Week 1: Lecture Notes

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0.1 Lecture 1

0.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial x^2+1 does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\,dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

0.1.2 What is the goal?

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2+1=0$. Then we define the following set
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$
Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:
$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$
- Multiplication:
$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. **To do.** ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. **To do.** ■