Homework-6 - Math-234

Problem-1: (Holomorphic functions are uniquely determined by their values on a line segment or an open subset)

Let $D \subseteq \mathbb{C}$ open and connected, $f : D \longrightarrow \mathbb{C}$ be holomorphic.

- Use the following fact (without proof, we will prove this later: Let $B(a, r)$ be the open ball with center at $a \in \mathbb{C}$ and radius $r > 0$. Let $g : B(a,r) \longrightarrow \mathbb{C}$ be holomorphic. Let L be a line segment contained in the ball $B(a,r)$. If $g(z) = 0$ for all $z \in L$, then $g(z) = 0$ for all $z \in B(a,r)$.

  (This says that if g is a holomorphic function in an open ball and $g(z) = 0$ for all z is a line segment inside the ball, then g is identically on on that ball.)

- You can also use the following fact: Let L be a polygonal path joining $p, q \in D$, then we can find open balls $B(z_1, r_1), \dots, B(z_n, r_n)$ such that

  (i) $z_1, z_2, \dots, z_n$, $z_1 = p$ and $z_n = q$

  (ii) $B(z_i, r_i) \cap B(z_{i+1}, r_{i+1}) \neq \phi$ for all $i = 1, 2, \dots, n-1$

(This holds for any polygonal path in D)

(a) Let L be a polygonal path in D. Assume that $f(z) = 0$ for all $z \in L$. Show that $f(z) = 0$ for all $z \in D$.

Hint: Let $p \in D$ be arbitrary. Show that $f(p) = 0$ as follows: Add a small line segment to L and show that $f(z) = 0$ in this new polygonal path. Now use induction and connectedness of D together with facts above.

(b) Let $D' \subseteq D$, D' open. Assume that $f(z)=0$ for all $z \in D'$. Show that $f(z)=0$ for all $z \in D$.

(c) Let $f_1, f_2 : D \to \mathbb{C}$, holomorphic. Assume that
$f_1(z) = f_2(z)$ for all $z \in L$ where $L \subseteq D$ is a line segment. Show that $f_1(z) = f_2(z)$ for all $z \in D$.

(d) Let $D = \mathbb{C}$, $f_1(z) = e^z$. Let $f_2 : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

$$f_2(x + i\cdot 0) = e^x \quad \text{for all } x \in \mathbb{R}.$$

Prove that $f_2(z) = f_1(z)$ for all $z \in \mathbb{C}$.

Problem-2: Let D be an open rectangle such that its sides are parallel to real and imaginary axes. Let $u : D \to \mathbb{R}$ be a harmonic function. Fix $x_0 + iy_0 \in D$. In the lecture, we sketched that if we define

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\,ds \longrightarrow (*)$$

then $f = u + iv$ is holomorphic on D. In this problem, we fill in some details.

Fact: (Leibniz rule): Let $g:[a,b]\times[c,d]\to\mathbb{R}$ be continuous. Suppose that $(x,y)\mapsto \frac{\partial g}{\partial x}(x,y)$ exists and is continuous. Then the function

$$h(x)=\int_c^d g(x,y)\,dy \quad \text{is differentiable and}$$

$$h'(x)=\int_c^d \frac{\partial g}{\partial x}(x,y)\,dy$$

Use this fact to establish the followings

(i) Show that $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists are continuous functions and C-R equations hold.

(ii) Conclude that f is holomorphic. Make sure to justify $\hat{f}=\begin{bmatrix} u \\ v \end{bmatrix}$ is differentiable in the sense of Calculus.

(iii) Let us fix $x_1+iy_1\in D$ and defined v_1 by

$$v_1(x,y)=\int_{y_1}^{y}\frac{\partial u}{\partial x}(x,t)\,dt-\int_{x_1}^{x}\frac{\partial u}{\partial y}(s,y_1)\,ds$$

Let $f_1=u+iv_1$. Then we know f_1 is holomorphic by above. Show that $f-f_1$ is constant on D.

[Note: Thus changing the initial point changes the holomorphic function by a constant]

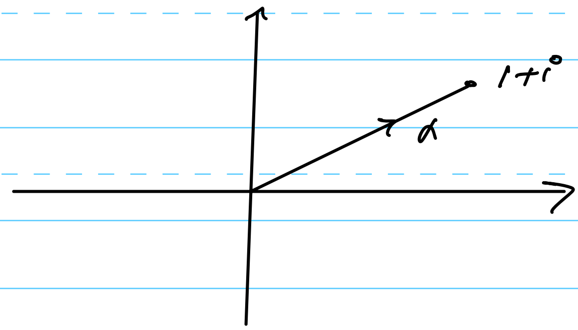
Problem 3: Define $u:\mathbb{C}\to\mathbb{R}$ by $u(x,y)=x^3-3xy^2$ (Here we are thinking of $\mathbb{C}$ as $\mathbb{R}^2$).

(i) Show that u is harmonic.

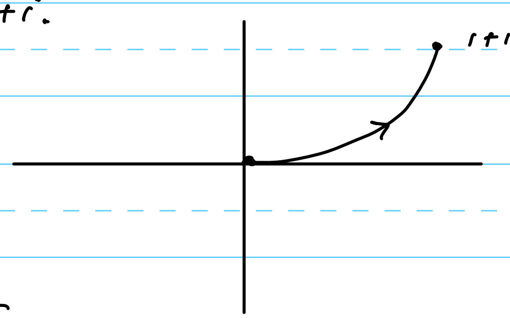
(ii) Find a holomorphic $f: \mathbb{C} \to \mathbb{C}$ such that $Re(f) = u$.

Problem 4: Compute $\int_{\alpha} z\, e^{z^2}\, dz$ where

(i) α is the line segment joining the point 0 to $1+i$.

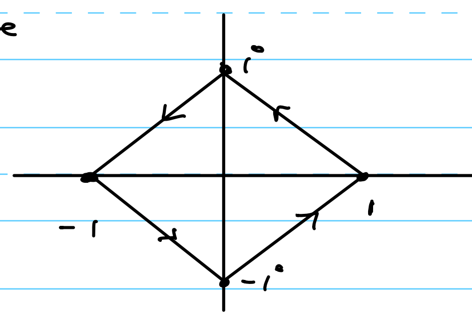


(ii) α is a piece of parabola $y = x^2$ that lies between 0 and $1+i$.



(iii) What did you observe? Is there a direct way to justify your observation?

Problem-5: Consider the curve α as shown in the picture



(i) Compute $\frac{1}{2\pi i} \int_{\alpha} \frac{1}{z}\, dz$

(ii) $\frac{1}{2\pi i} \int_{\alpha} \frac{1}{z^2}\, dz$

Problem 6: (i) Let $D \subseteq \mathbb{C}$, $f: D \to \mathbb{C}$ continuous, let $\alpha: [a,b] \to \mathbb{C}$ be a smooth curve. Assume that $\alpha([a,b]) \subseteq D$ and $\int_\alpha f(z)\,dz$ exists. Let $\phi: [c,d] \to [a,b]$ a continuously differentiable function such that $\phi(c) = a$ and $\phi(d) = b$.

Show that $\int_\alpha f(z)\,dz = \int_{\alpha\circ\phi} f(w)\,dw$

(ii) Assume that α is piecewise smooth instead of smooth in part (i). Show that

$$\int_\alpha f(z)\,dz = \int_{\alpha\circ\phi} f(w)\,dw$$

(This problem says that the complex line integral only depends on the image or trace of α)

Problem 7: (i) Let $D = \mathbb{C}^\bullet$. Show that we can not find a holomorphic $f: D \to \mathbb{C}$ such that

$$f'(z) = \frac{1}{z} \quad \text{for all } z \in \mathbb{C}^\bullet.$$

(ii) Does it contradict with $\frac{d}{dz}(\operatorname{Log} z) = \frac{1}{z}$? Why or why not? Justify your answer.

Problem-8: Let $\mathcal{F} = \{ f : [a,b] \to \mathbb{C} \,/\, f \text{ is integrable} \}$

Define $I : \mathcal{F} \to \mathbb{C}$ by

$$I(f) = \int_a^b f(t)\,dt$$

(i) Show that I is $\mathbb{C}$-linear.

(ii) Let $f, g \in \mathcal{F}$ and assume that $Re(f)$, $Re(g)$, $Im(f)$, $Im(g)$ are continuously differentiable. Show that

$$\int_a^b f(t)\,g'(t)\,dt = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f(t)\,g'(t)\,dt$$

Hint: You can apply fundamental theorem of Calculus without any proof.