Math 230A: Homework 1

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1. Mark each statement True or False.

   1-1) If x and y are elements of an ordered field, then either $x \leq y$ or $y < x$. **True.**

   1-2) Every ordered field has the least upper bound property. **False.**

   1-3) If $E \subseteq \mathbb{R}$ is bounded above and $\alpha = \sup E$, then $\alpha \in E$. **False.**

   1-4) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, then $\alpha = \sup E$. **False.**

   1-5) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \in E$, then $\alpha = \sup E$. **True.**

   1-6) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \notin E$, then $\alpha \neq \sup(E)$. **True.**

2. Prove the following: Suppose α is an upper bound for $E \subset \mathbb{R}$. Then $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   Proof. Suppose α is an upper bound for $E \subseteq \mathbb{R}$. For the forwards direction, suppose $\alpha = \sup E$ and let $\varepsilon > 0$. Note that $\alpha - \varepsilon < \alpha$ implies that $\alpha - \varepsilon$ is NOT an upper bound of E. By definition, there must exist an element $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$. Thus, $\alpha < x_0 + \varepsilon$.

   For the backwards direction, let $\varepsilon = 1$ and let α be an upper bound of E. We need to show $\alpha = \sup E$; that is, we need to show that α is an upper bound of E and that any $\gamma < \alpha$ implies γ is NOT an upper bound of E. Notice that the first property is satisfied by our assumption. Thus, all that is left to show is the second property.

   Suppose $\gamma < \alpha$. Our goal is to show that there exists some element $x \in E$ such that $x > \gamma$. We can use the fact that there exist $x_0 \in E$ such that $x_0 > \alpha - 1$ to say that

   $$\gamma < \alpha < x_0 + 1.$$

   Set $x = x_0 + 1$ and thus, we have $x > \gamma$ for some $x \in E$. Therefore, γ is NOT an upper bound of E and so we conclude that

   $$\alpha = \sup E.$$

   ■

3. Let $E = \{n/n + 1 : n \in \mathbb{N}\}$. Prove that $\sup E = 1$.

   Proof. To show that 1 is the supremum of E, we need to show that 1 is an upper bound for E and that it satisfies the lemma found in Problem 2. First, we will show that 1 is an upper bound of E. Observe that for any $n \in \mathbb{N}$, we have

   $$\frac{n}{n+1} < \frac{n}{n} = 1.$$

   Thus, E is bounded above by 1.

Let $\varepsilon > 0$. Since $\frac{1}{n+1} > 0$ for any $n \in \mathbb{N}$, there exists $k \in \mathbb{Z}^+$ such that

$$\frac{k}{n+1} > 1 > 1 - \varepsilon$$

by the Archimedean Property. Set $x_0 = k/n + 1$. By the lemma found in Problem 2, we must have $\sup(E) = 1$.

■

4. State the analogue of the Very Useful Theorem for infimums.

Solution. Suppose β is a lower bound for $E \subseteq \mathbb{R}$. Then $\beta = \inf E$ if and only if for all $\varepsilon > 0$, there exists $y_0 \in E$ such that $y_0 < \beta + \varepsilon$. ■

5. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Prove $a \leq b$.

Proof. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Suppose for sake of contradiction that $a > b$. Observe that

$$b < a \leq b + \varepsilon \Rightarrow b \leq b + \varepsilon.$$

Then subtracting b on both sides gives us $\varepsilon \geq 0$. But this is a contradiction because we had assumed that $\varepsilon > 0$. Thus, it must be the case that $a \leq b$. ■

6. Complete the following proof.

Theorem (Greatest-lower-bound property of $\mathbb{R}$). Every nonempty subset of A of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf A$ exists and is a real number.

Proof. Continuation of the proof presented in homework. Set $\beta = -\alpha$ where $\alpha = \sup(-A)$. Thus, there exists $a \in A$ such that $-\gamma < -a$. Multiplying by a negative on both sides of this inequality, we get $\gamma > a$ for some $a \in A$. If γ was a lower bound of A, then $\gamma \leq a$ for all $a \in A$. However, we have the negation of this, so we must have that γ is NOT a lower bound of A, which satisfies part (ii). Since $\beta \leq \alpha$ for all $a \in A$ and $\beta < \gamma$ implies γ is not a lower bound for A, we conclude that β must be the infimum of A. ■

7. (i) Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many elements of A.

Proof. Since A is nonempty and bounded above, we know that A has the least-upper-bound property. Set $\sup A = \alpha$. Suppose $\sup A \notin A$. Suppose for sake of contradiction that there exists an $\varepsilon > 0$ such that the open interval $(\alpha - \varepsilon, \alpha)$ contains a finitely many elements of A. Thus, for every $x_i \in A$ with $1 \leq i \leq n$, we see that $x_i \in (\alpha - \varepsilon, \alpha)$ for some $\varepsilon > 0$. Then we see that

$$\alpha - \varepsilon < x_i < \alpha \Rightarrow \alpha < x_i < \alpha + \varepsilon.$$

But note that for all $x_i \in A$, we have $x_i > \alpha$. This tells us that α is NOT an upper bound of A, but instead that $\sup A \in A$ which is a contradiction. Thus, it must be the case that $(\alpha - \varepsilon, \alpha)$ must contain infinitely many points. ■

(ii) Let $B \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below. Show that if $\inf B \notin B$, then for all $\varepsilon > 0$ the open interval $(\inf B, \inf B + \varepsilon)$ contains infinitely many elements of B.

Proof. Since B is nonempty and bounded below, we know that B must contain the greatest upper bound property. Set this greatest upper bound to be $\beta = \inf B$. Suppose for sake of contradiction that there exists an $\varepsilon > 0$ such that the open interval $(\beta, \beta + \varepsilon)$ contains finitely many elements of B. Thus, for every $x_i \in B$ with $1 \leq i \leq n$, we see that $x_i \in (\beta, \beta + \varepsilon)$. Then we see that

$$\beta < x_i < \beta + \varepsilon \Rightarrow \beta - \varepsilon < x_i < \beta.$$

But note from this inequality that $x_i < \beta$, implying that β is not a lower bound of B. Which means that $\beta \in B$ which is a contradiction of our assumption that $\beta \notin B$. Hence, B must contain infinitely many points. ■

8. Suppose the construction of $\mathbb{R}$ and the proof of the least-upper-bound property for $\mathbb{R}$ can be completed without directly using the well-ordering principle for $\mathbb{N}$. (Of course, statements equivalent to the well-ordering principle might have been used, but we are assuming that the nontrivial fact of their equivalence to the well-ordering principle was not utilized in the construction of $\mathbb{R}$ or in proving its Dedekind completeness.) Use the greatest-lower-bound property of $\mathbb{R}$ and the result of the previous exercises to prove the well-ordering principle for $\mathbb{N}$; that is, prove that every nonempty subset of $\mathbb{N}$ has a minimum.

Proof. ■

9. Let $A, B \subseteq \mathbb{R}$ be nonempty, bounded sets and let $c \in \mathbb{R}$. Define the following sets:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{ab : a \in A, b \in B\} \\ cA &= \{ca : a \in A\}. \end{aligned}$$

Prove that

9-1) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Since A and B are both nonempty and bounded below, we know that both $\inf(A)$ and $\inf(B)$ exists. Thus, $a \geq \inf(A)$ and $b \geq \inf(B)$ for all $a \in A$ and $b \in B$, respectively. Thus,

$$a + b \geq \inf(A) + \inf(B) \quad \text{for all } a + b \in A + B.$$

implies that $A + B$ is bounded below. Since $A + B \neq \emptyset$ (since A and B are both nonempty), we see that $\inf(A + B)$ exists.
Now, we will show that

$$\inf(A + B) = \inf(A) + \inf(B);$$

that is, we need to show that

$$\inf(A + B) \geq \inf(A) + \inf(B) \tag{1}$$

and

$$\inf(A + B) \leq \inf(A) + \inf(B). \tag{2}$$

To show (1), let $\varepsilon > 0$. Using the lemma found in Problem 2, there exists an $\alpha \in A$ and $\beta \in \beta$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{2}$$

and

$$\beta < \inf(B) + \frac{\varepsilon}{2},$$

respectively. Adding these two inequalities, we get

$$\alpha + \beta < \inf(A) + \inf(B) + \varepsilon.$$

Since $A + B$ is bounded below, we have $\inf(A + B) \leq \alpha + \beta$ such that

$$\inf(A + B) \leq \alpha + \beta < \inf(A) + \inf(B) + \varepsilon.$$

Using Problem 5, we conclude that

$$\inf(A + B) \leq \inf(A) + \inf(B).$$

To show (2), let $\varepsilon > 0$ again. Using the same lemma, there exiss $\varphi \in A + B$ with $\varphi = \gamma + \lambda$ where $\gamma \in A$ and $\lambda \in B$ such that

$$\gamma + \lambda = \varphi < \inf(A + B) + \varepsilon.$$

Since A and B are both bounded below, we can see that

$$\inf(A) + \inf(B) \leq \gamma + \lambda.$$

Thus, we have

$$\inf(A) + \inf(B) \leq \inf(A + B)$$

by Problem 5. ■

9-2) $\sup(A + B) = \sup(A) + \sup(B)$

Proof. Since A and B are both nonempty, we can see that $A+B$ is also nonempty. Furthermore, A and B are both bounded above, so $a \leq \sup(A)$ for all $a \in A$ and $b \leq \sup(B)$ for all $b \in B$ implies that

$$a + b \leq \sup(A) + \sup(B) \quad \text{for all } a \in A \text{ and } b \in B.$$

Thus, $A + B$ must be bounded above, and so by definition 1.10 in the textbook, we see that the $\sup(A + B)$ exists. Our goal is to show that

$$\sup(A + B) = \sup(A) + \sup(B);$$

that is, it suffices to show that both

$$\sup(A + B) \leq \sup(A) + \sup(B) \tag{1}$$

and

$$\sup(A + B) \geq \sup(A) + \sup(B). \tag{2}$$

To prove (1), let $\varepsilon > 0$. By the lemma found in Problem 2, there exists $\varphi \in A+B$ such that $\varphi > \sup(A+B) - \varepsilon$ with $\varphi = \alpha + \beta$ for some $\alpha \in A$ and $\beta \in B$. Thus, we have $\alpha \leq \sup(A)$ and $\beta \leq \sup(B)$, and so

$$\alpha + \beta \leq \sup(A) + \sup(B)$$

implies

$$\sup(A) + \sup(B) \geq \alpha + \beta > \sup(A + B) - \varepsilon.$$

Using Problem 5, we can see that

$$\sup(A) + \sup(B) \geq \sup(A + B)$$

which proves (1).
To prove (2), let $\varepsilon > 0$ again. Since $\sup(A)$ and $\sup(B)$ exists, we see that there exists $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{2}$$

and

$$\beta > \sup(B) - \frac{\varepsilon}{2}$$

by the lemma found in Problem 2. Adding these two inequalities results in

$$\alpha + \beta > \sup(A) + \sup(B) - \varepsilon.$$

Note that $\alpha + \beta \in A + B$ and $A + B$ is bounded above, we have that

$$\sup(A + B) \geq \alpha + \beta > \sup(A) + \sup(B) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can see that

$$\sup(A + B) \geq \sup(A) + \sup(B)$$

by Problem 5. Since (1) and (2) are satisfied, we can conclude that

$$\sup(A + B) = \sup(A) + \sup(B).$$

■

9-3) $\sup(-A) = -\inf(A)$

Proof. Since $A \neq \emptyset$ and A is bounded above, we can say that $-A$ is also nonempty and bounded above as well from Problem 6. Thus, we know that the supremum of $-A$ exists. Our goal is to show that both

$$\sup(-A) \leq -\inf(A) \tag{1}$$

and

$$\sup(-A) \geq -\inf(A). \tag{2}$$

Let $\varepsilon > 0$. To prove (1), we can use the lemma from Problem 2 to state that there exists an $-a \in -A$ such that

$$\sup(-A) < -a - \varepsilon.$$

Since A is bounded below, we know that $a \geq \inf(A)$. Multiplying through with a negative gives us

$$-a \leq -\inf(A)$$

which implies that

$$\sup(-A) < -a - \varepsilon \leq -\inf(A) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$-\inf(A) \geq \sup(-A)$$

which proves (1).

Now, to prove (2), we can use the lemma from problem 6, there exists a $\alpha \in A$ such that

$$\alpha < \inf(A) + \varepsilon.$$

Multiplying through by a negative on this inequality, we have

$$-\alpha > -\inf(A) - \varepsilon.$$

But A is bounded below, so we must have $-\alpha \leq \sup(-A)$. Thus, we have

$$\sup(-A) \geq -\alpha > -\inf(A) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\sup(-A) \geq -\inf(A)$$

by Problem 5, which proves (2).

■

9-4) $\inf(-A) = -\sup(A)$.

Proof. Our goal is to show that $\inf(-A) = -\sup(A)$; that is, we need to show that

$$\inf(-A) \leq -\sup(A) \tag{1}$$

and

$$\inf(-A) \geq -\sup(A). \tag{2}$$

First, we show (1). Let $\varepsilon > 0$. Using the lemma found in Problem 2, there exists $-\alpha - A$ such that

$$-\alpha < \inf(-A) + \varepsilon$$

. Since A is bounded above, $\alpha \leq \sup(A)$. Multiplying by a negative on this inequality produces $-\alpha \geq -\sup(A)$. Thus, we have

$$-\sup(A) \leq -\alpha < \inf(-A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$-\sup(A) \leq \inf(-A)$$

by Problem 5, showing (2). ■

9-5) $\sup(A - B) = \sup(A) - \inf(B)$.

Proof. Observe that

$$\begin{aligned} \sup(A-B) = \sup(A+(-B)) &= \sup(A)+\sup(-B) && \text{(9-2))} \\ &= \sup(A)-\inf(B) && \text{(9-3))} \end{aligned}$$

■

9-6) $\inf(A-B) = \inf(A) - \sup(B)$.

Proof. Observe that

$$\begin{aligned} \inf(A-B) = \inf(A+(-B)) &= \inf(A)+\inf(-B) && \text{(9-1))} \\ &= \inf(A)-\sup(A). && \text{(9-4))} \end{aligned}$$

■

9-7) $\sup(cA) = c\sup(A)$ if $c > 0$.

Proof. Let $c > 0$. We will show that $c\sup(A)$ is the supremum of cA; that is, we need to show that $c\sup(A)$ is an upper bound of cA and the least upper bound of cA. Since the supremum of A exists and A is bounded above, we have $a \leq \sup(A)$ for all $a \in A$. Since $c > 0$, we can multiply by $c > 0$ to get $ca \leq c\sup(A)$. Thus, cA is bounded above by $c\sup(A)$.

Now, let γ be any upper bound of A. Since $\sup(A)$ exists, we know that $\sup(A) \leq \gamma$. By multiplying by $c > 0$, we have

$$c\sup(A) \leq c\gamma.$$

But $c\gamma$ is an upper bound of cA, so $c\sup(A)$ must be the least upper bound of cA. Thus, we have that $c\sup(A)$ is the supremum of cA and that $\sup(cA) = c\sup(A)$. ■

9-8) $\inf(cA) = c\inf(A)$ if $c > 0$.

Proof. Let $c > 0$. Since A is bounded below and $\inf(A)$ exists, we know that $a \geq \inf(A)$ for all $a \in A$. Multiplying by c gives us

$$ca \geq c\inf(A).$$

This tells us that every $ca \in cA$ is bounded above by $c\inf(A)$. Now, suppose λ is any lower bound of A. Since $\inf(A)$ exists, we know that $\inf(A) \geq \lambda$. Then multiplying by $c > 0$ on both sides of this inequality gives

$$c\inf(A) \geq c\lambda,$$

proving that $c\inf(A)$ is the greatest lower bound of cA. Thus, we conclude that

$$\inf(cA) = c\inf(A).$$

■

9-9) $\sup(cA) = c\inf(A)$ if $c < 0$.

Proof. Let $c < 0$. We will show that $c\inf(A)$ is the supremum of cA. Since A is bounded below and $\inf(A)$ exists, we have $a \geq \inf(A)$ for all $a \in A$. Since $c < 0$, we have $ca \leq c\inf(A)$ for all $ca \in cA$. Thus, cA is bounded above by $c\inf(A)$.

Let $c\varphi$ be any upper bound of cA. Let $\varepsilon > 0$. Since $\inf(A)$ exists, there exists $\lambda_0 \in A$ such that

$$\lambda_0 < \inf(A) + \frac{\varepsilon}{c}$$

by the lemma found in Problem 3. Since $c < 0$, we can multiply c on both sides of the inequality above to get

$$c\lambda_0 \geq c\inf(A) + \varepsilon.$$

But note that $c\varphi$ is an upper bound of cA. Thus, $c\lambda_0 \leq c\varphi$ and so we have

$$c\varphi \geq c\lambda_0 > c\inf(A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can use the result from Problem 5 to state that

$$c\varphi \geq c\inf(A).$$

Which tells us that $c\inf(A)$ is the least upper bound of cA. Hence, we conclude that $\sup(cA) = c\inf(A)$. ■

9-10) $\inf(cA) = c\sup(A)$ if $c < 0$.

Proof. Let $c < 0$. We will show that $c\sup(A)$ is the infimum of cA. First, we will show that $c\sup(A)$ is a lower bound of cA. Since $\sup(A)$ exists, we have $a \leq \sup(A)$ for all $a \in A$. Since $c < 0$, we have

$$ca \geq c\sup(A) \quad \text{for all } ca \in cA.$$

Thus, $c\sup(A)$ is a lower bound for cA.

Now, suppose $c\omega$ is any lower bound of cA. Since $\sup(A)$ exists, there exists an $a_0 \in A$ such that

$$a_0 > \sup(A) - \frac{\varepsilon}{c}.$$

Multiplying by $c < 0$, we can write

$$ca_0 < c\sup(A) - \varepsilon$$

by the lemma found in Problem 2. Since $c\omega$ is a lower bound of cA, we have $ca_0 \geq c\omega$. Thus, we have

$$\begin{aligned} c\omega \leq ca_0 < c\sup(A) - \varepsilon &\Rightarrow c\omega \leq c\sup(A) - \varepsilon \\ &\Rightarrow c\omega \leq c\sup(A). && \text{(Problem 5)} \end{aligned}$$

Since $c\omega$ is an arbitrary lower bound of cA, this shows that $c\sup(A)$ must be the greatest lower bound of A. Thus, we conclude that

$$\inf(cA) = c\sup(A).$$

■

9-11) Is it true that $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

Proof. This is false. Consider the sets $A = \{1, 2, 4, 8\}$ and $B = \{-5, -2, -3, -1\}$. Then $AB = \{-5, -4, -12, -8\}$. Observe that $\sup(A) \cdot \sup(B) = 8 \cdot -1 = -8$, but $\sup(AB) = -5 \neq -8 = \sup(A) \cdot \sup(B)$. ■

10. Recall that $\mathbb{Q}$ is a field; in particular, $\mathbb{Q}$ is closed under addition and multiplication.

10-1) Prove that if $p \in \mathbb{Q}$ and t is an irrational number, then $p + t$ is an irrational number.

Proof. Suppose $p \in \mathbb{Q}$ and t is an irrational number. Suppose for sake of contradiction that $p + t$ is an rational number. Then there exists $x, y \in \mathbb{Z}$ with $y \neq 0$ such that

$$p + t = \frac{x}{y}.$$

Subtracting p on both sides, we get

$$t = \frac{x}{y} - p.$$

Since $\mathbb{Q}$ is a field and $\frac{x}{y}, p \in \mathbb{Q}$, we get that $\frac{x}{y} - p \in \mathbb{Q}$. But this tells us that t is a rational number which is a contradiction. Thus, t must be irrational. ∎

10-2) Complete the following proof.

Theorem. Given any two real numbers $x < y$, there exists an irrational number t satisfying $x < t < y$.

Proof. It follows from $x < y$ that $x - \sqrt{2} < y - \sqrt{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $p \in \mathbb{Q}$ such that $x - \sqrt{2} < p < y - \sqrt{2}$. Adding $\sqrt{2}$ on both sides gives us

$$x < p + \sqrt{2} < y.$$

Since $p \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, we get that $t = p + \sqrt{2}$ is irrational from 10-2). Thus, $x < t < y$ for some irrational number t.

∎

11. Prove the following:

Theorem (Nested Interval Property). For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, that is $\cap_n^\infty I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ because $a_n \in A$. Since $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$ and that each I_{n+1} is contained within each I_n , we see that each $a_n \in A$ must be bounded above by b_n for all $n \in \mathbb{N}$. Since $A \neq \emptyset$ and bounded above, there must exists a number x such that $x = \sup(A)$. Since x is an upper bound of A, we have $a_n \leq x \leq b_n$. Thus, $x \in I_n$ for all $n \in \mathbb{N}$ which means that

$$x \in \bigcap_{n=1}^{\infty} I_n,$$

proving that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

∎

12. Complete the following proof:

    Prove that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.

    Proof. Suppose for sake of contradiction $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. So there exists $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. This implies that $x > 0$ and $x < 1/n$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $n_0 \in \mathbb{N}$ such that

    $$\frac{1}{n_0} < x.$$

    But this contradicts our assumption that $x < 1/n$ holds for all $n \in \mathbb{N}$. Thus, the intersection

    $$\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$$

    must be empty. ∎