

# Linear Algebra Notes

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# Chapter 1

## Vector Spaces

### 1.1 Vector Spaces

#### 1.1.1 Basics

**Definition 1.1.1 (Vector Spaces).** A **vector space** (or **linear space**) over a field  $F$  consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements  $x, y$ , in  $V$  there is a unique element  $ax$  in  $V$ , such that the following conditions hold:

- (VS 1) For all  $x, y \in V$ ,  $x + y = y + x$  (commutativity of addition).
- (VS 2) For all  $x, y, z \in V$ ,  $(x + y) + z = x + (y + z)$  (associativity of addition).
- (VS 3) There exists an element in  $V$  denoted by  $O$  such that  $x + O = x$  for each  $x \in V$ .
- (VS 4) For each element  $x \in V$ , there exists an element  $y \in V$  such that  $x + y = O$ .
- (VS 5) For each element  $x \in V$ , we have  $1x = x$ .
- (VS 6) For each  $a, b \in F$  and each element  $x \in V$ , then  $(ab)x = a(bx)$ .
- (VS 7) For each element  $a \in F$  and each pair  $x, y \in V$ , we have  $a(x + y) = ax + ay$ .
- (VS 8) For each pair  $a, b \in F$  and each  $x \in V$ , we have  $(a + b)x = ax + bx$ .

The elements  $x + y$  and  $ax$  are called the **sum** of  $x$  and  $y$  and the **product** of  $a$  and  $x$ , respectively.

- The elements of a field  $F$  are called **scalars** and the elements of a vector space  $V$  are called **vectors** (these should not be confused!).
- Every vector space will always be defined over a given field, mostly defined over the real numbers  $\mathbb{R}$  or the complex numbers  $\mathbb{C}$  unless otherwise noted.
- Every vector space should specify the operations of addition and scalar multiplication.

**Definition 1.1.2 (n-tuples).** An object of the form  $(a_1, a_2, \dots, a_n)$ , where the entries  $a_1, a_2, \dots, a_n$  are elements of a field  $F$ , is called an **n-tuple** with entries from  $F$ . The elements  $a_1, a_2, \dots, a_n$  are called **entries** or **components** of the  $n$ -tuple.

**Definition 1.1.3.** We say that two  $n$ -tuples,  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$ , are **equal** if  $a_i = b_i$  for  $i = 1, 2, \dots, n$ .

**Example 1.1.1.** The set of all  $n$ -tuples with entries from a field  $F$  denoted by  $F_n$  is a vector space. To see why, suppose  $u, v \in F_n$  where  $u = (a_1, a_2, \dots, a_n)$  and  $v = (b_1, b_2, \dots, b_n)$ . If we take

term-by-term addition of the entries in both  $u$  and  $v$ , then we end up with

$$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and likewise,

$$cu = (ca_1, ca_2, \dots, ca_n).$$

These same set of operations define  $\mathbb{R}^3$  as a vector space over  $\mathbb{R}$  and likewise,  $\mathbb{C}^2$  is a vector space over  $\mathbb{C}$ .

- Note that vectors in  $F^n$  can be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

rather than **row vectors**  $(a_1, a_2, \dots, a_n)$ .

- 1-tuples are just scalars or an just an element from  $F$ .

**Definition 1.1.4.** An  $m \times n$  **matrix** with entries from a field  $F$  is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry  $a_{ij}$  with  $(1 \leq i \leq m, 1 \leq j \leq n)$  is an element of  $F$ . We call the entries  $a_{ij}$  with  $i = j$  the **diagonal entries** of the matrix. The entries  $a_{i1}, a_{i2}, \dots, a_{in}$  compose the  $i$ **th row** of the matrix, and the entries  $a_{1j}, a_{2j}, \dots, a_{mj}$  compose the  $j$ **th column** of the matrix.

- The rows make a vector space which we denote  $F^n$ .
- Likewise, the columns make a vector space we denote  $F^m$ .

**Definition 1.1.5 (Zero Matrix).** The  $m \times n$  matrix in which each entry equals zero is called the **zero matrix** and is denoted by  $O$ .

**Definition 1.1.6 (Square Matrix).** A matrix is **square** if the number of rows and columns of a matrix are equal.

- Just like our tuple example, the set of all  $m \times n$  matrices with entries from a field  $F$  form a vector space. Denote this vector space as  $M_{m \times n}(F)$  endowed with two operations; that is, **matrix addition** and **scalar multiplication**. Suppose for  $A, B \in M_{m \times n}(F)$  and  $c \in F$ , we have

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

and

$$(cA)_{ij} = cA_{ij}$$

for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . In other words, the two operations can be performed entry-wise.

- The operations from our tuple case extends very naturally to  $M_{m \times n}(F)$ . In other words, if we add two matrices  $A_{ij}$  and  $B_{ij}$ , then we would expect to that  $A_{ij} + B_{ij} \in M_{m \times n}(F)$  as well and likewise for the scalar multiplication case.

**Definition 1.1.7 (Set of All Functions).** Let  $S$  be any nonempty set and  $F$  be any field, and let  $\mathcal{F}(S, F)$  denote the set of all functions from  $S$  to  $F$ .

**Definition 1.1.8.** Two functions  $f, g \in \mathcal{F}(S, F)$  are called **equal** if  $f(s) = g(s)$  for each  $s \in S$ .

The set of functions above forms a vector space with the operations of addition and scalar multiplication defined for every  $f, g \in \mathcal{F}(S, F)$  and  $c \in F$  with

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$

**Definition 1.1.9 (Set of All Polynomials).** A **polynomial** with coefficients from a field  $F$  is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where  $n$  is a nonnegative integer and each  $a_k \in F$  is called the **coefficient** of  $x_k$ .

**Definition 1.1.10 (Zero Polynomial).** We call  $f(x) = 0$  the **zero polynomial** if  $a_n = a_{n-1} = \cdots = a_0 = 0$ .

**Definition 1.1.11 (Degree).** The **degree** of a given polynomial  $f$  is defined to be the largest exponent of  $x$  that appears in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

**Definition 1.1.12 (Equality of Polynomials).** We call two polynomials  $f, g$ , where

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

are **equal** if  $m = n$  and  $a_i = b_i$  for all  $i = 0, 1, \dots, n$ .

Suppose we have  $c \in F$  and say we evaluated the polynomial  $f \in F$  at  $c$ . Then we would have the following

$$f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

where  $f(c) \in F$ .

**Definition 1.1.13 (Basic Operations of Polynomials).** Define polynomial **addition**  $f + g$  as the following:

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Let  $c \in F$ . Let scalar **multiplication** be defined by

$$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0.$$

The operations above form a vector space for  $P(F)$  (the set of all polynomials).

**Definition 1.1.14 (Sequences).** A **sequence** in  $F$  is a function  $\sigma : \mathbb{Z}^+ \rightarrow F$ . A given sequence  $\sigma$  such that  $\sigma(n) = a_n$  for  $n = 1, 2, \dots$  is denoted  $(a_n)$ .

Let  $V$  be the set of all sequences  $\sigma(n) \in F$ . For every  $(a_n), (b_n) \in V$  with  $t \in F$ , we have the following operations

$$(a_n) + (b_n) = (a_n + b_n) \text{ and } t(a_n) = (ta_n).$$

### 1.1.2 Non-examples

**Example 1.1.2.** Let  $S = \{(a_1, a_2) : a_1, a_2 \in R\}$  where  $R$  is a field. For every  $(a_1, a_2), (b_1, b_2) \in S$  and  $c \in R$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Note that  $S$  is **NOT** a vector space since (VS 1), (VS 2), and (VS 8) fail.

**Example 1.1.3.** Let  $S$  be the same set as in the last example. For  $(a_1, a_2), (b_1, b_2) \in S$  and  $c \in R$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$$

and

$$c(a_1, a_2) = (ca_1, 0).$$

Note that  $S$  is **NOT** a vector space with these operations since (VS 3), (VS 4), and (VS 5) fail.

### 1.1.3 Basic Extensions from Definition

**Theorem 1.1.1** (Cancellation Law for Vector Addition). If  $x, y, z \in V$  such that  $x + z = y + z$ , then  $x = y$ .

**Proof.** There exists a vector  $v \in V$  such that  $z + v = O$  (VS 4). Thus, we have

$$\begin{aligned} x &= x + O \\ &= x + (z + v) \\ &= (x + z) + v \\ &= (y + z) + v \\ &= y + (z + v) \\ &= y + O \\ &= y. \end{aligned}$$

Hence, we have  $x = y$ . ■

**Corollary.** The vector described  $O$  described in (VS 3) is unique.

**Proof.** Let  $x \in V$ . Suppose there exists two elements  $O, O' \in V$  such that

$$x + O = x \tag{1}$$

and

$$x + O' = x. \tag{2}$$

Our objective is to show that  $O = O'$ . Equating (1) and (2), we find that

$$x + O = x + O' \tag{3}$$

Using the Cancellation Law for Vector Addition, (3) implies that  $O = O'$ . Hence,  $O$  is a unique vector in  $V$ . ■

**Corollary.** The vector  $y$  described in (VS 4) is unique.

**Proof.** Let  $x \in V$ . Suppose there exists  $y, y' \in V$  such that

$$x + y = O \text{ and } x + y' = O.$$



Equating the two equations above, we get that

$$x + y = x + y'.$$

Applying the Cancellation Law for Vector Addition once again, we get that  $y = y'$ . Hence,  $y$  is a unique vector in  $V$ . ■

**Theorem 1.1.2.** In any vector space  $V$ , the following statements are true:

- (a)  $0x = O$  for any  $x \in V$ .
- (b)  $(-a)x = -(ax) = a(-x)$  for each  $a \in F$  and each  $x \in V$ .
- (c)  $a0 = 0$  for any  $a \in F$ .

**Proof.** (a) Using (VS 8), (VS 3), and (VS 1), it follows that

$$\begin{aligned} 0x + 0x &= (0 + 0)x \\ &= 0x \\ &= 0x + O \\ &= O + 0x. \end{aligned}$$

Hence,  $0x = O$  by Theorem 1.1.

- (b) The vector  $-(ax)$  is the unique element of  $V$  such that  $ax + [-(ax)] = O$ . If we know that  $ax + (-a)x = O$ , we can use Corollary 2 to Theorem 1.1 to state that  $(-a)x = -(ax)$ . But by (VS 8), we have

$$ax + (-ax) = [a + (-a)]x = 0x = O$$

by (a). Hence, we have that  $(-a)x = -(ax)$ . In particular, we have  $(-1)x = -x$ . Thus, (VS 6) implies that

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

- (c) The proof of (c) is similar to the proof of (a). ■

## 1.2 Subspaces

**Definition 1.2.1 (Subspaces).** A subset  $W$  of a vector space  $V$  over a field  $F$  is called a **subspace** of  $V$  if  $W$  is a vector space over  $F$  with the operations of addition and scalar multiplication.

The most simple examples of subspaces of  $V$  is  $V$  itself and the set containing just the zero vector. The latter is denoted as the **zero subspace** of  $V$ .

We don't have to check all the vector space properties to prove that a subset  $W$  of  $V$  is a subspace of  $V$ . This is because vectors in any subset of  $V$  already satisfy all the properties of a vector space. Hence, we need only check that  $W$  is closed under

1. **Addition:**  $x + y \in W$  whenever  $x \in W$  and  $y \in W$ .
2. **Scalar Multiplication:**  $cx \in W$  whenever  $c \in F$  and  $x \in W$ .
3.  $W$  contains a zero vector.
4. Each vector in  $W$  has an additive inverse in  $W$ .

**Theorem 1.2.1 (Subspaces).** Let  $V$  be a vector space and  $W$  a subset of  $V$ . Then  $W$  is a subspace of  $V$  if and only if the following three conditions hold for the operations defined in  $V$ .

- (a)  $O \in W$ .

(b)  $x + y \in W$  whenever  $x \in W$  and  $y \in W$ .

(c)  $cx \in W$  whenever  $c \in F$  and  $x \in W$ .

**Proof.** Suppose  $W$  is a subspace of  $V$ . Since  $W$  is also a vector space with the operations of addition and scalar multiplication defined in  $V$ . Hence, we know that  $W$  is closed under addition and scalar multiplication. Note that  $W$  contains a zero vector  $0' \in W$  such that for any  $x \in W$ , we have  $x + 0' = x$ . Since  $x$  is also in  $V$  (since  $W \subseteq V$ ), we know that  $x + 0 = x$ . Using the cancellation, we can see that  $x + 0' = x + 0$  implies  $0' = 0$ . Hence, properties (a), (b), and (c) are satisfied.

Conversely, suppose conditions (a), (b), and (c) hold. We need to show that  $W$  is a subspace. This tells us that  $W$  is closed under addition and scalar multiplication in addition to containing the zero vector. Let  $x \in W$ . Since  $W$  is closed under scalar multiplication, we know that  $(-1)x \in W$ . By part (b) of theorem 2, we know that  $1(-x) = -x \in W$ . Hence,  $W$  contains an additive inverse and we are done. ■

The theorem above provides a simpler way to determine whether a given subset of a vector space is a subspace. Instead of using the definition to show that a give subset is a subspace, it is more common to use the result above.

**Definition 1.2.2 (Transpose).** The **transpose** of  $A^t$  of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix obtained from  $A$  by interchanging the rows with the columns; that is,  $(A^t)_{ij} = A_{ji}$ .

**Example 1.2.1.**

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

**Definition 1.2.3 (Symmetric Matrices).** A **symmetric matrix** is a matrix  $A$  such that  $A^t = A$ .

- The easiest example of a symmetric matrix would be a square matrix where  $i = j$ .
- The set  $W$  of all symmetric matrices in  $M_{n \times n}(F)$  is a subspace of  $M_{n \times n}(F)$  since the conditions of Theorem 1.3 hold.

We can show that  $W$  is indeed a subspace.

1. The zero matrix is equal to its transpose and hence belongs to  $W$ .
2. Suppose  $A \in W$  and  $B \in W$ . Hence,  $A^t = A$  and  $B^t = B$ . Hence, we have

$$(A + B)^t = A^t + B^t = A + B$$

which implies that  $A + B \in W$ .

3. We have  $A \in W$  implies  $A^t = A$ . Now, let  $a \in F$ . Then we have that  $(aA)^t = aA^t = aA$ . Hence,  $aA \in W$ .

**Example 1.2.2.** Let  $n$  be non-negative integer, and let  $P_n(F)$  consist of all polynomials in  $P(F)$  having degree less than or equal to  $n$ . We get that  $P_n(F)$  is a subspace because:

1. Zero polynomial has degree -1, it is in  $P_n(F)$ .
2. The sum of two polynomials with degrees less than or equal to  $n$  is another a polynomial less than or equal to  $n$ .
3. The product of a scalar and a polynomial of degree less than or equal to  $n$  is a polynomial less than or equal to  $n$ .

This tells us that  $P_n(F)$  is a subspace of the space of all polynomials.

**Example 1.2.3.** Let  $C(\mathbb{R})$  denote the set of all continuous real-valued functions defined on  $\mathbb{R}$ . We know that  $C(\mathbb{R})$  is a subset of the vector space  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  defined in Example 3. We will show that  $C(\mathbb{R})$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .

1. Note that the zero function  $f(x) = 0$  for all  $x \in \mathbb{R}$  of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  is a continuous real-valued function.
2. Let  $f, g \in C(\mathbb{R})$ . Since the sum of  $f$  and  $g$  is also continuous for all  $x \in \mathbb{R}$ , we have that  $f + g \in C(\mathbb{R})$ .
3. Let  $c \in \mathbb{R}$  and  $f \in C(\mathbb{R})$ . We have that for any  $x \in \mathbb{R}$ ,  $(cf)(x) = cf(x)$  is a continuous function. Hence, property (c) is satisfied.

Hence,  $C(\mathbb{R})$  is a subspace.

**Definition 1.2.4 (Upper Triangular).** An  $m \times n$  matrix  $A$  is called **upper triangular** if all its entries below the diagonal entries are zero; that is, if  $A_{ij} = 0$  whenever  $i > j$ .

**Example 1.2.4.** Let  $B$  be an upper triangular  $3 \times 4$  matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{pmatrix}$$

**Definition 1.2.5 (Diagonal Matrix).** An  $n \times n$  matrix  $M$  is called a **diagonal matrix** if  $M_{ij} = 0$  whenever  $i \neq j$ ; that is, if all its non-diagonal entries are zero.

**Example 1.2.5.** Let  $A$  be diagonal  $3 \times 3$  matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

**Example 1.2.6.** The set of diagonal matrices is a subspace of  $M_{n \times n}(F)$ . To see why, we have

1. Let  $O$  be the zero matrix of  $M_{n \times n}(F)$ . Since  $O$  is also a diagonal matrix, we know that  $O$  must be in the set of diagonal matrices.
2. Let  $A, B$  in the set of diagonal matrices. Let  $i \neq j$  such that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0.$$

Hence,  $A + B$  is a diagonal matrix and so addition is closed.

3. Let  $A$  be a diagonal matrix as before and let  $c \in F$ . Let  $i \neq j$  again, and observe that

$$(cA)_{ij} = cA_{ij} = c \cdot 0 = 0.$$

Hence,  $cA$  is a diagonal matrix and so scalar multiplication is closed.

Since the set of diagonal matrices satisfies all properties of theorem 3, we conclude that it is indeed a subspace of  $M_{n \times n}(F)$ .

**Definition 1.2.6 (Trace).** The **trace** of an  $n \times n$  matrix  $M$ , denoted  $\text{tr}(M)$ , is the sum of the diagonal entries of  $M$ ; that is,

$$\text{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

The set of all  $n \times n$  matrices that have a trace equal to zero is a subspace of  $M_{n \times n}(F)$  (proved in Exercise 6).

**Example 1.2.7 (Non-example).** Denote  $V$  as the set of matrices in  $M_{m \times n}(\mathbb{R})$  having non-negative entries. The subset  $V$  is not a subspace because it is not closed under scalar multiplication. We can see this by multiplying any matrix in  $V$  by a negative number and observe that the entries of said matrix are no longer all non-negative.

We can create subspaces out of other subspaces.

**Theorem 1.2.2.** Any intersection of subspaces of a vector space  $V$  is a subspace of  $V$ .

**Proof.** Let  $C$  be a collection of subspaces of  $V$ , and let  $W$  denote the intersection of the subspaces in  $C$ . Since every subspace contains the zero vector and the intersection  $W \neq \emptyset$ , the zero vector  $0 \in W$ . Let  $a \in F$  and  $x, y \in W$ . Since each subspace of  $C$  is closed under addition and scalar multiplication, it follows that  $x + y$  and  $ax$  are contained each subspace in  $C$  and hence  $W$  must be closed under addition and closed under scalar multiplication. Thus,  $W$  is a subspace of  $V$  by theorem 3. ■

A natural question to ask is whether or not the union of subspaces of a vector space  $V$  is a subspace of  $V$  too. We can see that indeed the union of subspaces of  $V$  satisfy the first first and third property of theorem 3. However, property 2 need not be satisfied all the time. In fact, the union can only be a subspace of  $V$  if and only if one the subspaces is a subset of the other.

### 1.3 Linear Combinations and Systems of Linear Equations

**Definition 1.3.1 (Linear Combinations).** Let  $V$  be a vector space and  $S$  a nonempty subset of  $V$ . A vector  $v \in V$  is called **linear combination** of vectors of  $S$  if there exist a finite number of vectors  $u_1, u_2, \dots, u_n \in S$  and scalars  $a_1, a_2, \dots, a_n \in F$  such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

In this case, we say that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n \in V$  and call  $a_1, a_2, \dots, a_n \in F$  the **coefficients** of the linear combination.

- An easy example of a vector expressed as a linear combination is the zero vector  $O$  where  $0v = O$  for each  $v \in V$ .
- Determining whether a vector is a linear combination of other vectors often involves solving a system of linear equations.

**Example 1.3.1.** Suppose we wanted to express the vector  $(2, 6, 8) \in \mathbb{R}^3$  as a linear combination of

$$u_1 = (1, 2, 1), u_2 = (-2, -4, -2), u_3 = (0, 2, 3), u_4 = (2, 0, -3), \text{ and } u_5 = (-3, 8, 16).$$

Our goal is to find scalars  $a_1, a_2, a_3, a_4$  and  $a_5$  such that

$$(2, 6, 8) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5.$$

Doing a bit of algebra, we would need to solve the following system of linear equations

$$\begin{aligned}a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\2a_1 - 4a_2 + 2a_3 + 8a_5 &= 6 \\a_1 - 2a_2 + 3a_3 + 16a_5 &= 8\end{aligned}$$

Solving the system of equations above involves three types of operations:

1. Interchanging the order of any two equations in the system;
2. multiplying any equation in the system by some non-zero constant;
3. adding a constant multiple of any equation to another equation in the system.

We will learn in a later section that the operations listed above do not change the set of solutions to the original system.

**Example 1.3.2.** We claim that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \text{ and } 3x^3 - 5x^2 - 4x - 9$$

in  $P_3(\mathbb{R})$ , but that

$$3x^3 - 2x^2 + 7x + 8$$

is not. In the first case we wish to find scalars  $a$  and  $b$  such that

$$\begin{aligned}2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) \\&\quad + b(3x^3 - 5x^2 - 4x - 9) \\&= (a + 3b)x^3 + (-2a - 5b)x^2 \\&\quad + (-5a - 4b)x + (-3a - 9b).\end{aligned}$$

Thus, we have the following system of linear equations:

$$\begin{aligned}a + 3b &= 2 \\-2a - 5b &= -2 \\-5a - 4b &= 12 \\-3a - 9b &= -6.\end{aligned}$$

Adding the appropriate multiples of the first equation to the others in order to eliminate  $a$ , we find that

$$\begin{aligned}a + 3b &= 2 \\b &= 2 \\11b &= 22 \\0b &= 0.\end{aligned}$$

Then we get that  $a = -4, b = 2, 0 = 0, 0 = 0$ . Hence, we have

$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$

Looking at the second case now, we observe that using the preceding technique leads us to the following system of linear equations

$$\begin{aligned}a + 3b &= 3 \\ -2a - 5b &= -2 \\ -5a - 4b &= 7 \\ -3a - 9b &= 8.\end{aligned}$$

Eliminating  $a$  as before yields the following:

$$\begin{aligned}a + 3b &= 3 \\ b &= 4 \\ 11b &= 22 \\ 0 &= 17.\end{aligned}$$

The presence of the non-sensical result in the last equality tells us that there are no solutions to the system and that the second polynomial cannot be written as a linear combination of the two polynomials.

We can denote the set of all linear combinations of some set of vectors in the following:

**Definition 1.3.2 (Span).** Let  $S$  be a nonempty subset of a vector space  $V$ . The **span** of  $S$ , denoted  $\text{span}(S)$ , is the set consisting of all linear combinations of the vectors in  $S$ . For convenience, we define  $\text{span}(\emptyset) = \{0\}$ .

Some immediate examples of spans are:

**Example 1.3.3.** In  $\mathbb{R}^3$ , the span of the set  $S = \{(1, 0, 0), (0, 1, 0)\}$  consist of all vectors in  $\mathbb{R}^3$  such that for some scalars  $a, b \in \mathbb{R}$ , we have

$$a(1, 0, 0) + b(0, 1, 0) = (a, b, 0).$$

This tells us that  $S$  contains all the points in the  $xy$ -plane. One can show that  $S$  is a subspace of  $\mathbb{R}^3$ .

**Theorem 1.3.1.** The span of any subset  $S$  of a vector space  $V$  is a subspace of  $V$  that contains  $S$ . Moreover, any subspace of  $V$  that contains  $S$  must also contain the span of  $S$ .

**Proof.** Suppose  $S = \emptyset$ . Then the span of  $S$  is just  $\text{span}(\emptyset) = \{0\}$  which is a subspace in which  $S$  is contained in. Moreover,  $\text{span}(\emptyset) = \{0\}$  is always contained in any subspace  $X$  of  $V$  because every subspace contains the zero vector  $0$ . Suppose  $S \neq \emptyset$ , then  $S$  contains a vector  $z$ . We need to show that  $\text{span}(S)$  is a subspace of  $V$ .

- (a) Since  $S \neq \emptyset$ , we know that  $S$  contains a vector  $v$  such that  $0v = 0$ . Hence,  $0 \in \text{span}(S)$ .
- (b) Let  $x, y \in \text{span}(S)$ . We need to show that  $x + y \in \text{span}(S)$ . If  $x \in \text{span}(S)$ , then we can find  $a_1, a_2, \dots, a_n \in F$  and  $x_1, x_2, \dots, x_n \in S$  such that

$$a_1x_1 + a_2x_2 + \dots + a_nx_n.$$

Likewise,  $y \in \text{span}(S)$  implies that we can find scalars  $b_1, b_2, \dots, b_m$  and vectors  $y_1, y_2, \dots, y_m$  such that

$$b_1y_1 + b_2y_2 + \dots + b_my_m.$$

Then

$$x + y = a_1x_1 + a_2x_2 + \dots + a_nx_n + b_1y_1 + b_2y_2 + \dots + b_my_m$$

is a linear combination and thus  $x + y \in \text{span}(S)$ .

(c) Note that  $c \in F$  implies that

$$\begin{aligned} cx &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (ca_1)x_1 + (ca_2)x_2 + \cdots + (ca_n)x_n \end{aligned}$$

is a linear combination and thus  $cx \in \text{span}(S)$ .

Hence,  $\text{span}(S)$  is a subspace of  $V$ . Now we need to show that  $S$  is contained within  $\text{span}(S)$ ; that is,  $S \subseteq \text{span}(S)$ . Let  $v \in S$ . Then using (VS 5), we can see that  $1 \cdot v = v$  is a linear combination; so we have  $v \in \text{span}(S)$  and hence, the span of  $S$  contains  $S$ .

Now let  $W$  be any subspace of  $V$  that contains  $S$ . We need to show that  $\text{span}(S) \subseteq W$ . Let  $v \in \text{span}(S)$ . Then we can find scalars  $a_1, a_2, \dots, a_n \in F$  and  $x_1, x_2, \dots, x_n \in S$  such that

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Since  $S \subseteq W$ , we know that  $x_1, x_2, \dots, x_n \in W$ . Using exercise 20 from section 1.3 and using the same set of scalars  $a_1, a_2, \dots, a_n \in F$ , we have  $v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$ . Hence,  $\text{span}(S) \subseteq W$ . ■

**Definition 1.3.3.** A subset  $S$  of a vector space  $V$  **generates** (or **spans**)  $V$  if  $\text{span}(S) = V$ . In this case, we also say that the vectors of  $S$  generates (or span)  $V$ .

**Example 1.3.4 (Vectors in  $\mathbb{R}^3$ ).** The vectors  $(1, 1, 0)$ ,  $(1, 0, 1)$ ,  $(0, 1, 1)$  generate  $\mathbb{R}^3$  since any given vector  $v \in \mathbb{R}^3$  is a linear combination of the three given vectors. Furthermore, there exists scalars  $r, s, t \in \mathbb{R}$  such that

$$r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3) = v$$

where

$$r = \frac{1}{2}(a_1 + a_2 - a_3), s = \frac{1}{2}(a_1 - a_2 + a_3) \text{ and } t = \frac{1}{2}(-a_1 + a_2 + a_3).$$

**Example 1.3.5 (Polynomials).** The polynomials  $x^2 + 3x - 2$ ,  $2x^2 + 5x - 3$ , and  $-x^2 - 4x + 4$  generate  $P_2(\mathbb{R})$  because each of the three given polynomials belongs to  $P_2(\mathbb{R})$  and each polynomial  $ax^2 + bx + c \in P_2(\mathbb{R})$  is a linear combination of these three. It can be shown that we can find  $a, b, c \in \mathbb{R}$  such that

$$a(x^2 + 3x - 2) + b(2x^2 + 5x - 3) + c(-x^2 - 4x + 4) = ax^2 + bx + c.$$

**Example 1.3.6 (Matrices).** Note that the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate  $M_{2 \times 2}(\mathbb{R})$  because an every  $A \in M_{2 \times 2}(\mathbb{R})$  can be expressed as a linear combination of the four given matrices found below where there exists scalars  $a, b, c, d \in \mathbb{R}$  such that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where it can be show that

$$\begin{aligned} a &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} - \frac{2}{3}a_{22} \\ b &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} - \frac{2}{3}a_{21} + \frac{1}{3}a_{22} \\ c &= \frac{1}{3}a_{11} - \frac{2}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22} \\ d &= -\frac{2}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}. \end{aligned}$$

A non-example set of matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

do not generate  $M_{2 \times 2}(\mathbb{R})$  since all of the matrices have equal diagonal entries. Thus, every the set of matrices above cannot generate every  $A \in M_{2 \times 2}(\mathbb{R})$ .

## 1.4 Linear Dependence and Linear Independence

### 1.4.1 Motivation

Suppose  $V$  is a vector space over a field  $F$  and that  $W \subseteq V$  is a subspace of  $V$ .

- Suppose we have a set  $S$  that will generate every vector in  $V$  as a linear combination of vectors in  $S$ .
- It is desirable to find a subset of  $S$  that is as small as possible or rather just enough to generate each vector in  $V$ .
- The reason for this is to reduce the amount of computations done to represent a vector in  $V$ .
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in  $S$ .
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in  $S$ .
- Doing this, guarantees that we can write any one of the vectors in  $S$  as a linear combination of the others.

**Definition 1.4.1 (Linearly Dependence).** A subset  $S$  of a vector space  $V$  is called **linearly dependent** if there exists a finite number of distinct vectors in  $u_1, u_2, \dots, u_n \in S$  and scalars  $a_1, a_2, \dots, a_n$  not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case, we also say that the vectors of  $S$  are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars  $a_1, a_2, \dots, a_n \in F$  and distinct vectors  $v_1, v_2, \dots, v_n \in S$  where for all  $1 \leq i \leq n$ , we have  $a_i = 0$ .
- This tells us that our definition of **linear dependence** implies that 0 is a non-trivial linear combination.
- Any subset of  $V$  that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is,  $1 \cdot 0 = 0$ .



### 1.4.2 Examples of Linearly Dependent Sets

**Example 1.4.1.** Consider a subset in  $\mathbb{R}^4$  defined by

$$S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}.$$

To show that  $S$  is a linear dependent set, we need to find scalars  $a_1, a_2, a_3, a_4 \in \mathbb{R}$  such that  $a_i$  for all  $1 \leq i \leq 4$  not all zero such that

$$a_1(1, 3, -4, 2) + a_2(2, 2, -4, 0) + a_3(1, -3, 2, -4) + a_4(-1, 0, 1, 0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients  $a_1 = 4, a_2 = -3, a_3 = 2$ , and  $a_4 = 0$ . Thus, we have that  $S$  is linearly dependent subset of  $\mathbb{R}^4$  and hence we can write any vector in  $S$  as a linear combination of the other vectors contained in  $S$ .

**Example 1.4.2.** Define a subset of  $M_{2 \times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients  $a_1, a_2, a_3$  such that  $a_1 = 5, a_2 = 3$ , and  $a_3 = -2$  where

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**Definition 1.4.2 (Linear Independence).** A subset  $S$  of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of  $S$  are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

### 1.4.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is,  $\{v\}$  is linearly independent.
- If  $\{v\}$  is linearly dependent, then it the singleton has to be the zero vector  $0$ . This is because  $au = 0$

$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

**Example 1.4.3 (A Set of Vectors in  $\mathbb{R}^4$ ).** It can be shown that the set

$$S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of  $0$  represent the trivial representation; that is, we have scalars  $a_1 = a_2 = a_3 = a_4 = 0$ .

**Example 1.4.4.** For  $k = 0, 1, \dots, n$ , let  $p_k = x^k + x^{k+1} + \dots + x^n$ . The set

$$\{p_0(x), p_1(x), \dots, p_n(x)\}$$

is linearly independent in  $P_n(F)$ . It can be shown that for some scalars  $a_0, a_1, \dots, a_n \in F$ , the following equation

$$a_0 + (a_0 + a_1 + a_2)x^2 + \dots + (a_0 + a_1 + \dots + a_n)x^n = 0$$

has the trivial-representation; that is,  $a_i = 0$  for all  $1 \leq i \leq n$ .

**Theorem 1.4.1.** Let  $V$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_1$  is linearly dependent, then  $S_2$  is linearly dependent.

**Proof.** Let  $V$  be a vector space. Suppose  $S_1$  is linearly dependent. Then there exists a finite number of distinct vectors  $v_1, v_2, \dots, v_n \in S_1$  and scalars  $a_1, a_2, \dots, a_n \in F$  not all zero such that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0.$$

Since  $S_1 \subseteq S_2$ , we must have  $v_1, v_2, \dots, v_n \in S_2$  as well. If these vectors are linearly dependent, then these vectors are also linearly dependent in  $S_2$ ; that is, we have scalars  $a_1, a_2, \dots, a_n \in F$  not all zero such that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0.$$

Hence,  $S_2$  is linearly dependent. ■

**Corollary.** Let  $V$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_2$  is linearly independent, then  $S_1$  is linearly independent.

**Proof.** Note that this corollary is just the contrapositive of the theorem before it. Hence,  $S_1$  is linearly independent. ■

- Determining whether there exists a minimal generating set for the span of  $S$  is related to solving the problem of finding whether some vector in  $S$  that can be written in terms of a linear combination of other vectors in  $S$ .
- We can see that in a given subset of  $\mathbb{R}^3$  defined by

$$S = \{u_1, u_2, u_3, u_4\}$$

where  $u_1 = (2, -1, 4)$ ,  $u_2 = (1, -1, 3)$ ,  $u_3 = (1, 1, -1)$ , and  $u_4 = (1, -2, 1)$ . Note that this  $S$  is linearly dependent. This tells us that any vector in  $S$  can be written as a linear combination of the others. Suppose we pick  $u_3$  and write as a linear combination of the vectors  $u_1, u_2$ , and  $u_4$ . We would find that the span of these three vectors (denote this set as  $S'$ ) generates the same spanning set  $S$  but with a cardinality one less than  $S$ ; that is,  $\text{span}(S) = \text{span}(S')$ .

- If we find that there does not exist a proper subset that is equivalent to the span of  $S$ , then  $S$  must be a linearly independent set.

**Theorem 1.4.2.** Let  $S$  be a linearly independent subset of a vector space  $V$ , and let  $v \in V$  but not in  $S$ . Then  $S \cup \{v\}$  is linearly dependent if and only if  $v \in \text{span}(S)$ .

**Proof.** ( $\Rightarrow$ ) Suppose  $S \cup \{v\}$  is a linearly dependent set. Then there exists a finite number of scalars  $a_1, a_2, \dots, a_n \in F$  and vectors  $u_1, u_2, \dots, u_n \in S \cup \{v\}$  such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

Since  $S$  is a linearly independent set, we must have  $u_i = v$  for some  $1 \leq i \leq n$ . Choose  $i = 1$  (any choice of  $i$  will do) such that

$$a_1v + a_2u_2 + \dots + a_nu_n = 0.$$

Solving for  $v$  by subtracting  $a_1v$  on both sides, multiplying by  $a_1^{-1}$  on both sides of the equation,

and distributing by  $a_1^{-1}$  yields the following:

$$v = (-a_1^{-1}a_2)u_2 + (-a_1^{-1}a_3)u_3 + \cdots + (-a_1^{-1}a_n)u_n.$$

Since  $v$  is a linear combination of vectors  $u_1, u_2, \dots, u_n \in S$ , we know that  $v \in \text{span}(S)$ .

( $\Leftarrow$ ) Conversely, suppose  $v \in \text{span}(S)$ . This implies that there exists a finite amount of scalars  $a_1, a_2, \dots, a_n \in F$  and  $u_1, u_2, \dots, u_n \in S$  such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Subtracting  $v$  from both sides of the equation above gives us the following equation:

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - v = 0.$$

Note that  $v$  is not contained in  $S$ , so  $v \neq u_i$  for all  $1 \leq i \leq n$ . Since  $S$  is a linearly independent set, we know that  $a_i = 0$  for all  $1 \leq i \leq n$ . This implies that the only coefficient that is non-zero is with  $-v = -1v$ . Hence, the set of vectors  $S' = \{u_1, u_2, \dots, u_n, v\}$  is linearly dependent. Since  $S' \subseteq S \cup \{v\}$ , we know that  $S \cup \{v\}$  is also linearly independent by Theorem 6. ■

## 1.5 Bases And Dimension

### 1.5.1 Bases

- Recall that  $S$  is a generating set for a subspace  $W$  and no proper subset of  $S$  is a generating set for  $W$ , then  $S$  must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

**Definition 1.5.1 (Basis).** A **basis**  $\beta$  for a vector space  $V$  is linearly independent subset of  $V$  that generates  $V$ . If  $\beta$  is a basis for  $V$ , we also say that the vectors of  $\beta$  form a basis for  $V$ .

**Example 1.5.1.** • Recall that the empty set  $\emptyset$  is linearly independent and that  $\text{span}(\emptyset) = \{0\}$ . The empty set  $\emptyset$  in this case is the basis for the zero vector space.

- Note that in  $F^n$ , the vectors  $e_1 = (1, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $e_n = (0, 0, \dots, 0, 1)$  form a basis for  $F^n$ .
- The basis for  $M_{m \times n}(F)$  is the set of matrices  $E^{ij}$  such that the only nonzero entry is a 1 in the  $i$ th and  $j$ th column.
- As we have seen in the last section, the set  $\{1, x, x^2, \dots, x^n\}$  is a basis for  $P_n(F)$ .
- In  $P(F)$ , the set  $\{1, x, x^2, \dots\}$  is a basis. *Bases are not limited to finite sets. They can be infinite.*

**Theorem 1.5.1.** Let  $V$  be a vector space and  $u_1, u_2, \dots, u_n$  be distinct vectors in  $V$ . Then  $\beta = \{u_1, u_2, \dots, u_n\}$  is a basis for  $V$  if and only if each  $v \in V$  can be unique expressed as a linear combination of vectors in  $\beta$ , that is, expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars  $a_1, a_2, \dots, a_n$ .

**Proof.** ( $\Rightarrow$ ) Suppose  $\beta = \{u_1, u_2, \dots, u_n\}$  is a basis for  $V$ . Then  $\text{span}(\beta) = V$ . If  $v \in V$ , then  $v \in \text{span}(\beta)$ . Hence, we can write  $v$  as a linear combination of vectors in  $\beta$  such that choosing

scalars  $a_1, a_2, \dots, a_n \in F$  leads to

$$v = \sum_{i=1}^n a_i u_i.$$

Suppose there exists another representation of  $v \in V$  such that

$$v = \sum_{i=1}^n b_i x_i$$

Hence, observe that

$$\begin{aligned} \sum_{i=1}^n a_i x_i &= \sum_{i=1}^n b_i y_i \\ \Rightarrow \sum_{i=1}^n (a_i - b_i) x_i &= 0. \end{aligned}$$

Since  $\beta$  is linearly independent, we know that  $a_i - b_i = 0$  which implies  $a_i = b_i$  for all  $1 \leq i \leq n$ . Hence,  $v$  can be expressed as a unique linear combination of vectors in  $\beta$ .

( $\Leftarrow$ ) Conversely, let  $v \in V$  be expressed as a unique linear combination of vectors in  $\beta$  such that

$$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \quad (1)$$

for unique scalars  $a_1, a_2, \dots, a_n$ . We want to show that  $\beta$  is a basis for  $V$ ; that is, we want to show that  $\beta$  is a spanning set for  $V$  and  $\beta$  is linearly independent. To show that  $\beta$  is a spanning set for  $V$ , we need to show that  $\text{span}(\beta) \subseteq V$  and  $V \subseteq \text{span}(\beta)$ . Note that  $\beta \subseteq V$  and  $\text{span}(\beta)$  is a subspace for  $V$ . Hence,  $\text{span}(\beta) \subseteq V$ . On the other hand,  $V \subseteq \text{span}(\beta)$  follows immediately from (1). To show that  $\beta$  is linearly independent, we need to show that for scalars  $\delta_1, \delta_2, \dots, \delta_n$ , we have

$$\delta_1 u_1 + \delta_2 u_2 + \dots + \delta_n u_n = 0$$

such that  $u_i = 0$  for all  $1 \leq i \leq n$ . Note that  $v + 0 = v$ . Hence, we can write

$$\begin{aligned} \sum_{i=1}^n a_i u_i + \sum_{i=1}^n \delta_i u_i &= \sum_{i=1}^n a_i u_i \\ \sum_{i=1}^n (a_i + \delta_i) u_i &= \sum_{i=1}^n a_i u_i \end{aligned}$$

Equating each term in the equation above yields the following equality:

$$a_i + \delta_i = a_i \Rightarrow \delta_i = 0$$

for all  $1 \leq i \leq n$ . But this tells us that  $\beta$  is linearly independent and we are done. ■

- Any vector  $v \in V$  can be written as a linear combination of vectors from the basis containing  $u_1, u_2, \dots, u_n \in V$ .
- This determines a unique  $n$ -tuple of scalars  $(a_1, a_2, \dots, a_n)$  and conversely, each  $n$ -tuple of scalars determines a unique vector  $v \in V$  such that each coefficient from the linear combination of  $u_1, u_2, \dots, u_n$  is an entry from said tuple.
- For example, in our vector space  $F^n$ ,  $n$  is the number of vectors that should be in the basis for  $F^n$  which is indeed the case.
- In this book, we are only concerned with finite bases.

**Theorem 1.5.2 (Finite Spanning Set For a Vector Space).** If a vector space  $V$  is generated by a finite set  $S$ , then some subset of  $S$  is a basis for  $V$ . Hence,  $V$  has a finite basis.

**Proof.** Suppose  $S = \emptyset$  or  $S = \{0\}$ , then  $V = \{0\}$  and  $\emptyset$  is a subset of  $S$  that is a basis for  $V$ . If  $S$  neither of these choices, then  $S$  must contain at least one nonzero vector  $u_1$ ; that is,  $S = \{u_1\}$ . Since  $u_1$  is nonzero, it follows that  $S$  is a linearly independent set. We can continue this process of adding vectors  $u_2, \dots, u_k$  into  $S$  such that  $S$  is a linearly independent set of  $k$  vectors. Since  $S$  is a finite set, we must end with the linearly independent set  $\beta = \{u_1, u_2, \dots, u_n\}$ . There are two cases for which this occurs, either  $\beta = S$  or  $\beta \subseteq S$ :

- (i) Suppose that  $\beta = S$  (remember that  $\beta$  is a finite set by construction). Then we have  $S$  is a linearly independent set and spanning set for  $V$  (since  $S$  is a finite set that generates  $V$ ). Hence,  $S$  is a finite basis for  $V$ .
- (ii) Suppose  $\beta \subseteq S$  is a linearly independent set such that adding  $v \in S$  where  $v \notin \beta$  makes a linearly dependent set. We claim that  $\beta$  is the desired subset of  $S$  that is a basis for  $V$ . Then we have two cases; that is, either  $v \in \beta$  or  $v \notin \beta$ . Since  $\beta \subseteq S$ , we know by Theorem 1.5 that  $\text{span}(\beta) \subseteq S$  (This applies for both cases). It suffices to show that  $S \subseteq \text{span}(\beta)$ . If  $v \in \beta$ , then surely  $v \in \text{span}(\beta)$ . Hence, we have  $S \subseteq \text{span}(\beta)$ . Suppose  $v \notin \beta$ . Since  $\beta$  is a linearly independent set, then by Theorem 1.7, we have that  $\beta \cup \{v\}$  being linearly dependent implies that  $v \in \text{span}(\beta)$ . Hence,  $S \subseteq \text{span}(\beta)$ . Thus, that both cases implies that  $\beta$  is a spanning set for  $V$ . ■

This theorem tells us that any spanning set of a vector space  $V$  can be reduced to a finite basis for  $V$ . This is illustrated in the following examples.

**Example 1.5.2.** Define

$$S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$$

which can be shown to generate  $\mathbb{R}^3$ . The idea is to create a proper subset of  $S$  such that none of the vectors in  $\beta$  are multiples of each other. In other words, we want a set that is linearly independent; that is, our choices of vectors determine whether a set will be linearly dependent or independent (we want the latter to hold). Say, we pick  $(2, -3, 5)$  as our first vector in our subset  $\beta$ . Right away, we can exclude  $(8, -12, 20)$  since it is a multiple of  $(2, -3, 5)$ . Otherwise, including it would make  $\beta$  linearly dependent (see exercise 9 from section 1.5). Next, add the vectors  $(1, 0, -2)$  and  $(0, 2, -1)$  since they are not multiples of the other. Narrowing down our set to

$$\beta = \{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}.$$

Note that adding the vector  $(7, 2, 0)$  makes  $\beta$  linearly dependent, so  $(7, 2, 0)$  is excluded from the list. Hence, we have arrived at a subset of  $S$  such that  $\beta \subseteq S$  is both a linearly independent set and spanning set for  $\mathbb{R}^3$ .

The following theorem and its corollaries are the most important results in the Chapter 1.

## 1.5.2 Dimensions

**Theorem 1.5.3 (Replacement Theorem).** Let  $V$  be a vector space that is generated by a set  $G$  containing exactly  $n$  vectors, and let  $L$  be a linearly independent subset of  $V$  containing exactly  $m$  vectors. Then  $m \leq n$  and there exists a subset  $H$  of  $G$  containing exactly  $n - m$  vectors such that  $L \cup H$  generates  $V$ .

**Proof.** Let us proceed the proof via induction on  $m$ . Let  $m = 0$  be our base case. Then we find that  $L = \emptyset$  is linearly independent set with exactly 0 vectors. Letting  $H = G$  gives us the desired result (since  $G$  contains exactly  $n$  vectors). Now suppose that the theorem holds for some integer  $m \geq 0$ . We will show that the theorem holds for the  $m + 1$  case. Let  $L = \{v_1, v_2, \dots, v_{m+1}\}$  be a

linearly independent subset of  $V$  consisting of exactly  $m + 1$  vectors. By the corollary to Theorem 1.6, we find that  $L' = \{v_1, v_2, \dots, v_m\}$  is a linearly independent set (because  $L' \subseteq L$  and  $L$  is linearly independent). Using our induction hypothesis, we can conclude that  $m \leq n$  and that there exists a subset  $H' = \{u_1, u_2, \dots, u_{n-m}\}$  of  $G$  such that  $L' \cup H'$  generates  $V$ . Thus there exists scalars  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_{n-m}$  such that

$$v_{m+1} = a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + b_2 u_2 + \dots + b_{n-m} u_{n-m}. \quad (1)$$

Note that  $n - m > 0$ , **unless**  $v_{m+1}$  is a linear combination of  $v_1, v_2, \dots, v_m$  which by Theorem 1.7 contradicts the assumption that  $L$  is a linearly independent set. Hence,  $n > m$ ; that is,  $n \geq m + 1$ . Furthermore, some  $b_i$ , say  $b_1$  is nonzero, for otherwise we obtain the same contradiction. Solving (1) for  $u_1$ , we get

$$\begin{aligned} u_1 &= (-b_1^{-1} a_1) v_1 + (-b_1^{-1} a_2) v_2 + \dots + (-b_1^{-1} a_m) v_m + (b_1^{-1}) v_{m+1} \\ &\quad + (-b_1^{-1} b_2) u_2 + \dots + (-b_1^{-1} b_{n-m}) u_{n-m}. \end{aligned}$$

Let  $H = \{u_2, u_3, \dots, u_{n-m}\}$ . Then  $u_1 \in \text{span}(L \cup H)$  and because  $v_1, v_2, \dots, v_m, u_2, \dots, u_{n-m}$  are clearly in  $\text{span}(L \cup H)$ , we have that

$$L' \cup H' \subseteq \text{span}(L \cup H).$$

Since  $L' \cup H'$  generates  $V$  and the fact that  $L' \cup H' \subseteq \text{span}(L \cup H)$  (note that  $\text{span}(L \cup H)$  is also a subspace), we know by Theorem 1.5 that  $\text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$ . Since  $\text{span}(L' \cup H')$  generates  $V$ , we know that  $V \subseteq \text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$ . Observe that  $\text{span}(L \cup H) \subseteq V$  is true by default. Hence,  $\text{span}(L \cup H)$  generates  $V$  and that  $H$  contains  $(n - m) - 1 = n - (m + 1)$  vectors which concludes our induction proof. ■

**Corollary.** Let  $V$  be a vector space having a finite basis. Then all bases for  $V$  are finite, and every basis for  $V$  contains the same number of vectors.

**Proof.** Let  $\beta$  be a finite basis for  $V$  that contains exactly  $n$  vectors. Let  $\gamma$  be any other basis that contains more than  $n$  vectors. Suppose we pick a subset  $S$  of  $\gamma$  such that this subset contains exactly  $n + 1$  vectors. Since  $\beta$  is a finite basis for  $V$ , we know that  $\beta$  is a linearly independent set. By Theorem 1.6, we know that  $S$  is also a linearly independent set. By Replacement Theorem, we have that  $n + 1 \leq n$  which is a contradiction. Therefore, we must have  $\gamma$  is finite, and  $\gamma$  contains exactly  $m$  amount of vectors in  $\gamma$  which satisfies  $m \leq n$ . To show that  $n \leq m$ , we can reverse the roles of  $\beta$  and  $\gamma$  and use the same argument as above. Hence,  $n \leq m$  and thus  $m = n$ . ■

The main takeaway from the corollary above is that the number of vectors in any basis for  $V$  is an inherent property of  $V$ .

**Definition 1.5.2 (Finite-dimensional).** A vector space is **finite-dimensional** if it has a basis consisting of a finite number of vectors.

**Definition 1.5.3 (Dimension of a Vector Space).** The unique integer  $n$  such that every basis for  $V$  contains exactly  $n$  elements is called the **dimension** of  $V$  and is denoted by  $\dim(V)$ .

**Definition 1.5.4 (Infinite-dimensional).** A vector space that is not finite-dimensional is called **infinite-dimensional**.

**Example 1.5.3.** • The vector space  $\{0\}$  has dimension zero. This is because  $\{0\}$  is generated by the empty set  $\emptyset$ .

- The vector space  $F^n$  has dimension  $n$ .
- The vector space  $M_{m \times n}(F)$  has dimension  $mn$ .

- The vector space  $P_n(F)$  has dimension  $n + 1$ .

It turns out that the dimension of a vector space depends on the choice of the field it's defined on.

**Example 1.5.4.** • Over the field of complex numbers, the vector space of complex numbers has dimension 1. (A basis is  $\{1\}$ ).

- On the other hand, if  $\mathbb{C}$  is defined over the real numbers  $\mathbb{R}$ , then the vector space has dimension 2.

The first conclusion of the replacement theorem states that given a finite-dimensional vector space, a linearly independent subset of  $V$  cannot contain no more than  $\dim(V)$  amount of vectors.

**Example 1.5.5 (Example of an Infinite-Dimensional Vector Space).** The vector space  $P(F)$  is infinite-dimensional since it contains a linearly independent set  $\{1, x, x^2, \dots\}$  that is infinite. This is peculiar because all of the results in this section thus far does not guarantee that an infinite-dimensional vector space containing a basis. In fact, it is proven in section 1.7 that infinite-dimensional vector space are guaranteed to have a basis.

The next corollary from the Replacement Theorem refers to the size of the generating set.

**Corollary.** Let  $V$  be a vector space with dimension  $n$ .

- Any finite generating set for  $V$  contains at least  $n$  vectors, and a generating set for  $V$  that contains exactly  $n$  vectors is a basis for  $V$ .
- Any linearly independent subset of  $V$  that contains exactly  $n$  vectors is a basis for  $V$ .
- Every linearly independent subset of  $V$  can be extended to a basis for  $V$ , that is, if  $L$  is a linearly independent subset of  $V$ , then there is a basis  $\beta$  of  $V$  such that  $L \subseteq \beta$ .

**Proof.** Let  $\beta$  be a basis for  $V$ .

- Let  $\beta$  be a finite generating set for  $V$ . By Theorem 1.9,  $\beta$  contains a subset  $S$  such that  $S$  is a finite basis for  $V$ . Since  $V$  is a vector space with dimension  $n$ , we know that  $S$  contains exactly  $n$  vectors. This means that  $\beta$  must contain at least  $n$  vectors since  $S \subseteq \beta$ .
- Let  $L$  be a linearly independent subset of  $V$  that contains exactly  $n$  vectors. Since  $\beta$  is a generating set for  $V$ , the Replacement Theorem states that we can find a subset of  $\beta$ , say  $L'$ , such that  $L'$  contains exactly  $n - n = 0$  vectors implies that  $L' = \emptyset$  and  $L \cup L' = L \cup \emptyset = L$  generates  $V$ . Since  $L$  is also linearly independent, we conclude that  $L$  is a basis for  $V$ .
- Let  $L$  be a linearly independent subset of  $V$  containing  $m$  vectors. Then the Replacement Theorem implies that there exists  $H \subseteq \beta$  containing  $n - m$ , where  $\beta$  is the generating set of  $V$  with exactly  $n$  amount of vectors, such that  $L \cup H$  generates  $V$  and is finite. This implies that  $L \cup H$  must contain at most  $n$  vectors, but part (a) also tells us that  $L \cup H$  contains at least  $n$  vectors, so  $L \cup H$  must contain exactly  $n$  vectors. Hence,  $L \cup H$  is a basis for  $V$ . ■

**Example 1.5.6.** • In Example 1.3.5 the set

$$\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$$

is a generating set for  $P_2(\mathbb{R})$  that contains exactly 3 vectors. Hence, the set above is a basis for  $P_2(\mathbb{R})$  by part (a).

- It follows from Example 1.3.6 and (a) of the second corollary to the Replacement Theorem that the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis for  $M_{2 \times 2}(\mathbb{R})$ .

- The set in Example 1.4.3 is a basis for  $\mathbb{R}^4$ .
- See Example 1.4.4. This set is a basis for  $P_n(F)$ .

### 1.5.3 An Overview of Dimension and Its Consequences

The main takeaways from our results are:

- We can reduce a finite generating set into a basis for a vector space  $V$ .
- A finite generating set contains at least  $n$  amount of vectors.
- A linearly independent set of vectors contain at most  $n$  amount of vectors.
- We can Extend a linearly independent set into a basis for  $V$ .
- A finite basis for  $V$  must contain exactly  $n$  vectors.
- Any basis for  $V$  contains the same number of vectors. In other words, if a basis for  $V$ , say  $\beta$ , contains  $n$  amount of vectors then any other basis must contain the same amount.
- The number of vectors in a basis is the dimension of a vector space.

### 1.5.4 The Dimension of Subspaces

We can relate the dimension of a vector space  $V$  to a subspace of  $V$ .

**Theorem 1.5.4.** Let  $W$  be a subspace of a finite-dimensional vector space  $V$ . Then  $W$  is a finite-dimensional and  $\dim(W) \leq \dim(V)$ . Moreover, if  $\dim(W) = \dim(V)$ , then  $V = W$ .

**Proof.** Let  $\dim(V) = n$  since  $V$  is finite-dimensional. Suppose  $W = \{0\}$ . Then  $W$  contains the empty set  $\emptyset$  such that  $\text{span}(\emptyset) = W$  and  $\emptyset$  is linearly independent. Hence,  $\emptyset$  is a subset of  $W$  that is a basis for  $W = \{0\}$  that contains 0 vectors. Hence,  $W$  is finite-dimensional. Otherwise,  $W$  contains a nonzero vector  $= x_1$  such that  $W' = \{x_1\}$  is a linearly independent set. Continue this process of adding vectors  $x_2, x_3, \dots, x_k$  into  $W'$  such that  $W'$  is linearly independent. Since no linearly independent subset of  $V$  cannot contain no more than  $n$  vectors, this process must stop at  $k \leq n$ . Note adding a vector from  $w \in W$  into  $W'$  such that  $w \notin W'$  will make a linearly dependent set. By Theorem 1.4.2, we get that  $w \in \text{span}(W')$ . Hence,  $W'$  is a basis for  $W$  and that  $\dim(W) = k \leq n = \dim(V)$ . Suppose  $\dim(W) = n$ , then there exists a basis  $\beta$  for  $W$  such that  $\beta$  contains exactly  $n$  amount of vectors. Since  $\beta$  is also linearly independent and contains the same amount of vectors as a basis for  $V$ , then we must have that  $\beta$  must also be a basis for  $V$  by Corollary 2 of the Replacement Theorem. Hence,  $W = \text{span}(\beta) = V$ . ■

**Example 1.5.7.** • Let

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 + a_3 + a_5 = 0, a_2 = a_4\}.$$

It can be easily shown that  $W$  is a subspace of  $F^5$  having

$$\{(-1, 0, 1, 0, 0), (-1, 0, 0, 0, 1), (0, 1, 0, 1, 0)\}$$

as a basis. Hence,  $\dim(W) = 3$

- The set of diagonal  $n \times n$  matrices is a subspace  $W$  of  $M_{n \times n}(F)$ . A basis for  $W$  is the following set

$$\{E^{11}, E^{22}, \dots, E^{nn}\},$$

where  $E^{ij}$  is the matrix in which the only nonzero entry is a 1 in the  $i$ th row and  $j$ th column. Thus, we have  $\dim(W) = n$ .



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**Corollary.** If  $W$  is a subspace of a finite-dimensional vector space  $V$ , then any basis for  $W$  can be extended to a basis for  $V$ .

**Proof.** Let  $\beta$  be a basis for the subspace  $W$ . Since  $\beta \subseteq W \subseteq V$  is a linearly independent subset of  $V$ , we know that  $\beta$  can be extended as a basis for  $V$  by Corollary 2 of the Replacement Theorem. ■

### 1.5.5 Geometrical View of Subspaces in Euclidean Space

- The subspaces of  $\mathbb{R}^2$  include  $\{0\}$  of dimension 0,  $\mathbb{R}$  of dimension 1, and  $\mathbb{R}^2$  itself.
  - The set  $\{0\}$  of zero dimension can be visualized as the origin in  $\mathbb{R}^2$ .
  - The set  $\mathbb{R}$  is the field itself and can be visualized as the line  $y = cx$  with  $c \in \mathbb{R}$  where  $x \in \mathbb{R}^2$ .
  - The set  $\mathbb{R}^2$  is the whole cartesian space itself.
- The subspaces of  $\mathbb{R}^3$  must have subspaces with dimensions 0, 1, 2, or 3. The same logic can be followed from the three bullet points above.



## Chapter 2

# Linear Transformations and Matrices

### 2.1 Linear Transformations, Null Spaces, and Ranges

Suppose we have a function  $T$  with domain  $V$  and codomain  $W$  denoted by  $T : V \rightarrow W$ .

**Definition 2.1.1 (Linear Transformation).** Let  $V$  and  $W$  be vector spaces (over  $F$ ). We call a function  $T : V \rightarrow W$  a **linear transformation from  $V$  to  $W$** , for all  $x, y \in V$  and  $c \in F$ , we have

- (a)  $T(x + y) = T(x) + T(y)$  and
- (b)  $T(cx) = cT(x)$ .

**Remark.** If  $F = \mathbb{Q}$ , then (a) implies (b) in the definition above. Otherwise, (a) and (b) are logically independent statements.

The following are a list of properties for linear functions:

1. If  $T$  is linear, then  $T(0) = 0$ .

**Proof.** Suppose  $T$  is linear, then  $T(0 \cdot 0) = 0T(0) = 0$ . ■

2. We have  $T$  is linear if and only if  $T(cx + y) = cT(x) + T(y)$  for all  $x, y \in V$  and  $c \in F$ .

**Proof.** Suppose  $T$  is linear. Let  $x, y \in V$  and  $c \in F$ . Then

$$T(cx + y) = T(cx) + T(y) = cT(x) + T(y).$$

Conversely, if  $c = 1$  then

$$T(x + y) = T(x) + T(y).$$

If  $y = 0$ , then

$$T(cx) = cT(x).$$

Hence,  $T$  is a linear transformation. ■

3. If  $T$  is linear, then  $T(x - y) = T(x) - T(y)$  for all  $x, y \in V$ .

**Proof.** Let  $x, y \in V$ . Suppose  $T$  is linear, then

$$T(x - y) = T(x) + T(-y) = T(x) - T(y).$$
 ■

4.  $T$  is linear if and only if, for  $x_1, x_2, \dots, x_n \in V$  and  $a_1, a_2, \dots, a_n \in F$ , we have

$$T\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i T(x_i).$$

**Proof.** Suppose  $T$  is linear. Let  $x_1, x_2, \dots, x_n \in V$  and  $a_1, a_2, \dots, a_n \in F$  such that

$$v = \sum_{i=1}^n a_i x_i.$$

Then observe that

$$T\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n T(a_i x_i) = \sum_{i=1}^n a_i T(x_i).$$

Conversely, for  $i = 2$  and denote  $a_i x_i = y_i$ . Then

$$T\left(\sum_{i=1}^2 a_i x_i\right) = T(a_1 x_1) + T(a_2 x_2) = T(y_1) + T(y_2)$$

and so property 1 is satisfied. If  $i = 1$ , then

$$T\left(\sum_{i=1}^1 a_i x_i\right) = a_1 T(x_1)$$

and so property 2 is satisfied. Hence,  $T$  is a linear transformation. ■

**Example 2.1.1.** Define

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ by } T(a_1, a_2) = (2a_1 + a_2, a_1).$$

Show that  $T$  is linear. Let  $c \in \mathbb{R}$  and  $x, y \in \mathbb{R}^2$ , where  $x = (a_1, a_2)$  and  $y = (b_1, b_2)$ . Then

$$\begin{aligned} cx + y &= c(a_1, a_2) + (b_1, b_2) \\ &= (ca_1, ca_2) + (b_1, b_2) \\ &= (ca_1 + b_1, ca_2 + b_2). \end{aligned}$$

So, we have

$$\begin{aligned} T(cx + y) &= T(ca_1 + b_1, ca_2 + b_2) \\ &= (2(ca_1 + b_1) + ca_2 + b_2, ca_1 + b_1) \\ &= ((2ca_1 + ca_2) + (2b_1 + b_2), ca_1 + b_1) \\ &= (2ca_1 + ca_2, ca_1) + (2b_1 + b_2, b_1) \\ &= c(2a_1 + a_2, a_1) + (2b_1 + b_2, b_1) \\ &= cT(a_1, a_2) + T(b_1, b_2) \\ &= cT(x) + T(y). \end{aligned}$$

Hence, we have that  $T(cx + y) = cT(x) + T(y)$  so  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear.

### 2.1.1 Examples of Linear Transformations

The most common linear transformations come from geometry:

- Rotations:  $T_\theta(a_1, a_2) = (a_1, a_2)$  by performing a counter-clockwise rotation by an angle  $\theta$  if  $(a_1, a_2) \neq (0, 0)$ . This is called **rotation by  $\theta$**  and  $T_\theta(0, 0) = (0, 0)$  otherwise.
- Reflections about the  $x$ -axis:  $T(a_1, a_2) = (a_1, -a_2)$ .
- Projections on the  $x$ -axis:  $T(a_1, a_2) = (a_1, 0)$ .

**Example 2.1.2 (Rotations).** Define the rotation transformation above by  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T_\theta(a_1, a_2)$  where it is described as above. Our goal in this example is to define an explicit formula of this transformation. Let us fix a nonzero vector  $(a_1, a_2) \in \mathbb{R}^2$ . Let  $\alpha$  be the angle such that  $(a_1, a_2)$  makes with the positive  $x$ -axis, and let  $r = \sqrt{a_1^2 + a_2^2}$ . Then using some trigonometry, we get that  $a_1 = r \cos(\alpha)$  and  $a_2 = r \sin(\alpha)$ . Note that  $T_\theta(a_1, a_2)$  has length  $r$  and makes an angle  $\alpha + \theta$  with the positive  $x$ -axis. Using some trigonometric identities, we get that

$$\begin{aligned} T_\theta(a_1, a_2) &= (r \cos(\alpha + \theta), r \sin(\alpha + \theta)) \\ &= (r \cos(\alpha) \cos(\theta) - r \sin(\alpha) \sin(\theta), r \cos(\alpha) \sin(\theta) + r \sin(\alpha) \cos(\theta)) \\ &= (a_1 \cos(\theta) - a_2 \sin(\theta), a_1 \sin(\theta) + a_2 \cos(\theta)). \end{aligned}$$

Observe that this formula holds for  $(a_1, a_2) = (0, 0)$ . One can show that  $T_\theta$  is linear.

**Example 2.1.3 (Transpose).** Define  $T : M_{m \times n}(F) \rightarrow M_{n \times m}(F)$  by  $T(A) = A^t$ , where  $A^t$  is the transpose of  $A$ , defined in Section 1.3.

**Example 2.1.4 (Derivatives of Polynomials).** Define  $T : P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$  by  $T(f(x)) = f'(x)$ , where  $f'(x)$  denotes the derivative of  $f(x)$ . We can show that  $T$  is linear. Let  $g(x), h(x) \in P_n(\mathbb{R})$  and  $a \in \mathbb{R}$ . Assuming that the differentiation rules holds for any  $T$ , we have

$$\begin{aligned} T(ag(x) + h(x)) &= (ag(x) + h(x))' \\ &= ag'(x) + h'(x) \\ &= aT(g(x)) + T(h(x)). \end{aligned}$$

By the second property of linearity, we have that  $T$  is linear.

**Example 2.1.5 (Integrals).** Let  $V = C(\mathbb{R})$  be the vector space of continuous real-valued functions on  $\mathbb{R}$ . Let  $a, b \in \mathbb{R}$ ,  $a < b$ . Define  $T : V \rightarrow \mathbb{R}$  by

$$T(f) = \int_a^b f(t) dt$$

for all  $f \in V$ . We know that  $T$  is a linear transformation because the definite integral of a linear combination of functions is the same as the linear combination of the definite integrals of the functions; that is,

$$\int_a^b \sum_{i=1}^n \gamma_i f(t_i) dt = \sum_{i=1}^n \gamma_i \int_a^b f(t_i) dt = \sum_{i=1}^n \gamma_i T(f_i)$$

for scalars  $\gamma_i$  where  $1 \leq i \leq n$  and

$$T(f_i) = \int_a^b f(t_i) dt.$$

## 2.1.2 Identity and Zero Transformations

**Remark.** Two very important examples of linear transformations are the **identity** and **zero transformations**.

**Definition 2.1.2 (Identity Transformation).** The **identity transformation** is denoted by  $I_V : V \rightarrow V$  by  $I_V(x) = x$  for all  $x \in V$ . From now on, we will denote this transformation as  $I$  instead of  $I_V$ .

**Definition 2.1.3 (Zero Transformation).** The **zero transformation** is denoted by  $T_0 : V \rightarrow W$  by

$$T_0(x) = 0 \text{ for all } x \in V.$$

### 2.1.3 Range and Null Spaces

The *Range* and *Null* spaces give us important insights on the intrinsic properties of a linear transformation.

**Definition 2.1.4 (Null Space).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. The **null space** (or **kernel**), denoted by  $N(T)$  of  $T$ , is the set of all vectors  $x \in V$  such that  $T(x) = 0$ ; that is,  $N(T) = \{x \in V : T(x) = 0\}$ .

**Definition 2.1.5 (Range).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. The **range** (or **image**)  $R(T)$  of  $T$  is the subset of  $W$  consisting of all images (under  $T$ ) of vectors in  $V$ ; that is,  $R(T) = \{T(x) : x \in V\}$ .

**Example 2.1.6.** Let  $V$  and  $W$  be vector spaces, and let  $I : V \rightarrow V$  and  $T_0 : V \rightarrow W$  be the identity and zero transformations, respectively. Then  $N(I) = \{0\}$ ,  $R(I) = V$ ,  $N(T_0) = V$ , and  $R(T_0) = \{0\}$ .

**Example 2.1.7.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

One can verify that

$$N(T) = \{(a, a, 0) : a \in \mathbb{R}\} \text{ and } R(T) = \mathbb{R}^2.$$

**Theorem 2.1.1.** Let  $V$  and  $W$  be vector spaces and  $T : V \rightarrow W$  be linear. Then  $N(T)$  and  $R(T)$  are subspaces of  $V$  and  $W$ , respectively.

**Proof.** Let  $T : V \rightarrow W$  be linear. First, we show that  $N(T)$  is a subspace of  $V$ .

- (a) Note that  $0_V \in V$  implies that  $T(0_V) = 0_W$  since  $T$  is linear (property (1) of linearity). Hence,  $0_V \in N(T)$ .
- (b) Let  $x, y \in N(T)$ . By definition, of the null space we have  $T(x) = 0_W$  and  $T(y) = 0_W$ . Since  $T$  is linear, we get that

$$T(x + y) = T(x) + T(y) = 0_W + 0_W = 0_W \in N(T).$$

Hence,  $N(T)$  is closed under addition.

- (c) Let  $x \in N(T)$  and let  $c \in F$ . Then

$$T(cx) = cT(x) = c0_W = 0_W \in N(T).$$

Hence, Theorem 1.3 tells us that  $N(T)$  is a subspace of  $V$ . Now, we will show that  $R(T)$  is a subspace of  $W$ .

- (a) Note that  $0_V \in V$  implies that  $T(0_V) = 0_W \in R(T)$  by property (1) of linearity. So,  $0_W \in R(T)$ .
- (b) Now let  $x, y \in R(T)$ . Then there exists elements  $z, w \in W$  such that  $T(x) = z$  and  $T(y) = w$ . Then by linearity, we have  $T(x + y) = T(x) + T(y) = z + w \in W$ . Hence, we must have  $x + y \in R(T)$  and thus  $R(T)$  is closed under addition.

(c) Let  $x \in R(T)$  and  $c \in F$ . Then we have  $T(cx) = cT(x) \in W$  since  $T(x) \in W$ . Hence,  $cx \in R(T)$ .

Thus, Theorem 1.3 also tells us that  $R(T)$  is a subspace of  $W$ . ■

The next theorem allows us to find a spanning set for the range of a linear transformation.

**Theorem 2.1.2 (Spanning Set for a Linear Transformation).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. If  $\beta = \{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ , then

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\}).$$

**Proof.** Let  $T : V \rightarrow W$  be linear where  $V$  and  $W$  are vector spaces. To show that  $T(\beta)$  generates  $R(T)$ ; we need to show that

$$\text{span}(T(\beta)) \subseteq R(T) \text{ and } R(T) \subseteq \text{span}(T(\beta)).$$

Observe that  $T(v_i) \in R(T)$  for all  $i$ . Hence,  $T(\beta) \subseteq R(T)$ . By theorem 1.5, we know that  $R(T)$  also contains the span of  $T(\beta)$ . Hence,  $\text{span}(T(\beta)) \subseteq R(T)$ .

Let  $w \in R(T)$ . Then for some  $v \in W$ , we have  $T(v) = w$ . Since  $\beta$  is a basis for  $V$ , we choose scalars  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that

$$v = \sum_{i=1}^n \alpha_i v_i.$$

Since  $T$  is linear, we have that

$$w = T(v) = T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i T(v_i).$$

This tells us that  $w \in \text{span}(T(\beta))$ . Hence,  $T(\beta)$  generates  $R(T)$ . ■

**Example 2.1.8.** Define the linear transformation  $T : P_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$  by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Note that

$$\beta = \{1, x, x^2\} \text{ is a basis for } P_2(\mathbb{R}) \text{ and } T(\beta) = \{T(1), T(x), T(x^2)\}.$$

Then observe that

$$\begin{aligned} R(T) &= \text{span}(T(\beta)) \\ &= \text{span}\left(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right\}\right). \end{aligned}$$

This tells us that  $R(T)$  contains the basis found in the second equality above. Hence, we must have  $\dim(R(T)) = 2$ .

Just like how the 'size' of a given subspace is denoted by its dimension, we can also determine the size of a null spaces and ranges. However, we will attach some special names associated with these sets.

**Definition 2.1.6 (Nullity).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. If  $N(T)$  is finite-dimensional, then we define **nullity** of  $T$  by  $\text{nullity}(T)$  to be the dimension of  $N(T)$ .

**Definition 2.1.7 (Rank).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$ . If  $R(T)$  is finite-dimensional, then we denote the **rank** of  $T$  by  $\text{rank}(T)$  to be the dimension of  $R(T)$ .

From these definitions, we can intuit the following relationships between Nullity and Rank of a linear transformation:

- 
- The larger the nullity, the smaller the rank of a linear transformation.
  - the larger the rank, the smaller the nullity.

This relationship between the two spaces is encompassed in the next theorem.

**Theorem 2.1.3 (Dimension Theorem).** Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. If  $V$  is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

**Proof.** ■