Definition 0.0.1 (Metric Space). A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$ called the *distance* from p to q, such that

(a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;

(b) $d(p,q) = d(q,p)$;

(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

- Suppose a subset Y of X is a metric space, with the same distance function.
- Y must also be a metric space with the same distance function as X (metric).

Definition 0.0.2.

- By the *segment* (a,b) we mean the set of all $x \in \mathbb{R}$ such that $a < x < b$.
- We call an *interval* $[a,b]$ to mean the set of all $x \in \mathbb{R}$ such that $a \leq x \leq b$ for $a, b \in \mathbb{R}$ with $a < b$.
- We call a *half-open interval* either $[a,b)$ or $(a,b]$ to mean $a \leq x < b$ and $a < x \leq b$, respectively.
- We call a *k-cell* to mean that if $a_i < b_i$, for $i = 1, \dots, k$, the set of all points $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy $a_i \leq x_i \leq b_i (1 \leq i \leq k)$.
- If $x \in \mathbb{R}^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at x and radius r is defined to be the set
$$B(x,r) = \{y \in \mathbb{R}^k : |y - x| < r\}$$
or alternatively, $|y - x| \leq r$.
- We call a set $E \subset \mathbb{R}^k$ *convex* if
$$\lambda x + (1 - \lambda) y \in E$$
whenever $x, y \in E$, and $0 < \lambda < 1$.

Definition 0.0.3. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$ for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.

(d) E is closed if every limit point of E is a point of E.

(e) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \neq E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E.

(i) E is *bounded* if there exists a $M \in \mathbb{R}$ and $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(j) E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

Remark. In $\mathbb{R}^1$, neighborhoods are segments and in $\mathbb{R}^2$ neighborhoods are interiors of circles.

Theorem 0.0.1 (Neighborhoods are Open). Every neighborhood is an open set.

Proof. Let $p \in X$. Consider the neighborhood $N_r(p)$ for some $r > 0$. Let $y \in X$. Similarly, we can construct a neighborhood $N_h(y)$ for some $h > 0$. Observe that the distance between p and y is

$$d(p, y) = r - h.$$

Our goal is to show that $N_h(y) \subset N_r(p)$ in order for $N_r(p)$ to be open. Let $x \in N_h(y)$. Using the triangle inequality, we can see that

$$\begin{aligned} d(p, x) &\leq d(p, y) + d(y, x) \\ &< (r - h) + h \\ &= r. \end{aligned}$$

This tells us that $x \in N_r(p)$, proving that $N_r(p)$ is an open set. ■

Theorem 0.0.2. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof. Suppose for sake of contradiction that there exists a neighborhood N of p which contains only a finite number of points of E. Let $q_1, q_2, \dots, q_n$ be the points of $N \cap E$ such that $q_m \neq p$ for all m. Observe that

$$r = \min_{1 \leq m \leq n} d(p, q_m) > 0$$

since each $d(p, q_m) > 0$. Since each $d(p, q_m) < \delta_m$ and not $d(p, q_m) = \delta_m$, we have that none of the $q_m \in N_r(p)$ where $q_m \neq p$. So, p must not be a limit point of E which is a contradiction. Thus, every neighborhood of p must contain infinitely many points of E. ■

Corollary. A finite point set has no limit points.

Example 0.0.1 (Examples of Closed, Open, Perfect, Bounded Sets). (a) The set of all $z \in \mathbb{C}$ such that $|z| < 1$. **Open and Bounded**

(b) The set of all $z \in \mathbb{C}$ such that $|z| \leq 1$. **Closed, Perfect, Bounded**

(c) A nonempty finite set. **(Closed, Bounded)**

(d) The set of all integers. **(Closed)**

(e) The set
$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$
Note that no point of E is a limit point of E; that is, there are no limit points contained in E. **Bounded**

(f) The set of all complex numbers (that is, $\mathbb{R}^2$).

(g) The segment (a, b). **(Bounded)**

Note that (g) is not open in $\mathbb{R}^1$ but open in $\mathbb{R}^2$.

Theorem 0.0.3. Let $\{E_\alpha\}$ be a (either finite or infinite) collection of sets E_α. Then
$$\left(\bigcup_\alpha E_\alpha \right)^c = \bigcap_\alpha (E_\alpha^c).$$

Theorem 0.0.4. A set E is open if and only if its complement is closed.

Proof. For the forwards direction, let x be a limit point of E^c. Thus, for all neighborhoods $N(x)$, $N(x)$ contains a point $q \neq x$ such that $q \in E^c$. Since E is open, $N(e) \subset E$ for every $e \in E$. Since $q \notin E^c$, q must NOT be an interior point of E. Thus, $x \in E^c$, proving that E^c is closed.

For the backwards direction, suppose E^c is closed. Let $x \in E$. Since E^c is closed, all the limit points of E^c must be contained in E^c. But $x \notin E^c$, so x must not be a limit point of E^c. Thus, there exists a neighborhood $N(x)$ such that $N(x) \cap E^c$ is empty. Thus, x must be an interior point of E; that is, $N(x) \subset E$. Hence, E must be open. ■

Corollary. A set F is closed if and only if its complement is open.

Theorem 0.0.5. (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

c For any finite collection $G_1, G_2, \ldots, G_n$ of open sets $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{i=1}^n F_i$ is closed.

Proof. ■

Example 0.0.2.
- The finiteness found in part (c) and (d) of the preceding theorem is essential.
- Suppose $G_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$. We can see that G_n is an open subset of $\mathbb{R}$. But observe that
$$G = \bigcap_{n=1}^{\infty} (-1/n, 1/n)$$
only contains one point, namely 0, which implies that G is not an open subset of $\mathbb{R}$.

- The intersection of an infinite collection of open sets **need not** be open.
- The union of an infinite collection of closed sets need not be closed.

Definition 0.0.4 (Closure). If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\overline{E} = E \cup E'$.

Theorem 0.0.6. If X is a metric space and $E \subset X$, then

(a) $\overline{E}$ is closed,

(b) $E = \overline{E}$ if and only if E is closed.

(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Proof. ■

Definition 0.0.5 (Open Relative to Y). Suppose $E \subset Y \subset X$, where X is a metric space. To say that the set X is *open relative* to Y is to say that for each $p \in E$, there is an associated $r > 0$ such that $q \in E$ whenever

$$d(p, q) < r \text{ and } q \in Y.$$

Theorem 0.0.7. Suppose $Y \subset X$. A subset of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof. For the forwards direction, suppose E is open relative to Y. For each $p \in E$, there exists a positive number r_p such that $q \in E$ whenever

$$d(p, q) < r_p \text{ and } q \in Y.$$

Let V_p be the set

$$\{q \in Y : d(p, q) < r_p\}$$

by definition. Since each V_p is just a neighborhood, we know that each V_p has to be an open subset of X. Thus, we can set

$$G = \bigcup_{p \in E} V_p$$

which implies G is open by part (a) of Theorem 2.24. Since $p \in V_p$ for all $p \in E$ and $p \in Y$ (since $E \subset Y$), we have $E \subset Y \cap G$. (**This is using the result from set theory that states $E \subset Y$ and $E \subset V_p$ implies $E \subset Y \cap V_p$.**) With our choice of V_p, we can see that $Y \cap V_p \subset E$. Since each $V_p \subset G$, we see that $Y \cap G \subset E$, which shows that $E = Y \cap G$.

For the backwards direction, suppose $E = Y \cap G$ for some open subset of G of X. Since G is open, for every $p \in E$, we can construct a neighborhood V_p such that $V_p \subset G$. So, $V_p \cap Y \subset E$. Thus, E is open relative to Y. ■