Math 234A: Homework 2

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1 Problem 1

Definition (Cauchy Sequence). A sequence (z_n) is called a **Cauchy Sequence** if for all $\varepsilon > 0$, there exists a positive interger $\mathbb{N}$ such that for all $m, n \geq N$,

$$|z_m - z_n| < \varepsilon.$$

Show that a sequence (z_n) in $\mathbb{C}$ is convergent if and only if it is Cauchy.

Proof. ($\Rightarrow$) Suppose the sequence (z_n) is convergent. We want to show that (z_n) is Cauchy. Let $\varepsilon > 0$. We want to show that there exists an $N \in \mathbb{Z}^+$ such that whenever $n.m \geq N$, we have

$$|z_n - z_m| < \varepsilon.$$

Since (z_n) converges, we can find an $N_1 \in \mathbb{Z}^+$ such that whenever $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2}. \tag{1}$$

Likewise, we can find an $N_2 \in \mathbb{Z}^+$ such that whenever $m \geq N_2$, we have

$$|z_m - z| < \frac{\varepsilon}{2}. \tag{2}$$

So, choosing $N = \max N_1, N_2$, we see that for any $m, n \geq N$, we have

$$\begin{aligned} |z_n - z_m| &= |z_n - z + z - z_m| \\ &\leq |z_n - z| + |z - z_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that (z_n) is Cauchy.

($\Leftarrow$) Suppose (z_n) is Cauchy. Let $\varepsilon > 0$. We want to show that there exists an $N \in \mathbb{Z}^+$ such that for any $n \geq N$, we have

$$|z_n - z| < \varepsilon.$$

Since (z_n) is Cauchy, we know that (z_n) must be bounded. Thus, (z_n) must contain a subsequence (z_{n_k}) that is convergent. Thus, we can find an $N_1 \in \mathbb{Z}^+$ such that whenever $n_k \geq N_1$, we have

$$|z_{n_k} - z| < \frac{\varepsilon}{2}. \tag{3}$$

Since (z_n) is Cauchy, we can find an $N_2 \in \mathbb{Z}^+$ such that whenever $n, n_k \geq N_2$, we have

$$|z_n - z_{n_k}| < \frac{\varepsilon}{2}. \tag{4}$$

Choose $N = \max N_1, N_2$. Then for any $n, n_k \geq N$, we have

$$\begin{aligned}|z_n - z| &= |z_n - z_{n_k} + z_{n_k} - z| \\ &\leq |z_n - z_{n_k}| + |z_{n_k} - z| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Thus, we conclude that $(z_n) \to z$. ■

2 Problem 2

(i) Let $z_0 = x_0 + i\zeta_0 \in \mathbb{C}$: Define a sequence (z_n) by $z_{n+1} = \frac{1}{2}(z_n + \frac{1}{z_n})$, $n \geq 1$, and $z_1 = \frac{1}{2}(z_0 + \frac{1}{z_0})$. Show that

$$\lim_{n\to\infty} z_n = \begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0 \end{cases}$$

Proof. ■

(ii) Discuss convergence and divergence of

$$z_n = 1 + i\frac{(-1)^n}{n^2} \text{ for} n = 1, 2, \ldots .$$

Let $\Phi = \operatorname{Arg}(z_n)$. Show that $(\Phi_n) \to 0$.

Proof. ■

(iii) Assume that $0 < \gamma < 1$. Show that

$$\begin{aligned}\sum_{n=1}^{\infty} \gamma^n \cos n\theta &= \frac{\gamma\cos\theta - \gamma^2}{1 - 2\gamma\cos\theta + \gamma^2} \\ \sum_{n=1}^{\infty} \gamma^n \sin n\theta &= \frac{\gamma\sin\theta}{1 - 2\gamma\cos\theta + \gamma^2}.\end{aligned}$$

Proof. ■

3 Problem 3

Let (z_n) be a sequence of non-zero complex numbers. Suppose that

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L.$$

Show that $\lim\limits_{n\to\infty} \sqrt[n]{|z_n|} = L$.

Proof. ■

4 Problem 4

5 Problem 5

6 Problem 6

7 Problem 7

8 Problem 8