1 Topics

(1) Topological Continuity

(2) Continuity and compactness

(3) Continuity and connectedness

(4) Continuity and the inverse

(5) Uniform Continuity

So far we have learnt two equivalent descriptions of the concept of continuity for functions $f : (X, d) \to (Y, \tilde{d})$:

(1) f is continuous if and only if

$$\forall c \in X\ \forall \varepsilon > 0\ \exists \delta_{\varepsilon,c} > 0 \text{ such that if } d(x, c) < \delta_{\varepsilon,c} \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

(2) f is continuous if and only if $\forall c \in X$, $a_n \to c$ implies $f(a_n) \to f(c)$.

In undergraduate analysis, most of the sequences we study are sequences in $\mathbb{R}$. In terms of convergence, we say that

$$a_n \to c \iff \forall\ \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N \quad |a_n - c| < \varepsilon.$$

For graduate analysis, convergence can be viewed more generally; that is, we have

$$a_n \to c \iff \forall N_\varepsilon(c)\ \exists N \in \mathbb{N} \text{ such that} \forall n > N\ a_n \in N_\varepsilon(c).$$

Theorem (Topological Characterization Continuity). Let (X, d) and $(Y, \tilde{d})$ be metric spaces and let $f : X \to Y$. The following statements are equivalent:

(i) f is continuous

(ii) For every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X.

Proof. ($(i) \implies (ii)$) Suppose f is continuous. Our goal is to show that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X. Let B be an open set in Y. Our goal is to show that $f^{-1}(B)$ is open in X. That is, we want to show that every point of $f^{-1}(B)$ is an interior point. Let $p \in f^{-1}(B)$. We need to find an $\delta > 0$ such that $N^X_\delta(p) \subseteq f^{-1}(B)$.

Since f is continuous at p, there exists $\hat{\delta} > 0$ such that

$$\forall x \in N^X_{\hat{\delta}}(p) \quad f(x) \in N^Y_\varepsilon(f(p)) \subseteq B.$$

Clearly, we have $N^X_{\hat{\delta}}(p) \subseteq f^{-1}(B)$, so we can use this $\hat{\delta}$ as the δ we were looking for.

($(ii) \implies (i)$) Assume that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X. Our goal is to show that f is continuous. We need to show that f is continuous at every point of X. Let $c \in X$. We will prove that f is continuous at c. That is,

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if} x \in N^X_\delta(c) \text{ then } f(x) \in N^Y_\varepsilon(f(c)).$$

Let $\varepsilon > 0$ be given. We need to show that there exists $\delta > 0$ such that

$$N^X_\delta(c) \subseteq f^{-1}(N^Y_\varepsilon(f(c))). \tag{*}$$

Since $N^Y_\varepsilon(f(c))$ is open in Y, it follows from the assumption that $f^{-1}(N^Y_\varepsilon(f(c)))$ is open in X. Since $f^{-1}(N^Y_\varepsilon(f(c)))$ is open in X and $c \in f^{-1}(N^Y_\varepsilon(f(c)))$, we have c is an interior point of $f^{-1}(N^Y_\varepsilon(fc))$. Hence, there exists $\delta > 0$ such that $N^X_\delta(c) \subseteq f^{-1}(N^Y_\varepsilon(f(c)))$ as desired. ■

Note that continuous functions does NOT necessarily preserve the following properties:

- open

- closed
- bounded

Some examples of maps that do not preserve the properties above are

(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Note that $E = (-2, 2)$ is open but $f(E) = [0, 4)$ is not.

(2) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$. Note that $E = [0, \infty)$ is closed but $f(E) = (0, 1]$ is not.

(3) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Note that $E = (0, 1)$ is a bounded set but $f(E) = (1, \infty)$ is not.

Theorem. Let (X, d) and $(Y, \tilde{d})$ are metric spaces and let $f : X \to Y$ be continuous and let E be a compact set in X. Then $f(E)$ is compact in Y.

Proof. Let $\{O_\alpha\}_{\alpha\in I}$ be an open cover of $F(E)$. Our goal is to show that this open cover has a finite subcover. We have

$$f(E) \subseteq \bigcup_{\alpha\in I} O_\alpha.$$

So,

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\Big(\bigcup_{\alpha\in I} O_\alpha\Big) = \bigcup_{\alpha\in E} f^{-1}(O_\alpha).$$

Since $E \subseteq f^{-1}(f(E))$ and $f^{-1}\Big(\bigcup_{\alpha\in I} O_\alpha\Big) = \bigcup_{\alpha\in I} f^{-1}(O_\alpha)$, we can conclude that

$$E \subseteq \bigcup_{\alpha\in I} f^{-1}(O_\alpha).$$

Now, we have $f : X \to Y$ is continuous and for each $\alpha \in I$, O_α is open in Y implies that for all $\alpha \in I$, we have $f^{-1}(O_\alpha)$ is open in X. Thus, $\{f^{-1}(O_\alpha)\}_{\alpha\in I}$ is an open cover for E. Since E is compact, we have

$$\exists \alpha_1, \dots, \alpha_n \in I \text{ such that } E \subseteq \bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i}).$$

Consequently, we have

$$\begin{aligned} f(E) &\subseteq f\Big(\bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i})\Big) \\ &= \bigcup_{i=1}^{n} f(f^{-1}(O_{\alpha_i})) \\ &\subseteq \bigcup_{i=1}^{n} O_{\alpha_i}. \end{aligned}$$

Thus, we have $\{O_{\alpha_i}\}_{1\le i\le n}$ is a finite subcover for $f(E)$. ∎

Theorem (Extreme Value Theorem). Let (X, d) be a compact metric space.

(i) If $f : (X, d) \to (Y, \tilde{d})$ is continuous, then $f(X)$ is a closed and bounded set in Y.

(ii) If $f : (X, d) \to \mathbb{R}$ is continuous, then f attains a maximum values and a minimum value. More precisely, $M = \sup_{x\in X} f(x)$

and $m = \inf_{x\in X} f(x)$ exists, and there exists points $a \in X$ and $b \in X$ such that $f(a) = M$ and $f(b) = m$.

Proof. (i) By the previous theorem, we can see that $f(X)$ must be compact in Y. As we know, every compact set in any metric space is closed and bounded.

(ii) By part (i), $f(X)$ is closed and bounded subset of $\mathbb{R}$. Since $f(X)$ is a bounded set in $\mathbb{R}$, $M = \sup f(X) = \sup_{x\in X} f(x)$ and $m = \inf f(X) = \inf_{x\in X} f(x)$ exists. Recall from Theorem 2.28 that $M \in \overline{F(X)}$ and $m \in \overline{f(X)}$. Since $\overline{f(X)} = f(X)$, we can conclude that $M \in f(X)$ and $m \in f(X)$. That is, there exists $a \in X$ such that $f(a) = M$ and there exists $b \in X$ such that $f(b) = m$.

■

Theorem (Preservation of Connectedness). Let (X, d) and $(Y, \tilde{d})$ be metric spaces and $f : X \to Y$ be a continuous map, and let $E \subseteq X$ be a connected set. Then $f(E)$ is connected in Y.

Proof. Assume for contradiction that $f(E)$ is NOT connected. Thus, we can write $f(E)$ as a union of two (nonempty) separated sets A and B:

$$f(E) = A \cup B,\ \overline{A} \cap B = \emptyset,\ A \cap \overline{B} = \emptyset.$$

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. In what follows, we will show that G and H form a separation for the set E, which contradicts the assumption that E is connected. We will show that

(1) G and H are both nonempty

(2) $\overline{G} \cap H = \emptyset$ (and similarly, $G \cap \overline{H} = \emptyset$)

(3) $E = G \cup H$

To show (1), we will prove that $f(G) = A$ (similarly, show that $f(H) = B$). We have

(1)

$$\begin{aligned} f(G) = f(E \cap f^{-1}(A)) &\subseteq f(E) \cap f(f^{-1}(A)) \\ &\subseteq f(E) \cap A && (f(E) = A \cup B) \\ &= A. \end{aligned}$$

Then we have $f(G) \subseteq A$. Now, we will show that $A \subseteq f(G)$. Let $y \in A$. Then $y \in f(E)$ and so there exists

$$x \in E \text{ such that } f(x) = y. \tag{*}$$

Then we have

$$f(x) = y \in A \Longrightarrow x \in f^{-1}(A). \tag{**}$$

Then (*) and (**) imply that

$$\begin{aligned} x \in E \cap f^{-1}(A) &\Longrightarrow f(x) \in f(E \cap f^{-1}(A)) = f(G) \\ &\Longrightarrow y \in f(G). \end{aligned}$$

Thus, $A \subseteq f(G)$.

(2) Observe that

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap [f^{-1}(A) \cap f^{-1}(B)] \\ &= E \cap [f^{-1}(A \cup B)] \\ &= E \cap [f^{-1}(f(E))] && (f(E) = A \cup B) \\ &= E && (E \cap f^{-1}(f(E))) \end{aligned}$$

(3) To this end, it is enough to show that $f(\overline{G}) \cap f(H) = \emptyset$. Note that $f(H) = B$. So, our goal is to show that $f(\overline{G}) \cap B = \emptyset$. Since $\overline{A} \cap B$ is empty, and so it suffices to show that $f(\overline{G}) \subseteq \overline{A}$. Notice that

$$G = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}).$$

Since f is continuous and $\overline{A}$ is a closed set in Y, we can see that $f^{-1}(\overline{A})$ is closed in X. Thus, we can write

$$G \subseteq f^{-1}(\overline{A}) \Longrightarrow \overline{G} \subseteq \overline{f^{-1}(\overline{A})} = f^{-1}(\overline{A}).$$

Therefore, we have

$$f(\overline{G}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}.$$

A similar argument shows that $f(\overline{H}) \cap f(G) = \emptyset$.

■

Theorem (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous map and let $f(a) \neq f(b)$. Let $L \in \mathbb{R}$ be such that $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = L$.

Proof. Since $f : [a, b] \to \mathbb{R}$ is continuous and $[a, b]$ is connected. Then, by the preservation of connectedness, we have $f([a, b])$ is connected in $\mathbb{R}$. This implies that $f([a, b])$ is either a singleton or an interval I in $\mathbb{R}$. Since $f(a) \neq f(b)$, we know that $f([a, b])$ cannot be a singleton. That is, $f([a, b])$ is an interval I in $\mathbb{R}$. Note that $f(a), f(b) \in I$ and L is between $f(a)$ and $f(b)$. So, $L \in f([a, b])$. Thus, there exists $c \in [a, b]$ such that $f(c) = L$. But neither $f(a) \neq L$ nor $f(b) \neq L$, so there exists $c \in (a, b)$ such that $f(c) = L$. ■

Note that if $f : X \to Y$ is continuous and bijective =, it is NOT necessarily true that $f^{-1} : Y \to X$ is continuous.

Example. $f : (-1, 0] \cup [1, 2] \to [0, 4]$ given by $f(x) = x^2$ is continuous and bijective. However, $f^{-1} : [0, 4] \to (-1, 0] \cup [1, 2]$ is NOT continuous. Indeed, we see that $[0, 4]$ is connected, but $f^{-1}([0, 4]) = (-1, 0] \cup [1, 2]$ is NOT.

Theorem (Continuous and Bijective maps on Compact Sets). Let (X, d) and $(Y, \tilde{d})$ are metric spaces, X is compact, and $f : X \to Y$ is continuous and bijective. Then $f^{-1} : Y \to X$ is continuous.

Proof. It suffices to show that for every open set $B \subseteq X$ $B \subseteq X$, $(f^{-1})^{-1}(B)$ is open in Y. That is, we will show that $f(B)$ is open in Y. Let B be an open set in X. Thus, B^c is closed in X. Since X is a compact set and $B^c \subseteq X$, B^c is compact in X. Since continuity preserves compactness, we have $f(B^c)$ is compact in Y. Since compact sets are closed and bounded, $f(B^c)$ is closed in Y. Now, $[f(B^c)]^c$ is open in Y. Since $[f(B^c)]^c$ is open in Y. Since f is bijective, we have

$$f(B^c) = [f(B)]^c.$$

Thus,

$$f(B) = [(f(B))^c]^c = [f(B^c)]^c$$

is open in Y. ■

Remark (Invariance of Domain). Let $U \subseteq \mathbb{R}^n$ is open and $V \subseteq \mathbb{R}$. Since $f : U \to V$ is continuous and bijective, then V is open and $f^{-1} : V \to U$ is continuous.

2 Uniform Continuity

Let A be a proper subset of (X, d).

Consider the following three questions:

(1) Given a continuous function $f : A \to \mathbb{R}$, is it possible to extend f to a continuous function that is defined on all of X.

(2) Under what conditions can we be certain that a continuous function $f : A \to \mathbb{R}$ has a continuous

extension to all of X.

Theorem (A special case of Tietze Extension Theorem)**.** Let (X, d) be a metric space. Let A be a nonempty closed set in X. If $f : A \to \mathbb{R}$ is continuous, then f has a continuous extension to all of X.

Theorem. Let (X, d) be a metric space. Let A be a nonempty set in X. If $f : A \to \mathbb{R}$ is **uniformly continuous on** A, then f can be extended to a continuous function $\overline{f} : \overline{A} \to \mathbb{R}$.

Recall that we say that $f : A \subseteq (X, d) \to (Y, \tilde{d})$ is continuous at $c \in A$:

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x, c) < \delta \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

We say that $f : A \subseteq (X, d) \to (Y, \tilde{d})$ is continuous on A, we have

$$\forall c \in A\ \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x, c) < \delta_{\varepsilon,c} \text{ then } d(f(\tilde{x)}, f(c)) < \varepsilon.$$

In general, the δ in the above statement depends on both ε and c. If the function the desirable property that given an arbitrary $\varepsilon > 0$, one can find a single δ_ε that holds (or stays constant) for all points $c \in A$, then we say that f is uniformly continuous on A.

Definition (Uniform Continuity)**.** Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$ be a function. We say that f is **uniformly continuous on** A if

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 \text{ such that } \forall x, y \in A \text{ if } d(x, c) < \delta_\varepsilon, \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

Remark. We say that f is NOT uniformly continuous on A if

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0\ \exists x, y \in A \text{ satisfying } d(x, y) < \delta \text{ but } \tilde{d}(f(x), f(y)).$$

Clearly, from the above, we can see that uniform continuity is a stronger property than continuity in the sense that if $f : A \to Y$ is uniformly continuous, then $f : A \to Y$ is continuous. Note that it is not really productive to talk about uniform continuity at a specific point. Uniform continuity is always discussed in the context to a particular domain. For example, we'll see that $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$. However, $f(x) = x^2$ is uniformly continuous on $[0, 1]$. This implies determining whether a function is uniform continuous is a matter of analyzing the domain on which the function is defined on.

Example. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is uniformly continuous on $\mathbb{R}$.

Our goal is to show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \in \mathbb{R} \text{ if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Clearly, we can take $\delta = \frac{\varepsilon}{2}$ (or any positive number less than $\frac{\varepsilon}{2}$).

The definition of uniform continuity directly implies that if some function $f : A \to Y$ is uniformly continuous on A, then it is also uniformly continuous on $B \subseteq A$.

Note that tells us that not all continuous functions are uniformly continuous. The following theorem gives a simple criterion for proving the absence of uniform continuity.

Theorem. Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$. If we can find $\varepsilon_0 > 0$ and two sequences (x_n) and (c_n) in A such that

$$d(x_n, c_n) \to 0 \text{ and } \forall n \tilde{d}(f(x_n), f(c_n)) \geq \varepsilon_0.$$

Then f is NOT uniformly continuous on A.

Proof. Recall that f is NOT uniformly continuous if and only if there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ satisfying $d(x, c) < \delta$ but $\tilde{d}(f(x), f(c)) \geq \varepsilon$. If this holds, then we can set $\varepsilon = \varepsilon_0$ such that for any $\delta > 0$, there exists N such that $d(x_N, c_N) < \delta$, but $\tilde{d}(f(x_n), f(c_n)) \geq \varepsilon$. ■

Example. Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. Let $x_n = n$ and $c_n = n + \frac{1}{n}$. We have

$$\lim_{n\to\infty} |x_n - c_n| = \lim_{n\to\infty} \left|\frac{-1}{n}\right| = 0.$$

Also, for all n, we have

$$\begin{aligned} |f(x_n) - f(c_n)| &= \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| \\ &= \left|n^2 - \left(n^2 + 2 + \frac{1}{n^2}\right)\right| \\ &= \left|-\left(2 + \frac{1}{n^2}\right)\right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2. \end{aligned}$$

Hence, we conclude that $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$. ■

Example. Prove that $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Solution. Use $x = \frac{1}{2n\pi}$ and $c_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Now, observe that $\lim x_n = 0$ and $\lim c_n = 0$ implies $\lim(x_n - c_n) = 0$. Thus, $\lim |x_n - c_n| = 0$. But for all n, we have

$$|f(x_n) - f(c_n)| = |\sin(2n\pi) - \sin\left(2n\pi + \frac{\pi}{2}\right)| = |0 - 1| = 1.$$

So, f is NOT uniformly continuous. ■

Theorem (Continuous Functions on Compact Sets are Uniformly Continuous)**.** Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$ is continuous and let A be a compact set. Then f is uniformly continuous on A.

Proof. Assume for sake of contradiction that f is NOT uniformly continuous; that is, there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \; \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \tilde{d}(f(x), f(c)) \geq \varepsilon.$$

In particular,

$$\begin{aligned} &\text{For}\delta = 1 \exists x_1, c_1 \in A \text{ satisfying } d(x_1, c_1) < 1 \text{ but } \tilde{d}(f(x_1), f(c_1)) \geq \varepsilon \\ &\text{For} \exists x_2, c_2 \in A \text{ satisfying } d(x_2, c_2) < \frac{1}{2} \text{ but } \tilde{d}(f(x_2), f(c_2)) \geq \varepsilon \\ &\text{For} \exists x_3, c_3 \in A \text{ satisfying } d(x_3, c_3) < \frac{1}{3} \text{ but } \tilde{d}(f(x_3), f(c_3)) \geq \varepsilon \\ &\vdots \end{aligned}$$

In this way, we will obtain two sequences (x_n) and (c_n) in A such that

(i) $0 \leq d(x_n, c_n) < \frac{1}{n}$ for all n. This implies that $\lim_{n\to\infty} d(x_n, c_n) = 0$.

(ii) $\tilde{d}(f(x_n), f(c_n)) \geq \varepsilon$ for all n.

Note that A is compact, so it is sequentially compact. If (x_n) is a sequence in A, then (x_n) contains a subsequence (x_{n_k}) that converges to a point in A. Let $x = \lim_{k\to\infty} x_{n_k}$. Let (c_{n_k}) be the corresponding subsequence of (c_n). We have

$$0 \leq d(c_{n_k}, x) \leq d(c_{n_k}, x_{n_k}) + d(x_{n_k}, x).$$

If we let $k \to \infty$ on the right-hand side of the above inequality, we have $d(c_{n_k}, x_{n_k}) \to 0$ and $d(x_{n_k}, x) \to 0$. Using the Squeeze Theorem, we have $\lim_{k\to\infty} c_{n_k} = x$. Therefore, (x_{n_k}) and (c_{n_k}) are two sequences in A that converge to $x \in A$. Since f is continuous and $x_{n_k} \to x$, we have

$$f(x_{n_k}) \to f(x). \tag{1}$$

Similarly, if $c_{n_k} \to x$, we have

$$f(c_{n_k}) \to f(x). \tag{2}$$

So, there exists $N_0 \in \mathbb{N}$ such that for all $k > N_0$

$$\forall k > N_0 \quad \tilde{d}(f(x_{n_k}), f(x)) < \frac{\varepsilon}{4} \text{ and } \tilde{d}(f(c_{n_k}), f(x)) < \frac{\varepsilon}{4}.$$

As a consequence, for all $k > N_0$, we have

$$\begin{aligned} \tilde{d}(f(x_{n_k}), f(c_{n_k})) &\leq \tilde{d}(f(x_{n_k}), f(x)) + \tilde{d}(f(x), f(c_{n_k})) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

But this contradicts (ii). ■