x)\ dx > 0.$$

But this contradicts our assumption. Hence, it must follow that $f(x) = 0$ for all $x \in [a, b]$.

(b) Let f be continuous on $[a, b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt\ \forall x \in [a, b].$$

Denote $F(x) = \int_a^x f(t)\ dt$ and $G(t) = \int_x^b f(t)\ dt = -\int_b^x f(t)\ dt$. Suppose for sake of contradiction that there exists a $c \in [a, b]$ such that $f(c) \neq 0$. Then by FTC II, we have $F'(c) = f(c)$ and $G'(c) = -f'(c)$. But by assumption, $f(c) = -f(c)$ which is absurd. Hence, it must be the case that $f(x) = 0$ for all $x \in [a, b]$.

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Problem 6. Suppose $f \in C^1[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)\ dx = 1$. Prove that

$$\int_a^b x f(x) f'(x)\ dx = \frac{-1}{2}$$

and

$$\int_a^b [f'(x)]^2\ dx \cdot \int_a^b x^2 f^2(x)\ dx \geq \frac{1}{4}.$$

Proof. Using Integration by Parts, we have

$$\begin{aligned}
\int_a^b x f(x) f'(x)\ dx &= x f^2(x)\Big]_a^b - \int_a^b f(x)[f(x) + x f'(x)]\ dx \\
&= x f^2(x)\Big]_a^b - \int_a^b f^2(x)\ dx - \int_a^b x f(x) f'(x)\ dx \\
&= [b f^2(b) - a f^2(a)] - 1 - \int_a^b x f(x) f'(x)\ dx \\
&= -1 - \int_a^b x f(x) f'(x)\ dx.
\end{aligned}$$

Hence, we have

$$2\int_a^b x f(x) f'(x)\ dx = -1 \Longrightarrow \int_a^b x f(x) f'(x)\ dx = \frac{-1}{2}.$$

From our result, we can see that

$$\int_a^b -(xf(x)f'(x))\ dx = \frac{1}{2}.$$

By Holder's Inequality for Integrals, we have

$$\begin{aligned}
&\int_a^b |xf(x)f'(x)|\ dx \le \Big(\int_a^b [f'(x)]^2\ dx\Big)^{1/2}\Big(\int_a^b (xf(x))^2\ dx\Big)^{1/2} \\
&\implies \int_a^b -xf(x)f'(x)\ dx \le \Big(\int_a^b [f'(x)]^2\ dx\Big)^{1/2}\Big(\int_a^b (xf(x))^2\ dx\Big)^{1/2} \\
&\implies \frac{1}{4} \le \int_a^b [f'(x)]^2\ dx \cdot \int_a^b (xf(x))^2\ dx.
\end{aligned}$$

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Problem 7 (A Substitution Formula For Ordinary Riemann Integrals)**.** Suppose that

- $g \in C^1[a,b]$ and $f \in C^0[c,d]$.
- The range of $g, g([a,b]) = \{g(x) : x \in [a,b]\}$, is contained in $[c,d]$ (so that the composition $f \circ g$ is defined).

7-1) Explain why f is the derivative of some function.

7-2) Explain why $(f \circ g)g'$ is a derivative of some function.

7-3) Prove the substitution formula:

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Proof. 7-1) Define the function $F : [g(a), g(b)] \to \mathbb{R}$ by

$$F(x) = \int_{g(a)}^{x} f(t)\ dt.$$

The above integral holds because $f \in R[c,d]$ by assumption. Since f is continuous for any $x \in [c,d]$ it follows that $F'(x) = f(x)$ for any $x \in [g(a), g(b)]$ by the Second Fundamental Theorem of Calculus.

7-2) Since $g \in C^1[a,b]$, we have g is continuous on $[a,b]$ and that g' exists and is continuous on $[a,b]$. Since $f \in C[a,b]$, it follows that $f \circ g \in C[a,b]$. Hence, by a Theorem proved in class, we have $f \circ g \in R[a,b]$. Furthermore, since $g' \in C[a,b]$, we have that $g' \in R[a,b]$. So, the product $(f \circ g)g' \in R[a,b]$. Define the function $H : [a,b] \to \mathbb{R}$ by

$$H(x) = \int_a^x (f \circ g)(x)g'(x)\ dx.$$

Clearly, we also have that $(f \circ g)g' \in C[a,b]$. The Fundamental Theorem of Calculus implies that for any $x \in [a,b]$, $H'(x) = (f \circ g)g'(x)$.

7-3) Our goal is to show that

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

It suffices to show that $H(b) - H(a) = F(g(b)) - F(g(a))$. Since $[g(a), g(b)] \subseteq [c,d]$, we can use (1) to write $(F(g(x)))' = f(g(x))g'(x)$ for all $x \in [g(a), g(b)]$. Using the First Fundamental

Theorem of Calculus, we have

$$H(b) - h(a) = \int_a^b f(g(x))g'(x)\ dx = \int_a^b (F(g(x)))'\ dx = F(g(b)) - F(g(a)).$$

Likewise, we have

$$\int_{g(a)}^{g(b)} f(u)\ du = \int_{g(a)}^{g(b)} F'(u)\ du = F(g(b)) - F(g(a))$$

which is our desired result.

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Problem 8. Prove the following integration by parts for "improper" Riemann Integrals:

Theorem (Integration by Parts of Improper Riemann Integrals)**.** Let $a \in \mathbb{R}$, $u : [a, \infty) \to \mathbb{R}$ and $v : [a, \infty) \to \mathbb{R}$ are differentiable, $\forall b > a\ u', v' \in R[a, b]$. Additionally, assume that $\int_a^\infty vu'\ dx$ exists in $\mathbb{R}$ and

$$\lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] \text{ exists (in } \mathbb{R}).$$

Then $\int_a^\infty uv'\ dx$ exists in $\mathbb{R}$ and

$$\int_a^\infty uv'\ dx = \lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] - \int_a^\infty vu'\ dx.$$

Proof. By assumption u' exists and so $u \in C[a, b]$. Hence, $u \in R[a, b]$ as an immediate consequence. Since $v' \in R[a, b]$, we can conclude that the product $v'u \in R[a, b]$. By the Ordinary Riemann Integration by Parts, we have that

$$\int_a^b u(x)v'(x)\ dx = [u(b)v(b) - u(a)v(b)] - \int_a^b u'(x)v(x)\ dx.$$

By assumption, the limit as $b \to \infty$ of each term on the right-hand side of the above equation holds. Thus, we have that

$$\lim_{b\to\infty} \int_a^b uv'\ dx \text{ exists.}$$

Hence, we have that

$$\begin{aligned}\lim_{b\to\infty} \int_a^b u(x)v'(x)\ dx &= \lim_{b\to\infty} \left[(u(b)v(b) - u(a)v(a)) - \int_a^b u'(x)v(x)\ dx\right] \\ &= \lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] - \lim_{b\to\infty} \int_a^b u'(x)v(x)\ dx \quad \text{(ALT for Functions)}\end{aligned}$$

which can be re-written into

$$\int_a^\infty u'v\ dx = \lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] - \int_a^\infty uv'\ dx$$

as our desired result. ■

Problem 9. Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Let $c > a$. Prove that the improper integral $\int_a^\infty f(x)\ dx$ converges if and only if the improper integral $\int_c^\infty f(x)\ dx$ converges.

Proof. ($\Longrightarrow$) Suppose that $\int_a^\infty f(x)\, dx$ converges. Our goal is to show that

$$\int_c^\infty f(x)\, dx \text{ exists}$$

We will consider two cases; that is, either $b < c$ or $c < b$. Suppose $b < c$. Then by the segment addition property of the Riemann Integral, we have that

$$\int_a^c f = \int_a^b f + \int_b^c f = \int_a^b f - \int_c^b f.$$

Hence, we have

$$\int_c^b f = \int_a^b f - \int_a^c f.$$

By the Algebraic Limit Theorem for Functions, we have that

$$\begin{aligned}\int_c^\infty f = \lim_{b\to\infty} \int_c^b f &= \lim_{b\to\infty} \Big[\int_a^b f - \int_a^c f \Big] \\ &= \lim_{b\to\infty} \int_a^b f - \lim_{b\to\infty} \int_a^c f && \text{(ALT for Functions)} \\ &= \int_a^\infty f - \int_a^c f.\end{aligned}$$

Hence, we see that $\int_c^\infty f$ exists. Now, suppose that $c < b$. Then by the segment addition property of integration, we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Hence, we have

$$\int_c^b f = \int_a^b f - \int_a^c f$$

and so applying the limit as $b \to \infty$, we get

$$\begin{aligned}\int_c^\infty f = \lim_{b\to\infty} \int_c^b f &= \lim_{b\to\infty} \Big[\int_a^b f - \int_a^c f\ d \Big] \\ &= \lim_{b\to\infty} \int_a^b f - \lim_{b\to\infty} \int_a^c && \text{(ALT for Functions)} \\ &= \int_a^\infty f - \int_a^c f.\end{aligned}$$

Hence, we have that $\int_c^\infty f$ exists.

$\Longleftarrow$ Suppose $\int_c^\infty f(x)\, dx$ converges. Then by the segment addition property of integration, we have that

$$\int_a^x f(t)\ dt = \int_a^c f(t)\ dt + \int_c^x f(t)\ dt.$$

Then applying the limit as $x \to \infty$ on both sides, we have

$$\begin{aligned}\lim_{x\to\infty} \int_a^x f(t)\ dt &= \lim_{x\to\infty} \Big[\int_a^c f(t)\ dt + \int_c^x f(t)\ dt \Big] \\ &= \lim_{x\to\infty} \int_a^c f(t)\ dt + \lim_{x\to\infty} \int_c^x f(t)\ dt \\ &= \int_a^c f(t)\ dt + \int_c^\infty f(t)\ dt\end{aligned}$$

Note that the first term is just a constant so the limit always exists and the second exists by assumption. Hence, we can conclude that

$$\int_a^\infty f(t)\ dt \text{ converges.}$$

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Problem 10. Let $a > 0$. Prove that $\displaystyle\int_a^\infty \frac{1}{x^p}\ dx \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if} p \leq 1 \end{cases}.$

Proof. Observe by the chain rule that

$$\frac{d}{dx}\ln(x^p) = \frac{1}{x^p}px^{p-1} \implies \frac{1}{x^p} = \frac{1}{px^{p-1}}\frac{d}{dx}\ln(x^p).$$

So, we have

$$\int_a^t \frac{1}{x^p}\ dx = \int_a^t \frac{1}{px^{p-1}}\frac{d}{dx}\ln(x^p)\ dx.$$

Applying Integration by Parts to the right-hand side of the above equation, we have

$$\int_a^t \frac{1}{x^p}\ dx = \frac{\ln(x^p)}{px^{p-1}}\bigg]_a^t - \int_a^t \frac{1-p}{p}x^{-p}\ln(x^p)\ dx$$

Applying Integration by Parts again on the second term of the above equation, we have

$$\int_a^t x^{-p}\ln(x^p)\ dx = \frac{1}{1-p}x^{1-p}\ln(x^p)\bigg]_a^t - \int_a^t \frac{1}{1-p}x^{1-2p}\ dx.$$

Now, rearranging terms we can see that

$$\begin{aligned}\int_a^t \frac{1}{x^p}\ dx &= \frac{\ln(x^p)}{px^{p-1}}\bigg]_a^t - \frac{1}{p}x^{1-p}\ln(x^p)\bigg]_a^t + \frac{1}{p}\int_a^t x^{1-2p}\ dx \\ &= \frac{\ln(x^p)}{px^{p-1}}\bigg]_a^t - \frac{1}{p}x^{1-p}\ln(x^p)\bigg]_a^t + \frac{1}{p}\cdot\frac{1}{2-2p}x^{2-2p}\bigg]_a^t.\end{aligned}$$

In what follows, we will show that each term with limit as $t \to \infty$ in the sum above converges. Starting with the first term, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln(t^p)}{pt^{p-1}} &=_{\frac{\infty}{\infty}} \lim_{t\to\infty}\frac{\frac{1}{t^p}pt^{p-1}}{p(p-1)t^{p-2}} \\ &= \frac{p}{p-1}\lim_{t\to\infty}\frac{1}{t^{p-1}} \\ &= 0.\end{aligned}$$

With the second term, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{p}t^{1-p}\ln(t^p) &= \lim_{t\to\infty}\frac{\ln(t^p)}{pt^{p-1}} \\ &=_{\frac{\infty}{\infty}} \lim_{t\to\infty}\frac{\frac{1}{t^p}pt^{p-1}}{pt^{p-1}} \\ &= \lim_{t\to\infty}\frac{1}{t^p} \\ &= 0.\end{aligned}$$

Lastly, the third term we have

$$\lim_{t\to\infty} \frac{1}{2p(1-p)} t^{2-2p} = \lim_{t\to\infty} \frac{1}{2p(1-p)t^{2(p-1)}} = 0.$$

So, we have

$$\int_a^t \frac{1}{x^p}\, dx = \frac{\ln(a^p)}{pa^{p-1}} + \frac{1}{p} a^{1-p} \ln(a^p) + \frac{1}{p} \cdot \frac{1}{2-2p} a^{2-2p}.$$

■

Problem 11 (Cauchy Criterion For Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\displaystyle\int_a^\infty f(x)\, dx$ converges if and only if

$$\forall \varepsilon > 0\ \exists M > a \text{ such that } \forall A, B > M \quad \Big| \int_A^B f(x)\, dx \Big| < \varepsilon.$$

Proof. ($\Longrightarrow$) Suppose $\int_a^\infty f(x)\, dx$ converges. Our goal is to show that for all $\varepsilon > 0$, there exists $M > a$ such that for all $A, B > M$

$$\Big| \int_A^B f(x)\, dx \Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Define

$$F(x) = \int_a^x f(t)\, dt.$$

Since $\int_a^\infty f(x)\, dx$, there exists an $\hat{M} > a$ such that for all $t > \hat{M}$ we have

$$|F(t) - L| < \frac{\varepsilon}{2}$$

for some $L \in \mathbb{R}$. Note that for any $B > \hat{M}$, we have

$$\Big| \int_B^t f(x)\, dx - L \Big| < \frac{\varepsilon}{2}$$

and for any $A > \hat{M}$, we have

$$\Big| \int_A^t f(x)\, dx - L \Big| < \frac{\varepsilon}{2}.$$

Using the segment addition property of integration, we have

$$\begin{aligned} \int_A^B f(x)\, dx &= \int_A^t f(x)\, dx + \int_t^B f(x)\, dx \\ &= \int_A^t f(x)\, dx - \int_B^t f(x)\, dx. \end{aligned}$$

We claim that $\hat{M}$ is the same M we were looking for. Then we have for any $A, B > \hat{M}$

$$\begin{aligned} \Big| \int_A^B f(x)\, dx \Big| &= \Big| \int_A^t f(x)\, dx - \int_B^t f(x)\, dx \Big| \\ &\leq \Big| \int_A^t f(x)\, dx - L \Big| + \Big| L - \int_B^t f(x)\, dx \Big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which is our desired result.

($\Longleftarrow$) Suppose the converse. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges; that is, for all $\varepsilon > 0$, there exists $M > a$ such that for all $t > M$,

$$\Big| \int_a^t f(x)\ dx - L \Big| < \varepsilon$$

for some $L \in \mathbb{R}$. Let $\varepsilon > 0$ be given. By assumption, (with our given ε) there exists an $\hat{M} > a$ such that for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| < \frac{\varepsilon}{2}. \tag{1}$$

We claim that $\hat{M}$ is the same M we were looking for. Then for any $t > B > \hat{M}$

$$\Big| \int_t^B f(x)\ dx \Big| < \frac{\varepsilon}{2}. \tag{2}$$

Suppose $t > B > A > \hat{M}$. By the segment addition property of integration, we have

$$\int_a^t f(x)\ dx = \int_a^A f(x)\ dx + \int_A^B f(x)\ dx + \int_B^t f(x)\ dx.$$

Define $L = \int_a^A f(x)\ dx$ to be a candidate for our limit. Then for any $t > B > A > \hat{M}$, we have

$$\begin{aligned}
\Big| \int_a^t f(x)\ dx - \int_a^A f(x)\ dx \Big| &= \Big| \int_A^B f(x)\ dx + \int_B^t f(x)\ dx \Big| \\
&\leq \Big| \int_A^B f(x)\ dx \Big| + \Big| \int_B^t f(x)\ dx \Big| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

■

Problem 12 (Absolute Convergence Implies Convergence)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\int_a^\infty |f(x)|\ dx$ converges, then the improper integral $\int_a^\infty f(x)\ dx$ also converges.

Proof. Suppose $f \in R[a, b]$ for every $b > a$. Our goal is to show that for all $\varepsilon > 0$ such that there exists $M > a$ such that for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\int_a^\infty |f(x)|\ dx$ converges, there exists an $\hat{M} > a$ such that for all $A, B > \hat{M}$

$$\Big| \int_A^B |f(x)|\ dx \Big| < \varepsilon.$$

We claim that $\hat{M}$ can be used as the same M we were looking for. Indeed, we have for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| \leq \int_A^B |f(x)|\ dx = \Big| \int_A^B |f(x)|\ dx \Big| < \varepsilon. \tag{*}$$

Note that the first inequality holds by a theorem proven in class. Also, the second equality holds because $|f(x)| \geq 0$ (note that $|f| \in R[A, B]$) and that

$$\int_A^B f(x)\ dx \geq 0$$

by another theorem proven in class. Hence, we see that (*) is our desired result. ■

Problem 13 (Comparison Test For Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Prove that the improper integral $\int_a^\infty g(x)\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.

Proof. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges given that $\int_a^\infty g(x)\ dx$ converges; that is, we want to show that for all $\varepsilon > 0$, there exists an $M > a$ such that for all $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Then $\int_a^\infty g(x)\ dx$ converges implies that there exists an $\hat{M} > a$ such that for any $A, B > \hat{M}$, we have

$$\Big| \int_A^B g(x)\ dx \Big| < \varepsilon.$$

Also, there exists a $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $M = \max\{k, \hat{M}\} + 1$. Note that with this constructed M, we have, by the order property of the integral that

$$0 \leq \int_A^B f(x)\ dx \leq \int_A^B g(x)\ dx.$$

Then for any $A, B > M$ (and using the fact that $|\cdot|$ is a continuous function), we have that

$$\Big| \int_A^B f(x)\ dx \Big| \leq \Big| \int_A^B g(x)\ dx \Big| < \varepsilon$$

which is our desired result. ■

Problem 14 (Limit Comparison Test for Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $L = \lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$. Prove that

(i) If $0 < L < \infty$, then $\int_a^\infty f(x)\ dx$ converges if and only if $\int_a^b g(x)\ dx$ converges.

(ii) If $L = \infty$ and $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$ converges.

(iii) If $L = 0$ and $\int_a^\infty g(x)\ dx$ converges, then $\int_a^\infty f(x)\ dx$ converges.

Proof. Since $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = L$, we know that for all $\varepsilon > 0$, there exists $\hat{M} > 0$ such that for any $x \geq M$, we have

$$\Big| \frac{f(x)}{g(x)} - L \Big| < \varepsilon$$

which can be further written as

$$-\varepsilon < \frac{f(x)}{g(x)} - L < \varepsilon$$

and so

$$L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon. \qquad (*)$$

If $L = 0$, then the above inequality can be re-written as

$$-\varepsilon < \frac{f(x)}{g(x)} < \varepsilon. \tag{**}$$

If $L = \infty$, then for all $C > 0$, there exists $\kappa > 0$ such that for all $x > \kappa$, we have

$$\frac{f(x)}{g(x)} > C. \tag{***}$$

(i) Suppose $0 < L < \infty$. ($\Longrightarrow$) Our goal is to show that $\int_a^\infty g(x)\ dx$ converges given that $\int_a^\infty f(x)\ dx$ converges. From the left-hand side of (*), we can see that for all $x > \hat{M}$, we have

$$g(x) < \frac{1}{L-\varepsilon} f(x).$$

Since there exists a $K \in \mathbb{R}$ such that for all $x \geq K$, $f(x) \geq 0$ and $g(x) \geq 0$. So, if we take $M = \max\{\hat{M}, k\}$, then for any $x > M > a$, we have

$$0 < g(x) < \frac{1}{L-\varepsilon} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, we can see that $\int_a^\infty \frac{1}{L-\varepsilon} f(x)\ dx$ converges. By the Comparison Test for Integrals, we have $\int_a^\infty g(x)\ dx$ converges.

($\Longleftarrow$) Suppose $\int_a^\infty g(x)\ dx$ converges. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges. From the right-hand side of (*), we can see that for any $x \geq \hat{M}$, we have

$$f(x) < (L+\varepsilon) g(x).$$

Since there exists $K \in \mathbb{R}$ such that $f(x) \geq 0$ and $g(x) \geq 0$, define $M = \max\{K, \hat{M}\}$. Then for any $x \geq M > a$, we have

$$0 < f(x) < (L+\varepsilon) g(x).$$

Now, since $\int_a^\infty g(x)\ dx$ converges, it follows that $\int_a^\infty (L+\varepsilon) g(x)\ dx$ converges. Hence, the Comparison Test implies that $\int_a^\infty f(x)\ dx$ converges.

(ii) Our goal is to show that if $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$ converges. From (**), we can see that

$$g(x) < \frac{1}{C} f(x).$$

Define $\tilde{M} = \max\{\kappa, \hat{M}\}$. Then we have for any $x > \tilde{M}$

$$0 < g(x) < \frac{1}{C} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, also have that $\int_a^\infty \frac{1}{C} f(x)\ dx$ converge. By the Comparison Test, we can see that $\int_a^\infty g(x)\ dx$ converges.

(iii) Suppose $L = 0$. Then by (**), we can see that for any $x > \hat{M}$ that

$$f(x) < \varepsilon g(x).$$

In this case, if we let $\varepsilon = 1$, we have that

$$f(x) < g(x)$$

for all $x > \hat{M}$. Since there exists $K \in \mathbb{R}$ such that for any $x \geq K$, we have $f(x) \geq 0$ and $g(x) \geq 0$. Note that in this case, we require that $g(x) > 0$. Otherwise, the ratio above will not be defined. So, using the same M from part (i), we have that for any $x > M$,

$$0 < f(x) < g(x).$$

Since $\int_a^\infty g(x)\ dx$ converges, it follows from the Comparison Test that $\int_a^\infty f(x)\ dx$ also converges.

■

Problem 15. Determine all values α and β for which $\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx$ is convergent.

Proof. Using integration by parts, we have

$$\begin{aligned}\int_2^t \frac{1}{x^\alpha (\ln x)^\beta}\ dx &= \frac{1}{\alpha+1} \cdot \frac{x^{\alpha+1}}{(\ln x)^\beta}\Big]_2^t + \frac{\beta}{\alpha+1}\int_2^t \frac{x^\alpha}{(\ln x)^{\beta+1}}\ dx \\ &= \frac{1}{\alpha+1} \cdot \frac{x^{\alpha+1}}{(\ln x)^\beta}\Big]_2^t + \frac{\beta}{\alpha+1}\int_2^t \frac{x^\alpha}{(\ln x)^{\beta+1}}\ dx \\ &= \frac{1}{\alpha+1} \cdot \frac{t^{\alpha+1}}{(\ln t)^\beta} - \frac{1}{\alpha+1} \cdot \frac{2^{\alpha+1}}{(\ln 2)^\beta} + \frac{\beta}{\alpha+1}\int_2^t \frac{x^\alpha}{(\ln x)^{\beta+1}}\ dx.\end{aligned}$$

In what follows, we will show that each term as $t \to \infty$ in the integral above converges based on the value of α. Indeed, the fact that was given to us implies that for $\alpha < 0$, we have

$$\lim_{t\to\infty} \frac{t^{1+\alpha}}{(\ln t)^\beta} = 0.$$

Otherwise, the limit above diverges. Clearly, the second term converges because it is just a fixed number. For the last term, note that

$$0 < \frac{x^\alpha}{(\ln x)^{\beta+1}} \leq x^\alpha$$

for all $x \geq 2$. By the fact found in Exercise 10, we can see that

$$\int_2^\infty x^\alpha\ dx \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha \geq 1 \end{cases}$$

So, it follows that

$$\int_2^\infty \frac{x^\alpha}{(\ln x)^{\beta+1}}\ dx$$

converges. Hence, for any $\alpha < 0$,

$$\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx \text{ converges.}$$

■

Problem 16. (a) Prove that $\int_0^\infty \frac{\sin x}{(1+x)^2}\ dx$ is absolutely convergent.

(b) Use the result of Exercise 8 to prove that

$$\int_0^\infty \frac{\cos x}{1+x}\ dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\ dx.$$

Proof. (a) Our goal is to show that $\int_0^\infty \frac{\sin x}{(1+x)^2}\ dx$ is absolutely convergent; that is, we need to show that $\int_a^\infty |\frac{\sin x}{(1+x)^2}|\, dx$ converges. Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we have that

$$0 \leq \left|\frac{\sin x}{(1+x)^2}\right| \leq \frac{1}{(1+x)^2} \tag{*}$$

for all $x \in \mathbb{R}$. If we consider the right-hand side of the inequality above, then by using a change of variables, we have the following integral

$$\int_0^t \frac{1}{(1+x)^2}\, dx = \int_1^u \frac{1}{u^2}\, du.$$

Since $p = 2$, we have that $\int_1^u \frac{1}{u^2}\, du$ converges as $u \to \infty$ by Exercise 10. Hence, $\int_0^t \frac{1}{(1+x)^2}\, dx$ converges. Applying the Comparison Test to the inequality in (*), we can see that $\int_0^\infty |\frac{\sin x}{(1+x)^2}|\, dx$ converges which is our desired result.

(b) Consider the integral on the right-hand side of our result

$$\int_0^t \frac{\cos x}{1+x}\, dx.$$

Applying integration by parts, we have

$$\begin{aligned}
\int_0^t \frac{\cos}{1+x}\, dx &= \frac{\sin x}{1+x}\Big]_0^t + \int_0^t \frac{\sin x}{(1+x)^2}\, dx \\
&= \Big[\frac{\sin t}{1+t} - 0\Big] + \int_0^t \frac{\sin x}{1+x}\, dx \\
&= \frac{\sin t}{1+t} + \int_0^t \frac{\sin x}{(1+x)^2}\, dx.
\end{aligned}$$

Note that the limit as $t \to \infty$ on second term on the right-hand side of the equation above holds by part (a). We also have that the first term on the right-hand side also converges. Indeed, using the fact that $|\sin t| \leq 1$ for all $t \in \mathbb{R}$, we have that

$$0 \leq \left|\frac{\sin t}{1+t}\right| \leq \frac{1}{|1+t|}.$$

Note that as $t \to \infty$ on the right-hand side of the inequality above, we have

$$\frac{1}{|1+t|} \to 0.$$

By squeeze theorem, we can see that

$$\lim_{t\to\infty} \left|\frac{\sin t}{1+t}\right| = 0$$

and so

$$\lim_{t\to\infty} \frac{\sin t}{1+t} = 0.$$

By the Algebraic Limit Theorem for Functions, we have that

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{1+x}\, dx &= \lim_{t\to\infty} \int_0^t \frac{\cos x}{1+x}\, dx \\
&= \lim_{t\to\infty} \frac{\sin t}{1+t} + \int_0^\infty \frac{\sin x}{(1+x)^2}\, dx \\
&= \int_0^\infty \frac{\sin x}{(1+x)^2}\, dx
\end{aligned}$$

which is our desired result. ■