0.1 Lecture 7

0.1.1 Topics

Our goal for this lecture is to discuss complex powers of a non-zero complex number.

0.1.2 Complex Powers

Let $z \in \mathbb{C}^\bullet$. We want to define z^w for $w \in \mathbb{C}$. Recall from real number system that, for $x \in \mathbb{R}_+$, $x^a = e^{a \ln x}$. We want our complex powers to extend this idea.

Definition. We define $z^w = \exp(w \log z)$. Note that

$$\begin{aligned} \log z &= \operatorname{Log} z + 2\pi i k \\ &= \ln|z| + i(\operatorname{Arg}(z) + 2\pi k), \;\; k \in \mathbb{Z} \end{aligned}$$

is a multi-valued function.

For this reason, the complex power of a non-zero complex number will also be multi-valued. Hence, by choosing a fixed k, we can find a specific value of z^w.

Example. (i) Compute i^i. Recall that $\log i = i\left(\frac{\pi}{2} + 2\pi k\right)$. Hence, we have

$$i^i = \exp(i \log i) = \{e^{\frac{-\pi}{2} + 2\pi k} : k \in \mathbb{Z}\}.$$

If we are just referring to the principal branch, then

$$i^i = e^{\frac{-\pi}{2}}.$$

(ii) Compute: $(-1)^{1+i}$. Note that

$$\log(-1) = i\pi + 2\pi i k = i(2k+1)\pi.$$

Then we see that for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} (-1)^{1+i} &= \exp((2k+1)\pi(i-1)) \\ &= e^{-(2k+1)\pi} \exp(e^{(2k+1)\pi i}) \\ &= e^{-(2k+1)\pi}\Big[\cos(2k+1)\pi + i\sin(2k+1)\pi\Big] \\ &= -e^{-(2k+1)\pi}. \end{aligned}$$

Let us summarize some facts about $\exp(z)$.

- For all $x \in \mathbb{R}$, we have $\exp(z) = e^x$. Thus, from now on, we write e^z in place of $\exp(z)$.
- For $z \in \mathbb{C}^\bullet$, we have $\log z$ is a multi-valued function. That is, we have

$$\log z = \ln|z| + i(\operatorname{Arg} z + 2\pi k), \;\; k \in \mathbb{Z}.$$

- For $z \in \mathbb{C}^\bullet$, we have $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ (this is the principal branch log).
- For $z \in \mathbb{C}^\bullet$, we have $z^w = e^{w \log z}$ is also a multi-valued function.

Example.
- Find all solutions of $\sin z = 1$.
- Find all possible solutions of $\cos z = i$.

Problem 1. Find all solutions of $e^z = i$.

Solution. ∎

0.2 Lecture 8

- Limits of a function
- Continuity
- Derivative

0.2.1 Functional Limit

Definition (Functional Limit)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. Let a be an accumulation point of D. Let $\ell \in \mathbb{C}$. We say that ℓ is **the limit of** $f(z)$ as z approaches to a if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $z \in D$ with $0 < |z - a| < \delta$, we have

$$|f(z) - \ell| < \varepsilon;$$

that is, $z \in D \cap (B(a, \delta) \setminus \{a\})$ implies $f(z) \in B(\ell, \varepsilon)$.

Example. Let $f : B(0, 1) \to \mathbb{C}$ defined by $f(z) = \frac{iz}{2}$ then $\lim_{z \to i} f(z) = -\frac{1}{2}$.

Proof. Observe that

$$|f(z) - \ell| = \left|\frac{i}{2}z - \left(-\frac{1}{2}\right)\right| = \left|\frac{i}{2}\right||z - i| = \frac{1}{2}|z - i|.$$

So, given any $\varepsilon > 0$, we can take any $0 < \delta < 2\varepsilon$ such that

$$0 < |z - i| < \delta$$

implies

$$|f(z) - \ell| < \varepsilon.$$

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Some facts about limits are:

Theorem. Let $D \subseteq \mathbb{C}$ with $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ be two functions let us write $f = u + iv$.

(i) Let $a = \alpha + i\beta$. Then $\lim_{z \to a} f(x) = \ell$ if and only if

$$\lim_{(x,y) \to (\alpha,\beta)} u(x, y) = \Re(\ell) \text{ and } \lim_{(x,y) \to (\alpha,\beta)} v(x, y) = \Im(\ell)\text{; that is,}$$

$$\lim_{z \to a} f(z) = \lim_{z \to a} u(z) + i \lim_{z \to a} v(z).$$

(ii) Let $\lim_{z \to a} f(z) = \ell$ and $\lim_{z \to a} g(z) = \ell'$ and $c, d \in \mathbb{C}$. Then

$$\lim_{z \to a}(cf(z) + dg(z)) = c\ell + d\ell'.$$

(iii) $\lim_{z \to a}(f(z)g(z)) = \lim_{z \to a} f(z) \cdot \lim_{z \to a} g(z)$.

(iv) We have

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$$

whenever $\lim_{z\to a} g(z) \neq 0$.

Problem 2. Let $\lim_{z\to i} f(z) = 2+i$ and $\lim_{z\to i} g(z) = 1-i$. Compute the following:

$$\lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2} \right].$$

Solution. Observe that

$$\begin{aligned}
\lim_{z\to i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2} \right] &= \lim_{z\to i} (f(z))^3 + \lim_{z\to i} \frac{(1+i)g(z)}{z^2} \\
&= (2+i)^3 + \frac{(1+i)(1-i)}{i^2} \\
&= (2+i)^3 - 2.
\end{aligned}$$

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Definition (Limits at Infinity). (i) Let $f : D \to \mathbb{C}$ be a function and $a \in \mathbb{C}$ be an accumulation point of D. Then we say $\lim_{z\to a} f(z) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that

$$z \in D \cap (B(a,\delta) \setminus \{a\})$$

implies $|f(z)| \geq M$; that is, f is unbounded as z approaches to a.

(ii) Let f be a complex function defined on the complement of a ball in $\mathbb{C}$. We say $\lim_{z\to\infty} f(z) = \ell$ if for all $\varepsilon > 0$, there exists $R > 0$ such that $|z| > R$ implies

$$|f(z) - \ell| < \varepsilon.$$

Example. (i) Consider $\lim_{z\to 0} \frac{1}{z} = \infty$. Let $M > 0$. Let $\delta = \frac{1}{M}$. Then $0 < |z| < \delta$ implies that

$$\left|\frac{1}{z}\right| > \frac{1}{\delta} = M.$$

(ii) Consider $\lim_{z\to\infty} \frac{1}{z} = 0$. Let $\varepsilon > 0$. Choose $R = \frac{1}{\varepsilon} > 0$. Then $|z| > R$ implies

$$\left|\frac{1}{z}\right| < \frac{1}{R} = \varepsilon.$$

Problem 3. (i) Show that $\lim_{z\to\infty} f(z) = \ell$ if and only if

$$\lim_{z\to\infty} f\left(\frac{1}{z}\right) = \ell.$$

(ii) $\lim_{z\to a} f(z) = \infty$ if and only if $\lim_{z\to a} \frac{1}{f(z)} = 0$.

(iii) Give a definition of $\lim_{z\to\infty} f(z) = \infty$. Show that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \infty$.

(iv) Compute $\lim_{z\to\infty} \frac{z-1}{z+i}$.

(v) Compute $\lim_{z\to\infty} \frac{gz+i}{z^2+z+1}$.

(vi) Can you compute $\lim_{z\to\infty} e^{-z}$?

0.2.2 Continuity

Definition (Continuity)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. We say f **is continuous at** $a \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(a, \delta) \cap D$, we have $f(z) \in B(f(a), \varepsilon)$.