Measure Theory Notes

Lance Remigio

July 4, 2024

Contents

List of Theorems

Chapter 1

Riemann Integration

1.1 Review: Riemann Integral

Let $\mathbb{R}$ denote the complete ordered field of real numbers.

Definition 1.1.1 (Partition). Suppose $a, b \in \mathbb{R}$ with $a < b$. A **partition** of $[a, b]$ is a finite list of the form $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We can think of closed intervals as union of closed subintervals $[x_{n-1}, x_n]$ for $n \geq 1$; that is,

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$

Definition 1.1.2 (Infimum and Supremum of Functions). If f is a real-valued function and A is a subset of the domain of f, then

$$\inf_A f = \inf\{f(x) : x \in A\} \quad \text{and} \quad \sup_A f = \sup\{f(x) : x \in A\}$$

The definition above approximates the area under the graph of a nonnegative function or the signed area corresponding to a real-valued function.

Definition 1.1.3 (Lower and Upper Riemann Sums). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition $x_0, \ldots, x_n$ of $[a, b]$. The **Lower Riemann Sum** $L(f, P, [a, b])$ and the **Upper Riemann Sum** $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

- The supremums represent an over approximation in terms of the height of each rectangle with $[x_{j-1}, x_j]$ being the width.
- Infimums are an under approximation compared to the supremum with $[x_{j-1}, x_j]$ being the width as before.

- The next result describes how adjoining more points to a partition *increases* the **lower Riemann sum** and *decreases* the **upper Riemann sum**.

Proposition 1.1.1 (Inequalities with Riemann Sums). Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a,b]$ such that the list defining P is a sublist of the list defining P' (that is, $P \subseteq P'$). Then

$$L(f,P,[a,b]) \leq L(f,P',[a,b]) \leq U(f,P',[a,b]) \leq U(f,P,[a,b]).$$

Proof. Suppose P is the partition $x_0, \dots, x_n$ and P' is the partition $x'_0, \dots, x'_N$ of $[a,b]$. For each $j = 1, \dots, n$ (that is, for each subinterval $[x_{j-1}, x_j]$), we can find $k \in \{0, \dots, N-1\}$ and a positive integer m such that

$$x_{j-1} = x'_k < x'_{k+1} < \cdots < x'_{k+m} = x_j.$$

Observe that

$$\begin{aligned}
L(f,P,[a,b]) = (x_j - x_{j-1}) \inf_{[x_{j-1},x_j]} f &= \sum_{i=1}^{m} (x'_{k+i} - x'_{(k+i)-1}) \inf_{[x_{j-1},x_j]} f \\
&\leq \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x'_{k+i-1},x'_{k+i}]} f \\
&= L(f,P',[a,b]).
\end{aligned}$$

Thus, we have that $L(f,P,[a,b]) \leq L(f,P',[a,b])$. The middle inequality can be proven by observing that the infimum of any set is always less than or equal to the supremum of that set. In regards to the partition P', we have $L(f,P',[a,b]) \leq U(f,P',[a,b])$. To get the last inequality, we can follow the same argument we used to prove the first inequality. Hence, we have

$$\begin{aligned}
U(f,P,[a,b]) = (x_j - x_{j-1}) \sup_{[x_{j-1},x_{j-1}]} f &= \sum_{j=1}^{m} (x_{k+i} - x_{(k+i)-1}) \sup_{[x_{j-1},x_j]} f \\
&\geq \sum_{j=1}^{m} (x_{k+i} - x_{(k+i)-1}) \sup_{[x_{k+i},x_{(k+i)-1}]} f \\
&= U(f,P',[a,b]).
\end{aligned}$$

Thus, we have that $U(f,P',[a,b]) \leq U(f,P,[a,b])$ in which we now conclude that

$$L(f,P,[a,b]) \leq L(f,P',[a,b]) \leq U(f,P',[a,b]) \leq U(f,P,[a,b]).$$

■

Proposition 1.1.2 (Lower Riemann Sums $\leq$ Upper Riemman Sums). Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a,b]$. Then

$$L(f,P,[a,b]) \leq U(f,P',[a,b]).$$

Proof. Let P'' be the partition of $[a,b]$ obtained by unioning partitions P and P'; that is,

$P'' = P \cup P'$. Since $P'' \subseteq P$ and $P'' \subseteq P'$, we have that

$$\begin{aligned} L(f, P, [a, b]) &\leq L(f, P'', [a, b]) \\ &\leq U(f, P'', [a, b]) \\ &\leq U(f, P', [a, b]) \end{aligned}$$

by the result before. ■

Next, we define the upper and lower Riemann integrals.

Definition 1.1.4 (lower and upper Riemann integrals). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. The **lower Riemann integral** $L(f, [a, b])$ and the **upper Riemann integral** $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_P L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_P U(f, P, [a, b]),$$

where the supremum and infimum above are taken over all partitions P of $[a, b]$.

- For the first equation above, we are taking the supremum of all the **Lower Riemann Sums**. This makes sense considering the lower Riemann sums become larger as we add more points to our partition.
- Likewise, adding more points to a partition causes the upper Riemann sum to become smaller, leading to the infimum over all the partitions.
- In both of these cases, we reach a more accurate estimate of the area under the graph of a function.

Proposition 1.1.3 (Lower Riemann Integral $\leq$ Upper Riemann Integral). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Then

$$L(f, [a, b]) \leq U(f, [a, b]).$$

Proof. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Notice that, by definition,

$$L(f, [a, b]) = \sup_P L(f, P, [a, b]);$$

that is, $L(f, [a, b])$ is a supremum over all partitions P of all the lower Riemann sums. Using a lemma from undergrad analysis, $L(f, [a, b]) = \sup_P L(f, P, [a, b])$ if and only if for all $\varepsilon > 0$, there exists $L(f, P_\varepsilon, [a, b]) \in \{L(f, P, [a, b]) : \text{for all } P \in P\}$ such that

$$L(f, [a, b]) - \varepsilon < L(f, P_\varepsilon, [a, b]).$$

Now, consider the upper Riemann integral $U(f, [a, b])$. Since

$$U(f, [a, b]) = \inf\{U(f, P, [a, b]) : \text{for all } P \in P\}$$

and the fact that

$$\{U(f, P, [a, b]) : \text{for all } P \in P\}$$

is a set of supremums (upper bounds of f over each partition P), we can see that by Proposition 1.1.1 that

$$L(f,[a,b]) - \varepsilon < L(f,P_\varepsilon,[a,b]) \leq U(f,[a,b]).$$

Since $\varepsilon > 0$ is arbitrary, we see that $L(f,[a,b]) \leq U(f,[a,b])$. ∎

Definition 1.1.5 (Riemann integrable; Riemann Integral).

- A bounded function on a closed interval is called **Riemann Integrable** if its lower Riemann integral equals its upper Riemann integral.
- If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then the **Riemann integral** $\int_a^b f$ is defined by

$$\int_a^b f = L(f,[a,b]) = U(f,[a,b]).$$

Proposition 1.1.4. Every continuous real-valued function on each closed bounded interval is Riemann integrable.

Proof. Let $a, b \in \mathbb{R}$ with $[a,b]$ being a compact interval. Let $f : [a,b] \to \mathbb{R}$ and let P be a partition of $[a,b]$. Let $\varepsilon > 0$. Since f is continuous on the compact interval $[a,b]$, f is uniformly continuous and must, therefore, attain its maximum and minimum on $[a,b]$. Thus, assign $M_j = f(y_j)$ and $m_j = f(z_j)$ where

$$M_j = \sup_{x \in [x_{j-1},x_j]} f \quad \text{and} \quad m_j = \inf_{x \in [x_{j-1},x_j]} f.$$

Furthermore, there exists a $\delta > 0$ such that whenever $|y_j - z_j| < \delta$, we have

$$|M_j - m_j| = |f(y_j) - f(z_j)| < \frac{\varepsilon}{b-a}.$$

Then we have

$$\begin{aligned} U(f,P,[a,b]) - L(f,P,[a,b]) &= \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{j=1}^{n} (x_j - x_{j-1}) \\ &= \frac{\varepsilon}{(b-a)} \cdot (b-a) = \varepsilon. \end{aligned}$$

Since $U(f,[a,b]) \leq U(f,P,[a,b])$ and $L(f,[a,b]) \geq L(f,P,[a,b])$, we have

$$U(f,[a,b]) - L(f,[a,b]) \leq U(f,P,[a,b]) - L(f,P,[a,b]) < \varepsilon.$$

Since ε is arbitrary, we must have $U(f,[a,b]) = L(f,[a,b])$. Thus, f is integrable. ∎

Proposition 1.1.5 (Bounds on Rieman Integral). Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Then

$$(b-a) \inf_{[a,b]} f \leq \int_a^b f \leq (b-a) \sup_{[a,b]} f.$$

Proof. Let P be the partition consisting of points $x_0 = a$ and $x_1 = b$. Since $f : [a, b] \to \mathbb{R}$ is Riemann integrable, we have

$$\int_a^b f = L(f, [a, b]) = U(f, [a, b]).$$

Starting with the first inequality, observe that

$$\int_a^b f = L(f, [a, b]) \geq L(f, P, [a, b]) = (b - a) \inf_{[a,b]} f.$$

Observe that the second inequality can be proven by a similar argument

$$\int_a^b f = U(f, [a, b]) \leq U(f, P, [a, b]) = (b - a) \sup_{[a,b]} f.$$

Thus, we have reached our desired result that

$$(b - a) \inf_{[a,b]} f \leq \int_a^b f \leq (b - a) \sup_{[a,b]} f.$$

■

1.1.1 Exercises

1. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function such that

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

   for some partition P of $[a, b]$. Prove that f is a constant function on $[a, b]$.

   Proof. ■

2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \varepsilon.$$

   Proof. ■

1.2 Riemann Integral Is Not Good Enough

There are three disadvantages of using Riemann integration; that is, Riemann integration does not handle

- Functions with many discontinuities;
- Unbounded functions;
- Does not work well with limits.

Example 1.2.1 (A function that is not Riemann integrable). Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

If $[a, b] \subset [0, 1]$ with $a < b$, then we see that

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

because $[a, b]$ contains an irrational number and contains a rational number. So,

$$L(f, P, [0, 1]) = 0 \quad \text{and} \quad U(f, P, [0, 1]) = 1$$

for every partition P of $[0, 1]$. This immediately implies that $L(f, [0, 1]) = 0 \neq 1 = U(f, [0, 1])$ and thus, f is not Riemann integrable.

Example 1.2.2 (Rieman integration with unbounded functions). Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

If $x_0, x_1, \ldots, x_n$ is a partition of $[0, 1]$, then $\sup_{[x_0,x_1]} = \infty$. This would mean that in every partition P of $[0, 1]$, we would have $U(f, P, [0, 1]) = \infty$ if we had applied Riemann integration. But note that the area of this function is 2 since

$$\lim_{a \to 0} \int_a^1 f - \lim_{a \to 0} (2 - 2\sqrt{a}) = 2.$$

This can be remedied by other methods seen in introductory Calculus courses, but in general, using Riemann integration over subdomains tends to fail, especially with more complicated functions than the one shown in the example above.

Example 1.2.3 (Area seems to make sense, but Riemann integral is not defined). Let $r_1, r_2, \ldots$ be a sequence that includes each rational number in $(0, 1)$ exactly once and that includes no other numbers. For $k \in \mathbb{Z}^+$, define $f_k : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{\sqrt{x - r_k}} & \text{if } x > r_k \\ 0 & \text{if } x \leq r_k. \end{cases}$$

Define $f : [0, 1] \to [0, \infty]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}.$$

Since every nonempty open interval of $[0, 1]$ contains a rational number (Density of $\mathbb{Q}$ in $\mathbb{R}$), we see that f is unbounded on every such subinterval (contains a sequence of rational numbers). However, the area under the graph of each f_k is less than 2. The formula defining f then shows that we should expect the area under the graph of f to be less than 2 rather than being undefined.

Proposition 1.2.1 (Interchanging Riemann integral and limit). Suppose $a, b, M \in \mathbb{R}$ with $a < b$. Suppose $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a, b]$ such that

$$|f_k(x)| \leq M$$

for all $k \in \mathbb{Z}^+$ and all $x \in [a, b]$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in [a, b]$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \lim_{k\to\infty f_k(x)}.$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f = \lim_{k\to\infty} \int_a^b f_k.$$

We have the following problems with the result above:

- We have to assume that f is Riemann integrable.
- The result follows even if f is not Riemann integrable (Of course, assuming all the other hypotheses hold true).
- The proof of the result above is intricate and delicate. We should find that the proof should be "simple".

Chapter 2

Measures

2.1 Outer Measure on $\mathbb{R}$

2.1.1 Motivation and Definition of Outer Measure

- This section focuses on developing the notion of assigning a "size" to subintervals when it comes to more complicated unions of open intervals.
- The hope is to create an integration theory that will cover a broader class of functions than covered by Riemann integration.
- We start off by giving a definition of the length of an open interval.

Definition 2.1.1 (Length of Open Interval; $\ell(I)$). The **length** $\ell(I)$ of an open interval I is defined bt

$$\ell(I) = \begin{cases} b - a & \text{if } I = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ with } a < b, \\ 0 & \text{if } I = \emptyset, \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R}, \\ \infty & \text{if } I = (-\infty, \infty). \end{cases}$$

Suppose we have a subset $A \subset \mathbb{R}$. Then the size of A should be the sum of the lengths of a sequence of open intervals whose union contains A. The infimum of all possible sums gives the definition of the size of A, denoted $|A|$ which we call the **outer measure** of A.

Definition 2.1.2 (Outer Measure; $|A|$). The **outer measure** $|A|$ of a set $A \subset \mathbb{R}$ is defined by

$$|A| = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

- Notice that the definition of outer measure involves an infinite sum.
- Recall that the infinite sum converges if the sequence of partial sums converges.
- If one of the elements in the sequence is ∞ (that is, $t_k = \infty$) for some k, then the infinite sum is ∞.

- Otherwise, the infinite sum $\sum_{k=1}^{\infty t_k}$ is defined to be the limit of the increasing sequence of partial sums where

$$\sum_{k=1}^{\infty} t_k = \lim_{n\to\infty} \sum_{k=1}^{n} t_k.$$

2.1.2 Properties of Outer Measure

Proposition 2.1.1 (Countable sets have outer measure 0). Every countable subset of $\mathbb{R}$ has outer measure 0.

Proof. Suppose $A = \{a_1, a_2, \dots\}$ is a countable subset of $\mathbb{R}$. Let $\varepsilon > 0$. For $k \in \mathbb{Z}^+$, let

$$I_k = \left(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\right).$$

Then $I_1, I_2, \dots$ is a sequence of open intervals whose union contains A. Since

$$\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon,$$

we have $|A| \leq 2\varepsilon$. Since ε is an arbitrary positive number, we must have $|A| = 0$. ∎

- Note that $\mathbb{Q}$ is a countable set, and thus contains a measure of 0.

Proposition 2.1.2 (Outer Measure Preserves Order). Suppose A and B are subsets of $\mathbb{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Proof. Suppose $I_1, I_2, \dots$ is a sequence of open intervals such that

$$B \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Since $A \subseteq B$, we can see that A is also contained within

$$\bigcup_{k=1}^{\infty} I_k.$$

Thus, we have

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k).$$

If we take the infimum over all the sequences of open intervals whose union contains B, we see that $|A| \leq |B|$. ∎

Definition 2.1.3 (Translation; $t + A$). If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then the translation $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

- If $t > 0$, then translating A by adding t moves A by $|t|$ units to the right.
- If $t < 0$, then translating A by adding t moves A to the left by $|t|$ units.

- In a similar manner to how functions do not change their shape when translated either horizontally or vertically, intervals do not change their shape/length when moved either left or right on the number line.
- By standard convention, we take $t + (-\infty) = -\infty$ and $t + \infty = \infty$.

Proposition 2.1.3 (Outer Measure is Translation Invariant). Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$. Then $|t + A| = |A|$.

Proof. Let $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. Suppose $I_1, I_2, \ldots$ is a sequence of open intervals such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Then $t + I_1, t + I_2, \ldots$ is a sequence of open intervals such that

$$t + A \subseteq \bigcup_{k=1}^{\infty} t + I_k.$$

Since translation by t of intervals I_k of A does not change, we have that

$$\ell(t + I_k) = \ell(I_k).$$

So, we see that

$$|t + A| \leq \sum_{k=1}^{\infty} \ell(t + I_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

But this tells us that the union of the sequence of intervals I_k of A contains $t + A$. So, taking the infimum over all such sequences, we have $|t + A| \leq |A|$.

Now, observe that $A = -t + (t + A)$ for $t \in \mathbb{R}$. Then we see that

$$|A| = |-t + (t + A)| \leq |t + A|.$$

Thus, $|A| = |t + A|$. ■

Proposition 2.1.4 (Countable Subadditivity of Outer Measure). Suppose $A_1, A_2, \ldots, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \leq \sum_{k=1}^{\infty} |A_k|.$$

Proof. If $|a_k| = \infty$ for some $k \in \mathbb{Z}^+$, then the inequality above holds. Thus, assume $|A_k| < \infty$ for all $k \in \mathbb{Z}^+$.

Let $\varepsilon > 0$. Using a lemma found in introductory real analysis textbooks, we can write that for each $k \in \mathbb{Z}^+$, let $I_{1,k}, I_{2,k}, \ldots$ be a sequence of open intervals whose union contains A_k such that

$$\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq \frac{\varepsilon}{2^k} + |A_k|.$$

Taking the infinite sum of both sides of this inequality, we get

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\ell(I_{j,k}) \leq \varepsilon + \sum_{k=1}^{\infty}|A_k|.$$

Note that

$$\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k} = \varepsilon$$

by using the geometric sum formula. Consider the doubly indexed collection of open intervals $\{I_{j,k} : j,k \in \mathbb{Z}^+\}$ into a sequence of open intervals such that

$$\bigcup_{k=1}^{\infty}\ell(I_{j,k}) \subseteq \bigcup\{I_{j,k} : j,k \in \mathbb{Z}^+\}.$$

Suppose in step k, starting with $k = 2$, we adjoin the $k-1$ intervals whose indices add up to k; that is,

$$\underbrace{I_{1,1}}_{2}, \underbrace{I_{1,2}, I_{2,1}}_{3}, \underbrace{I_{1,3}, I_{2,2}}_{4}, \underbrace{I_{1,5}, 1_{2,4}, I_{3,3}, I_{4,2}, I_{5,1}}_{6} \ldots.$$

Thus, we must have

$$\left|\bigcup_{k=1}^{\infty}A_k\right| \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\ell(I_{j,k}) \leq \varepsilon + \sum_{k=1}^{\infty}|A_k|.$$

Since $\varepsilon > 0$ is arbitrary, we must have that

$$\left|\bigcup_{k=1}^{\infty}A_k\right| \leq \sum_{k=1}^{\infty}|A_k|.$$

■

2.1.3 Outer Measure of Closed Bounded Interval

The next property we will prove is that the length of each closed interval $[a,b]$ with $a < b$ and $a,b \in \mathbb{R}$ is equal to $[a,b] = b-a$. If $\varepsilon > 0$, we see that $(a-\varepsilon, b+\varepsilon), \emptyset, \emptyset, \ldots$ is a sequence of open intervals whose union contains $[a,b]$. Then $|[a,b]| \leq b-a+2\varepsilon$. Since this inequaltiy holds for all $\varepsilon > 0$, we must have that

$$|[a,b]| \leq b-a.$$

Definition 2.1.4 (Open Cover). Suppose $A \subseteq \mathbb{R}$.

- A collection $\mathcal{C}$ of open subsets of $\mathbb{R}$ is called an **open cover** of A if A is contained in the union of all the sets in $\mathbb{C}$.
- An open cover $\mathcal{C}$ of A is said to have a **finite subcover** if A is contained in the union of some finite list of sets in $\mathcal{C}$

Proof. ■

Proposition 2.1.5 (Heine-Borel Theorem). Every open cover of a closed bounded subset of $\mathbb{R}$ has a finite subcover.

Proof. Suppose F is a closed bounded subset of $\mathbb{R}$ and $\mathcal{C}$ is an open cover of F (Note that $\mathcal{C}$ is the collection of open subsets in $\mathbb{R}$).

First, suppose $F = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Thus, $\mathcal{C}$ is an open cover of $[a, b]$. Let

$$D = \{d \in [a, b] : [a, d] \text{ has a finite subcover from } \mathcal{C}\}.$$

Let's show that this set is nonempty first. Note that $a \in D$ since $a \in G$ for some $G \in \mathcal{C}$. Thus, $D \neq \emptyset$ and so we can say that a supremum exists for D (clearly, D is bounded above). Let

$$s = \sup D.$$

So, $s \in [a, b]$. Hence, there exists an open set $G \in \mathcal{C}$ such that $s \in G$. Let $\delta > 0$ be such that $(s - \delta, s + \delta) \subset G$. Since $s = \sup D$, there exists $d \in (s - \delta, s]$ and $n \in \mathbb{Z}^+$ and $G_1, G_2, \ldots, G_n \in \mathcal{C}$ such that

$$[a, d] \subset \bigcup_{k=1}^{n} G_k.$$

Now, let $d' \in [s, s + \delta)$ be arbitrary. Then

$$[a, d'] \subset G \cup \bigcup_{k=1}^{n} G_k.$$

Since $[a, d']$ contains a finite subcover from $\mathcal{C}$, we get that $d' \in D$ for all $d' \in [s, s+\delta) \cap [a, b]$. Furthermore, this implies that $b = s$. So, with $d' = b$, $[a, b]$ must contain a finite subcover from $\mathcal{C}$, completing the proof in the case that $F = [a, b]$.

Now, suppose F is an arbitrary closed bounded subset of $\mathbb{R}$ and that $\mathcal{C}$ is an open cover of F. Let $a, b \in \mathbb{R}$ be such that $F \subseteq [a, b]$. Now $\mathcal{C} \cup \{\mathbb{R} \setminus F\}$ is an open cover of $\mathbb{R}$ and hence it is an open cover of $[a, b]$. By our first case, there exists $G_1, \ldots, G_n \in \mathcal{C}$ such that

$$[a, b] \subset \left(\bigcup_{k=1}^{n} G_k\right) \cup (\mathbb{R} \setminus F).$$

Thus,

$$F \subset \bigcup_{k=1}^{n} G_k,$$

completing the proof. ■

Proposition 2.1.6 (Outer Measure of a Closed Interval). Suppose $a, b \in \mathbb{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proof. The proof of $|[a, b]| \leq b - a$ can be found here.

Now, we want to show that $|[a, b]| \geq b - a$. Suppose $I_1, I_2, \ldots$ is a sequence of open intervals such that $[a, b] \subset \bigcup_{k=1}^{\infty} I_k$. Using the Heine-Borel Theorem, there exists $n \in \mathbb{Z}^+$ such that

$$[a, b] \subset \bigcup_{k=1}^{n} I_k.$$

We will now show, by induction on n, that the inclusion above implies that

$$\sum_{k=1}^{n} \ell(I_k) \geq b - a. \tag{1}$$

Proving this will show that

$$\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) \geq b - a, \tag{2}$$

completing the proof that $|[a, b]| \geq b - a$. If $n = 1$, we have that (1) implies (2). Now, suppose $n > 1$. Then (1) implies (2) for all choices of $a, b \in \mathbb{R}$ with $a < b$. Suppose $I_1, \ldots, I_n, I_{n+1}$ are open intervals such that

$$[a, b] \subset \bigcup_{k=1}^{n+1} I_k.$$

Thus, b must be in at least one of these intervals. By relabeling, we can assume that $b \in I_{n+1}$. Suppose $I_{n+1} = (c, d)$. If $c \leq a$, then $\ell(I_{n+1}) \geq b - a$ and there is nothing further to prove; thus, assume that $a < c < b < d$. Thus, we can assume that

$$[a, c] \subset \bigcup_{k=1}^{n} I_k.$$

Using our induction hypothesis, we have

$$\sum_{k=1}^{n} \ell(I_k) \geq c - a.$$

Since $d > b$ and $c \in [a, b]$, we can see that $\ell(I_{n+1}) = d - c$. Thus, we have that

$$\begin{aligned}
\sum_{k=1}^{n+1} \ell(I_k) &= \ell(I_{n+1}) + \sum_{k=1}^{n} \ell(I_k) \\
&\geq (c - a) + (d - c) \\
&= d - a \\
&\geq b - a,
\end{aligned}$$

completing the proof. ∎

Proposition 2.1.7 (Nontrivial Intervals Are Uncountable). Every interval in $\mathbb{R}$ that contains at least two distinct elements is uncountable.

Proof. ∎

2.1.4 Outer Measure is Not Additive

Let us state a result that will be used in the upcoming proof of this result.

The result states that a nonempty intersection of two equivalence classes implies that the two classes are equal to each other.

Proposition 2.1.8. If $a, b \in [-1, 1]$ and $\tilde{a} \cap \tilde{b} \neq \emptyset$ where

$$\tilde{a} = \{c \in [-1, 1] : a - c \in \mathbb{Q}\},$$

($\tilde{b}$ is defined similarly) then $\tilde{a} = \tilde{b}$.

Proof. Let $d \in \tilde{a} \cap \tilde{b}$. Then $d \in \tilde{a}$ and $d \in \tilde{b}$. By definition of $\tilde{a}$ and $\tilde{b}$, we have $a - d \in \mathbb{Q}$ and $b - d \in \mathbb{Q}$, respectively. Since $\mathbb{Q}$ is a field, we can use subtraction to get

$$a - d - (b - d) = a - b.$$

This implies that $a - b \in \mathbb{Q}$ since $a - d \in \mathbb{Q}$ and $b - d \in \mathbb{Q}$. Thus, we have

$$a - c = (a - b) + (b - c)$$

is a rational number if and only $b - c \in \mathbb{Q}$. But, $\tilde{a} \cap \tilde{b} \neq \emptyset$. So, we must have

$$a - c = b - c \Rightarrow \tilde{a} = \tilde{b}.$$

■

Proposition 2.1.9 (Nonadditivity of Outer Measure). There exists disjoint subsets of A and B of $\mathbb{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

Proof. Continuing from the proof before this result, we can see that $a \in \tilde{a}$ for each $a \in [-1, 1]$. Thus, we have

$$[-1, 1] = \bigcup_{a \in [-1,1]} \tilde{a}.$$

■

2.1.5 Exercise 2A

1. Prove that if A and B are subsets of $\mathbb{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

   Proof. Let A and B be subsets of $\mathbb{R}$ and $|B| = 0$. Hence, by finite subadditivity, we must have

   $$|A \cup B| \leq |A| + |B| = |A|.$$

   Since $A \subseteq A \cup B$, we can use result 2.5 to conclude that

   $$|A| \leq |A \cup B|.$$

   Thus, we can conclude that $|A \cup B| = |A|$. ■

2. Suppose $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t||A|$.

   Proof. Let $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. We proceed by showing the following two inequalities:

   $$|tA| \leq |t||A| \text{ and } |tA| \geq |t||A|.$$

Let $I_1, I_2, \ldots$ be a sequence of open intervals such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Thus, we have

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k) \tag{1}$$

Now, let $tI_1, tI_2, \ldots$ be a sequence of open interval such that

$$tA \subseteq \bigcup_{k=1}^{\infty} tI_k.$$

Thus, we have

$$|tA| \leq \sum_{k=1}^{\infty} \ell(tI_k). \tag{2}$$

Noting that each the length of each subinterval of tA is just the length of each subinterval of A but scaled by $|t|$ where $t > 0$, we must have that $\ell(tI_k) = |t|\ell(I_k)$. To show the first inequality, we can multiply (1) by $|t|$ to get

$$|t||A| \leq |t| \sum_{k=1}^{\infty} |t|\ell(I_k) = \sum_{k=1}^{\infty} \ell(tI_k).$$

Then taking the infimum over all sequences of open intervals containing tA, we have that $|t||A| \leq |tA|$.

Now, let $\varepsilon > 0$. Then we have

$$\sum_{k=1}^{\infty} \ell(tI_k) \leq |tA| + \varepsilon.$$

By (1), we get that

$$\sum_{k=1}^{\infty} \ell(tI_k) = |t| \sum_{k=1}^{\infty} \ell(I_k) \geq |t||A|.$$

Since $\varepsilon > 0$ is arbitrary, we must have $|t||A| \leq |tA|$. Thus, we conclude that $|tA| = |t||A|$.

■

3. Prove that if $A, B \subset \mathbb{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

Proof. Let $A, B \subset \mathbb{R}$ and $|A| < \infty$. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be a sequence of open intervals such that

$$B \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}_k.$$

Thus, we see that

$$|B| \leq \sum_{k=1}^{\infty} \ell(\mathcal{I}_k).$$

Similarly, let $I_1, I_2, \ldots$ and $\mathcal{J}_1, \mathcal{J}_2, \ldots$ be sequences of open intervals for $B \setminus A$ and A, respectively. Then we have

$$B \setminus A \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and } A \subseteq \bigcup_{k=1}^{\infty} \mathcal{J}_k.$$

Observe that

$$\sum_{k=1}^{\infty} \ell(I_k) + \sum_{k=1}^{\infty} \ell(\mathcal{J}_k) \geq \sum_{k=1}^{\infty} \ell(\mathcal{I}_k) \geq |B|.$$

Now, taking the infimum over all sequences I_k and $\mathcal{J}_k$ for $k \in \mathbb{N}$, we must have

$$|B| \leq |B \setminus A| + |A|$$

which can be rewritten to get our result that

$$|B \setminus A| \geq |B| - |A|.$$

■

2.2 The Nonexistence of a Function that generalizes length to all subsets of $\mathbb{R}$

Proposition 2.2.1 (Nonexistence of Extension of Length to all subsets of $\mathbb{R}$). There does not exists a function μ with all the following properties:

(a) μ is a function from the set of subsets of $\mathbb{R}$ to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval of $\mathbb{R}$.

(c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of $\mathbb{R}$.

(d) $\mu(t + A) = \mu(A)$ for every $A \subset \mathbb{R}$ and every $t \in \mathbb{R}$.

Proof. We proceed via contradiction. Our goal is to show that if the function μ has all the properties that were used to prove proposition 2.18, then we should reach the conclusion that for any disjoint sets A, B, we have

$$\mu(A \cup B) \neq \mu(A) + \mu(B).$$

Those properties in particular are Starting with the empty set, we see that $\mu(\emptyset) = 0$ by property (b) since the empty set is an open interval with a length of 0.

Suppose $A \subset B \subset \mathbb{R}$. Assume that we can write B as the union of the disjoint of the sequence $A, B \setminus A, \emptyset, \emptyset, \ldots$. Then we can see that

$$\mu(B) = \mu(A) + \mu(B \setminus A) + 0 + 0 + \cdots = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

So, we conclude that μ preserves order.

Let $a, b \in \mathbb{R}$ with $a < b$. Now, we want to show that $\mu([a, b]) = b - a$. Observe that $(a, b) \subset [a, b] \subset (a - \varepsilon, b + \varepsilon)$ for every $\varepsilon > 0$. Since μ preserves order, we have

$$b - a \leq \mu([a, b]) \leq b - a + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu([a, b]) = b - a$.

If $A_1, A_2, \dots$ is a sequence of subsets of $\mathbb{R}$, then $A_1, A_2 \setminus A_1, A_3 \setminus (A_1 \cup A_2), \dots$ is a disjoint sequence of subsets of $\mathbb{R}$ whose union is $\bigcup_{k=1}^{\infty} A_k$. Thus, we have

$$\begin{aligned}
\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) &= \mu\Big(A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2)) \cup \dots\Big) \\
&= \mu(A_1) + \mu(A_2 \setminus A_1) + \mu(A_3 \setminus (A_1 \cup A_2)) + \cdots \\
&\leq \sum_{k=1}^{\infty} \mu(A_k),
\end{aligned}$$

where the second equality follows from the countable additivity of μ. Since μ implies that the length of empty sets are zero, preserves order, and contains a countable subadditivity. Performing the same process used to prove result 2.18, we see that

$$\mu(A \cup B) \neq \mu(A) + \mu(B)$$

where $A, B \subset \mathbb{R}$ are disjoint. But note that this contradicts property (c) of μ. Thus, $A, B, \emptyset, \emptyset$ is a disjoint sequence of open intervals of $\mathbb{R}$ does not satisfy the countable additivity property of property (c) which completes our proof. ■

2.2.1 $\sigma-$Algebras

In order to generalize the notion of size to all subsets of $\mathbb{R}$, we need to give up or make one of the conditions from the proposition above more relaxed. We see that parts (b), (c), and (d) cannot be taken out since:

- The size of an interval needs to be its length;
- Countable additivity is needed to prove results about limits;
- We cannot have sizes of intervals change if we translate them.

So, the only reasonable condition to relax is part (a). In order to have the notion of size generalized to more general subsets of $\mathbb{R}$, we need to develop a notion of size allows sets to be closed under complementation and closed under countable unions.

Definition 2.2.1 ($\sigma-$algebra)**.** Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then S is called a $\sigma-$algebra on X if the following three conditions are satisfied:

- $\emptyset \in S$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

Example 2.2.1.
- Suppose X is a set. Then clearly $\{\emptyset, X\}$ is a $\sigma-$algebra on X.
- Suppose X is a set. The set of all subsets of X is a $\sigma-$algebra on X.
- Suppose X is a set. All the subsets of E of X such that E is countable or $X \setminus E$ is countable is a $\sigma-$algebra on X.

Proposition 2.2.2 ($\sigma-$algebras are closed under countable intersection). Suppose S is a $\sigma-$algebra on a set X. Then

(a) $X \in \mathcal{S}$;

(b) If $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$, $D \cap E \in \mathcal{S}$, and $D \setminus E \in \mathcal{S}$;

(c) If $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

Proof. Suppose $\mathcal{S}$ is a $\sigma-$algebra, we know that $\emptyset \in \mathcal{S}$. Note that $X = X \setminus \emptyset$ and thus, $X \setminus \emptyset \in \mathcal{S}$. Hence, $X \in S$ completing the proof for part (a).

For part (b), suppose $D, E \in \mathcal{S}$. Note that $D \cup E$ is the union of the sequence of $D, E, \emptyset, \emptyset, \ldots$ of elements of $\mathcal{S}$. By definition of $\sigma-$algebra, we see that $D \cup E \in \mathcal{S}$.

Using De Morgan's Laws, we can see that

$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E).$$

If $D, E \in \mathcal{S}$, then the right side of the equation above tells us that $X \setminus (D \cap E) \in \mathcal{S}$, and so $D \cap E \in \mathcal{S}$. Since $D \setminus E = D \cap (X \setminus E)$, we have $D \setminus E \in \mathcal{S}$. Furthermore, $D \setminus E = D \cap (X \setminus E)$ implies $D \setminus E \in \mathcal{S}$, completing the proof of part (b).

Finally, suppose $E_1, E_2, \ldots$ is a sequence of elements in $\mathcal{S}$. Using De Morgan's Laws again, we see that

$$X \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (X \setminus E_k).$$

Since the right side of the equation above is in $\mathcal{S}$, this tells us that the left side of the equation is also in $\mathcal{S}$. Therefore, $X \setminus (X \setminus \bigcap_{k=1}^{\infty} E_k) \in \mathcal{S}$, proving (c). ■

Definition 2.2.2 (Measurable Space; Measurable Set).
- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a $\sigma-$algebra on X.
- An element of $\mathcal{S}$ is called an $\mathcal{S}-$*measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context; that is, a set E is measurable if $E \in \mathcal{S}$.

Example 2.2.2. If $X = \mathbb{R}$ and $\mathcal{S}$ is the set of all subsets of $\mathbb{R}$ that are countable or have a countable complement, then the set of rational numbers is $\mathcal{S}-$measurable but the set of positive real numbers

2.2.2 Borel Subsets of $\mathbb{R}$

Proposition 2.2.3 (smallest $\sigma-$algebra containing a collection of subsets). Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all $\sigma-$algebras on X that contain $\mathcal{A}$ is a $\sigma-$algebra on X.

Proof. There is at least one $\sigma-$algebra on X that contains $\mathcal{A}$because the $\sigma-$algebra consisting of all subsets of X contains $\mathcal{A}$.

Let $\mathcal{S}$ be the intersection of all $\sigma-$algebras on X that contain $\mathcal{A}$. Since $\emptyset$ is in every $\sigma-$algebra on X, we see that $\emptyset \in \mathcal{S}$ as well.

Suppose $E \in \mathcal{S}$. Thus, E is in each $\sigma-$algebra on X. Therefore, $X \setminus E$ is in each $\sigma-$algebra on X and so $X \setminus E \in \mathcal{S}$

Let $E_1, E_2, \ldots$ be a sequence of elements of $\mathcal{S}$. ■

Definition 2.2.3 (Borel Sets)**.** A subset S of X is a Borel set if it is in the $\sigma-$algebra generated by the open subsets of X.

The "generating" means that S can be formed by a countable union of open subsets of X.

Definition 2.2.4 (Collection of Borel Sets)**.** The smallest $\sigma-$algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$ is called the collection of *Borel subsets* of $\mathbb{R}$. An element of this $\sigma-$algebra is called a *Borel Set.*

- The open subsets in the set above refer to the Borel sets.

Example 2.2.3 (Borel Sets)**.**

- Every closed subset of $\mathbb{R}$ is a Borel set because every closed subset of $\mathbb{R}$ is the complement of an open subset of $\mathbb{R}$. In other words, for every $a < b$ with $a, b \in \mathbb{R}$, we have $(a, b) \subseteq [a, b]$.
- Every countable subset of $\mathbb{R}$ is a Borel set because if $B = \{x_1, x_2, \dots\}$, then $B = \bigcup_{k=1}^{\infty}\{x_k\}$, which is a Borel set because each $\{x_k\}$ is a closed set.
- In $f : \mathbb{R} \to \mathbb{R}$ is a function, then the set of points at which f is continuous is the intersection of a sequence of open sets and thus is a Borel set.

- The intersection of every sequence of open subsets of $\mathbb{R}$ is a Borel set.
- On the other hand, the set of all such intersections is NOT the Borel sets since it is not closed under countable unions.
- Furthermore, the set of all countable unions of countable intersections is also not a set of Borel sets.

2.2.3 Inverse Images

Definition 2.2.5 (Inverse Image; $f^{(-1)}(A)$)**.** If $f : X \to Y$ is a function and $A \subset Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Proposition 2.2.4 (Algebra of Inverse Images)**.** Suppose $f : X \to Y$ is a function. Then

(a) $f^{-1}(\mathcal{Y} \setminus A) = X \setminus f^{-1}(A)$ for every $A \subset \mathcal{Y}$;

(b) $f^{-1}(\bigcup_{A\in\mathcal{A}} A) = \bigcup_{A\in\mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of $\mathcal{Y}$.

(c) $f^{-1}(\bigcap_{A\in\mathcal{A}} A) = \bigcap_{A\in\mathcal{A}} A$ for every set $\mathcal{A}$ of subsets of $\mathcal{Y}$.

Proof. Suppose $f : X \to \mathcal{Y}$ is a function.

(a) Suppose $A \subset \mathcal{Y}$. For $x \in X$, we have

$$\begin{aligned} x \in f^{-1}(\mathcal{Y} \setminus A) &\Leftrightarrow f(x) \in \mathcal{Y} \setminus A \\ &\Leftrightarrow f(x) \notin A \\ &\Leftrightarrow x \notin f^{-1}(A) \\ &\Leftrightarrow x \in X \setminus f^{-1}(A). \end{aligned}$$

Thus, we have $f^{-1}(\mathcal{Y} \setminus A) = X \setminus f^{-1}(A)$ which finishes the proof for part (a).

(b) Suppose $A \subset \mathcal{Y}$. For $x \in X$, we have

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{A \in \mathcal{A}} A\Big) &\Leftrightarrow f(x) \in \bigcup_{A \in \mathcal{A}} A \\ &\Leftrightarrow f(x) \in A && \text{(for some } A \in \mathcal{A}) \\ &\Leftrightarrow x \in f^{-1}(A) && \text{(for some } A \in \mathcal{A}) \\ &\Leftrightarrow x \in \bigcup_{A \in \mathcal{A}} f^{-1}(A). \end{aligned}$$

Thus, we conclude that

$$f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A).$$

(c) Again, suppose $A \subset \mathcal{Y}$. For $x \in X$, we have

$$\begin{aligned} x \in f^{-1}\Big(\bigcap_{A \in \mathcal{A}} A\Big) &\Leftrightarrow f(x) \in \bigcap_{A \in \mathcal{A}} A \\ &\Leftrightarrow f(x) \in A && \text{(for all } A \in \mathcal{A}) \\ &\Leftrightarrow x \in f^{-1}(A) && \text{(for all } A \in \mathcal{A}) \\ &\Leftrightarrow x \in \bigcap_{A \in \mathcal{A}} f^{-1}(A). \end{aligned}$$

Thus, we conclude that

$$f^{-1}\Big(\bigcap_{A \in \mathcal{A}} A\Big) = \bigcap_{A \in \mathcal{A}} f^{-1}(A).$$

■

Proposition 2.2.5 (Inverse Image of a Composition). Suppose $f : X \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{W}$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$$

for every $A \subset \mathcal{W}$.

Proof. Suppose $A \subset \mathcal{W}$. For $x \in X$, we have

$$\begin{aligned} x \in (g \circ f)^{-1}(A) &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow x \in f^{-1}(g^{-1}(A)). \end{aligned}$$

Thus, we have

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)).$$

■

2.2.4 Measurable Functions

Definition 2.2.6 (Measurable Function). Suppose $(X, \mathcal{S})$ is a measurable space. A function $f : X \to \mathbb{R}$ is called $\mathcal{S}-$*measurable* (or just *measurable* if $\mathcal{S}$ is clear from context) if

$$f^{-1}(B) \in S$$

for every Borel set $B \subset \mathbb{R}$.

Example 2.2.4 (Measurable Functions).
- If $\mathcal{S} = \{\emptyset, X\}$, then the only $\mathcal{S}-$measurable functions from X to $\mathbb{R}$ are the constant functions.
- If $\mathcal{S}$ is the set of all subsets of X, then every function X to $\mathbb{R}$ is $\mathcal{S}-$measurable.
-

Definition 2.2.7 (Characteristic Function χ_E). Suppose E is a subset of a set X. The *characteristic function* of E is the function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that X is not explicitly mentioned in the notation of the characteristic function since it always be clear from context.

- In order for a function to be $\mathcal{S}-$measurable, we require the inverse image of every Borel set to be in $\mathcal{S}$.
- The next result allows us to check that this is the case using a much smaller collection of subsets of $\mathbb{R}$.

Proposition 2.2.6 (Condition for Measurable Function). Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is a function such that

$$f^{-1}((a, \infty)) \in \mathcal{S}$$

for all $a \in \mathbb{R}$. Then f is an $\mathcal{S}-$measurable function.

Note that
$$f^{-1}((a, \infty)) = \{x \in X : f(x) > a\}.$$

Proof. Let
$$\mathcal{T} = \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{S}\}.$$

We want to show that every Borel subset of $\mathbb{R}$ is in $\mathcal{T}$. In order to do this, we need to show that $\mathcal{T}$ is a $\sigma-$algebra on $\mathbb{R}$. Certainly, $\emptyset \in \mathcal{T}$ since $f^{-1}(\emptyset) = \emptyset \in \mathcal{S}$.

If $A \in \mathcal{T}$, then $f^{-1}(A) \in \mathcal{S}$. Thus, we have

$$f^{-1}(\mathbb{R} \setminus A) = X \setminus f^{-1}(A)$$

by part (a) of Proposition 2.1.4. Since $\mathcal{S}$ is a $\sigma-$algebra, we have $X \setminus f^{-1}(A) \in \mathcal{S}$. Thus, $X \setminus f^{-1}(A) \in \mathcal{S}$ which tells us that $\mathbb{R} \setminus A \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under complementation.

Now, suppose $A_1, A_2, \dots \in \mathcal{T}$, then $f^{-1}(A_1), f^{-1}(A_2), \dots \in \mathcal{S}$. Since $\mathcal{S}$ is a $\sigma-$algebra, we can see that

$$f^{-1}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) \in \mathcal{S}. \qquad \text{(part (b) of Proposition 2.1.4)}$$

Thus, we see that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{T}$; that is, $\mathcal{T}$ is closed under countable unions. Hence, $\mathcal{T}$ is $\sigma-$algebra on $\mathbb{R}$.

By hypothesis, $\mathcal{T}$ contains $\{(a,\infty) : a \in \mathbb{R}\}$. Since $\mathcal{T}$ is closed under complementation, $\mathcal{T}$ also contains $\{(-\infty, b] : b \in \mathbb{R}\}$. Since $\mathcal{T}$ is closed under finite intersections, we see that $\mathcal{T}$ contains $\{(a,b] : a, b \in \mathbb{R}\}$. Since $(a,b) = \bigcup_{k=1}^{\infty}(a, b-\frac{1}{k}]$ and $(-\infty, b) = \bigcup_{k=1}^{\infty}(-k, b-\frac{1}{k}]$ and $\mathcal{T}$ is closed under countable unions, we can conclude that $\mathcal{T}$ contains every open subset of $\mathbb{R}$.

Thus, the $\sigma-$algebra $\mathcal{T}$ contains the smallest $\sigma-$algebra on $\mathbb{R}$ that contains all open subsets of $\mathbb{R}$; that is, $\mathcal{T}$ contains every Borel subset of $\mathbb{R}$. Thus, f is an $\mathcal{S}-$measurable function. ■

- To generalize the result above, we can replace the collection of sets $\{(a,\infty)\} : a \in \mathbb{R}$ by any collection of subsets of $\mathbb{R}$ such that the smallest $\sigma-$algebra containing that collection contains the Borel subsets of $\mathbb{R}$.

Definition 2.2.8 (Borel Measurable Function). Suppose $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called *Borel Measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbb{R}$.

- Suppose $X \subset \mathbb{R}$ and that there exists a Borel measurable function $f : X \to \mathbb{R}$. Thus, X is a Borel set (since $X = f^{-1}(\mathbb{R})$).
- If $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function, then f is a Borel measurable function if and only if $f^{-1}((a,\infty))$ is a Borel set for every $a \in \mathbb{R}$.

Proposition 2.2.7 (Every Continuous Function is Borel Measurable). Every continuous real-valued function defined on a Borel subset of $\mathbb{R}$ is Borel measurable function.

Proof. Suppose $X \subset \mathbb{R}$ is a Borel set and $f : X \to \mathbb{R}$ is continuous. To show that f is Borel Measurable., fix $a \in \mathbb{R}$.

If $x \in X$ and $f(x) > a$, then (by the continuity of f) there exists $\delta_x > 0$ such that $f(y) > a$ for all $y \in (x-\delta_x, x+\delta_x) \cap X$. Thus,

$$f^{-1}((a,\infty)) = \Big(\bigcup_{x\in f^{-1}\big((a,\infty)\big)} (x-\delta_x, x+\delta_x)\Big) \cap X.$$

Note that the union inside the large parentheses above is an open subset of $\mathbb{R}$; hence its intersecton with X is a Borel set. Thus, f is a Borel measurable function. ■

Definition 2.2.9 (Increasing Function). Suppose $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function.

- f is called *increasing* if $f(x) \le f(y)$ for all $x, y \in X$ with $x < y$.
- f is called *strictly increasing* if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

Proposition 2.2.8 (Every increasing function is Borel measurable). Every increasing function defined on a Borel subset of $\mathbb{R}$ of $\mathbb{R}$ is a Borel measurable function.

Proof. Suppose $X \subset \mathbb{R}$ is a Borel set and $f : X \to \mathbb{R}$ is increasing. To prove that f is Borel measurable, fix $a \in \mathbb{R}$. Let $b = \inf f^{-1}((a, \infty))$. Then we have either

$$f^{-1}((a,\infty)) = (b,\infty) \cap X \text{ or } f^{-1}((a,\infty)) = [b,\infty) \cap X.$$

Either way, we can conclude that $f^{-1}((a,\infty))$ is a Borel set. Thus, f is a Borel measurable function via result 2.39. ■

Proposition 2.2.9 (Composition of Measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is an $\mathcal{S}-$measurable function. Suppose g is a real-valued Borel measurable function defined on a subset of $\mathbb{R}$ that includes the range of f. Then $g \circ f : X \to \mathbb{R}$ is an $\mathcal{S}-$measurable function.

Proof. Suppose $B \subset \mathbb{R}$ is a Borel set. Since g is a real-valued Borel measurable function defined on a subset of $\mathbb{R}$ that includes the range of f, we can see that $g^{-1}(B)$ is a Borel set as well. Because f is $\mathcal{S}-$measurable and $g^{-1}(B)$ is a Borel set, we can see that $f^{-1}(g^{-1}(B)) \in \mathcal{S}$. By proposition 2.34, we see that

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \in \mathcal{S}.$$

Thus, $g \circ f$ is an $\mathcal{S}-$measurable function. ■

Proposition 2.2.10 (Algebraic Operations with Measurable Functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f, g : X \to \mathbb{R}$ are $\mathcal{S}-$measurable. Then

(a) $f + g, f - g$ and fg are $\mathcal{S}-$measurable functions;

(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}-$measurable function.

Proof. Suppose $a \in \mathbb{R}$. Our goal is to show that

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r \in \mathbb{Q}} \Big(f^{-1}((r,\infty)) \cap g^{-1}((a-r),\infty)\Big),$$

which implies that $(f+g)^{-1}((a,\infty)) \in \mathcal{S}$. Suppose

$$x \in (f+g)^{-1}((a,\infty)).$$

Then $(f+g)(x) \in (a,\infty)$. Thus, $a < f(x) + g(x)$. Hence, the open interval $(a - g(x), f(x))$ is nonempty, and therefore it contains some $r \in \mathbb{Q}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$). This implies that $r < f(x)$ which means $x \in f^{-1}((r,\infty))$ and $a - g(x) < r$, which implies that $x \in g^{-1}((a-r,\infty))$. Thus, we see that

$$x \in \bigcup_{r \in \mathbb{Q}} \Big(f^{-1}((r,\infty)) \cap g^{-1}((a-r),\infty)\Big).$$

Thus, we have $x \in f^{-1}((r,\infty))$ and $x \in g^{-1}((a-r,\infty))$ which implies

$$r < f(x) \text{ and } a - r < g(x),$$

respectively. We can see that adding these two inequalities gives us $a < f(x)+g(x)$. Hence, $(f+g)(x) > a$ and thus $x \in (f+g)^{-1}((a,\infty))$. So, $f+g$ is an $\mathcal{S}-$measurable function.

Note that $-g$ is an $\mathcal{S}-$measurable function since $-g$ can be written as a composition with a continuous function. Thus, we can write

$$f - g = f + (-g)$$

and conclude that the difference is also an $\mathcal{S}-$measurable function.

To show that fg is $\mathcal{S}-$measurable, we take note that fg can written as

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

Note that f^2, g^2, and $\frac{1}{2}f$ is also $\mathcal{S}-$measurable by the same reasoning that showed $-f$ as an $\mathcal{S}-$measurable function. Since the sum and difference of $\mathcal{S}-$measurable functions is $\mathcal{S}-$measurable, we see that fg is also $\mathcal{S}-$measurable, thereby completing the proof of (a).

Suppose $g(x) \neq 0$ for all $x \in X$. Since g defined on $\mathbb{R} \setminus \{0\}$, which is a Borel subset of $\mathbb{R}$, that takes x to $\frac{1}{x}$ is continuous, we can conclude that $\frac{1}{g}$ is an $\mathcal{S}-$measurable function. Since the product of two $\mathcal{S}-$measurable functions is $\mathcal{S}-$measurable, we can see that $\frac{f}{g}$ is an $\mathcal{S}-$ function as well, proving (b). ■

- The next result states that the pointwise limit of a sequence of $\mathcal{S}-$measurable functions is $\mathcal{S}-$measurable.
- This property is highly desirable for integration since the sequence of Riemann integrable functions is not closed under taking pointwise limits.

Proposition 2.2.11 (Limit of $\mathcal{S}-$measurable functions). Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}-$measurable functions from X to $\mathbb{R}$. Suppose $\lim\limits_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Then f is an $\mathcal{S}-$measurable function.

Proof. Suppose $a \in \mathbb{R}$. Our goal is to show that

$$f^{-1}((a,\infty)) = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1}((a+\frac{1}{j},\infty)), \tag{1}$$

which implies that $f^{-1}((a,\infty)) \in \mathcal{S}$. To prove (1), let $x \in f^{-1}((a,\infty))$. Thus, there exists $j \in \mathbb{Z}^+$ such that $f(x) > a + \frac{1}{j}$. Using the definition of limit, we can find an $m \in \mathbb{Z}^+$ such that $f_k(x) > a + \frac{1}{j}$ for all $k \geq m$. Thus, we have $x \in f_k^{-1}((a+\frac{1}{j},\infty))$, and so

$$x \in \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1}((a+\frac{1}{j},\infty)).$$

To show the other inclusion, suppose the above. Then there exists $j, m \in \mathbb{Z}^+$ such that $f_k(x) > a+\frac{1}{j}$ for all $k \geq m$. Taking the limit as $k \to \infty$, we see that $f(x) \geq a+\frac{1}{j} > a$ (taking the limit here involves using the order limit theorem). Thus, implying that $x \in f^{-1}((a,\infty))$.

Thus, we conclude that

$$f^{-1}((a,\infty)) = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1}((a+\frac{1}{j},\infty)),$$

and so f must be an $\mathcal{S}-$measurable function. ■

Definition 2.2.10. A subset of $[-\infty,\infty]$ is called a **Borel set** if its intersection with $\mathbb{R}$ is a Borel set.

- That is, we have $C \subset [-\infty,\infty]$ is a Borel set if and only if we can find a Borel set $B \subset \mathbb{R}$ such that $C = B$ or $C = B \cup \{\infty\}$ or $C = B \cup \{-\infty\}$ or $C = B \cup \{\infty, -\infty\}$.
- The set of Borel subsets of $[-\infty,\infty]$ forms a $\sigma-$algebra on $[-\infty,\infty]$.

Definition 2.2.11 (Measurable Function)**.** Suppose $(X,\mathcal{S})$ is a measurable space. A function $f : X \to [-\infty,\infty]$ is called $\mathcal{S}-$measurable if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset [-\infty,\infty]$.

Proposition 2.2.12 (Condition for Measurable Function)**.** Suppose $(X,\mathcal{S})$ is a measurable space and $f : X \to [-\infty,\infty]$ is a function such that

$$f^{-1}((a,\infty]) \in \mathcal{S}$$

for all $a \in \mathbb{R}$. Then f is an $\mathcal{S}-$measurable function.

The final result of this section states that the pointwise limit of the infimum and supremum of a sequence of $\mathcal{S}-$measurable functions is $\mathcal{S}-$measurable.

Proposition 2.2.13 (Infimum and Supremum of a Sequence of $\mathcal{S}-$measurable functions)**.** Suppose $(X,\mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}-$measurable functions from X to $[-\infty,\infty]$. Define $g, h : X \to [-\infty,\infty]$ by

$$g(x) = \inf\{f_k(x) : k \in \mathbb{Z}^+\} \text{ and } h(x) = \sup\{f_k : k \in \mathbb{Z}^+\}$$

Then g and h are $\mathcal{S}-$measurable functions.

Proof. Let $a \in \mathbb{R}$. To show that $h(x)$ is $\mathcal{S}-$measurable, we can use the definition of the supremum to imply that

$$h^{-1}((a,\infty]) = \bigcup_{k=1}^{\infty} f_k^{-1}((a,\infty]).$$

Let $x \in h^{-1}((a,\infty])$. Then we have $h(x) \in (a,\infty]$. Let $\varepsilon > 0$. By definition of $h(x)$, we can find a $k \in \mathbb{Z}^+$ such that

$$f_k(x) > h(x) - \varepsilon > a - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we find that $f_k(x) > a$ and thus $f_k(x) \in (a, \infty]$. Thus, we have

$$x \in \bigcup_{k=1}^{\infty} f_k^{-1}((a, \infty]).$$

To get the other inclusion, suppose x is in

$$\bigcup_{k=1}^{\infty} f_k^{-1}((a, \infty]).$$

Then $x \in f_k^{-1}((a, \infty])$ for some $k \in \mathbb{Z}^+$. Thus $f_k(x) \in (a, \infty]$. By definition of $h(x)$, we see that $h(x) \geq f_k(x) > a$. But this means that $x \in h^{-1}((a, \infty])$, which completes the proof for the other inclusion. This now implies that h is an $\mathcal{S}-$measurable function. Note that

$$g(x) = -\sup\{-f_k(x) : k \in \mathbb{Z}^+\}$$

for all $x \in X$. Hence, the result about the supremum also implies that g is an $\mathcal{S}-$measurable function. ■