0.1 Lecture 6

0.1.1 A few examples of Metrics

Example 0.1.1. Consider $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

This is a metric on $(\mathbb{R}, d)$. **Prove this on homework!**

Remark. If (X, D) is a metric space, then (X, D) is also a metric space where

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

We can define a metric that will always be less than or equal to 1.

Example 0.1.2 (Taxi Cab Metric). Consider $(\mathbb{R}^2, d)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ is defined by

$$d((a, b), (x, y)) = |a - x| + |b - y|$$

We want to show that this is a metric.

(i) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have $d((a, b), (x, y)) = |a - x| + |b - y| \geq 0$ by property of the absolute value $| \cdot |$.

(ii) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} d((a, b), (x, y)) = 0 &\Leftrightarrow |a - x| + |b - y| = 0 \\ &\Leftrightarrow |a - x| = 0 \text{ and } |b - y| = 0 \\ &\Leftrightarrow a - x = 0 \text{ and } b - y = 0 \\ &\Leftrightarrow (a, b) = (x, y). \end{aligned}$$

(iii) For all $(a, b), (x, y) \in \mathbb{R}^2$.

$$d((a, b), (x, y)) = |a - x| + |b - y| = |x - a| + |y - b| = d((x, y), (a, b)).$$

(iv) For all $(a, b), (x, y), (t, s) \in \mathbb{R}^2$, we want to show that

$$d((a, b), (x, y)) \leq d((a, b), (t, s)) + d((t, s), (x, y)).$$

We have

$$\begin{aligned} d((a, b), (t, s)) + d((t, s), (x, y)) &= |a - t| + |b - s| + |t - x| + |s - y| \\ &= (|a - t| + |t - x|) + (|b - s| + |s - y|) \\ &\geq |a - x| + |b - y| \\ &= d((a, b), (x, y)) \end{aligned}$$

Example 0.1.3 (Discrete Metric). Consider $X \to$ any nonempty set. Consider $d : X \times X \to [0, \infty)$,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

We want to show that this is a metric on $(X, \text{any non-empty set})$.

(i) For all $x, y \in X$, we have $d(x, y) \geq 0$ because either the $d(x, y) = 1$ or $d(x, y) = 0$.

(ii) Let $x, y \in X$. Clearly, we have $x = y$ if and only if $d(x, y) = 0$ by definition of $d(x, y)$.

(iii) Let $x, y \in X$. Clearly, we have $d(x, y) = d(y, x)$ by definition.

(iv) Let $x, y, z \in X$. We want to show that

$$d(x, y) \leq d(x, z) + d(z, y)$$

Let us consider two cases:

(1) $x = y$. Thus, both sides of the triangle inequality clearly hold since $d(x, y) = 0$.

(2) $x \neq y$. In this case, we have $d(x, y) = 1$. So, we need to show that

$$d(x, z) + d(z, y) \geq 1.$$

Since $x \neq y$, at least one of the statements $z \neq y$ or $z \neq x$ is true. If $z \neq x$, then $d(z, x) = 1$ and so

$$d(x, z) + d(z, y) = 1 + d(z, y) \geq 1.$$

If $z \neq y$, then $d(z, y) = 1$, and so

$$d(x, z) + d(z, y) = d(x, z) + 1 \geq 1.$$

Example 0.1.4. Consider $(V, \|\cdot\|) \to$ any normed vector space with the metric $d : V \times V \to [0, \infty)$ and $d(x, y) = \|x - y\|$. We wan to show that this is a metric on V.

(i) Let $x, y \in V$. By the property of the norm, we have $d(x, y) = \|x - y\| \geq 0$.

(ii) Let $x, y \in V$. By the 2nd property of the norm, we have

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow \|x - y\| = 0 \\ &\Leftrightarrow x - y = 0 \\ &\Leftrightarrow x = y. \end{aligned}$$

(iii) Let $x, y \in V$. We have

$$d(x, y) = \|x - y\| = \| - (y - x)\| = | - 1|\|y - x\| = \|y - x\| = d(y, x).$$

(iv) Let $x, y, z \in V$. We want to show that

$$d(x, y) \leq d(x, z) + d(z, y).$$

We have,

$$\begin{aligned} d(x, z) + d(z, y) &= \|x - z\| + \|z - y\| \\ &\geq \|(x - z) + (z - y)\| \\ &= \|x - y\| \\ &= d(x, y). \end{aligned}$$

This example illustrates how every time we have a norm, we immediately have a metric space. But the other around is not necessarily true!

Example 0.1.5 (Standard Distance in $\mathbb{R}^n$). Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we have

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

Note that if we define for all $x \in \mathbb{R}^n$, we have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

Then

$$d(x, y) = \|x - y\|_2.$$

Proving that the above is a norm is enough to show that the original in question is a metric.

Let $x, y \in \mathbb{R}^n$.

(i) We have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \geq 0.$$

(ii) We have

$$\begin{aligned} \|x_2\|_2 = 0 &\Leftrightarrow \sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0 \\ &\Leftrightarrow |x_1|^2 + \cdots + |x_n|^2 = 0 \\ &\Leftrightarrow |x_1| = 0, |x_2| = 0, \ldots, |x_n| = 0 \\ &\Leftrightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0. \end{aligned}$$

Thus, we have $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|\alpha x\|_2 &= \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2 (x_1^2 + \cdots + x_n^2)} \\ &= |\alpha| \sqrt{x_1^2 + \cdots + x_n^2} \\ &= |\alpha| \|x\|_2. \end{aligned}$$

(iv) Now, we want to show that

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

That is, we want to show that

$$\sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} \leq \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}.$$

We will show this later!

Example 0.1.6. Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$, we have

$$d_p(x, y) = \Big[|x_1 - y_1|^p + \cdots + |x_n - y_n|^p \Big]^{\frac{1}{p}}.$$

0.1.2 Inequalities

In what follows we will list several key inequalities that can be used in proving that, for example, a given expression is a metric or a norm.

(1) The triangle inequality for the standard norm in $\mathbb{R}$. We have for all $x, y \in \mathbb{R}$, we have

$$|x + y| \leq |x| + |y|.$$

More generally, we have for all $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\Big| \sum_{i=1}^n x_i \Big| \leq \sum_{i=1}^n |x_i|.$$

(2) Reverse triangle inequality for the standard norm in $\mathbb{R}$; that is, we have for all $x, y \in \mathbb{R}$,

$$||x| - |y|| \leq |x - y|.$$

(3) For all $a, b \geq 0$ and for all $\rho > 0$, we have

$$ab \leq \frac{1}{2}\Big(\rho a^2 + \frac{1}{\rho}b^2\Big).$$

Note that we are dealing with real numbers here!

(4) Cauchy-Schwarz Inequality. For al $x, y \in \mathbb{R}^n$, we have

$$|x_1y_1 + \cdots + x_ny_n| \leq \Big(\sqrt{x_1^2 + \cdots + x_n^2}\Big)\Big(\sqrt{y_1^2 + \cdots + y_n^2}\Big). \tag{*}$$

0.2 Lecture 7

0.2.1 Topics

- Review of Inequalities
- Neighborhood of a point
- Limit point of a set, isolated point of a set
- Closed set
- Interior point of a set
- Open set
- Bounded set
- Closure
- Dense

0.2.2 Inequalities

0.2.3 Minkowski

We wan to show the triangle inequality for $\|\cdot\|_2$ in $\mathbb{R}^n$; that is, we want to show that

Proposition. Let $\|\cdot\|_2$ be a norm in $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$, $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$.

Proof. We see that $x \cdot y \leq |x \cdot y| \leq \|x\|_2\|y\|_2$ by the Cauchy Cauchy-Schwarz Inequality. Thus, we have

$$\begin{aligned}
\|x + y\|_2^2 &= (x + y) \cdot (x + y) \\
&= x \cdot x + x \cdot y + y \cdot x + y \cdot y \\
&= \|x\|_2^2 + 2x \cdot y + \|y\|_2^2 \\
&\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
&= (\|x\|_2 + \|y\|_2)^2.
\end{aligned}$$

Hence, we have

$$\|x + y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2.$$

Therefore, we have

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

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0.2.4 Minkowski for General p

For general p, we have Holder's Inequality which is

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

for any fixed real number $p \geq 1$ and for any $x, y \in \mathbb{R}^n$.

0.2.5 $(x+1)^n$

Recall that for all $a, b \in \mathbb{R}$, for all $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

So, in particular, if $x \geq 0$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\
&\geq 1 + nx.
\end{aligned}$$

Hence, we have for all $x \geq 0$ and for all $n \in \mathbb{N}$, we have

$$(x+1)^n \geq 1 + nx.$$

0.2.6 $p-$means

Let $x_1, \ldots, x_n$ be positive real numbers. Let $p \in \mathbb{N} \cup \{0\}$. By the $p-$mean of $x_1, \ldots, x_n$ denoted by $A_p(x_1, \ldots, x_n)$, we mean

$$A_p(x_1, \ldots, x_n) = \begin{cases} \frac{\sqrt[p]{x_1^p + \cdots + x_n s^p n}}{} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 \ldots x_n} & \text{if } p = 0. \end{cases} .$$

For example, if $p = 1$, we have

$$A_1(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

which is the Arithmetic Mean. If $p = 2$, we have

$$A_2(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

If $p = 0$, we have

$$A_0(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$$

which is the geometric mean. It can be shown that

$$A_0(x_1, \ldots, x_n) \leq A_1(x_1, \ldots, x_n) \leq A_2(x_1, \ldots, x_n) \leq \ldots .$$

In particular, we have $A_0 \leq A_1$; that is,

$$\sqrt[n]{x_1 \ldots x_n} \leq \frac{x_1 + \cdots + x_n}{n}. \qquad \text{(AM-GM Inequality)}$$

We can prove this inequality using differentiation. Without differentiation, we can prove the same inequality using **Cauchy Induction**.

0.2.7 Jensen's Inequality

Suppose $f : (a, b) \to \mathbb{R}$ is a convex function ($f"(x) \geq 0$ for all $x \in (a, b)$). Let $x_1, \ldots, x_n$ be points in (a, b). Let $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

We want to show this inequality holds for $n = 2$; that is,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

where $\lambda_1 + \lambda_2 = 1$. Note that

$$f((1 - \lambda_2)x_1 + \lambda_2 x_2) \leq (1 - \lambda_2)f(x_2) + \lambda_2 f(x_2).$$

Remark. If we want to have an expression that defines a function that gives us any number in between two points $e < h$, we can have

$$f(\lambda) = (1 - \lambda)e + \lambda h$$

for any $0 \leq \lambda \leq 1$.

0.2.8 Neighborhood of a point

Definition (Neighborhood). Let (X, d) be a metric space. Let $p \in X$. For any $\varepsilon > 0$, we call

$$N_\varepsilon(p) = \{x \in X : d(p, x) < \varepsilon\}.$$

the **neighborhood of p of radius ε**.

Example 0.2.1. Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. The neighborhood of any $p \in \mathbb{R}$ with radius $\varepsilon > 0$ is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Note that $|x - p| < \varepsilon$ is the same thing as $p - \varepsilon < x < p + \varepsilon$ or that $x \in (p - \varepsilon, p + \varepsilon)$.

Example 0.2.2. Let $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$. Let $(a, b) \in \mathbb{R}^2$ with $\varepsilon > 0$. Then

$$\begin{aligned} N_\varepsilon((a, b)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y)(a, b)) < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \varepsilon^2 .\} \end{aligned}$$

Thus, $N_\varepsilon((a, b))$ consists of the points inside the circle of radius ε centered in (a, b).

Example 0.2.3. $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = |a - x| + |b - y|$ and let $\varepsilon = 1$. We have

$$\begin{aligned} N_1((0, 0)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x - 0| + |y - 0| < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \end{aligned}$$

If we graph this out in $\mathbb{R}^2$, then the shape of the neighborhood will take on a rhombus.

Example 0.2.4. Let $(\mathbb{R}, d)$ with the discrete metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $p \in \mathbb{R}$. Let $\varepsilon > 0$. Let us consider two cases:

(1) Let $\varepsilon \leq 1$. Note that if

$$d(x,p) < \varepsilon \leq 1,$$

then $d(x,p) < 1$, and so $d(x,p) = 0$. Hence, $x = p$. Then the neighborhood is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x,p) < \varepsilon\} = \{p\}.$$

(2) Let $\varepsilon > 1$. Clearly, for all $x \in \mathbb{R}$, we have $d(x,p) \leq 1 < \varepsilon$. So,

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x,p) < \varepsilon\} = \mathbb{R}.$$

0.2.9 Limit Points

Definition (Limit Points, Isolated Points). Let (X, d) is a metric space with $E \subseteq X$. Then we call

(1) A point $p \in X$ is said to be a **limit point of** E if for all $\varepsilon > 0$,

$$N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset.$$

(2) The **collection of all the limit points of** E is denoted by E'; that is,

$$E' = \{p \in X : \text{for all } \varepsilon > 0, N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset\}.$$

(3) A point $p \in E$ is said to be an **isolated point of** E if p is NOT a limit point; that is, $p \in E$ but $p \notin E'$. Another way of saying this is $E \setminus E'$.

(4) If p is NOT a limit point, we have $p \notin E'$ if and only if there exists $\varepsilon > 0$ such that

$$N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset.$$

Remark. The statement for all $\varepsilon > 0$, $N_\varepsilon(p)$ is equivalent to for all $N_\varepsilon(p)$.

Example 0.2.5. Let $(\mathbb{R}, d)$ with $d(x,y) = |x - y|$. Note that $0 \notin E$. Also, recall that $0 \in E'$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(0) \cap (E \setminus \{0\}) \neq \emptyset$. If $0 \in E'$, we just need to show that for all $\varepsilon > 0$, $N_\varepsilon(0) \cap E \neq \emptyset$; that is, we need to show that

$$\text{for all } \varepsilon > 0,\ (-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Clearly, $1/m \in (-\varepsilon, \varepsilon) \cap E$.