Math 234A Take Home Quiz

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1. Indicate whether the following statements are True or False. You **do not** need to justify your answer.

   (a) Let $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$ is open. Let $a \in D$ and suppose that f is holomorphic at a. Then, we can find an open set $D' \subseteq D$ such that $a \in D'$ and f is complex differentiable at each $z \in D'$. **True**.

   (b) Let $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$ is an open set. Let $f(z) = u + iv$ with $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$. Suppose that u and v satisfy the Cauchy-Riemann Equations on D. Then, f is holomorphic on D. **False.**

   (c) Let $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \sin(\overline{z})$. Then, f is holomorphic at $0 \in \mathbb{C}$. **False.**

   (d) Let $f : [0, 1] \to \mathbb{C}$ be defined by $f(t) = 2z(1 - t) + 2tw$ where $z, w \in \mathbb{C}$ are fixed complex numbers. Then $\int_0^1 f(t)\ dt = w - z$. **False.**

2. Give definitions of the following terms.

   (a) Cauchy-Riemann equations for a pair of functions $u, v : D \to \mathbb{R}$ where D is an open subset of $\mathbb{R}^2$.

   Solution. Suppose that $u, v : D \to \mathbb{R}$ where D is an open subset of $\mathbb{R}^2$ and u and v contain partial derivatives that exists and are continuous on every point of D. Then the Cauchy-Riemann equations are

   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   ■

   (b) A piecewise smooth curve in $\mathbb{C}$.

   Solution. A curve $\alpha : [a, b] \to \mathbb{C}$ is **piecewise smoooth** if there is a partition

   $$a = a_0 < a_1 < \cdots < a_n = b$$

   such that $\alpha\big|_{[a_{i-1}, a_i]}$ is smooth for $i = 1, 2, \dots, n$. ■

   (c) A complex line integral of a continuous function $f : D \to \mathbb{C}$ over a piecewise smooth curve in D, where $D \subseteq \mathbb{C}$.

   Solution. Assume $\alpha : [a, b] \to \mathbb{C}$ is a piecewise smooth curve with partition

   $$a = a_0 < a_1 < \dots, < a_n = b$$

   such that $\alpha\big|_{[a_{i-1}, a_i]}$ smooth for $i = 1, 2, \dots, n$. Let $f : D \to \mathbb{C}$ continuous and $\alpha([a, b]) \subseteq D$. Then we define

   $$\int_\alpha f(z)\ dz = \sum_{i=1}^n \int_{\alpha|_{[a_{i-1}, a_i]}} f(z)\ dz$$

   to be the **complex line integral of f over a piecewise smoooth curve in D, where $D \subseteq \mathbb{C}$.**

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3. Suppose that $D \subseteq \mathbb{C}$ is open and connected. Assume that $f : D \to \mathbb{C}$ is holomorphic. Assume that $g : D \to \mathbb{C}$ defined by $g(z) = \overline{f(z)}$ is also holomorphic. Prove that f and g are both constant functions. Use this to deduce that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{\sin z}$ is not holomorphic.

Proof. Our goal is to show that f and g are both constant. It suffices to show that $\Re(f)$ is constant and $\Re(g) = \Re(\overline{f})$ is constant. Note that $f = u+iv$ with u, v are real-valued functions. Since f and g are holomorphic, we

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\partial \overline{f}}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$$

Note that if f and g are holomorphic, then we have

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\Big[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big] = 0 \tag{1}$$
$$\frac{\partial \overline{f}}{\partial \overline{z}} = \frac{1}{2}\Big[\frac{\partial \overline{f}}{\partial x} + i\frac{\partial \overline{f}}{\partial y}\Big] = 0. \tag{2}$$

Adding equations (1) and (2), we have

$$\frac{1}{2}\Big[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big] = -\frac{1}{2}\Big[\frac{\partial \overline{f}}{\partial x} + i\frac{\partial \overline{f}}{\partial y}\Big] \Longrightarrow \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = -\Big[\frac{\partial \overline{f}}{\partial x} + i\frac{\partial \overline{f}}{\partial y}\Big].$$

Furthermore, we have

$$\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big) + i\Big(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\Big) = -\Big(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\Big) - i\Big(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\Big)$$

and hence,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \Longrightarrow 2\frac{\partial u}{\partial x} = 0 \Longrightarrow \frac{\partial u}{\partial x} = 0.$$

Since $u = \Re(f)$, we can see that $\Re(f)$ is a constant function. Since $\Re(f) = \Re(\overline{f})$ (By Exercise 4(a) of Homework 5), we also have that $\Re(\overline{f})$ is a constant function. Thus, f and g are constant functions.

To show that $f(z) = \overline{\sin z}$ is not holomorphic, we can just show that the real and imaginary parts of $\overline{\sin z}$ are not constant. By Exercise 6 (a) of homework 5, we see that

$$h(z) = \sin z = \frac{1}{2}(e^{-y} + e^{y})\sin x + i\Big[-\frac{1}{2}(e^{-y} + e^{y})\cos x\Big]$$

is a holomorphic function on $\mathbb{C}$. Note that $f(z) = \overline{h(z)}$. Hence, we see that

$$\overline{\sin z} = \frac{1}{2}(e^{-y} + e^{y})\sin x + i\Big[\frac{1}{2}(e^{-y} + e^{y})\cos x\Big].$$

Clearly, both real and imaginary parts of $h(z) = \overline{\sin z}$ and $f(z) = \sin z$ are NOT constant. Hence, $f(z) = \overline{\sin z}$ is NOT a holomorphic function. $\blacksquare$

4. Assume that $f : [a, b] \to \mathbb{C}$ be integrable.

   (a) Prove that $\left|\int_a^b f(t)\, dt\right| \leq \int_a^b |f(t)|\, dt$.

Proof. Let $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$ such that $f(t) = x(t) + iy(t)$. Our goal is to show that

$$\Big|\int_a^b f(t)\, dt\Big| \leq \int_a^b |f(t)|\, dt.$$

It suffices to show that

$$\Big|\int_a^b f(t)\ dt\Big|^2 \leq \Big(\int_a^b |f(t)|\ dt\Big)^2.$$

Observe that

$$\begin{aligned}
\Big|\int_a^b f(t)\ dt\Big|^2 &= \Big(\int_a^b f(t)\ dt\Big)\overline{\Big(\int_a^b f(s)\ ds\Big)} \\
&= \Big[\int_a^b (x(t)+iy(t))\ dt\Big]\cdot\Big[\int_a^b (x(s)-iy(s))\ ds\Big] \\
&= \int_a^b\int_a^b (x(t)+iy(t))(x(s)-iy(s))\ dtds \\
&= \int_a^b\int_a^b (x(t)x(s)+i(y(t)x(s)-x(t)y(s))+y(t)y(s))\ dtds \\
&= \int_a^b\int_a^b [x(t)x(s)+y(s)x(t)]\ dtds + i\int_a^b\int_a^b [y(t)x(s)-x(t)y(s)]\ dtds.
\end{aligned}$$

Note that

$$\int_a^b\int_a^b [y(t)x(s)-x(t)y(s)]\ dt\ ds = \int_a^b\int_a^b y(t)x(s)\ dtds - \int_a^b\int_a^b x(t)y(s)\ dt\ ds = 0.$$

Hence, we see that

$$\Big|\int_a^b f(t)\ dt\Big|^2 = \int_a^b\int_a^b [x(t)x(s)+y(s)x(t)]\ dtds. \tag{*}$$

Now, notice we have

$$\begin{aligned}
|x(t)x(s)+y(s)x(t)| &\leq \sqrt{((x(t))^2+(y(t))^2)((x(s))^2+(y(s))^2)} \\
&= \sqrt{((x(t))^2+(y(t))^2)}\cdot\sqrt{((x(s))^2+(y(s))^2)}
\end{aligned}$$

by the Cauchy-Schwarz inequality. So, (*) implies that

$$\begin{aligned}
\Big|\int_a^b f(t)\ dt\Big|^2 &= \int_a^b\int_a^b [x(t)x(s)+y(s)x(t)]\ dtds \\
&\leq \int_a^b\int_a^b \sqrt{((x(t))^2+(y(t))^2)}\cdot\sqrt{((x(s))^2+(y(s))^2)}\ dtds \\
&= \Big(\int_a^b \sqrt{((x(t))^2+(y(t))^2)}\ dt\Big)\cdot\Big(\int_a^b \sqrt{(x(s))^2+(y(s))^2}\ ds\Big) \\
&= \Big(\int_a^b \sqrt{(x(t))^2+(y(t))^2}\ dt\Big)^2 \\
&= \Big(\int_a^b |f(t)|\ dt\Big)^2.
\end{aligned}$$

Hence, squaring both sides will gives us our desired result; that is,

$$\Big|\int_a^b f(t)\ dt\Big|^2 \leq \int_a^b |f(t)|\ dt.$$

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(b) Prove that $\int_a^b f(t)\ dt = -\int_a^b f(s)\ ds$.

Proof. By definition, $f(t) = x(t) + iy(t)$ with $x, y : [a, b] \to \mathbb{R}$. Our goal is to show that

$$\int_a^b f(t)\ dt = -\int_a^b f(s)\ ds. \tag{*}$$

By definition of the complex integral, we see that

$$\int_a^b f(t)\ dt = \int_a^b x(t)\ dt + i\int_a^b y(t)\ dt.$$

Since $x(t)$ and $y(t)$ are real-integrable functions for all $t \in [a, b]$, we see that

$$-\int_a^b x(s)\ ds = \int_a^b x(t)\ dt \quad \text{and} \quad -\int_a^b y(s)\ ds = \int_a^b y(t)\ dt.$$

Starting with the right-hand side of (*), we get

$$\begin{aligned} -\int_a^b f(s)\ ds &= -\Big[\int_a^b x(s)\ ds + i\int_a^b y(s)\ ds\Big] \\ &= -\int_a^b x(s)\ ds + i\Big(-\int_a^b y(s)\ ds\Big) \\ &= \int_a^b x(t)\ dt + i\int_a^b y(t)\ dt \\ &= \int_a^b f(t)\ dt. \end{aligned}$$

Thus, we have established (*). ■

5. Compute the integral $\displaystyle\int_0^2 f(t)\ dt$, where $f : [0, 2] \to \mathbb{C}$ is defined by

$$f(t) = \begin{cases} (1+i)t & \text{if } 0 \le t \le 1 \\ 1 + it^2 & \text{if } 1 < t \le 2. \end{cases}$$

Solution. Observe that

$$\begin{aligned} \int_0^2 f(t)\ dt &= \int_0^1 (1+i)t\ dt + \int_1^2 (1+i)t^2\ dt \\ &= \Big[\frac{(1+i)t^2}{2}\Big]_0^1 + \Big[\frac{(1+i)t^3}{3}\Big]_1^2 \\ &= \frac{1+i}{2} + \Big(\frac{(1+i)8}{3} + \frac{1+i}{3}\Big) \\ &= \frac{17(1+i)}{6}. \end{aligned}$$

Hence, we have

$$\int_0^2 f(t)\ dt = \frac{17(1+i)}{6}.$$

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