Math 241A Project Notes

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Chapter 1

Preliminaries

Lemma (Equality (3.8-2))**.** If $\langle v_1, w\rangle = \langle v_2, w\rangle$ for all w in an inner product space X, then $v_1 = v_2$. In particular, $\langle v_1, w\rangle = 0$ for all $w \in X$ implies $v_1 = 0$.

Chapter 2

Spectral Properties

2.1 Spectral Properties of Bounded Self-adjoint Linear Operators

First, we will recall two relevant definitions from our study of Hilbert Spaces from chapter 3.

Definition (Hilber-Adjoint Operator). Let $T : H \to H$ be a bounded linear operartor on a complex Hilber space H. Then the **Hilbert-adjoint** operator $T^* : H \to H$ is defined to be the operator satisfying

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \quad \forall x, y \in H. \tag{*}$$

Note that T is said to be **self-adjoint** or **Hermitian** if

$$T = T^*.$$

If the above holds, then it follows from (*) that

$$\langle T(x), y\rangle = \langle x, T(y)\rangle \quad \forall x, y \in H.$$

In developing the theory for the Spectral Theorem, we first need to make sure that eigenvalues are real so that we are able to respect the geometry induced by inner product spaces.

Theorem (Eigenvalues, eigenvectors). Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilber space H. Then:

(a) All the eigenvalues of T (if they exist) are real.

(b) Eigenvectors corresponding to (numerically) different eigenvalues of T are orthogonal.

Proof. (a) Our goal is to show that $\lambda = \overline{\lambda}$ for all eigenvalues λ of T. To this end, Let λ be an eigenvalue of T and x be the corresponding eigenvector. Then $x \neq 0$ and $Tx = \lambda x$. Since T is self-adjoint, it follows that

$$\begin{aligned}\lambda\langle x, x\rangle = \langle \lambda x, x\rangle &= \langle Tx, x\rangle \\ &= \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle.\end{aligned}$$

Since $\langle x, x\rangle \neq 0$, then it follows that $\lambda = \overline{\lambda}$, implying that λ must be real.

(b) Our goal is to show that for any two (distinct) eigenvalues λ and μ corresponding to eigenvectors x and y, respectively, we have $\langle x, y\rangle = 0$. To this end, let λ and μ be eigenvalues of T, and let x and y be corresponding eigenvectors. Then it follows that $Tx = \lambda x$ and $Ty = \mu y$. Since T is self-adjoint and μ is real, we have

$$\begin{aligned}\lambda\langle x, y\rangle = \langle \lambda x, y\rangle &= \langle Tx, y\rangle \\ &= \langle x, Ty\rangle = \langle x, \mu y\rangle = \overline{\mu}\langle x, y\rangle = \mu\langle x, y\rangle.\end{aligned}$$

Since we have assumed that $\lambda \neq \mu$, it follows that $\langle x, y\rangle = 0$. Hence, x and y are orthogonal.

■

Theorem (Resolvent Set). Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c > 0$ such that for every $x \in H$,

$$\|T_\lambda(x)\| \geq c\|x\|. \qquad (T_\lambda = T - \lambda I)$$

Proof. ($\Longrightarrow$) Our goal is to show that there exists a $c > 0$ such that for every $x \in H$, we have

$$\|T_\lambda x\| \geq c\|x\| \tag{2}$$

where $T_\lambda = T - \lambda I$. Since $\lambda \in \rho(T)$, it follows that $R_\lambda = T_\lambda^{-1}$ exists and is bounded. That is, $\|R_\lambda\| = k$, where $k > 0$ since $R_\lambda \neq 0$. Now, we have $I = R_\lambda T_\lambda$ such that every $x \in H$, we have

$$\|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\|\|T_\lambda x\| = k\|T_\lambda x\|$$

which gives us the following inequality

$$\|T_\lambda x\| \geq c|x|$$

where $c = \frac{1}{k}$.

($\Longleftarrow$) Our goal is to show that $\lambda \in \rho(T)$. We need to show that λ satisfies the following three properties:

(1) $T_\lambda : H \to T_\lambda(H)$ is bijective.

(2) $T_\lambda(H)$ is dense in H;

(3) $T_\lambda(H)$ is closed in H.

(1) Clearly, T is onto by definition. Our goal is to show that for any $x_1, x_2 \in H$ such that $T_\lambda x_1 = T_\lambda x_2$, we have $x_1 = x_2$. Since T is linear and our assumption, it follows that

$$0 = \|T_\lambda x_1 - T_\lambda x_2\| = \|T_\lambda(x_1 - x_2)\| \geq c\|x_1 - x_2\|$$

for some $c > 0$. Since $\|x_1 - x_2\| \geq 0$, it follows from the above inequality that $x_1 = x_2$. Hence, T is injective and thus T is bijective.

(2) Our goal is to show that $T_\lambda(H)$ is dense in H; that is, $\overline{T_\lambda(H)} = H$. It suffices to show via the Projection Theorem in Chapter 3 that $x_0 \perp \overline{T_\lambda(H)}$ implies $x_0 = 0$. Let $x_0 \perp \overline{T_\lambda(H)}$. Then we have $x_0 \perp T_\lambda(H)$. Hence, for all $x \in H$, we have

$$\begin{aligned} 0 = \langle T_\lambda x, x_0\rangle &= \langle (T - \lambda I)x, x_0\rangle \\ &= \langle Tx, x_0\rangle - \lambda\langle x, x_0\rangle. \end{aligned}$$

Since T is self-adjoint, it follows that

$$\langle x, Tx_0\rangle = \langle Tx, x_0\rangle = \langle x, \overline{\lambda}x_0\rangle.$$

Hence, we have $Tx_0 = \overline{\lambda}x_0$ by (3.8-2). Note that we would need to have $x_0 = 0$ because otherwise if $x_0 \neq 0$, it would mean that $\overline{\lambda}$ is an eigenvalue of T so that $\overline{\lambda} = \lambda$ by (9.1-1) and $Tx_0 - \lambda x_0 = T_\lambda x_0 = 0$, and (2) would imply

$$0 = \|T_\lambda x_0\| \geq c\|x_0\| > 0$$

since $c > 0$ which is absurd. Hence, we can see that $\overline{T_\lambda(H)}^\perp = \{0\}$ where x_0 is an arbitrary vector orthogonal to $T_\lambda(H)$. Hence, it follows that $\overline{T_\lambda(H)} = H$ by the Projection Theorem and so we can conclude that $T_\lambda(H)$ is dense in H.

(3) Finally, we will show that $T_\lambda(H)$ is closed in H. In what follows, we will show $\overline{T_\lambda(H)} = T_\lambda(H)$. Clearly, we can see that $T_\lambda(H) \subseteq \overline{T_\lambda(H)}$. So, it suffices to show that $\overline{T_\lambda(H)} \subseteq T_\lambda(H)$. Let $y \in \overline{T_\lambda(H)}$. Then there is a sequence (y_n) in $T_\lambda(H)$j such that $y_n \to y$. Note that $y_n \in T_\lambda(H)$ and so $y_n T_\lambda x_n$ for some $x_n \in H$. By (2), we obtain

$$\|x_n - x_m\| \leq \frac{1}{c}\|T_\lambda(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|.$$

Since $y_n \to y$, it follows that y_n is a Cauchy sequence. From the inequality above, we can see that (x_n) must also be Cauchy when we let $m, n \to \infty$. Since H is complete, $x_n \to x$ for some $x \in H$. Since T is continuous (because it is bounded), we have that

$$y_n = T_\lambda x_n \to T_\lambda x.$$

Since limits are unique, it follows that $y = T_\lambda x$ and so we have $y \in T_\lambda(H)$. Hence, $T_\lambda(H)$ must be closed. As a consequence, we have $T_\lambda(H) = H$ from (2).

This tells us that $R_\lambda = T_\lambda^{-1}$ is defined on all of H, and is bounded, which follows from the Bounded Inverse Theorem (4.12-2). Thus, we see that $\lambda \in \rho(T)$. ∎

Theorem (Spectrum)**.** The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilber space H is real.

Proof. Using the previous theorem, we will show that for every $\lambda = \alpha + i\beta \in \sigma(T)$ where $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ that $\lambda \in \rho(T)$. Since T is self-adjoint, it follows from (9.1-1) that $\sigma(T) \subseteq \mathbb{R}$. Hence, it suffices to show that there exists a $c > 0$ such that

$$\|T_\lambda x\| \geq c\|x\|. \tag{*}$$

For every $x \neq 0$ in H, we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle.$$

Since $\langle x, x\rangle$ and $\langle Tx, x\rangle$ are real and so

$$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle.$$

Note that $\overline{\lambda} = \alpha - i\beta$. Subtracting the two quantities above, we can see that

$$-2i\Im\langle T_\lambda x, x\rangle = \overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \overline{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2$$

which imply that

$$-\langle T_\lambda x, x\rangle = \beta\|x\|^2.$$

Applying the Cauchy-Schwarz inequality, we can see that

$$|\beta|\|x\|^2 = |\Im\langle T_\lambda x, x\rangle| \leq |\langle T_\lambda x, x\rangle| = |\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\|x\|.$$

Since $\|x\| \neq 0$, we see that $|\beta|\|x\| \leq \|T_\lambda x\|$. If $\beta \neq 0$, then $\lambda \in \rho(T)$ by (9.1-2). Hence, for $\lambda \in \sigma(T)$ we see that $\beta = 0$, and so λ is real.

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2.2 Further Spectral Properties

In the previous section, we saw that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator T is real and that it is a compact set by chapter 4.

Theorem (Spectrum)**.** The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where

$$m = \inf_{\|x\|=1} \langle T(x), x\rangle \quad \text{and} \quad M = \sup_{\|x\|=1} \langle T(x), x\rangle. \tag{1}$$

Proof. By (9.1-3), we see that $\sigma(T)$ lies on the real axis. We will show that for any real $\lambda = M + c$ where M is defined above lies in the resolvent set $\rho(T)$. For every $x \neq 0$, define $v = \|x\|^{-1}x$ and so $x = \|x\|v$. As a consequence, we have

$$\langle Tx, x\rangle = \|x\|^2\langle Tv, v\rangle \leq \|x\|^2 \sup_{\|\tilde{v}\|=1} \langle T\tilde{v}, \tilde{v}\rangle = \langle x, x\rangle M.$$

Hence, we see that

$$-\langle Tx, x\rangle \geq -\langle x, x\rangle M$$

and so by the Schwarz Inequality we obtain

$$\begin{aligned}\|T_\lambda x\|\|x\| \geq -\langle T_\lambda x, x\rangle &= -\langle Tx, x\rangle + \lambda\langle x, x\rangle \\ &\geq (-M+\lambda)\langle x, x\rangle \\ &= c\|x\|^2\end{aligned}$$

where $c = \lambda - M > 0$ by assumption. Since $x \neq 0$, it follows by division of $\|x\|$ on both sides of the above inequality yields

$$\|T_\lambda x\| \geq c\|x\|.$$

By (9.1-2), $\lambda \in \rho(T)$. If we assume $\lambda < m$, then a similar argument will lead to the same result. ■

Theorem (Norm)**.** For any bounded self-adjoint linear operator T on a complex Hilbert space H we have

$$\|T\| = \max(|m|, |M|) = \sup_{\|x\|=1} |\langle T(x), x\rangle|.$$

Proof. Our goal is to show that for any bounded self-adjoint linear operator T, we have

$$\|T\| = \max(|m|, |M|) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Denote $K = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. It suffices to show that $\|T\| \leq K$ and $K \leq \|T\|$. Using the Schwarz Inequality, it follows that

$$K \leq \sup_{\|x\|=1} \|Tx\|\|x\| = \|T\|.$$

Hence, the second inequality is satisfied. Now, we show $\|T\| \leq K$. Observe that if $Tz = 0$ for all z such that $\|z\| = 1$, then $T = 0$ and we are done. Suppose, otherwise that for any z such that $\|z\| = 1$, we have $Tz \neq 0$. Set $v = \|Tz\|^{1/2}z$ and $w = \|Tz\|^{-1/2}Tz$. Then we have

$$\|v\|^2 = \|w\|^2 = \|Tz\|.$$

Furthermore, set

$$y_1 = v + w \quad \text{and} \quad y_2 = v - w.$$

By a straight-forward calculation and the fact that T is self-adjoint, we obtain the following

$$\begin{aligned}\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle &= 2(\langle Tv, w\rangle + \langle Tw, v\rangle) \\ &= 2(\langle Tz, Tz\rangle + \langle T^2 z, z\rangle) \\ &= 4\|Tz\|^2.\end{aligned}$$

Now, for every $y \neq 0$ and $x = \|y\|^{-1}y$, we have $y = \|y\|x$ and so

$$|\langle Ty, y\rangle| = \|y\|^2|\langle Tx, x\rangle| \leq \|y\|^2 \sup_{\|\tilde{x}\|=1} |\langle T\tilde{x}, \tilde{x}\rangle| = K\|y\|^2.$$

Using the triangle inequality, it follows that

$$\begin{aligned}|\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle| &\leq |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle| \\ &\leq K(\|y_1\|^2 + \|y_2\|^2) \\ &= 2K(\|v\|^2 + \|w\|^2) \\ &= 4K\|Tz\|.\end{aligned}$$

Note that the left-hand side of the above inequality is equal to $4\|Tz\|^2$. As a consequence with $\|Tz\| \neq 0$, we have

$$4\|Tz\|^2 \leq 4K\|Tz\| \Longrightarrow \|Tz\| \leq K.$$

Taking the supremum over all z of norm 1, it follows that $\|T\| \leq K$. ■

This theorem is important in the following ways:

- It tells us how the operator norm is governed by the spectrum of T.
- Even in cases where the eigenvalues of T do not exist, we have a direct way to estimate the spectrum and understand the behavior of T.
- It gives us a way to approximate infinite-dimensional operators.

Theorem (m and M are spectral values)**.** Let H and T be as in Theorem 9.2-1 and $H \neq \{0\}$. Then m and M are defined in (1) are spectral values of T.

Proof. Our goal is to show that m, M are contained in $\sigma(T)$. The proof to show that $m \in \sigma(T)$ is directly analogous. By the Spectral Mapping Theorem, the spectrum of $T + kI$, where $K \in \mathbb{R}$ is a constant, can be obtained from that of T via a translation, and so we have

$$M \in \sigma(T) \iff M + k \in \sigma(T).$$

We will show that $\lambda = M$ cannot belong to the resolvent set of T by 9.1-2. Indeed, without loss of generality assume $0 \leq m \leq M$. Using the previous theorem, we see that

$$M = \sup_{\|x\|=1} \langle Tx, x \rangle = \|T\|.$$

Using the definition of a supremum, we can find a sequence (x_n) such that $\|x_n\| = 1$ and

$$\langle Tx_n, x_n \rangle = M - \delta_n$$

where $\delta_n \geq 0$ and that $\delta_n \to 0$. Then we have

$$\|Tx_n\| \leq \|T\|\|x_n\| = \|T\| = M.$$

Since T is self-adjoint, we have

$$\begin{aligned}
\|Tx_n - Mx_n\| &= \langle Tx_n - Mx_n, Tx_n - Mx_n \rangle \\
&= \|Tx_n\|^2 - 2M\langle Tx_n, x_n \rangle + M^2\|x_n\|^2 \\
&\leq M^2 - 2M(M - \delta_n) + M^2 \\
&= 2M\delta_n.
\end{aligned}$$

Since $\delta_n \to 0$, it follows from the Squeeze Theorem that

$$\|Tx_n - Mx_n\| \to 0.$$

Hence, there is no positive c such that

$$\|T_M x_n\| = \|Tx_n - Mx_n\| \geq c = c\|x_n\|.$$

Using 9.1-2 , we now have that $M \notin \rho(T)$ and so we must have $M \in \sigma(T)$ which is our desired result. ■

Theorem (Residual Spectrum)**.** The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert Space H is empty.

Proof. Suppose for sake of contradiction that $\sigma_r(T) \neq \emptyset$ leads to a contradiction. Let $\lambda \in \sigma_r(T)$. By definition, the inverse of T_λ exists, but its domain $D(T_\lambda^{-1})$ is not dense in H. By the Projection Theorem there exist a $y \neq 0$ in H such that y is orthogonal to $D(T_\lambda^{-1})$. However, $D(T_\lambda^{-1})$ is the range of T_λ. Thus,

$$\langle T_\lambda x, y \rangle = 0$$

for all $x \in H$. Since λ is real by 9.1-3 and T is self-adjoint, we obtain

$$\langle x, T_\lambda y \rangle = 0$$

for all x. Since $y \neq 0$, we see that λ is an eigenvalue of T. But this contradicts the assumption that $\lambda \in \sigma_r(T)$, making $\sigma_r(T) \neq \emptyset$ absurd. Hence, it must follow that $\sigma_r(T) = \emptyset$. ■

2.3 Positive Operators

In this section, we develop the framework that enables us to do more analysis on our self-adjoint linear operators. To do this, we will create a partial order on our operators which will be an extension of the ordering defined on the set of real numbers in real analysis.

Definition (Partial Order). We define a **partial order** on the set of bounded self-adjoint linear operators defined over the complex Hilbert Space H by $T_1 \leq T_2$ if and only if $\langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle$.

Definition (Positive Operators). Let H be a complex Hilbert Space. We say that $T : H \to H$ is **positive**; that is, $T \geq 0$ if $\langle Tx, x\rangle \geq 0$.

In what follows, we will mention two facts that follow immediately from the definitions above.

Lemma. Let H be a Complex Hilbert Space and let $T_1, T_2 : H \to H$ be two linear operators such that $T_1 \leq T_1$. We say that $T_2 - T_1$ is positive if $T_2 - T_1 \geq 0$.

Lemma. The sum of two positive operators is positive.

Theorem (Product Of Positive Operators). If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute ($ST = TS$), then their product ST is positive.

Proof. Our goal is to show that $\langle STx, x\rangle \geq 0$ for all $x \in H$. Clearly, if $S = 0$, then the result holds. Suppose $S \neq 0$. We will show the following:

(a) If we consider $S_1 = \frac{1}{\|S\|}S$, $S_{n+1} = S_n - S_n^2$ for all $n \in \mathbb{N}$, then we will show via induction that

$$0 \leq S_n \leq I. \tag{*}$$

(b) Then we conclude that $\langle STx, x\rangle \geq 0$ for all $x \in H$.

(a) For $n = 1$, then the inequality in (*) holds. Indeed, using the Cauchy-Schwarz inequality, we have

$$\langle S_1 x, x\rangle = \frac{1}{\|S\|}\langle Sx, x\rangle \leq \frac{1}{\|S\|}\|Sx\|\|x\| \leq \|x\|^2 = \langle Ix, x\rangle \Longrightarrow S_1 \leq I.$$

Suppose (*) holds for all $n = k$; that is,

$$0 \leq S_k \leq I \Longrightarrow 0 \leq I - S_k \leq I.$$

Since each S_k is self-adjoint for every $x \in H$ and $y = S_k x$, we obtain

$$\begin{aligned}\langle S_k^2(I - S_k)x, x\rangle &= \langle (I - S_k)S_k x, S_k x\rangle \\ &= \langle (I - S_k)y, y\rangle \geq 0.\end{aligned}$$

By definition, we see that

$$S_k^2(I - S_k) \geq 0 \tag{1}$$

and similarly, we have

$$S_k(I - S_k)^2 \geq 0. \tag{2}$$

Adding (1) and (2) together, we have

$$0 \leq S_k^2(I - S_k) + S_k(I - S_k)^2 = S_k - S_k^2 = S_{k+1}.$$

Hence, we see that $0 \leq S_{k+1}$. And $S_{k+1} \leq I$ follows from $S_k^2 \geq 0$ and $I - S_k \geq 0$ by addition. Indeed, we see that

$$0 \leq I - S_k + S_k^2 = I - S_{k+1}$$

which completes the induction proof of (a).

■

2.4 Square Roots of a Positive Operator

Theorem (Positive Square Root). Let $T : H \to H$ be a positive bounded self-adjoint linear operator on a complex Hilbert space H. Then a bounded self-adjoint linear operator A is called a **square root** of T if

$$A^2 = T.$$

If, in addition, $A \geq 0$, then A is called a **positive square root** of T and is denoted by

$$A = T^{1/2}.$$

That is, $T^{1/2}$ exists and unique.

Theorem (Positive Square Root). Every positive bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert Space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

2.5 Projection Operators

Definition (Projections). The map $P : H \to Y$ defined by $x \mapsto y = Px$ is called the projection of H onto Y.

Proposition. A linear operator $P : H \to H$ is a projection on H is there is a closed subspace Y of H such that Y is the range of P and $Y^\perp$ is the null space of P and $P|_Y$ is the identity operator on Y.

The above implies that we can write any $x \in H$ in the following way

$$x = y + z \tag{1}$$

where $y \in Y$ and $z \in Y^\perp$. Furthermore, we can write H in terms of Y and $Y^\perp$ as a direct sum; that is,

$$H = Y \oplus Y^\perp.$$

We can also rewrite (1) in the following way

$$x = y + z = Px + (I - P)x.$$

Another way to characterize the projection on H is the presented in the following theorem below:

Theorem (Projection). A bounded linear operator $P : H \to H$ on a Hilbert Space H is a projection if and only if P is self-adjoint and idempotent (That is, $P^2 = P$).

Proof. ($\Longrightarrow$) Suppose that $P_1P_2 = P_2P_1$. Then by Theorem 3.10-4, P is self-adjoint. Also, we see that P is idempotent because

$$P^2 = (P_1P_2)(P_1P_2) = P_1^2P_2^2 = P_1P_2 = P]$$

where P_1 and P_2 are projections on H. By 9.5-1 , P is a projection, and for every $x \in H$, we have

$$Px = P_1(P_2x) = P_2(P_1x).$$

Since P_1 is a projects H onto Y_1, we must have $P_1(P_2x) \in Y_1$. Similarly, we see that $P_2(P_1x \in Y_2)$. Hence, we see that $Px \in Y_1 \cap Y_2$. Note that this projection is onto since

$$Py = P_1P_2y = P_1y = y.$$

($\Longleftarrow$) If $P = P_1P_2$ is a projection defined on H, then P is self-adjoint by 9.5-1. Also, we see that P_1 and P_2 commute by Theorem 3.10-4. ■

Theorem (Sum of Projections). Let P_1 and P_2 be projections on a Hilbert space H. Then:

(a) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(b) If $P = P_1 + P_2$ is a projection, P projects H onto $Y = Y_1 \oplus Y_2$.

(a) ($\Longrightarrow$) Suppose that $P = P_1 + P_2$ is a projection on H. Our goal is to show that Y_1 and Y_2 are orthogonal; that is, it suffices to show that $P_1P_2 = 0$ by Theorem 9.5-3(b). Observe that P is idempotent. Hence, we have

$$\begin{aligned} P_1 + P_2 &= (P_1 + P_2)^2 \\ &= P_1^2 + P_1P_2 + P_2P_1 + P_2^2 \\ &= P_1 + P_1P_2 + P_2P_1 + P_2. \end{aligned}$$

Hence, we are left with

$$P_1P_2 + P_2P_1 = 0.$$

Multiplying the quantity above by P_2 on the left, we get

$$P_2P_1P_2 + P_2^2P_1 = 0 \Longleftrightarrow P_2P_1P_2 + P_2P_1 = 0.$$

Multiplying the quantity above by P_2 on the right side, we obtain

$$P_2P_1P_2^2 + P_2P_1P_2 = 0 \Longrightarrow 2P_2P_1P_2 = 0.$$

Hence, it follows that

$$P_1P_2 = 0$$

which is our desired result. ($\Longleftarrow$) Suppose $Y_1 \perp Y_2$. To show that $P = P_1 + P_2$ is a projection on H, it suffices to show that $P^2 = P$ and P is self-adjoint. Starting with the first result, since P_1 and P_2 are both idempotent, we have that

$$\begin{aligned} P^2 &= (P_1 + P_2)^2 \\ &= P_1^2 + P_1P_2 + P_2P_1 + P_2^2 \\ &= P_1^2 + P_2^2 \\ &= P_1 + P_2 \\ &= P. \end{aligned}$$

Hence, we see that $P^2 = P$ and so P is idempotent. Also, since P_1 and P_2 are also self-adjoint, we can see immediately that $P = P_1 + P_2$ is also idempotent. Thus, we can conclude based on Theorem 9.5-3 that P is indeed a projection on H.

(b) Suppose $P = P_1 + P_2$ is a projection. Note that Y is the closed subspace onto which P projects to. By definition of P, it follows that for all $x \in H$, we have

$$\begin{aligned} Px &= (P_1 + P_2)x \\ &= P_1x + P_2x. \end{aligned}$$

Since P is an onto projection, we have

$$y = Px = P_1x + P_2x \tag{*}$$

where $P_1x \in Y_1$ and $P_2x \in Y_2$. We will show now that $Y = Y_1 \oplus Y_2$. Let $y \in Y$. From (*), it follows that $y \in Y_1 \oplus Y_2$ (clearly, P being a projection on Y means that $Y_1 \perp Y_2$ and so $Y_1 \cap Y_2 = \{0\}$). Now, let $v \in Y_1 \oplus Y_2$. Then it follows that

$$\begin{aligned} P(v) &= P_1v + P_2v \\ &= P_1(y_1) + P_2(y_2) \\ &= Y_1 + Y_2 \\ &= v. \end{aligned}$$

Hence, $v \in Y$ and so we have $Y_1 \oplus Y_2 \subseteq Y$. Thus, we conclude that $Y = Y_1 \oplus Y_2$.

2.6 Further Properties of Projections

Theorem (Partial Order)**.** Let P_1 and P_2 be projections defined on a Hilbert space H. Denote by $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$, the subspaces onto which H is projected by P_1 and P_2, and let $N(P_1)$ and $N(P_2)$ be the null spaces of these projections. Then the following conditions are equivalent.

(1) $P_2P_1 = P_1P_2 = P_1$

(2) $Y_1 \subseteq Y_2$

(3) $N(P_1) \supseteq N(P_2)$

(4) $\|P_1x\| \leq \|P_2x\|$ for all $x \in H$

(5) $P_1 \leq P_2$.

Proof. ■

Theorem (Monotone Increasing Sequence)**.** Let (P_n) be a monotone increasing sequence of projections P_n defined on a Hilbert space H. Then:

(a) (P_n) is strongly operator convergent, say, $P_nx \to Px$ for every $x \in H$, and the limit operator P is a projection defined on H.

(b) P projects H onto

$$P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}.$$

(c) P has the null space

$$N(P) = \bigcap_{n=1}^{\infty} N(P_n).$$