1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Concepts such as
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

3 The First Defining Property

The set of real numbers is a field.

Definition (Fields). A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

4 The Second Defining Property

Definition (Ordered Fields). An **ordered field** is a field F equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

Remark. We say that x is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

Definition (Upper Bounds). Suppose F is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call β an **upper bound of** A.

Remark. We call the collection of upper bounds of A by $\mathrm{UP}(A)$. If $\mathrm{UP}(A) \neq \emptyset$, then we say that A **is bounded above**.

Similarly, we define the lower bounds of a set.

Definition (Lower Bounds). Suppose F is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then α is called the **lower bound of** A.

Remark. Similarly, we denote the set of lower bounds of A by $\mathrm{LO}(A)$. We say that A is bounded below if $\mathrm{LO}(A) \neq \emptyset$.

Example 4.1. Suppose we have $A = [0, 1)$. We have

$$\begin{aligned} \mathrm{UP}(A) &= [1, \infty) \\ \mathrm{LO}(A) &= (-\infty, 0]. \end{aligned}$$