Linear Algebra Notes

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Chapter 1

Vector Spaces

1.1 Vector Spaces

1.1.1 Basics

Definition 1.1.1 (Vector Spaces). A **vector space** (or **linear space**) over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in V there is a unique element ax in V, such that the following conditions hold:

(VS 1) For all $x, y \in V$, $x + y = y + x$ (commutativity of addition).

(VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in V denoted by O such that $x + O = x$ for each $x \in V$

(VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = O$.

(VS 5) For each element $x \in V$, we have $1x = x$.

(VS 6) For each $a, b \in F$ and each element $x \in V$, then $(ab)x = a(bx)$.

(VS 7) For each element $a \in F$ and each pair $x, y \in V$, we have $a(x + y) = ax + ay$.

(VS 8) For each pair $a, b \in F$ and each $x \in V$, we have $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.

- The elements of a field F are called **scalars** and the elements of a vector space V are called **vectors** (these should not be confused!).
- Every vector space will always be defined over a given field, mostly defined over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ unless otherwise noted.
- Every vector space should specify the operations of addition and scalar multiplication.

Definition 1.1.2 (n-tuples). An object of the form $(a_1, a_2, \ldots, a_n)$, where the entries $a_1, a_2, \ldots, a_n$ are elements of a field F, is called an **n-tuple** with entries from F. The elements $a_1, a_2, \ldots, a_n$ are called **entries** or **components** of the n-tuple.

Definition 1.1.3. We say that two n-tuples, $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, are **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Example 1.1.1. The set of all n-tuples with entries from a field F denoted by F_n is a vector space. To see why, suppose $u, v \in F_n$ where $u = (a_1, a_2, \ldots a_n)$ and $v = (b_1, b_2, \ldots, b_n)$. If we take term-by-term addition of the entries in both u and v, then we end up with

$$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and likewise,

$$cu = (ca_1, ca_2, \dots, ca_n).$$

These same set of operations define $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ and likewise, $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.

- Note that vectors in F^n can be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

  rather than **row vectors** $(a_1, a_2, \dots, a_n)$.

- 1-tuples are are just scalars or an just an element from F.

Definition 1.1.4. An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry a_{ij} with $(1 \leq i \leq m, 1 \leq j \leq n)$ is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \dots a_{in}$ compose the i**th row**of the matrix, and the entries $a_{1j}, a_{2j}, \dots a_{mj}$ compose the jth column of the matrix.

- The rows make a vector space which we denote F^n.
- Likewise, the columns make a vector space we denote F^m

Definition 1.1.5 (Zero Matrix)**.** The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

Definition 1.1.6 (Square Matrix)**.** A matrix is **square** if the number of rows and columns of a matrix are equal.

- Just like our tuple example, the set of all $m \times n$ matrices with entries from a field F form a vector space. Denote this vector space as $M_{m \times n}(F)$ endowed with two operations; that is, **matrix addition** and **scalar multiplication**. Suppose for $A, B \in M_{m \times n}(F)$ and $c \in F$, we have

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

  and

$$(cA)_{ij} = cA_{ij}$$

  for $1 \leq i \leq m$ and $1 \leq j \leq n$. In other words, the two operations can be performed entry-wise.

- The operations from our tuple case extends very naturally to $M_{m \times n}(F)$. In other words, if we add two matrices A_{ij} and B_{ij}, then we would expect to that $A_{ij} + B_{ij} \in M_{m \times n}(F)$ as well and likewise for the scalar multiplication case.

Definition 1.1.7 (Set of All Functions)**.** Let S be any nonempty set and F be any field, and let $\mathcal{F}(S,F)$ denote the set of all functions from S to F.

Definition 1.1.8. Two functions $f,g \in \mathcal{F}(S,F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$.

The set of functions above forms a vector space with the operations of addition and scalar multiplication defined for every $f,g \in \mathcal{F}(S,F)$ and $c \in F$ with

$$(f+g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$

Definition 1.1.9 (Set of All Polynomials)**.** A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each $a_k \in F$ is called the **coefficient** of x_k.

Definition 1.1.10 (Zero Polynomial)**.** We call $f(x) = 0$ the **zero polynomial** if $a_n = a_{n-1} = \cdots = a_0 = 0$.

Definition 1.1.11 (Degree)**.** The **degree** of a given polynomial f is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Definition 1.1.12 (Equality of Polynomials)**.** We call two polynomials f,g, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

are **equal** if $m = n$ and $a_i = b_i$ for all $i = 0, 1, \ldots, n$.

Suppose we have $c \in F$ and say we evaluated the polynomial $f \in F$ at c. Then we would have the following

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$$

where $f(c) \in F$.

Definition 1.1.13 (Basic Operations of Polynomials)**.** Define polynomial **addition** $f + g$ as the following:

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Let $c \in F$. Let scalar **multiplication** be defined by

$$cf(x) = ca_n x^n + ca_{n-1}x^{n-1} + \cdots + ca_1 x + ca_0.$$

The operations above form a vector space for $P(F)$ (the set of all polynomials).

Definition 1.1.14 (Sequences)**.** A **sequence** in F is a function $\sigma : \mathbb{Z}^+ \to F$. A given sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \ldots$ is denoted (a_n).

Let V be the set of all sequences $\sigma(n) \in F$. For every $(a_n), (b_n) \in V$ with $t \in F$, we have the following operations

$$(a_n) + (b_n) = (a_n + b_n) \text{ and } t(a_n) = (ta_n).$$

1.1.2 Non-examples

Example 1.1.2. Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$ where R is a field. For every $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Note that S is **NOT** a vector space since (VS 1), (VS 2), and (VS 8) fail.

Example 1.1.3. Let S be the same set as in the last example. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$$

and

$$c(a_1, a_2) = (ca_1, 0).$$

Note that S is **NOT** a vector space with these operations since (VS 3), (VS 4), and (VS 5) fail.

1.1.3 Basic Extensions from Definition

Theorem 1.1.1 (Cancellation Law for Vector Addtion). If $x, y, z \in V$ such that $x + z = y + z$, then $x = y$.

Proof. There exists a vector $v \in V$ such that $z + v = O$ (VS 4). Thus, we have

$$\begin{aligned} x &= x + O \\ &= x + (z + v) \\ &= (x + z) + v \\ &= (y + z) + v \\ &= y + (z + v) \\ &= y + O \\ &= y. \end{aligned}$$

Hence, we have $x = y$. ■

Corollary. The vector described O described in (VS 3) is unique.

Proof. Let $x \in V$. Suppose there exists two elements $O, O' \in V$ such that

$$x + O = x \tag{1}$$

and

$$x + O' = x. \tag{2}$$

Our objective is to show that $O = O'$. Equating (1) and (2), we find that

$$x + O = x + O' \tag{3}$$

Using the Cancellation Law for Vector Addition, (3) implies that $O = O'$. Hence, O is a unique vector in V. ■

Corollary. The vector y described in (VS 4) is unique.

Proof. Let $x \in V$. Suppose there exists $y, y' \in V$ such that

$$x + y = O \text{ and } x + y' = O.$$

Equating the two equations above, we get that

$$x + y = x + y'.$$

Applying the Cancellation Law for Vector Addition once again, we get that $y = y'$. Hence, y is a unique vector in V. ■

Theorem 1.1.2 (Properties of a Vector Space)**.** In any vector space V, the following statements are true:

(a) $0x = O$ for any $x \in V$.

(b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.

(c) $a0 = 0$ for any $a \in F$.

Proof. (a) Using (VS 8), (VS 3), and (VS 1), it follows that

$$\begin{aligned} 0x + 0x &= (0+0)x \\ &= 0x \\ &= 0x + O \\ &= O + 0x. \end{aligned}$$

Hence, $0x = O$ by Theorem 1.1.

(b) The vector $-(ax)$ is the unique element of V such that $ax + [-(ax)] = O$. If we know that $ax + (-a)x = O$, we can use Corollary 2 to Theorem 1.1 to state that $(-a)x = -(ax)$. But by (VS 8), we have

$$ax + (-ax) = [a + (-a)]x = 0x = O$$

by (a). Hence, we have that $(-a)x = -(ax)$. In particular, we have $(-1)x = -x$. Thus, (VS 6) implies that

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

(c) The proof of (c) is similar to the proof of (a).

■

1.2 Subspaces

Definition 1.2.1 (Subspaces)**.** A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication.

The most simple examples of subspaces of V is V itself and the set containing just the zero vector. The latter is denoted as the **zero subspace** of V.

We don't have to check all the vector space properties to prove that a subset W of V is a subspace of V. This is because vectors in any subset of V already satisfy all the properties of a vector space. Hence, we need only check that W is closed under

1. **Addition**: $x + y \in W$ whenever $x \in W$ and $y \in W$.
2. **Scalar Multiplication**: $cx \in W$ whenever $c \in F$ and $x \in W$.
3. W contains a zero vector.
4. Each vector in W has an additive inverse in W.

Theorem 1.2.1 (Subspaces)**.** Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

(a) $O \in W$.

(b) $x + y \in W$ whenever $x \in W$ and $y \in W$.

(c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Proof. Suppose W is a subspace of V. Since W is also a vector space with the operations of addition and scalar multiplication defined in V. Hence, we know that W is closed under addition and scalar multiplication. Note that W contains a zero vector $O' \in W$ such that for any $x \in W$, we have $x + 0' = x$. Since x is also in V (since $W \subseteq V$), we know that $x + 0 = x$. Using the cancellation , we can see that $x + 0' = x + 0$ implies $0' = 0$. Hence, properties (a), (b), and (c) are satisfied.

Conversely, suppose conditions (a), (b), and (c) hold. We need to show that W is a subspace. This tells us that W is closed under addition and scalar multiplication in addition to containing the zero vector. Let $x \in W$. Since W is closed under scalar multiplication, we know that $(-1)x \in W$. By part (b) of theorem 2, we know that $1(-x) = -x \in W$. Hence, W contains an additive inverse and we are done. ■

The theorem above provides a simpler way to determine whether a given subset of a vector space is a subspace. Instead of using the definition to show that a give subset is a subspace, it is more common to use the result above.

Definition 1.2.2 (Tranpose). The **transpose** of A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$.

Example 1.2.1.

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Definition 1.2.3 (Symmetric Matrices). A **symmetric matrix** is a matrix A such that $A^t = A$.

- The easiest example of a symmetric matrix would be a square matrix where $i = j$.
- The set W of all symmetric matrices in $M_{n\times n}(F)$ is a subspace of $M_{n\times n}(F)$ since the conditions of Theorem 1.3 hold.

We can show that W is indeed a subspace.

1. The zero matrix is equal to its transpose and hence belongs to W.
2. Suppose $A \in W$ and $B \in W$. Hence, $A^t = A$ and $B^t = B$. Hence, we have
$$(A + B)^t = A^t + B^t = A + B$$
which implies that $A + B \in W$.
3. We have $A \in W$ implies $A^t = A$. Now, let $a \in F$. Then we have that $(aA)^t = aA^t = aA$. Hence, $aA \in W$.

1.2.1 Examples of Subspaces

Example 1.2.2. Let n be non-negative integer, and let $P_n(F)$ consist of all polynomials in $P(F)$ having degree less than or equal to n. We get that $P_n(F)$ is a subspace because:

1. Zero polynomial has degree -1, it is in $P_n(F)$.
2. The sum of two polynomials with degrees less than or equal to n is another a polynomial less

than or equal to n.

3. The product of a scalar and a polynomial of degree less than or equal to n is a polynomial less than or equal to n.

This tells us that $P_n(F)$ is a subspace of the space of all polynomials.

Example 1.2.3. Let $C(\mathbb{R})$ denote the set of all continuous real-valued functions defined on $\mathbb{R}$. We know that $C(\mathbb{R})$ is a subset of the vector space $\mathcal{F}(\mathbb{R},\mathbb{R})$ defined in Example 3. We will show that $C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$.

1. Note that the zero function $f(x) = 0$ for all $x \in \mathbb{R}$ of $\mathcal{F}(\mathbb{R},\mathbb{R})$ is a continuous real-valued function.

2. Let $f, g \in C(\mathbb{R})$. Since the sum of f and g is also continuous for all $x \in \mathbb{R}$, we have that $f + g \in C(\mathbb{R})$.

3. Let $c \in \mathbb{R}$ and $f \in C(\mathbb{R})$. We have that for any $x \in \mathbb{R}$, $(cf)(x) = cf(x)$ is a continuous function. Hence, property (c) is satisfied.

Hence, $C(\mathbb{R})$ is a subspace.

Definition 1.2.4 (Upper Triangular)**.** An $m \times n$ matrix A is called **upper triangular** if all its entries below the diagonal entries are zero; that is, if $A_{ij} = 0$ whenever $i > j$.

Example 1.2.4. Let B be an upper triangular 3×4 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{pmatrix}$$

Definition 1.2.5 (Diagonal Matrix)**.** An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$; that is, if all its non-diagonal entries are zero.

Example 1.2.5. Let A be diagonal 3×3 matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Example 1.2.6. The set of diagonal matrices is a subspace of $M_{n\times n}(F)$. To see why, we have

1. Let O be the zero matrix of $M_{n\times n}(F)$. Since O is also a diagonal matrix, we know that O must be in the set of diagonal matrices.

2. Let A, B in the set of diagonal matrices. Let $i \neq j$ such that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0.$$

   Hence, $A + B$ is a diagonal matrix and so addition is closed.

3. Let A be a diagonal matrix as before and let $c \in F$. Let $i \neq j$ again, and observe that

$$(cA)_{ij} = cA_{ij} = c \cdot 0 = 0.$$

   Hence, cA is a diagonal matrix and so scalar multiplication is closed.

Since the set of diagonal matrices satisfies all properties of theorem 3, we conclude that it is indeed a subspace of $M_{n\times n}(F)$.

Definition 1.2.6 (Trace)**.** The **trace** of an $n \times n$ matrix M, denoted $\mathrm{tr}(M)$, is the sun of the diagonal entries of M; that is,

$$\mathrm{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

The set of all $n \times n$ matrices that have a trace equal to zero is a subspace of $M_{n\times n}(F)$ (proved in Exercise 6).

Example 1.2.7 (Non-example)**.** Denote V as the set of matrices in $M_{m\times n}(\mathbb{R})$ having non-negative entries. The subset V is not a subspace because it is not closed under scalar multiplication. We can see this by multiplying any matrix in V be a negative number and observe that the entries of said matrix are no longer all non-negative.

We can create subspaces out of other subspaces.

1.2.2 Constructing Subspaces Out of Existing Subspaces

Theorem 1.2.2 (Intersection of Subspaces is a Subspace)**.** Any intersection of subspaces of a vector space V is a subspace of V.

Proof. Let C be a collection of subspaces of V, and let W denote the intersection of the subspaces in C. Since every subspace contains the zero vector and the intersection $W \neq \emptyset$, the zero vector $0 \in W$. Let $a \in F$ and $x, y \in W$. Since each subspace of C is closed under addition and scalar multiplication, it follows that $x + y$ and ax are contained each subspace in C and hence W must be closed under addition and closed under scalar multiplication. Thus, W is a subspace of V by theorem 3. ■

A natural question to ask is whether or not the union of subspaces of a vector space V is a subspace of V too. We can see that indeed the union of subspaces of V satisfy the first first and third property of theorem 3. However, property 2 need not be satisfied all the time. In fact, the union can only be a subspace of V if and only if one the subspaces is a subset of the other.

1.3 Linear Combinations and Systems of Linear Equations

1.3.1 Linear Combinations

Definition 1.3.1 (Linear Combinations)**.** Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called **linear combination** of vectors of S if there exist a finite number of vectors $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

In this case, we say that v is a linear combination of $u_1, u_2, \ldots, u_n \in V$ and call $a_1, a_2, \ldots, a_n \in F$ the **coefficients** of the linear combination.

- An easy example of a vector expressed as a linear combination is the zero vector O where $0v = O$ for each $v \in V$.
- Determining whether a vector is a linear combination of other vectors often involves solving a system of linear equations.

Example 1.3.1. Suppose we wanted to express the vector $(2, 6, 8) \in \mathbb{R}^3$ as a linear combination of

$$u_1 = (1, 2, 1),\ u_2 = (-2, -4, -2),\ u_3 = (0, 2, 3),\ u_4 = (2, 0, -3),\ \text{and}\ u_5 = (-3, 8, 16).$$

Our goal is to find scalars a_1, a_2, a_3, a_4 and a_5 such that

$$(2,6,8) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5.$$

Doing a bit of algebra, we would need to solve the following system of linear equations

$$\begin{aligned} a_1 - 2a_2 \phantom{{}+2a_3} + 2a_4 - 3a_5 &= 2 \\ 2a_1 - 4a_2 + 2a_3 \phantom{{}+2a_4} + 8a_5 &= 6 \\ a_1 - 2a_2 + 3a_3 + 16a_5 &= 8 \end{aligned}$$

Solving the system of equations above involves three types of operations:

1. Interchanging the order of any two equations in the system;
2. multiplying any equation in the system by some non-zero constant;
3. adding a constant multiple of any equation to another equation in the system.

We will learn in a later section that the operations listed above do not change the set of solutions to the original system.

Example 1.3.2. We claim that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \text{ and } 3x^3 - 5x^2 - 4x - 9$$

in $P_3(\mathbb{R})$, but that

$$3x^3 - 2x^2 + 7x + 8$$

is not. In the first case we wish to find scalars a and b such that

$$\begin{aligned} 2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) \\ &\quad + b(3x^3 - 5x^2 - 4x - 9) \\ &= (a + 3b)x^3 + (-2a - 5b)x^2 \\ &\quad + (-5a - 4b)x + (-3a - 9b). \end{aligned}$$

Thus, we have the following system of linear equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= -2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6. \end{aligned}$$

Adding the appropriate multiples of the first equation to the others in order to eliminate a, we find that

$$\begin{aligned} a + 3b &= 2 \\ b &= 2 \\ 11b &= 22 \\ 0b &= 0. \end{aligned}$$

Then we get that $a = -4, b = 2, 0 = 0, 0 = 0$. Hence, we have

$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$

Looking at the second case now, we observe that using the preceding technique leads us to the following system of linear equations

$$\begin{aligned} a + 3b &= 3 \\ -2a - 5b &= -2 \\ -5a - 4b &= 7 \\ -3a - 9b &= 8. \end{aligned}$$

Eliminating a as before yields the following:

$$\begin{aligned} a + 3b &= 3 \\ b &= 4 \\ 11b &= 22 \\ 0 &= 17. \end{aligned}$$

The presence of the non-sensical result in the last equality tells us that there are no solutions to the system and that the second polynomial cannot be written as a linear combination of the two polynomials.

We can denote the set of all linear combinations of some set of vectors in the following:

1.3.2 The Span

Definition 1.3.2 (Span)**.** Let S be a nonempty subset of a vector space V. The **span** of S, denoted $\operatorname{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $\operatorname{span}(\emptyset) = \{0\}$.

Some immediate examples of spans are:

Example 1.3.3. In $\mathbb{R}^3$, the span of the set $S = \{(1,0,0),(0,1,0)\}$ consist of all vectors in $\mathbb{R}^3$ such that for some scalars $a, b \in \mathbb{R}$, we have

$$a(1,0,0) + b(0,1,0) = (a,b,0).$$

This tells us that S contains all the points in the xy-plane. One can show that S is a subspace of $\mathbb{R}^3$.

Theorem 1.3.1 (The Span of Subset is a Subspace)**.** The span of any subset S of a vector space v is a subspace of V that contains S. Moreover, any subspace of V that contains S must also contain the span of S.

Proof. Suppose $S = \emptyset$. Then the span of S is just $\operatorname{span}(\emptyset) = \{0\}$ which is a subspace in which S is contained in. Moreover, $\operatorname{span}(\emptyset) = \{0\}$ is always contained in any subspace X of V because every subspace contains the zero vector 0. Suppose $S \neq \emptyset$, then S contains a vector z. We need to show that $\operatorname{span}(S)$ is a subspace of V.

(a) Since $S \neq \emptyset$, we know that S contains a vector v such that $0z = 0$. Hence, $0 \in \operatorname{span}(S)$.

(b) Let $x, y \in \operatorname{span}(S)$. We need to show that $x + y \in \operatorname{span}(S)$. If $x \in \operatorname{span}(S)$, then we can find $a_1, a_2, \ldots, a_n \in F$ and $x_1, x_2, \ldots, x_n \in S$ such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Likewise, $y \in \operatorname{span}(S)$ implies that we can find scalars $b_1, b_2, \ldots, b_m$ and vectors $y_1, y_2, \ldots, y_m$ such that

$$b_1y_1 + b_2y_2 + \cdots + b_my_m.$$

Then

$$x + y = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_1y_1 + b_2y_2 + \cdots + b_my_m$$

is a linear combination and thus $x + y \in \text{span}(S)$.

(c) Note that $c \in F$ implies that

$$\begin{aligned} cx &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (ca_1)x_1 + (ca_2)x_2 + \cdots + (ca_n)x_n \end{aligned}$$

is a linear combination and thus $cx \in \text{span}(S)$.

Hence, $\text{span}(S)$ is a subspace of V. Now we need to show that S is contained within $\text{span}(S)$; that is, $S \subseteq \text{span}(S)$. Let $v \in S$. Then using (VS 5), we can see that $1 \cdot v = v$ is a linear combination; so we have $v \in \text{span}(S)$ and hence, the span of S contains S.

Now let W be any subspace of V that contains S. We need to show that $\text{span}(S) \subseteq W$. Let $v \in \text{span}(S)$. Then we can find scalars $a_1, a_2, \ldots, a_n \in F$ and $x_1, x_2, \ldots, x_n \in S$ such that

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Since $S \subseteq W$, we know that $x_1, x_2, \ldots, x_n \in W$. Using exercise 20 from section 1.3 and using the same set of scalars $a_1, a_2, \ldots, a_n \in F$, we have $v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$. Hence, $\text{span}(S) \subseteq W$. ■

1.3.3 Generating a Vector Space

Definition 1.3.3 (Generating/Spanning Sets). A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we also say that the vectors of S generates (or span) V.

Example 1.3.4 (Vectors in $\mathbb{R}^3$). The vectors $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ generate $\mathbb{R}^3$ since any given vector $v \in \mathbb{R}^3$ is a linear combination of the three given vectors. Furthermore, there exists scalars $r, s, t \in \mathbb{R}$ such that

$$r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3) = v$$

where

$$r = \tfrac{1}{2}(a_1 + a_2 - a_3), s = \tfrac{1}{2}(a_1 - a_2 + a_3) \text{ and } t = \tfrac{1}{2}(-a_1 + a_2 + a_3).$$

Example 1.3.5 (Polynomials). The polynomials $x^2 + 3x - 2, 2x^2 + 5x - 3$, and $-x^2 - 4x + 4$ generate $P_2(\mathbb{R})$ because each of the three given polynomials belongs to $P_2(\mathbb{R})$ and each polynomial $ax^2 + bx + c \in P_2(\mathbb{R})$ is a linear combination of these three. It can be shown that we can find $a, b, c \in \mathbb{R}$ such that

$$a(x^2 + 3x - 2) + b(2x^2 + 5x - 3) + c(-x^2 - 4x + 4) = ax^2 + bx + c.$$

Example 1.3.6 (Matrices). Note that the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate $M_{2\times2}(\mathbb{R})$ because an every $A \in M_{2\times2}(\mathbb{R})$ can be expressed as a linear combination of the four given matrices found below where there exists scalars $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where it can be show that

$$\begin{aligned} a &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} - \frac{2}{3}a_{22} \\ b &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} - \frac{2}{3}a_{21} + \frac{1}{3}a_{22} \\ c &= \frac{1}{3}a_{11} - \frac{2}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22} \\ d &= -\frac{2}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}. \end{aligned}$$

A non-example set of matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

do not generate $M_{2\times 2}(\mathbb{R})$ since all of the matrices have equal diagonal entries. Thus, every the set of matrices above cannot generate every $A \in M_{2\times 2}(\mathbb{R})$.

1.4 Linear Dependence and Linear Independence

1.4.1 Motivation

Suppose V is a vector space over a field F and that $W \subseteq V$ is a subspace of V.

- Suppose we have a set S that will generate every vector in V as a linear combination of vectors in S.
- It is desirable to find a subset of S that is as small as possible or rather just enough to generate each vector in V.
- The reason for this is to reduce the amount of computations done to represent a vector in V.
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in S.
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in S.
- Doing this, guarantees that we can write any one of the vectors in S as a linear combination of the others.

Definition 1.4.1 (Linearly Dependence). A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors in $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case, we also say that the vectors of S are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $v_1, v_2, \ldots, v_n \in S$ where for all $1 \leq i \leq n$, we have $a_i = 0$.
- This tells us that our definition of **linear dependence** implies that 0 is a non-trivial linear combination.

- Any subset of V that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is, $1 \cdot 0 = 0$.

1.4.2 Examples of Linearly Dependent Sets

Example 1.4.1. Consider a subset in $\mathbb{R}^4$ defined by

$$S = \{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4),(-1,0,1,0)\}.$$

To show that S is a linear dependent set, we need to find scalars $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that a_i for all $1 \leq i \leq 4$ not all zero such that

$$a_1(1,3,-3,2) + a_2(2,2,-4,0) + a_3(1,-3,2,-4) + a_4(-1,0,1,0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients $a_1 = 4, a_2 = -3, a_3 = 2$, and $a_4 = 0$. Thus, we have that S is linearly dependent subset of $\mathbb{R}^4$ and hence we can write any vector in S as a linear combination of the other vectors contained in S.

Example 1.4.2. Define a subset of $M_{2\times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} \}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients a_1, a_2, a_3 such that $a_1 = 5, a_2 = 3$, and $a_3 = -2$ where

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 1.4.2 (Linear Independence)**.** A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

1.4.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is, $\{v\}$ is linearly independent.
- If $\{v\}$ is linearly dependent, then it the singleton has to be the zero vector 0. This is because $au = 0$

$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

Example 1.4.3 (A Set of Vectors in $\mathbb{R}^4$)**.** It can be shown that the set

$$S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of 0 represent the trivial representation; that is, we have scalars $a_1 = a_2 = a_3 = a_4 = 0$.

Example 1.4.4. For $k = 0, 1, \ldots, n$, let $p_k = x^k + x^{k+1} + \cdots + x^n$. The set

$$\{p_0(x), p_1(x), \ldots p_n(x)\}$$

is linearly independent in $P_n(F)$. It can be shown that for some scalars $a_0, a_1, \ldots, a_n \in F$, the following equation

$$a_0 + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n = 0$$

has the trivial-representation; that is, $a_i = 0$ for all $1 \leq i \leq n$.

Theorem 1.4.1 (Linearly Dependent Subsets)**.** Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Proof. Let V be a vector space. Suppose S_1 is linearly dependent. Then there exists a finite number of distinct vectors $v_1, v_2, \ldots, v_n \in S_1$ and scalars $a_1, a_2, \ldots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Since $S_1 \subseteq S_2$, we must have $v_1, v_2, \ldots, v_n \in S_2$ as well. If these vectors are linearly dependent, then these vectors are also linearly dependent in S_2; that is, we have scalars $a_1, a_2, \ldots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Hence, S_2 is linearly dependent. ■

Corollary (Linearly Independent Subsets)**.** Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof. Note that this corollary is just the contrapositive of the theorem before it. Hence, S_1 is linearly independent. ■

- Determining whether there exists a minimal generating set for the span of S is related to solving the problem of finding whether some vector in S that can be written in terms of a linear combination of other vectors in S.
- We can see that in a given subset of $\mathbb{R}^3$ defined by

$$S = \{u_1, u_2, u_3, u_4\}$$

  where $u_1 = (2, -1, 4), u_2 = (1, -1, 3), u_3 = (1, 1, -1)$, and $u_4 = (1, -2, 1)$. Note that this S is linearly dependent. This tells us that any vector in S can be written as a linear combination of the others. Suppose we pick u_3 and write as a linear combination of the vectors u_1, u_2, and u_4. We would find that the span of these three vectors (denote this set as S') generates the same spanning set S but with a cardinality one less than S; that is, $\text{span}(S) = \text{span}(S')$.
- If we find that there does not exist a proper subset that is equivalent to the span of S, then S must be a linearly independent set.

Theorem 1.4.2 (Adjoining Elements not in Linearly Independent Sets)**.** Let S be a linearly independent subset of a vector space V, and let $v \in V$ but not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof. ($\Rightarrow$) Suppose $S \cup \{v\}$ is a linearly dependent set. Then there exists a finite number of scalars $a_1, a_2, \ldots, a_n \in F$ and vectors $u_1, u_2, \ldots, u_n \in S \cup \{v\}$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

Since S is a linearly independent set, we must have $u_i = v$ for some $1 \leq i \leq n$. Choose $i = 1$ (any choice of i will do) such that

$$a_1 v + a_2 u_2 + \cdots + a_n u_n = 0.$$

Solving for v by subtracting $a_1 v$ on both sides, multiplying by a_1^{-1} on both sides of the equation, and distributing by a_1^{-1} yields the following:

$$v = (-a_1^{-1} a_2) u_2 + (-a_1^{-1} a_3) u_3 + \cdots + (-a_1^{-1} a_n) u_n.$$

Since v is a linear combination of vectors $u_1, u_2, \ldots, u_n \in S$, we know that $v \in \text{span}(S)$.

($\Leftarrow$) Conversely, suppose $v \in \text{span}(S)$. This implies that there exists a finite amount of scalars $a_1, a_2, \ldots, a_n \in F$ and $u_1, u_2, \ldots, u_n \in S$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Subtracting v from both sides of the equation above gives us the following equation:

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n - v = 0.$$

Note that v is not contained in S, so $v \neq u_i$ for all $1 \leq i \leq n$. Since S is a linearly independent set, we know that $a_i = 0$ for all $1 \leq i \leq n$. This implies that the only coefficient that is non-zero is with $-v = -1v$. Hence, the set of vectors $S' = \{u_1, u_2, \ldots, u_n, v\}$ is linearly dependent. Since $S' \subseteq S \cup \{v\}$, we know that $S \cup \{v\}$ is also linearly independent by Theorem 6. ■

1.5 Bases And Dimension

1.5.1 Bases

- Recall that S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

Definition 1.5.1 (Basis). A **basis** β for a vector space V is linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Example 1.5.1.

- Recall that the empty set $\emptyset$ is linearly independent and that $\text{span}(\emptyset) = \{0\}$. The empty set $\emptyset$ in this case is the basis for the zero vector space.
- Note that in F^n, the vectors $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0) \ldots, e_n = (0, 0, \ldots, 0, 1)$ form a basis for F^n.
- The basis for $M_{m \times n}(F)$ is the set of matrices E^{ij} such that the only nonzero entry is a 1 in the ith and jth column.
- As we have seen in the last section, the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$.
- In $P(F)$, the set $\{1, x, x, x^2, \ldots\}$ is a basis. *Bases are not limited to finite sets. They can be infinite.*

Theorem 1.5.1 (Unique Linear Combinations from Bases). Let V be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if each $v \in V$ can be unique expressed as a linear combination of vectors in β, that is, expressed in the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof. ($\Rightarrow$) Suppose $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V. Then $\text{span}(\beta) = V$. If $v \in V$, then $v \in \text{span}(\beta)$. Hence, we can write v as a linear combination of vectors in β such that choosing scalars $a_1, a_2, \dots, a_n \in F$ leads to

$$v = \sum_{i=1}^{n} a_i u_i.$$

Suppose there exists another representation of $v \in V$ such that

$$v = \sum_{i=1}^{n} b_i x_i$$

Hence, observe that

$$\begin{aligned} \sum_{i=1}^{n} a_i x_i &= \sum_{i=1}^{n} b_i y_i \\ \Rightarrow \sum_{i=1}^{n} (a_i - b_i) x_i &= 0. \end{aligned}$$

Since β is linearly independent, we know that $a_i - b_i = 0$ which implies $a_i = b_i$ for all $1 \leq i \leq n$. Hence, v can be expressed as a unique linear combination of vectors in β.

($\Leftarrow$) Conversely, let $v \in V$ be expressed as a unique linear combination of vectors in β such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \tag{1}$$

for unique scalars $a_1, a_2, \dots, a_n$. We want to show that β is a basis for V; that is, we want to show that β is a spanning set for V and β is linearly independent. To show that β is a spanning set for V, we need to show that $\text{span}(\beta) \subseteq V$ and $V \subseteq \text{span}(\beta)$. Note that $\beta \subseteq V$ and $\text{span}(\beta)$ is a subspace for V. Hence, $\text{span}(\beta) \subseteq V$. On the other hand, $V \subseteq \text{span}(\beta)$ follows immediately from (1). To show that β is linearly independent, we need to show that for scalars $\delta_1, \delta_2, \dots, \delta_n$, we have

$$\delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_n u_n = 0$$

such that $u_i = 0$ for all $1 \leq i \leq n$. Note that $v + 0 = v$. Hence, we can write

$$\begin{aligned} \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n} \delta_i u_i &= \sum_{i=1}^{n} a_i u_i \\ \sum_{i=1}^{n} (a_i + \delta_i) u_i &= \sum_{i=1}^{n} a_i u_i \end{aligned}$$

Equating each term in the equation above yields the following equality:

$$a_i + \delta_i = a_i \Rightarrow \delta_i = 0$$

for all $1 \leq i \leq n$. But this tells us that β is linearly independent and we are done. ■

- Any vector $v \in V$ can be written as a linear combination of vectors from the basis containing $u_1, u_2, \dots, u_n \in V$.
- This determines a unique n-tuple of scalars $(a_1, a_2, \dots, a_n)$ and conversely, each n-tuple of scalars determines a unique vector $v \in V$ such that each coefficient from the linear combination of $u_1, u_2, \dots, u_n$ is an entry from said tuple.
- For example, in our vector space F^n, n is the number of vectors that should be in the basis for F^n which is indeed the case.
- In this book, we are only concerned with finite bases.

Theorem 1.5.2 (Finite Spanning Set For a Vector Space)**.** If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence, V has a finite basis.

Proof. Suppose $S = \emptyset$ or $S = \{0\}$, then $V = \{0\}$ and $\emptyset$ is a subset of S that is a basis for V. If S neither of these choices, then S must contain at least one nonzero vector u_1; that is, $S = \{u_1\}$. Since u_1 is nonzero, it follows that S is a linearly independent set. We can continue this process of adding vectors $u_2, \dots, u_k$ into S such that S is a linearly independent set of k vectors. Since S is a finite set, we must end with the linearly independent set $\beta = \{u_1, u_2, \dots, u_n\}$. There are two cases for which this occurs, either $\beta = S$ or $\beta \subseteq S$:

(i) Suppose that $\beta = S$ (remember that β is a finite set by construction). Then we have S is a linearly independent set and spanning set for V (since S is a finite set that generates V). Hence, S is a finite basis for V.

(ii) Suppose $\beta \subseteq S$ is a linearly independent set such that adding $v \in S$ where $v \notin \beta$ makes a linearly dependent set. We claim that β is the desired subset of S that is a basis for V. Then we have two cases; that is, either $v \in \beta$ or $v \notin \beta$. Since $\beta \subseteq S$, we know by Theorem 1.5 that $\text{span}(\beta) \subseteq S$ (This applies for both cases). It suffices to show that $S \subseteq \text{span}(\beta)$. If $v \in \beta$, then surely $v \in \text{span}(\beta)$. Hence, we have $S \subseteq \text{span}(\beta)$. Suppose $v \notin \beta$. Since β is a linearly independent set, then by Theorem 1.7 , we have that $\beta \cup \{v\}$ being linearly dependent implies that $v \in \text{span}(\beta)$. Hence, $S \subseteq \text{span}(\beta)$. Thus, that both cases implies that β is a spanning set for V.

■

This theorem tells us that any spanning set of a vector space V can reduced to a finite basis for V. This is illustrated in the following examples.

Example 1.5.2. Define

$$S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$$

which can be shown to generate $\mathbb{R}^3$. The idea is to create a proper subset of S such that none of the vectors in β are a multiples of each other. In other words, we want a set that is linearly independent; that is, our choices of vectors determine whether a set will be linearly dependent or independent (we want the latter to hold). Say, we pick $(2, -3, 5)$ as our first vector in our subset β. Right away, we can exclude $(8, -12, 20)$ since it is a multiple of $(2, -3, 5)$. Otherwise, including it would make β linearly dependent (see exercise 9 from section 1.5). Next, add the vectors $(1, 0, -2)$ and $(0, 2, -1)$ since they are not multiples of the other. Narrowing down our set to

$$\beta = \{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}.$$

Note that adding the vector $(7, 2, 0)$ makes β linearly dependent, so $(7, 2, 0)$ is excluded from the list. Hence, we have arrived at a subset of S such that $\beta \subseteq S$ is both a linearly independent set and spanning set for $\mathbb{R}^3$.

The following theorem and its corollaries are the most important results in the Chapter 1.

1.5.2 Dimensions

Theorem 1.5.3 (Replacement Theorem)**.** Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Proof. Let us proceed the proof via induction on m. Let $m = 0$ be our base case. Then we find that $L = \emptyset$ is linearly independent set with exactly 0 vectors. Letting $H = G$ gives us the desired result (since G contains exactly n vectors). Now suppose that the theorem is holds for some integer $m \geq 0$. We will show that the theorem holds for the $m + 1$ case.Let $L = \{v_1, v_2, \dots, v_{m+1}\}$ be a linearly independent subset of V consisting of exactly $m+1$ vectors. By the corollary to Theorem 1.6 , we find that $L' = \{v_1, v_2, \ldots, v_m\}$ is a linearly independent set (because $L' \subseteq L$ and L is linearly independent). Using our induction hypothesis, we can conclude that $m \leq n$ and that there exists a subset $H' = \{u_1, u_2, \ldots, u_{n-m}\}$ of G such that $L' \cup H'$ generates V. Thus there exists scalars $a_1, a_2, \ldots a_m, b_1, b_2, \ldots, b_{n-m}$ such that

$$v_{m+1} = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + b_1 u_1 + b_2 u_2 + \cdots + b_{n-m} u_{n-m}. \tag{1}$$

Note that $n - m > 0$, **unless** v_{m+1} is a linear combination of $v_1, v_2, \ldots, v_m$ which by Theorem 1.7 contradicts the assumption that L is a linearly independent set. Hence, $n > m$; that is, $n \geq m+1$. Furthermore, some b_i, say b_1 is nonzero, for otherwise we obtain the same contradiction. Solving (1) for u_1, we get

$$\begin{aligned} u_1 = (-b_1^{-1}a_1)v_1 + (-b_1^{-1}a_2)v_2 + \cdots + (-b_1^{-1})v_m + (b_1^{-1})v_{m+1} \\ + (-b_1^{-1}b_2)u_2 + \cdots + (-b_1^{-1}b_{n-m})u_{n-m}. \end{aligned}$$

Let $H = \{u_2, u_3, \ldots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$ and because $v_1, v_2, \ldots, v_m, u_2, \ldots, u_{n-m}$ are clearly in $\text{span}(L \cup H)$, we have that

$$L' \cup H' \subseteq \text{span}(L \cup H).$$

Since $L' \cup H'$ generates V and the fact that $L' \cup H' \subseteq \text{span}(L \cup H)$ (note that $\text{span}(L \cup H)$ is also a subspace), we know by Theorem 1.5 that $\text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$. Since $\text{span}(L' \cup H')$ generates V, we know that $V \subseteq \text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$. Observe that $\text{span}(L \cup H) \subseteq V$ is true by default. Hence, $\text{span}(L \cup H)$ generates V and that H contains $(n-m) - 1 = n - (m+1)$ vectors which concludes our induction proof. ■

Corollary. Let V be a vector space having a finite basis. Then all bases for V are finite, and every basis for V contains the same number of vectors.

Proof. Let β be a finite basis for V that contains exactly n vectors. Let γ be any other basis that contains more than n vectors. Suppose we pick a subset S of γ such that this subset contains exactly $n+1$ vectors. Since β is a finite basis for V, we know that β is a linearly independent set. By Theorem 1.6, we know that S is also a linearly independent set. By Replacement Theorem, we have that $n+1 \leq n$ which is a contradiction. Therefore, we must have γ is finite, and γ contains exactly m amount of vectors in γ which satisfies $m \leq n$. To show that $n \leq m$, we can reverse the roles of β and γ and use the same argument as above. Hence, $n \leq m$ and thus $m = n$. ■

The main takeaway from the corollary above is that the number of vectors in any basis for V is an inherent property of V.

Definition 1.5.2 (Finite-dimensional)**.** A vector space is **finite-dimensional** if it has a basis consisting of a finite number of vectors.

Definition 1.5.3 (Dimension of a Vector Space)**.** The unique integer n such that every basis for V contains exactly n elements is called the **dimension** of V and is denoted by $\dim(V)$.

Definition 1.5.4 (Infinite-dimensional)**.** A vector space that is not finite-dimensional is called **infinite-dimensional**.

Example 1.5.3.

- The vector space $\{0\}$ has dimension zero. This is because $\{0\}$ is generated by the empty set $\emptyset$.
- The vector space F^n has dimension n.
- The vector space $M_{m \times n}(F)$ has dimension mn.

- The vector space $P_n(F)$ has dimension $n+1$.

It turns out that the dimension of a vector space depends on the choice of the field it's defined on.

Example 1.5.4.
- Over the field of complex numbers, the vector space of complex numbers has dimension 1. (A basis is $\{1\}$).
- On the other hand, if $\mathbb{C}$ is defined over the real numbers $\mathbb{R}$, then the vector space has dimension 2.

The first conclusion of the replacement theorem states that given a finite-dimensional vector space, a linearly independent subset of V cannot contain no more than $\dim(V)$ amount of vectors.

Example 1.5.5 (Example of an Infinite-Dimensional Vector Space)**.** The vector space $P(F)$ is infinite-dimensional since it contains a linearly independent set $\{1, x, x^2, \dots\}$ that is infinite. This is peculiar because all of the results in this section thus far does not guarantee that an infinite-dimensional vector space containing a basis. In fact, it is proven in section 1.7 that infinite-dimensional vector space are guaranteed to have a basis.

The next corollary from the Replacement Theorem refers to the size of the generating set.

Corollary. Let V be a vector space with dimension n.

(a) Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis for V, that is, if L is a linearly independent subset of V, then there is a basis β of V such that $L \subseteq \beta$.

Proof. Let β be a basis for V.

(a) Let β be a finite generating set for V. By Theorem 1.9, β contains a subset S such that S is a finite basis for V. Since V is a vector space with dimension n, we know that S contains exactly n vectors. This means that β must contain at least n vectors since $S \subseteq \beta$.

(b) Let L be a linearly independent subset of V that contains exactly n vectors. Since β is a generating set for V, the Replacement Theorem states that we can find a subset of β, say L', such that L' contains exactly $n - n = 0$ vectors implies that $L' = \emptyset$ and $L \cup L' = L \cup \emptyset = L$ generates V. Since L is also linearly independent, we conclude that L is a basis for V.

(c) Let L be a linearly independent subset of V containing m vectors. Then the Replacement Theorem implies that there exists $H \subseteq \beta$ containing $n - m$, where β is the generating set of V with exactly n amount of vectors, such that $L \cup H$ generates V and is finite. This implies that $L \cup H$ must contain at most n vectors, but part (a) also tells us that $L \cup H$ contains at least n vectors, so $L \cup H$ must contain exactly n vectors. Hence, $L \cup H$ is a basis for V.

■

Example 1.5.6.
- In Example 1.3.5 the set

$$\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$$

is a generating set for $P_2(\mathbb{R})$ that contains exactly 3 vectors. Hence, the set above is a basis for $P_2(\mathbb{R})$ by part (a).

- It follows from Example 1.3.6 and (a) of the second corollary to the Replacement Theorem that the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis for $M_{2\times 2}(\mathbb{R})$.

- The set in Example 1.4.3 is a basis for $\mathbb{R}^4$.
- See Example 1.4.4. This set is a basis for $P_n(F)$.

1.5.3 An Overview of Dimension and Its Consequences

The main takeaways from our results are:

- We can reduce a finite generating set into a basis for a vector space V.
- A finite generating set contains at least n amount of vectors.
- A linearly independent set of vectors contain at most n amount of vectors.
- We can Extend a linearly independent set into a basis for V.
- A finite basis for V must contain exactly n vectors.
- Any basis for V contains the same number of vectors. In other words, if a basis for V, say β, contains n amount of vectors then any other basis must contain the same amount.
- The number of vectors in a basis is the dimension of a vector space.

1.5.4 The Dimension of Subspaces

We can relate the dimension of a vector space V to a subspace of V.

Theorem 1.5.4 (Dimension of Subspaces)**.** Let W be a subspace of a finite-dimensional vector space V. Then W is a finite-dimensional and $\dim(W) \le \dim(V)$. Moreover, if $\dim(V) = \dim(W)$, then $V = W$.

Proof. Let $\dim(V) = n$ since V is finite-dimensional. Suppose $W = \{0\}$. Then W contains the empty set $\emptyset$ such that $\text{span}(\emptyset) = W$ and $\emptyset$ is linearly independent. Hence, $\emptyset$ is a subset of W that is a basis for $W = \{0\}$ that contains 0 vectors. Hence, W is finite-dimensional. Otherwise, W contains a nonzero vector $= x_1$ such that $W' = \{x_1\}$ is a linearly independent set. Continue this process of adding vectors $x_2, x_3, \dots, x_k$ into W' such that W' is linearly independent. Since no linearly independent subset of V cannot contain no more than n vectors, this process must stop at $k \le n$. Note adding a vector from $w \in W$ into W' such that $w \notin W'$ will make a linearly dependent set. By Theorem 1.4.2, we get that $w \in \text{span}(W')$. Hence, W' is a basis for W and that $\dim(W) = k \le n = \dim(V)$. Suppose $\dim(W) = n$, then there exists a basis β for W such that β contains exactly n amount of vectors. Since β is also linearly independent and contains the same amount of vectors as a basis for V, then we must have that β must also be a basis for V by Corollary 2 of the Replacement Theorem. Hence, $W = \text{span}(\beta) = V$. ■

Example 1.5.7.

- Let
$$W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 + a_3 + a_5 = 0, a_2 = a_4\}.$$
It can be easily shown that W is a subspace of F^5 having
$$\{(-1, 0, 1, 0, 0), (-1, 0, 0, 0, 1), (0, 1, 0, 1, 0)\}$$
as a basis. Hence, $\dim(W) = 3$
- The set of diagonal $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. A basis for W is the following set
$$\{E^{11}, E^{22}, \dots, E^{nn}\},$$

where E^{ij} is the matrix in which the only nonzero entry is a 1 in the ith row and jth column. Thus, we have $\dim(W) = n$.

Corollary. If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

Proof. Let β be a basis for the subspace W. Since $\beta \subseteq W \subseteq V$ is a linearly independent subset of V, we know that β can be extended as a basis for V by Corollary 2 of the Replacement Theorem. ■

1.5.5 Geometrical View of Subspaces in Euclidean Space

- The subspaces of R^2 include $\{0\}$ of dimension 0, $\mathbb{R}$ of dimension 1, and R^2 itself.
  - The set $\{0\}$ of zero dimension can be visualized as the origin in R^2.
  - The set $\mathbb{R}$ is the field itself and can be visualized as the line $y = cx$ with $c \in \mathbb{R}$ where $x \in \mathbb{R}^2$.
  - The set $\mathbb{R}^2$ is the whole cartesian space itself.
- The subspaces of $\mathbb{R}^3$ must have subspaces with dimensions $0, 1, 2$, or 3. The same logic can be followed from the three bullet points above.

Chapter 2

Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

Suppose we have a function T with domain V and codomain W denoted by $T : V \to W$.

Definition 2.1.1 (Linear Transformation). Let V and W be vector spaces (over F). We call a function $T : V \to W$ a **linear transformation from V to W**, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x + y) = T(x) + T(y)$ and

(b) $T(cx) = cT(x)$.

Remark. If $F = \mathbb{Q}$, then (a) implies (b) in the definition above. Otherwise, (a) and (b) are logically independent statements.

The following are a list of properties for linear functions:

Proposition 2.1.1. Let V and W be vector spaces (over F). If $T : V \to W$ is a **linear transformation from V to W**, then the following properties hold:

(a) If T is linear, then $T(0) = 0$.

(b) T is linear if and only if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$ and $c \in F$.

(c) If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$.

(d) T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} a_i T(x_i).$$

Proof. Let V and W be vector spaces such that $T : V \to W$.

(a) If T is linear, then $T(0) = 0$. Suppose T is linear, then $T(0 \cdot 0) = 0T(0) = 0$.

(b) We have T is linear if and only if $T(cx+y) = cT(x)+T(y)$ for all $x, y \in V$ and $c \in F$. Suppose T is linear. Let $x, y \in V$ and $c \in F$. Then

$$T(cx + y) = T(cx) + T(y) = cT(x) + T(y).$$

Conversely, if $c = 1$ then

$$T(x + y) = T(x) + T(y).$$

If $y = 0$, then

$$T(cx) = cT(x).$$

Hence, T is a linear transformation.

(c) If T is linear, then $T(x-y) = T(x) - T(y)$ for all $x, y \in V$. Let $x, y \in V$. Suppose T is linear, then

$$T(x-y) = T(x) + T(-y) = T(x) - T(y).$$

(d) T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} a_i T(x_i).$$

Suppose T is linear. Let $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$ such that

$$v = \sum_{i=1}^{n} a_i x_i.$$

Then observe that

$$T\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} T(a_i x_i) = \sum_{i=1}^{n} a_i T(x_i).$$

Conversely, for $i = 2$ and denote $a_i x_i = y_i$. Then

$$T\Big(\sum_{i=1}^{2} a_i x_i\Big) = T(a_1 x_1) + T(a_2 x_2) = T(y_1) + T(y_2)$$

and so property 1 is satisfied. If $i = 1$, then

$$T\Big(\sum_{i=1}^{1} a_i x_i\Big) = a_1 T(x_1)$$

and so property 2 is satisfied. Hence, T is a linear transformation.

■

Example 2.1.1. Define

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ by } T(a_1, a_2) = (2a_1 + a_2, a_1).$$

Show that T is linear. Let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, where $x = (a_1, a_2)$ and $y = (b_1, b_2)$. Then

$$\begin{aligned} cx + y &= c(a_1, a_2) + (b_1, b_2) \\ &= (ca_1, ca_2) + (b_1, b_2) \\ &= (ca_1 + b_1, ca_2 + b_2). \end{aligned}$$

So, we have

$$\begin{aligned} T(cx + y) &= T(ca_1 + b_1, ca_2 + b_2) \\ &= (2(ca_1 + b_1) + ca_2 + b_2, ca_1 + b_1) \\ &= ((2ca_1 + ca_2) + (2b_1 + b_2), ca_1 + b_1) \\ &= (2ca_1 + ca_2, ca_1) + (2b_1 + b_2, b_1) \\ &= c(2a_1 + a_2, a_1) + (2b_1 + b_2, b_1) \\ &= cT(a_1, a_2) + T(b_1, b_2) \\ &= cT(x) + T(y). \end{aligned}$$

Hence, we have that $T(cx + y) = cT(x) + T(y)$ so $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear.

2.1.1 Examples of Linear Transformations

The most common linear transformations come from geometry:

- Rotations: $T_\theta(a_1, a_2) = (a_1, a_2)$ by performing a counter-clockwise rotation by an angle θ if $(a_1, a_2) \neq (0, 0)$. This is called **rotation by** θ and $T_\theta(0, 0) = (0, 0)$ otherwise.
- Reflections about the $x-$axis: $T(a_1, a_2) = (a_1, -a_2)$.
- Projections on the $x-$axis: $T(a_1, a_2) = (a_1, 0)$.

Example 2.1.2 (Rotations). Define the rotation transformation above by $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_\theta(a_1, a_2)$ where it is described as above. Our goal in this example is to define an explicit formula of this transformation. Let us fix a nonzero vector $(a_1, a_2) \in \mathbb{R}^2$. Let α be the angle such that (a_1, a_2) makes with the positive $x-$axis, and let $r = \sqrt{a_1^2 + a_2^2}$. Then using some trigonometry, we get that $a_1 = r\cos(\alpha)$ and $a_2 = r\sin(\alpha)$. Note that $T_\theta(a_1, a_2)$ has length r and makes an angle $\alpha + \theta$ with the positive $x-$axis. Using some trigonometric identities, we get that

$$\begin{aligned} T_\theta(a_1, a_2) &= (r\cos(\alpha + \theta), r\sin(\alpha + \theta)) \\ &= (r\cos(\alpha)\cos(\theta) - r\sin(\alpha)\sin(\theta), r\cos(\alpha)\sin(\theta) + r\sin(\alpha)\cos(\theta)) \\ &= (a_1\cos(\theta) - a_2\sin(\theta), a_1\sin(\theta) + a_2\cos(\theta)). \end{aligned}$$

Observe that this formula holds for $(a_1, a_2) = (0, 0)$. One can show that T_θ is linear.

Example 2.1.3 (Transpose). Define $T : M_{m\times n}(F) \to M_{n\times m}(F)$ by $T(A) = A^t$, where A^t is the transpose of A, defined in Section 1.3.

Example 2.1.4 (Derivatives of Polynomials). Define $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. We can show that T is linear. Let $g(x), h(x) \in P_n(\mathbb{R})$ and $a \in \mathbb{R}$.Assuming that the differentiation rules holds for any T, we have

$$\begin{aligned} T(ag(x) + h(x)) &= (ag(x) + h(x))' \\ &= ag'(x) + h'(x) \\ &= aT(g(x)) + T(h(x)). \end{aligned}$$

By the second property of linearity , we have that T is linear.

Example 2.1.5 (Integrals). Let $V = C(\mathbb{R})$ be the vector space of continuous real-valued functions on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, $a < b$. Define $T : V \to \mathbb{R}$ by

$$T(f) = \int_a^b f(t)\ dt$$

for all $f \in V$. We know that T is a linear transformation because the definite integral of a linear combination of functions is the same as the linear combination of the definite integrals of the functions; that is,

$$\int_a^b \sum_{i=1}^n \gamma_i f(t_i)\ dt = \sum_{i=1}^n \gamma_i \int_a^b f(t_i)\ dt = \sum_{i=1}^n \gamma_i T(f_i)$$

for scalars γ_i where $1 \leq i \leq n$ and

$$T(f_i) = \int_a^b f(t_i)\ dt.$$

2.1.2 Identity and Zero Transformations

Remark. Two very important examples of linear transformations are the **identity and zero transformations**.

Definition 2.1.2 (Identity Transformation). The **identity transformation** is denoted by $I_V : V \to V$ by $I_V(x) = x$ for all $x \in V$. From now on, we will denote this transformation as I instead of I_V.

Definition 2.1.3 (Zero Transformation). The **zero transformation** is denoted by $T_0 : V \to W$ by $T_0(x) = 0$ for all $x \in V$.

2.1.3 Range and Null Spaces

The *Range* and *Null* spaces give us important insights on the intrinsic properties of a linear transformation.

Definition 2.1.4 (Null Space). Let V and W be vector spaces, and let $T : V \to W$ be linear. The **null space** (or **kernel**), denoted by $N(T)$ of T, is the set of all vectors $x \in V$ such that $T(x) = 0$; that is, $N(T) = \{x \in V : T(x) = 0\}$.

Definition 2.1.5 (Range). Let V and W be vector spaces, and let $T : V \to W$ be linear. The **range** (or **image**) $R(T)$ of T is the subset of W consisting of all images (under T) of vectors in V; that is, $R(T) = \{T(x) : x \in V\}$.

Example 2.1.6. Let V and W be vector spaces, and let $I : V \to V$ and $T_0 : V \to W$ be the identity and zero transformations, respectively. Then $N(I) = \{0\}$, $R(I) = V$, $N(T_0) = V$, and $R(T_0) = \{0\}$.

Example 2.1.7. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

One can verify that

$$N(T) = \{(a, a, 0) : a \in \mathbb{R}\} \text{ and } R(T) = \mathbb{R}^2.$$

Theorem 2.1.1 (The Null and Range are Both Subspaces). Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Proof. Let $T : V \to W$ be linear. First, we show that $N(T)$ is a subspace of V.

(a) Note that $0_V \in V$ implies that $T(0_V) = 0_W$ since T is linear (property (1) of linearity). Hence, $0_V \in N(T)$.

(b) Let $x, y \in N(T)$. By definition, of the null space we have $T(x) = 0_W$ and $T(y) = 0_W$. Since T is linear, we get that

$$T(x + y) = T(x) + T(y) = 0_W + 0_W = 0_W \in N(T).$$

Hence, $N(T)$ is closed under addition.

(c) Let $x \in N(T)$ and let $c \in F$. Then

$$T(cx) = cT(x) = c0_V = 0_W \in N(T).$$

Hence, Theorem 1.3 tells us that $N(T)$ is a subspace of V. Now, we will show that $R(T)$ is a subspace of W.

(a) Note that $0_V \in V$ implies that $T(0_V) = 0_W \in R(T)$ by property (1) of linearity. So, $0_W \in R(T)$.

(b) Now let $x, y \in R(T)$. Then there exists elements $z, w \in W$ such that $T(x) = z$ and $T(y) = w$. Then by linearity, we have $T(x + y) = T(x) + T(y) = z + y \in W$. Hence, we must have $x + y \in R(T)$ and thus $R(T)$ is closed under addition.

(c) Let $x \in R(T)$ and $c \in F$. Then we have $T(cx) = cT(x) \in W$ since $T(x) \in W$. Hence, $cx \in R(T)$.

Thus, Theorem 1.3 also tells us that $R(T)$ is a subspace of W. ■

The next theorem allows us to find a spanning set for the range of a linear transformation.

Theorem 2.1.2 (Spanning Set for a Linear Transformation)**.** Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, v_2, \dots v_n\}$ is a basis for V, then

$$R(T) = \operatorname{span}(T(\beta)) = \operatorname{span}(\{T(v_1), T(v_2), \dots, T(v_n)\}).$$

Proof. Let $T : V \to W$ be linear where V and W are vector spaces. To show that $T(\beta)$ generates $R(T)$; we need to show that

$$\operatorname{span}(T(\beta)) \subseteq R(T) \text{ and } R(T) \subseteq \operatorname{span}(T(\beta)).$$

Observe that $T(v_i) \in R(T)$ for all i. Hence, $T(\beta) \subseteq R(T)$. By theorem 1.5, we know that $R(T)$ also contains the span of $T(\beta)$. Hence, $\operatorname{span}(T(\beta)) \subseteq R(T)$.

Let $w \in R(T)$. Then for some $v \in W$, we have $T(v) = w$. Since β is a basis for V, we choose scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

Since T is linear, we have that

$$w = T(v) = T\Big(\sum_{i=1}^{n} a_i v_i\Big) = \sum_{i=1}^{n} a_i T(v_i).$$

This tells us that $v \in \operatorname{span}(T(\beta))$. Hence, $T(\beta)$ generates $R(T)$. ■

Example 2.1.8. Define the linear transformation $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Note that

$$\beta = \{1, x, x^2\} \text{ is a basis for } P_2(\mathbb{R}) \text{ and } T(\beta) = \{T(1), T(x), T(x^2)\}.$$

Then observe that

$$\begin{aligned} R(T) &= \operatorname{span}(T(\beta)) \\ &= \operatorname{span}\left(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right\}\right). \end{aligned}$$

This tells us that $R(T)$ contains the basis found in the second equality above. Hence, we must have $\dim(R(T)) = 2$.

Example 2.1.9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

One can show that

$$N(T) = \{(a,a,0)\} : a \in \mathbb{R} \text{ and } R(T) = \mathbb{R}^2.$$

Just like how the 'size' of a given subspace is denoted by its dimension, we can also determine the size of a null spaces and ranges. However, we will attach some special names associated with these sets.

Definition 2.1.6 (Nullity)**.** Let V and W be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ is finite-dimensional, then we define **nullity** of T by $\text{nullity}(T)$ to be the dimension of $N(T)$.

Definition 2.1.7 (Rank)**.** Let V and W be vector spaces, and let $T : V \to W$. If $R(T)$ is finite-dimensional, then we denote the **rank** of T by $\text{rank}(T)$ to be the dimension of $R(T)$.

From these definitions, we can intuit the following relationships between Nullity and Rank of a linear transformation:

- The larger the nullity, the smaller the rank of a linear transformation.
- the larger the rank, the smaller the nullity.

This relationship between the two spaces is encompassed in the next theorem.

2.1.4 Dimension Theorem

Theorem 2.1.3 (Dimension Theorem)**.** Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof. Suppose $\dim(V) = n$ and $\dim(N(T)) = k$, and $\{u_1, u_2, \ldots, u_k\}$ is a basis for $N(T)$. By corollary to Theorem 1.11 β can be extended to be a basis for V. Denote this basis as

$$\{u_1, u_2, \ldots, u_n\}.$$

We claim that that $S = \{T(u_{k+1}), T(u_{k+2}), \ldots, T(u_n)\}$ is a basis for $R(T)$. First, we show that S generates $R(T)$. Using The Theorem 2.1 and the fact that $T(u_i) = 0$ for $1 \leq i \leq k$, we get that

$$\begin{aligned} R(T) &= \text{span}(\{T(u_1), T(u_2), \ldots, T(u_n)\}) \\ &= \text{span}(\{T(u_{k+1}), T(u_{k+2}), \ldots, T(u_n)\}) \\ &= \text{span}(T(S)). \end{aligned}$$

Hence, S generates $R(T)$.

Now, we want to show that S is linearly independent. Then choose scalars $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n$ such that

$$\sum_{i=k+1}^{n} \alpha_i T(u_i) = 0. \tag{1}$$

Since T is linear, we can re-write (1) into the following form

$$T\Big(\sum_{i=k+1}^{n} \alpha_i u_i\Big) = 0.$$

Hence, we find that

$$\sum_{i=k+1}^{n} \alpha_i u_i \in N(T). \tag{2}$$

Since $N(T)$ contains $\{u_1, u_2, \dots, u_k\}$ as a basis, we can express (2) as a linear combination of vectors in this set. Hence, we have

$$\sum_{i=k+1}^{n} \alpha_i u_i = \sum_{i=1}^{k} \beta_i u_i \tag{3}$$

for some scalars $\beta_1, \beta_2, \dots, \beta_k$. Now, we have

$$\sum_{i=k+1}^{n} \alpha_i u_i - \sum_{i=1}^{k} \beta_i u_i = 0.$$

Since $\{u_1, u_2, \dots, u_n\}$ is linearly independent, we find that both α_i's and β_i's are all zero. Hence, S is also linearly independent. Thus, S is a basis for $R(T)$ and that $\operatorname{rank}(T) = n - k$ and so we get our desired result

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T).$$

■

Applying the dimension theorem to example 9 allows us to conclude that $\operatorname{nullity}(T) + 2 = 3$ implies $\operatorname{nullity}(T) = 1$.

Before we move on, let us recall two key definitions needed for the next topic.

Definition 2.1.8 (Injective Functions). Let $f : A \to B$ and $x, y \in A$. We call f **one-to-one** if $f(x) = f(y)$ implies $x = y$ or, equivalently, if $x \neq y$ implies $f(x) \neq f(y)$ (this latter part is the contrapositive of the definition).

Definition 2.1.9 (Surjective Functions). Let $f : A \to B$. We call f **onto** if $f(A) = B$; that is, for any $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

As we will see, these two definitions will give insights into the nullity and rank of linear transformations.

Theorem 2.1.4 (Injectivity $\Leftrightarrow$ Null Space is $\{0\}$). Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is **one-to-one** if and only if $N(T) = \{0\}$.

Proof. ($\Rightarrow$) Suppose T is injective. Let $x \in V$ be arbitrary. Then $T(x) = T(0_V)$ implies $x = 0_V$. This tells us that $N(T) = \{0\}$. ($\Leftarrow$) Conversely, let $N(T) = \{0\}$. Let $x, y \in V$ be arbitrary and assume $T(x) = T(y)$. By using linearity, we have

$$\begin{aligned} T(x) = T(y) &\Leftrightarrow T(x) - T(y) = 0_W \\ &\Leftrightarrow T(x - y) = 0_W \end{aligned}$$

Since $x - y \in N(T)$ and $N(T) = \{0\}$, we have $x - y = 0$ if and only if $x = y$. Hence, T is an injective transformation. ■

Referring back to example 9, we find that $\operatorname{nullity}(T) = 1$implies that example 9 is not injective.

Theorem 2.1.5 (Equal Finite Dimensions Between Vector Spaces). Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent.

(a) T is injective.

(b) T is surjective.

(c) $\operatorname{rank}(T) = \dim(V)$.

Proof. Using the dimension theorem, we have

$$\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V).$$

By theorem 2.5, We know that T is injective if and only if $N(T) = \{0\}$. By definition, we have nullity$(T) = 0$. By the dimension theorem, this is true if and only if

$$\text{rank}(T) = \dim(V) \Leftrightarrow \text{rank}(T) = \dim(W)$$

where V and W have equal dimensions. This is true if and only if $\dim(R(T)) = \dim(W)$. This is true if and only if $R(T) = W$ by theorem 1.11. By definition, we know that T is surjective. ■

Remark. If V is not finite-dimensional then we find that neither (a) nor (b) follows from each other.

Example 2.1.10. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\ dt.$$

Observe that

$$R(T) = \text{span}(T(1), T(x), T(x^2)) = \text{span}(\{3x, 2 + \tfrac{3}{2}x^2, 4x + x^3\}).$$

One can show that the set $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$ is linearly independent and that rank$(T) = 3$. Since rank$(T) \neq \dim(P_3(\mathbb{R}))$, we have that T is not surjective. Since $\dim(P_2(\mathbb{R})) = 3$, we know that

$$\text{rank}(T) = \dim(P_2(\mathbb{R}))$$

by the Dimension Theorem. Thus, nullity$(T) = \{0\}$ and hence T is injective.

Example 2.1.11. Let $T : F^2 \to F^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1).$$

One can show that T is injective so that $N(T) = \{0\}$. The dimension theorem tells us that T must also be onto.

Example 2.1.12. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2).$$

One can easily show that T is linear and one-to-one. We find that $S = \{2 - x + 3x^2, x + x^2, 1 - 2x^2\}$ is linearly independent in $P_2(\mathbb{R})$ since

$$T(S) = \{(2, -1, 3), (0, 1, 1), (1, 0, -2)\}$$

is linearly independent in $\mathbb{R}^3$.

Theorem 2.1.6 (Unique Linear Transformations)**.** Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n \in W$, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $1 \leq i \leq n$.

Proof. Let $x \in V$. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, we have

$$\sum_{i=1}^{n} \gamma_i v_i$$

for some scalars $\gamma_1, \gamma_2, \ldots, \gamma_n$. Define the map $T : V \to W$ by

$$T(x) = \sum_{i=1}^{n} \gamma_i w_i$$

for $w_i \in W$ and $x \in V$.

First, we show that T is linear. Let $x, y \in V$. Then

$$x = \sum_{i=1}^{n} \gamma_i v_i$$

and

$$y = \sum_{i=1}^{n} \delta_i v_i$$

for some scalars γ_i and δ_i for all $1 \leq i \leq n$. Then by definition of T, we get that

$$T(x) = \sum_{i=1}^{n} \gamma_i w_i$$

and

$$T(y) = \sum_{i=1}^{n} \delta_i w_i.$$

By definition of T, we get that

$$T(x+y) = \sum_{i=1}^{n} (\gamma_i + \delta_i) w_i = \sum_{i=1}^{n} \gamma_i w_i + \sum_{i=1}^{n} \delta_i w_i = T(x) + T(y).$$

Hence, the first property of linearity holds. For the second property, let $c \in F$ and observe that

$$T(cx) = \sum_{i=1}^{n} \gamma_i (c w_i) = c \sum_{i=1}^{n} \gamma_i w_i = cT(x).$$

Hence, the second property of linearity holds. We conclude that T is linear and that $T(u_i) = w_i$ for all $1 \leq i \leq n$.

Now suppose there exists another linear map $U : V \to W$ such that $U(v_i) = w_i$ for all $1 \leq i \leq n$ defined by.

$$U(x) = \sum_{i=1}^{n} \gamma_i U(v_i).$$

Then observe that

$$U(x) = \sum_{i=1}^{n} \gamma_i U(v_i) = \sum_{i=1}^{n} \gamma_i w_i = T(x).$$

Thus, $U = T$ for all $x \in V$ and so, we conclude that T is a unique linear map. ■

Corollary. Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \ldots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for all $1 \leq i \leq n$, then $U = T$.

Example 2.1.13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$$

and suppose that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Let $\{(1,2), (1,1)\}$ be a basis for $\mathbb{R}^2$. If $U(1,2) = (3,3)$ and $U(1,1) = (1,3)$, then we find that $U(x_i) = T(x_i)$ $i = 1, 2$. Thus $U = T$ from the corollary to theorem 2.6.