Math 234A: Homework 3

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Problem 1 (Complex Logarithms). Compute the following:

(i) $\mathrm{Log}(i)$ and $\log(i)$

(ii) $\mathrm{Log}(1+i)$ and $\log(1+i)$.

(iii) $\mathrm{Log}(-1)$ and $\log(-1)$.

Note: for $z \in \mathbb{C}^\bullet$, $\log z$ is a set not a single number.

Solution. ■

Problem 2 (Complex Powers). Compute the following:

(i) $(1+i)^{3+i}$

(ii) $\left(\frac{1+i}{1-i}\right)^i$

(iii) $(-e)^{i/2}$.

Solution. ■

Problem 3. (a) Let $A \subseteq \mathbb{C}$. Show that the following statements are equivalent.

(i) A is closed.

(ii) For any sequence (a_n) in A such that $a_n \to a \in \mathbb{C}$ implies that $a \in A$.

(iii) A contains all its accumulation points; that is, if $a \in \mathbb{C}$ is an accumulation point of A, then $a \in A$.

(b) Given a set $A \subseteq \mathbb{C}$, we define

$$\mathcal{F}_A = \{F \subseteq \mathbb{C} : F \text{ is closed and } A \subseteq F\}.$$

Define $\overline{A} = \bigcup_{F \in \mathcal{F}_A} F$. Show that $\overline{A} = A \cup A'$ where

$$A' = \{z \in \mathbb{C} : z \text{ is an accumulation point}\}.$$

Proof. (a) To show that all the statements are equivalent, we will show that $(i) \implies (ii) \implies (iii) \implies (i)$.

$(i) \implies (ii)$ Suppose A is closed. Let (a_n) be a sequence in A where $a_n \to a \in \mathbb{C}$ (note that $a_n \neq a$. Our goal is to show that $a \in A$. Suppose for sake of contradiction that $a \notin A$. Then there exists some $\varepsilon > 0$ such that $N_\varepsilon(a) \cap E = \emptyset$; that is, $N_\varepsilon(a) \subseteq A^c$. Hence, we have that $a \in A^c$. But $(a_n) \to a \in \mathbb{C}$ implies that there exists at least one $a_n \neq a$ such that $a_n \in A^c$. However, the sequence (a_n) must be entirely contained in A by assumption which is a contradiction. Thus, $a \in A$.

$(ii) \implies (iii)$ Let (a_n) be a sequence in A where $a_n \neq a \in \mathbb{C}$ where $a \in A$. Our goal is to show that A contains all of its limit points. Let a be a limit point of A. Choose $\varepsilon = 1/n$ and choose $a_n \neq a$ to be a sequence of points in A. Then by assumption, the sequence $(a_n) \to a \in \mathbb{C}$ implies that $a \in A$; that is, we have that

$$N_{1/n}(a) \cap A \neq \emptyset.$$

Because $a \in A$, we can conclude that A must contain all of it's accumulation points.

$(iii) \implies (i)$ Suppose A contains all of its accumulation points. Our goal is to show that A is closed. It suffices to show that A^c is an open set; that is, we need to find an $\delta > 0$ such that $N_\delta(x) \subseteq A^c$ for all $x \in A^c$. To this end, let $x \in A^c$. Then $x \notin A$. This tells us that x cannot be a limit point of A. That is, there exists an $\delta > 0$ such that $N_\delta(x) \cap A = \emptyset$. This implies that $N_\delta(x) \subseteq A^c$ for some $\delta > 0$, and so A^c must be open. Hence, A must be closed.

(b) Our goal is to show that $\overline{A} = A \cup A'$. First, we would like to show two lemmas:

(*) $A \cup A'$ is a closed set.

(**) If F is a closed set and $A \subseteq F$, then $A \cup A' \subseteq F$ as well.

To show that (*) holds, let x be a accumulation point of $A \cup A'$. Our goal is to show that this accumulation point is contained in $A \cup A'$. By definition, we see that for all $\varepsilon > 0$, we have

$$B(x, \varepsilon) \cap ((A \cup A') \setminus \{x\}) \neq \emptyset.$$

To this end, pick a point in this intersection, say, a such that $a \in B(x, \varepsilon)$ and $a \in (A \cup A') \setminus \{x\}$. That is, we have $a \in A$ or $a \in A'$. If $a \in A$, then x is a accumulation point of A, and so $x \in A \cup A'$. If $a \in A'$, then a is a accumulation point of A'. That is, for all $\delta > 0$, we have

$$B(a, \delta) \cap A' \setminus \{a\} \neq \emptyset.$$

Pick a point in this intersection, say, $p \neq a$ such that $p \in A'$. But this implies that x must be a limit point of A, and so $x \in A'$ and thus $A \cup A'$ must be a closed set.

To show that (**) holds, suppose F is a closed set and that $A \subseteq F$. Our goal is to show that $A \cup A' \subseteq F$. Let $x \in A \cup A'$. Then either $x \in A$ or $x \in A'$. If $x \in A$, then $x \in F$ since $A \subseteq F$. On the other hand, if $x \in A'$, then x is a limit point of A. That is, for all $\delta > 0$, we have

$$B(x, \delta) \cap (A \setminus \{x\}) \neq \emptyset.$$

Since $A \subseteq F$, we can see that

$$B(x, \delta) \cap (F \setminus \{x\}) \neq \emptyset$$

which implies that x is a limit point of F. But F is closed, so x must be contained in F. Thus, we have $A \cup A' \subseteq F$ in both cases.

In what follows, we will show that $\overline{A} = A \cup A'$. To do this, we need to show two inclusions:

(1) $\overline{A} \subseteq A \cup A'$

(2) $A \cup A' \subseteq \overline{A}$.

Starting with (1), we see that $A \cup A' \subseteq F$ by (*). But this implies that $A \cup A'$ is the smallest closed set containing F, we must have that

$$A \cup A' \subseteq \bigcap_{F \in \mathcal{F}_A} F = \overline{A}$$

which satisfies (1).

With (2), we want to show that $\overline{A} \subseteq A \cup A'$.

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