0.1 Lecture 4

0.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

0.1.2 Class Exercises

(i) Show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z.

(ii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ converges for all z.

(iii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all z.

Definition (Exponential, Cosine, and Sine)**.** We define

$$\begin{aligned}\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

Our main goal for this lecture is to show that $\exp(z+w) = \exp(z)\exp(w)$.

0.1.3 Cauchy Multiplication Theorem

Theorem (Cauhcy Multiplication Theorem)**.** Assume that $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converges absolutely. Then

$$\sum_{n=0}^{\infty} = \Big(\sum_{n=0}^{\infty} z_n\Big)\Big(\sum_{n=0}^{\infty} w_n\Big) \tag{1}$$

where the series in the left converges absolutely.

Proof. We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is bounded. Observe that

$$\begin{aligned}\sum_{n=0}^{N} \Big|\sum_{i+j=n} z_i w_j\Big| &\leq \sum_{n=0}^{N} \sum_{i+j=n} |z_i w_j| \\ &= \sum_{0 \leq i+j \leq N} |z_i w_j| \\ &\leq \sum_{0 \leq i,j \leq N} |z_i w_j| \\ &= \sum_{i=0}^{N} |z_i| \sum_{j=0}^{N} |w_j|.\end{aligned}$$

By our assumption, we can see that the sequence of partial sums of $\sum_{n=0}^{\infty}$ and $\sum_{n=0}^{\infty} w_n$ are bounded, and thus the left side of the equation above is bounded. Thus, we see that

$$\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \Big|\sum_{n=0}^{2N}\sum_{i+j=n} z_i w_j - \sum_{i=0}^{N} z_i \sum_{j=0} w_j\Big| \to 0$$

as $N \to \infty$. Let us define the following sets

$$\begin{aligned} T_N &= \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : i \geq 0, j \geq 0, 0 \leq i+j \leq N\} \\ t_n &= \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : 0 \leq i \leq N, 0 \leq j \leq N\}. \end{aligned}$$

Observe that $T_N \subseteq t_n$ and $t_N \subseteq T_{2N} \subseteq t_{2N}$. Thus,

$$\begin{aligned} \alpha_N = \Big|\sum_{(i,j)\in T_{2N}\setminus t_N} z_i w_j\Big| &\leq \sum_{(i,j)\in T_{2N}\setminus t_N} |z_i w_j| \\ &\leq \sum_{(i,j)\in t_{2N}\setminus t_N} |z_i w_j| \\ &= \sum_{i=0}^{2N}\sum_{j=0}^{2N}|z_i w_j| - \sum_{i=0}^{N}\sum_{j=0}^{N}|z_i w_j| \\ &= \sum_{i=0}^{2N}|z_i|\sum_{j=0}^{2N}|w_j| - \sum_{i=0}^{N}|z_i|\sum_{j=0}^{N}|w_j| \to 0 \text{ as } N \to \infty. \end{aligned}$$

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Corollary. For any $z, w \in \mathbb{C}$, we have $\exp(z)\cdot\exp(w) = \exp(z+w)$.

Proof. Let

$$C_n = \sum_{k=0}^{n}\frac{z^k}{k!}\cdot\frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by $n!$ on both sides of the equation above, we see that

$$n!C_n = \sum_{k=0}^{n}\binom{n}{k} z^k w^{n-k} = (z+w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^{N} C_n = \sum_{n=0}^{N}\frac{n!C_n}{n!} = \sum_{n=0}^{N}\frac{(z+w)^n}{n!}.$$

Let $n \to \infty$. Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} = \exp(z+w).$$

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0.2 Lecture 5