# Measure Theory Notes

Lance Remigio

June 17, 2024

# Contents

# List of Theorems

# Chapter 1

# Riemann Integration

## 1.1 Review: Riemann Integral

Let $\mathbb{R}$ denote the complete ordered field of real numbers.

**Definition 1.1.1** (Partition). Suppose $a, b \in \mathbb{R}$ with $a < b$. A **partition** of $[a, b]$ is a finite list of the form $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We can think of closed intervals as union of closed subintervals $[x_{n-1}, x_n]$ for $n \geq 1$; that is,

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$

**Definition 1.1.2** (Infimum and Supremum of Functions). If $f$ is a real-valued function and $A$ is a subset of the domain of $f$, then

$$\inf_A f = \inf\{f(x) : x \in A\} \quad \text{and} \quad \sup_A f = \sup\{f(x) : x \in A\}$$

The definition above approximates the area under the graph of a nonnegative function or the signed area corresponding to a real-valued function.

**Definition 1.1.3** (Lower and Upper Riemann Sums). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and $P$ is a partition $x_0, \ldots, x_n$ of $[a, b]$. The **Lower Riemann Sum** $L(f, P, [a, b])$ and the **Upper Riemann Sum** $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

- The supremums represent an over approximation in terms of the height of each rectangle with $[x_{j-1}, x_j]$ being the width.
- Infimums are an under approximation compared to the supremum with $[x_{j-1}, x_j]$ being the width as before.

- The next result describes how adjoining more points to a partition *increases* the **lower Riemann sum** and *decreases* the **upper Riemann sum**.

**Proposition 1.1.1** (Inequalities with Riemann Sums). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and $P, P'$ are partitions of $[a, b]$ such that the list defining $P$ is a sublist of the list defining $P'$ (that is, $P \subseteq P'$). Then

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

**Proof.** Suppose $P$ is the partition $x_0, \ldots, x_n$ and $P'$ is the partition $x'_0, \ldots, x'_N$ of $[a, b]$. For each $j = 1, \ldots, n$ (that is, for each subinterval $[x_{j-1}, x_j]$), we can find $k \in \{0, \ldots, N-1\}$ and a positive integer $m$ such that

$$x_{j-1} = x'_k < x'_{k+1} < \cdots < x'_{k+m} = x_j.$$

Observe that

$$\begin{aligned} L(f, P, [a, b]) = (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f &= \sum_{i=1}^{m} (x'_{k+i} - x'_{(k+i)-1}) \inf_{[x_{j-1}, x_j]} f \\ &\leq \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x'_{k+i-1}, x'_{k+i}]} f \\ &= L(f, P', [a, b]). \end{aligned}$$

Thus, we have that $L(f, P, [a, b]) \leq L(f, P', [a, b])$. The middle inequality can be proven by observing that the infimum of any set is always less than or equal to the supremum of that set. In regards to the partition $P'$, we have $L(f, P', [a, b]) \leq U(f, P', [a, b])$. To get the last inequality, we can follow the same argument we used to prove the first inequality. Hence, we have

$$\begin{aligned} U(f, P, [a, b]) = (x_j - x_{j-1}) \sup_{[x_{j-1}, x_{j-1}]} f &= \sum_{j=1}^{m} (x_{k+i} - x_{(k+i)-1}) \sup_{[x_{j-1}, x_j]} f \\ &\geq \sum_{j=1}^{m} (x_{k+i} - x_{(k+i)-1}) \sup_{[x_{k+i}, x_{(k+i)-1}]} f \\ &= U(f, P', [a, b]). \end{aligned}$$

Thus, we have that $U(f, P', [a, b]) \leq U(f, P, [a, b])$ in which we now conclude that

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

■

**Proposition 1.1.2** (Lower Riemann Sums $\leq$ Upper Riemman Sums). Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and $P, P'$ are partitions of $[a, b]$. Then

$$L(f, P, [a, b]) \leq U(f, P', [a, b]).$$

**Proof.** Let $P''$ be the partition of $[a, b]$ obtained by unioning partitions $P$ and $P'$; that is,

$P'' = P \cup P'$. Since $P'' \subseteq P$ and $P'' \subseteq P'$, we have that

$$\begin{aligned} L(f, P, [a,b]) &\leq L(f, P'', [a,b]) \\ &\leq U(f, P'', [a,b]) \\ &\leq U(f, P', [a,b]) \end{aligned}$$

by the result before. ■

Next, we define the upper and lower Riemann integrals.

**Definition 1.1.4** (lower and upper Riemann integrals). Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. The **lower Riemann integral** $L(f, [a,b])$ and the **upper Riemann integral** $U(f, [a,b])$ of $f$ are defined by

$$L(f, [a,b]) = \sup_P L(f, P, [a,b])$$

and

$$U(f, [a,b]) = \inf_P U(f, P, [a,b]),$$

where the supremum and infimum above are taken over all partitions $P$ of $[a,b]$.

- For the first equation above, we are taking the supremum of all the **Lower Riemann Sums**. This makes sense considering the lower Riemann sums become larger as we add more points to our partition.
- Likewise, adding more points to a partition causes the upper Riemann sum to become smaller, leading to the infimum over all the partitions.
- In both of these cases, we reach a more accurate estimate of the area under the graph of a function.

**Proposition 1.1.3** (Lower Riemann Integral $\leq$ Upper Riemann Integral). Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. Then

$$L(f, [a,b]) \leq U(f, [a,b]).$$

**Proof.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Notice that, by definition,

$$L(f, [a,b]) = \sup_P L(f, P, [a,b]);$$

that is, $L(f, [a,b])$ is a supremum over all partitions $P$ of all the lower Riemann sums. Using a lemma from undergrad analysis, $L(f, [a,b]) = \sup_P L(f, P, [a,b])$ if and only if for all $\varepsilon > 0$, there exists $L(f, P_\varepsilon, [a,b]) \in \{L(f, P, [a,b]) : \text{for all } P \in P\}$ such that

$$L(f, [a,b]) - \varepsilon < L(f, P_\varepsilon, [a,b]).$$

Now, consider the upper Riemann integral $U(f, [a,b])$. Since

$$U(f, [a,b]) = \inf\{U(f, P, [a,b]) : \text{for all } P \in P\}$$

and the fact that

$$\{U(f, P, [a,b]) : \text{for all } P \in P\}$$

is a set of supremums (upper bounds of $f$ over each partition $P$), we can see that by Proposition 1.1.1 that

$$L(f,[a,b]) - \varepsilon < L(f,P_\varepsilon,[a,b]) \leq U(f,[a,b]).$$

Since $\varepsilon > 0$ is arbitrary, we see that $L(f,[a,b]) \leq U(f,[a,b])$. ■

**Definition 1.1.5** (Riemann integrable; Riemann Integral).

- A bounded function on a closed interval is called **Riemann Integrable** if its lower Riemann integral equals its upper Riemann integral.
- If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then the **Riemann integral** $\int_a^b f$ is defined by

$$\int_a^b f = L(f,[a,b]) = U(f,[a,b]).$$

**Proposition 1.1.4.** Every continuous real-valued function on each closed bounded interval is Riemann integrable.

**Proof.** Let $a,b \in \mathbb{R}$ with $[a,b]$ being a compact interval. Let $f : [a,b] \to \mathbb{R}$ and let $P$ be a partition of $[a,b]$. Let $\varepsilon > 0$. Since $f$ is continuous on the compact interval $[a,b]$, $f$ is uniformly continuous and must, therefore, attain its maximum and minimum on $[a,b]$. Thus, assign $M_j = f(y_j)$ and $m_j = f(z_j)$ where

$$M_j = \sup_{x\in[x_{j-1},x_j]} f \quad \text{and} \quad m_j = \inf_{x\in[x_{j-1},x_j]} f.$$

Furthermore, there exists a $\delta > 0$ such that whenever $|y_j - z_j| < \delta$, we have

$$|M_j - m_j| = |f(y_j) - f(z_j)| < \frac{\varepsilon}{b-a}.$$

Then we have

$$\begin{aligned} U(f,P,[a,b]) - L(f,P,[a,b]) &= \sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) \\ &< \frac{\varepsilon}{b-a}\sum_{j=1}^n (x_j - x_{j-1}) \\ &= \frac{\varepsilon}{(b-a)} \cdot (b-a) = \varepsilon. \end{aligned}$$

Since $U(f,[a,b]) \leq U(f,P,[a,b])$ and $L(f,[a,b]) \geq L(f,P,[a,b])$, we have

$$U(f,[a,b]) - L(f,[a,b]) \leq U(f,P,[a,b]) - L(f,P,[a,b]) < \varepsilon.$$

Since $\varepsilon$ is arbitrary, we must have $U(f,[a,b]) = L(f,[a,b])$. Thus, $f$ is integrable. ■

**Proposition 1.1.5** (Bounds on Rieman Integral). Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Then

$$(b-a)\inf_{[a,b]} f \leq \int_a^b f \leq (b-a)\sup_{[a,b]} f.$$

**Proof.** Let $P$ be the partition consisting of points $x_0 = a$ and $x_1 = b$. Since $f : [a, b] \to \mathbb{R}$ is Riemann integrable, we have

$$\int_a^b f = L(f, [a, b]) = U(f, [a, b]).$$

Starting with the first inequality, observe that

$$\int_a^b f = L(f, [a, b]) \geq L(f, P, [a, b]) = (b - a) \inf_{[a,b]} f.$$

Observe that the second inequality can be proven by a similar argument

$$\int_a^b f = U(f, [a, b]) \leq U(f, P, [a, b]) = (b - a) \sup_{[a,b]} f.$$

Thus, we have reached our desired result that

$$(b - a) \inf_{[a,b]} f \leq \int_a^b f \leq (b - a) \sup_{[a,b]} f.$$

■

### 1.1.1 Exercises

1. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function such that

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

for some partition $P$ of $[a, b]$. Prove that $f$ is a constant function on $[a, b]$.

**Proof.** ■

2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that $f$ is Riemann integrable if and only if for each $\varepsilon > 0$, there exists a partition $P$ of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \varepsilon.$$

**Proof.** ■

## 1.2 Riemann Integral Is Not Good Enough

There are three disadvantages of using Riemann integration; that is, Riemann integration does not handle

- Functions with many discontinuities;
- Unbounded functions;
- Does not work well with limits.

**Example 1.2.1** (A function that is not Riemann integrable). Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

If $[a,b] \subset [0,1]$ with $a < b$, then we see that

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

because $[a,b]$ contains an irrational number and contains a rational number. So,

$$L(f, P, [0,1]) = 0 \quad \text{and} \quad U(f, P, [0,1]) = 1$$

for every partition $P$ of $[0,1]$. This immediately implies that $L(f,[0,1]) = 0 \neq 1 = U(f,[0,1])$ and thus, $f$ is not Riemann integrable.

**Example 1.2.2** (Rieman integration with unbounded functions). Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

If $x_0, x_1, \ldots, x_n$ is a partition of $[0,1]$, then $\sup_{[x_0,x_1]} = \infty$. This would mean that in every partition $P$ of $[0,1]$, we would have $U(f,P,[0,1]) = \infty$ if we had applied Riemann integration. But note that the area of this function is 2 since

$$\lim_{a\to 0} \int_a^1 f - \lim_{a \to 0} (2 - 2\sqrt{a}) = 2.$$

This can be remedied by other methods seen in introductory Calculus courses, but in general, using Riemann integration over subdomains tends to fail, especially with more complicated functions than the one shown in the example above.

**Example 1.2.3** (Area seems to make sense, but Riemann integral is not defined). Let $r_1, r_2, \ldots$ be a sequence that includes each rational number in $(0,1)$ exactly once and that includes no other numbers. For $k \in \mathbb{Z}^+$, define $f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{\sqrt{x - r_k}} & \text{if } x > r_k \\ 0 & \text{if } x \leq r_k. \end{cases}$$

Define $f : [0,1] \to [0,\infty]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}.$$

Since every nonempty open interval of $[0,1]$ contains a rational number (Density of $\mathbb{Q}$ in $\mathbb{R}$), we see that $f$ is unbounded on every such subinterval (contains a sequence of rational numbers). However, the area under the graph of each $f_k$ is less than 2. The formula defining $f$ then shows that we should expect the area under the graph of $f$ to be less than 2 rather than being undefined.

**Proposition 1.2.1** (Interchanging Riemann integral and limit). Suppose $a, b, M \in \mathbb{R}$ with $a < b$. Suppose $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a, b]$ such that

$$|f_k(x) \leq M|$$

for all $k \in \mathbb{Z}^+$ and all $x \in [a, b]$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in [a, b]$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \lim_{k\to\infty f_k(x)} .$$

If $f$ is Riemann integrable on $[a, b]$, then

$$\int_a^b f = \lim_{k\to\infty} \int_a^b f_k.$$

We have the following problems with the result above:

- We have to assume that $f$ is Riemann integrable.
- The result follows even if $f$ is not Riemann integrable (Of course, assuming all the other hypotheses hold true).
- The proof of the result above is intricate and delicate. We should find that the proof should be "simple".

# Chapter 2

# Measures

## 2.1 Outer Measure on $\mathbb{R}$

### 2.1.1 Motivation and Definition of Outer Measure

- This section focuses on developing the notion of assigning a "size" to subintervals when it comes to more complicated unions of open intervals.
- The hope is to create an integration theory that will cover a broader class of functions than covered by Riemann integration.
- We start off by giving a definition of the length of an open interval.

**Definition 2.1.1** (Length of Open Interval; $\ell(I)$). The **length** $\ell(I)$ of an open interval $I$ is defined bt

$$\ell(I) = \begin{cases} b-a & \text{if } I = (a,b) \text{ for some } a,b \in \mathbb{R} \text{ with } a < b, \\ 0 & \text{if } I = \emptyset, \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R}, \\ \infty & \text{if } I = (-\infty, \infty). \end{cases}$$

Suppose we have a subset $A \subset \mathbb{R}$. Then the size of $A$ should be the sum of the lengths of a sequence of open intervals whose union contains $A$. The infimum of all possible sums gives the definition of the size of $A$, denoted $|A|$ which we call the **outer measure** of $A$.

**Definition 2.1.2** (Outer Measure; $|A|$). The **outer measure** $|A|$ of a set $A \subset \mathbb{R}$ is defined by

$$|A| = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

- Notice that the definition of outer measure involves an infinite sum.
- Recall that the infinite sum converges if the sequence of partial sums converges.
- If one of the elements in the sequence is $\infty$ (that is, $t_k = \infty$) for some $k$, then the infinite sum is $\infty$.

- Otherwise, the infinite sum $\sum_{k=1}^{\infty t_k}$ is defined to be the limit of the increasing sequence of partial sums where

$$\sum_{k=1}^{\infty} t_k = \lim_{n\to\infty} \sum_{k=1}^{n} t_k.$$

### 2.1.2 Properties of Outer Measure

**Proposition 2.1.1** (Countable sets have outer measure 0). Every countable subset of $\mathbb{R}$ has outer measure 0.

**Proof.** Suppose $A = \{a_1, a_2, \dots\}$ is a countable subset of $\mathbb{R}$. Let $\varepsilon > 0$. For $k \in \mathbb{Z}^+$, let

$$I_k = \left(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\right).$$

Then $I_1, I_2, \dots$ is a sequence of open intervals whose union contains $A$. Since

$$\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon,$$

we have $|A| \leq 2\varepsilon$. Since $\varepsilon$ is an arbitrary positive number, we must have $|A| = 0$. ■

- Note that $\mathbb{Q}$ is a countable set, and thus contains a measure of 0.

**Proposition 2.1.2** (Outer Measure Preserves Order). Suppose $A$ and $B$ are subsets of $\mathbb{R}$ with $A \subset B$. Then $|A| \leq |B|$.

**Proof.** Suppose $I_1, I_2, \dots$ is a sequence of open intervals such that

$$B \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Since $A \subseteq B$, we can see that $A$ is also contained within

$$\bigcup_{k=1}^{\infty} I_k.$$

Thus, we have

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k).$$

If we take the infimum over all the sequences of open intervals whose union contains $B$, we see that $|A| \leq |B|$. ■

**Definition 2.1.3** (Translation; $t + A$). If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then the translation $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

- If $t > 0$, then translating $A$ by adding $t$ moves $A$ by $|t|$ units to the right.
- If $t < 0$, then translating $A$ by adding $t$ moves $A$ to the left by $|t|$ units.

- In a similar manner to how functions do not change their shape when translated either horizontally or vertically, intervals do not change their shape/length when moved either left or right on the number line.
- By standard convention, we take $t + (-\infty) = -\infty$ and $t + \infty = \infty$.

**Proposition 2.1.3 (Outer Measure is Translation Invariant).** Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$. Then $|t + A| = |A|$.

**Proof.** Let $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. Suppose $I_1, I_2, \dots$ is a sequence of open intervals such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

Then $t + I_1, t + I_2, \dots$ is a sequence of open intervals such that

$$t + A \subseteq \bigcup_{k=1}^{\infty} t + I_k.$$

Since translation by $t$ of intervals $I_k$ of $A$ does not change, we have that

$$\ell(t + I_k) = \ell(I_k).$$

So, we see that

$$|t + A| \leq \sum_{k=1}^{\infty} \ell(t + I_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

But this tells us that the union of the sequence of intervals $I_k$ of $A$ contains $t + A$. So, taking the infimum over all such sequences, we have $|t + A| \leq |A|$.

Now, observe that $A = -t + (t + A)$ for $t \in \mathbb{R}$. Then we see that

$$|A| = |-t + (t + A)| \leq |t + A|.$$

Thus, $|A| = |t + A|$. ■

**Proposition 2.1.4 (Countable Subadditivity of Outer Measure).** Suppose $A_1, A_2, \dots, \dots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \leq \sum_{k=1}^{\infty} |A_k|.$$

**Proof.** ■