Math 234A Lecture Notes

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Contents

Chapter 1

Week1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial $x^2 + 1$ does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\ dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system $\mathbb{R}$.

(i) $(\mathbb{R}, +, \cdot)$ is a field.

(ii) We have an order relation on $\mathbb{R}$.

(iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in $\mathbb{R}$.

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$
Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that $\mathbb{C}$ forms a field.

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane $\mathbb{C}$ as a working model for complex numbers.
- Discuss the Euclidean Topology on $\mathbb{C}$.
- Discuss polar representation of a complex number.

1.3 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

Lemma. Let F be a field containing $\mathbb{R}$ and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then $\mathbb{C}$ is a **subfield** of F.

(ii) Let F be another field containing $\mathbb{R}$ and containing a solution i' of $x^2 + 1 = 0$ and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then $\mathbb{C}$ and $\mathbb{C}'$ are isomorphic as fields.

The second part of this lemma is simply saying that $\mathbb{C}$ is a unique subfield of F.

1.3.1 A model for $\mathbb{C}$

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication $\cdot$ as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers $\mathbb{C}$ have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \to \mathbb{C}$ by $x \to (x, 0)$. Then $\mathcal{J}$ is a subfield and forms an isomorphism between $\mathbb{R}$ and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

(i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.

(ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homorphism.

(iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that $\mathcal{J}$ is a bijective map.

■

Definition (Constructing a solution for $x^2 + 1 = 0$). Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number). Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate). Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\overline{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the $x-$axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(iii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iv) $\Re(z) = \frac{1}{2}(z + \overline{z})$.

(v) $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $z\overline{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number). Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

Proposition (More Properties of Complex Numbers). Given $z, w \in \mathbb{C}$, we have the following properties:

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|zw| = |z||w|$.

(iii) $|\Re(z)| \leq |z|$.

(iv) $|\Im(z)| \leq |z|$.

(v) $|z + w| \leq |z| + |w|$.

(vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

1.3.2 Viewing $\mathbb{C}$ as a Metric Space

In $\mathbb{C}$, the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space). For $z, w \in \mathbb{C}$, we call $\mathbb{C}$ a metric space if it satisfies the following properties:

(i) $d(z, w) = d(w, z)$.

(ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have $\mathbb{C}$ is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of complex numbers.
- Convergence of sequences in $\mathbb{C}$.

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^{\cdot} = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_{+} = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos\varphi, \sin\varphi)$$

with $\tan\varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos\psi, \sin\psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_{+} \times \mathbb{R} \to \mathbb{C}^{\cdot}$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^{\cdot}$, then

$$z = \gamma(\cos\varphi + i\sin\varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument)**.** Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\mathrm{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\ &= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')). \end{aligned}$$

∎

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^{\cdot}$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi). \end{aligned}$$

Corollary (De Moivre's Theorem)**.** Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example 2.1.1 (An example of De Moivre's Theorem)**.** Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i\sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned}
|\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\
&\leq \alpha|z_n - z| + \beta|w_n - w| \\
&< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\
&\leq |w_n||z_n - z| + |z||w_n - w| \\
&\leq M|z_n - z| + |z||w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we conclude that

$$z_n w_n \to zw.$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned}
|z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\
&< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\
&= \varepsilon.
\end{aligned}$$

Thus, we conclude that $z_n^{-1} \to z^{-1}$.

■

Chapter 3

Week3

3.1 Lecture 4

3.1.1 Topics

(i) Continue discussion of convergence of sequence/series.

(ii) Discuss exponential, sine, and cosine function.

3.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

Definition (Convergence of Series)**.** If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example 3.1.1. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i}\Big[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\Big]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$

$$\frac{1}{2}is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider

$$\frac{1}{2}i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow s_n = \frac{1}{i}\left[1 - \frac{1}{2^n}\right].$$

Thus, taking the limit as $n \to \infty$ gives us

$$\lim_{n\to\infty} s_n = \frac{1}{i}.$$

Example 3.1.2. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since $|z| < 1$, we know that

$$\sum_{n-1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n\to\infty} \sum_{k=1}^{n} z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as $n \to \infty$, we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series)**.** Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example 3.1.3. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in $\mathbb{C}$.

3.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function)**.** For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series)**.** We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z+w) = \exp(z)\cdot\exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \textbf{ Show this!}.$$

3.2 Lecture 5

3.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

3.2.2 Sine and Cosine Functions

Recall that we defined

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C}$$
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}.$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2$.

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
&= \cos z + i \sin z.
\end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x+iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.

- $\Im(\exp(z)) = e^x \sin y, \ |\exp(z)| = e^x$.
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and $w = u + iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x + iy$. Let $z = \gamma(\cos\varphi + i \sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ∎

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\cdot}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\cdot} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Proposition. Let $z \in \mathbb{C}^{\cdot}$. Find all possible solutions of $\exp(w) = z$.

Lemma. For $z \in \mathbb{C}^{\cdot}$, we have

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

3.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \mathrm{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \operatorname{Log} z + 2\pi i k$$

is a **multivalued function** and $\log z = \operatorname{Log} z$ if we want the solution to be in S.

Chapter 4

Week4