# Homework 4

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**Problem 1.** Prove the following theorem.

**Theorem** (Generalize Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ is integrable and either $g \geq 0$ on $[a, b]$ or $g \leq 0$ on $[a, b]$. Then there exists a $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g. \tag{*}$$

**Proof.** Our goal is to find a $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

Since $f : [a, b] \to \mathbb{R}$ is a continuous function and $[a, b]$ is a compact set, $f$ attains its maximum and minimum on $[a, b]$. Then

$$m = \sup_{x \in [a,b]} f(x) \text{ and } m = \inf_{x \in [a,b]} f(x)$$

and hence,

$$m \leq f(x) \leq M \Longrightarrow mg(x) \leq f(x)g(x) \leq Mg(x) \ \forall x \in [a, b]. \tag{†}$$

Now, note that the inequality in follows immediately if $g(x) = 0$ for all $x \in [a, b]$. Hence, suppose that $g(x) \neq 0$ on $[a, b]$. By the Order Theorem for Integrals, (†) implies

$$\int_a^b mg \leq \int_a^b fg \leq \int_a^b Mg$$

which further implies (by the algebraic theorem for integrals) that

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g. \tag{1}$$

From (1), we may consider two cases; that is, either $\int_a^b g \neq 0$ or $\int_a^b g = 0$. If $\int_a^b g \neq 0$, then we can divide by $\int_a^b g$ on (1). So, we have

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

But note that $f$ is continuous on $[a, b]$ and so, by the Intermediate Value Theorem, there exists a $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b fg}{\int_a^b g} \Longrightarrow \int_a^b fg = f(c) \int_a^b g$$

which is our desired result.

If $\int_a^b g = 0$, then it follows from (1) that

$$\int_a^b fg = 0$$

and so

$$\int_a^b fg = f(c) \int_a^b g$$

for all $c \in [a, b]$. ■

**Problem 2.** Prove the following theorem.

**Theorem** (Rudin, Theorem 6.15). If $a < s < b$, $f$ is bounded on $[a, b]$, $f$ is continuous at $s$, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f \; d\alpha = f(s).$$

**Proof.** Our goal is to show that $\int_a^b f \; d\alpha = f(s)$. Define the partition

$$P_n = \left\{a, s - \frac{1}{n}, s + \frac{1}{n}, b\right\}.$$

Since $f \in R_\alpha[a, b]$, we have

$$\int_a^b f \; d\alpha = \lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n). \tag{*}$$

So, it suffices to show that

$$\lim_{n\to\infty} U(f, \alpha, P_n) = \lim_{n\to\infty} L(f, \alpha, P_n) = f(s).$$

Note that

$$\alpha(x) = I(x - s) = \begin{cases} 1 & \text{if } x > s \\ 0 & \text{if } x \leq s \end{cases}$$

By definition of $\alpha$, we have

$$\begin{aligned} k &= 1; \Delta\alpha_1 = \alpha(s - \delta) - \alpha(a) = 0 - 0 = 0 \\ k &= 2; \Delta\alpha_2 = \alpha(s + \delta) - \alpha(s - \delta) = 1 - 0 = 1 \\ k &= 3; \Delta\alpha_3 = \alpha(b) - \alpha(s + \delta) = 1 - 1 = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} U(f, \alpha, P_n) &= \sum_{k=1}^{3} M_k \Delta\alpha_k \\ &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3 \\ &= 0 + M_2 \cdot 1 + 0 \\ &= M_2 \cdot 1 \\ &= \sup_{x \in [s - \frac{1}{n}, s + \frac{1}{n}]} f(x) \end{aligned}$$

and similarly,

$$L(f, \alpha, P_n) = \sum_{k=1}^{3} m_k \Delta\alpha_k = m_2 = \inf_{x \in [s - \frac{1}{n}, s + \frac{1}{n}]} f(x).$$

Because $f$ is continuous at $s$, we know by exercise 4 of homework 3 that

$$\lim_{n\to\infty} U(f,\alpha,P_n) = \lim_{n\to\infty} \sup_{x\in[s-\frac{1}{n},s+\frac{1}{n}]} f(x) = f(s)$$

and

$$\lim_{n\to\infty} L(f,\alpha,P_n) = \lim_{n\to\infty} \inf_{x\in[s-\frac{1}{n},s+\frac{1}{n}]} f(x) = f(s)$$

which is our desired result. ■

**Problem 3.** Prove the following theorem.

**Theorem** (Rudin, Theorem 6.16). (a) Let $N \in \mathbb{N}$. Let $c_1,\dots,c_N$ be nonnegative numbers. Suppose $s_1,\dots,s_N$ are distinct points in $(a,b)$, and let $\alpha(x) = \sum_{n=1}^N c_n I(x-s_n)$. Let $f$ be continuous at $s$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^N c_n f(s_n).$$

(b) Suppose $c_n \geq 0$ for $n \in \mathbb{N}$, $\sum_{n=1}^\infty c_n$ converges, $(s_n)$ is a sequence of distinct points in $(a,b)$, and $\alpha(x) = \sum_{n=1}^\infty c_n I(x-s_n)$. Let $f$ be continuous at $s$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^\infty c_n f(s_n).$$

**Proof.** (i) Denote $\alpha_n = I(x-s_n)$ for $1 \leq n \leq N$. Note that we can easily show that each $\alpha_n$ is an increasing function by definition of $I(x-s_n)$. Since $f$ is continuous on $[a,b]$, we see that $f \in R_{\alpha_n}[a,b]$ for each $1 \leq n \leq N$. Furthermore, we can easily show, through an induction argument and by an exercise 5 from homework 3, that

$$\int_a^b f \, d\Big(\sum_{n=1}^N c_n a_n\Big) = \sum_{n=1}^N \int_a^b f \, d(c_n a_n) \tag{1}$$

Also, note that for all $1 \leq n \leq N$

$$\int_a^b f \, d\alpha_n = f(s_n) \tag{2}$$

by applying the previous exercise for each $1 \leq n \leq N$. Then (1) and (2) imply that

$$\begin{aligned}
\int_a^b f \, d\alpha &= \int_a^b \, d\Big(\sum_{n=1}^N c_n\alpha_n\Big) \\
&= \sum_{n=1}^N \int_a^b f \, d(c_n\alpha_n) \\
&= \sum_{n=1}^N c_n \int_a^b f \, d\alpha_n && \text{(Exercise 5 from HW3)} \\
&= \sum_{n=1}^N c_n f(s_n).
\end{aligned}$$

Hence, we have that

$$\int_a^b f \, d\alpha = \sum_{n=1}^N c_n f(s_n).$$

(ii) Our goal is to show that

$$\int_a^b f\ d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

where $c_n \geq 0$ for all $n \in \mathbb{N}$. That is, we want to show that for any $\varepsilon > 0$, there exists $K$ such that for any $N > K$ that

$$\Big|\sum_{n=1}^{N} c_n f(s_n) - \int_a^b f\ d\alpha\Big| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Note that $I(x - s_n) \leq 1$ for all $x \in [a, b]$ and so we have

$$0 \leq c_n I(x - s_n) \leq c_n.$$

Since $\sum c_n$ is a convergent series, it follows from applying the comparison test to the above inequality that $\sum c_n I(x-s_n)$ converges. Observe that $\alpha$ is also an increasing function. Indeed, we can easily see that this is the case by using the fact that $\sum c_n I(x - s_n)$ is a convergent series and applying the order limit theorem for any $x < y$ in $[a, b]$. Since $\alpha$ is an increasing function on $[a, b]$ and $f$ is a continuous function on $[a, b]$, we have that $f \in R_\alpha[a, b]$. Now, consider $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ where

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n) \ \text{ and } \ \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n).$$

Also, note that by an exercise in homework 3, we have

$$\int_a^b f\ d\alpha = \int_a^b f\ d\alpha_1 + \int_a^b f\ d\alpha_2. \tag{1}$$

By part (a), we see that

$$\int_a^b f\ d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n).$$

Hence, the left-hand side of (*) can be written as (by using (1) and the equality above)

$$\begin{aligned}\Big|\sum_{n=1}^{N} c_n f(s_n) - \int_a^b f\ d\alpha\Big| &= \Big|\int_a^b f\ d\alpha_2 - \int_a^b f\ d\alpha\Big| \\ &= \Big|\int_a^b f\ d\alpha_2\Big|.\end{aligned}$$

Since $f$ is bounded on $[a, b]$ (because it is continuous on the compact interval $[a, b]$), we have that for some $\tilde{M} > 0$, we have $|f(x)| \leq \tilde{M}$ for all $x \in [a, b]$. By the triangle inequality for integrals, we can see that

$$\Big|\int_a^b f\ d\alpha_2\Big| \leq \int_a^b |f|\ d\alpha_2 \leq \tilde{M}(\alpha_2(b) - \alpha_2(a))$$

by a Theorem proven in lecture. Note that on the above inequality, we see that $|f| \in R_{\alpha_2}[a, b]$ since $f \in R_{\alpha_2}[a, b]$. Next, consider the difference $\alpha_2(b) - \alpha_2(a)$. Then observe that

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^{\infty} c_n$$

by definition of $I(x - s_n)$. Indeed, we see that $\alpha_2(a) = 0$ since $a < s_n$ and $\alpha_2(b) = 1$ since $a > s_n$. Since $\sum c_n$ converges, it follows from an exercise done in 230A that

$$\lim_{N \to \infty} \sum_{n=N+1}^{N} c_n = 0.$$

As a consequence, we can find a $\hat{K}$ such that for any $n > \hat{K}$

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^{\infty} c_n < \frac{\varepsilon}{\tilde{M}}.$$

Note that we dropped the absolute value on the above quantity because of the fact that $c_n$ is nonnegative for all $n \in \mathbb{N}$. We claim that this is the desired $K$ we were looking for. Indeed, for any $n > \hat{K}$, we have

$$\begin{aligned} \Big| \sum_{n=1}^{N} c_n f(s_n) - \int_a^b f \, d\alpha \Big| &= \Big| \int_a^b f \, d\alpha_2 \Big| \\ &\leq \tilde{M}(\alpha_2(b) - \alpha_2(a)) \\ &< \tilde{M} \cdot \frac{\varepsilon}{\tilde{M}} \\ &= \varepsilon \end{aligned}$$

which is our desired result.

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**Problem 4.** Let $p, q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

4-1) Prove that if $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

then $\int_a^b fg \, d\alpha \leq 1$.

4-2) Prove that if $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, then

$$\int_a^b |fg| \, d\alpha \leq \Big[ \int_a^b |f|^p \, d\alpha \Big]^{\frac{1}{p}} \Big[ \int_a^b |g|^q \, d\alpha \Big]^{\frac{1}{q}}.$$

**Proof.** 4-1) Since $f \geq$, $g \geq 0$, and $p, q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}.$$

Note that if $f, g = 0$ on $[a, b]$, then the result immediately holds. So, suppose $f, g > 0$ on $[a, b]$. Since $f \in R_\alpha[a, b]$ and $g \in R_\alpha[a, b]$, we can use the Order Theorem for Integrals and the

Algebraic Theorem for integrals to write

$$\begin{aligned}\int_a^b |fg|\ d\alpha &= \int_a^b fg\ d\alpha && (fg>0)\\ &\le \int_a^b \left[\frac{f^p}{p}+\frac{g^q}{q}\right]\ d\alpha\\ &= \int_a^b \frac{f^p}{p}\ d\alpha + \int_a^b \frac{g^q}{q}\ d\alpha\\ &= \frac{1}{p}\int_a^b f^p\ d\alpha + \frac{1}{q}\int_a^b g^q\ d\alpha\\ &= \frac{1}{p}+\frac{1}{q}\\ &= 1.\end{aligned}$$

Hence, we have

$$\int_a^b |fg|\ d\alpha \le 1.$$

4-2) Our goal is to show that

$$\int_a^b |fg|\ d\alpha \le \Big(\int_a^b |f|^p\ d\alpha\Big)^{\frac{1}{p}}\Big(\int_a^b |g|^q\ d\alpha\Big)^{\frac{1}{q}}.$$

Let

$$A = \Big(\int_a^b |f|^p\ d\alpha\Big)^{\frac{1}{p}} \text{ and } B = \Big(\int_a^b |g|^q\ d\alpha\Big)^{\frac{1}{q}}.$$

We will consider two cases:

(i) $(A, B \neq 0)$ Let

$$\tilde{f(x)} = \frac{f(x)}{A} = \frac{f(x)}{\Big(\int_a^b |f|^p\ d\alpha\Big)^{\frac{1}{p}}}$$

and

$$\tilde{g(x)} = \frac{g(x)}{B} = \frac{g(x)}{\Big(\int_a^b |g|^q\ d\alpha\Big)^{\frac{1}{q}}}.$$

From here, it suffices to show that

$$\int_a^b |\tilde{f}||\tilde{g}|\ d\alpha \le 1.$$

Note that

$$\begin{aligned}\int_a^b |\tilde{f}|^p\ d\alpha &= \int_a^b \frac{|f|^p}{\int_a^b |f|^p\ d\alpha}\ d\alpha\\ &= \frac{1}{\int_a^b |f|^p\ d\alpha}\int_a^b |f|^p\ d\alpha = 1.\end{aligned}$$

Similarly, we have

$$\int_a^b |\tilde{g}|^q\ d\alpha = 1.$$

So, by (4-1), we have

$$\int_a^b |\tilde{f}||\tilde{g}|\ d\alpha \le 1.$$

(ii) ($A = 0$ or $B = 0$) From this case, we can see that

$$\int_a^b |f|^p \, d\alpha = 0 \quad \text{or} \quad \int_a^b |g|^q \, d\alpha = 0.$$

Our goal is to show that

$$\int_a^b |fg| \, d\alpha \leq \Big( \int_a^b |f|^p \, d\alpha \Big)^{\frac{1}{p}} \Big( \int_a^b |g|^q \, d\alpha \Big)^{\frac{1}{q}}.$$

It suffices to show that the left-hand side of the above inequality is zero. Suppose that

$$\int_a^b |f|^p \, d\alpha = 0.$$

The proof for the other case follows analogously. Indeed, by using Young's Inequality and a linearity property of the R.S integral, we have

$$\begin{aligned} 0 \leq \int_a^b |fg| \, d\alpha &\leq \int_a^b \Big( \frac{|f|^p}{p} + \frac{|g|^q}{q} \Big) \, d\alpha \\ &= \frac{1}{p} \int_a^b |f|^p \, d\alpha + \frac{1}{q} \int_a^b |g| \, d\alpha \\ &= \frac{1}{q} \int_a^b |g|^q \, d\alpha. \end{aligned}$$

Hence,

$$0 \leq \int_a^b |fg| \, d\alpha \leq \frac{1}{q} \int_a^b |g|^q \, d\alpha.$$

Note that for all $r > 0$, we have

$$0 \leq \int_a^b |r \cdot fg| \, d\alpha \leq \frac{1}{q} \int_a^b r^q |g|^q \, d\alpha.$$

That is,

$$0 \leq r \int_a^b |fg| \, d\alpha \leq \frac{1}{q} r^q \int_a^b |g|^q \, d\alpha \Longrightarrow 0 \leq \int_a^b |fg| \, d\alpha \leq \frac{r^{q-1}}{q} \int_a^b |g|^q \, d\alpha.$$

If we let $r \to 0$, we will get that

$$\frac{r^{q-1}}{q} \int_a^b |g|^q \, d\alpha \to 0$$

and similarly, the left-hand side of the above inequality also goes to 0 as $r \to 0$. Hence, the Squeeze Theorem implies that

$$\int_a^b |fg| \, d\alpha \to 0$$

as $r \to 0$. That is,

$$\int_a^b |fg| \, d\alpha = 0.$$

■

**Problem 5.** Suppose $f \in C^1[a,b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)\ dx = 1$. Prove that

$$\int_a^b xf(x)f'(x)\ dx = \frac{-1}{2}$$

and

$$\int_a^b [f'(x)]^2\ dx \cdot \int_a^b x^2 f^2(x)\ dx \geq \frac{1}{4}.$$

**Proof.** Using Integration by Parts, we have

$$\begin{aligned}\int_a^b xf(x)f'(x)\ dx &= xf^2(x)\Big]_a^b - \int_a^b f(x)[f(x) + xf'(x)]\ dx \\ &= xf^2(x)\Big]_a^b - \int_a^b f^2(x)\ dx - \int_a^b xf(x)f'(x)\ dx \\ &= [bf^2(b) - af^2(a)] - 1 - \int_a^b xf(x)f'(x)\ dx \\ &= -1 - \int_a^b xf(x)f'(x)\ dx.\end{aligned}$$

Hence, we have

$$2\int_a^b xf(x)f'(x)\ dx = -1 \Longrightarrow \int_a^b xf(x)f'(x)\ dx = \frac{-1}{2}.$$

From our result, we can see that

$$\int_a^b -(xf(x)f'(x))\ dx = \frac{1}{2}.$$

As a consequence of the above equality, we have

$$\left|\frac{-1}{2}\right| = \left|\int_a^b xf(x)f'(x)\ dx\right| \leq \int_a^b |xf(x)f'(x)|\ dx$$

by the Triangle Inequality for integrals. Let $p = q = 2$. Then $\frac{1}{p} + \frac{1}{q} = 1$. Let $u = f'(x)$ and $v = xf(x)$. By applying Holder's Inequality for Integrals, we have

$$\int_a^b |xf(x)f'(x)|\ dx \leq \Big(\int_a^b (f'(x))^2\ dx\Big)^{\frac{1}{2}} \Big(\int_a^b x^2 f^2(x)\ dx\Big)^{\frac{1}{2}} \tag{2}$$

Using (1) and (2), we conclude that

$$\frac{1}{2} \leq \Big(\int_a^b (f'(x))^2\ dx\Big)^{\frac{1}{2}} \Big(\int_a^b x^2 f^2(x)\ dx\Big)^{\frac{1}{2}}$$

and so

$$\frac{1}{4} \leq \int_a^b (f'(x))^2\ dx \cdot \int_a^b x^2 f^2(x)\ dx$$

as our desired result. ∎

**Problem 6.** (a) Let $f$ be continuous on $[a,b]$ such that for any subinterval $[c,d] \subseteq [a,b]$

$$\int_c^d f(t)\ dt = 0.$$

Prove that $f(x) = 0$ on $[a,b]$.

(b) Let $f$ be continuous on $[a, b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt,\ \forall x \in [a, b].$$

**Proof.** (a) Let $f$ be continuous on $[a, b]$ such that for any subinterval $[c, d] \leq [a, b]$

$$\int_c^d f(t)\ dt = 0.$$

Suppose for sake of contradiction that $f(c) \neq 0$ for some $c \in [a, b]$. Since $f$ is continuous on $[a, b]$, it follows that $f$ is continuous at $c$. We have two cases to consider; that is, either

(1) $f(c) > 0$

(2) $f(c) < 0$

We will prove the case where $f(c) > 0$. The other case will follow analogously. Indeed, since $f$ is continuous at $c$, we have

$$f(c) > 0 \implies \exists \delta > 0 \text{ such that } \forall x \in (c - \delta, c + \delta)\ f(x) > 0.$$

By the Order Theorem for Integrals, we have

$$f(x) > 0 \implies \int_{c-\delta}^{c+\delta} f(x)\ dx > 0.$$

But this contradicts our assumption. Hence, it must follow that $f(x) = 0$ for all $x \in [a, b]$.

(b) Let $f$ be continuous on $[a, b]$ such that

$$\int_a^x f(t)\ dt = \int_x^b f(t)\ dt\ \forall x \in [a, b].$$

Denote $F(x) = \int_a^x f(t)\ dt$ and $G(x) = \int_x^b f(t)\ dt = -\int_b^x f(t)\ dt$. Suppose for sake of contradiction that there exists a $c \in [a, b]$ such that $f(c) \neq 0$. Since $f$ is continuous on $[a, b]$ and, in particular, $f$ is continuous at $c \in [a, b]$, then by FTC II, we have $F'(c) = f(c)$ and $G'(c) = -f'(c)$. But by assumption, $f(c) = -f(c)$ which implies that

$$2f(c) = 0 \implies f(c) = 0$$

which contradicts our assumption that $f(c) \neq 0$. Hence, it must be the case that $f(x) = 0$ for all $x \in [a, b]$.

■

**Problem 7** (A Substitution Formula For Ordinary Riemann Integrals)**.** Suppose that

- $g \in C^1[a, b]$ and $f \in C^0[c, d]$.
- The range of $g, g([a, b]) = \{g(x) : x \in [a, b]\}$, is contained in $[c, d]$ (so that the composition $f \circ g$ is defined).

7-1) Explain why $f$ is the derivative of some function.

7-2) Explain why $(f \circ g)g'$ is a derivative of some function.

7-3) Prove the substitution formula:

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

**Proof.** 7-1) Define the function $F : [c, d] \to \mathbb{R}$ by

$$F(x) = \int_c^x f(t)\ dt.$$

The above integral holds because $f \in R[c, d]$ by assumption. Since $f$ is continuous for any $x \in [c, d]$ it follows that $F'(x) = f(x)$ for any $x \in [g(a), g(b)]$ by the Second Fundamental Theorem of Calculus.

7-2) Note that $(f \circ g)g'$ is the derivative of $F \circ g$.

7-3) Our goal is to show that

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Note that we can use (7-2) to write $((F \circ g)(x))' = f(g(x))g'(x)$ for all $x \in [a, b]$. Since $[g(a), g(b)] \subseteq [c, d]$, we can use the First Fundamental Theorem of Calculus to get

$$\begin{aligned}\int_a^b f(g(x))g'(x)\ dx &= \int_a^b ((F \circ g)(x))'\ dx \\ &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} F'(u)\ du \\ &= \int_{g(a)}^{g(b)} f(u)\ du\end{aligned}$$

which is our desired result.

■

**Problem 8.** Prove the following integration by parts for "improper" Riemann Integrals:

**Theorem** (Integration by Parts of Improper Riemann Integrals)**.** Let $a \in \mathbb{R}$, $u : [a, \infty) \to \mathbb{R}$ and $v : [a, \infty) \to \mathbb{R}$ are differentiable, $\forall b > a\ u', v' \in R[a, b]$. Additionally, assume that $\int_a^\infty vu'\ dx$ exists in $\mathbb{R}$ and

$$\lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] \text{ exists (in } \mathbb{R}).$$

Then $\int_a^\infty uv'\ dx$ exists in $\mathbb{R}$ and

$$\int_a^\infty uv'\ dx = \lim_{b\to\infty} [u(b)v(b) - u(a)v(a)] - \int_a^\infty vu'\ dx.$$

**Proof.** By assumption $u'$ exists and so $u \in C[a, b]$. Hence, $u \in R[a, b]$ as an immediate consequence. Since $v' \in R[a, b]$, we can conclude that the product $v'u \in R[a, b]$. By the Ordinary Riemann Integration by Parts, we have that

$$\int_a^b u(x)v'(x)\ dx = [u(b)v(b) - u(a)v(b)] - \int_a^b u'(x)v(x)\ dx.$$

By assumption, the limit as $b \to \infty$ of each term on the right-hand side of the above equation holds. Thus, we have that

$$\lim_{b\to\infty} \int_a^b uv'\ dx \text{ exists.}$$

Hence, we have that

$$\begin{aligned}\lim_{b\to\infty}\int_a^b u(x)v'(x)\ dx &= \lim_{b\to\infty}\Big[(u(b)v(b)-u(a)v(a))-\int_a^b u'(x)v(x)\ dx\Big]\\ &= \lim_{b\to\infty}[u(b)v(b)-u(a)v(a)]-\lim_{b\to\infty}\int_a^b u'(x)v(x)\ dx \quad \text{(ALT for Functions)}\end{aligned}$$

which can be re-written into

$$\int_a^\infty u'v\ dx = \lim_{b\to\infty}[u(b)v(b)-u(a)v(a)]-\int_a^\infty uv'\ dx$$

as our desired result. ■

**Problem 9.** Let $a\in\mathbb{R}$ be a fixed number. Suppose $f\in R[a,b]$ for every $b>a$. Let $c>a$. Prove that the improper integral $\int_a^\infty f(x)\ dx$ converges if and only if the improper integral $\int_c^\infty f(x)\ dx$ converges.

**Proof.** ($\Longrightarrow$) Suppose that $\int_a^\infty f(x)\ dx$ converges. Our goal is to show that

$$\int_c^\infty f(x)\ dx \text{ exists}$$

Without loss of generality, suppose that $b<c$ Suppose $b<c$. Then by the segment addition property of the Riemann Integral, we have that

$$\int_a^c f = \int_a^b f + \int_b^c f = \int_a^b f - \int_c^b f.$$

Hence, we have

$$\int_c^b f = \int_a^b f - \int_a^c f.$$

By the Algebraic Limit Theorem for Functions, we have that

$$\begin{aligned}\int_c^\infty f = \lim_{b\to\infty}\int_c^b f &= \lim_{b\to\infty}\Big[\int_a^b f - \int_a^c f\Big]\\ &= \lim_{b\to\infty}\int_a^b f - \lim_{b\to\infty}\int_a^c f \quad \text{(ALT for Functions)}\\ &= \int_a^\infty f - \int_a^c f.\end{aligned}$$

Hence, we see that $\int_c^\infty f$ exists. ($\Longleftarrow$) Suppose $\int_c^\infty f(x)\ dx$ converges. Then by the segment addition property of integration, we have that

$$\int_a^x f(t)\ dt = \int_a^c f(t)\ dt + \int_c^x f(t)\ dt.$$

Then applying the limit as $x\to\infty$ on both sides, we have

$$\begin{aligned}\lim_{x\to\infty}\int_a^x f(t)\ dt &= \lim_{x\to\infty}\Big[\int_a^c f(t)\ dt + \int_c^x f(t)\ dt\Big]\\ &= \lim_{x\to\infty}\int_a^c f(t)\ dt + \lim_{x\to\infty}\int_c^x f(t)\ dt\\ &= \int_a^c f(t)\ dt + \int_c^\infty f(t)\ dt\end{aligned}$$

Note that the first term is just a constant so the limit always exists and the second exists by assumption. Hence, we can conclude that

$$\int_a^\infty f(t)\, dt \text{ converges.}$$

■

**Problem 10.** Let $a > 0$. Prove that $\displaystyle\int_a^\infty \frac{1}{x^p}\, dx \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$

**Proof.** We will consider three cases:

(i) ($p = 1$) Consider the integral $\int_a^t \frac{1}{x}\, dx$. Since $(\ln x)' = \frac{1}{x}$ on $[a, t]$ for any $t > 0$, we can see by FTC I that

$$\int_a^t \frac{1}{x}\, dx = \ln(t) - \ln(a).$$

Since $\lim_{t\to\infty} \ln(t) = \infty$, it follows that when we apply the limit as $t \to \infty$ to the above equation that $\int_a^\infty \frac{1}{x}\, dx$ diverges.

(ii) ($p > 1$) Consider the integral $\int_a^t \frac{1}{x^p}\, dx$. Note that

$$\frac{d}{dx}\left[\frac{1}{1-p}x^{1-p}\right] = \frac{1}{x^p}$$

for all $x \in [a, t]$ where $t > a$. Then by FTC I, we can see that

$$\int_a^t \frac{1}{x^p}\, dx = \frac{1}{1-p}t^{1-p} + \frac{1}{1-p}a^{t-p}. \tag{*}$$

Since $1 - p < 0$ by assumption, we can see that $\lim_{t\to\infty} t^{1-p} = 0$ by the fact given to us and so applying the limit as $t \to \infty$ to (*), we can see that

$$\int_a^\infty \frac{1}{x^p}\, dx = 0 - \frac{1}{1-p}a^{t-p} = -\frac{1}{1-p}a^{t-p}.$$

by the Algebraic Limit Theorem.

(iii) ($p < 1$) From the fact that was given to us, we can see that $1 - p > 0$ implies $\lim_{t\to\infty} t^{1-p} = \infty$. Hence, from the equality in (*), we can see that $\int_a^\infty \frac{1}{x^p}\, dx$ diverges.

■

**Problem 11** (Cauchy Criterion For Improper Integrals)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\displaystyle\int_a^\infty f(x)\, dx$ converges if and only if

$$\forall \varepsilon > 0\ \exists M > a \text{ such that } \forall A, B > M \quad \left|\int_A^B f(x)\, dx\right| < \varepsilon.$$

**Proof.** ($\Longrightarrow$) Suppose $\int_a^\infty f(x)\, dx$ converges. Our goal is to show that for all $\varepsilon > 0$, there exists $M > a$ such that for all $A, B > M$

$$\left|\int_A^B f(x)\, dx\right| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Without loss of generality, assume that $A < B$. Since $\int_a^\infty f(x)\, dx$ converges, it follows from our given $\varepsilon$ that there exists an $M_1 > B$ such that for any $t > M_1$, we have

$$\Big| \int_B^t f(x)\ dx - L \Big| < \frac{\varepsilon}{2} \tag{1}$$

Similarly, there exists an $M_2 > A$ such that for any $t > M_2$, we have

$$\Big| \int_A^t f(x)\ dx - L \Big| < \frac{\varepsilon}{2}. \tag{2}$$

Using the segment addition property of integration, we have

$$\begin{aligned} \int_A^B f(x)\ dx &= \int_A^t f(x)\ dx + \int_t^B f(x)\ dx \\ &= \int_A^t f(x)\ dx - \int_B^t f(x)\ dx. \end{aligned}$$

Let $M = \max\{M_1, M_2\}$. Then we have for any $B > A > M$, (1) and (2) imply

$$\begin{aligned} \Big| \int_A^B f(x)\ dx \Big| &= \Big| \int_A^t f(x)\ dx - \int_B^t f(x)\ dx \Big| \\ &\le \Big| \int_A^t f(x)\ dx - L \Big| + \Big| L - \int_B^t f(x)\ dx \Big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which is our desired result.

($\Longleftarrow$) Suppose that for any $\varepsilon > 0$, there exists $M > a$ such that for all $A, B > M$,

$$\Big| \int_A^B f(x)\ dx \Big| < \varepsilon.$$

Consider the sequence $(c_n)$ in $\mathbb{R}$ defined by

$$c_n = \int_a^n f(x)\ dx.$$

First, we will show that $c_n$ converges to some $L \in \mathbb{R}$. To this end, we will show that $c_n$ is a Cauchy sequence. We claim that the same $M$ can be used to do this. Then for any $n > m > M$, we have

$$\begin{aligned} |c_n - c_m| &= \Big| \int_a^n f(x)\ dx - \int_a^m f(x)\ dx \Big| \\ &= \Big| - \int_n^a f(x)\ dx - \int_a^m f(x)\ dx \Big| \\ &= \Big| \int_n^m f(x)\ dx \Big| \\ &< \varepsilon. \end{aligned}$$

Hence, we see that $c_n$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $c_n \to L$ for some $L \in \mathbb{R}$. As a consequence, we need to show that

$$\lim_{x \to \infty} \int_a^x f(t)\ dt = L.$$

Our goal is to show that for any $\varepsilon > 0$, there exists an $M > a$ such that for any $x > M$, we have

$$\Big| \int_a^x f(t)\ dt - L \Big| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Observe by segment addition property of the integral, we can see that

$$\int_a^x f(t)\ dt = \int_a^{\lfloor x \rfloor} f(t)\ dt + \int_{\lfloor x \rfloor}^x f(t)\ dt.$$

Since $\lim_{n\to\infty} \int_a^n f(x)\ dx = L$, we know there exists an $M_1 > a$ (with our given $\varepsilon$) such that for any $n > M_1$, we have that

$$\Big| \int_a^n f(t)\ dt - L \Big| < \frac{\varepsilon}{2}.$$

In particular, since $\lfloor x \rfloor \in \mathbb{N}$ and $\lfloor x \rfloor > M_1$, we have that

$$\Big| \int_a^{\lfloor x \rfloor} f(t)\ dt - L \Big| < \frac{\varepsilon}{2}. \tag{1}$$

By assumption, there also exists an $M_2 > a$ such that for any $A > B > M_2$, we have that

$$\Big| \int_A^B f(t)\ dt \Big| < \frac{\varepsilon}{2}.$$

In particular, since $x > \lfloor x \rfloor > M_2$, we have that

$$\Big| \int_{\lfloor x \rfloor}^x f(t)\ dt \Big| < \frac{\varepsilon}{2}. \tag{2}$$

Now, let $M = \max\{M_1, M_2\}$. Then using (1) and (2), we have that for any $x > M$,

$$\begin{aligned}
\Big| \int_a^x f(t)\ dt - L \Big| &= \Big| \int_a^{\lfloor x \rfloor} f(t)\ dt + \int_{\lfloor x \rfloor}^x f(t)\ dt - L \Big| \\
&\leq \Big| \int_a^{\lfloor x \rfloor} f(t)\ dt - L \Big| + \Big| \int_{\lfloor x \rfloor}^x f(t)\ dt - L \Big| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we conclude that

$$\lim_{x\to\infty} \int_a^x f(t)\ dt = L.$$

■

**Problem 12** (Absolute Convergence Implies Convergence)**.** Let $a \in \mathbb{R}$ be a fixed number. Suppose $f \in R[a, b]$ for every $b > a$. Prove that the improper integral $\int_a^\infty |f(x)|\ dx$ converges, then the improper integral $\int_a^\infty f(x)\ dx$ also converges.

**Proof.** Suppose $f \in R[a, b]$ for every $b > a$. Our goal is to show that for all $\varepsilon > 0$ such that there exists $M > a$ such that for any $A, B > M$, we have

$$\Big| \int_A^B f(x)\ dx \Big| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\int_a^\infty |f(x)|\,dx$ converges, there exists an $\hat{M} > a$ such that for all $A, B > \hat{M}$

$$\left|\int_A^B |f(x)|\,dx\right| < \varepsilon.$$

We claim that $\hat{M}$ can be used as the same $M$ we were looking for. Let $M = \hat{M}$. Indeed, we have for any $A, B > M$, we have

$$\left|\int_A^B f(x)\,dx\right| \leq \int_A^B |f(x)|\,dx = \left|\int_A^B |f(x)|\,dx\right| < \varepsilon \tag{*}$$

by the triangle inequality for integrals. Note that the second equality holds because $|f(x)| \geq 0$ and that

$$\int_A^B |f(x)|\,dx \geq 0$$

by another theorem proven in class. Hence, we see that (*) is our desired result. ■

**Problem 13** (Comparison Test For Improper Integrals). Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Prove that the improper integral $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.

**Proof.** Our goal is to show that $\int_a^\infty f(x)\,dx$ converges given that $\int_a^\infty g(x)\,dx$ converges; that is, we want to show that for all $\varepsilon > 0$, there exists an $M > a$ such that for all $A, B > M$, we have

$$\left|\int_A^B f(x)\,dx\right| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Then $\int_a^\infty g(x)\,dx$ converges implies that there exists an $\hat{M} > a$ such that for any $A, B > \hat{M}$, we have

$$\left|\int_A^B g(x)\,dx\right| < \varepsilon.$$

Also, there exists a $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $M = \max\{k, \hat{M}\} + 1$. Note that with this constructed $M$, we have, by the order property of the integral that

$$0 \leq \int_A^B f(x)\,dx \leq \int_A^B g(x)\,dx.$$

Then for any $A, B > M$, we have that

$$\left|\int_A^B f(x)\,dx\right| = \int_A^B f(x)\,dx \leq \int_A^B g(x)\,dx = \left|\int_A^B g(x)\,dx\right| < \varepsilon$$

which is our desired result. ■

**Problem 14** (Limit Comparison Test for Improper Integrals). Let $a \in \mathbb{R}$ be a fixed number. Suppose $f, g \in R[a, b]$ for every $b > a$ and that there exists $K \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x > K$. Let $L = \lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$. Prove that

(i) If $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^b g(x)\,dx$ converges.

(ii) If $L = \infty$ and $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

(iii) If $L = 0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

**Proof.** Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$, we know that for all $\varepsilon > 0$, there exists $\hat{M} > 0$ such that for any $x \geq M$, we have

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$$

which can be further written as

$$-\varepsilon < \frac{f(x)}{g(x)} - L < \varepsilon$$

and so

$$L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon. \tag{*}$$

If $L = 0$, then the above inequality can be re-written as

$$-\varepsilon < \frac{f(x)}{g(x)} < \varepsilon. \tag{**}$$

If $L = \infty$, then for all $C > 0$, there exists $\kappa > 0$ such that for all $x > \kappa$, we have

$$\frac{f(x)}{g(x)} > C. \tag{***}$$

(i) Suppose $0 < L < \infty$. ($\Longrightarrow$) Our goal is to show that $\int_a^\infty g(x)\ dx$ converges given that $\int_a^\infty f(x)\ dx$ converges. From the left-hand side of (*), we can see that for all $x > \hat{M}$, we have

$$g(x) < \frac{1}{L - \varepsilon} f(x).$$

In particular, if $\varepsilon = \frac{L}{2}$, then we have

$$g(x) < \frac{2}{L} f(x).$$

Since there exists a $K \in \mathbb{R}$ such that for all $x \geq K$, $f(x) \geq 0$ and $g(x) \geq 0$. So, if we take $M = \max\{\hat{M}, k\}$, then for any $x > M > a$, we have

$$0 < g(x) < \frac{2}{L} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, we can see that $\int_a^\infty \frac{2}{L} f(x)\ dx$ converges. By the Comparison Test for Integrals, we have $\int_a^\infty g(x)\ dx$ converges.

($\Longleftarrow$) Suppose $\int_a^\infty g(x)\ dx$ converges. Our goal is to show that $\int_a^\infty f(x)\ dx$ converges. From the right-hand side of (*), we can see that for any $x \geq \hat{M}$, we have

$$f(x) < (L + \varepsilon) g(x).$$

In particular, if $\varepsilon = L$, then

$$f(x) < 2Lg(x).$$

Since there exists $K \in \mathbb{R}$ such that $f(x) \geq 0$ and $g(x) \geq 0$, define $M = \max\{K, \hat{M}\}$. Then for any $x \geq M > a$, we have

$$0 < f(x) < 2Lg(x).$$

Now, since $\int_a^\infty g(x)\ dx$ converges, it follows that $\int_a^\infty 2Lg(x)\ dx$ converges. Hence, the Comparison Test implies that $\int_a^\infty f(x)\ dx$ converges.

(ii) Our goal is to show that if $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$ converges. From (**), we can see that

$$g(x) < \frac{1}{C} f(x).$$

Define $\tilde{M} = \max\{\kappa, \hat{M}\}$. Then we have for any $x > \tilde{M}$

$$0 < g(x) < \frac{1}{C} f(x).$$

Since $\int_a^\infty f(x)\ dx$ converges, also have that $\int_a^\infty \frac{1}{C} f(x)\ dx$ converge. By the Comparison Test, we can see that $\int_a^\infty g(x)\ dx$ converges.

(iii) Suppose $L = 0$. Then by (**), we can see that for any $x > \hat{M}$ that

$$f(x) < \varepsilon g(x).$$

In this case, if we let $\varepsilon = 1$, we have that

$$f(x) < g(x)$$

for all $x > \hat{M}$. Since there exists $K \in \mathbb{R}$ such that for any $x \geq K$, we have $f(x) \geq 0$ and $g(x) \geq 0$. Note that in this case, we require that $g(x) > 0$. Otherwise, the ratio above will not be defined. So, using the same $M$ from part (i), we have that for any $x > M$,

$$0 < f(x) < g(x).$$

Since $\int_a^\infty g(x)\ dx$ converges, it follows from the Comparison Test that $\int_a^\infty f(x)\ dx$ also converges.

■

**Problem 15.** Determine all values $\alpha$ and $\beta$ for which $\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\ dx$ is convergent.

**Proof.** We will consider three cases.

(i) ($\alpha = 1$) Consider the improper integral below

$$\int_2^\infty \frac{1}{x(\ln x)^\beta}\ dx. \tag{*}$$

Using a change of variables (using $u = \ln x$ and so $u' = \frac{1}{x}$), we have that

$$\int_2^\infty \frac{1}{x(\ln x)^\beta}\ dx = \int_2^\infty \frac{1}{u^\beta}\ du.$$

From Exercise 10, we can see that the improper integral above converges if $\beta > 1$ and diverges if $\beta \leq 1$. In this case, if $\alpha = 1$ and $\beta > 1$, then the improper integral in (*) converges.

(ii) ($\alpha < 1$) Choose $p \in (1, \alpha)$. Using Exercise 10 again, we have that

$$\int_2^\infty \frac{1}{x^p}\ dx = \lim_{t \to \infty} \int_2^t \frac{1}{x^p}\ dx$$

converges if $p > 1$ and diverges if $p \leq 1$. Define the function $f : [2, t] \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^\alpha (\ln x)^\beta}$$

and $g : [2, t] \to \mathbb{R}$ by

$$g(x) = \frac{1}{x^p}.$$

Since $p - \alpha < 0$, it follows from the fact given to us that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^{p-\alpha}}{(\ln x)^\beta} = 0.$$

By the Limit Comparison Test, since $\int_2^\infty g(x)\,dx$ converges, we can conclude that $\int_2^\infty f(x)\,dx$ converges. That is, if $\alpha > 1$, then regardless of the value of $\beta$, the improper integral

$$\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\,dx$$

converges.

(iii) ($\alpha > 1$) Let $p \in (\alpha, 1)$. Since $p > 1$, it follows that $\int_2^\infty g(x)\,dx$ diverges. Since $p - \alpha > 0$, we see that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{p-\alpha}}{(\ln x)^\beta} = \infty$$

By the Limit Comparison Test, we can conclude that

$$\int_2^\infty \frac{1}{x^\alpha (\ln x)^\beta}\,dx$$

diverges.

■

**Problem 16.** (a) Prove that $\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ is absolutely convergent.

(b) Use the result of Exercise 8 to prove that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

**Proof.** (a) Our goal is to show that $\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ is absolutely convergent; that is, we need to show that $\int_a^\infty |\frac{\sin x}{(1+x)^2}|\,dx$ converges. Please note that every $b > 0$

$$\Big|\frac{\sin x}{(1+x)^2}\Big| = \frac{|\sin x|}{(1+x)^2} \in C[0,b]$$

and so it must be contained in $R[0,b]$. Furthermore, for all $x \in [1,\infty)$

$$\frac{|\sin x|}{(1+x)^2} \leq \frac{1}{(1+x)^2} \leq \frac{1}{x^2} \tag{*}$$

Since $p = 2 > 1$, we have that the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges}$$

by exercise 10. Using the comparison on the inequality on (*), we conclude that

$$\int_1^\infty \Big|\frac{\sin x}{(1+x)^2}\Big|\,dx \text{ converges}$$

and so the improper integral

$$\int_1^\infty \frac{\sin x}{(1+x)^2}\,dx$$

converges absolutely.

(b) By part (a), we see that

$$\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx \tag{1}$$

converges and that

$$\lim_{b\to\infty}\left[\frac{-\sin x}{1+x}\right]_0^b = \frac{\sin(0)}{1+0} - \lim_{b\to\infty}\frac{\sin b}{1+b} = 0 - 0 = 0. \tag{2}$$

From (1) and (2), we can use Exercise 8 to write

$$\int_0^\infty \frac{\sin x}{(1+x)^2}\ dx = \lim_{b\to\infty}\left[\frac{-\sin x}{1+x}\right]_0^b - \int_0^\infty \frac{-\cos x}{1+x}\ dx$$

which implies that

$$\begin{aligned}\int_0^\infty \frac{-\cos x}{1+x}\ dx &= \lim_{b\to\infty}\left[\frac{-\sin x}{1+x}\right]_0^b - \int_0^\infty \frac{\sin x}{(1+x)^2}\ dx \\ &= -\int_0^\infty \frac{\sin x}{(1+x)^2}\ dx.\end{aligned}$$

Hence, we see that

$$\int_0^\infty \frac{\cos x}{1+x}\ dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\ dx.$$

■