Homework 7

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Problem 1. Let $(V, \|\cdot\|_V)$ be a finite dimensional normed space.

(i) Let W be a subspace of V such that $\dim(W) < \dim(V)$. Let $f : W \to \mathbb{F}$ be a linear functional.

(ii) Let $v \in V \setminus \{0\}$. Show that there is $f \in V^*$ such that $f(v) = 1$ and $\|f\| = \|v\|$.

Proof. *(i)* Since V is finite dimensional and f is a linear functional, it follows that f is also bounded. That is, f is a continuous linear functional. Hence, for any $v_n \to v$, we have $f(v_n) \to f(v)$. So, define a functional $\tilde{f} : V \to \mathbb{F}$ by

$$\tilde{f}(v) = \lim_{n\to\infty} f(v_n).$$

It follows from the algebraic properties of the limit and the linearity of f that $\tilde{f}$ is also linear. Indeed, for any $x, y \in V$ such that $x_n \to x$ and $y_n \to y$ with $\alpha \in \mathbb{F}$, we have

$$\begin{aligned}
\tilde{f}(x + \alpha y) &= \lim_{n\to\infty} f(x_n + \alpha y_n) \\
&= \lim_{n\to\infty} [f(x_n) + \alpha f(y_n)] && (f \text{ is linear}) \\
&= \lim_{n\to\infty} f(x_n) + \alpha \lim_{n\to\infty} f(y_n) && (\text{Algebraic Limit Theorem}) \\
&= \tilde{f}(x) + \alpha \tilde{x}(y).
\end{aligned}$$

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Proof. *(ii)* Using the linear functional f we constructed from part (i), it follows from its linearity that for every $v \in V \setminus \{0\}$, we have

$$\begin{aligned}
\left\| f\Big(\frac{v}{\|f(v)\|}\Big) \right\| &= \left\| f\Big(\frac{v}{\|f(v)\|}\Big) \right\| \\
&= \left\| \frac{1}{\|f(v)\|} f(v) \right\| \\
&= \frac{1}{\|f(v)\|} \cdot \|f(v)\| \\
&= 1.
\end{aligned}$$

So, define $\hat{f} : V \to \mathbb{F}$ by

$$\hat{f}(v) = \left\| f\Big(\frac{v}{\|f(v)\|}\Big) \right\| = 1.$$

Now, our goal is to show that $\|\hat{f}\| = \|v\|$. Clearly, we see that $\hat{f}$ is bounded and so

$$\|\hat{f}(v)\| \leq \|\hat{f}\| \|v\| = 1 \cdot \|v\| = \|v\|.$$

Taking the supremum of the left-hand side above over all v such that $\|v\| = 1$, we have

$$\|\hat{f}\| \leq \|v\|.$$

Now, we want to show that $\|\hat{f}\| \geq \|v\|$. Indeed, we have

$$\begin{aligned}\|\hat{f}\| = \sup_{\|v\|=1} \|\hat{f}(v)\| &\geq \|\hat{f}(v)\| \\ &= \Big\| \|v\| \hat{f}(v) \Big\| \\ &= \|v\| \|\hat{f}(v)\| \\ &= \|v\|.\end{aligned}$$

Hence, we conclude that $\|\hat{f}\| = \|v\|$. ■

Problem 2. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Let $B(V, W) = \{T : V \to W : T \text{ is bounded and linear}\}$. We proved that $B(V, W)$ is a normed space. Assume that $B(V, W)$ is Banach. The goal of this exercise is to show $B(V, W)$ is a Banach space.

(i) Let (T_n) be a Cauchy sequence in $B(V, W)$ is a Banach Space.
Proof. Suppose (T_n) is a Cauchy sequence in $B(V, W)$. That is, for $n, m \to \infty$, we have

$$\|T_n - T_m\| \to 0.$$

Our goal is to show that $(T_n v)$ is convergent in W. It suffices to show that Cauchy in W since W is a Banach normed space; that is, we need to show that

$$\|T_n v - T_m v\|_W \to 0$$

as $n, m \to \infty$. Note that for all $n \in \mathbb{N}$, T_n is bounded since (T_n) is a sequence in $B(V, W)$. Hence, we have

$$0 \leq \|(T_n - T_m)(v)\|_W \leq \|T_n - T_m\| \|v\|_V \to 0$$

as $n, m \to \infty$. Hence, we have that $\|(T_n - T_m)(v)\|_W \to 0$ as $n, m \to \infty$. Thus, $(T_n v)$ is Cauchy in W which is our desired result. ■

(ii) Use (i) to define $T : V \to W$ and show that T is linear.
Proof. Define $T : V \to W$ by

$$T(v) = \lim_{n\to\infty} T_n(v).$$

We will show that T is linear. Indeed, we see that for any $x, y \in V$ and $\alpha \in \mathbb{F}$, we have

$$\begin{aligned}T(x + \alpha y) &= \lim_{n\to\infty} T_n(x + \alpha y) \\ &= \lim_{n\to\infty} [T_n(x) + \alpha T_n(y)] \\ &= \lim_{n\to\infty} T_n(x) + \alpha \lim_{n\to\infty} T_n(y) \\ &= T(x) + \alpha T(y).\end{aligned}$$

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(iii) Prove that for large n, $T_n - T \in B(V, W)$ and $\|T_n - T\| \to 0$ as $n \to \infty$.
Proof. Note that $T_n - T_m \in B(V, W)$. Using the continuity of the norm and that $\|T_n - T_m\| \to 0$, we have

$$\begin{aligned}\|(T_n - T)v\|_W &= \|T_n v - \lim_{m\to\infty} T_m v\|_W \\ &= \lim_{m\to\infty} \|(T_n - T_m)v\|_W \\ &\leq \Big(\lim_{m\to\infty} \|T_n - T_m\|\Big) \cdot \|v\|_V.\end{aligned}$$

Dividing by $\|v\|_V$ (assuming that $v \neq 0$) and taking the supremum of the left-hand side of the inequality,

$$\|T_n - T\| \leq \lim_{m\to\infty} \|T_n - T_m\| \to 0.$$

Hence, $\|T_n - T\| \to 0$. Using the fact that $\|\cdot\|$ is continuous and $T_m(v) \to T(v)$, it follows that for some $R > 0$

$$\begin{aligned} &\|(T_n - T_m)v\|_W \leq R\|v\|_V \\ \implies &\|(T_n - T)v\|_W \leq R\|v\|_V. \end{aligned}$$

for all $v \in V$. Hence, $T_n - T \in B(V,W)$.

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(iv) Prove that $T \in B(V,W)$ and conclude that $B(V,W)$ is complete.
Proof. Since T_n for all $n \in \mathbb{N}$ is bounded, we can find $C_1 > 0$ such that

$$\|T_n v\|_W \leq C_1\|v\|_V \tag{1}$$

Moreover, $T - T_n$ is bounded from part (iii), so we can find a $C_2 > 0$ such that

$$\|(T - T_n)v\|_W \leq C_2\|v\|_V.$$

Using (1) and (2), we have

$$\begin{aligned} \|T(v)\|_W &= \|(T - T_n + T_n)v\|_W \\ &= \|(T - T_n)v + T_n v\|_W \\ &\leq \|(T - T_n)v\|_W + \|T_n v\|_W \\ &\leq C_2\|v\|_V + C_1\|v\|_V \\ &= (C_1 + C_2)\|v\|_V \end{aligned}$$

where $C = C_1 + C_2$ is our desired constant. ■

Problem 3. Let $p > 1$. Prove that $((\ell^r)', \|\cdot\|)$ is isomorphic to $(\ell^q, \|\cdot\|_q)$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the Schauder basis for ℓ^p is $(e^{(k)})$ where $e^{(k)} = (\delta_{kj})$. Since $x \in \ell^p$, we have

$$x = \sum_{k=1}^{\infty} x_k e^{(k)}.$$

Let $f \in (\ell^p)'$. Since f is linear and bounded, we have

$$f(x) = \sum_{k=1}^{\infty} x_k \alpha_k \tag{1}$$

where $\alpha_k = f(e^{(k)})$. Since $\frac{1}{p} + \frac{1}{q} = 1$ and define

$$x_k^{(n)} = \begin{cases} |\alpha_k|^1/\alpha_k & \text{if } k \leq n \text{ and } \alpha_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \alpha_k = 0 \end{cases}$$

Using (1) and our expression above, we have

$$f(x_n) = \sum_{k=1}^{\infty} x_k^{(n)} \alpha_k = \sum_{k=1}^{n} |\alpha_k|^q.$$

Furthermore, since $(q-1)p = q$, we have

$$\begin{aligned} f(x_n) \leq \|f\|\|x_n\| &= \|f\|\Big(\sum_{k=1}^{n} |x_k^{(n)}|^p\Big)^{1/p} \\ &= \|f\|\Big(\sum_{k=1}^{n} |\alpha_k|^{(q-1)p}\Big)^{1/p} \\ &= \|f\|\Big(\sum_{k=1}^{n} |\alpha_k|^q\Big)^{1/p}. \end{aligned}$$

Hence, we have

$$f(x_n) = \sum_{k=1}^{n} |\alpha_k|^q \leq \|f\| \Big(\sum_{k=1}^{n} |\alpha_k|^q \Big)^{1/q} \leq \|f\|.$$

Letting $n \to \infty$, we obtain

$$\Big(\sum_{k=1}^{n} |\alpha_k|^{1/q} \Big) \leq \|f\|. \tag{*}$$

Thus, $(\alpha_k) \in \ell^q$.

To obtain the other inequality, observe that for any $b = (\beta_k) \in \ell^q$, we associate a bounded linear functional g on ℓ^p. Hence, we may define g on ℓ^p by setting

$$g(x) = \sum_{k=1}^{\infty} x_k \beta_k$$

where $x = (x_k)$ in ℓ^p. Since g is linear and bounded (which follows from applying the Holder's Inequality on (1)), it follows that $g \in (\ell^p)'$. From (1) and the Holder's Inequality, we have

$$\begin{aligned} |f(x)| = \Big| \sum_{k=1}^{\infty} x_k \alpha_k \Big| &\leq \Big(\sum_{k=1}^{\infty} |\alpha_k|^p \Big)^{1/p} \Big(\sum_{k=1}^{\infty} |\alpha_k| \Big)^{1/q} \\ &= \|x\| \Big(\sum_{k=1}^{\infty} |\alpha_k|^q \Big)^{1/q} \end{aligned}$$

and so by taking the supremum over all x such that $\|x\| = 1$, we have

$$\|f\| \leq \Big(\sum_{k=1}^{\infty} |\alpha_k|^q \Big)^{1/q}. \tag{**}$$

Using (*) and (**), we can conclude that

$$\|f\| = \|\alpha\|_q$$

where $\alpha = (\alpha_k)$ and $\alpha_k = f(e^{(k)})$. Moreover, the mapping of $(\ell^p)'$ onto ℓ^q defined by $f \mapsto \alpha$ is linear and bijective, so it follows that ℓ^q and $(\ell^p)'$ are isomorphic. ■

Problem 4. Let V be a vector space.

(i) Let β be a basis for V. Show that for each $b \in \beta$ there is $f_b \in V^*$ such that $f_b(b) = 1$.
Proof. We will consider two different cases. Suppose V is finite dimensional and so denote $\dim(V) = n$. Then β is finite and denote $\beta = \{x_1, x_2, \ldots, x_n\}$. Since β is a basis, we know that for any $x \in V$

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

for some $\alpha_i \in \mathbb{F}$ for all $1 \leq i \leq n$. Note that if we are trying to represent an x_i in β, we have

$$x_i = \sum_{i=1}^{n} \alpha_i x_i = 0 \cdot x_1 + \cdots + 1 \cdot x_i + \cdots + 0 \cdot x_n.$$

Hence, for each $1 \leq i \leq n$ we can define a map $f : V \to \mathbb{F}$ such that $f_{x_i}(x_i) = 1$ for each $1 \leq i \leq n$. More generally, for any $i \in \mathbb{N}$ where $x_i \in \beta$, we have $f_{x_i}(x_i) = 1$. ■

(ii) Let $v \in V \setminus \{0\}$. Show that there is $f \in V^*$ such that $f(v) \neq 0$.
Proof. Let $v \in V \setminus \{0\}$. Since $v = 1 \cdot v$, we can define a functional $f : V \to \mathbb{F}$, we have $f(v) = 1 \neq 0$. Clearly, this is linear. ■

(iii) Use (ii) to prove the Canonical map $C : V \to V^{**}$ is injective.

Proof. Suppose $v \neq u$ where $v, u \in V$. Our goal is to show that $C(v) \neq C(u)$. By assumption, we have

$$v - u \neq 0 \implies \exists f \in V^* \text{ such that } f(v-u) \neq 0.$$

From the linearity of f, we have $f(v) - f(u) \neq 0$ implies $f(v) \neq f(u)$. But this tells us that $C(v) \neq C(u)$. ■

Problem 5. Let V be an infinite dimensional normed space. Note that V' is a subspace of V^*. Prove that there is $f \in V^*$ such that $f \notin V'$.

Proof. Since V is infinite dimensional, we can find a sequence of vectors v_n in V such that

(i) $\|v_n\| = 1$

(ii) $\|v_n - v_m\| \geq \frac{1}{2}$ for all $n \neq m$.

Define $T(v_n) = \|v_n\|$ which is linear. From the above properties, we can find a subsequence (v_{n_k}) such that $T(v_{n_k})$ does not converge which means that T is an unbounded linear operator. Hence, $T \notin V'$. ■

Problem 6. Let $(V, \|\cdot\|)$ be an infinite dimensional normed space.

(i) Assume that $(V, \|\cdot\|)$ is Banach. Let (v_n) be a sequence in V. Assume that $\sum_{n=1}^{\infty} \|v_n\|$ converges in $\mathbb{R}$. Prove that $\sum_{n=1}^{\infty} v_n$ converges in V.
Proof. Let $n > m$. Assume that $\sum_{n=1}^{\infty} \|v_n\|$ converges in V. Our goal is to show that $\sum_{n=1}^{\infty} v_n$ converges in V. Since V is a Banach space, it suffices to show that the sequence of partial sums $s_n = \sum_{k=1}^{n} v_k$ is Cauchy in V. Denote the sequence of partial sums (ξ_n) by

$$\xi_n = \sum_{k=1}^{n} \|v_k\|.$$

Since (ξ_n) converges by assumption, it follows that (ξ_n) is a Cauchy sequence in V. Hence,

$$|\xi_n - \xi_m| = \left| \sum_{k=m+1}^{n} \|v_k\| \right| \to 0 \text{ as } n, m \to \infty.$$

Now, we can see from the triangle inequality property of $\|\cdot\|$ that

$$0 \leq \|s_n - s_m\| \leq \left\| \sum_{k=m+1}^{n} v_k \right\| \leq \sum_{k=m+1}^{n} \|v_k\| = |\xi_n - \xi_m| \to 0 \text{ as } n, m \to \infty.$$

Hence, we can see that (s_n) is a Cauchy sequence in V which is our desired result. ■

(ii) Let $V \subseteq \ell'$ that consists of sequences $x = (x_n)$ such that $x_n = 0$ for all $n \geq N$ for some N; that is, V consists of all sequences for which all terms are zero after some Nth term. Define $y^{(n)} \in \ell^{\infty}$ by setting

$$y_j^{(n)} = \begin{cases} \frac{1}{2^n} & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}.$$

That is, the nth term of $y^{(n)}$ is $\frac{1}{2^n}$ and all other terms are zero. Then $y^{(n)} \in V$ for all n.

Show that

(a) $\sum_{n=1}^{\infty} \|y^{(n)}\| < \infty$.
Proof. Let $n > m$. Define the sequence of partial sums $(\xi^{(n)})$ by

$$\xi^{(n)} = \sum_{k=1}^{n} \|y^{(k)}\|.$$

Since $\mathbb{R}$ is complete, it suffices to show that $\xi^{(n)}$ is a Cauchy sequence in $\mathbb{R}$. This will show that the infinite series of the terms $\|y^{(n)}\|$ converges. Note that

$$\begin{aligned} 0 \le |\xi^{(n)} - \xi^{(m)}| &= \Big| \sum_{k=1}^{n} \|y^{(k)}\| - \sum_{k=1}^{n} \|y^{(k)}\| \Big| \\ &= \Big| \|y^{(n)}\| - \|y^{(m)}\| \Big| \\ &= \|y^{(n)} - y^{(m)}\| \\ &\le \|y^{(n)} - y^{(m)}\|_\infty \\ &\le \frac{1}{2^m} \to 0. \qquad (n, m \to \infty) \end{aligned}$$

Hence, we see that $(\xi^{(n)})$ is a Cauchy sequence in $\mathbb{R}$ which is our desired result.

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(b) $\sum_{n=1}^{\infty} y^{(n)}$ does not converge in V.

Proof. Note that by $\lim\limits_{n\to\infty} y^{(n)} \neq 0$; that is, $y^{(n)} \to \frac{1}{2^i}$ for all $i \in \mathbb{N}$. Clearly, the limit $\left(\frac{1}{2^i}\right)$ for all $i \in \mathbb{N}$ is non-zero. Hence, $\sum_{n=1}^{\infty} y^{(n)}$ does not converge. ■

> **Problem 7** (Extra Credit). Let $(V, \|\cdot\|)$ be a normed space in which for any sequence (v_n) in V
>
> $$\sum_{n=1}^{\infty} \|v_n\| < \infty \Longrightarrow \sum_{n=1}^{\infty} v_n \text{ converges in } V.$$
>
> Prove that $(V, \|\cdot\|)$ is Banach.

Proof. Suppose that every absolutely convergent series is convergent. Our goal is to show that $(V, \|\cdot\|)$ is a Banach space. To do this, we will show that every Cauchy sequence in V converges. Let (v_n) be a Cauchy sequence in V. From here, our strategy is to find a subsequence (v_{n_k}) of (v_n) such that (v_{n_k}) converges in V (by the lemma). By definition, (v_n) being Cauchy implies that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$\|v_n - v_m\| < \varepsilon.$$

For $\varepsilon = 1$, there exists an $n_1 \in \mathbb{N}$ such that for any $n > m > n_1$, we have

$$\|v_n - v_m\| < 1.$$

Furthermore, if $\varepsilon = \frac{1}{2}$. So, there exists an $n_2 > n_1$ by the Archimedean Property such that for any $n > m > n_2$, we have

$$\|v_n - v_m\| < \frac{1}{2}.$$

In particular, if $\varepsilon = \frac{1}{2^{k-1}}$ for all $k \in \mathbb{N}$, then we can find an $n_k \in \mathbb{N}$ such that for any $n > m > n_k$, we have

$$\|v_n - v_m\| < \frac{1}{2^{k-1}}.$$

Moreover, by the Archimedean Property we can find an $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k > n_{k-1}$. Hence, it follows that (v_{n_k}) is a subsequence in V such that

$$0 \le \|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^{k-1}}. \qquad (*)$$

Note that since $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ is a geometric series it follows from the Comparison Test that

$$\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\|$$

converges to some $v \in V$. By assumption, this tells us that

$$\sum_{k=1}^{\infty}(v_{n_{k+1}} - v_{n_k})$$

converges to some v in V. Now, observe that

$$\begin{aligned} v_{n_1} + \sum_{j=1}^{k-1}(v_{n_{j+1}} - v_{n_j}) &= v_{n_1} + (v_{n_2} - v_{n_1}) + (v_{n_3} - v_{n_2}) + \cdots + (v_{n_k} - v_{n_{k-1}}) \\ &= v_{n_k}. \end{aligned}$$

Taking the limit on both sides of the above equality, we see that

$$\begin{aligned} \lim_{k\to\infty} v_{n_k} &= \lim_{k\to\infty} \left[v_{n_k} + \sum_{j=1}^{k-1}(v_{n_{j-1}} - v_{n_j})\right] \\ &= v_{n_1} + \lim_{k\to\infty} \sum_{j=1}^{k-1}(v_{n_{j+1}} - v_{n_j}) \\ &= v_{n_1} + v. \end{aligned}$$

Thus, we now see that (v_{n_k}) converges in V which tells us that (v_n) is a converges in V. Hence, ■

Lemma. Let $(V, \|\cdot\|)$ be a normed space. Suppose (v_n) is a Cauchy sequence, and some subsequence (v_{n_k}) converges to a point v in V. Then (v_n) converges to v in V.

Proof. Let $n > m$. Since (v_n) is a Cauchy sequence in V, it follows that

$$\|v_n - v_m\| \to 0$$

as $n, m \to \infty$. Also, (v_{n_k}) converges to some $v \in V$. So, for $k \to \infty$, we have

$$\|v_{n_k} - v\| \to 0.$$

Using the triangle inequality, it follows that

$$0 \leq \|v_n - v\| \leq \|v_n - v_{n_k}\| + \|v_{n_k} - v\| \to 0.$$

Using the Squeeze Theorem, we have

$$\|v_n - v\| \to 0$$

as $n \to \infty$ and we are done. ■