1 Lecture 18-19

1.1 Topics

- $\limsup$ and $\liminf$ (Two equivalent characterizations)
- Theorem: $\liminf a_n \leq \limsup a_n$.
- Theorem: $\lim_{n\to\infty} a_n$ exists in $\overline{\mathbb{R}}$ if and only if $\liminf a_n = \limsup a_n \in \overline{\mathbb{R}}$.
- Theorem: $\limsup(a_n+b_n) \leq \limsup a_n + \limsup b_n$ provided that the right-hand side is not $\infty - \infty$.
- Some special sequences

1.2 First Characterization of Limsup and Liminf

Let (x_n) be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \text{there exists a subsequence } (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \to x\}.$$

We define,

$$\limsup x_n = \sup S$$
$$\liminf x_n = \inf S.$$

1.3 Second Characterization of Limsup and Liminf

Let (x_n) be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \geq n\}$. Clearly, we have

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots .$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \cdots \qquad \text{(A decreasing sequence in } \overline{\mathbb{R}})$$

and similarly, we have

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \dots . \qquad \text{(An increasng sequence in } \overline{\mathbb{R}})$$

By the Monotone Convergence Theorem (in $\overline{\mathbb{R}}$), we know that $\lim_{n\to\infty} \sup F_n$ and $\lim_{n\to\infty} \inf F_n$ exists in $\overline{\mathbb{R}}$. We define

$$\limsup x_n = \lim_{n\to\infty} \sup F_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf F_n.$$

That is, we have

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf_n(\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup_n(\inf F_n).$$

Take note of the following notation:

$$\limsup x_n = \lim_{n\to\infty} \sup x_n = \overline{\lim} x_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf x_n = \underline{\lim} x_n.$$

Example. (i) $x_n = (-1)^n$

Notice that

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \sup\{-1, 1\} = \lim_{n\to\infty} 1 = 1$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 1\} = \lim_{n\to\infty} -1 = -1.$$

(ii) Consider $(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots)$

Then we have

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \{-1, 2, 3\} = \lim_{n\to\infty} 3 = 3$$
$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 2, 3\} = \lim_{n\to\infty} -1 = -1.$$

(iii) Consider $a_n = n$

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \sup\{n, n+1, n+2, \dots\} = \lim_{n\to\infty} n = \infty.$$

and similarly, we have

$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{n, n+1, n+2, \dots\} = \lim_{n\to\infty} n = \infty.$$

Remark. (i) $\liminf x_n = \sup_n \inf\{x_k : k \geq n\}$

(ii) $\limsup x_n = \inf_n \sup\{x_k : k \geq n\}$

Theorem. Let (a_n) be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n.$$

Proof. Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}.$$

Since we already proved that the limits of both sides exists (in $\overline{\mathbb{R}}$), it follows from the order limit theorem (in $\overline{\mathbb{R}}$) that

$$\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

That is, we have

$$\liminf a_n \leq \limsup a_n.$$

■

Theorem. Let (a_n) be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n \text{ exists in } \overline{\mathbb{R}} \text{ if and only if } \limsup a_n = \liminf a_n.$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.

Proof. ($\Longleftarrow$) Let $A = \limsup a_n = \liminf a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\lim a_n = A$. We may consider three cases; that is,

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), note that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}.$$

Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that

$\lim_{n\to\infty} a_n = A$.

For (2) ($A = \infty$), we have for all $n \in \mathbb{N}$ that $\inf\{a_k : k \geq n\} \leq a_n$ and $\lim_{n\to\infty} \inf\{a_k : k \geq n\} = \infty$ implies $\lim_{n\to\infty} a_n = \infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

For (3) ($A = -\infty$), we know that for all $n \in \mathbb{N}$ that $a_n \leq \sup\{a_k : k \geq n\}$ and $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = -\infty$ implies that $\lim_{n\to\infty} a_n = -\infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

($\Longrightarrow$) Let $A = \lim_{n\to\infty} a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\limsup a_n = A$ and $\liminf a_n = A$. We may consider three cases:

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), suppose that $A \in \mathbb{R}$. Our goal is to show that

$$A \leq \liminf a_n \text{ and } \limsup a_n \leq A,$$

and so

$$A \leq \liminf a_n \leq \limsup a_n \leq A.$$

Thus, it suffices to show that for all $\varepsilon > 0$

$$A - \varepsilon \leq \liminf a_n \text{ and } \limsup a_n \leq A + \varepsilon.$$

To this end, let $\varepsilon > 0$ be given. Since $a_n \to A$, there exists an $N \in \mathbb{N}$ such that

$$\forall n > N \quad |a_n - A| < \varepsilon;$$

that is,

$$\forall n > N \quad A - \varepsilon < a_n < A + \varepsilon.$$

Now, observe that

$$\begin{aligned}
\forall n > N \quad a_n < A + \varepsilon &\Longrightarrow A + \varepsilon \text{ is an upper bound of } \{a_k : k \geq n\} \\
&\Longrightarrow \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \varepsilon \\
&\Longrightarrow \lim_{n\to\infty} \sup\{a_k : k \geq n\} \leq \lim_{n\to\infty} (A + \varepsilon) \qquad \text{(Order Limit Theorem)} \\
&\Longrightarrow \limsup a_n \leq A + \varepsilon
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
\forall n > N \quad A - \varepsilon < a_n &\Longrightarrow A - \varepsilon \text{ is a lower bound of} \{a_k : k \geq n\} \\
&\Longrightarrow \forall n > N \ \inf\{a_k : k \geq n\} \geq A - \varepsilon \\
&\Longrightarrow \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} A - \varepsilon \\
&\Longrightarrow \liminf a_n \geq A - \varepsilon.
\end{aligned}$$

Now, suppose (2). Our goal is to show that $\liminf a_n = \infty$ so that $\liminf a_n \leq \limsup a_n$ will imply that $\limsup a_n = \infty$. In order to show that $\liminf a_n = \infty$, it suffices to show that

$$\forall M > 0 \quad M \leq \liminf a_n.$$

To this end, let $M > 0$ be given. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
\forall n > N \quad a_n > M &\Longrightarrow \forall n > N \ \inf\{a_k : k \geq n\} \geq M \\
&\Longrightarrow \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} M \\
&\Longrightarrow \liminf a_n \geq M.
\end{aligned}$$

Note that an analogous process to the above is used to prove (3). ■

Theorem. Let (a_n) and (b_n) be the two sequences of real numbers. Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that the right-hand side is not of the form $\infty - \infty$ or $-\infty + \infty$.

Proof. First note that, by our assumption, $\limsup a_n + \liminf a_n$ is not of the form $\infty - \infty$ or $(-\infty + \infty)$, there exists n_0 such that

$$\forall n \geq n_0 \quad \sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\} \text{ is not of the form } \infty - \infty \text{ or } -\infty + \infty$$

For each $n \geq n_0$, we have

$$\forall k \geq n \quad a_k \leq \sup\{a_\ell : \ell \geq n\}$$
$$\forall k \geq n \quad b_k \leq \sup\{b_m : m \geq n\}.$$

Thus, we have

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Therefore,

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Now, label $R_n = \sup\{a_k + b_k : k \geq n\}$, $L_n = \sup\{a_\ell : \ell \geq n\}$ and $S_n = \sup\{b_m : m \geq n\}$. From the above, we can see that $\lim_{n\to\infty} R_n$, $\lim_{n\to\infty} L_n$, and $\lim_{n\to\infty} S_n$ all exists in $\overline{\mathbb{R}}$. Since $\lim_{n\to\infty} L_n + \lim_{n\to\infty} S_n$ is not of the form $\infty - \infty$, it follows form the Algebraic Limit Theorem that $\lim_{n\to\infty}(L_n + S_n)$ exists and is equal to that of $\lim L_n + \lim S_n$. By the Order Limit Theorem, we see that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

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Theorem ((e))**.** If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. Clearly, if x = 0, then the claim holds. So, let's assume $x \in (-1, 1)$ and $x \neq 0$. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ |x^n - 0| < \varepsilon.$$

That is, we need to show, given the setup above, that $|x^n| < \varepsilon$. Since $0 < |x| < 1$, there exists $y > 0$ such that $|x| = \frac{1}{1+y}$. Note that

$$|x|^n < \varepsilon \iff \frac{1}{(1+y)^n} < \varepsilon.$$

Using the Binomial Theorem ($(1 + y)^n \geq 1 + ny$), we can see that

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}.$$

Therefore, in order to ensure that $|x|^n < \varepsilon$, we just need to choose n large enough so that $\frac{1}{ny} < \varepsilon$. To this end, it suffices to choose n larger than $\frac{1}{\varepsilon y}$; that is, we can take $N = \frac{1}{\varepsilon y}$ and the result follows. ■

Theorem ((b))**.** If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Proof. If $p = 1$, the claim obviously holds. If $p \neq 1$, we may consider two cases.

For the first case, assume that $p > 1$. Then let $x_n = \sqrt[n]{p} - 1$. It suffices to show that $\lim_{n\to\infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also, we have

$$\begin{aligned} \sqrt[n]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n}. \end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim_{n\to\infty} x_n = 0$.

Now, suppose that $0 < p < 1$. Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by the previous case, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1 \iff \lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = 1.$$

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Theorem ((c)). $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Proof. Let $x_n = \sqrt[n]{n} - 1$. Observe that, by the binomial formula, we have for all $n \geq 2$,

$$\begin{aligned} \sqrt[n]{n} = 1 + x_n &\implies n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}. \end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \to 0$ and so $\sqrt[n]{n} \to 1$. ■