Linear Algebra Notes

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Chapter 1

Vector Spaces

1.1 Vector Spaces

Definition 1. A **vector space** (or **linear space**) over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in V there is a unique element ax in V, such that the following conditions hold:

(VS 1) For all $x, y \in V$, $x + y = y + x$ (commutativity of addition).

(VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in V denoted by O such that $x + O = x$ for each $x \in V$

(VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = O$.

(VS 5) For each element $x \in V$, we have $1x = x$.

(VS 6) For each $a, b \in F$ and each element $x \in V$, then $(ab)x = a(bx)$.

(VS 7) For each element $a \in F$ and each pair $x, y \in V$, we have $a(x + y) = ax + ay$.

(VS 8) For each pair $a, b \in F$ and each $x \in V$, we have $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.

- The elements of a field F are called **scalars** and the elements of a vector space V are called **vectors** (these should not be confused!).
- Every vector space will always be defined over a given field, mostly defined over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ unless otherwise noted.
- Every vector space should specify the operations of addition and scalar multiplication.

Definition 2. An object of the form $(a_1, a_2, \dots, a_n)$, where the entries $a_1, a_2, \dots, a_n$ are elements of a field F, is called an **n-tuple** with entries from F. The elements $a_1, a_2, \dots, a_n$ are called **entries** or **components** of the n-tuple.

Definition 3. We say that two n-tuples, $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$, are **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

Example. The set of all n-tuples with entries from a field F denoted by F_n is a vector space. To see why, suppose $u, v \in F_n$ where $u = (a_1, a_2, \dots a_n)$ and $v = (b_1, b_2, \dots, b_n)$. If we take term-by-term addition of the entries in both u and v, then we end up with

$$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and likewise,

$$cu = (ca_1, ca_2, \ldots, ca_n).$$

These same set of operations define $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ and likewise, $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.