Week 1: Lecture Notes

Lance Remigio

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1 Lecture 1

1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial $x^2 + 1$ does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\, dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

1.2 What is the goal?

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have
$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$
Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:
$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$
- Multiplication:
$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. **To do.** ■

Lemma (Existence of a Square Root)**.** Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. **To do.** ■

2 Lecture 2

3 Topics

- Discuss the complex plane $\mathbb{C}$ as a working model for complex numbers.
- Discuss the Euclidean Topology on $\mathbb{C}$.
- Discuss polar representation of a complex number.

4 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

Lemma. Let F be a field containing $\mathbb{R}$ and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let
$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ is a **subfield** of F.

(ii) Let F be another field containing $\mathbb{R}$ and containing a solution i' of $x^2 + 1 = 0$ and
$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ and $\mathbb{C}'$ are isomorphic as fields.

The second part of this lemma is simply saying that $\mathbb{C}$ is a unique subfield of F.

4.1 A model for $\mathbb{C}$

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication $\cdot$ as
$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$
and
$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$
respectively.

Lemma. The complex numbers $\mathbb{C}$ have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \to \mathbb{C}$ by $x \to (x, 0)$. Then $\mathcal{J}$ is a subfield and forms an isomorphism between $\mathbb{R}$ and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

(i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.

(ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homorphism.

(iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that $\mathcal{J}$ is a bijective map. ■

Definition (Constructing a solution for $x^2 + 1 = 0$)**.** Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. **To do.** ■

Definition (Real and Imaginary part of Complex Number)**.** Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the $x-$axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(iii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iv) $\Re(z) = \frac{1}{2}(z + \overline{z})$.

(v) $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $z\overline{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. **To do.** ■

Definition (Modulus of a Complex Number)**.** Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

Proposition (More Properties of Complex Numbers)**.** Given $z, w \in \mathbb{C}$, we have the following properties:

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|zw| = |z||w|$.

(iii) $|\Re(z)| \leq |z|$.

(iv) $|\Im(z)| \leq |z|$.

(v) $|z + w| \leq |z| + |w|$.

(vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

5 Viewing $\mathbb{C}$ as a Metric Space

In $\mathbb{C}$, the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space)**.** For $z, w \in \mathbb{C}$, we all $\mathbb{C}$ a metric space if it satisfies the following properties:

(i) $d(z, w) = d(w, z)$.

(ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have $\mathbb{C}$ is a metric space endowed with the metric $d(z, w) = |z - w|$.