Math 234A Lecture Notes

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Contents

Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial $x^2 + 1$ does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\ dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system $\mathbb{R}$.

(i) $(\mathbb{R}, +, \cdot)$ is a field.

(ii) We have an order relation on $\mathbb{R}$.

(iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in $\mathbb{R}$.

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2+1 = 0$. Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \operatorname{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$)**.** Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that $\mathbb{C}$ forms a field.

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. To do. ∎

Lemma (Existence of a Square Root)**.** Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ∎

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane $\mathbb{C}$ as a working model for complex numbers.
- Discuss the Euclidean Topology on $\mathbb{C}$.
- Discuss polar representation of a complex number.

1.3 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

Lemma. Let F be a field containing $\mathbb{R}$ and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let
$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ is a **subfield** of F.

(ii) Let F be another field containing $\mathbb{R}$ and containing a solution i' of $x^2 + 1 = 0$ and
$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ and $\mathbb{C}'$ are isomorphic as fields.

The second part of this lemma is simply saying that $\mathbb{C}$ is a unique subfield of F.

1.3.1 A model for $\mathbb{C}$

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication $\cdot$ as
$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$
and
$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers $\mathbb{C}$ have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \to \mathbb{C}$ by $x \to (x, 0)$. Then $\mathcal{J}$ is a subfield and forms an isomorphism between $\mathbb{R}$ and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

(i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.

(ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homorphism.

(iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that $\mathcal{J}$ is a bijective map.

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Definition (Constructing a solution for $x^2 + 1 = 0$)**.** Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

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Definition (Real and Imaginary part of Complex Number)**.** Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate)**.** Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\overline{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the $x-$axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(iii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iv) $\Re(z) = \frac{1}{2}(z + \overline{z})$.

(v) $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $z\overline{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

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Definition (Modulus of a Complex Number)**.** Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

Proposition (More Properties of Complex Numbers)**.** Given $z, w \in \mathbb{C}$, we have the following properties:

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|zw| = |z||w|$.

(iii) $|\Re(z)| \leq |z|$.

(iv) $|\Im(z)| \leq |z|$.

(v) $|z + w| \leq |z| + |w|$.

(vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

1.3.2 Viewing $\mathbb{C}$ as a Metric Space

In $\mathbb{C}$, the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space)**.** For $z, w \in \mathbb{C}$, we call $\mathbb{C}$ a metric space if it satisfies the following properties:

(i) $d(z, w) = d(w, z)$.

(ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have $\mathbb{C}$ is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of Complex Numbers
- Convergence of Sequences in $\mathbb{C}$

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^\bullet = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Definition (Polar Representation of Complex Numbers)**.** Let $(\alpha, \beta) \in \mathbb{R}^2$. The polar representation of (α, β) is

$$(\alpha, \beta) = \gamma(\cos\varphi, \sin\varphi)$$

with $\tan\varphi = \frac{\beta}{\alpha}$. Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos\psi, \sin\psi)$$

where γ is uniquely defined and φ is defined up to the addition of a multiple of 2π.

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^\cdot$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^\cdot$, then

$$z = \gamma(\cos\varphi + i\sin\varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument)**.** Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\mathrm{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\ &= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')). \end{aligned}$$

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The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^{\cdot}$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi). \end{aligned}$$

Corollary (De Moivre's Theorem)**.** Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example (An example of De Moivre's Theorem)**.** Suppose we have the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i \sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i \sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we can conclude that
$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that
$$|z_n w_n - zw| < \varepsilon.$$
Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have
$$|z_n - z| < \frac{\varepsilon}{2M}$$
where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have
$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$
Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have
$$\begin{aligned} |z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\ &\leq |w_n||z_n - z| + |z||w_n - w| \\ &\leq M|z_n - z| + |z||w_n - w| \\ &< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
Thus, we conclude that
$$z_n w_n \to zw.$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have
$$|z_n^{-1} - z^{-1}| < \varepsilon.$$
Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have
$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$
Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have
$$|z_n| > \frac{|z|}{2}.$$
If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have
$$\begin{aligned} |z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\ &< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\ &= \varepsilon. \end{aligned}$$
Thus, we conclude that $z_n^{-1} \to z^{-1}$.

(iv) ($\Longrightarrow$) Let $\varepsilon > 0$. Since $(z_n) \to z$, we can choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have
$$\begin{aligned} |\Re(z_n) - \Re(z)| = |\Re(z_n - z)| \leq |z_n - z| < \varepsilon \\ |\Im(z_n) - \Im(z)| = |\Im(z_n - z)| \leq |z_n - z| < \varepsilon. \end{aligned}$$
Hence, the real and imaginary part of (z_n) converge.

($\Longleftarrow$) Letting $\varepsilon > 1$ again. Our goal is to find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n - z| < \varepsilon.$$

Since the real and imaginary part of z_n converge, we know that there exists $N_1, N_2 \in \mathbb{N}$ such that, we have

$$|\Re(z_n) - \Re(z)| < \frac{\varepsilon}{2} \tag{1}$$

$$|\Im(z_n) - \Im(z)| < \frac{\varepsilon}{2} \tag{2}$$

whenever $n \geq N_1$ and $n \geq N_2$, respectively. Now, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|z_n - z| &= |(\Re(z_n) - \Re(z)) + i(\Im(z_n) - \Im(z))| \\
&\leq |\Re(z_n) - \Re(z)| + |\Im(z_n) - \Im(z)| && (|i| = 1) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, we see that $(z_n) \to z$.

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Chapter 3

Week 3

3.1 Lecture 4

3.1.1 Topics

(i) Convergence of Series

(ii) Exponential, Sine, and Cosine functions

3.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

Definition (Convergence of Series)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Define

$$s_k = \sum_{k=1}^{n} z_k = z_1 + \cdots + z_n$$

where (s_n) is called the **sequence of partial sums** of $\{z_n\}$. If $s_n \to s$, then we say that the series $\sum_{n=1}^{\infty} z_n$ converges and write $\sum_{n=1}^{\infty} z_n = s$.

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i}\Big[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\Big]. \end{aligned}$$

$$i s_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$

$$\frac{1}{2} i s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider
$$\frac{1}{2}i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$
Then subtracting (1) and (2), we see that
$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Longrightarrow s_n = \frac{1}{i}\left[1 - \frac{1}{2^n}\right].$$
Thus, taking the limit as $n \to \infty$ gives us
$$\lim_{n\to\infty} s_n = \frac{1}{i}.$$

Example. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute
$$\sum_{n=1}^{\infty} z^n.$$
Since $|z| < 1$, we know that
$$\sum_{n-1}^{\infty} z^n$$
is a geometric series and that it converges. Using the geometric series sum formula, we see that
$$\sum_{n=1}^{\infty} z^n = \lim_{n\to\infty} \sum_{k=1}^{n} z^k = \frac{z}{1-z}.$$
Consider this alternative way. Suppose we have
$$s_n = z + z^2 + z^3 + \cdots + z^n.$$
and
$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$
Then subtracting these two equations, we see that
$$(1-z)s_n = z - z^{n+1}.$$
Then we have
$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$
Then taking the limit as $n \to \infty$, we must have
$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series)**.** Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers
$$\sum_{n=1}^{\infty} |z_n|$$
converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in $\mathbb{C}$.

3.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function)**.** For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series)**.** We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define
$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z+w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \textbf{ Show this!}.$$

Chapter 4

Week 4

4.1 Lecture 4

4.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

4.1.2 Class Exercises

(i) Show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z.

(ii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ converges for all z.

(iii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all z.

Definition (Exponential, Cosine, and Sine). We define

$$\begin{aligned} \exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \end{aligned}$$

Our main goal for this lecture is to show that $\exp(z+w) = \exp(z)\exp(w)$.

4.1.3 Cauchy Multiplication Theorem

Theorem (Cauchy Multiplication Theorem). Assume that $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converges absolutely. Then

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} z_k w_{n-k} \Big) = \Big(\sum_{n=0}^{\infty} z_n \Big) \Big(\sum_{n=0}^{\infty} w_n \Big) \tag{1}$$

where the series on the left-hand side of (1) converges absolutely.

Proof. We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is bounded. Observe that

$$\begin{aligned}\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big| &\le \sum_{n=0}^{N}\sum_{i+j=n}|z_i w_j|\\ &= \sum_{0\le i+j\le N}|z_i w_j|\\ &\le \sum_{0\le i,j\le N}|z_i w_j|\\ &= \sum_{i=0}^{N}|z_i|\sum_{j=0}^{N}|w_j|.\end{aligned}$$

By our assumption, we can see that the sequence of partial sums of $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ are bounded, and thus the left side of the inequality above is bounded. Thus, we see that

$$\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \Big|\sum_{n=0}^{2N}\sum_{i+j=n} z_i w_j - \sum_{i=0}^{N} z_i \sum_{j=0}^{N} w_j\Big| \to 0$$

as $N \to \infty$. Let us define the following sets

$$\begin{aligned}T_N &= \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : i\ge 0, j\ge 0, 0\le i+j\le N\}\\ t_n &= \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : 0\le i\le N, 0\le j\le N\}.\end{aligned}$$

Observe that $T_N \subseteq t_n$ and $t_N \subseteq T_{2N} \subseteq t_{2N}$. Thus,

$$\begin{aligned}\alpha_N = \Big|\sum_{(i,j)\in T_{2N}\setminus t_N} z_i w_j\Big| &\le \sum_{(i,j)\in T_{2N}\setminus t_N}|z_i w_j|\\ &\le \sum_{(i,j)\in t_{2N}\setminus t_N}|z_i w_j|\\ &= \sum_{i=0}^{2N}\sum_{j=0}^{2N}|z_i w_j| - \sum_{i=0}^{N}\sum_{j=0}^{N}|z_i w_j|\\ &= \sum_{i=0}^{2N}|z_i|\sum_{j=0}^{2N}|w_j| - \sum_{i=0}^{N}|z_i|\sum_{j=0}^{N}|w_j| \to 0 \text{ as } N\to\infty.\end{aligned}$$

■

Corollary. For any $z, w \in \mathbb{C}$, we have $\exp(z)\cdot\exp(w) = \exp(z+w)$.

Proof. Let

$$C_n = \sum_{k=0}^{n}\frac{z^k}{k!}\cdot\frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by $n!$ on both sides of the equation above, we see that

$$n!C_n = \sum_{k=0}^{n}\binom{n}{k} z^k w^{n-k} = (z+w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^{N} C_n = \sum_{n=0}^{N} \frac{n! C_n}{n!} = \sum_{n=0}^{N} \frac{(z+w)^n}{n!}.$$

Let $n \to \infty$. Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w).$$

■

4.2 Lecture 5

4.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

4.2.2 Sine and Cosine Functions

Recall that we defined

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C}$$
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}.$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2.$

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
&= \cos z + i \sin z.
\end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x+iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.
- $\Im(\exp(z)) = e^x \sin y, \ |\exp(z)| = e^x$.
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and $w = u + iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x + iy$. Let $z = \gamma(\cos\varphi + i \sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ■

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\bullet}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\bullet} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Problem 1. Let $z \in \mathbb{C}^{\bullet}$. Find all possible solutions of $\exp(w) = z$.

Solution. Taking the log of both sides of $\exp(w) = z$, we see that

$$w = \log(z) = \ln|z| + i\Big(\operatorname{Arg}(z) + 2\pi k\Big), \ k \in \mathbb{Z}.$$

■

Lemma. For $z \in \mathbb{C}^{\bullet}$, we have

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

4.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \operatorname{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \operatorname{Log} z + 2\pi ik$$

is a **multivalued function** and $\log z = \operatorname{Log} z$ if we want the solution to be in S.

Chapter 5

Week 5

5.1 Lecture 6

5.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on $\mathbb{C}$:

$$d(z, w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ using the map $\mathbb{C} \to \mathbb{R}^2$ with $\alpha + i\beta \to (\alpha, \beta)$. Thus, we can visualize $\mathbb{C}$ using $\mathbb{R}^2$.

Recall that definition of continuity for functions in $\mathbb{R}$.

Definition (continuity in $\mathbb{R}$)**.** Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **continuous** for each $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say $|x - a| < \delta$, we can also say that $x \in (a - \delta, a + \delta)$. Similarly, we have $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

5.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x, a) < \delta\}$$

which is a more general statement than the one found in the definition above.

Definition (Open Ball)**.** Consider the usual metric $d(z, w) = |z - w|$ in $\mathbb{C}$. Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a, and radius $\varepsilon > 0$, we mean the set

$$B(a, \varepsilon) = \{z \in \mathbb{C} : d(z, a) < \varepsilon\}.$$

Definition (Open Set)**.** Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Example. (i) $\mathbb{C}$ is open.

(ii) Let $\varepsilon > 0$ and $a \in \mathbb{C}$. Then $B(a, \varepsilon)$ is also open; that is, open balls are open.

(iii) Let D_1 and D_2 be open sets. Show that $D_1 \cap D_2$ is open. Let $x \in D_1 \cap D_2$ To show that $D_1 \cap D_2$ is open, we need to find $\delta > 0$ such that $B(x, \delta) \subseteq D_1 \cap D_2$. Since $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$. Since D_1 and D_2 are open sets, we can find $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$B(x, \varepsilon) \subseteq D_1 \text{ and } B(x, \varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that $d(x, p) < \varepsilon$ for all $p \in D_1$ and $d(x, q) < \varepsilon'$. Then pick $\delta = \min\{d(x, p), d(x, q)\}$. Since x is in both D_1 and D_2, we have that

$$B(x, \delta) \subseteq D_1 \cap D_2.$$

Hence, $D_1 \cap D_2$ is open.

1. The empty set is open.
2. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a collection of open ets, where Λ is an indexing set. Define the set

$$\bigcup_{\lambda \in \Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that $\bigcup_{\lambda \in \Lambda} D_\lambda$ is also open.

5.1.3 Interior Points

Definition (Interior Point)**.** Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Proposition. D is open if and only if each $a \in D$ is an interior point of D.

Example. Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton $\{0\}$. Show that $\{0\}$ is not open. Note that the intersection of arbitrary many open sets may not be open.

5.1.4 Closed sets

Definition (Closed Set)**.** We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

Example. Define a closed ball with center at $a \in \mathbb{C}$ and radius $\varepsilon > 0$ by

$$\overline{B}(a, \varepsilon) = \{z \in \mathbb{C} : |z - a| \leq \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that $\overline{B}(a, \varepsilon)$ is closed.

Example.
- For any $a \in \mathbb{C}$, we say that the singleton $\{a\}$ is closed.
- If $A_1, A_2, \ldots, A_n$ are closed sets. Then

$$\bigcup_{i=1}^{n} A_i \text{ is also closed.}$$

  We can just show that the complement of this set is open.
- Let $\{A_\Lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Define

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\Lambda \ \forall \lambda \in \Lambda\},$$

  then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.

Definition (Boundary Points)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a, \varepsilon) \cap D \neq \emptyset$ and $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a, \varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.

Example. Any point on the circle $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$ is a boundary point of $B(a, \varepsilon)$, where $\varepsilon > 0$.

Example (Boundary Points of Singletons)**.**
- Note that the boundary point of the singleton is just the singleton itself.
- Let $D = \mathbb{C} \setminus \{a\}$. Just like the first item, the boundary points of this set is just $\{a\}$.

Definition (Accumulation Point)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,

$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

Example.
(i) Let $D = \mathbb{C}$. Then $\{0\}$ is an accumulation point of D.
(ii) Let a such that $|a| = 1$. Then a is an accumulation point of $D = \{z : |z| < 1\}$.
(iii) Let $D = \{1, i\}$. Then D has no accumulation points.
(iv) If $D = \{w\}$ for some nonzero $w \in \mathbb{C}$. Thus, $D' = \emptyset$. That is, every singleton set contains no accumulation points. Furthermore, any finite set does not have any accumulation points.
(v) Let $D = \mathbb{R}$. Then the accumulation points of D is just $\mathbb{R}$ itself. That is, any point of $\mathbb{R}$ is an accumulation point.
(vi) Let $D = B(0, 1)$. The accumulation points are contained in the following set:

$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Another way to define the set of accumulation points of D is the following:

$$\{a \in \mathbb{C} : \forall \varepsilon > 0, B(a, \varepsilon) \cap (D \setminus \{a\})\}.$$

Definition (Closure)**.** Let $D \subseteq \mathbb{C}$. We denote the **closure of** D by $\mathbf{D}$ and define $\overline{D}$ as

$$\overline{D} = D \cup \{x \in D : \forall \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

The closure is the smallest closed set that contains D!

Lemma. (i) $\overline{D}$ is closed in $\mathbb{C}$.

(ii) The following statements are equivalent:

(a) $a \in \mathbb{C}$ is an accumulation point of D

(b) There is a sequence (z_n) with $z_n \in D$, $z_n \neq a$ for all $n \in \mathbb{Z}_+$ and $z_n \to a$.

Proof. Proof is in the homework! ■

Definition (Bounded)**.** We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$; that is, $D \subseteq \overline{B}(0, R)$.

Chapter 6

Week 6

6.1 Lecture 7

6.1.1 Topics

Our goal for this lecture is to discuss complex powers of a non-zero complex number.

6.1.2 Complex Powers

Let $z \in \mathbb{C}^{\bullet}$. We want to define z^w for $w \in \mathbb{C}$. Recall from real number system that, for $x \in \mathbb{R}_+$, $x^a = e^{a \ln x}$. We want our complex powers to extend this idea.

Definition. We define $z^w = \exp(w \log z)$. Note that

$$\begin{aligned} \log z &= \operatorname{Log} z + 2\pi i k \\ &= \ln |z| + i(\operatorname{Arg}(z) + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$

is a multi-valued function.

For this reason, the complex power of a non-zero complex number will also be multi-valued. Hence, by choosing a fixed k, we can find a specific value of z^w.

Example. (i) Compute i^i. Recall that $\log i = i\Big(\frac{\pi}{2} + 2\pi k\Big)$. Hence, we have

$$i^i = \exp(i \log i) = \{e^{\frac{-\pi}{2} + 2\pi k} : k \in \mathbb{Z}\}.$$

If we are just referring to the principal branch, then

$$i^i = e^{\frac{-\pi}{2}}.$$

(ii) Compute: $(-1)^{1+i}$. Note that

$$\log(-1) = i\pi + 2\pi i k = i(2k+1)\pi.$$

Then we see that for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} (-1)^{1+i} &= \exp((2k+1)\pi(i-1)) \\ &= e^{-(2k+1)\pi} \exp(e^{(2k+1)\pi i}) \\ &= e^{-(2k+1)\pi} \Big[\cos(2k+1)\pi + i \sin(2k+1)\pi\Big] \\ &= -e^{-(2k+1)\pi}. \end{aligned}$$

Let us summarize some facts about $\exp(z)$.

- For all $x \in \mathbb{R}$, we have $\exp(z) = e^x$. Thus, from now on, we write e^z in place of $\exp(z)$.
- For $z \in \mathbb{C}^\bullet$, we have $\log z$ is a multi-valued function. That is, we have

$$\log z = \ln|z| + i(\operatorname{Arg} z + 2\pi k), \ \ k \in \mathbb{Z}.$$

- For $z \in \mathbb{C}^\bullet$, we have $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ (this is the principal branch log).
- For $z \in \mathbb{C}^\bullet$, we have $z^w = e^{w \log z}$ is also a multi-valued function.

Example.
- Find all solutions of $\sin z = 1$.
- Find all possible solutions of $\cos z = i$.

Problem 2. Find all solutions of $e^z = i$.

Solution. ■

6.2 Lecture 8

- Limits of a function
- Continuity
- Derivative

6.2.1 Functional Limit

Definition (Functional Limit)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. Let a be an accumulation point of D. Let $\ell \in \mathbb{C}$. We say that ℓ is **the limit of** $f(z)$ as z approaches to a if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $z \in D$ with $0 < |z - a| < \delta$, we have

$$|f(z) - \ell| < \varepsilon;$$

that is, $z \in D \cap (B(a, \delta) \setminus \{a\})$ implies $f(z) \in B(\ell, \varepsilon)$.

Example. Let $f : B(0, 1) \to \mathbb{C}$ defined by $f(z) = \frac{iz}{2}$ then $\lim\limits_{z \to i} f(z) = -\frac{1}{2}$.

Proof. Observe that

$$|f(z) - \ell| = \left|\frac{i}{2}z - \left(-\frac{1}{2}\right)\right| = \left|\frac{i}{2}\right||z - i| = \frac{1}{2}|z - i|.$$

So, given any $\varepsilon > 0$, we can take any $0 < \delta < 2\varepsilon$ such that

$$0 < |z - i| < \delta$$

implies

$$|f(z) - \ell| < \varepsilon.$$

■

Some facts about limits are:

Theorem. Let $D \subseteq \mathbb{C}$ with $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ be two functions let us write $f = u + iv$.

(i) Let $a = \alpha + i\beta$. Then $\lim\limits_{z \to a} f(x) = \ell$ if and only if

$$\lim_{(x,y)\to(\alpha,\beta)} u(x,y) = \Re(\ell) \text{ and } \lim_{(x,y)\to(\alpha,\beta)} v(x,y) = \Im(\ell); \text{ that is,}$$

$$\lim_{z\to a} f(z) = \lim_{z\to a} u(z) + i \lim_{z\to a} v(z).$$

(ii) Let $\lim_{z\to a} f(z) = \ell$ and $\lim_{z\to a} g(z) = \ell'$ and $c, d \in \mathbb{C}$. Then

$$\lim_{z\to a}(cf(z) + dg(z)) = c\ell + d\ell'.$$

(iii) $\lim_{z\to a}(f(z)g(z)) = \lim_{z\to a} f(z) \cdot \lim_{z\to a} g(z)$.

(iv) We have

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$$

whenever $\lim_{z\to a} g(z) \neq 0$.

Problem 3. Let $\lim_{z\to i} f(z) = 2 + i$ and $\lim_{z\to i} g(z) = 1 - i$. Compute the following:

$$\lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2}\right].$$

Solution. Observe that

$$\begin{aligned}\lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2}\right] &= \lim_{z\to i}(f(z))^3 + \lim_{z\to i} \frac{(1+i)g(z)}{z^2} \\ &= (2+i)^3 + \frac{(1+i)(1-i)}{i^2} \\ &= (2+i)^3 - 2.\end{aligned}$$

■

Definition (Limits at Infinity). (i) Let $f : D \to \mathbb{C}$ be a function and $a \in \mathbb{C}$ be an accumulation point of D. Then we say $\lim_{z\to a} f(z) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that

$$z \in D \cap (B(a,\delta) \setminus \{a\})$$

implies $|f(z)| \geq M$; that is, f is unbounded as z approaches to a.

(ii) Let f be a complex function defined on the complement of a ball in $\mathbb{C}$. We say $\lim_{z\to\infty} f(z) = \ell$ if for all $\varepsilon > 0$, there exists $R > 0$ such that $|z| > R$ implies

$$|f(z) - \ell| < \varepsilon.$$

Example. (i) Consider $\lim_{z\to 0} \frac{1}{z} = \infty$. Let $M > 0$. Let $\delta = \frac{1}{M}$. Then $0 < |z| < \delta$ implies that

$$\left|\frac{1}{z}\right| > \frac{1}{\delta} = M.$$

(ii) Consider $\lim_{z\to\infty} \frac{1}{z} = 0$. Let $\varepsilon > 0$. Choose $R = \frac{1}{\varepsilon} > 0$. Then $|z| > R$ implies

$$\left|\frac{1}{z}\right| < \frac{1}{R} = \varepsilon.$$

Problem 4. (i) Show that $\lim_{z\to\infty} f(z) = \ell$ if and only if

$$\lim_{z\to\infty} f\left(\frac{1}{z}\right) = \ell.$$

(ii) $\lim_{z\to a} f(z) = \infty$ if and only if $\lim_{z\to a} \frac{1}{f(z)} = 0$.

(iii) Give a definition of $\lim_{z\to\infty} f(z) = \infty$. Show that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \infty$.

(iv) Compute $\lim_{z\to\infty} \frac{z-1}{z+i}$.

(v) Compute $\lim_{z\to\infty} \frac{gz+i}{z^2+z+1}$.

(vi) Can you compute $\lim_{z\to\infty} e^{-z}$?

6.2.2 Continuity

Definition (Continuity)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. We say f **is continuous at** $a \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(a,\delta) \cap D$, we have $f(z) \in B(f(a), \varepsilon)$.

Example. Compute $\lim_{z\to\infty} \frac{z-i}{z+i}$. Observe that

$$\lim_{w\to 0} \frac{\frac{1}{w} - i}{\frac{1}{w} + i} = \lim_{w\to 0} \frac{1 - wi}{1 + wi} = 1.$$

Lemma. Let $f : D \to \mathbb{C}$ be a function. Then the following statements are equivalent.

(i) If f is continuous at $a \in D$.

(ii) For any sequence (a_n) and $a_n \in D$ such that $a_n \to a$, we have

$$f(a_n) \to f(a).$$

We say that $f : D \to \mathbb{C}$ is continuous if f is continuous if f is continuous for every $a \in D$.

Some facts that we can prove as an exercise:

Lemma. Let $f : D \to \mathbb{C}$ with $D \subseteq \mathbb{C}$. Then we have

(1) f is continuous if and only if $\Re(f)$ and $\Im(f)$ are continuous (Note that $\Re(f)$ and $\Im(f)$ are functions of two variables in $\mathbb{R}$; that is, $\Re(f) : D \to \mathbb{R}$ an $\Im(f) : D \to \mathbb{R}$).

(2) Linear combinations and products of continuous functions are continuous.

(3) Composition of functions are continuous.

(4) If f is continuous on D and $f(z) \neq 0$ for all $z \in D$, then $\frac{1}{f}$ is continuous on D.

(5) If f is continuous, then $|f|$ is also continuous (note that $|f| = \sqrt{(\Re(f))^2 + (\Im(f))^2}$).

Proof. Left as an exercise. ■

Example. (1) $\exp : \mathbb{C} \to \mathbb{C}^\bullet$ is continuous.

(2) $\sin, \cos, \sinh, \cosh$ are all continuous.

(3) $\text{Arg} : \mathbb{C}^\bullet \to (-\pi, \pi]$ is continuous on $\mathbb{C}_- = \mathbb{C}^\bullet \setminus \{\text{negative real axis}\}$ **(this will be on the**

next homework).

(4) Log : $\mathbb{C}^{\bullet} \to \{x + iy : -\pi < y \leq \pi\}$ is continuous on $\mathbb{C}_{-}$ (**this will be on the next homework**)

(5) Arg and Log are both not continuous at any $z < 0$ (that is, points on negative axis).

(6) Fix $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Then

$$f(z) = a_0 + a_1 z + \cdots + a_n z_n$$

is continuous on $\mathbb{C}$; that is, any polynomial with complex coefficients will be continuous.

Note that Arg : $\mathbb{C}^{\bullet} \to (-\pi, \pi] \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a discontinuous.

Chapter 7

Week 7

7.1 Lecture 10

7.1.1 Topics

- Discuss few leftover limit topics from last lecture.
- Discuss differentiability of a function $f : D \to \mathbb{C}$.

Recall the following lemma:

Lemma. Let $f : D \to \mathbb{C}$ be a function where $D \subseteq \mathbb{C}$, and $\ell \in \mathbb{C}$. Then the following statements are equivalent:

(1) $\lim_{z \to a} f(z) = \ell$

(2) Define $\tilde{f} : D \cup \{a\} \to \mathbb{C}$ by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ \ell & \text{if } z = a. \end{cases}$$

Then $\tilde{f}$ is continuous at a.

7.1.2 Complex Differentiability

Definition (Complex Differentiability). Let $f : D \to \mathbb{C}$ be a function, and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. We say that f is **complex differentiable at** a if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists.}$$

Remark. Note that D may not always be open!

If f is complex differentiable at a, we write

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

We call f is complex differentiable on D if the limit above exists for every $a \in D$. Furthermore, we can define a function f' by mapping $z \in D$ to $f'(z) \in \mathbb{C}$. This f' is called the **complex derivative** of f.

In our definition of complex differentiability, we are allowed to choose

$$D = [c, d] \subseteq \mathbb{R}$$

which allows us to write f in terms of real functions $u(x)$ and $v(x)$; that is, we have

$$f(x) = u(x) + iv(x), \quad x \in [c, d].$$

Example. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z$. Let $a \in \mathbb{C}$. We will compute $f'(a)$. Clearly, we have

$$f'(a) = 1.$$

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \overline{z}$. If we fix $\Im(z) = 0$ and $z \to 0$ along the real axis, we have

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\overline{z}}{z} = 1.$$

If we fix $\Re(z) = 0$ and let $z \to 0$ along the imaginary axis, we have

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\overline{z}}{z} = -1.$$

Thus, we see that the function $f(z) = \overline{z}$ is not complex differentiable at 0.

Remark. The definition of complex differentiability depends on the domain D. In most textbooks, the domain D of a function in $\mathbb{C}$ is frequently stated to be an open set. Different properties can arise when we compare real and complex functions defined on open sets in terms of looking at their differentiability.

Lemma. Let $f : D \to \mathbb{C}$ and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. Suppose that f is complex differentiable at a. Define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}$$

Then g is continuous at $z = a$.

Proof. Left as an exercise. ■

The main take away from this lemma is that we now have the ability to rewrite our function in a different way; that is, we can write f in terms of

$$f(z) = f(a) + (z - a)g(z)$$

where g is a continuous at a.

Corollary. If the function f is complex differentiable at $a \in D$, then f is continuous at a.

Problem 5. Let $a \in D$ be a accumulation point of $D \setminus \{a\}$ and $\ell \in \mathbb{C}$. Then the following statements are equivalent:

(1) f is complex differentiable at a and $f'(a) = \ell$.

(2) Define $\gamma : D \to \mathbb{C}$ by

$$f(z) = f(a) + \ell(z - a) + \gamma(z);$$

that is,

$$r(z) = [f(z) - f(a)] + \ell(z - a),$$

then

$$\lim_{z\to a}\left|\frac{\gamma(z)}{z-a}\right| = 0.$$

In this case, $\ell = f'(a)$.

Theorem. Assume that f and g are complex differentiable at a.

(i) $f+g, \lambda f$ where $\lambda \in \mathbb{C}$ are also complex complex differentiable at a and

$$(f+g)' = f'(a) + g'(a)$$

and

$$(\lambda f)'(a) = \lambda f'(a).$$

(ii) The product fg is complex differentiable and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

(iii) Assume that $f(a) \neq 0$, then $\frac{1}{f}$ is also complex differentiable at a and

$$\left(\frac{1}{f}\right)'(a) = \frac{f'(a)}{(f(a))^2}.$$

Proof. Left as an exercise. ■

Theorem (Chain Rule). Let $f : D \to \mathbb{C}$ and $g : D' \to \mathbb{C}$ such that $f(D) \subseteq D'$. Let $a \in D$. Assume that f is complex differentiable at $a \in D$ and g is complex differentiable at $f(a)$. Then $g \circ f$ is complex differentiable at $a \in D$ and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. ■

Example. (i) Let $f : \mathbb{C} \to \mathbb{C}$ and n is a positive integer such that f is defined as $f(z) = z^n$. Then f is complex differentiable at any $z \in \mathbb{C}$. Show that $f'(z) = nz^{n-1}$.

(ii) Let $f : \mathbb{C}^\bullet \to \mathbb{C}$ be defined by

$$f(z) = \frac{1}{z}.$$

Then we have

$$f'(z) = -\frac{1}{z^2}.$$

(iii) Let $f : \mathbb{C}^\cdot \to \mathbb{C}$ defined by $f(z) = z^{-n}$ where n is a positive integer. Then f is complex differentiable on $\mathbb{C}^\cdot$ and

$$f'(z) = -zz^{-n-1}.$$

(iv) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Then f is complex differentiable on $\mathbb{C}$ and

$$f'(z) = \sum_{k=1}^{n} k a_k z^{k-1}.$$

(v) Let $f : \mathbb{C} \to \mathbb{C}$ and $f(z) = e^z$.

Regarding example (v) from above, we have that $f'(z) = f(z)$.