Linear Algebra Exercises

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Chapter 1

Vector Spaces

1.1 Vector Spaces

Exercise 1.2.1

Label the following statements as true or false.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

Exercise 1.2.7

Let $S = \{0, 1\}$ and $F = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R}$, show that $f = g$ and where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

Proof. To show that $f = g$, we have to show that for each $s \in S$ that $f(s) = g(s)$. Since $S = \{0, 1\}$, we can just evaluate both f and g for elements in S. Note that

$$f(0) = 2(0) + 1 = 1$$

and likewise,

$$g(0) = 1 + 4(0) - 2(0)^2.$$

Hence, $f(0) = g(0)$. Now let us evaluate both functions f and g at $s = 1$. Hence, we have

$$f(1) = 2(1) + 1 = 3$$

and

$$g(1) = 1 + 4(1) - 2(1)^2 = 3.$$

Thus, we must have $f(s) = g(s)$ for all $s \in S$.

Now, we need to show that $f + g = h$. Like we did above, we have to show that this is the case for all $s \in S$. Note that

$$(f + g)(s) = f(s) + g(s).$$

Hence, we have

$$f(s) + g(s) = 2 + 6s - 2s^2.$$

Evaluating at $s = 0$, we have

$$f(0) + g(0) = 2.$$

and likewise,

$$h(0) = 5^0 + 1 = 2.$$

Hence, $(f+g)(0) = h(0)$. Now let us evaluate $f+g$ at $s=1$

$$f(1) + g(1) = 2 + 6(1) - 2(1)^2 = 6$$

and likewise, we have

$$h(1) = 5^1 + 1 = 6.$$

Hence, we have $(f+g)(1) = h(1)$. Thus, we have that $f+g = h$ for all $s \in S$.

■

Exercise 1.2.8

In any vector space V, we have

$$(a+b)(x+y) = ax + ay + bx + by$$

for any $x, y \in V$ and any $a, b \in F$.

Proof. Observe the following set equalities:

$$\begin{aligned} (a+b)(x+y) &= a(x+y) + b(x+y) && \text{(VS 8)} \\ &= ax + ay + bx + by. && \text{(VS 7)} \end{aligned}$$

Hence, we have that

$$(a+b)(x+y) = ax + ay + bx + by$$

for any $x, y \in V$ and any $a, b \in F$. ■

Exercise 1.2.10

Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Let V denote the set of all differentiable real-valued functions defined on the real line. We need to show that V is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined in Example 3.

(VS 1) Let $f, g \in V$. We need to show that $f + g = g + f$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Since $f(x)$ and $g(x)$ are also real numbers, we have

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x).$$

Hence, we have $f + g = g + f$.

(VS 2) Let $f, g, h \in V$. We need to show that $f + (g+h) = (f+g) + h$. Let $x \in \mathbb{R}$. Since $f(x), g(x), h(x) \in \mathbb{R}$, we can see that

$$\begin{aligned} (f + (g+h))(x) &= f(x) + (g+h)(x) \\ &= f(x) + g(x) + h(x) \\ &= (f+g)(x) + h(x) \\ &= ((f+g)+h)(x). \end{aligned}$$

Hence, we have $f + (g+h) = (f+g) + h$.

(VS 3) Observe that $f_0(x) = 0$ for all $x \in \mathbb{R}$ is also a real-valued function that is differentiable. We need to show that $f + f_0 = f$. Hence, we have

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x).$$

Hence, $f + f_0 = f$ for all $x \in \mathbb{R}$.

(VS 4) Take $c = -1$ and perform a scalar operation with a $f \in V$. Observe that $(-1 \cdot f)(x) = (-1)f(x) = -f(x)$. Denote $g = -f$. Since additive inverses exists in $\mathbb{R}$, we have

$$\begin{aligned}(f - g)(x) &= (f - f)(x) \\ &= f(x) - f(x) \\ &= 0 \\ &= f_0(x).\end{aligned}$$

Hence, $f - g = f_0$ for all $x \in \mathbb{R}$.

(VS 5) Let $f \in V$ and let $x \in \mathbb{R}$. We need to show that $1 \cdot f = f$. Since $f(x) \in \mathbb{R}$, we can see that multiplicative identities in $\mathbb{R}$ are also preserved in V; that is, we have

$$(1f)(x) = 1 \cdot f(x) = f(x).$$

Hence, we have $1f = f$ for all $x \in \mathbb{R}$.

(VS 6) Let $a, b \in \mathbb{R}$ and $f \in V$. Let $x \in \mathbb{R}$ be arbitrary. Then observe that

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a(bf)(x).$$

Hence, we have $(ab)f = a(bf)$ for all $x \in \mathbb{R}$.

(VS 7) Let $a \in \mathbb{R}$ and let $f, g \in V$. We need to show that $a(f + g) = af + ag$. Let $x \in \mathbb{R}$. Observe that $f(x), g(x) \in \mathbb{R}$ imply

$$\begin{aligned}(a(f + g))(x) &= a(f + g)(x) \\ &= a(f(x) + g(x)) \\ &= af(x) + ag(x) \\ &= (af)(x) + (ag)(x).\end{aligned}$$

Hence, we conclude that $a(f + g) = af + ag$ for all $x \in \mathbb{R}$.

(VS 8) Let $a, b \in \mathbb{R}$ and let $f \in V$. Let $x \in \mathbb{R}$. Then we have

$$\begin{aligned}((a + b)f)(x) &= (a + b)f(x) \\ &= af(x) + bf(x) \\ &= (af)(x) + (bf)(x).\end{aligned}$$

Hence, we have $(a + b)f = af + bf$ for all $x \in \mathbb{R}$.

Since all the properties of a vector space have been satisfied, we conclude that V is a vector space. ∎

Exercise 1.2.11

Let $V = \{0\}$ consist of a single vector O and define $O + O = O$ and $cO = O$ for each scalar $c \in F$. Prove that V is a vector space over F.

Proof. Let $x, y \in O$. Since $V = \{0\}$, we know that x and y are both the 0 vector. By using the addition defined on V, we can see that

$$x + y = O + O = y + x.$$

Note that $x + y \in V$ implies that $x + y = O + O$ and likewise $y + z = O + O$ for every $x, y, z \in V$. Hence, we have that

$$x + (y + z) = O + (O + O) = (O + O) + O = (x + y) + z.$$

Since V consists of only the zero vector O, we know that (VS 3) and (VS 4) are satisfied. By the same reasoning, (VS 5) is satisfied because

$$1 \cdot O = O.$$

Let $a, b \in F$. Then (VS 6) and (VS 7) are satisfied because

$$(ab)O = 0 = a \cdot 0 = a(bO) \text{ and } a(O+O) = 0 = O + O = aO + aO \text{ respectively.}$$

Let $a, b \in F$ again. Then we have

$$\begin{aligned}(a+b)O &= 0 \\ &= O + O \\ &= aO + bO.\end{aligned}$$

Hence, (VS 8) is satisfied. ■

Exercise 1.2.12

A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for all $t \in \mathbb{R}$. Prove that the set of even functions defined on $\mathbb{R}$ with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof. First, let us show that for every $t \in \mathbb{R}$ that $(f+g)(t)$ is also an even function for every pair of even functions f, g and likewise $(cf)(t)$ is an even function for every $c \in \mathbb{R}$. Observe that

$$\begin{aligned}(f+g)(-t) &= f(-t) + g(-t) \\ &= f(t) + g(t) \\ &= (f+g)(t).\end{aligned}$$

Hence, the function $f + g$ is also even. Now observe that

$$\begin{aligned}(cf)(-t) &= cf(-t) \\ &= cf(t) \\ &= (cf)(t).\end{aligned}$$

Hence, cf is also even function.

Now we can show that the set of even functions is a vector space.

(VS 1) Let f, g be a pair of real-valued even functions. Let $t \in \mathbb{R}$ be arbitrary. We need to show that $f + g = g + f$. Since $f(t)$ and $g(t)$ are real numbers, observe that

$$(f+g)(t) = f(t) + g(t) = g(t) + f(t) = (g+f)(t).$$

Hence, we have that $f + g = g + f$.

(VS 2) Let f, g, h be even functions and let $t \in \mathbb{R}$ be arbitrary. We need to show that $f + (g + h) = (f + g) + h$. By the same reasoning we used to prove (VS 1), observe that

$$\begin{aligned}f(t) + ((g+h)(t)) &= f(t) + (g(t) + h(t)) \\ &= (f(t) + g(t)) + h(t) \\ &= ((f+g)(t)) + h(t)\end{aligned}$$

Hence, we have that $f + (g + h) + (f + g) + h$.

(VS 3) Let f be an even function. Note that the zero function is an even function. Denote the zero function as f_0. For every $t \in \mathbb{R}$, we have $f_0(x) = O$. We need to show that $f + f_0 = f$. Since $f_0(x)$ is just a real number, we have

$$(f + f_0)(t) = f(t) + f_0(t) = f(t) + O = f(t).$$

(VS 4) Let f be an even function and let f_0 be the zero function defined above. We need to show that there exists a even function g such that $f+g=f_0$. Let $t \in \mathbb{R}$ be arbitrary. Note that

$$(f+g)(t) = f(t) + g(t)$$

Since $f(t)$ and $g(t)$ are real numbers and there exists an additive identity in the real numbers, we have that $f(t)+g(t) = f_0(t)$. Hence, (VS 5) is satisfied.

(VS 5) Let $x \in \mathbb{R}$ and f an even function. Then we immediately have $(1 \cdot f)(t) = 1f(t) = f(t)$.

(VS 6) Let $a, b \in \mathbb{R}$. We need to show that $(ab)f = a(bf)$. Hence, let $t \in \mathbb{R}$. Then we have

$$(abf)(t) = a(bf)(t).$$

(VS 7) Let $a \in \mathbb{R}$ and let f, g be even functions. Let $t \in \mathbb{R}$. We need to show that $a(f+g) = af + ag$. Then we have

$$\begin{aligned} a(f+g)(t) &= a[f(t)+g(t)] \\ &= af(t) + ag(t). \end{aligned}$$

Hence, (VS 7) is satisfied.

(VS 8) Now let $a, b \in \mathbb{R}$ and let f be an even function. We need to show that $(a+b)f = af + bf$. Let $t \in \mathbb{R}$ be arbitrary. Observe that

$$\begin{aligned} (a+b)f(t) &= af(t) + bf(t) \\ &= (af)(t) + (bf)(t) \end{aligned}$$

Hence, (VS 8) is satisfied.

■

Exercise 1.2.13

Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, a_2 b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2).$$

Is V is a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We have that V is not a vector space of $\mathbb{R}$. To see why, let $(2,1), (4,2) \in V$ where $x = (2,1)$ and $y = (4,2)$. We will show that (VS 1) does not hold; that is, $x + y \neq y + x$. Hence, observe that

$$(2,1) + (4,2) = (2+1, 2) = (3,2)$$

and

$$(4,2) + (2,1) = (4+2, 2) = (6,2)$$

Hence, we have $x + y \neq y + x$ and so V is **NOT** a vector space. ■

Exercise 1.2.14

Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in C \text{ for } i = 1, 2, \ldots, n)\}$; so V is a vector space over C by Example 1. Is V is a vector space over the field of real numbers with the operations of coordinate-wise addition and multiplication?

Proof. Yes, V where
$$V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$$
is a vector space of $\mathbb{R}$.

(VS 1) Let $x, y \in V$ such that $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$. Since addition is entry-wise in V and each entry in both x and y are elements of $\mathbb{R}$ (where $\mathbb{R}$ is a field), we have $a_i + b_i = b_i + a_i$ for all $i = 1, 2, \ldots, n$. Hence, $x + y = y + x$.

(VS 2) Let $x, y, z \in V$ with x and y as defined as before where z contains entries c_i for all $i = 1, 2, \ldots n$. We can see that the entries of x, y, z are elements of $\mathbb{R}$ so associativity is preserved; that is, $a_i + (b_i + c_i) = (a_i + b_i) + c_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $x + (y + z) = (x + y) + z$.

(VS 3) Since $\mathbb{R}$ contains the zero element 0 and V is the set of n-tuples, there exists an element denoted by O such that this element consisting of entries that only have the zero element 0; that is, $O = (0, 0, \ldots, 0)$. Take $x \in V$. Hence, we have $a_i + 0 = a_i$ for every $i = 1, 2, \ldots, n$. Thus, we must have $x + O = x$.

(VS 4) Let $x \in V$ be arbitrary as defined before. Since every entry in x is an element of $\mathbb{R}$; that is, every $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we know that every entry contains an element c_i such that $a_i + c_i = 0$ for every $i = 1, 2, \ldots, n$. Denote $x' = (c_1, c_2, \ldots, c_n)$. Hence, we have $x + x' = O$.

(VS 5) Let $x \in V$. Every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $1 \cdot a_i = a_i$ which holds for all i. Denote this identity element as I with entries consisting only of 1. Hence, we have $I \cdot x = x$.

(VS 6) Let $e, r \in \mathbb{R}$ and let $x \in V$. For every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $(er)a_i = e(ra_i)$ for all $i = 1, 2, \ldots, n$. By using the operations of scalar multiplication for n-tuples, this tells us that $(er)x = e(rx)$. Hence, (VS 6) is satisfied.

(VS 7) Let $e \in \mathbb{R}$ and let $x, y \in V$. We need to show that $e(x + y) = ex + ey$. Note that $a_i, b_i \in \mathbb{R}$ implies that $e(a_i + b_i) = ea_i + eb_i$. Hence, we have $e(x + y) = ex + ey$.

(VS 8) Let $e, r \in \mathbb{R}$ and $x \in V$ as defined before. We need to show that $(e + r)x = ex + rx$. Since $a_i \in \mathbb{R}$ with $e, r \in \mathbb{R}$, we are guaranteed to have $(e + r)a_i = ea_i + ra_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $(e + r)x = ex + rx$.

■

Exercise 1.2.16

Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over $\mathbb{R}$ by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

Proof. (VS 1) Let $A, B \in V$. Since A, B consist of elements $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $A_{ij} + B_{ij} = B_{ij} + A_{ij}$. Hence, $A + B = B + A$.

(VS 2) Let $A, B, C \in V$ with A, B defined as before and C containing real entries C_{ij}. With the same reasoning used to prove (VS 1), we know that $A_{ij} + (B_{ij} + C_{ij}) = (A_{ij} + B_{ij}) + C_{ij}$. Hence, we have $A + (B + C) = (A + B) + C$.

(VS 3) Let $A \in V$ once again. Since the entries of A imply that there exists an element O such that $A_{ij} + 0 = A_{ij}$, we know that $A + O = A$ where O is the **zero matrix** of V.

(VS 4) Since the real entries of A also consists of an element A'_{ij} such that $A_{ij} + A'_{ij} = 0$, this implies that $A + A' = O$ where A' is the additive inverse matrix of V.

(VS 5) Let $x \in V$ as defined as before. Every entry of A, $A_{ij} \in \mathbb{R}$, has the following property: $1 \cdot A_{ij} = A_{ij}$ for all for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. The matrix whose entries consists of only one we can define as the **identity matrix** denoted by I where $I_{ij} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, we have $A \cdot I = A$.

(VS 6) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. We need to show that $(rt)A = r(tA)$. Since $A_{ij} \in \mathbb{R}$, entry-wise scalar multiplication implies that $(rt)A_{ij} = r(tA_{ij})$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, we must have $(rt)A = r(tA)$.

(VS 7) Let $r \in \mathbb{Q}$ and $A, B \in V$ as defined before. Since $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $r(A_{ij} + B_{ij}) = rA_{ij} + rB_{ij}$. Hence, we have $r(A + B) = rA + rB$.

(VS 8) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. Since $A_{ij} \in \mathbb{R}$ for all i, j, we must have $(r + t)A_{ij} = rA_{ij} + tA_{ij}$. Hence, we have $(r + t)A = rA + tA$.

Hence, V is a vector space over $\mathbb{Q}$. ■

Exercise 1.2.17

Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinate-wise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (ca_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

Proof. We claim that V is not a vector space over F because V fails to satisfy (VS 5). To see why, let $(1, 2) \in V$. Using (VS 5), we have

$$1 \cdot (1, 2) = (1, 0) \neq (1, 2).$$

Hence, V cannot be a vector space. ■

Exercise 1.2.18

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $F = \mathbb{R}$ with these operations? Justify your answer?

Proof. We claim that V is not a vector space over $\mathbb{R}$ and we will use (VS 1) to show this. Let $x, y \in V$ be defined by $x = (1, 2)$ and $y = (3, 4)$. Observe that

$$x + y = (1, 2) + (3, 4) = (7, 14)$$

and

$$y + x = (3, 4) + (1, 2) = (5, 10).$$

Clearly, we have $x + y = (7, 14) \neq (5, 10) = y + x$ and so (VS 1) does not hold. ■

Exercise 1.2.19

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinate-wise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We claim that V is not a vector space over $\mathbb{R}$. To see why, consider (VS 8). If we let $(0, 1) \in V$ with $c = 2 + 1 = 3$. Observe that

$$(2 + 1)(0, 1) = \left(0, \frac{1}{2 + 1}\right) = \left(0, \frac{1}{3}\right).$$

Likewise, we have

$$2(0, 1) + 1(0, 1) = \left(0, \frac{1}{2}\right) + \left(0, 1\right) = \left(0, \frac{3}{2}\right).$$

Notice that $(2+1)(0,1) \neq 2(0,1) + 1(0,1)$. Hence, V cannot be a vector space over $\mathbb{R}$. ■

Exercise 1.2.20

Let V denote the set of all real-valued functions f defined on the real line such that $f(1) = 0$. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Define V as a vector space with the operations of addition and scalar multiplication defined in Example 3. We must show that V is a vector space.

(VS 1) Let $f, g \in V$. This means that $f(1) = 0$ and $g(1) = 0$. We need to show that $f + g = g + f$. Since $f(1), g(1) \in \mathbb{R}$ and commutativity holds in $\mathbb{R}$, we can write

$$\begin{aligned}(f+g)(1) &= f(1) + g(1)\\ &= g(1) + f(1)\\ &= (g+f)(1)\end{aligned}$$

Hence, we have $f + g = g + f$.

(VS 2) Let $f, g, h \in V$ then $f(1) = g(1) = h(1) = 0$. We need to show that $f + (g+h) = (f+g) + h$. Observe that

$$\begin{aligned}\Big(f + (g+h)\Big)(1) &= f(1) + (g+h)(1)\\ &= f(1) + g(1) + h(1)\\ &= (f+g)(1) + h(1)\\ &= \Big((f+g) + h\Big)(1).\end{aligned}$$

Hence, we have $f + (g+h) = (f+g) + h$.

(VS 3) Let $f \in V$. We need to show that $f + f_0 = f$ for some $f_0 \in V$. Since V contains elements of $f \in V$ such that $f(1) = 0$, we can choose f_0 such that $f_0(1) = 0$. We can show that this is indeed the additive inverse of V by writing the following:

$$(f + f_0)(1) = f(1) + f_0(1) = f(1) + 0 = f(1).$$

Hence, we have $f + f_0 = f$.

(VS 4) Let $f \in V$. We need to find an element $g \in V$ such that $f + g = f_0$ By definition of V, $f(1) = 0$. We need to show that $f + g = f_0$ with f_0 defined as before. Choose $g = -f$ as our additive inverse and observe that

$$\begin{aligned}(f+g)(1) &= (f-f)(1)\\ &= f(1) - f(1)\\ &= 0 - 0\\ &= 0\\ &= f_0(1).\end{aligned}$$

Hence, $g = -f$ an element such that $f + g = f_0$.

(VS 5) Let $f \in V$. By definition of V, we have $f(1) = 0$. Since $f(1) \in \mathbb{R}$, we know that $1 \cdot f(1) = f(1)$. We need to show that $1f = f$. Observe that

$$(1f)(1) = 1 \cdot f(1) = f(1).$$

Hence, (VS 5) holds.

(VS 6) Let $a, b \in \mathbb{R}$ and let $x \in V$. We need to show that $(ab)f = a(bf)$. By using scalar multiplication, we can see that

$$(ab \cdot f)(1) = (ab)f(1) = a(bf(1)) = a(b \cdot f)(1).$$

Hence, (VS 6) must hold.

(VS 7) Let $a \in \mathbb{R}$ and let $f, g \in V$. We need to show that $a(f+g) = af + ag$. Observe that

$$\begin{aligned} a(f+g)(1) &= a(f(1)+g(1)) \\ &= af(1) + ag(1) \\ &= (af)(1) + (ag)(1). \end{aligned}$$

Hence, $a(f+g) = af + ag$ and so (VS 7) is satisfied.

(VS 8) Let $a, b \in \mathbb{R}$ and let $f \in V$. Observe that

$$\begin{aligned} (a+b)f(1) &= af(1) + bf(1) \\ &= (af)(1) + (bf)(1) \end{aligned}$$

Hence, $(a+b)f = af + bf$. Thus, V must be a vector space over $\mathbb{R}$. ■

Exercise 1.2.21

Let V and W be vector spaces over a field F. Let

$$Z = \{(v, w) : v \in v \text{ and } w \in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. Let V and W be vector spaces over a field F.

(VS 1) Let $x, y \in Z$ where $x = (v_1, w_1)$ and $y = (v_2, w_2)$. Since V and W are vector spaces, commutativity holds. Since the addition defined on both V and W is entry-wise, we can see that

$$\begin{aligned} x + y &= (v_1, w_1) + (v_2, w_2) \\ &= (v_1 + v_2, w_1 + w_2) \\ &= (v_2 + v_1, w_2 + w_1) \\ &= (v_2, w_2) + (v_1, w_1) \\ &= y + x. \end{aligned}$$

Hence, we can see that (VS 1) is satisfied.

(VS 2) Let $x, y, z \in Z$ with x and y as defined before as well as $z = (z_3, w_3)$. Using the entry-wise addition defined for Z, we can see that

$$\begin{aligned} x + (y + z) &= (v_1, w_1) + \Big((v_2, w_2) + (v_3, w_3)\Big) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (x + y) + z. \end{aligned}$$

(VS 3) Since V and W are vector spaces, we know that there exists an additive identity O_V and O_W respectively. Hence, we have $(O_V, O_W) \in Z$ and denote $O_Z = (O_V, O_W)$. Now, let $x \in Z$ as defined before. Observe that

$$\begin{aligned} x + O_Z &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + O_V, w_1 + O_W) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 4) Observe that V and W contain additive inverses for each $v \in V$ and $w \in W$ respectively. Since $x = (v_1, w_1)$ with $v_1 \in V$ and $w_1 \in W$, there exists an additive inverse $v_1' \in V$ and $w_1' \in W$ such that $v_1 + v_1' = O_V$ and $w_1 + w_1' = O_W$. This implies that $(v_1', w_1') \in Z$ which we will denote by x' such that

$$\begin{aligned} x + x' &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + v_1', w_1 + w_1') \\ &= (O_V, O_W) \\ &= O_Z. \end{aligned}$$

(VS 5) Let $x \in Z$. Since V and W are vector spaces (VS 5) implies that $1 \cdot v_1 = v_1$ and $1 \cdot w_1 = w_1$ respectively. Then observe that

$$\begin{aligned} 1 \cdot x &= 1 \cdot (v_1, w_1) \\ &= (1 \cdot v_1, 1 \cdot w_1) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 6) Let $x \in Z$ and $a, b \in F$. Then

$$\begin{aligned} (ab)x &= (ab)(v_1, w_1) \\ &= ((ab)v_1, (ab)w_1) \\ &= (a(bv_1), a(bw_1)) && (V, W \text{ vector space}) \\ &= a(bv_1, bw_1) \\ &= a(bx) \end{aligned}$$

(VS 7) Let $x, y \in Z$ as defined before. Let $a \in F$. Then

$$\begin{aligned} a(x + y) &= a\Big((v_1, w_1) + (v_2, w_2)\Big) \\ &= a\Big((v_1 + v_2, w_1 + w_2)\Big) \\ &= (a(v_1 + v_2), a(w_1 + w_2)) \\ &= (av_1 + av_2, aw_1 + aw_2) \\ &= (av_1, av_2) + (aw_1, aw_2) \\ &= a(v_1, v_2) + a(w_1, w_2) \\ &= ax + ay. \end{aligned}$$

(VS 8) Let $a, b \in F$ and let $x \in V$ as defined before. Since V and W are vector spaces, we know that

(VS 8) holds for bot entries $v_1 \in V$ and $w_1 \in W$. Hence, observe that

$$\begin{aligned}
(a+b)x &= (a+b)(v_1, w_1) \\
&= \Big((a+b)v_1, (a+b)w_1\Big) \\
&= (av_1 + bv_1, aw_1 + bw_1) \\
&= (av_1, aw_1) + (bv_1, bw_1) \\
&= a(v_1, w_1) + b(v_1, w_1) \\
&= ax + bx.
\end{aligned}$$

Hence, Z is a vector space.

■

1.2 Subspaces

Exercise 1.3.3

Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$.

Proof. Let $A, B \in M_{m\times n}(F)$ and let $a, b \in F$ be arbitrary. Using scalar multiplication defined on $M_{m\times n}(F)$, we have

$$\begin{aligned}
(aA + bB)^t &= (aA)^t + (bB)^t \\
&= aA^t + bB^t.
\end{aligned}$$

Hence, we are done. ■

Exercise 1.3.5

Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. Let A be an arbitrary square matrix. Since square matrices are symmetric, we have that $A^t = A$. We need to show that $(A + A^t)^t$. Observe that

$$\begin{aligned}
(A + A^t)^t &= A^t + (A^t)^t \\
&= A + A^t.
\end{aligned}$$

Hence, we have $A + A^t$ is symmetric. ■

Exercise 1.3.

Prove that $\mathrm{tr}(aA + bB) = a\mathrm{tr}(A) + b\mathrm{tr}(B)$ for any $A, B \in M_{n\times n}(F)$.

Proof. Let $A, B \in M_{n\times n}(F)$ and let $a, b \in F$ be arbitrary. Now, let $i = j$ and observe that

$$\begin{aligned}
\operatorname{tr}(aA + bB) &= \sum_{i,j\in\mathbb{N}}^{n} (aA + aB)_{ij} \\
&= \sum_{i,j\in\mathbb{N}}^{n} (aA)_{ij} + (bB)_{ij} \\
&= \sum_{i,j\in\mathbb{N}}^{n} (aA)_{ij} + \sum_{i,j\in\mathbb{N}}^{n} (bB)_{ij} \\
&= \sum_{i,j\in\mathbb{N}}^{n} aA_{ij} + \sum_{i,j\in\mathbb{N}}^{n} bB_{ij} \\
&= a\sum_{i,j\in\mathbb{N}}^{n} A_{ij} + b\sum_{i,j\in\mathbb{N}}^{n} B_{ij} \\
&= a\operatorname{tr}(A) + b\operatorname{tr}(B).
\end{aligned}$$

Hence, we conclude

$$\operatorname{tr}(aA + bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B)$$

for any $A, B \in M_{n\times n}(F)$. ■

Exercise 1.3.7

Prove that diagonal matrices are symmetric matrices.

Proof. Let $A \in M_{n\times n}(F)$ be diagonal. Let $i \neq j$ where $A_{ij} = 0$. We need to show that $A^t = A$. If we apply a transpose on A, we get that $A_{ji} = 0$ since A^t is also diagonal and square. Since $A_{ij} = A_{ji} = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$. Hence, $A^t = A$ ■

Exercise 1.3.8

Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

Proof. We claim that W_1 is a subspace of $\mathbb{R}^3$.

(a) Note that $O_{\mathbb{R}^3} \in W_1$ where $O_{\mathbb{R}^3} = (0, 0, 0)$ because $0 = 3 \cdot 0$ and $0 = -1 \cdot 0$.

(b) Let $x, y \in W_1$ where $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$. We need to show that $x + y \in W_1$. Since $a_1 = 3a_2$ and $a_3 = -a_2$ as well as $b_1 = 3b_2$ and $b_3 = -b_2$, we can write $a_1 + b_1 = 3(a_2 + b_2)$ and $a_3 + b_3 = -(a_2 + b_2)$. Hence, $x + y \in W_1$.

(c) Let $c \in \mathbb{R}$ and $x \in W_1$ with x defined as before. Then observe that $ca_1 = c(3a_2) = 3(ca_2)$ and $ca_3 = c(-a_2) = -(ca_2)$. Hence, $cx \in W_1$.

Since all the properties of a Theorem 3 have been satisfied, we can conclude that W_1 is a subspace of $\mathbb{R}^3$. ■

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - 3a_3 = 1\}$

(e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

(f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Exercise 1.3.9

Let W_1, W_3, W_4 be as in Exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$ and observe that each is a subspace of $\mathbb{R}^3$.

Proof. ■

Exercise 1.3.11

Prove that the set $W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1, a_2, \dots, a_n)\} \in F^n : a_1 + a_2 + \cdots + a_n = 1$ is not.

Proof. We need to show that W_1 is a subspace of F^n. We proceed by satisfying the properties of Theorem 3 to do this.

(a) Note that $O_{F^n} \in W_1$ since $0 + 0 + \cdots + 0 = 0$ n times.

(b) Let $x, y \in W_1$ with $x = (a_1, a_2, \dots, a_3)$ and $y = (b_1, b_2, \dots, b_n)$. By definition of W_1, we can see that

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Hence, $x + y \in W_1$ which tells us that W_1 is closed under addition.

(c) Let $x \in W_1$ and $c \in F$. Then observe that

$$\sum_{i=1}^{n}(ca_i) = c\sum_{i=1}^{n} a_i = c \cdot 0 = 0.$$

Hence, we have $cx \in F^n$.

We claim that W_2 is not a subspace because W_2 is not closed under addition. Let $(0,1), (1,0) \in F^2$. Observe that $0 + 1 = 1$ and $1 + 0 = 1$, but $(0 + 1) + (1 + 0) = 1 + 1 = 2$. Hence, $(0,1) + (1,0) \notin W_2$.

■

Exercise 1.3.11

Is the set $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Exercise 1.3.12

Prove that the set of $m \times n$ upper triangular matrices is a subspace of $M_{m\times n}(F)$.

Proof. Let V denote the set of $m \times n$ upper triangular matrices. We will show that V is a subspace of $M_{m\times n}(F)$ using Theorem 3.

(a) The zero matrix O from $M_{m\times n}(F)$ contains entries $O_{ij} = 0$ whenever $i > j$. Hence, $O \in V$.

(b) Let $A, B \in V$. By definition of V, A and B are upper triangular where $A_{ij} = 0$ and $B_{ij} = 0$ whenever $i > j$. Observe that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$$

whenever $i > j$. Hence, $A + B \in V$.

(c) Let $c \in F$ and $A \in V$ as defined before. Let $i > j$ and observe that $(cA)_{ij} = cA_{ij} = c \cdot 0 = 0$. Hence, $cA \in V$.

Since all the properties of theorem 3 have been satisfied, we conclude that V is indeed a subspace of $M_{m\times n}(F)$. ■

Exercise 1.3.13

Let S be nonempty set and F is a field. Prove that for any $s_0 \in S$, the set $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, F)$.

Proof. We will proceed to prove that S is a subspace of $\mathcal{F}(S, F)$ over the field F by satisfying the properties of Theorem 3. Let $V = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$.

(a) Note that the zero function $f_0 \in \mathcal{F}(S, F)$ where $f_0(s_0) = 0$ for any $s_0 \in S$ implies that $f_0 \in V$.

(b) Let $f, g \in V$. By definition of V, $f(s_0) = 0$ and $g(s_0) = 0$ for any $s_0 \in S$. We have $f + g \in V$ since

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0.$$

Hence, V is closed under addition.

(c) Let $f \in V$ and $c \in F$. We have $cf \in V$ since

$$(cf)(s_0) = cf(s_0) = c \cdot 0 = 0$$

for any $s_0 \in S$. Hence, V is closed under scalar multiplication.

■

Exercise 1.3.14

Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

Proof. We proceed by using Theorem 3 to prove that $\mathcal{C}(S, F)$ is a subspace where S is a nonempty set and F is a field.

(a) Note that the zero vector $f_0 \in \mathcal{F}(S, F)$ is in $\mathcal{C}(S, F)$ because $f_0(x_n) = 0$ where $x_n \in S$ for finitely many n.

(b) Let $f, g \in \mathcal{C}(S, F)$. We need to show that $f + g \in \mathcal{C}(S, F)$. Let $x_n \in S$ for finitely many n. Then using the addition defined on $\mathcal{F}(S, F)$, we can write

$$(f + g)(x_n) = f(x_n) + g(x_n) = 0 + 0 = 0.$$

(c) Let $f \in \mathcal{C}(S, F)$ and $c \in F$. We need to show that $cf \in \mathcal{C}(S, F)$. Let $x_n \in S$ for finitely many n. Using the scalar operation defined on $\mathcal{F}(S, F)$, we can write

$$(cf)(x_n) = cf(x_n) = c \cdot 0 = 0.$$

Since all the properties of Theorem 3 have been satisfied, we conclude that $\mathcal{C}(S, F)$ is indeed a subspace of $\mathcal{F}(S, F)$. ■

Exercise 1.3.15

Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$?

Proof. We claim that the set of all differentiable real-valued functions defined on $\mathbb{R}$ is a subspace of $C(\mathbb{R})$. Denote this set as V.

(a) Note that the zero function f_0 is differentiable for all $x \in \mathbb{R}$ and continuous for all $x \in R$.

Hence, $f_0 \in V$.

(b) Let $f, g \in V$. Using the addition operation defined on $C(\mathbb{R})$, we get that the sum $(f+g)(x) = f(x) + g(x)$ differentiable which implies that the sum of functions f, g is also continuous. Hence, $f + g \in V$.

(c) Let $f \in V$ and let $c \in \mathbb{R}$. Then $(cf)(x) = cf(x)$ is differentiable for all $x \in \mathbb{R}$ which means that cf is also continuous. Hence, $cf \in V$.

Hence, V is a subspace of $C(\mathbb{R})$. ■

Exercise 1.3.16

Let $C^n(\mathbb{R})$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $C^n(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof. ■

Exercise 1.3.17

Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. For the forwards direction, let $W \subseteq V$ where V is a vector space and W is a subspace of V. Let $a \in F$ and $x, y \in W$. Since W is a subspace, we know that $O_V \in W$. So, W is nonempty. Since W is closed under addition and multiplication, we get that $x + y \in W$ and $ax \in W$ and we are done.

For the backwards direction, let $W \neq \emptyset$ and $W \subseteq V$. Let $a \in F$ and $x, y \in W$ be arbitrary such that $ax \in W$ and $x + y \in W$. We need to show that W is a subspace of V. We need only show that $O_V \in W$ since W is closed under addition and scalar multiplication. Let $x \in W$. We can pick any $c \in F$ such that $c = 0$. So, we have $c \cdot x = 0 \cdot x = O_w$. Since the zero vector $O_W \in W$ is unique, we must have $O_V = O_W$. Hence, $O_V \in W$ and we conclude that W is a subspace of V. ■

Exercise 1.3.18

Prove that a subset W of a vector space V is a subspace of V if and only if $O \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Proof. ($\Rightarrow$) Let $W \subseteq V$ where W is a subspace of V. Since W is a subspace of V, we know that W is closed under addition and scalar multiplication. Let $a \in F$ and $x, y \in W$. Using the third property of Theorem 3, we can see that $ax \in W$. Since W is closed under addition, we can take $y \in W$ and $ax \in W$ such that $ax + y \in W$. Since W is also a vector space by definition, we know that $O_W \in W$. But $O_W = O_V$ so $O_V \in W$.

($\Leftarrow$) Let $a \in F$ and $x, y \in W$. We want to show that $W \subseteq V$ is a subspace of V. We can do this by using Theorem 3.

(a) By assumption, the zero vector $O_V \in W$.

(b) Let $x, y \in W$. Choose $a = 1$ such that $ax + y = x + y$. Since $ax + y \in W$ and $ax + y = x + y$, we also have $x + y \in W$. Hence, W is closed under addition.

(c) Let $x \in W$ and $O_V \in W$. Let $a \in F$. Then we have $ax + O_V = ax \in W$.

Hence, W is a subspace of V by Theorem 3. ■

Exercise 1.3.19

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. ($\Rightarrow$) Let $W_1 \cup W_2$ is a subspace of V. We need to show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We proceed by showing the contrapositive. Assume $W_2 \not\subseteq W_1$ and $W_1 \not\subseteq W_2$. We need to show that $W_1 \cup W_2$ is **NOT** a subspace of V. By assumption, $x \in W_1$ is not contained in W_2 as well as $y \in W_2$ is not contained in W_1. This implies that $W_1 \cup W_2 = \emptyset$. Since $W_1 \cup W_2$ is empty where $W_1 \cup W_2$ does not contain O_V, it cannot possibly be a subspace of V.

($\Leftarrow$) Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We need to show that $W_1 \cup W_2$ is a subspace of V. We proceed by using Theorem 3 to do this. Without loss of generality, assume $W_1 \subseteq W_2$. The proof will be the same if we use $W_2 \subseteq W_1$.

(a) Since W_1 is a subspace of V, we get that $O_W \in W$. Furthermore, $W_1 \subseteq W_2$ implies that $O_V \in W$. Since $O_W \in W_1$ and $O_V \in W_2$, we get that $O_V \in W_1 \cup W_2$ by definition of union.

(b) Let $x, y \in W_1$. Since W_1 is a subspace, we get that $x + y \in W_1$. Since $W_1 \subseteq W_2$, we also get that $x + y \in W_2$. Since both $x + y \in W_1$ and $x + y \in W_2$, we know that $x + y \in W_1 \cup W_2$ by definition of the union.

(c) Let $x \in W_1$ and $c \in F$. Since W_1 is closed under scalar multiplication, we have that $cx \in W_1$. But $W_1 \subseteq W_2$ so W_2 also contains $cx \in W_1$. So we must have $cx \in W_1 \cup W_2$.

■

Exercise 1.3.20

Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n$.

Proof. ■