Homework 3

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1. Prove that for all $a, b \in \mathbb{R}$, we have $|ab| = |a||b|$.

Proof. Let $a, b \in \mathbb{R}$. Our goal is to show that $|ab| = |a||b|$. There are a few cases to consider to which we will list below:

(1) If $a = 0$ and $b = 0$.

(2) $a > 0$ and $b > 0$

(3) $a < 0$ and $b > 0$

(4) $a > 0$ and $b < 0$.

(5) $a < 0$ and $b < 0$.

Note that, by definition of absolute value, we have

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

$$|b| = \begin{cases} b & \text{if } b > 0 \\ -b & \text{if } b \leq 0 \end{cases}$$

$$|ab| = \begin{cases} ab & \text{if } ab > 0 \\ -ab & \text{if } ab \leq 0 \end{cases}.$$

Now, we proceed with each case as follows:

(1) Suppose $a = 0$ and $b = 0$, then the result immediately follows.

(2) Suppose $a > 0$ and $b > 0$. Then $ab > 0$ which implies that

$$|ab| = a \cdot b = |a| \cdot |b|.$$

(3) Suppose $a < 0$ and $b > 0$, then we have $ab < 0$. Thus,

$$|ab| = -ab = (-a)(b) = |a||b|.$$

(4) Suppose $a > 0$ and $b < 0$. Then we have $ab < 0$ which tells us that

$$|ab| = -ab = (a)(-b) = |a||b|.$$

(5) Suppose $a < 0$ and $b < 0$. Then we have $ab > 0$ which tells us that

$$|ab| = ab = (-a)(-b) = |a||b|.$$

Thus, we conclude that for all $a, b \in \mathbb{R}$, we have $|ab| = |a||b|$. ■

2. Prove that for all $c \in \mathbb{R}$, we have $-|c| \leq c \leq |c|$.

Proof. Let $c \in \mathbb{R}$. We will show that $-|c| \leq c \leq |c|$. There are two cases to consider:

(1) $c = 0$,

(2) $c > 0$,

(2) $c < 0$.

We proceed with each case as follows:

(1) If $c = 0$, then the result immediately follows.

(2) Suppose $c > 0$. Then $|c| = c$ which implies that

$$c \leq c \Rightarrow c \leq |c|.$$

For the other inequality, note that $c \geq -c$. So, we have

$$c \geq -c = -|c|.$$

Thus, we have $-|c| \leq c \leq |c|$.

(3) Suppose $c < 0$. Note that $|c| = -c$. If $c < 0$, then we see that $-c < c$. Clearly, we have $c \leq c$, and so we have

$$-|c| = -(-c) = c \leq c \Rightarrow -|c| \leq c.$$

Since $c < -c$, we see that

$$c \leq -c = |c|.$$

Hence, we have $-|c| \leq c \leq |c|$.

■

3. Prove that for all $c, d \in \mathbb{R}$, we have

$$|c| \leq d \iff -d \leq c \leq d.$$

Proof. Let $c, d \in \mathbb{R}$. We will show that $|c| \leq d$ if and only if $-d \leq c \leq d$.

($\Rightarrow$) Suppose $|c| \leq d$. Note that we have $|c| = c$ if $c > 0$ or $|c| = -c$ if $c < 0$ or $|c| = 0$ if $c = 0$. Clearly, if $c = 0$, then the result immediately follows. Thus, suppose $c > 0$. Then

$$|c| \leq d \implies c \leq d$$

and

$$\begin{aligned} |c| \leq d &\implies c \leq d \\ &\implies -(-c) \leq d \\ &\implies -c \geq -d. \end{aligned}$$

Note that $c > -c$, so $c \geq -d$. Thus, we see that

$$-d \leq c \leq d.$$

If $c < 0$, then $|c| = -c$. So,

$$\begin{aligned} |c| \leq d &\Rightarrow -c \leq d \\ &\Rightarrow c \geq -d. \end{aligned}$$

If $c < 0$, then $-c > 0$ implies $-c > c$. So, we have

$$\begin{aligned} c \geq -d &\Rightarrow -(-c) \geq -d \\ &\Rightarrow -c \leq d \\ &\Rightarrow c < -c \leq d \\ &\Rightarrow c \leq d. \end{aligned}$$

Thus, we see that

$$-d \leq c \leq d.$$

($\Leftarrow$) Suppose $-d \leq c \leq d$. Note that if $c = 0$, then the result immediately follows. Thus, we either have $c > 0$ or $c < 0$. If $c > 0$, then $|c| = c$. Then by assumption, we have

$$|c| = c \leq d.$$

If $c < 0$, then $|c| = -c$. By assumption, we see that

$$-d \leq c \Rightarrow -c \leq d.$$

Thus, we have

$$|c| = -c \leq d.$$

■

4. (i) Use induction to prove that $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ for any n real numbers $x_1, \dots, x_n$.

Proof. We proceed with induction on n. Let $n = 1$, then we clearly we have $|x_1| \leq |x_1|$. If $n = 2$, then our claim is that

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

to which we will show that

$$(x_1 + x_2)^2 \leq (|x_1| + |x_2|)^2.$$

Using problem 1 and 2, we can see that

$$\begin{aligned} (x_1 + x_2)^2 &= x_1^2 + 2x_1x_2 + x_2^2 \\ &\leq |x_1|^2 + 2|x_1x_2| + |x_2|^2 \\ &= |x_1|^2 + 2|x_1||x_2| + |x_2|^2 \\ &= (|x_1| + |x_2|)^2. \end{aligned}$$

Then we have

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

by taking the square root of both sides. Thus, case $n = 2$ holds.

Now, suppose the result holds for the nth case. We will show that the $n+1$ case holds. Note that $p = (x_1 + x_2 + \cdots + x_n)$. So, applying the result for $n = 2$ yields the following result:

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &= |(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \\ &= |p + x_{n+1}| \\ &\leq |p| + |x_{n+1}|. \end{aligned}$$

By our induction hypothesis, we see that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Thus, we have

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |z_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

and we are done. ■

(ii) Prove $||x| - |y|| \leq |x - y|$ for any two real numbers x and y.

Proof. Let $x, y \in \mathbb{R}$. To show that $||x| - |y|| \leq |x - y|$, it is enough to show that

$$-(|x - y|) \leq |x| - |y| \leq |x - y|.$$

Observe that $|x| \leq |x|$. Then using the part (i), we see that

$$\begin{aligned} |x| \leq |x| &\iff |x| \leq |(x - y) + y| \leq |x - y| + |y| \\ &\iff |x| - |y| \leq |x - y|. \end{aligned}$$

Similarly, observe that $|y| \leq |y|$ implies that

$$\begin{aligned} |y| \leq |y| &\iff |y| \leq |(y - x) + x| \leq |y - x| + |x| \\ &\iff |y| - |x| \leq |y - x| \\ &\iff -(|x| - |y|) \leq |x - y|. \end{aligned}$$

Thus, we see that

$$-(|x - y|) \leq |x| - |y| \leq |x - y|$$

which is our desired result. ■

5. Recall that according to the AM-GM inequality (for the case $n = 2$) we have

$$\forall x, y > 0 \quad \frac{x + y}{2} \geq \sqrt{xy}.$$

Use the inequality above to show that for all real numbers $a > 0$ we have $a + \frac{1}{a} \geq 2$.

Proof. Let $a > 0$. Then by the $n = 2$ case of the AM-GM inequality, we have

$$\begin{aligned} a + \frac{1}{a} = \frac{2a^2}{2a} + \frac{2}{2a} &= \frac{2a^2/a}{2} + \frac{2/a}{2} \\ &\geq \sqrt{\frac{2a^2}{a} \cdot \frac{2}{a}} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

Thus, we conclude that

$$a + \frac{1}{a} \geq 2.$$

■

6. Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for all nonnegative real numbers a and b, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \in \mathbb{R}$ be nonnegative. We have the following cases:

(1) $a = 0$ and $b = 0$

(2) $a = 0$ and $b > 0$

(3) $a > 0$ and $b = 0$

(4) $a > 0$ and $b > 0$.

We proceed with the proof of the result with the following cases.

(1) If $a = 0$ and $b = 0$, then the result is immediate.

(2) If $a = 0$ and $b > 0$, then we immediately have

$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{b^q}{q}.$$

(3) If $b = 0$ and $a > 0$, then we similarly have

$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{a^p}{p}.$$

(4) Suppose $a > 0$ and $b > 0$. By the property of logarithms, we see that

$$ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b}.$$

Also, we see that

$$\ln(a^p) = p \ln a \text{ and } \ln(b^q) = q \ln b.$$

Now, observe that

$$e^{\ln a + \ln b} = e^{\frac{p}{p} \ln a + \frac{q}{q} \ln b} = e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)}.$$

Notice that e^t, when differentiated twice, is a strictly positive function. Thus, e^t is convex for all $t \in \mathbb{R}$ our knowledge of calculus. Thus, we can use Jensen's inequality to conclude that

$$e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)} \leq \frac{1}{p} e^{\ln(a^p)} + \frac{1}{q} e^{\ln(b^q)} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

■

7. **(Holder's Inequality)** Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two points in $\mathbb{R}^n$. Prove that

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

(Note that the Cauchy-Schwarz inequality is a special case of the inequality above where $p = q = 2$)

Proof. Let $a = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ and $b = \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}$.As a continuation of the proof presented in the homework sheet, we see that $a, b \neq 0$ implies

$$\begin{aligned}\sum_{i=1}^n \left|\left(\frac{x_i}{a}\right)\left(\frac{y_i}{b}\right)\right| \leq 1 &\implies \sum_{i=1}^n \left|\frac{x_i}{a}\right|\left|\frac{y_i}{b}\right| \leq 1 \\ &\implies \frac{1}{ab}\sum_{i=1}^n |x_i||y_i| \\ &\implies \sum_{i=1}^n |x_i y_i| \leq ab = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}.\end{aligned}$$

■

8. **(Minkowski's Inequality)** Let $p \geq 1$ be a real number. Suppose $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are two points in $\mathbb{R}^n$. Prove that

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

Proof. As a continuation of proof presented in the homework sheet, we have

$$\sum_{i=1}^n |x_i + y_i|^p \leq \left(\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}.$$

Dividing $\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}$ by both sides, we see that

$$\frac{\sum_{i=1}^n |x_i + y_i|^p}{\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}} \leq \left(\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right)$$

which can be re-written to

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

Now, observe that

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p}.$$

If we set

$$A = \sum_{i=1}^n |x_i + y_i|^p,$$

then we see that

$$A^{1-\frac{1}{q}} = A^{1-\left(1-\frac{1}{p}\right)} = A^{\frac{1}{p}}.$$

Thus, we see that

$$\left(\sum_{i=1}^n |x_i + y_i|\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

■

9. Let $n \geq 2$ where $n \in \mathbb{N}$. Prove that

$$y^n - x^n < n(y-x)y^{n-1}.$$

Proof. Using the identity that

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2} + \cdots + x^{n-1})$$

and the assumption that $x < y$, we can see that

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + \cdots + x^{n-1}) \\ &< (y-x)(y^{n-1} + y^{n-2} \cdot y + \cdots + y^{n-1}) \\ &= (y-x)(y^{n-1} + y^{n-1} + \cdots + y^{n-1}) \\ &= (y-x)ny^{n-1}. \end{aligned}$$

Thus, we conclude that

$$y^n - x^n < n(y-x)y^{n-1}.$$

■

10. **Every normed space is a metric space.** Let $(V, \|\cdot\|)$ be a normed space (in particular, V is a vector space). Prove that $d : V \times V \to \mathbb{R}$ defined by $d(x,y) = \|x - y\|$ is a metric space on V.

Proof. To show that $d(x,y)$ is a metric on V, we need to show the following properties:

(i) For all $x, y \in V$, $d(x,y) \geq 0$.

(ii) For all $x, y \in V$, $d(x,y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in V$, $d(x,y) = d(y,x)$.

(iv) For all $x, y, z \in V$, we have
$$d(x,y) \leq d(x,z) + d(z,y).$$

We proceed with the following proof of each property listed above.

(i) Let $x, y \in V$. Since $(V, \|\cdot\|)$ is a normed space, we know that $\|x - y\| \geq 0$ which satisfies property (i).

(ii) Let $x, y \in V$. Suppose $d(x,y) = 0$. By definition of $d(x,y)$ and by property (2) of norms, we have

$$\begin{aligned} d(x,y) = 0 &\Longrightarrow \|x - y\| = 0 \\ &\Longrightarrow x - y = 0 \\ &\Longrightarrow x = y. \end{aligned}$$

This shows property (ii).

(iii) Let $x, y \in V$. Then by property (3) of norms, we see that

$$d(x,y) = \|x - y\| = \|-(y-x)\| = |-1|\|y - x\| = d(y,x).$$

Thus, property (iii) is satisfied.

(iv) Let $x, y, z \in V$. Then by the triangle inequality property of norms, we see that

$$\begin{aligned} d(x,y) = \|x-y\| &= \|(x-z)+(z-y)\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Thus, property (iv) is satisfied.

Since all four properties of a metric have been satisfied, we can now conclude that, indeed, $d(x,y) = \|x-y\|$ is a metric on V. ■

11. Let $p \geq 1$ be a real number. Define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \quad d_p(x,y) = \Big(\sum_{i=1}^n |x_i - y_i|^p\Big)^{\frac{1}{p}}.$$

Prove that d_p is a metric on $\mathbb{R}^n$. (Note that the special case $p = 2$ gives the standard metric on $\mathbb{R}^n$.)

Proof. Let us define the function $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ by $\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}}$. We will show that $\|\cdot\|_p$ is a norm on the vector space $\mathbb{R}^n$ and then define $d_p(x,y) = \|x-y\|_p$ and then show that this is a metric using the last exercise. Firstly, we proceed by showing that $\|\cdot\|_p$ satisfies the properties of norms. Thus, we have:

(i) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Note that $|x_i|^p \geq 0$ for all x_i's if and only if $|x_i| \geq 0$ which is true if and only if $x_i \geq 0$ for all $1 \leq i \leq n$. This tells us that

$$\sum_{i=1}^n |x_i|^p \geq 0.$$

Since $p \geq 1$ is a real number, we must have

$$\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} \geq 0.$$

Thus, the first property of norms are satisfied.

(ii) Let $x \in \mathbb{R}^n$. Then we see that

$$\begin{aligned} \|x\|_p = 0 &\iff \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} = 0 \\ &\iff \sum_{i=1}^n |x_i|^p = 0 \\ &\iff |x_i|^p = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff |x_i| = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff x_i = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff x = 0. \end{aligned}$$

Hence, property (ii) is satisfied.

(iii) Let $\alpha \in \mathbb{R}$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then we see that

$$\begin{aligned} \|\alpha x\|_p &= \Big(\sum_{i=1}^{n} |\alpha x_i|^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{i=1}^{n} |\alpha|^p |x_i|^p\Big)^{\frac{1}{p}} \\ &= |\alpha| \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} \\ &= |\alpha| \|x\|_p. \end{aligned}$$

Thus, we see that property (iii) has been satisfied.

(iv) Let $x, y \in V$. By using Minkowski's inequality, we see that

$$\begin{aligned} \|x+y\|_p &= \Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p. \end{aligned}$$

Thus, the triangle inequality property of norms is satisfied.

This tells us that $\|\cdot\|_p$ is a norm on the vector space $\mathbb{R}^n$. Defining the function $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_p(x, y) = \|x - y\|_p$, we see that d_p must be a metric on $\mathbb{R}^n$ by Exercise 10. ∎

12. Define $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \quad d_\infty = \max\{|x_i - y_i| : 1 \leq i \leq n\}.$$

Prove that d_∞ is a metric on $\mathbb{R}^n$.

Proof. Let us define the function $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ by

$$\|x\|_\infty = \max\{|x_i| : i = 1, \dots, n\} = \max_{1 \leq i \leq n} |x_i|.$$

We will prove that this is, indeed, a norm on the vector space $\mathbb{R}^n$ and thereby show that, by defining the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_\infty(x, y) = \|x - y\|_\infty$, it defines a metric on $\mathbb{R}^n$. In what follows, we will show the four properties of norms:

(i) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Since the absolute value $|\cdot|$ is always nonnegative, we can see that $|x_i| \geq 0$ for all $1 \leq i \leq n$. Thus, we have

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \geq 0.$$

Thus, the first property of norms is satisfied.

(ii) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Observe that

$$\begin{aligned}
\|x\|_\infty = 0 &\implies \max_{1\le i\le n} |x_i| = 0 \\
&\implies |x_i| \le 0 && (\forall 1 \le i \le n) \\
&\implies 0 \le |x_i| \le 0 && (\forall 1 \le i \le n) \\
&\implies |x_i| = 0 && (\forall 1 \le i \le n) \\
&\implies x_i = 0 && (\forall 1 \le i \le n) \\
&\implies x = 0.
\end{aligned}$$

Now, suppose $x = (x_1, x_2, \dots, x_n) = 0$. Then for all $1 \le i \le n$, we see that $|x_i| = 0$ and so $\max_{1\le i\le n} |x_i| = 0$. By definition of $\|\cdot\|_\infty$, we see that $\|x\|_\infty = 0$.

(iii) Let $\alpha \in \mathbb{R}$ and let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then observe that by problem 9-7) of homework 1 that

$$\begin{aligned}
\|\alpha x\|_\infty &= \max_{1\le i\le n} |\alpha x_i| \\
&= \max_{1\le i\le n} |\alpha||x_i| && \text{(Problem 1)} \\
&= |\alpha| \max_{1\le i\le n} |x_i| && \text{(Problem 9-7) of Homework 1)} \\
&= |\alpha|\|x\|_\infty.
\end{aligned}$$

Thus, we see that $\|\cdot\|_\infty$ satisfies the third property of norms.

(iv) Let $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. By the triangle inequality of the absolute value $|\cdot|$, we can see that

$$\begin{aligned}
\|x+y\|_\infty &= \max_{1\le i\le n} |x_i + y_i| \\
&\le \max_{1\le i\le n} |x_i| + |y_i| \\
&= \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i| \\
&= \|x\|_\infty + \|y\|_\infty.
\end{aligned}$$

Thus, property (iv) is satisfied.

Hence, we conclude that $\|\cdot\|_\infty$ defines a norm on $\mathbb{R}^n$. So, we can define the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_\infty(x, y) = \|x - y\|_\infty$. Applying Exercise 10, we can see that this function defines a metric on $\mathbb{R}^n$. ■

Remark. Alternatively, we can show the triangle inequality above in the following way. By definition, we know that $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ and $\|y\|_\infty = \max_{1\le i\le n} |y_i|$. Then by definition of maximums, we see that for all $1 \le i \le n$, we have

$$|y_i| \le \|y\|_\infty \tag{1}$$

and

$$|x_i| \le \|x\|_\infty. \tag{2}$$

Adding (1) and (2) together and then using the triangle inequality we see that

$$|x_i + y_i| \le |x_i| + |y_i| \le \|x\|_\infty + \|y\|_\infty.$$

This shows that that $\|x\|_\infty + \|y\|_\infty$ is an upper bound for the set

$$\{|x_i + y_i| : 1 \leq i \leq n\}.$$

Since the set above is clearly finite, bounded above, and nonempty, we see that the maximum for this set exists and thus

$$\max_{1\leq i\leq n} |x_i + y_i| \leq \|x\|_\infty + \|y\|_\infty.$$

Hence, we have

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty.$$

13. Let $X = \mathbb{R}^2$. Is $d(x, y) = |x_1x_2 + y_1y_2|$ a metric on X?

Solution. No, $d(x, y) = |x_1x_2 + y_1y_2|$ does not define a metric on $\mathbb{R}^2$. Consider $(0, 1), (1, 0) \in \mathbb{R}^2$ with $x = (0, 1)$ and $(1, 0)$. Clearly, $(0, 1) \neq (1, 0)$ but

$$|x_1x_2 + y_1y_2| = |0 \cdot 1 + 1 \cdot 0| = |0 + 0| = |0| = 0.$$

■

14. Let $X = C([0, 1])$ be the set of continuous real-valued functions on $[0, 1]$. Is $d(f, g) = |f(1) - g(1)|$ a metric on X?

Solution. This is not a metric on X. Consider $f(x) = e^x$ and $g(x) = e$. We have $f \neq g$, but $d(f, g) = |f(1) - g(1)| = 0$. ■

15. Let $X = \{0\}$. Can you define a metric on X?

Solution. Yes, consider the function $d : X \times X \to X$ defined by $d(x, y) = 0$ for any $x, y \in X$. This function satisfies all the properties of a metric. ■

16. Let $X = C[0, 1]$. It can be proved that X is a vector space. Also, it can be shown that every real-valued continuous function on a closed and bounded interval attains a maximum. Define

$$\|f\|_\infty = \max\{|f(x)| : x \in [0, 1]\} = \max_{x\in[0,1]} |f(x)|.$$

Prove that $\|\cdot\|_\infty$ is a norm on X. What is the corresponding metric?

Proof. We will show that $\|\cdot\|_\infty$ defines a norm on X. We will do so by showing that $\|\cdot\|_\infty$ satisfies the properties of a norm.

(i) Let $f \in X$. Then for all $x \in [0, 1]$, we see that $|f(x)| \geq 0$ by the nonnegative property of the absolute value $|\cdot|$. Thus, we see that

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)| \geq 0.$$

(ii) Let $f \in X$. We have that

$$\begin{aligned}
\|f\|_\infty = 0 &\implies \max_{x\in[0,1]} |f(x)| = 0 \\
&\implies 0 \leq |f(x)| \leq 0 && (\forall x \in [0,1]) \\
&\implies |f(x)| = 0 && (\forall x \in [0,1]) \\
&\implies f(x) = 0 && (\forall x \in [0,1]) \\
&\implies f = 0.
\end{aligned}$$

Now, suppose $f = 0$. Then for all $x \in [0,1]$, we see that $f(x) = 0$. So, $|f(x)| = 0$ and thus $\max_{1\leq i\leq n} |x_i| = 0$. Hence, $\|x\|_\infty = 0$. Thus, property (ii) is satisfied.

(iii) Let $\alpha \in \mathbb{R}$ and $f \in X$. Then by exercise 9-7) of homework 1, we can see that

$$\begin{aligned}
\|\alpha f\|_\infty &= \max_{x\in[0,1]} |(\alpha f)(x)| \\
&= \max_{x\in[0,1]} |\alpha \cdot f(x)| \\
&= \max_{x\in[0,1]} |\alpha||f(x)| && \text{(Problem 1)} \\
&= |\alpha| \max_{x\in[0,1]} |f(x)| && \text{(Problem 9-7) of hw1)} \\
&= |\alpha|\|f\|_\infty.
\end{aligned}$$

Thus, we see that property (iii) is satisfied.

(iv) Let $f, g \in X$. Then observe that

$$\begin{aligned}
\|f+g\|_\infty &= \max_{x\in[0,1]} |(f+g)(x)| \\
&= \max_{x\in[0,1]} |f(x)+g(x)| \\
&\leq \max_{x\in[0,1]} |f(x)| + |g(x)| && \text{(Triangle Inequality)} \\
&= \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)| && \text{(Problem 9-2) of hw1)} \\
&= \|f\|_\infty + \|g\|_\infty.
\end{aligned}$$

Thus, the triangle inequality property is satisfied.

Since all four properties have been satisfied, we can now conclude that $\|\cdot\|_\infty$ is a norm on the vector space $X = C[0,1]$. The corresponding metric is the function $d_\infty : X \times X \to \mathbb{R}$ defined by $d_\infty(f,g) = \|f-g\|_\infty$. ■

Remark. We can appeal to the same methods in the remark of problem 12 to show the triangle inequality for the $\|\cdot\|_\infty$. By definition of maximums, we see that $|f(x)| \leq \|f\|_\infty$ and $|g(x)| \leq \|g\|_\infty$ for all $x \in [0,1]$. Adding these two inequalities together and using the triangle inequality of the $|\cdot|$, we see that

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

which holds for all $x \in [0,1]$. Thus, we see that the set

$$|f(x)+g(x)| : x \in [0,1]$$

is bounded above by $\|f\|_\infty + \|g\|_\infty$, and so we have

$$\|f+g\|_\infty \max_{1\leq i\leq n} |f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$

17. Let (X,d) be a metric space. Define $D(x,y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x,y \in X$. Prove that (X,D) is a metric space.

Proof. Let (X,d) be a metric space. We will show that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is a metric space.

(i) Let $x,y \in X$. Since (X,d) is a metric space, we know that $d(x,y) \geq 0$ for every $x,y \in X$ by property (i) of metric spaces. Thus, we see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0 \ \forall x,y \in X.$$

Hence, property (i) is satisfied.

(ii) Let $x,y \in X$. Since (X,d) is a metric space, we know that $d(x,y) = 0$ if and only if $x = y$. Thus, we see that

$$\begin{aligned} D(x,y) = 0 &\iff \frac{d(x,y)}{1+d(x,y)} = 0 \\ &\iff d(x,y) = 0 \\ &\iff x = y. \end{aligned}$$

Hence, property (ii) is satisfied.

(iii) Let $x,y \in X$. Since $d(x,y) = d(y,x)$ for all $x,y \in X$ (because (X,d) is a metric space), we see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = D(y,x).$$

Thus, property (iii) is satisfied.

(iv) To show that $D(x,y)$ satisfies the triangle inequality, we first need to show that $D(x,y)$ is a monotonically increasing function; that is, we need to show that for any $a < b$ where $a,b \in X$, that

$$\frac{a}{1+a} < \frac{b}{1+b}.$$

Now, let $a < b$. Then observe that

$$\begin{aligned} a < b &\iff ab + a < ab + b \\ &\iff a(b+1) < b(a+1) \\ &\iff \frac{a}{1+a} < \frac{b}{1+b}. \end{aligned}$$

Thus, we can see that $D(x,y)$ is a monotonically increasing function. Let $x,y,z \in X$. We need to consider a few cases when proving the triangle inequality:

(i) $d(x,y) \leq d(x,z)$.

(ii) $d(x,y) \leq d(z,y)$

(iii) $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$.

We proceed with each case as follows:

(i) If $d(x,y) \leq d(x,z)$, then by using the monotonicity of $D(x,y)$, we can see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)}{1+d(x,z)} = D(x,z) \leq D(x,z) + D(z,y).$$

(ii) Similarly, if $d(x,y) \leq d(z,y)$, we have by using the monotonicity of $D(x,y)$ that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(z,y)}{1+d(z,y)} = D(z,y) \leq D(x,z) + D(z,y).$$

(iii) If both $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$, then by the triangle inequality property of (X,d), we see that

$$\begin{aligned} D(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &\leq \frac{d(x,z)+d(z,y)}{1+d(x,y)} \\ &= \frac{d(x,z)}{1+d(x,y)} + \frac{d(z,y)}{1+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= D(x,z) + D(z,y). \end{aligned}$$

Thus, we can see that $D(x,y)$ satisfies the triangle inequality.

We can, therefore, conclude that $D(x,y)$ defines a metric on X. ■

Extra Credit Problems

1. Use the AM-GM inequality to prove that the sequence $(a_n)_{n\geq 1}$ given by $a_n = \left(1+\frac{1}{n}\right)^n$ is an increasing sequence. That is, show that for all $n \in \mathbb{N}$

$$\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Proof. Our goal is use the AM-GM inequality to show that the sequence $(a_n)_{n\geq 1}$ given by

$$a_n = \left(1+\frac{1}{n}\right)^n$$

is an increasing sequence; that is, we want to show that for all $n \in \mathbb{N}$ that $a_n \leq a_{n+1}$. From the AM-GM inequality, if we take $n+1$ points, then we see that

$$(x_1 \ldots x_{n+1})^{\frac{1}{n+1}} \leq \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}$$

where

$$(xy^n)^{\frac{1}{n+1}} \leq \frac{x+ny}{n+1}$$

with $x = x_1$ and $y = x_2 = \cdots = x_{n+1}$. Now, taking $x = 1$ and $y = 1 + \frac{1}{n}$, we see that

$$\frac{1 + n\Big(1 + \frac{1}{n}\Big)}{n+1} = \frac{1 + (n+1)}{n+1} = \frac{1}{n+1} + 1$$

and so,

$$\begin{aligned} (xy^n)^{\frac{1}{n+1}} \leq \frac{x + ny}{n+1} &\iff \left(\left(1 + \frac{1}{n}\right)^n\right)^{\frac{1}{n+1}} \leq \frac{1 + n\Big(1 + \frac{1}{n}\Big)}{n+1} \\ &\iff \left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}} \leq 1 + \frac{1}{n+1} \\ &\iff \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} \\ &\iff a_n \leq a_{n+1}. \end{aligned}$$

Thus, we conclude that the sequence $(a_n)_{n\geq 1}$ is increasing. ■

2. Let (X, d) be a metric space. Define $D(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Prove that (X, D) is a metric space.

Proof. Let (X, d) be a metric space. Define $D(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Our goal is to show that (X, D) is a metric space.

(i) Let $x, y \in X$. Since $d(x, y) \geq 0$ by property (i) of metric spaces, we can see immediately that

$$D(x, y) = \min\{1, d(x, y)\} \geq 0.$$

Thus, the property (i) of metric spaces is satisfied.

(ii) Let $x, y \in X$. Then

$$\begin{aligned} D(x, y) = 0 &\iff \min\{1, d(x, y)\} = 0 \\ &\iff d(x, y) = 0 \\ &\iff x = y. && ((X, d) \text{ is a metric space}) \end{aligned}$$

Thus, property (ii) of metric spaces is satisfied.

(iii) Let $x, y \in X$. Then we see that

$$D(x, y) = \min\{1, d(x, y)\} = \min\{1, d(y, x)\} = D(y, x).$$

Thus, property (iii) of metric spaces is satisfied.

(iv) Let $x, y, z \in X$. Our goal is to show that

$$D(x, y) \leq D(x, z) + D(z, y). \tag{1}$$

We have some cases to consider; that is, either

(1) $d(x, z) = 0$, $d(z, y) = 0$, and $d(x, y) = 0$.

(2) $d(x, z) \geq 1$ or

(3) $d(z, y) \geq 1$ or

(4) both $d(x, z) \geq 1$ and $d(z, y) \geq 1$

(5) otherwise, $d(x, z) < 1$ and $d(z, y) < 1$.

Thus, we proceed by proving (1) with the following cases in mind:

(1) If $d(x,z) = 0$, $d(z,y) = 0$, and $d(x,y) = 0$, then we get

$$D(x,y) = 0 \leq 0 = 0 + 0 = D(x,z) + D(z,y).$$

(2) Suppose $d(x,z) \geq 1$. By definition of $D(\cdot,\cdot)$, we see that $D(x,z) = \min\{1, d(x,z)\} = 1$. Then observe that

$$D(x,z) + D(z,y) = 1 + D(z,y) \geq 1 \geq D(x,y).$$

(3) Suppose $d(z,y) \geq 1$. Then similarly, $D(z,y) = \min\{1, d(z,y)\} = 1$. Thus, we see that

$$D(x,z) + D(z,y) = D(x,z) + 1 \geq 1 \geq D(x,y).$$

(4) Suppose both $d(x,z) \geq 1$ and $d(z,y) \geq 1$. Then we have both $D(x,z) = \min\{1, d(x,z)\} = 1$ and $D(z,y) = \min\{1, d(z,y)\} = 1$. Thus, we have

$$D(x,z) + D(z,y) = 1 + 1 \geq 1 \geq D(x,y).$$

(5) Now, suppose $d(x,z) < 1$ and $d(z,y) < 1$. Then by definition of $D(\cdot,\cdot)$, we see that $D(x,z) = \min\{1, d(x,z)\} = d(x,z)$ and $D(z,y) = \min\{1, d(z,y)\} = d(z,y)$. Since (X,d) is a metric space, we know that the triangle inequality for the metric $d(\cdot,\cdot)$ holds. Thus, we have that

$$D(x,z) + D(z,y) = d(x,z) + d(z,y) \geq d(x,y) \geq D(x,y).$$

Thus, we conclude that the $D(x,y) \leq D(x,z) + D(z,y)$ for all of these cases. ■