1 Lecture 16

1.1 Topics

- Diameter of a set
- Theorem: $\operatorname{diam} \overline{E} = \operatorname{diam} E$
- Theorem: Nested sequence of nonempty compact sets K_n and if $\operatorname{diam} K_n \to 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.
- Theorem: Every compact metric space is complete.
- Theorem: $\mathbb{R}^k$ is a complete metric space.

Definition (Diameter of a Set). Let (X, d) be a metric space. Let E be a nonmempty set in X. The diameter of E, denoted by $\operatorname{diam} E$, is defined as follows:

$$\operatorname{diam} E = \sup\{d(a, b) : a, d \in E\}.$$

Remark. Note that if $\emptyset \neq A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Thus, we have

$$\sup\{d(a, b) : a, b \in A\} \leq \sup\{d(a, b) : a, b \in B\}.$$

That is,

$$\operatorname{diam} A \leq \operatorname{diam} B.$$

Theorem (Interesting Observation). Let (X, d) be a metric space. Let (x_n) be a sequence in X. For all $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+1}, \dots\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n\to\infty} \operatorname{diam} E_n = 0.$$

Proof. ($\Longrightarrow$) Assume that (x_n) is a Cauchy sequence. Note that

$$\begin{aligned} E_1 &= \{x_2, x_3, x_4, x_5, \dots\} \\ E_2 &= \{x_3, x_4, x_5, \dots\} \\ E_3 &= \{x_4, x_5, \dots\} \\ E_4 &= \dots \\ &\vdots \end{aligned}$$

Clearly, we have

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq \cdots.$$

So, we have

$$\operatorname{diam} E_1 \geq \operatorname{diam} E_2 \geq \operatorname{diam} E_3 \geq \cdots.$$

Our goal is to show that

$$\forall\, \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \underbrace{|\operatorname{diam} E_n - 0|}_{\operatorname{diam} E_n \geq 0} < \varepsilon. \tag{*}$$

So, it suffices to show that $\operatorname{diam} E_n < \varepsilon$. To this end, let $\varepsilon > 0$ be given. Since (x_n) is Cauchy, there exists $\hat{N} \in \mathbb{N}$ such that

$$\forall n, m > \hat{N}\quad d(x_n, x_m) < \frac{\varepsilon}{2}.$$

We claim that this $\hat{N}$ is the same N we were looking for. Letting $N = \hat{N}$, we have

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}.$$

Hence, we have

$$\forall a, b \in E_{\hat{N}} \quad d(a,b) < \frac{\varepsilon}{2}.$$

Thus, we have

$$\operatorname{diam} E_{\hat{N}} = \sup\{d(a,b) : a, b \in E_{\hat{N}}\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

If $n > \hat{N}$, then

$$\operatorname{diam} E_n \leq \operatorname{diam} E_{\hat{N}} < \varepsilon$$

as desired.

($\Longleftarrow$) Assume that $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$. Our goal is to show that (x_n) is Cauchy; that is,

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N \ d(x_n, x_m) < \varepsilon. \tag{*}$$

Since $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$, for this given ε, there exists $\hat{N}$ such that

$$\forall n > \hat{N} \quad \operatorname{diam} E_n < \varepsilon.$$

In particular, we have $\operatorname{diam} E_{\hat{N}+1} < \varepsilon$. Now, we claim that $N = \hat{N} + 1$ can be used as the same N we were looking for. Indeed, if we let $N = \hat{N} = 1$, we have

$$\forall n, m > \hat{N} + 1, \ x_n, x_m \in E_{\hat{N}+1}$$

and thus

$$d(x_n, x_m) \leq \operatorname{diam} E_{\hat{N}+1} < \varepsilon.$$

■

Theorem. Let (X, d) be a metric space. Let E be a nonempty subset of X. Then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

Proof. Note that, since $E \subseteq \overline{E}$, we have $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$. In what follows, we will prove that $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$ by showing that

$$\forall \varepsilon > 0 \quad \operatorname{diam} \overline{E} \leq \operatorname{diam} E + \varepsilon.$$

Let $\varepsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a,b) : a, b \in \overline{E}\} \leq \operatorname{diam} E + \varepsilon.$$

To this end, it suffices to show that $\operatorname{diam} E + \varepsilon$ is an upper bound for $\{d(a,b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$a \in \overline{E} \Longrightarrow N_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset \Longrightarrow \exists x \in E \text{ such that } d(x,a) < \frac{\varepsilon}{2}$$
$$b \in \overline{E} \Longrightarrow N_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset \Longrightarrow \exists y \in E \text{ such that } d(y,b) < \frac{\varepsilon}{2}.$$

Therefore, we have that

$$\begin{aligned} d(a,b) &\le d(a,x) + d(x,y) + d(y,b) \\ &< \frac{\varepsilon}{2} + d(x,y) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \operatorname{diam} E + \frac{\varepsilon}{2} \\ &= \varepsilon + \operatorname{diam} E \end{aligned}$$

which is our desired result. ∎

Theorem. Let (X,d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a nested sequence of nonempty compact sets where $\lim_{n\to\infty} \operatorname{diam} K_n = 0$. Then

$\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.3.6, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose that $a,b \in K$ and we will prove $a = b$. In order to show that $a = b$, we will prove that $d(a,b) = 0$. Showing this is equivalent to showing that

$$\forall \varepsilon > 0 \quad d(a,b) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n > N \quad \operatorname{diam} K_n < \varepsilon.$$

In particular, $\operatorname{diam} K_{N+1} < \varepsilon$. Now, we have

$$a \in \bigcap_{n=1}^{\infty} \implies a \in K_{N+1}$$

and

$$b \in \bigcap_{n=1}^{\infty} K_n \implies b \in K_{N+1}$$

which implies further that

$$d(a,b) \le \operatorname{diam} K_{N+1} < \varepsilon$$

which is our desired result. ∎

Theorem (Compact Space $\implies$ Complete Space)**.** Any compact metric space is complete.

Proof. Let (X,d) be a compact metric space. Let (x_n) be a Cauchy sequence in X. Our goal is to show that (x_n) converges in X. For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3} \dots\}$. We know that

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

(2) (x_n) is Cauchy $\implies \lim_{n\to\infty} \operatorname{diam} E_n = 0$.

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \cdots \tag{†}$$

Since closed subsets of a compact metric space are compact, we know that (†) is a nested sequence of nonempty compact sets. Since $\operatorname{diam} E_n = \operatorname{diam} \overline{E_n}$, it follows from (2) that $\lim_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$. Hence, we know by the previous theorem that $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let us denote this point as a where

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}.$$

In what follows, we will prove that $\lim_{n\to\infty} x_n = a$. To this end, it suffices to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ d(x_n, a) < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be give. Since $\lim_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$, for this given ε, there exists $\hat{N}$ such that

$$\forall n > \hat{N} \quad \operatorname{diam} \overline{E_n} < \varepsilon.$$

We claim that $\hat{N} + 1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N} + 1$, then (*) holds. The reason is as follows: If $n > \hat{N} + 1$, then

$$x_n \in E_{\hat{N}+1} \Longrightarrow x_n \in \overline{E_{\hat{N}+1}}. \tag{I}$$

Furthermore,

$$a \in \bigcap_{n=1}^{\infty} \overline{E_n} \Longrightarrow a \in \overline{E_{\hat{N}+1}}. \tag{II}$$

Thus, (I) and (II) imply that

$$d(x_n, a) \leq \operatorname{diam} \overline{E_{\hat{N}+1}} < \varepsilon.$$

■

Theorem ($\mathbb{R}^k$ is complete)**.** $\mathbb{R}^k$ is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^k$. By homework 7, we see that (x_n) must be bounded; that is,

$$\exists p \in \mathbb{R}^k, \varepsilon > 0 \text{ such that } \forall n \in \mathbb{N}\ x_n \in N_\varepsilon(p).$$

Note that $\overline{N_\varepsilon(p)}$ is a closed and bounded set in $\mathbb{R}^k$, and so it must be compact by the Heine-Borel Theorem. Thus, if $\overline{N_\varepsilon(p)}$ is a compact metric space and (x_n) is a Cauchy sequence in $\overline{N_\varepsilon(p)}$, we have that (x_n) converges to a point $x \in \overline{N_\varepsilon(p)}$ by the previous theorem. Since the metric in $\overline{N_\varepsilon(p)}$ is exactly the same as the metric in $\mathbb{R}^k$, we can conclude that $x_n \to x$ in $\mathbb{R}^k$.

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2 Lecture 17

2.1 Topics

- Algebraic Limit Theorem (for sequences of numbers)
- Divergent sequences
- Monotone sequences
- Monotone Convergence Theorem
- Extended real numbers
- Properties of limit in $\overline{\mathbb{R}}$ (Extended Real Numbers)

Theorem (Algebraic Limit Theorem)**.** Suppose (a_n) and (b_n) are sequences of real numbers, and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then

(i) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(ii) n $\lim_{n\to\infty} (ca_n) = ca$ for any real number c.

(iii) $\lim_{n\to\infty} (a_n b_n) = ab$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided that $b \neq 0$.

Proof. *Proof left to the reader.* ■

Definition. Consider $\mathbb{R}$ with its standard metric. Let (x_n) be a sequence of real numbers. If (x_n) does not converge, we say (x_n) **diverges**.

Divergence can come in three different forms:

(i) (x_n) becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n > M.$$

From this, we say that $x_n \to \infty$ or $\lim\limits_{n\to\infty} x_n = \infty$.

(ii) $(-x_n)$ becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ \underbrace{-x_n > M}_{x_n < -M}.$$

From this, we say that $x_n \to -\infty$ or $\lim\limits_{n\to\infty} x_n = -\infty$.

(iii) (x_n) is NOT convergent and it does NOT diverge to ∞ nor $-\infty$. For example,

$$(x_n) = ((-1)^n) = (-1, +1, -1, +1, \dots)$$

or

$$(x_n) = (n(-1)^n) = (-1, 2, -3, 4, -5, 6, \dots).$$

Remark (Strategies to Prove Divergence)**.** Let (a_n) be a sequence of real numbers.

(1) If (a_n) is unbounded, then (a_n) diverges.

(2) If (a_n) has a pair of subsequences converging to different limits, then (a_n) diverges.

(3) Let $a \in \mathbb{R}$ be fixed but arbitrary. Show that the assumption $a_n \to a$ leads to a contradiction.

Definition (Increasing, Decreasing, Monotone)**.** Consider $\mathbb{R}$ with the standard metric.

(i) (a_n) is said to be **increasing** if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$.

(ii) (a_n) is said to be **decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

(iii) (a_n) is said to be **monotone** if it is either increasing or decreasing or both.

(iv) (a_n) is said to be **strictly increasing** if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

(v) (a_n) is said to be **strictly decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} < a_n$.

Example. (i) $1, -1, 1, -1, 1, -1, \dots$ is neither an increasing nor decreasing sequence.

(ii) $2, 4, 6, 8, 10, \dots$ is strictly increasing.

(iii) $6, 6, 7, 7, 8, 8, \dots$ is an increasing sequence but not strictly increasing.

(iv) $-2, -4, -6, -8, -10, -12, \dots$ is a strictly decreasing sequence.

Theorem (Monotone Convergence Theorem)**.** Consider $\mathbb{R}$ with its standard metric.

(i) If (a_n) is increasing and bounded, then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$.

(ii) If (a_n) is decreasing and bounded, then (a_n) converges to $\inf\{a_n : n \in \mathbb{N}\}$.

(iii) If (a_n) is increasing and unbounded, then $a_n \to \infty$.

(iv) If (a_n) is decreasing and unbounded, then $a_n \to -\infty$.

Proof. Here will prove item (i). The rest of the parts follow analogously.

Suppose that (a_n) is increasing ($\forall n\ a_n \leq a_{n+1}$) and (a_n) is a bounded sequence. Our goal is to show that $a_n \to s$ where

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$

Notice that since $\{a_n : n \in \mathbb{N}\}$ is a bounded set, we know that $\{a_n : n \in \mathbb{N}\}$ must exists in $\mathbb{R}$. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ |a_n - s| < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. To show (*), it suffices to show that

$$\text{if } n > N, \text{ then } s - \varepsilon < a_n < s + \varepsilon.$$

Since $s = \sup\{a_n : n \in \mathbb{N}\}$, we see that $s - \varepsilon$ must NOT be an upper bound of $\{a_n : n \in \mathbb{N}\}$. Hence, there exists an element in $\{a_n : n \in \mathbb{N}\}$ that is larger than $s - \varepsilon$. That is, there exists $\hat{N} \in \mathbb{N}$ such that $a_{\hat{N}} > s - \varepsilon$. We claim that this $\hat{N}$ can be used as the N that were looking for. Indeed, if we let $N = \hat{N}$, then

(1) if $n > \hat{N}$, then $a_n \geq a_{\hat{N}} > s - \varepsilon$

(2) if $n > \hat{N}$, then $a_n \leq \sup\{a_n : n \in \mathbb{N}\} = s < s + \varepsilon$.

Thus, (1) and (2) imply that if $n > \hat{N}$, then $s - \varepsilon < a_n < s + \varepsilon$ as desired. ■

Remark. Consider $\mathbb{R}$ with its standard metric.

(*) Boundedness does not generally imply convergence. An example of such a bounded sequence that does not converge is $a_n = (-1)^n$.

(*) Monotone does not necessarily imply convergence. An example of such a sequence is $a_n = n$.

(*) Convergence does not generally imply being monotone; that is, consider $a_n = \frac{(-1)^n}{n}$ which converges to 0 but does not have a monotone behavior.

Example. Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Use induction to show that $a_n \leq 2$ for every n. We claim that $\forall n \geq 1$, $a_n \leq 2$. Let our base case be $n = 1$. Then $a_1 = 1 \leq 2$. Suppose for our induction hypothesis that the claim holds for $n = k (k \geq 1)$, that is, suppose that $a_k \leq 2$. Our goal is to show that the claim holds for $n = k + 1$, that is, prove that $a_{k+1} \leq 2$. Hence, we have

$$a_{k+1} = \frac{1}{2}a_k + 1 \underbrace{\leq}_{a_k \leq 2} \frac{1}{2}(2) + 1 = 1 + 1 = 2.$$

(ii) Use induction to show that (a_n) is an increasing sequence. Our claim that for all n, $a_n \leq a_{n+1}$. Let our base case be $n = 1$. Then $a_1 = 1$ and

$$a_2 = \frac{1}{2}a_1 + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2}.$$

Hence, this implies that $a_1 \leq a_2$. Suppose that the claim for $n = k + 1$, that is, prove that

$a_{k+1} \leq a_{k+2}$. Now, we have

$$\begin{aligned} a_k \leq a_{k+1} &\Longrightarrow \frac{1}{2}a_k \leq \frac{1}{2}a_{k+1} \\ &\Longrightarrow \frac{1}{2}a_k + 1 \leq \frac{1}{2}a_{k+1} + 1 \\ &\Longrightarrow a_{k+1} \leq a_{k+2}. \end{aligned}$$

(iii) We will show that (i) and (ii) will imply that (a_n) converges. We have

(1) $\forall n$, $1 = a_1 \leq a_n \leq 2 \Longrightarrow (a_n)$ is bounded,

(2) (a_n) is increasing.

By the Monotone Convergence Theorem, we see that (a_n) converges.

(iv) Now, we will show that $\lim_{n\to\infty} a_n = 2$. Let $A = \lim_{n\to\infty} a_n$. We have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[\frac{1}{2}a_n + 1\right] &\Longrightarrow \lim_{n\to\infty} a_{n+1} = \frac{1}{2}(\lim_{n\to\infty} a_n) + 1 \\ &\Longrightarrow A = \frac{1}{2}A + 1 \\ &\Longrightarrow \frac{1}{2}A = 1 \\ &\Longrightarrow A = 2. \end{aligned}$$

2.2 Extended Real Numbers

The set of extended real number, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols $+\infty, -\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

(*) $\overline{\mathbb{R}}$ is equipped with an order. We preserve the original order in $\mathbb{R}$ and we define

$$\forall x \in \mathbb{R} \quad -\infty < x < \infty.$$

(*) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{aligned} &\forall x \in \overline{\mathbb{R}} \text{ with } x > 0 \ x \cdot (+\infty) = +\infty \ \ x \cdot (-\infty) = -\infty \\ &\forall x \in \overline{\mathbb{R}} \text{ with } x < 0, x \cdot (+\infty) = -\infty \ x \cdot (-\infty) = +\infty \\ &\forall x \in \mathbb{R}, \ x + \infty = +\infty \\ &\forall x \in \mathbb{R}, \ x - \infty = -\infty. \end{aligned}$$

As for the last convention, we define

$$+\infty + \infty = +\infty \quad -\infty - \infty = -\infty.$$

Furthermore, we have

$$\forall x \in \mathbb{R} \quad \frac{x}{+\infty} = 0 \quad \frac{x}{-\infty} = 0.$$

Please notice that we did <u>not</u> define the following

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \ldots, 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0.$$

(*) $\sup A = \infty \Longleftrightarrow$ either $\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and A is not bounded above in $\mathbb{R} \cup \{-\infty\}$.

(*) $\inf A = -\infty \Longleftrightarrow -\infty$ or $A \subseteq \mathbb{R} \cup \{\infty\}$ and A is not bounded below in $\mathbb{R} \cup \{\infty\}$.

(*) $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Remark. Let (a_n) bea sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

(i) $\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ |a_n - a| < \varepsilon$

(ii) $\lim_{n\to\infty} a_n = \infty \iff \forall M > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ a_n > M$

(iii) $\lim_{n\to\infty} a_n = -\infty \iff \forall M > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ -a_n > M$.

Theorem (Algebraic Limit Theorem for $\overline{\mathbb{R}}$). Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $c \in \mathbb{R}$, then $ca_n \to ca$

(ii) $a_n + b_n \to a + b$ (provided that $\infty - \infty$ does not appear)

(iii) $a_n b_n \to ab$ (provided that $(\pm\infty)\cdot 0$ or $0\cdot(\pm\infty)$ do not appear).

(iv) If $a = \infty$, then $\frac{1}{a_n} \to 0$. If $a = -\infty$, then $\frac{1}{a_n} \to 0$.

(v) If $a_n \to 0$ and $a_n > 0$, then $\frac{1}{a_n} \to \infty$. If $a_n \to 0$ and $a_n < 0$, then $\frac{1}{a_n} \to -\infty$.

Theorem (Order Limit Theorem for $\overline{\mathbb{R}}$). Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $a_n \le b_n$, then $a \le b$.

(ii) If $a_n \le e_n$ and $a_n \to \infty$, then $e_n \to \infty$.

(iii) If $e_n \le a_n$ and $a_n \to -\infty$, then $e_n \to -\infty$.

Theorem (Monotone Convergence Theorem in $\overline{\mathbb{R}}$). Let (a_n) be a sequence in $\overline{\mathbb{R}}$.

(i) If (a_n) is increasing, then $a_n \to \sup\{a_n : n \in \mathbb{N}\}$.

(ii) If (a_n) is decreasing, then $a_n \to \inf\{a_n : n \in \mathbb{N}\}$.

Note that if $a_n \to \infty$, then $\frac{1}{a_n} \to 0$, however, $\frac{1}{a_n} \to 0$ does not imply that $a_n \to \infty$. For example, consider $(a_n) = (-n)$ or $(a_n) = ((-1)^n n)$.

Remark. (i) $\overline{\mathbb{R}}$ can be equipped with the following metric:

Let $f : \overline{\mathbb{R}} \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ be defined by

$$f(x) = \begin{cases} \frac{-\pi}{2} & x = \infty \\ \arctan(x) & -\infty < x < \infty \\ \frac{\pi}{2} & x = \infty \end{cases}$$

Define $\overline{d}(x,y) = |f(x) - f(y)|\ \forall x, y \in \overline{\mathbb{R}}$. The closure of $\mathbb{R}$ in $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(ii) One can show that if (a_n) is a sequence in $\mathbb{R}$, then

$$a_n \to a \in \overline{\mathbb{R}} \iff \text{The sequence } a_n \text{ converges to } a \text{ in the metric space } (\overline{\mathbb{R}}, \overline{d}).$$

(iii) The closure of $\mathbb{R}$ in the metric space $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(iv) Every set in $(\overline{\mathbb{R}}, \overline{d})$ is bounded:

$$\forall x, y \in \overline{\mathbb{R}}\ \ \overline{d}(x,y) \le \pi.$$