

Math 234A Lecture Notes

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Contents

1	Week1	2
1.1	Lecture 1	2
1.1.1	Why do we need complex analysis?	2
1.1.2	What is the goal?	2
1.2	Lecture 2	3
1.2.1	Topics	3
1.3	Complex Plane \mathbb{C} as a working model for Complex Numbers	3
1.3.1	A model for \mathbb{C}	3
1.3.2	Viewing \mathbb{C} as a Metric Space	5
2	Week2	6
2.1	Lecture 3	6
2.1.1	Topics	6
2.1.2	Polar Representation of Complex Numbers	6
2.1.3	Convergence of Sequences in \mathbb{C}	8
3	Week3	10
3.0.1	Topics	10

Chapter 1

Week1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either \mathbb{R} or \mathbb{C} . For example, the polynomial $x^2 + 1$ does not have a solution in \mathbb{R} , but it does have a solution in \mathbb{C} .
- Solving real integrals that may be difficult to deal with using standard techniques developed in \mathbb{R} ; that is, something like

$$\int_0^\infty \frac{\sin x}{x} dx.$$

- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system \mathbb{R} .

- (i) $(\mathbb{R}, +, \cdot)$ is a field.
- (ii) We have an order relation on \mathbb{R} .
- (iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in \mathbb{R} .

Our goal is to find the "smallest" field \mathbb{C} such that

- (i) \mathbb{R} is "contained" in \mathbb{C} .
- (ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in \mathbb{C} .

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of \mathbb{R} as a vector space, and using the operations defined on that vector space to define the operations of \mathbb{C} . Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i . In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on \mathbb{C} .

Definition (Operations on \mathbb{C}). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication \cdot , in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that \mathbb{C} forms a field.

Proposition. The defined operations of \mathbb{C} form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \mathbb{C}$. Then there exists $\gamma + i\delta \in \mathbb{C}$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane \mathbb{C} as a working model for complex numbers.
- Discuss the Euclidean Topology on \mathbb{C} .
- Discuss polar representation of a complex number.

1.3 Complex Plane \mathbb{C} as a working model for Complex Numbers

Lemma. Let F be a field containing \mathbb{R} and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

- (i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} is a **subfield** of F .

- (ii) Let F be another field containing \mathbb{R} and containing a solution i' of $x^2 + 1 = 0$ and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} and \mathbb{C}' are isomorphic as fields.

The second part of this lemma is simply saying that \mathbb{C} is a unique subfield of F .

1.3.1 A model for \mathbb{C}

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication \cdot as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers \mathbb{C} have the following properties

- (i) $(\mathbb{C}, +, \cdot)$ is a field.
- (ii) Let $\mathcal{J} : \mathbb{R} \rightarrow \mathbb{C}$ by $x \rightarrow (x, 0)$. Then \mathcal{J} is a subfield and forms an isomorphism between \mathbb{R} and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

- (i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.
- (ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homomorphism.
- (iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that \mathcal{J} is a bijective map.

■

Definition (Constructing a solution for $x^2 + 1 = 0$). Define $i = (0, 1)$ as our imaginary number in \mathbb{C} and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number). Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z , respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate). Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\bar{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the x -axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

- (i) $\bar{\bar{z}} = z$.
- (ii) $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
- (iii) $\overline{zw} = \bar{z} \cdot \bar{w}$.
- (iv) $\Re(z) = \frac{1}{2}(z + \bar{z})$.
- (v) $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
- (vi) $z\bar{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number). Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\bar{z}}$.

Proposition (More Properties of Complex Numbers). Given $z, w \in \mathbb{C}$, we have the following properties:

- (i) $|z| = 0$ if and only if $z = 0$.
- (ii) $|zw| = |z||w|$.
- (iii) $|\Re(z)| \leq |z|$.
- (iv) $|\Im(z)| \leq |z|$.
- (v) $|z + w| \leq |z| + |w|$.
- (vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

1.3.2 Viewing \mathbb{C} as a Metric Space

In \mathbb{C} , the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space). For $z, w \in \mathbb{C}$, we call \mathbb{C} a metric space if it satisfies the following properties:

- (i) $d(z, w) = d(w, z)$.
- (ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have \mathbb{C} is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of complex numbers.
- Convergence of sequences in \mathbb{C} .

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos \varphi, \sin \varphi)$$

with $\tan \varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos \psi, \sin \psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π .

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}^*$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos \varphi + i \sin \varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^*$, then

$$z = \gamma(\cos \varphi + i \sin \varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π .

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of z** .

Definition (Argument and Principle Argument). Let $z \in \mathbb{C}$ and $z = \gamma(\cos \varphi + i \sin \varphi)$ be a polar representation of z . Then φ is called **an argument of z** . If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of z** and it is denoted by $\text{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos \varphi + i \sin \varphi)$ and $w = \gamma'(\cos(\varphi') + i \sin(\varphi'))$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \gamma\gamma'[\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos \varphi + i \sin \varphi)(\cos \varphi' + i \sin \varphi') \\ &= \gamma\gamma'[(\cos \varphi \cos \varphi' - \sin \varphi \sin \varphi') + i(\sin \varphi \cos \varphi' + \sin \varphi \cos \varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')). \end{aligned}$$

■

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}$ with $z = \gamma(\cos \varphi + i \sin \varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i \sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos \varphi - i \sin \varphi). \end{aligned}$$

Corollary (De Moivre's Theorem). Let $z = \gamma(\cos \varphi + i \sin \varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i \sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the n th roots of a non-zero complex number.

Example 2.1.1 (An example of De Moivre's Theorem). Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10} . First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i \sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

- (i) The set of all positive real numbers $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$
- (ii) The set of all complex numbers excluding zero $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow r(\cos \varphi + i \sin \varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in \mathbb{C}

Definition (Convergence in \mathbb{C}). Let $\{z_n\}_{n=1}^\infty$ be a sequence in \mathbb{C} . We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z , then we write $z_n \rightarrow z$.

Proposition (Properties of Convergent Sequences). Assume $(z_n) \rightarrow z$ and $(w_n) \rightarrow w$.

- (i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w$.
- (ii) $z_n w_n \rightarrow zw$.
- (iii) $z_n^{-1} \rightarrow z^{-1}$.
- (iv) $(z_n) \rightarrow z$ if and only if $\Re(z_n) \rightarrow \Re(z)$ and $\Im(z_n) \rightarrow \Im(z)$ as a sequences in \mathbb{R} .

Proof. Suppose $(z_n) \rightarrow z$ and $(w_n) \rightarrow w$.

- (i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \rightarrow z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \rightarrow w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w.$$

- (ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \rightarrow z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \rightarrow w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned} |z_n w_n - z w| &= |z_n w_n - w_n z + w_n z - z w| \\ &\leq |w_n| |z_n - z| + |z| |w_n - w| \\ &\leq M |z_n - z| + |z| |w_n - w| \\ &< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we conclude that

$$z_n w_n \rightarrow z w.$$

- (iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \rightarrow z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \rightarrow z$, there must exist an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned} |z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z| |z_n|} \\ &< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that $z_n^{-1} \rightarrow z^{-1}$.

■

Chapter 3

Week3

3.0.1 Topics

- (i) Continue discussion of convergence of sequence/series.
- (ii) Discuss exponential, sine, and cosine function.

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^n z_k$$

Definition (Convergence of Series). If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example 3.0.1. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i} \left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$

$$\frac{1}{2} is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider

$$\frac{1}{2} i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow s_n = \frac{1}{i} \left[1 - \frac{1}{2^n} \right].$$

Thus, taking the limit as $n \rightarrow \infty$ gives us

$$\lim_{n \rightarrow \infty} s_n = \frac{1}{i}.$$

Example 3.0.2. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since $|z| < 1$, we know that

$$\sum_{n=1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n \rightarrow \infty} \sum_{k=1}^n z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as $n \rightarrow \infty$, we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series). Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example 3.0.3. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Definition (Complex Version of Exponential Function). For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series). We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z+w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \quad \textbf{Show this!}.$$