Math 230A Lecture Notes

Lance Remigio

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Contents

Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

1.1.2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

1.1.3 The First Defining Property

The set of real numbers is a field.

Definition (Fields)**.** A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

1.1.4 The Second Defining Property

Definition (Ordered Fields)**.** An **ordered field** is a field F equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

Remark. We say that x is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

Definition (Upper Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call β an **upper bound of** A.

Remark. We call the collection of upper bounds of A by $\text{UP}(A)$. If $\text{UP}(A) \neq \emptyset$, then we say that A **is bounded above**.

Similarly, we define the lower bounds of a set.

Definition (Lower Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then α is called the **lower bound of** A.

Remark. Similarly, we denote the set of lower bounds of A by $\text{LO}(A)$. We say that A is bounded below if $\text{LO}(A) \neq \emptyset$.

Example 1.1.1. Suppose we have $A = [0, 1)$. We have

$$\text{UP}(A) = [1, \infty)$$
$$\text{LO}(A) = (-\infty, 0].$$

1.2 Lecture 2

1.2.1 Review of Least Upper Bound Property

Definition (Supremum)**.** Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\beta \in F$ such that

(i) $\beta \in \mathrm{UP}(A)$

(ii) If $\gamma \in F$ and $\gamma < \beta$, then $\gamma \notin \mathrm{UP}(A)$.

We call β the **least upper bound** of A or the **supremum** of A. We denote the supremum of A as $\beta = \sup A$.

Remark. When we say THE supremum, we are implicitly stating that the supremum of A is unique.

Definition (Infimum)**.** Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\alpha \in F$ such that

(i) $\alpha \in \mathrm{LO}(A)$

(ii) If $\gamma \in F$ and $\gamma > \alpha$, then $\gamma \notin \mathrm{LO}(A)$.

We call α the **greatest upper bound** of A or the **infimum** of A, and write $\alpha = \inf A$.

Definition (Least Upper Bound Property)**.** An ordered field F is said to have the **least-upper-bound property** if the following is true:

Every nonempty set A in F that is bounded above has a least upper bound in F.

That is, if $A \neq \emptyset$ and $\mathrm{UP}(A) \neq \emptyset$, then $\sup(A)$ exists.

Theorem. There is exactly one ordered field that has the least-upper-bound bound property. The set $\mathbb{R}$ is the unique ordered field that contains $\mathbb{Q}$ as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$ is dedekind complete
- $\mathbb{R}$ satisfies the Axiom of Completeness.

Remark. Note that $\mathbb{Q}$ being an ordered field does not immediately imply that $\mathbb{Q}$ has the LUBP.

Definition (Maximums and Infimums)**.** Let $A \subseteq \mathbb{R}$.

- If $\sup A \in A$, then we call, $\sup A$, the **maximum of** A and we denote this by $\max A$.
- If $\inf A \in A$, we call, $\inf A$, the **minimum of** A and we denote this by $\min A$.

Lemma (Useful Fact for Supremum)**.** Let $A \subseteq \mathbb{R}$. Then $\beta = \sup A$ if and only if

(i) $\beta \in \mathrm{UP}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a > \beta - \varepsilon$.

Remark. We can restate property (ii) above as "for all $\varepsilon > 0$, $\beta - \varepsilon \notin \mathrm{UP}(A)$".

Lemma (Useful Fact for Infimums)**.** Let $A \subseteq \mathbb{R}$. Then $\alpha = \inf A$ if and only if

(i) $\alpha \in \mathrm{LO}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a < \alpha + \varepsilon$.

Remark. Similarly, we can restate property (ii) as "for all $\varepsilon > 0$, $\alpha + \varepsilon \notin \text{LO}(A)$".

Theorem (Greatest Lower Bound Property of $\mathbb{R}$)**.** Every nonempty subset A of $\mathbb{R}$ that is bounded below has a **greatest upper bound in** $\mathbb{R}$.

Another way to say this is the following:

$$\text{If } A \neq \emptyset \text{ and } \text{LO}(A) \neq \emptyset, \text{ then } \inf A \text{ exists in } \mathbb{R}.$$

1.2.2 Consequences of Least Upper Bound Property

Theorem (Archimedean Property)**.** If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.

Proof. Let $A = \{nx : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ since $1 \cdot x \in A$. Suppose for sake of contradiction that for all $n \in \mathbb{Z}^+$, $nx \leq y$. This means that y is an upper bound of A. Let $\beta = \sup A$. By the first useful fact, we have that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\beta - \varepsilon < nx$. Let $\varepsilon = x$. Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \Rightarrow \beta < x(n+1).$$

But this tells us that $x(n+1) \in A$ ($x \in A$ and $n + 1 \in \mathbb{N}$) and that β is NOT an upper bound which is a contradiction. Thus, it must be the case that $nx > y$ for some $n \in \mathbb{Z}^+$. ■

Remark. The well ordering property of $\mathbb{N}$ can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

Corollary. Let A be a nonempty subset of $\mathbb{R}$ that consists of only integers.

(i) If A is bounded above, then $\sup(A) \in A$.

(ii) If A is bounded below, then $\inf(A) \in A$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$ with $x < y$, there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Our goal is to find a $p \in \mathbb{Q}$ such that

$$x < p < y$$

with $p = \frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; that is, find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$nx < m < ny.$$

First, notice that $x < y$. This implies that $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \tag{1}$$

Choose $m \in \mathbb{Z}$ such that m to be the minimum element greater than nx; that is, choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m. \tag{2}$$

Let $A = \{k \in \mathbb{Z} : k > nx\}$ which is nonempty by the Archimedean Property. Furthermore, nx is a lower bound for A. By the Well-ordering property, A contains a minimum. Thus, $m = \min A$. Hence,we have

$$nx < m \Rightarrow x < \frac{m}{n}. \tag{3}$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \leq nx \Longrightarrow m \leq nx + 1 &< n\left(y - \frac{1}{n}\right) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \tag{4}$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \Longleftrightarrow x < p < y.$$

■

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

2.1.2 Review, Existence of Roots

Proposition. There is no rational number whose square is 2.

Theorem. There is a unique positive real number α satisfying $\alpha^2 = 2$.

Proof. (i) **Uniqueness:** Suppose there are two of them α_1 and α_2. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that A is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

■

Remark. A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number α such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

2.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

Definition (Usual Way of Defining Functions)**.** Let A and B be two sets. A **function** from A to B denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

Definition (The Correct Way of Defining Functions)**.** Let A and B be two sets. A function from A

to B is a triple (f, A, B) where f is a **relation** form A to B satisfying

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, A is called the **domain** of f and B is called the **codomain** of f)

Example 2.1.1. Let $A = \emptyset$ and B be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to B is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

Definition (Image, Range, Onto (Surjective)). Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of f as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of f as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say f is **Onto (Surjective)**.

Definition (Preimage). Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of D under f is denoted by

$$f^{-1}(D) = \{x \in A : f(x)\}$$

Definition (One-to-One (Injective)). Consider a function $f : A \to B$. We call f one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of A.

2.1.4 Equivalent Sets

Definition. Let A and B be two sets. We say that A and B have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

Remark.
- An injective and surjective mapping is a bijective mapping.
- A and B have the same cardinal number

  $= A$ and B have the same cardinality
  $= A$ and B can be put in the **one-to-one correspondence**
  $= \mathrm{card}\, A = \mathrm{card}\, B$
  $= A$ and B are equivalent
  $= A$ and B are equipotent

Example 2.1.2. Consider $\{1,2,3\} \sim \{a,b,c\}$. Indeed, the function $f : \{1,2,3\} \to \{a,b,c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

Example 2.1.3. $\mathbb{N} \sim \{2,4,6,\dots\}$. Indeed, the function $f : \mathbb{N} \to \{2,4,6,\dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

Example 2.1.4. $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

Example 2.1.5. $(-\infty,\infty) \sim (0,\infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty,\infty)$ and $(0,\infty)$.

Example 2.1.6. $(0,\infty) \sim (0,1)$. Indeed, the function $f : (0,\infty) \to (0,1)$ defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

Example 2.1.7. $[0,1) \sim (0,1)$. Indeed, the function $f : [0,1) \to (0,1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if} x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

Definition ($\sim$ is an equivalence relation)**.** Let A and B be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1,2,3,\dots,n\}$$
$$\mathbb{N} = \{1,2,3,\dots\}$$

2.1.5 Finite, Infinite, Countable, At most countable

Definition (Finite, Infinite, Countable, At most countable)**.** Let A be any set.

(a) We say that A is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number n.

(*) When $A \sim \mathbb{N}_n$, we say A has n elements and we write $\text{card}(A) = n$.

(*) Also, we set $\text{card}(\emptyset) = 0$.

(b) The set A is said to be **infinite** if it is not finite.

(c) The set A is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set A is said to be **uncountable** if it is neither countable or finite.

(e) The set A is said to be **at most countable** if it is either finite or countable.

Remark. Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$\begin{gathered} A \text{ is finite} \Leftrightarrow B \text{ is finite} \\ A \text{ is countable} \Leftrightarrow B \text{ is countable} \\ A \text{ is uncountable} \Leftrightarrow B \text{ is uncountable} \end{gathered}$$

(iii) The union of two finite sets is finite. If A is infinite and B is infinite, then $A \setminus B$ is infinite.

(iv) If A **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.

Chapter 3

Week 3

3.1 Lecture 4

Definition (Sequence). We call a **sequence**, we mean a function f on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence f by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in A.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

Theorem. Every infinite subset of a countable set is countable.

Proof. Let A be a countable set. Let $E \subseteq A$ and E is infinite. Our goal is to show that E is countable. Since A is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let n_1 be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let n_2

■

Remark.

Corollary.

Example 3.1.1 ($\mathbb{N} \times \mathbb{N}$ is countable).

Example 3.1.2 ($\mathbb{Q}$ is countable).

Theorem. Countable union of at most countable sets is at most countable.

Corollary.

Corollary.

Theorem. Finite product of countable sets is countable.

Example 3.1.3 ($\mathbb{Q}$ is countable).

3.2 Lecture 5

3.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks
- Inequalities

3.2.2 Summary of Last Lecture

1. How to prove A is at most countable:
   - Show that $A \subseteq S$ where S is countable.
   - Find an injective function $f : A \to S$ where S is countable.
2. How to prove A is infinite:
   - Prove that A has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subsetneq A$ such that $A \sim B$.
3. Find a proper subset $B \subsetneq A$ such that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

Theorem (D). Let A be the set of all sequences whose terms are the digits 0 and 1 (that is A is the collection of all binary sequences). This set A is uncountable.

Proof. First, notice that A is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

$$\text{for all } h(n) = \text{The binary sequence whose } n\text{th term is the digit 1 and all other terms are zero.}$$

Clearly, h is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, A is infinite

Suppose for sake of contradiction that A is NOT uncountable. Since A is infinite and not uncountable, this assumption tells us that A must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned} f(1) &= (a_1^1, a_2^1, \dots) \\ f(2) &= (a_1^2, a_2^2, \dots) \\ f(3) &= (a_1^3, a_2^3, \dots) \\ &\vdots \\ f(n) &= (a_1^n, a_2^n, \dots) \\ &\vdots \end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that A contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N} \; b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence (b_n) is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

3.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of a is defined to be $|a|$. But note that the **absolute value of** a is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers a and b in $\mathbb{R}$, the **standard distance** between a and b is

$$\text{dist}(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as h gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

Theorem (Basic Properties of Standard Size in $\mathbb{R}$)**.** Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Remark. Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

Definition. Let V be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on V if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space V equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

Theorem (Basic Properties of Standard Distance in $\mathbb{R}$)**.** For all $a, b \in \mathbb{R}$.

(i) $\operatorname{dist}(a, b) \geq 0$

(ii) $\operatorname{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\operatorname{dist}(a, b) = \operatorname{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\operatorname{dist}(a, c) \leq \operatorname{dist}(a, b) + \operatorname{dist}(b, c)$.

3.2.4 Metric Spaces

Definition (Metric Spaces)**.** Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X equipped with a metric d is called a **metric space**. (Sometimes we write (X, d) is a metric space)

Remark. X does **NOT** need to be a vector space. Furthermore, d is a more general notion of size than $\|\cdot\|$.

Example 3.2.1. $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)

3.2.5 Inequalities