Homework-1-241

Problem-1: (a) Define $f : [0,\infty) \to \mathbb{R}$ by $f(t) = \frac{t}{1+t}$

Show that f is an increasing function.

(b) Let (X, d) be a metric space. Define $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that $\tilde{d}$ is a metric on X

Hint: Think about using part (a) to triangle inequality for d.

Problem 2: Let $X = \mathbb{R}^n$. For $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

define $d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ and

$d_\infty(\vec{x}, \vec{y}) = \max\{ |x_i - y_i| \mid 1 \leq i \leq n \}$

Show that d_1 and d_∞ are metrics on $\mathbb{R}^n$.

Problem 3: Let $X = \mathbb{R}^n$ and d_1 and d_∞ be as defined in problem 2.

(a) (i) Show that $d_\infty(\vec{x}, \vec{y}) \leq d_1(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(ii) Let $\vec{x_0} \in \mathbb{R}^n$ and $r > 0$. Let

$$B_1 = \{ \vec{x} \in \mathbb{R}^n \mid d_1(\vec{x}, \vec{x_0}) < r \}$$

$$B_2 = \{ \vec{x} \in \mathbb{R}^n \mid d_\infty(\vec{x}, \vec{x_0}) < r \}.$$

Which one of the following holds and why?

$$B_1 \subseteq B_2 \quad \text{or} \quad B_2 \subseteq B_1$$

(b) Show that $d_1(\vec{x}, \vec{y}) \leq n \cdot d_\infty(\vec{x}, \vec{y})$.

(c) Fix $\vec{x_0} \in \mathbb{R}^n$ and $r > 0$.

Prove that $\{ \vec{x} \in \mathbb{R}^n \mid d_\infty(\vec{x}, \vec{x_0}) < r/n \}$

$$\subseteq \{ \vec{x} \in \mathbb{R}^n \mid d_1(\vec{x}, \vec{x_0}) < r \}.$$

(d) Prove that $M \subseteq \mathbb{R}^n$ is open with respect to the metric d_1 if and only if M is open with respect to the metric d_∞.

Remark: Let X be a non-empty set and d and $\tilde{d}$ be two metrics on X such that $\exists\ r_1 > 0,\ r_2 > 0$ such that

$$d(x,y) \leq r_1\, \tilde{d}(x,y) \text{ and } \tilde{d}(x,y) \leq r_2\, d(x,y)$$

for all $x, y \in X$. Then a set M is open with respect to d if and only if M is open with respect to $\tilde{d}$.

(Try to verify this, do not need to submit)

<u>Problem 4:</u> Notation: Let I be an indexing set and $\{X_i\}_{i\in I}$ be a collection of subsets of X.

We define

$$\bigcup_{i\in I} X_i = \{x \in X \,/\, x \in X_i \text{ for some } i \in I\}$$

$$\bigcap_{i\in I} X_i = \{x \in X \,/\, x \in X_i \text{ for all } i \in I\}$$

Let (X, d) be a metric space.

(i) Let $\{M_i\}_{i\in I}$ be a collection of open sets in X. Show that $\bigcup_{i\in I} M_i$ is also open.

(ii) Let M_1 and M_2 be two open sets in X. Prove that $M_1 \cap M_2$ is open.

(iii) Let $x_0 \in X$ and $r > 0$. Show that $B(x_0; r)$ is open.

(iv) Let $\{K_i\}_{i \in I}$ be a collection of closed sets in X. Prove that $\bigcap_{i \in I} K_i$ is closed.

(v) Let K_1 and K_2 be closed in X. Prove that $K_1 \cup K_2$ is closed.

(vi) Let $x_0 \in X$ and $r > 0$. Prove that $\bar{B}(x_0; r)$ is closed.

(vi) Let $M \subseteq X$. Prove that $\bar{M}$ is closed in X.

(vii) Let $x_0 \in X$ and $r > 0$. Prove that

$$\overline{B(x_0; r)} \subseteq \bar{B}(x_0; r)$$

closure of the ball. — closed ball

Is it always true that $\overline{B(x_0; r)} = \bar{B}(x_0, r)$?

Justify your answer.

(Thanks to Lance for this problem).

Problems: In this problem, we will establish some key inequalities that will be used later in the class.

(i) Young's Inequality: Let $a, b \in \mathbb{R}$, $a \geq 0, b \geq 0$ $p > 1$. Let $q = \frac{p}{p-1}$. Then

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}$$

Prove Young's inequality as follows:

Step 1: Show that

$$e^{t\alpha + (1-t)\beta} \leq t e^{\alpha} + (1-t) e^{\beta}$$

for all $0 \leq t \leq 1$, where α and β are fixed real numbers.

(Think about a geometrical way to verify this)

Step 2: (Note that the inequality is obvious if $a = 0$ or $b = 0$). Assume that $a > 0$ and $b > 0$ and then apply step 1 to $\alpha = \ln a$ and $\beta = \ln b$. You will have find appropriate t to be able to apply step 1.

(ii) Höder's Inequality: Let $p > 1$. For $\vec{x} \in \mathbb{R}^n$,

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, define $\|\vec{x}\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$. Let $q = \frac{p}{p-1}$.

For $\vec{x}, \vec{y} \in \mathbb{R}^n$, prove that

$$\sum_{i=1}^{n} |x_i y_i| \leq \|\vec{x}\|_p \|\vec{y}\|_q$$

where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

(iii) Minkowski's Inequality:

Let p, $\vec{x}$ and $\vec{y}$ be as in (ii).

Prove that

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$$

(iv) Let $X = \mathbb{R}^n$, $p \geq 1$. For $\vec{x}, \vec{y} \in \mathbb{R}^n$, define $d_p(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_p$.

Prove that d_p is a metric on $\mathbb{R}^n$.