Handout-2:

Plan: • Topological Terminologies II

- Discuss continuity
- Discuss Sequences in a metric space

Topological Terminologies II

Let (X, d) be a metric space.

Interior Point: Let $M \subseteq X$. We say $x_0 \in M$ is an interior point of M if we can find $r > 0$ such that $B(x_0; r) \subseteq M$.

Lemma: Let $M \subseteq X$. Let U be an open set such that $U \subseteq M$. Then each $x_0 \in U$ is an interior point of M.

Proof: Exercise.

Let $M \subseteq X$. We denote the set of all interior points of M by M°.

Exercise: Prove that M° is open.

<u>Accumulation Point:</u> Let $M \subseteq X$. We say $x_0 \in X$ is an accumulation point of M if $U \cap M \setminus \{x_0\} \neq \phi$ for any open set U such that $x_0 \in U$. In other words every open set U containing x_0 intersects M at points different from x_0.

<u>Closure of a set</u> Let $M \subseteq X$. The closure of M is denoted by $\bar{M}$ and it is defined as

$$\bar{M} = M \cup \{x_0 \in X \,/\, x_0 \text{ is an accumulation point of } M\}.$$

<u>Example</u> • Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$

$M = (0, 1)$. Then

$M^o = (0,1)$, 0 and 1 are limit points of M

$\bar{M} = [0,1]$

Exercise: Let $X = \mathbb{R}$, $d(x,y) = \sqrt{|x-y|}$

Let $M = (0,1)$, find M^o and $\bar{M}$.

<u>Example:</u> Let X be a nonempty set and d be the discrete metric on X.

Let $M \subseteq X$, $M \neq \phi$.

Then $M^o = M$ and $\bar{M} = M$.

Exercise: Let $X = \mathbb{R}^2$, $M = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle/ \begin{array}{l} -1 < x < 1, \\ -1 < y < 1 \end{array} \right\}$

(i) Compute M^o and $\bar{M}$ for

(i) d_{Euclid} (ii) d_∞ and (iii) d_1.

Hint: Draw picture

(ii) What did you observe? Think about reasoning behind your observation.

Continuity

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f: X \to Y$ be a function. We say f is continuous at $x_0 \in X$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$ i.e.

$$x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \epsilon)$$

$$\left(\text{Or } f(\underbrace{B(x_0, \delta)}_{\text{open ball in } X}) \subseteq \underbrace{B(f(x_0), \epsilon)}_{\text{open ball in } Y} \right)$$

(This is the generalization of ϵ-δ definition of continuity on $\mathbb{R}$)

We say f is continuous on X if f is continuous at $x_0 \in X$ for all $x_0 \in X$.

Class activity: $f: (X, d_X) \to (Y, d_Y)$ is continuous if and only if $f^{-1}(B)$ is open in X for any open ball B in Y.

Examples: • Usual continuous functions from Calculus are continuous functions.

• Let $f:(X,d_X)\to(\mathbb{R},|\cdot|)$ and $g:(X,d_X)\to(\mathbb{R},|\cdot|)$ be continuous functions. Then $\alpha f+\beta g:(X,d_X)\to(\mathbb{R},|\cdot|)$ is continuous for all $\alpha,\beta\in\mathbb{R}$.

<u>Theorem:</u> Let $f:(X,d_X)\to(Y,d_X)$. Then the following statements are equivalent.

(i) f is continuous on X.

(ii) $f^{-1}(V)$ is open in X for every open set V in Y.

Proof: Suppose that f is continuous. Let $V\subseteq Y$ be open in Y. We want to show $f^{-1}(V)$ is open in X. For this it suffices to show that for each $x_0\in f^{-1}(V)$, there is $\delta>0$ such that $B(x_0,\delta)\subseteq f^{-1}(V)$.

Now $x_0\in f^{-1}(V)\Rightarrow f(x_0)\in V$. Since V is open we can find $\epsilon>0$ such that $B(f(x_0),\epsilon)\subseteq V$.

By the class activity above $f^{-1}(B f(x_0),\epsilon)$ is

open in X. Since $x_0 \in f^{-1}(B(f(x_0), \epsilon))$, we can find $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(B f(x_0), \epsilon))$.
This immediately implies $B(x_0, \delta) \subseteq f^{-1}(V)$

$(\because f^{-1}(B(f(x_0), \epsilon)) \subseteq f^{-1}(V))$.

Conversely, assume that the given condition holds. Let B be an open ball in Y. Then B is open in Y. By our assumption, $f^{-1}(B)$ is open in X. Now the class activity $\Rightarrow f$ is continuous on X.

<u>Corollary:</u> Let $f : (X, d_X) \to (Y, d_Y)$ be continuous, and $g : (Y, d_Y) \to (Z, d_Z)$ be continuous. Then $g \circ f : (X, d_X) \to (Z, d_Z)$ is continuous.

Proof: Left as an exercise.

Sequences and Convergence of a Sequence in a metric space:

Let (X, d) be a metric space.

Recall the notion of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers. We usually write this as

$x_1, x_2, x_3, \ldots$ where x_i's are real numbers and some x_i's are allowed to be same. We can make more formal by thinking of a sequence as an assignments that assigns to each natural number i a unique real number x_i. Thus, a sequence in $\mathbb{R}$ is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$, $f(i) = x_i$. We will follow this to give a mathematical definition of a sequence in a metric space.

<u>Definition:</u> Let (X, d) be a metric space. A sequence in X is a function $f: \mathbb{N} \to X$. Let $f(i) = x_i$. It is a general tradition to use $(x_i)_{i=1}^{\infty}$ or (x_i) or $\{x_i\}$ to present a sequence.

The set $f(\mathbb{N}) \subseteq X$ is called the range of the sequence. Note that

$$f(\mathbb{N}) = \underbrace{\{x_i \mid i \in \mathbb{N}\}}_{\text{considered as a set.}}$$