0.1 Convergent Sequences

Definition (3.1). A sequence (P_n) in a metric space X is said to **converge** if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$ where d denotes the distance in X.

- Another way we say the statement above is to say that $(p_n) \to p$, or that p is the limit of (p_n), or that
$$\lim_{n\to\infty} p_n = p.$$
- If (p_n) does not converge, then it is said to **diverge**.
- We can be more specific about the convergence of a sequence by specifying the convergence of the sequence with respect to the metric space.

Theorem (3.2). Let (p_n) be a sequence in a metric space X.

(a) (p_n) converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.

(b) If $p \in X$, $p' \in X$, and if (p_n) converges to p and to p', then $p' = p$.

(c) If (p_n) converges, then (p_n) is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence (p_n) in E such that $p = \lim_{n\to\infty} p_n$.

Theorem (3.3). Suppose $(s_n), (t_n)$ are complex sequences and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$ for any number c;

(c) $\lim_{n\to\infty} s_n t_n = st$;

(d) $\lim_{n\to\infty} (1/s_n) = 1/s$, provided $s_n \neq 0 (n = 1, 2, 3, \dots)$ and $s \neq 0$.

Theorem (3.4). (a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, 3 \dots)$ and
$$x_n = (\alpha_{1,n}, \dots, \alpha_{k,n}).$$
Then $(x_n) \to x$ with $x = (\alpha_1, \alpha_2, \dots, \alpha_k)$ if and only if
$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k).$$

(b) Suppose $(x_n), (y_n)$ are sequences in $\mathbb{R}^k$, (β_n) is a sequence of real numbers, and $x_n \to$

x, $y_n \to y$, and $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y, \quad \lim_{n\to\infty} x_n \cdot y_n = x \cdot y, \quad \lim_{n\to\infty} \beta_n x_n = \beta x.$$

0.2 Subsequences

Definition (3.5). Given a sequence (p_n), consider a sequences (n_k) of positive integers such that $n_1 < n_2 < \dots$. Then the sequence (p_{n_i}) is called a **subsequence** of (p_n). If (p_{n_i}) converges, its limit is called the **subsequential limit** of (p_n).

Recall that if (p_n) converges to p iff every subsequence of (p_n) converges to p.

Theorem (3.6). (a) If (p_n) is a sequence in a compact metric space X, then some subsequence of (p_n) converges to a point of x.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Theorem (3.7). The subsequential limits of a sequence (p_n) in a metric space form a closed subset of X.

0.3 Cauchy Sequences

Definition (3.8). A sequence (p_n) in a metric space X is said to be a **Cauchy sequence** if for every $\varepsilon > 0$, there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Definition (3.9). Let E be a nonempty subset of a metric space X and let S be the set of all real numbers of the form $d(p, q)$ with $p \in E$ and $q \in E$. The supremum of S is called the *diameter* of E.

Theorem (3.10). (a) If $\overline{E}$ is the closure of a set E in a metric space X, then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

(b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \dots)$ and if

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$

then $\cap_{n=1}^{\infty} K_n$ consists of exactly one point.

Theorem (3.11). (a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if (p_n) is a Cauchy sequence in X, then (p_n) converges to some point of X.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

Definition (3.12). A metric space in which every Cauchy sequence converges is said to be *complete*.

- In Theorem 3.11, we see that *all compact metric spaces and all euclidean spaces are complete.*
- Referring to the same theorem again, we also add that *every closed subset E of a complete metric space is complete.*
- Convergent sequences in $\mathbb{R}^k$ are bounded, but the converse need not be true.

Definition (3.13). A sequence (s_n) of real numbers is said to be

(a) *monotonically increasing* if $s_n \leq s_{n+1}$ $(n \in \mathbb{N})$;

(b) *monotonically decreasing* if $s_n \geq s_{n+1}$ $(n \in \mathbb{N})$.

The following is the monotone convergence theorem.

Theorem (3.14). Suppose (s_n) is monotonic. Then (s_n) converges if and only if it is bounded.

0.4 Upper and Lower Limits

Definition (3.15). Let (S_n) be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there exists an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

Definition (3.16). Let (s_n) be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence (s_{n_k}). This set E contains all subsequential limits as in Definition 3.5 and possibly the numbers $+\infty, -\infty$. Set

$$\begin{aligned} s^* &= \sup E, \\ s_* &= \inf E. \end{aligned}$$

We denote the numbers s^*, s_* as the *upper* and *lower* limits of (s_n). We use the notation

$$\lim_{n\to\infty} \sup s_n = s^*, \quad \lim_{n\to\infty} \inf s_n = s_*.$$

Theorem (3.17). Let (s_n) be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 3.16. Then s^* has the following two properties:

(a) $s^* \in E$.

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Theorem (3.19). If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n,$$
$$\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n.$$

0.5 Some Special Sequences

Theorem (3.2). (a) If $p > 0$, then $\lim_{n\to\infty} 1/n^p = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} n^\alpha/(1+p)^n = 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.