Linear Algebra Notes

Lance Remigio

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Contents

Chapter 1

Vector Spaces

1.1 Linear Combinations and Systems of Linear Equations

Definition 1.1.1 (Linear Combinations). Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called **linear combination** of vectors of S if there exist a finite number of vectors $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

In this case, we say that v is a linear combination of $u_1, u_2, \ldots, u_n \in V$ and call $a_1, a_2, \ldots, a_n \in F$ the **coefficients** of the linear combination.

- An easy example of a vector expressed as a linear combination is the zero vector O where $0v = O$ for each $v \in V$.
- Determining whether a vector is a linear combination of other vectors often involves solving a system of linear equations.

Example 1. Suppose we wanted to express the vector $(2, 6, 8) \in \mathbb{R}^3$ as a linear combination of

$$u_1 = (1, 2, 1),\ u_2 = (-2, -4, -2),\ u_3 = (0, 2, 3),\ u_4 = (2, 0, -3), \text{ and } u_5 = (-3, 8, 16).$$

Our goal is to find scalars a_1, a_2, a_3, a_4 and a_5 such that

$$(2, 6, 8) = a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4 + a_5 u_5.$$

Doing a bit of algebra, we would need to solve the following system of linear equations

$$\begin{aligned} a_1 - 2a_2 \quad + 2a_4 - 3a_5 &= 2 \\ 2a_1 - 4a_2 + 2a_3 \quad + 8a_5 &= 6 \\ a_1 - 2a_2 + 3a_3 + 16a_5 &= 8 \end{aligned}$$

Solving the system of equations above involves three types of operations:

1. Interchanging the order of any two equations in the system;
2. multiplying any equation in the system by some non-zero constant;
3. adding a constant multiple of any equation to another equation in the system.

We will learn in a later section that the operations listed above do not change the set of solutions to the original system.

Example 2. We claim that
$$2x^3 - 2x^2 + 12x - 6$$
is a linear combination of
$$x^3 - 2x^2 - 5x - 3 \text{ and } 3x^3 - 5x^2 - 4x - 9$$
in $P_3(\mathbb{R})$, but that
$$3x^3 - 2x^2 + 7x + 8$$
is not. In the first case we wish to find scalars a and b such that
$$\begin{aligned} 2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) \\ &\quad + b(3x^3 - 5x^2 - 4x - 9) \\ &= (a + 3b)x^3 + (-2a - 5b)x^2 \\ &\quad + (-5a - 4b)x + (-3a - 9b). \end{aligned}$$
Thus, we have the following system of linear equations:
$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= -2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6. \end{aligned}$$
Adding the appropriate multiples of the first equation to the others in order to eliminate a, we find that
$$\begin{aligned} a + 3b &= 2 \\ b &= 2 \\ 11b &= 22 \\ 0b &= 0. \end{aligned}$$
Then we get that $a = -4, b = 2, 0 = 0, 0 = 0$. Hence, we have
$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$
Looking at the second case now, we observe that using the preceding technique leads us to the following system of linear equations
$$\begin{aligned} a + 3b &= 3 \\ -2a - 5b &= -2 \\ -5a - 4b &= 7 \\ -3a - 9b &= 8. \end{aligned}$$
Eliminating a as before yields the following:
$$\begin{aligned} a + 3b &= 3 \\ b &= 4 \\ 11b &= 22 \\ 0 &= 17. \end{aligned}$$
The presence of the non-sensical result in the last equality tells us that there are no solutions to the system and that the second polynomial cannot be written as a linear combination of the two polynomials.

We can denote the set of all linear combinations of some set of vectors in the following:

Definition 1.1.2 (Span). Let S be a nonempty subset of a vector space V. The **span** of S, denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $\text{span}(\emptyset) = \{0\}$.

Some immediate examples of spans are:

Example 3. In $\mathbb{R}^3$, the span of the set $S = \{(1,0,0),(0,1,0)\}$ consist of all vectors in $\mathbb{R}^3$ such that for some scalars $a, b \in \mathbb{R}$, we have

$$a(1,0,0) + b(0,1,0) = (a,b,0).$$

This tells us that S contains all the points in the xy-plane. One can show that S is a subspace of $\mathbb{R}^3$.

Theorem 1.1.1. The span of any subset S of a vector space v is a subspace of V that contains S. Moreover, any subspace of V that contains S must also contain the span of S.

Proof. Suppose $S = \emptyset$. Then the span of S is just $\text{span}(\emptyset) = \{0\}$ which is a subspace in which S is contained in. Moreover, $\text{span}(\emptyset) = \{0\}$ is always contained in any subspace X of V because every subspace contains the zero vector 0. Suppose $S \neq \emptyset$, then S contains a vector z. We need to show that $\text{span}(S)$ is a subspace of V.

(a) Since $S \neq \emptyset$, we know that S contains a vector v such that $0z = 0$. Hence, $0 \in \text{span}(S)$.

(b) Let $x, y \in \text{span}(S)$. We need to show that $x + y \in \text{span}(S)$. If $x \in \text{span}(S)$, then we can find $a_1, a_2, \dots, a_n \in F$ and $x_1, x_2, \dots, x_n \in S$ such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Likewise, $y \in \text{span}(S)$ implies that we can find scalars $b_1, b_2, \dots, b_m$ and vectors $y_1, y_2, \dots, y_m$ such that

$$b_1y_1 + b_2y_2 + \cdots + b_my_m.$$

Then

$$x + y = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_1y_1 + b_2y_2 + \cdots + b_my_m$$

is a linear combination and thus $x + y \in \text{span}(S)$.

(c) Note that $c \in F$ implies that

$$\begin{aligned} cx &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (ca_1)x_1 + (ca_2)x_2 + \cdots + (ca_n)x_n \end{aligned}$$

is a linear combination and thus $cx \in \text{span}(S)$.

Hence, $\text{span}(S)$ is a subspace of V. Now we need to show that S is contained within $\text{span}(S)$; that is, $S \subseteq \text{span}(S)$. Let $v \in S$. Then using (VS 5), we can see that $1 \cdot v = v$ is a linear combination; so we have $v \in \text{span}(S)$ and hence, the span of S contains S.

Now let W be any subspace of V that contains S. We need to show that $\text{span}(S) \subseteq W$. Let $v \in \text{span}(S)$. Then we can find scalars $a_1, a_2, \dots, a_n \in F$ and $x_1, x_2, \dots, x_n \in S$ such that

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Since $S \subseteq W$, we know that $x_1, x_2, \dots, x_n \in W$. Using exercise 20 from section 1.3 and using the same set of scalars $a_1, a_2, \dots, a_n \in F$, we have $v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$. Hence, $\text{span}(S) \subseteq W$. ■

Definition 1.1.3. A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we also say that the vectors of S generates (or span) V.

Example 4 (Vectors in $\mathbb{R}^3$)**.** The vectors $(1,1,0),(1,0,1),(0,1,1)$ generate $\mathbb{R}^3$ since any given vector $v \in \mathbb{R}^3$ is a linear combination of the three given vectors. Furthermore, there exists scalars $r,s,t \in \mathbb{R}$ such that

$$r(1,1,0)+s(1,0,1)+t(0,1,1)=(a_1,a_2,a_3)=v$$

where

$$r=\tfrac{1}{2}(a_1+a_2-a_3), s=\tfrac{1}{2}(a_1-a_2+a_3) \text{ and } t=\tfrac{1}{2}(-a_1+a_2+a_3).$$

Example 5 (Polynomials)**.** The polynomials $x^2+3x-2, 2x^2+5x-3$, and $-x^2-4x+4$ generate $P_2(\mathbb{R})$ because each of the three given polynomials belongs to $P_2(\mathbb{R})$ and each polynomial $ax^2+bx+c \in P_2(\mathbb{R})$ is a linear combination of these three. It can be shown that we can find $a,b,c \in \mathbb{R}$ such that

$$a(x^2+3x-2)+b(2x^2+5x-3)+c(-x^2-4x+4)=ax^2+bx+c.$$

Example 6 (Matrices)**.** Note that the matrices

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \text{ and } \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

generate $M_{2\times 2}(\mathbb{R})$ because an every $A \in M_{2\times 2}(\mathbb{R})$ can be expressed as a linear combination of the four given matrices found below where there exists scalars $a,b,c,d \in \mathbb{R}$ such that

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + b\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + c\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + d\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

where it can be show that

$$\begin{aligned} a &= \frac{1}{3}a_{11}+\frac{1}{3}a_{12}+\frac{1}{3}a_{21}-\frac{2}{3}a_{22}\\ b &= \frac{1}{3}a_{11}+\frac{1}{3}a_{12}-\frac{2}{3}a_{21}+\frac{1}{3}a_{22}\\ c &= \frac{1}{3}a_{11}-\frac{2}{3}a_{12}+\frac{1}{3}a_{21}+\frac{1}{3}a_{22}\\ d &= -\frac{2}{3}a_{11}+\frac{1}{3}a_{12}+\frac{1}{3}a_{21}+\frac{1}{3}a_{22}. \end{aligned}$$

A non-example set of matrices are

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \text{ and } \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

do not generate $M_{2\times 2}(\mathbb{R})$ since all of the matrices have equal diagonal entries. Thus, every the set of matrices above cannot generate every $A \in M_{2\times 2}(\mathbb{R})$.

1.2 Linear Dependence and Linear Independence

1.2.1 Motivation

Suppose V is a vector space over a field F and that $W \subseteq V$ is a subspace of V.

- Suppose we have a set S that will generate every vector in V as a linear combination of vectors in S.
- It is desirable to find a subset of S that is as small as possible or rather just enough to generate each vector in V.

- The reason for this is to reduce the amount of computations done to represent a vector in V.
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in S.
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in S.
- Doing this, guarantees that we can write any one of the vectors in S as a linear combination of the others.

Definition 1.2.1 (Linearly Dependence). A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors in $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case, we also say that the vectors of S are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $v_1, v_2, \ldots, v_n \in S$ where for all $1 \leq i \leq n$, we have $a_i = 0$.
- This tells us that our definition of **linear dependence** implies that 0 is a non-trivial linear combination.
- Any subset of V that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is, $1 \cdot 0 = 0$.

1.2.2 Examples of Linearly Dependent Sets

Example 7. Consider a subset in $\mathbb{R}^4$ defined by

$$S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}.$$

To show that S is a linear dependent set, we need to find scalars $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that a_i for all $1 \leq i \leq 4$ not all zero such that

$$a_1(1, 3, -3, 2) + a_2(2, 2, -4, 0) + a_3(1, -3, 2, -4) + a_4(-1, 0, 1, 0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients $a_1 = 4, a_2 = -3, a_3 = 2$, and $a_4 = 0$. Thus, we have that S is linearly dependent subset of $\mathbb{R}^4$ and hence we can write any vector in S as a linear combination of the other vectors contained in S.

Example 8. Define a subset of $M_{2\times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} \}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients a_1, a_2, a_3 such that $a_1 = 5, a_2 = 3$, and $a_3 = -2$ where

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 1.2.2 (Linear Independence). A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

1.2.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is, $\{v\}$ is linearly independent.
- If $\{v\}$ is linearly dependent, then it the singleton has to be the zero vector 0. This is because $au = 0$
$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

Example 9 (A Set of Vectors in $\mathbb{R}^4$). It can be shown that the set

$$S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of 0 represent the trivial representation; that is, we have scalars $a_1 = a_2 = a_3 = a_4 = 0$.

Example 10. For $k = 0, 1, \dots, n$, let $p_k = x^k + x^{k+1} + \cdots + x^n$. The set

$$\{p_0(x), p_1(x), \dots p_n(x)\}$$

is linearly independent in $P_n(F)$. It can be shown that for some scalars $a_0, a_1, \dots, a_n \in F$, the following equation

$$a_0 + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n = 0$$

has the trivial-representation; that is, $a_i = 0$ for all $1 \le i \le n$.

Theorem 1.2.1. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Proof. Let V be a vector space. Suppose S_1 is linearly dependent. Then there exists a finite number of distinct vectors $v_1, v_2, \dots, v_n \in S_1$ and scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Since $S_1 \subseteq S_2$, we must have $v_1, v_2, \dots, v_n \in S_2$ as well. If these vectors are linearly dependent, then these vectors are also linearly dependent in S_2; that is, we have scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Hence, S_2 is linearly dependent. ■

Corollary. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof. Note that this corollary is just the contrapositive of the theorem before it. Hence, S_1 is linearly independent. ■

- Determining whether there exists a minimal generating set for the span of S is related to solving the problem of finding whether some vector in S that can be written in terms of a linear combination of other vectors in S.
- We can see that in a given subset of $\mathbb{R}^3$ defined by

$$S = \{u_1, u_2, u_3, u_4\}$$

where $u_1 = (2, -1, 4), u_2 = (1, -1, 3), u_3 = (1, 1, -1)$, and $u_4 = (1, -2, 1)$. Note that this S is linearly dependent. This tells us that any vector in S can be written as a linear combination of the others. Suppose we pick u_3 and write as a linear combination of the vectors u_1, u_2, and u_4. We would find that the span of these three vectors (denote this set as S') generates the same spanning set S but with a cardinality one less than S; that is, $\text{span}(S) = \text{span}(S')$.
- If we find that there does not exist a proper subset that is equivalent to the span of S, then S must be a linearly independent set.

Theorem 1.2.2. Let S be a linearly independent subset of a vector space V, and let $v \in V$ but not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof. ($\Rightarrow$) Suppose $S \cup \{v\}$ is a linearly dependent set. Then there exists a finite number of scalars $a_1, a_2, \dots, a_n \in F$ and vectors $u_1, u_2, \dots, u_n \in S \cup \{v\}$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

Since S is a linearly independent set, we must have $u_i = v$ for some $1 \leq i \leq n$. Choose $i = 1$ (any choice of i will do) such that

$$a_1v + a_2u_2 + \cdots + a_nu_n = 0.$$

Solving for v by subtracting a_1v on both sides, multiplying by a_1^{-1} on both sides of the equation, and distributing by a_1^{-1} yields the following:

$$v = (-a_1^{-1}a_2)u_2 + (-a_1^{-1}a_3)u_3 + \cdots + (-a_1^{-1}a_n)u_n.$$

Since v is a linear combination of vectors $u_1, u_2, \dots, u_n \in S$, we know that $v \in \text{span}(S)$.

($\Leftarrow$) Conversely, suppose $v \in \text{span}(S)$. This implies that there exists a finite amount of scalars $a_1, a_2, \dots, a_n \in F$ and $u_1, u_2, \dots, u_n \in S$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Subtracting v from both sides of the equation above gives us the following equation:

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - v = 0.$$

Note that v is not contained in S, so $v \neq u_i$ for all $1 \leq i \leq n$. Since S is a linearly independent set, we know that $a_i = 0$ for all $1 \leq i \leq n$. This implies that the only coefficient that is non-zero is with $-v = -1v$. Hence, the set of vectors $S' = \{u_1, u_2, \dots, u_n, v\}$ is linearly dependent. Since $S' \subseteq S \cup \{v\}$, we know that $S \cup \{v\}$ is also linearly independent by Theorem 6. ■

1.3 Bases And Dimension

- Recall that S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

Definition 1.3.1 (Basis). A **basis** β for a vector space V is linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Example 11.

- Recall that the empty set $\emptyset$ is linearly independent and that $\text{span}(\emptyset) = \{0\}$. The empty set $\emptyset$ in this case is the basis for the zero vector space.
- Note that in F^n, the vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0) \dots, e_n = (0, 0, \dots, 0, 1)$ form a basis for F^n.
- The basis for $M_{m\times n}(F)$ is the set of matrices E^{ij} such that the only nonzero entry is a 1 in the ith and jth column.
- As we have seen in the last section, the set $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$.
- In $P(F)$, the set $\{1, x, x, x^2, \dots\}$ is a basis. *Bases are not limited to finite sets. They can be infinite.*

Theorem 1.3.1. Let V be a vector space and $u_1, u_2, \dots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V if and only if each $v \in V$ can be unique expressed as a linear combination of vectors in β, that is, expressed in the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

for unique scalars $a_1, a_2, \dots, a_n$.

Proof. ($\Rightarrow$) Suppose $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V. Then $\text{span}(\beta) = V$. If $v \in V$, then $v \in \text{span}(\beta)$. Hence, we can write v as a linear combination of vectors in β such that choosing scalars $a_1, a_2, \dots, a_n \in F$ leads to

$$v = \sum_{i=1}^{n} a_i u_i.$$

Suppose there exists another representation of $v \in V$ such that

$$v = \sum_{i=1}^{n} b_i x_i$$

Hence, observe that

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i y_i$$
$$\Rightarrow \sum_{i=1}^{n} (a_i - b_i) x_i = 0.$$

Since β is linearly independent, we know that $a_i - b_i = 0$ which implies $a_i = b_i$ for all $1 \leq i \leq n$. Hence, v can be expressed as a unique linear combination of vectors in β.

($\Leftarrow$) Conversely, let $v \in V$ be expressed as a unique linear combination of vectors in β such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \tag{1}$$

for unique scalars $a_1, a_2, \dots, a_n$. We want to show that β is a basis for V; that is, we want to show that β is a spanning set for V and β is linearly independent. To show that β is a spanning set for V, we need to show that $\text{span}(\beta) \subseteq V$ and $V \subseteq \text{span}(\beta)$. Note that $\beta \subseteq V$ and $\text{span}(\beta)$ is a subspace for V. Hence, $\text{span}(\beta) \subseteq V$. On the other hand, $V \subseteq \text{span}(\beta)$ follows immediately from (1). To show that β is linearly independent, we need to show that for scalars $\delta_1, \delta_2, \dots, \delta_n$, we have

$$\delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_n u_n = 0$$

such that $u_i = 0$ for all $1 \le i \le n$. Note that $v + 0 = v$. Hence, we can write

$$\begin{aligned}\sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n} \delta_i u_i &= \sum_{i=1}^{n} a_i u_i \\ \sum_{i=1}^{n} (a_i + \delta_i) u_i &= \sum_{i=1}^{n} a_i u_i\end{aligned}$$

Equating each term in the equation above yields the following equality:

$$a_i + \delta_i = a_i \Rightarrow \delta_i = 0$$

for all $1 \le i \le n$. But this tells us that β is linearly independent and we are done. ■

- Any vector $v \in V$ can be written as a linear combination of vectors from the basis containing $u_1, u_2, \ldots, u_n \in V$.
- This determines a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ and conversely, each n-tuple of scalars determines a unique vector $v \in V$ such that each coefficient from the linear combination of $u_1, u_2, \ldots, u_n$ is an entry from said tuple.
- For example, in our vector space F^n, n is the number of vectors that should be in the basis for F^n which is indeed the case.
- In this book, we are only concerned with finite bases.

Theorem 1.3.2 (Finite Spanning Set For a Vector Space)**.** If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence, V has a finite basis.

Proof. Suppose $S = \emptyset$ or $S = \{0\}$, then $V = \{0\}$ and $\emptyset$ is a subset of S that is a basis for V. If S neither of these choices, then S must contain at least one nonzero vector u_1; that is, $S = \{u_1\}$. Since u_1 is nonzero, it follows that S is a linearly independent set. We can continue this process of adding vectors $u_2, \ldots, u_k$ into S such that S is a linearly independent set of k vectors. Since S is a finite set, we must end with the linearly independent set $\beta = \{u_1, u_2, \ldots, u_n\}$. There are two cases for which this occurs, either $\beta = S$ or $\beta \subseteq S$:

(i) Suppose that $\beta = S$ (remember that β is a finite set by construction). Then we have S is a linearly independent set and spanning set for V (since S is a finite set that generates V). Hence, S is a finite basis for V.

(ii) Suppose $\beta \subseteq S$ is a linearly independent set such that adding $v \in S$ where $v \notin \beta$ makes a linearly dependent set. We claim that β is the desired subset of S that is a basis for V. Then we have two cases; that is, either $v \in \beta$ or $v \notin \beta$. Since $\beta \subseteq S$, we know by Theorem 1.5 that $\operatorname{span}(\beta) \subseteq S$ (This applies for both cases). It suffices to show that $S \subseteq \operatorname{span}(\beta)$. If $v \in \beta$, then surely $v \in \operatorname{span}(\beta)$. Hence, we have $S \subseteq \operatorname{span}(\beta)$. Suppose $v \notin \beta$. Since β is a linearly independent set, then by Theorem 1.7 , we have that $\beta \cup \{v\}$ being linearly dependent implies that $v \in \operatorname{span}(\beta)$. Hence, $S \subseteq \operatorname{span}(\beta)$. Thus, that both cases implies that β is a spanning set for V.

■

This theorem tells us that any spanning set of a vector space V can reduced to a finite basis for V. This is illustrated in the following examples.

Example 12. Define

$$S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$$

which can be shown to generate $\mathbb{R}^3$. The idea is to create a proper subset of S such that none of the vectors in β are a multiples of each other. In other words, we want a set that is linearly independent; that is, our choices of vectors determine whether a set will be linearly dependent or independent (we want the latter to hold). Say, we pick $(2, -3, 5)$ as our first vector in our subset β. Right away, we can exclude $(8, -12, 20)$ since it is a multiple of $(2, -3, 5)$. Otherwise, including it would make β linearly dependent (see exercise 9 from section 1.5). Next, add the vectors $(1, 0, -2)$ and $(0, 2, -1)$ since they are not multiples of the other. Narrowing down our set to

$$\beta = \{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}.$$

Note that adding the vector $(7, 2, 0)$ makes β linearly dependent, so $(7, 2, 0)$ is excluded from the list. Hence, we have arrived at a subset of S such that $\beta \subseteq S$ is both a linearly independent set and spanning set for $\mathbb{R}^3$.

The following theorem and its corollaries are the most important results in the Chapter 1.

Theorem 1.3.3 (Replacement Theorem)**.** Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Proof. ■