Exercises

Lance Remigio

April 25, 2025

1 Regular Curves

Problem 1. Show that it is not possible to parametrize the cissoid of Diocles

$$x(x^2 + y^2) = 2Ry^2$$

so that it is regular at the origin.

Proof. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then the cissoid of Diocles can be parametrized via polar coordinates:

$$\begin{aligned} &x(x^2 + y^2) = 2Ry^2 \\ &\implies r\cos\theta \cdot r^2 = 2R \cdot r^2 \sin^2\theta \\ &\implies r^3\cos\theta = 2R \cdot r^2 \sin^2\theta \\ &\implies r = 2R\sin\theta\tan\theta. \end{aligned}$$

Now, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we obtain

$$\begin{aligned} x &= 2R\sin\theta\tan\theta = 2R\sin^2\theta \\ y &= 2R\sin\theta\tan\theta \cdot \sin\theta \\ &= 2R\frac{\sin^3\theta}{\cos\theta}. \end{aligned}$$

Note that if we let $t = \sin\theta$, we get

$$q(t) = 2R(x(t), y(y)) = 2R\Big(t^2, \frac{t^3}{\sqrt{1-t^2}}\Big).$$

Differentiating, we get

$$\begin{aligned} \dot{x}(t) &= 4Rt \\ \dot{y}(t) &= 2R\frac{3t^2 - 2t^4}{(1-t^3)^{3/2}}. \end{aligned}$$

and so we have

$$|\dot{q}(t)| = 2R\sqrt{4t^2 + \frac{3t^2 - 2t^4}{(1-t^2)^{3/2}}}.$$

But note that if $t_0 = 0$, then $|\dot{q}(t)| = 0$ which tells us that the cissoid of Diocles is not regular at the origin. ■

2 Curvature

Problem 2. Let $q(t) = r(t)(\cos t, \sin t)$. Show that the speed is given by

$$\Big(\frac{ds}{dt}\Big)^2 = \Big(\frac{dr}{dt}\Big)^2 + r^2$$

and the curvature
$$\kappa = \frac{\left|2(\frac{dr}{dt})^2 + r^2 - r\frac{d^2r}{dt^2}\right|}{((\frac{dr}{dt})^2 + r^2)^{3/2}}.$$

Proof. To find the curvature of q, it suffices to use the following formula:
$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{|\dot{q}|^3}.$$
Differentiating $q(t)$ once, we obtain
$$\begin{aligned}\dot{q}(t) &= \dot{r}(\cos t, \sin t) + r(-\sin t, \cos t)\\ &= (\dot{r}\cos t - r\sin t, \dot{r}\sin t + r\cos t)\end{aligned}$$
and so the magnitude of $\dot{q}(t)$ is given by:
$$\begin{aligned}\dot{s}(t) = |\dot{q}(t)| &= \sqrt{(\dot{r}\cos t - r\sin t)^2 + (\dot{r}\sin t + r\cos t)^2}\\ &= \sqrt{\dot{r}^2\cos^2 t - 2r\dot{r}\sin t\cos t + 2r\dot{r}\sin t\cos t + r^2\sin^2 t + \dot{r}^2\sin^2 t + r^2\cos^2 t}\\ &= \sqrt{\dot{r}^2 + r^2}.\end{aligned}$$
But this implies that
$$|\dot{q}(t)|^2 = \dot{r}^2 + r^2.$$
Now, notice that
$$\begin{aligned}\dot{x}\ddot{y} &= (\dot{r}\cos t - r\sin t)(2\dot{r}\cos t + (\ddot{r} - r)\sin t)\\ &= 2\dot{r}^2 + \dot{r}\ddot{r}\sin t\cos t - \dot{r}r\sin t\cos t - 2\dot{r}r\sin t\cos t - r\ddot{r}\sin^2 t + r^2\sin^2 t\end{aligned}$$
and
$$\begin{aligned}\ddot{x}\dot{y} &= ((\ddot{r} - r)\cos t - 2\dot{r}\sin t)(\dot{r}\sin t + r\cos t)\\ &= \ddot{r}\dot{r}\sin t\cos t - \dot{R}r\sin t\cos t - 2\dot{r}^2\sin^2 t + r\ddot{r}\cos^2 t - r^2\cos^2 t - 2\dot{r}r\sin t\cos t.\end{aligned}$$
Subtracting $\dot{x}\ddot{y}$ and $\ddot{x}\dot{y}$, we get that
$$2\dot{r}^2 + r^2 - r\ddot{r}$$
and so the curvature of q is just
$$\begin{aligned}\kappa &= \frac{|\dot{x}\ddot{y} - \dot{x}\dot{y}|}{|\dot{q}|^3}\\ &= \frac{|2\dot{r}^2 + r^2 - r\ddot{r}|}{(\dot{r}^2 + r^2)^{3/2}}\end{aligned}$$
which is our desired result.

■

Problem 3. Compute the curvature of the logarithmic spiral
$$ae^{bt}(\cos t, \sin t).$$

Solution. Let $q(t) = ae^{bt}(\cos t, \sin t)$. To compute the curvature of the logarithmic spiral, we first need to differentiate $q(t)$ twice. Indeed, we have
$$\begin{aligned}\dot{q}(t) &= abe^{bt}(\cos t, \sin t) + ae^{bt}(-\sin t, \cos t)\\ &= ae^{bt}(b\cos t - \sin t, b\sin t + \cos t)\end{aligned}$$
and so
$$\begin{aligned}\|\dot{q}(t)\| &= ae^{bt}\sqrt{(b\cos t - \sin t)^2 + (b\sin t + \cos t)^2}\\ &= ae^{bt}\sqrt{b^2 + 1}.\end{aligned}$$

Now, differentiating one more time, we get

$$\begin{aligned}\ddot{q}(t) &= abe^{bt}(-b\sin t - \cos t, b\cos t - \sin t) + ae^{bt}(-n\sin t - \cos t, b\cos t - \sin t)\\ &= ae^{bt}((b^2-1)\cos t - 2b\sin t, (b^2-1)\sin t + 2b\cos t).\end{aligned}$$

Since curvature is given by the following formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{|\dot{q}|^3},$$

we can obtain the following

$$\begin{aligned}\dot{x}\ddot{y} &= ae^{bt}(b(b^2-1)\sin t\cos t + 2b^2\cos^2 t - (b^2-1)\sin^2 t - 2b\sin t\cos t)\\ \dot{y}\ddot{x} &= ae^{bt}(b(b^2-1)\sin t\cos t - 2b^2\sin^2 t + (b^2-1)\cos^2 t - 2b\sin t\cos t).\end{aligned}$$

Hence, we have the following

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = a^2e^{2bt}(b^2+1)$$

and so our curvature is just

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{q}|^3} = \frac{e^{-bt}}{a\sqrt{b^2+1}}.$$

■

3 Regular Surfaces

Definition (Regular Surfaces). A subset $S \subseteq \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $x : U \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ such that

(1) $\mathbf{x}$ is differentiable; that is, the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders in U.

(2) $\mathbf{x}$ is a homeomorphism; that is, $\mathbf{x}$ contains an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous.

(3) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

Problem 4. Show that the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$ is a regular surface.

Proof. Define the following parametrization for C by $x = \cos(u)$, $y = \sin(u)$, and $z = v$ for every $(u, v) \in U$ where $U = (a, b) \times (c, d) \subseteq \mathbb{R}^2$ is an open set. We will show that this parametrization of C satisfies properties (1) through (3) of the definition above. Indeed, we have

(1) Clearly, we see that $\cos(u)$, $\sin(u)$, and v are functions that have derivatives of all order. Hence, $\mathbf{x}(u, v) = (\cos(u), \sin(u), v)$ contains derivatives of all order on $C \cap U$.

(2) We can see that the component functions contain their respective inverses on $C \cap U$; that is, $\cos^{-1}(x) = u$, $\sin^{-1}(y) = v$, and $z = v$. Thus, $\mathbf{x}^{-1}$ exists and is continuous (since their respective components are also continuous).

(3) Note that

$$\frac{\partial \mathbf{x}}{\partial u} = (\cos(u), \sin(u), v)$$

and

$$\frac{\partial \mathbf{x}}{\partial v} = (0, 0, 1).$$

Hence, we have

$$\mathrm{d}\mathbf{X}_{(u,v)} = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that the matrix above has rank 2 and thus $\mathrm{d}\mathbf{x}_{(u,v)}$ must be an injective linear map.

Since properties (1)-(3) are satisfied, it follows that C is a regular surface. ■

4 Gauss Map and Fundamental Properties

Problem 5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^{\pi} k_n(\theta)\ d\theta \tag{*}$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Proof. Let k_1 and k_2 be the maximum normal curvature and minimum normal curvature, respectively. To show (*), it suffices to show that

$$\frac{1}{\pi}\int_0^{\pi} k_n(\theta)\ d\theta = \frac{k_1+k_2}{2}.$$

Choose $\{e_1, e_2\}$ as our basis for $T_p(S)$. Then for any $v \in T_p(S)$, we have

$$v = e_1\cos\theta + e_2\sin\theta$$

where θ is a fixed direction. Using the Second Fundamental Form, we have

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta.$$

Hence, we can see that

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} k_n(\theta)\ d\theta &= \frac{1}{\pi}\int_0^{\pi} [k_1\cos^2\theta + k_2\sin^2\theta]\ d\theta \\
&= \frac{1}{\pi}\int_0^{\pi} k_1\cos^2\theta\ d\theta + \frac{1}{\pi}\int_0^{\pi} k_2\sin^2\theta\ d\theta \\
&= \frac{k_1}{2\pi}\int_0^{\pi}[1+\cos 2\theta]\ d\theta + \frac{k_2}{2\pi}\int_0^{\pi}[1-\cos 2\theta]\ d\theta \\
&= \frac{k_1}{2\pi}\left[\pi + \frac{1}{2}\sin 4\pi\right] + \frac{k_2}{2\pi}\left[\pi - \frac{1}{2}\sin 4\pi\right] \\
&= \frac{k_1+k_2}{2} \\
&= H.
\end{aligned}$$

Thus, we have

$$H = \frac{1}{\pi}\int_0^{\pi} k_n(\theta)\ d\theta.$$

■

Problem 6. Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N : S \to \S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$.

Proof. Note that since $H \equiv 0$, then $k_1 = -k_2$. Also, since S has no planar points $dN_p \neq 0$ for all $p \in S$. Choose an orthonormal eigenbasis $\{e_1, e_2\}$. Then for $w_1, w_2 \in T_p(S)$, we have

$$w_1 = e_1\cos\theta + e_2\sin\theta \tag{1}$$

and

$$w_2 = e_1\cos\phi + e_2\sin\phi. \tag{2}$$

where ϕ and θ are angles formed with e_1. Hence, we have

$$\begin{aligned}
dN_p(w_1) &= e_1k_1\cos\theta + e_2k_2\sin\theta \\
dN_p(w_2) &= e_1k_1\cos\phi + e_2k_2\sin\phi.
\end{aligned}$$

Then we have

$$\begin{aligned}
-K(p)\langle w_1, w_2\rangle &= -k_1k_2\langle w_1, w_2\rangle \\
&= \langle -k_1w_1, k_2w_2\rangle \\
&= \langle -k_1[e_1\cos\theta + e_2\sin\theta], k_2[e_1\cos\phi + e_2\sin\phi]\rangle \\
&= \langle k_2e_1\cos\theta + k_2e_2\sin\theta, k_2e_1\cos\phi + k_2e_2\sin\phi\rangle \\
&= \langle dN_p(w_1), dN_p(w_2)\rangle.
\end{aligned}$$

Hence, we conclude that

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle.$$

■