1-1) $\overline{A}^Y = \overline{A}^X \cap Y$

proof

Our goal is to show that $\overline{A}^Y = \overline{A}^X \cap Y$.

Note $\overline{A}$ is the smallest closed set that contains A and the fact that we can write $\overline{A}$ in the following way:

$$\overline{A} = \bigcap_{\substack{A \subseteq F \\ F \text{ is closed} \\ \text{in } X}} F = \{\text{intersection of closed sets that contain } A\}.$$

Now, Y is a subspace of X. If F is a closed set in Y, then there must exists a closed set $G \subseteq X$ such that $F = G \cap Y$. Thus, we have

$$\overline{A}^Y = \bigcap_{\substack{A \subseteq F \\ F \text{ is} \\ \text{closed in} \\ Y}} F = \bigcap_{A \subseteq G \cap Y} (G \cap Y) = \Bigg(\bigcap_{\substack{A \subseteq G \\ G \text{ is} \\ \text{closed} \\ \text{in } X}} G\Bigg) \cap Y = (\overline{A})^X \cap Y$$

$\blacksquare$

1-2) $\mathrm{int}_Y A = \mathrm{int}_X([A \cup (X \setminus Y)]) \cap Y$.

<u>proof</u>

Our goal is to show that

$$\mathrm{int}_Y A = \mathrm{int}_X([A \cup (X \setminus Y)]) \cap Y.$$

Note that $\mathrm{int}_X A$ can be written in the following way:

$$\mathrm{int}_Y A = \bigcup_{\substack{G \text{ is open in } Y \\ G \subseteq A}} G .$$

Furthermore, if G is open relative to Y, there exists $\underset{\text{open}}{O} \subseteq X$ such that $\underset{\text{open}}{O} = G \cap Y$.

Also, if $A \subseteq Y$, then $O \cap Y \subseteq A$ is equivalent to $O \subseteq A \cup (X \setminus Y)$.

Then observe that

$$\mathrm{int}_Y A = \bigcup_{\substack{G \text{ is open in } Y, \\ G \subseteq A}} G = \bigcup_{\substack{O \text{ is open in } X, \\ O \cap Y \subseteq A}} (O \cap Y)$$

$$= \Big(\bigcup_{\substack{O \text{ is open}, \\ O \subseteq A \cup (X \setminus Y)}} O \Big) \cap Y$$

by definition

$$= \mathrm{int}_X [A \cup (X \setminus Y)] \cap Y.$$

$\blacksquare$

2- Let (X,d) be a metric space and suppose E is dense in X. Let Y be a nonempty subset of X.

2-1)

Let $X=\mathbb{R}$ and $Y=\{\sqrt{2}\}$. Let $E=\mathbb{Q}$ be dense in $\mathbb{R}$. Then we see that

$$\mathbb{Q}\cap Y = \emptyset$$

and so clearly $\mathbb{Q}\cap Y$ is NOT dense in Y.

$\blacksquare$

2-2) Prove that if Y is open, then $E \cap Y$ is dense in Y.

<u>proof</u>

Suppose Y is a nonempty open set and assume that E is dense in X. Our goal is to show that $E \cap Y$ is dense in Y; that is, any nonempty open set in Y has a nonempty intersection with $E \cap Y$.

To this end, let $G \subseteq Y$ be a nonempty open set in Y. Since G and Y are both nonempty open sets, the finite intersection $G \cap Y$ is open.

Since E is dense in X, $E \cap (G \cap Y) \neq \emptyset$.

Thus, $(E \cap Y) \cap G \neq \emptyset$.

So, $E \cap Y$ is dense in Y.

$\blacksquare$

3. Let (X,d) be a metric space and let $E \subseteq X$. Suppose $x \in X$. Prove that if there exists a sequence in $E \setminus \{x\}$ that converges to x, then $x \in E'$.

<u>proof</u>

Suppose there exists a sequence $(x_n) \subseteq E \setminus \{x\}$ such that $(x_n) \to x$. Our goal is to show that $x \in E'$. Let $\varepsilon > 0$. By assumption, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have $d(x_n, x) < \varepsilon$. (1) Note that $x_n \neq x$ since $E \setminus \{x\}$. (2) By (1) and (2), we see that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset .$$

But this implies that $x \in E'$.

$\blacksquare$

4- <u>proof</u> Suppose $(x_n) \subseteq E$. $(x_n \in E)$

Our goal is to show that $x \in E'$. Assume that there exists a sequence in E that converges to x. Note that $x_n = x$ or $x_n \neq x$. Suppose that $x_n = x$. Clearly, $x \in E'$. and so $x \in \overline{E}$ since $E' \subseteq \overline{E}$. Suppose $x_n \neq x$. Then use problem 3 to conclude that $x \in E'$ and thus $x \in \overline{E}$ since $E' \subseteq \overline{E}$.

$\blacksquare$

5- proof

Suppose that (x_n) is Cauchy. Our claim is that (x_n) is bounded. Now, we have for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, m > N$, $d(x_n, x_m) < \varepsilon$. If we let $\varepsilon = 1$, then we see that

$$d(x_n, x_m) < 1.$$

In particular, let us fix $m = N+1$. Thus, for all $n > N$

$$d(x_n, x_{N+1}) < 1.$$

If we take $r = \max\{d(x_1, x_{N+1}), d(x_2, x_{N+1}), \dots, d(x_n, x_{N+1}) + 1\}$ then we see that $d(x_n, x_{N+1}) < r$ for all $n \in \mathbb{N}$. Thus, $\forall n \in \mathbb{N}$ $x_n \in N_r(x_{N+1}) \Rightarrow \{x_n : n \in \mathbb{N}\} \subseteq N_r(x_{N+1})$.

$\blacksquare$

6- proof

Let $(x_n) \subseteq E$ be a Cauchy sequence and E is a closed set. Since $E \subseteq X$, $\forall n \in \mathbb{N}$ $x_n \in X$. Since X is complete, (x_n) converges to some $x \in X$. By problem 3, $x \in E'$. Note that E is closed, so $x \in E$. Hence, E is a complete.

$\blacksquare$

7. Let $(a_n) \to a$ and $(b_n) \to b$ with $a, b \in \mathbb{R}$.

<u>proof</u>

(i) $\lim_{n \to \infty} (a_n + b_n) = a + b$

Let $\varepsilon > 0$. Suppose that $(a_n) \to a$ and $(b_n) \to b$.

Since $(a_n) \to a$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|a_n - a| < \frac{\varepsilon}{2}. \qquad (1)$$

Since $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|b_n - b| < \frac{\varepsilon}{2}. \qquad (2)$$

Now, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, we see that $\lim_{n \to \infty} (a_n + b_n) = a + b$.

(ii) $\lim_{n\to\infty} (a_n b_n) = ab$

Our goal is to show that $\lim_{n\to\infty} (a_n b_n) = ab$. Let $\varepsilon > 0$.

Since $(a_n) \to a$, we know there exists $M > 0$ such that $|a_n| \leq M$ (*). Since $(b_n) \to b$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|b_n - b| < \frac{\varepsilon}{2M}. \tag{3}$$

Similarly, $(a_n) \to a$ implies there exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|a_n - a| < \frac{\varepsilon}{2|b|} \tag{4}$$

Hence, we see that (3) and (4) imply that

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&\leq |a_n||b_n - b| + |b||a_n - a|. \\
&\leq M|b_n - b| + |b||a_n - a| \\
&< M \cdot \frac{\varepsilon}{2M} + |b| \cdot \frac{\varepsilon}{2|b|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\Rightarrow \lim_{n\to\infty} a_n b_n = ab.$

(iii) $\lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{b}$; provided that $b \neq 0$.

Our goal is to show that $\lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{b}$. Let $\varepsilon > 0$.

Since $(b_n) \to b$, we know that there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$ we have

$$|b_n - b| < \frac{|b|^2 \varepsilon}{2} \qquad (1)$$

By the same assumption, we can choose $N_2 \in \mathbb{N}$ s.t for any $n > N_2$

$$|b - b_n| < \frac{|b|}{2} \iff |b_n| > \frac{|b|}{2}$$

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n b} \right| = \frac{|b_n - b|}{|b_n||b|}$$

$$= \frac{1}{|b|} \cdot \frac{1}{|b_n|} \cdot |b_n - b|$$

$$< \frac{1}{|b|} \cdot \frac{2}{|b|} \cdot \frac{|b|^2 \varepsilon}{2}$$

$$= \varepsilon$$

$\Rightarrow \lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{b}$. ∎

(iv) $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

Note that by part (ii) and part (iii), we see that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} \frac{1}{b_n}$$

$$= a \cdot \frac{1}{b}$$

$$= \frac{a}{b}.$$

8. Prove the following theorem.

Thm (Order Limit Theorem)

Consider $\mathbb{R}$ equipped with its standard metric. Assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

(i) If $a_n \geq 0$ $\forall n$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ $\forall n$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ s.t $b_n \geq c$ $\forall n$, then $b \geq c$.

(iv) If there exists $c \in \mathbb{R}$ s.t $a_n \leq c$ $\forall n$, then $a \leq c$.

proof

(i) Suppose $a_n \geq 0$ $\forall n \in \mathbb{N}$. Let $\varepsilon > 0$. Since $(a_n) \to a$, we know there exists $N \in \mathbb{N}$ such that for any $n > N$, we have

$$|a_n - a| < \varepsilon \iff |a_n| < |a| + \varepsilon .$$

Since $a_n \geq 0$ $\forall n \in \mathbb{N}$, we have

$$|a| + \varepsilon > |a_n| \geq 0 .$$

Since $\varepsilon > 0$ is arbitrary, we have $a \geq 0$.

(iii) Suppose $a_n \le b_n$ for all $n \in \mathbb{N}$. Our goal is to show that $a \le b$.

Since $(a_n) \to a$ and $(b_n) \to b$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \Rightarrow |a| < |a_n| + \frac{\varepsilon}{2} \quad \text{for any } n \ge N_1 \qquad (1)$$

$$|b_n - b| < \frac{\varepsilon}{2} \quad \text{for any } n \ge N_2 \qquad (2)$$

Our goal is to show that $|a| \le |b| + \varepsilon$, so that we can show that $|a| \le |b|$. Note that if $a_n = b_n$ $\forall n \in \mathbb{N}$, then clearly $a = b$. Suppose $a_n < b_n$ $\forall n \in \mathbb{N}$. Then we have

$$\begin{aligned}
|a| < |a_n| + \frac{\varepsilon}{2} &\le |b_n| + \frac{\varepsilon}{2} \\
&= |(b_n - b) + b| + \frac{\varepsilon}{2} \\
&\le |b_n - b| + |b| + \frac{\varepsilon}{2} \\
&< |b| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= |b| + \varepsilon \quad \text{as of hw1}
\end{aligned}$$

$$\Rightarrow |a| < |b| + \varepsilon \Longrightarrow |a| < |b|.$$

(iii) If there exists $c \in \mathbb{R}$ such that $b_n \geq c \ \forall n \in \mathbb{N}$, then $b \geq c$.

<u>proof</u>

Suppose there exists $c \in \mathbb{R}$ such that $b_n \geq c \ \forall n \in \mathbb{N}$. Our goal is to show $b \geq c$. By assumption, $(b_n) \to b$ implies that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|b_n - b| < \varepsilon \Leftrightarrow b_n - \varepsilon < b < b_n + \varepsilon$$

$$\Rightarrow b > b_n - \varepsilon \geq c - \varepsilon \quad \forall n \in \mathbb{N}$$

$$\Rightarrow b \geq c - \varepsilon.$$

Note that $\varepsilon > 0$ is arbitrary. Hence, $b \geq c$.

$\blacksquare$

(iv) If there exists $c \in \mathbb{R}$ such that $a_n \leq c \ \forall n \in \mathbb{N}$, then $a \leq c$.

<u>proof</u>

Suppose there exists $c \in \mathbb{R}$ such that $a_n \leq c \ \forall n \in \mathbb{N}$.

Since $(a_n) \to a$, we know that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - a| < \varepsilon \Leftrightarrow |a - a_n| < \varepsilon$$

$$\Leftrightarrow a_n - \varepsilon < a < a_n + \varepsilon$$

$$\Rightarrow a < a_n + \varepsilon \leq c + \varepsilon \ \forall n \in \mathbb{N}$$

$$\Rightarrow a \leq c + \varepsilon$$

$$\Rightarrow a \leq c$$ (since $\varepsilon > 0$ is arbitrary)

Problem 5 *
HW1

■

$\Rightarrow a < b$ ∎

9- proof

Assume that $x_n \leq y_n \leq z_n \ \forall n \in \mathbb{N}$. Assume that $x_n \to L$ and $z_n \to L$. Our goal is to show that $y_n \to L$. Let $\varepsilon > 0$. Our goal is to show that there exists some $N \in \mathbb{N}$ such that for any $n > N$,

$$|y_n - L| < \varepsilon .$$

By assumption, we see that $\forall n \in \mathbb{N}$

$$x_n \leq y_n \leq z_n \iff y_n - x_n \leq z_n - x_n .$$

Since $(x_n) \to L$, there exists some $N_1 \in \mathbb{N}$ s.t $\forall n > N_1$

$$|x_n - L| < \frac{\varepsilon}{4} . \qquad (1)$$

Since $(z_n) \to L$, there exists some $N_2 \in \mathbb{N}$ s.t $\forall n > N_2$

$$|z_n - L| < \frac{\varepsilon}{2} . \qquad (2)$$

Now, choose $N = \max\{N_1, N_2\}$ s.t $\forall n > N$, we have that

$$
\begin{aligned}
|Y_n - L| &= |Y_n - X_n + X_n - L| \\
&\leq |Y_n - X_n| + |X_n - L| \\
&\leq |Z_n - X_n| + |X_n - L| \\
&\leq |Z_n - L| + 2|X_n - L| \\
&< \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\end{aligned}
$$

$\Rightarrow (Y_n) \to L.$ $\blacksquare$

10-

<u>proof</u>

Assume for sake of contradiction that $(x_n) \not\to L$. So, we must have $\exists\, \varepsilon > 0$ such that for all $n > N$, we have

$$d(x_n, x) \geq \varepsilon.$$

To reach a contradiction, we will construct a subsequence of (x_n) that does not have any further subsequence that converges to L. If $N=1$, then $\exists\, n_1 > 1$ s.t. $d(x_{n_1}, L) \geq \varepsilon$.

$$\Rightarrow N = N_1 \Rightarrow \exists\, n_2 > n_1 \text{ s.t. } d(x_{n_2}, L) \geq \varepsilon$$

$$\Rightarrow N = N_2 \Rightarrow \exists\, n_3 > n_2 \text{ s.t. } d(x_{n_3}, L) \geq \varepsilon$$

$$\vdots$$

In this way, we have constructed a subsequence (x_{n_k}) of (x_n) with the property that

$$\forall k \in \mathbb{N} \quad d(x_{n_k}, L) \geq \varepsilon.$$

But this tells us that (x_{n_k}) must diverge, which is the contradiction we were aiming for. $\blacksquare$

11- Consider $\mathbb{R}$ equipped with its standard metric. Let $x_n \geq 0 \ \forall n \in \mathbb{N}$.

11-1) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

<u>proof</u>

Suppose $(x_n) \to 0$. Our goal is to show that $\sqrt{x_n} \to 0$; that is, we need to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have

$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \varepsilon. \qquad (*)$$

To this end, let $\varepsilon > 0$. Note that if $x_n = 0$ $\forall n \in \mathbb{N}$, then clearly $(\sqrt{x_n}) \to 0$. Suppose $x_n > 0$ $\forall n \in \mathbb{N}$. Since $(x_n) \to 0$, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$, we have

$$|x_n - 0| < \varepsilon^2 \qquad (1)$$

Using (1) and choosing $N = \hat{N} \in \mathbb{N}$, we can let any $n \geq N$,

$$|x_n| < \varepsilon^2 \Rightarrow x_n < \varepsilon^2$$

$$\Rightarrow \sqrt{x_n} < \varepsilon^2$$

$$\Rightarrow |\sqrt{x_n} - 0| < \varepsilon \quad (\varepsilon > 0)$$

Hence, $(\sqrt{x_n}) \to 0$.

11-2) If $(x_n) \to x$, show that $\sqrt{x_n} \to \sqrt{x}$.

<u>proof</u>

Note that if $x = 0$, then we can use 11-1) to conclude that $\sqrt{x_n} \to \sqrt{x}$. Suppose $x > 0$. Our goal is to show that $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, $|\sqrt{x_n} - \sqrt{x}| < \varepsilon$. Observe that

$$|\sqrt{x_n}-\sqrt{x}| = \left|\sqrt{x_n}-\sqrt{x}\cdot\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right| = \frac{|x_n-x|}{\sqrt{x_n}+\sqrt{x}}$$

$$\leq \frac{|x_n-x|}{\sqrt{x}}.$$

Note that $x>0$. Hence, $(x_n)\to x$ implies that there exists $\hat{N}\in\mathbb{N}$ such that for any $n>\hat{N}$, we have

$$|x_n-x| < \sqrt{x}\cdot\varepsilon. \qquad (*)$$

Choose $N=\hat{N}$. Then for any $n>N$, we have

$$|\sqrt{x_n}-\sqrt{x}| \leq \frac{|x_n-x|}{\sqrt{x}} < \frac{\sqrt{x}\cdot\varepsilon}{\sqrt{x}} = \varepsilon$$

$$\Rightarrow |\sqrt{x_n}-\sqrt{x}| < \varepsilon$$

$$\Rightarrow (\sqrt{x_n})\to x.$$

∎

12- Let (a_n) be a sequence in $\mathbb{R}$. Prove the following statements are equivalent.

(i) (a_n) bounded

(ii) There exists L and M in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $L \leq a_n \leq M$.

(iii) There exists $c > 0$ such that for all $n \in \mathbb{N}$, we have $|a_n| \leq c$.

<u>proof</u>

(i) $\Rightarrow$ (ii)

Suppose that (a_n) is bounded. Our goal is to find $L, M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $L \leq a_n \leq M$. Let $n \in \mathbb{N}$ be given. Since (a_n) is bounded, there exists $p \in \mathbb{R}$ and $r > 0$ such that

$$\{X_n : n \in \mathbb{N}\} \subseteq N_r(p) \subseteq \overline{N_r(p)}.$$

Note that in $\mathbb{R}$ $\overline{N_r(p)} = C_r(p)$. Thus, for every $n \in \mathbb{N}$, $|X_n - p| \leq r$. This implies that

$$|X_n - p| \leq r \iff p - r \leq X_n \leq p + r$$

Set $p-r=L$ and $p+r=M$. Then we see that

$$L \leq a_n \leq M.$$

(ii) $\Rightarrow$ (iii)

Suppose there exists L and M in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $L \leq a_n \leq M$. (*) Our goal is to show that there exists $C>0$ such that for all $n \in \mathbb{N}$, we have $|a_n| \leq C$; that is, $\exists\, C>0$ s.t $\forall n \in \mathbb{N}$, $-C \leq a_n \leq C$. To this end, let $n \in \mathbb{N}$. By assumption, we can set $L=-C$ and $M=C$. Hence, (*) implies that

$$-C \leq a_n \leq C \Leftrightarrow |a_n| \leq C.$$

(iii) $\Rightarrow$ (i)

Suppose there exists $C>0$ such that for all $n \in \mathbb{N}$, we have $|a_n| \leq C$. (+) Our goal is to show that (a_n) is bounded; that is, there exists $q \in \mathbb{R}$ and $r>0$ such that $d(a_n, q) < r$.

Note that (†) implies that

$$-c \leq a_n \leq c. \qquad (1)$$

Adding q on both sides of (1) implies

$$q-c \leq a_n \leq q+c \Leftrightarrow -c \leq a_n - q \leq c$$

$$\Leftrightarrow |a_n - q| \leq c < c+1$$

Set $r = c+1 > 0$. Clearly, the inequality above implies that $\forall n \in \mathbb{N}$, $d(a_n, q) < r$.

Thus, $\{a_n : n \in \mathbb{N}\} \subseteq N_r(q)$ and so (a_n) is bounded.

$\blacksquare$