Math 230A Lecture Notes

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Contents

Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

1.1.2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

1.1.3 The First Defining Property

The set of real numbers is a field.

Definition (Fields). A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

1.1.4 The Second Defining Property

Definition (Ordered Fields). An **ordered field** is a field F equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

Remark. We say that x is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

Definition (Upper Bounds). Suppose F is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call β an **upper bound of** A.

Remark. We call the collection of upper bounds of A by $\mathrm{UP}(A)$. If $\mathrm{UP}(A) \neq \emptyset$, then we say that A **is bounded above**.

Similarly, we define the lower bounds of a set.

Definition (Lower Bounds). Suppose F is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then α is called the **lower bound of** A.

Remark. Similarly, we denote the set of lower bounds of A by $\mathrm{LO}(A)$. We say that A is bounded below if $\mathrm{LO}(A) \neq \emptyset$.

Example 1.1.1. Suppose we have $A = [0, 1)$. We have

$$\mathrm{UP}(A) = [1, \infty)$$
$$\mathrm{LO}(A) = (-\infty, 0].$$

1.2 Lecture 2

1.2.1 Review of Least Upper Bound Property

Definition (Supremum). Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\beta \in F$ such that

(i) $\beta \in \mathrm{UP}(A)$

(ii) If $\gamma \in F$ and $\gamma < \beta$, then $\gamma \notin \mathrm{UP}(A)$.

We call β the **least upper bound** of A or the **supremum** of A. We denote the supremum of A as $\beta = \sup A$.

Remark. When we say THE supremum, we are implicitly stating that the supremum of A is unique.

Definition (Infimum). Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\alpha \in F$ such that

(i) $\alpha \in \mathrm{LO}(A)$

(ii) If $\gamma \in F$ and $\gamma > \alpha$, then $\gamma \notin \mathrm{LO}(A)$.

We call α the **greatest upper bound** of A or the **infimum** of A, and write $\alpha = \inf A$.

Definition (Least Upper Bound Property). An ordered field F is said to have the **least-upper-bound property** if the following is true:

Every nonempty set A in F that is bounded above has a least upper bound in F.

That is, if $A \neq \emptyset$ and $\mathrm{UP}(A) \neq \emptyset$, then $\sup(A)$ exists.

Theorem. There is exactly one ordered field that has the least-upper-bound bound property. The set $\mathbb{R}$ is the unique ordered field that contains $\mathbb{Q}$ as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$ is dedekind complete
- $\mathbb{R}$ satisfies the Axiom of Completeness.

Remark. Note that $\mathbb{Q}$ being an ordered field does not immediately imply that $\mathbb{Q}$ has the LUBP.

Definition (Maximums and Infimums). Let $A \subseteq \mathbb{R}$.

- If $\sup A \in A$, then we call, $\sup A$, the **maximum of** A and we denote this by $\max A$.
- If $\inf A \in A$, we call, $\inf A$, the **minimum of** A and we denote this by $\min A$.

Lemma (Useful Fact for Supremum). Let $A \subseteq \mathbb{R}$. Then $\beta = \sup A$ if and only if

(i) $\beta \in \text{UP}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a > \beta - \varepsilon$.

Remark. We can restate property (ii) above as "for all $\varepsilon > 0$, $\beta - \varepsilon \notin \text{UP}(A)$".

Lemma (Useful Fact for Infimums)**.** Let $A \subseteq \mathbb{R}$. Then $\alpha = \inf A$ if and only if

(i) $\alpha \in \text{LO}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a < \alpha + \varepsilon$.

Remark. Similarly, we can restate property (ii) as "for all $\varepsilon > 0$, $\alpha + \varepsilon \notin \text{LO}(A)$".

Theorem (Greatest Lower Bound Property of $\mathbb{R}$)**.** Every nonempty subset A of $\mathbb{R}$ that is bounded below has a **greatest upper bound in** $\mathbb{R}$.

Another way to say this is the following:

$$\text{If } A \neq \emptyset \text{ and } \text{LO}(A) \neq \emptyset, \text{ then } \inf A \text{ exists in } \mathbb{R}.$$

1.2.2 Consequences of Least Upper Bound Property

Theorem (Archimedean Property)**.** If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.

Proof. Let $A = \{nx : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ since $1 \cdot x \in A$. Suppose for sake of contradiction that for all $n \in \mathbb{Z}^+$, $nx \leq y$. This means that y is an upper bound of A. Let $\beta = \sup A$. By the first useful fact, we have that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\beta - \varepsilon < nx$. Let $\varepsilon = x$. Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \Rightarrow \beta < x(n+1).$$

But this tells us that $x(n+1) \in A$ ($x \in A$ and $n+1 \in \mathbb{N}$) and that β is NOT an upper bound which is a contradiction. Thus, it must be the case that $nx > y$ for some $n \in \mathbb{Z}^+$. ∎

Remark. The well ordering property of $\mathbb{N}$ can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

Corollary. Let A be a nonempty subset of $\mathbb{R}$ that consists of only integers.

(i) If A is bounded above, then $\sup(A) \in A$.

(ii) If A is bounded below, then $\inf(A) \in A$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$ with $x < y$, there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Our goal is to find a $p \in \mathbb{Q}$ such that

$$x < p < y$$

with $p = \frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; that is, find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$nx < m < ny.$$

First, notice that $x < y$. This implies that $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \tag{1}$$

Choose $m \in \mathbb{Z}$ such that m to be the minimum element greater than nx; that is, choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m. \tag{2}$$

Let $A = \{k \in \mathbb{Z} : k > nx\}$ which is nonempty by the Archimedean Property. Furthermore, nx is a lower bound for A. By the Well-ordering property, A contains a minimum. Thus, $m = \min A$. Hence,we have

$$nx < m \Rightarrow x < \frac{m}{n}. \tag{3}$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \leq nx \implies m \leq nx + 1 &< n\Big(y - \frac{1}{n}\Big) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \tag{4}$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \iff x < p < y.$$

■

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

2.1.2 Review, Existence of Roots

Proposition. There is no rational number whose square is 2.

Theorem. There is a unique positive real number α satisfying $\alpha^2 = 2$.

Proof. (i) **Uniqueness:** Suppose there are two of them α_1 and α_2. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that A is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

■

Remark. A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number α such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

2.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

Definition (Usual Way of Defining Functions)**.** Let A and B be two sets. A **function** from A to B denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

Definition (The Correct Way of Defining Functions)**.** Let A and B be two sets. A function from A to B is a triple (f, A, B) where f is a **relation** form A to B satisfying

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, A is called the **domain** of f and B is called the **codomain** of f)

Example 2.1.1. Let $A = \emptyset$ and B be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to B is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

Definition (Image, Range, Onto (Surjective))**.** Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of f as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of f as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say f is **Onto (Surjective)**.

Definition (Preimage)**.** Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of D under f is denoted by

$$f^{-1}(D) = \{x \in A : f(x)\}$$

Definition (One-to-One (Injective))**.** Consider a function $f : A \to B$. We call f one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of A.

2.1.4 Equivalent Sets

Definition. Let A and B be two sets. We say that A and B have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

Remark.

- An injective and surjective mapping is a bijective mapping.

- A and B have the same cardinal number

$$
\begin{aligned}
&= A \text{ and } B \text{ have the same cardinality}\\
&= A \text{ and } B \text{ can be put in the } \textbf{one-to-one correspondence}\\
&= \operatorname{card} A = \operatorname{card} B\\
&= A \text{ and } B \text{ are equivalent}\\
&= A \text{ and } B \text{ are equipotent}
\end{aligned}
$$

Example 2.1.2. Consider $\{1,2,3\} \sim \{a,b,c\}$. Indeed, the function $f : \{1,2,3\} \to \{a,b,c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

Example 2.1.3. $\mathbb{N} \sim \{2,4,6,\dots\}$. Indeed, the function $f : \mathbb{N} \to \{2,4,6,\dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

Example 2.1.4. $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

Example 2.1.5. $(-\infty,\infty) \sim (0,\infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty,\infty)$ and $(0,\infty)$.

Example 2.1.6. $(0,\infty) \sim (0,1)$. Indeed, the function $f : (0,\infty) \to (0,1)$ defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

Example 2.1.7. $[0,1) \sim (0,1)$. Indeed, the function $f : [0,1) \to (0,1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if} x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

Definition ($\sim$ is an equivalence relation)**.** Let A and B be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}$$
$$\mathbb{N} = \{1, 2, 3, \dots\}$$

2.1.5 Finite, Infinite, Countable, At most countable

Definition (Finite, Infinite, Countable, At most countable)**.** Let A be any set.

(a) We say that A is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number n.

(*) When $A \sim \mathbb{N}_n$, we say A has n elements and we write $\text{card}(A) = n$.

(*) Also, we set $\text{card}(\emptyset) = 0$.

(b) The set A is said to be **infinite** if it is not finite.

(c) The set A is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set A is said to be **uncountable** if it is neither countable or finite.

(e) The set A is said to be **at most countable** if it is either finite or countable.

Remark. Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$A \text{ is finite} \Leftrightarrow B \text{ is finite}$$
$$A \text{ is countable} \Leftrightarrow B \text{ is countable}$$
$$A \text{ is uncountable} \Leftrightarrow B \text{ is uncountable}$$

(iii) The union of two finite sets is finite. If A is infinite and B is infinite, then $A \setminus B$ is infinite.

(iv) If A **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.

Chapter 3

Week 3

3.1 Lecture 4

3.1.1 Topics

3.1.2 Sequences

Definition (Sequence). We call a **sequence**, we mean a function f on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence f by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in A.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

3.1.3 Infinite subset of a countable set is countable

Theorem. Every infinite subset of a countable set is countable.

Proof. Let A be a countable set. Let $E \subseteq A$ and E is infinite. Our goal is to show that E is countable. Since A is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let n_1 be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let n_2 be the smallest positive integer greater than n_1 such that $x_{n_2} \in E$.

$\vdots$

(k) Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Observe that the set

$$\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}$$

is both nonempty (since E is infinite) and bounded below (by the well ordering property of natural numbers). Thus, we know that

$$n_k = \min\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}.$$

Now, define the function $f : \mathbb{N} \to E$ as follows:

$$f(k) = x_{n_k}.$$

We claim that this is a bijective map. If we can prove this fact, then surely E is countable. We need to prove a two things:

(1) f is injective and

(2) f is surjective.

Starting with (1), suppose $k_1 \neq k_2$. Then we have $n_{k_1} \neq n_{k_2}$ and thus, $x_{n_{k_1}} \neq x_{n_{k_2}}$. Hence, we see that $f(k_1) \neq f(k_2)$. Therefore, f is injective which proves (1).

With (2), let $b \in E$. Since $E \subseteq A$, we must have $b \in A$ and thus there exists an $m \in \mathbb{N}$ such that $b = x_m$. Hence, there exists $1 \leq k \leq m$ such that $x_{n_k} = x_m = b$. Thus, $f(k) = b$ which shows that f is surjective and so (2) is satisfied.

Therefore, f must be a bijective map and we can now conclude that E is countable. ■

Remark. Consider the contrapositive of the theorem above:

Let $E \subseteq A$. If E is not countable, then A is not countable.

When we say that a set is NOT countable, we do not necessarily mean the set is uncountable. However, in the case in that E is an infinite set, we can say that E being not countable is equivalent to saying that E is uncountable. So, only in the case in which E is infinite then the contrapositive is

Let $E \subseteq A$ be infinite. If E is uncountable, then A is uncountable. By the theorem we just proved, we see that $f(A) \subseteq S$ and S being countable implies that $f(A)$ must be at most countable. Thus, A must be at most countable.

Corollary. Let A be any set and let S be a countable set. If there exists an injective mapping $f : A \to S$, then A is at most countable.

Proof. Let A be any set and let S be a countable set. Suppose there exists an injective mapping $f : A \to S$. We can restrict the codomain S of f to its range $f(A)$. Because f is injective, the mapping $f : A \to f(A)$ will be bijective. Thus, we have that $A \sim f(A)$. ■

Example 3.1.1 ($\mathbb{N} \times \mathbb{N}$ is countable). (1) The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(x, y) = 2^x 3^y$$

is injective. By the corollary we have just proved, we see taht $\mathbb{N} \times \mathbb{N}$ is at most countable.

(2) Notice that $g : \mathbb{N} \to \{1\} \times \mathbb{N}$ is a bijection $g(a) = (1, a)$. Hence, $\{1\} \times \mathbb{N}$ is countable. Now, we have that

(i) $\{1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$

(ii) $\{1\} \times \mathbb{N}$ is countable implies that $\{1\} \times \mathbb{N}$ is infinite.

Thus, (i) and (ii) imply that $\mathbb{N} \times \mathbb{N}$ is countable.

Proving that a set A is countable can be done by finding a subset B that is countable which forces A to be countable.

Example 3.1.2 ($\mathbb{Q}$ is countable). (1) The function $f : \mathbb{Q} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2(2^p 3^q) & \text{if } x = \frac{p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 2(2^p 3^q) + 1 & \text{if } x = \frac{-p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is an injective mapping. So, $\mathbb{Q}$ is at most countable.

(2) Observe that $\mathbb{N} \subseteq \mathbb{Q}$ and $\mathbb{N}$ being a countable (and thus $\mathbb{N}$ is infinite) set implies that $\mathbb{Q}$ is infinite. Thus, $\mathbb{Q}$ is is countable.

3.1.4 Countable union of at most countable sets is at most countable

Theorem. Countable union of at most countable sets is at most countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of at most countable sets; that is, for each $n \in \mathbb{N}$, A_n is at most countable. Our goal is to show that

$$K = \bigcup_{n \in \mathbb{N}} A_n \text{ is at most countable.}$$

To this end, it suffices to show that there exists an injective map $f : K \to \mathbb{N} \times \mathbb{N}$. Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\vdots \\ B_{n+1} &= A_{n+1} \setminus \Big(\bigcup_{k=1}^{n} A_k \Big). \end{aligned}$$

We leave as an exercise to show that

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \text{ are pairwise disjoint.}$$

Note that for all $n \in \mathbb{N}$, we have $B_n \subseteq A_n$ and A_n is at most countable. Then for each $n \in \mathbb{N}$. B_n must be at most countable. So, for each $n \in \mathbb{N}$, there exists an injective mapping $f_n : B_n \to \mathbb{N}$. Our goal is to show that

$$\bigcup_{n=1}^{\infty} B_n \text{ is at most countable.}$$

To this end, we define the function

$$f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N} \times \mathbb{N}$$

as follows:

For each $x \in \bigcup_{n=1}^{\infty} B_n$, there is exactly one $n \in \mathbb{N}$ such that $x \in B_n$; let us denote this n by n_x.

Thus, we define

$$f(x) = (n_x, f_{n_x}(x)).$$

This function is injective because

$$\begin{aligned} f(x) = f(y) &\implies (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y)) \\ &\implies n_x = n_y \ \wedge \ f_{n_x}(x) = f_{n_y}(y) \\ &\implies f_{n_k}(x) \wedge f_{n_x}(y) \\ &\implies x = y && \text{(Since } f_{n_x} \text{ is injective)} \end{aligned}$$

By the corollary, we see that the set

$$\bigcup_{n=1}^{\infty} B_n$$

is at most countable. ■

Corollary. A countable union of countable sets is countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. By the previous Theorem, we see that $\bigcup_{n\in\mathbb{N}} A_n$ is at most countable. Note that $A_1 \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and that A_1 is countable. Thus, A_1 must be infinite and so

$$\bigcup_{n\in\mathbb{N}} A_n \text{ is infinite.}$$

Thus, we see that

$$\bigcup_{n\in\mathbb{N}} A_n$$

must be countable. ■

Corollary. If A and B are at most countable, then $A \cup B$ at most countable.

Proof. Let $A_1 = A$ and $A_2 = B$ where $A_2 = A_3 = \cdots = A_n = \emptyset$. Then $A \cup B$ is at most countable by the previous Theorem. ■

Theorem. If A is countable, then $A \times A$ is countable.

Proof. (1) Note that $A \times A = \bigcup_{b\in A} \{b\} \times A$ for each $b \in A$.

(2) The function $f : A \to \{b\} \times A$ defined by $f(x) = (b, x)$ is bijective. So, $A \sim \{b\} \times A$. Hence, $\{b\} \times A$ is countable.

Since a countable union of countable sets is countable, we see that $A \times A$ must be countable by (1) and (2). ■

3.2 Lecture 5

3.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks
- Inequalities

3.2.2 Summary of Last Lecture

1. How to prove A is at most countable:
   - Show that $A \subseteq S$ where S is countable.
   - Find an injective function $f : A \to S$ where S is countable.
2. How to prove A is infinite:
   - Prove that A has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subsetneq A$ such that $A \sim B$.
3. Find a proper subset $B \subseteq A$ such that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

Theorem (D). Let A be the set of all sequences whose terms are the digits 0 and 1 (that is A is the collection of all binary sequences). This set A is uncountable.

Proof. First, notice that A is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

$$\text{for all } h(n) = \text{The binary sequence whose } n\text{th term is the digit 1 and all other terms are zero.}$$

Clearly, h is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, A is infinite

Suppose for sake of contradiction that A is NOT uncountable. Since A is infinite and not uncountable, this assumption tells us that A must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned}
f(1) &= (a_1^1, a_2^1, \dots) \\
f(2) &= (a_1^2, a_2^2, \dots) \\
f(3) &= (a_1^3, a_2^3, \dots) \\
&\vdots \\
f(n) &= (a_1^n, a_2^n, \dots) \\
&\vdots
\end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that A contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N} \ b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence (b_n) is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

3.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of a is defined to be $|a|$. But note that the **absolute value of** a is defined as follows:
$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$
- Given two numbers a and b in $\mathbb{R}$, the **standard distance** between a and b is
$$\operatorname{dist}(a, b) = |a - b|.$$
- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as h gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

Theorem (Basic Properties of Standard Size in $\mathbb{R}$)**.** Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Remark. Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

Definition. Let V be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on V if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space V equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

Theorem (Basic Properties of Standard Distance in $\mathbb{R}$)**.** For all $a, b \in \mathbb{R}$.

(i) $\mathrm{dist}(a, b) \geq 0$

(ii) $\mathrm{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\mathrm{dist}(a, b) = \mathrm{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\mathrm{dist}(a, c) \leq \mathrm{dist}(a, b) + \mathrm{dist}(b, c)$.

3.2.4 Metric Spaces

Definition (Metric Spaces)**.** Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X equipped with a metric d is called a **metric space**. (Sometimes we write (X, d) is a metric space)

Remark. X does **NOT** need to be a vector space. Furthermore, d is a more general notion of size than $\|\cdot\|$.

Example 3.2.1. $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)

Chapter 4

Week 4

4.1 Lecture 6

4.1.1 A few examples of Metrics

Example 4.1.1. Consider $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

This is a metric on $(\mathbb{R}, d)$. **Prove this on homework!**

Remark. If (X, D) is a metric space, then (X, D) is also a metric space where

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

We can define a metric that will always be less than or equal to 1.

Example 4.1.2 (Taxi Cab Metric)**.** Consider $(\mathbb{R}^2, d)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ is defined by

$$d((a, b), (x, y)) = |a - x| + |b - y|$$

We want to show that this is a metric.

(i) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have $d((a, b), (x, y)) = |a-x|+|b-y| \geq 0$ by property of the absolute value $|\cdot|$.

(ii) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} d((a, b), (x, y)) = 0 &\Leftrightarrow |a - x| + |b - y| = 0 \\ &\Leftrightarrow |a - x| = 0 \text{ and } |b - y| = 0 \\ &\Leftrightarrow a - x = 0 \text{ and } b - y = 0 \\ &\Leftrightarrow (a, b) = (x, y). \end{aligned}$$

(iii) For all $(a, b), (x, y) \in \mathbb{R}^2$.

$$d((a, b), (x, y)) = |a - x| + |b - y| = |x - a| + |y - b| = d((x, y), (a, b)).$$

(iv) For all $(a,b),(x,y),(t,s) \in \mathbb{R}^2$, we want to show that

$$d((a,b),(x,y)) \leq d((a,b),(t,s)) + d((t,s),(x,y)).$$

We have

$$\begin{aligned} d((a,b),(t,s)) + d((t,s),(x,y)) &= |a-t| + |b-s| + |t-x| + |s-y| \\ &= (|a-t| + |t-x|) + (|b-s| + |s-y|) \\ &\geq |a-x| + |b-y| \\ &= d((a,b),(x,y)) \end{aligned}$$

Example 4.1.3 (Discrete Metric). Consider $X \to$ any nonempty set. Consider $d : X \times X \to [0,\infty)$,

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

We want to show that this is a metric on $(X, \text{any non-empty set})$.

(i) For all $x,y \in X$, we have $d(x,y) \geq 0$ because either the $d(x,y) = 1$ or $d(x,y) = 0$.

(ii) Let $x,y \in X$. Clearly, we have $x = y$ if and only if $d(x,y) = 0$ by definition of $d(x,y)$.

(iii) Let $x,y \in X$. Clearly, we have $d(x,y) = d(y,x)$ by definition.

(iv) Let $x,y,z \in X$. We want to show that

$$d(x,y) \leq d(x,z) + d(z,y)$$

Let us consider two cases:

(1) $x = y$. Thus, both sides of the triangle inequality clearly hold since $d(x,y) = 0$.

(2) $x \neq y$. In this case, we have $d(x,y) = 1$. So, we need to show that

$$d(x,z) + d(z,y) \geq 1.$$

Since $x \neq y$, at least one of the statements $z \neq y$ or $z \neq x$ is true. If $z \neq x$, then $d(z,x) = 1$ and so

$$d(x,z) + d(z,y) = 1 + d(z,y) \geq 1.$$

If $z \neq y$, then $d(z,y) = 1$, and so

$$d(x,z) + d(z,y) = d(x,z) + 1 \geq 1.$$

Example 4.1.4. Consider $(V, \|\cdot\|) \to$ any normed vector space with the metric $d : V \times V \to [0,\infty)$ and $d(x,y) = \|x-y\|$. We wan to show that this is a metric on V.

(i) Let $x,y \in V$. By the property of the norm, we have $d(x,y) = \|x-y\| \geq 0$.

(ii) Let $x,y \in V$. By the 2nd property of the norm, we have

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow \|x-y\| = 0 \\ &\Leftrightarrow x - y = 0 \\ &\Leftrightarrow x = y. \end{aligned}$$

(iii) Let $x, y \in V$. We have

$$d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x).$$

(iv) Let $x, y, z \in V$. We want to show that

$$d(x,y) \leq d(x,z) + d(z,y).$$

We have,

$$\begin{aligned} d(x,z) + d(z,y) &= \|x-z\| + \|z-y\| \\ &\geq \|(x-z)+(z-y)\| \\ &= \|x-y\| \\ &= d(x,y). \end{aligned}$$

This example illustrates how every time we have a norm, we immediately have a metric space. But the other around is not necessarily true!

Example 4.1.5 (Standard Distance in $\mathbb{R}^n$)**.** Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we have

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

Note that if we define for all $x \in \mathbb{R}^n$, we have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

Then

$$d(x,y) = \|x-y\|_2.$$

Proving that the above is a norm is enough to show that the original in question is a metric.

Let $x, y \in \mathbb{R}^n$.

(i) We have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \geq 0.$$

(ii) We have

$$\begin{aligned} \|x_2\|_2 = 0 &\Leftrightarrow \sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0 \\ &\Leftrightarrow |x_1|^2 + \cdots + |x_n|^2 = 0 \\ &\Leftrightarrow |x_1| = 0, |x_2| = 0, \ldots, |x_n| = 0 \\ &\Leftrightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0. \end{aligned}$$

Thus, we have $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|\alpha x\|_2 &= \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2(x_1^2 + \cdots + x_n^2)} \\ &= |\alpha|\sqrt{x_1^2 + \cdots + x_n^2} \\ &= |\alpha|\|x\|_2. \end{aligned}$$

(iv) Now, we want to show that

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

That is, we want to show that

$$\sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} \leq \sqrt{x_1^2+\cdots+x_n^2} + \sqrt{y_1^2+\cdots+y_n^2}.$$

We will show this later!

Example 4.1.6. Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$, we have

$$d_p(x,y) = \Big[|x_1-y_1|^p + \cdots + |x_n-y_n|^p\Big]^{\frac{1}{p}}.$$

4.1.2 Inequalities

In what follows we will list several key inequalities that can be used in proving that, for example, a given expression is a metric or a norm.

(1) The triangle inequality for the standard norm in $\mathbb{R}$. We have for all $x, y \in \mathbb{R}$, we have

$$|x+y| \leq |x| + |y|.$$

More generally, we have for all $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\Big|\sum_{i=1}^n x_i\Big| \leq \sum_{i=1}^n |x_i|.$$

(2) Reverse triangle inequality for the standard norm in $\mathbb{R}$; that is, we have for all $x, y \in \mathbb{R}$,

$$||x| - |y|| \leq |x-y|.$$

(3) For all $a, b \geq 0$ and for all $\rho > 0$, we have

$$ab \leq \frac{1}{2}\Big(\rho a^2 + \frac{1}{\rho} b^2\Big).$$

Note that we are dealing with real numbers here!

(4) Cauchy-Schwarz Inequality. For al $x, y \in \mathbb{R}^n$, we have

$$|x_1y_1+\cdots+x_ny_n| \leq \Big(\sqrt{x_1^2+\cdots+x_n^2}\Big)\Big(\sqrt{y_1^2+\cdots+y_n^2}\Big). \tag{*}$$

4.2 Lecture 7

4.2.1 Topics

- Review of Inequalities
- Neighborhood of a point
- Limit point of a set, isolated point of a set
- Closed set
- Interior point of a set
- Open set

- Bounded set
- Closure
- Dense

4.2.2 Inequalities

4.2.3 Minkowski

We wan to show the triangle inequality for $\|\cdot\|_2$ in $\mathbb{R}^n$; that is, we want to show that

Proposition. Let $\|\cdot\|_2$ be a norm in $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$, $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.

Proof. We see that $x \cdot y \leq |x \cdot y| \leq \|x\|_2\|y\|_2$ by the Cauchy Cauchy-Schwarz Inequality. Thus, we have

$$\begin{aligned}
\|x+y\|_2^2 &= (x+y)\cdot(x+y) \\
&= x\cdot x + x\cdot y + y\cdot x + y\cdot y \\
&= \|x\|_2^2 + 2x\cdot y + \|y\|_2^2 \\
&\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
&= (\|x\|_2 + \|y\|_2)^2.
\end{aligned}$$

Hence, we have

$$\|x+y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2.$$

Therefore, we have

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

■

4.2.4 Minkowski for General p

For general p, we have Holder's Inequality which is

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

for any fixed real number $p \geq 1$ and for any $x, y \in \mathbb{R}^n$.

4.2.5 $(x+1)^n$

Recall that for all $a, b \in \mathbb{R}$, for all $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

So, in particular, if $x \geq 0$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\
&\geq 1 + nx.
\end{aligned}$$

Hence, we have for all $x \geq 0$ and for all $n \in \mathbb{N}$, we have

$$(x+1)^n \geq 1 + nx.$$

4.2.6 $p-$means

Let $x_1, \ldots, x_n$ be positive real numbers. Let $p \in \mathbb{N} \cup \{0\}$. By the $p-$mean of $x_1, \ldots, x_n$ denoted by $A_p(x_1, \ldots, x_n)$, we mean

$$A_p(x_1, \ldots, x_n) = \begin{cases} \frac{\sqrt[p]{x_1^p + \cdots + x_n s^p} n}{} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 \ldots x_n} & \text{if } p = 0. \end{cases}.$$

For example, if $p = 1$, we have

$$A_1(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

which is the Arithmetic Mean. If $p = 2$, we have

$$A_2(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

If $p = 0$, we have

$$A_0(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$$

which is the geometric mean. It can be shown that

$$A_0(x_1, \ldots, x_n) \leq A_1(x_1, \ldots, x_n) \leq A_2(x_1, \ldots, x_n) \leq \ldots .$$

In particular, we have $A_0 \leq A_1$; that is,

$$\sqrt[n]{x_1 \ldots x_n} \leq \frac{x_1 + \cdots + x_n}{n}. \tag{AM-GM Inequality}$$

We can prove this inequality using differentiation. Without differentiation, we can prove the same inequality using **Cauchy Induction**.

4.2.7 Jensen's Inequality

Suppose $f : (a, b) \to \mathbb{R}$ is a convex function ($f"(x) \geq 0$ for all $x \in (a, b)$). Let $x_1, \ldots, x_n$ be points in (a, b). Let $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

We want to show this inequality holds for $n = 2$; that is,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

where $\lambda_1 + \lambda_2 = 1$. Note that

$$f((1 - \lambda_2)x_1 + \lambda_2 x_2) \leq (1 - \lambda_2) f(x_2) + \lambda_2 f(x_2).$$

> **Remark.** If we want to have an expression that defines a function that gives us any number in between two points $e < h$, we can have
>
> $$f(\lambda) = (1 - \lambda)e + \lambda h$$
>
> for any $0 \leq \lambda \leq 1$.

4.2.8 Neighborhood of a point

Definition (Neighborhood). Let (X, d) be a metric space. Let $p \in X$. For any $\varepsilon > 0$, we call

$$N_\varepsilon(p) = \{x \in X : d(p, x) < \varepsilon\}.$$

the **neighborhood of** p **of radius** ε.

Example 4.2.1. Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. The neighborhood of any $p \in \mathbb{R}$ with radius $\varepsilon > 0$ is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Note that $|x - p| < \varepsilon$ is the same thing as $p - \varepsilon < x < p + \varepsilon$ or that $x \in (p - \varepsilon, p + \varepsilon)$.

Example 4.2.2. Let $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$. Let $(a, b) \in \mathbb{R}^2$ with $\varepsilon > 0$. Then

$$\begin{aligned} N_\varepsilon((a, b)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y)(a, b)) < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \varepsilon^2.\} \end{aligned}$$

Thus, $N_\varepsilon((a, b))$ consists of the points inside the circle of radius ε centered in (a, b).

Example 4.2.3. $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = |a - x| + |b - y|$ and let $\varepsilon = 1$. We have

$$\begin{aligned} N_1((0, 0)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x - 0| + |y - 0| < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \end{aligned}$$

If we graph this out in $\mathbb{R}^2$, then the shape of the neighborhood will take on a rhombus.

Example 4.2.4. Let $(\mathbb{R}, d)$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $p \in \mathbb{R}$. Let $\varepsilon > 0$. Let us consider two cases:

(1) Let $\varepsilon \leq 1$. Note that if

$$d(x, p) < \varepsilon \leq 1,$$

then $d(x, p) < 1$, and so $d(x, p) = 0$. Hence, $x = p$. Then the neighborhood is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{p\}.$$

(2) Let $\varepsilon > 1$. Clearly, for all $x \in \mathbb{R}$, we have $d(x, p) \leq 1 < \varepsilon$. So,

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \mathbb{R}.$$

4.2.9 Limit Points

Definition (Limit Points, Isolated Points). Let (X, d) is a metric space with $E \subseteq X$. Then we call

(1) A point $p \in X$ is said to be a **limit point of** E if for all $\varepsilon > 0$,

$$N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset.$$

(2) The **collection of all the limit points of** E is denoted by E'; that is,

$$E' = \{p \in X : \text{for all } \varepsilon > 0, N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset\}.$$

(3) A point $p \in E$ is said to be an **isolated point of** E if p is NOT a limit point; that is, $p \in E$ but $p \notin E'$. Another way of saying this is $E \setminus E'$.

(4) If p is NOT a limit point, we have $p \notin E'$ if and only if there exists $\varepsilon > 0$ such that

$$N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset.$$

Remark. The statement for all $\varepsilon > 0$, $N_\varepsilon(p)$ is equivalent to for all $N_\varepsilon(p)$.

Example 4.2.5. Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$. Note that $0 \notin E$. Also, recall that $0 \in E'$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(0) \cap (E \setminus \{0\}) \neq \emptyset$. If $0 \in E'$, we just need to show that for all $\varepsilon > 0$, $N_\varepsilon(0) \cap E \neq \emptyset$; that is, we need to show that

$$\text{for all } \varepsilon > 0,\ (-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Clearly, $1/m \in (-\varepsilon, \varepsilon) \cap E$.