Understanding Analysis Notes

Lance Remigio

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Contents

Chapter 1

The Real Numbers

1.1 The Axiom of Completeness

Theorem 1.1.1 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition 1.1.1. We call a set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Otherwise, a set is *bounded below* if there exists a $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for every $a \in A$.

Definition 1.1.2. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it satisfies the following criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We denote the least upper bound of a set A by calling it the *supremum* of A i.e $\sup(A)$. Similarly, we denote the greatest upper bound of set A by calling it the *infimum* of A i.e $\inf(A)$.

Note that a set can have many upper/lower bounds. But there can only exist one supremum and one infimum. In other words, these bounds are unique. Furthemore, the infimum and supremum need not be in the set.

Consider the following set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, ... \right\}$$

This set is bounded above and below. In addition, we can see that $\sup(A) = 1$ and $\inf(A) = 0$ (this is because each subsequent number in the sequence gets smaller and smaller).

Definition 1.1.3. We say that $a_0 \in \mathbb{R}$ is a *maximum* of the set A if $a_0 \in A$ and $a \leq a_0$ for all $a \in A$. Likewise, we say that $a_1 \in \mathbb{R}$ is a *minimum* of A if $a_1 \in A$ and $a \geq a_1$ for every $a \in A$.

If we have an open set $(0, 2)$ then the end points of this set are the infimum and supremum of the set respectively. Note that the maximum and the minimum do not exists because the infimum and the supremum are not in the set. If this set were to be closed, then the supremum and infimum would be in the set which implies that the max and min exists.

Now consider the Example

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

Notice that when we try and search for the supremum for this set, we cannot find one since we can always find a smaller number for an upper bound. One might say that $r = \sqrt{2}$ is the supremum of S but this is false since $r \notin \mathbb{Q}$ and is irrational.

Example. Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and is bounded above. Let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Prove that $\sup(c + A) = c + \sup(A)$

Proof. We use defintion 0.2 to prove this proposition. First, we need to prove that this $\sup(c + A)$ is an upper bound. We have $\sup(A) = s$ for some $s \in A$ if $s \geq a$ for all $a \in A$. We find that adding $c \in \mathbb{R}$ gives us

$$c + s \geq c + a.$$

Hence, we have that $c + s$ is an upper bound for the set $c + A$.

Next, we prove that $\sup(c + A) = c + s$ is the *least upper bound*. We know that $c + s \geq c + a$ for all $a \in A$. Suppose we have another upper bound $b \in A$ such that $c + a \leq b$ for all $a \in A$. Another manipulation gives us $a \leq b - c$ for all $a \in A$. Since $\sup(A) = s$ is the least upper bound for A, it follows that $s \leq b - c$. Hence, we have

$$c + s \leq b \implies \sup(c + A) = c + \sup(A).$$

■

There is another way to restate part (ii) of defintion 0.2 i.e

Lemma 1.1.1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

Proof. For the forward direction, suppose that $s = \sup A$ and consider $s - \epsilon$. Since s is an upper bound, we have that $s - \epsilon < s$. This means that $s - \epsilon$ is not an upper bound. Hence, we can find an element $a \in A$ such that $s - \epsilon < a$ because otherwise $s - \epsilon$ would be an upper bound. This concludes the forward direction.

For the backwards direction, assume s is an upper bound. We must satisfy part (ii) of defintion 0.2. Let $\epsilon > 0$, then $\epsilon = s - b$. But since any number smaller than s is not an upper bound, we have that $s \leq b$ if b is any other upper bound for S. Hence, $s = \sup A$. ■

1.1.1 Exercises

Exercise 1.3.3

(a) Let $A \neq \emptyset$ and bounded below, and define

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\},$$

Show that $\inf A = \sup B$.

Proof. Our goal is to show that both $\inf A \leq \sup B$ and $\inf A \geq \sup B$. Since $B \neq \emptyset$ and bounded above, we have that the $\sup B$ exists. First we want to show that $\inf A \leq \sup B$. By definition of $\sup B$, it is the greatest lower bound of B. Since $A \neq \emptyset$ and bounded below, we have that the $\sup B$ is greater than any lower bound of A. Hence, we have that $\inf A \leq \sup B$. Now we want to show that $\inf A \geq \sup B$. Suppose for sake of contradiction that $\inf A < \sup B$. Since $A \neq \emptyset$ and bounded below, we have that

$$a - \epsilon \geq \inf A \tag{1}$$

for some $a \in A$. Our goal is to show that there exists that some $a \in A$ is less than $\sup B$. Hence, choose $\epsilon = \sup B + a$ such that (1) and $\inf A < \sup B$ implies that

$$\begin{aligned} a - \epsilon &< \sup B \\ a - (\sup B + 3a) &< \sup B \\ a &< \sup B. \end{aligned}$$

But this is a contradiction since every element in A has to be bigger than B i.e $b > a$ for all $b \in B$. Hence, it must be the case that $\inf A \geq \sup B$. Hence, we have that $\inf A = \sup B$. ■

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. There is no need to assert that greatest lower bounds exist as part of the axiom because we can always seperate a set A that is bounded below into a set B that just consists of lower bounds from A. Since the infimum is just the greatest lower bound, it is equivalent to taking the supremum of a set of lower bounds. We can do this because every element in B is bounded above by every element in A which is permitted by the Axiom of Completeness. ■

Exercise 1.3.4

Let $A_1, A_2, A_3, ...$ be a collection of nonempty sets, each of which is bounded above.

1. Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\big(\cup_{k=1}^n A_k\big)$.

   Solution. For $\sup(A_1 \cup A_2)$, we have

   $$\sup(A_1 \cup A_2) = \sup\{A_1, A_2\}$$

   and for $\sup\big(\cup_{k=1}^n A_k\big)$, we have

   $$\sup\big(\cup_{k=1}^n A_k\big) = \sup\{A_k\}$$

   for $k \in \mathbb{N}$. ■

2. Consider $\sup\big(\cup_{k=1}^\infty A_k\big)$. Does the formula in (a) extend to the infinite case?

   Solution. No, because then $\cup_{k=1}^\infty A_k$ would be an unbounded set. ■

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set
$$cA = \{ca : a \in A\}.$$

(a) If $c \geq 0$, show that $\sup A(cA) = c \sup A$.

Proof. Suppose $c \geq 0$. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Denote $\alpha = \sup A$. By definition, we have that $\alpha \geq a$ for all $a \in A$. Multiplying by $c \geq 0$, we have that
$$c\alpha \geq ca$$
$$c \sup A \geq ca$$
for all $a \in A$. This shows that $c \sup A$ is an upper bound for cA.

Now we want to show that this upper bound is the least upper bound in cA. Hence, take any upper bound in $b \in A$ such that $ca \leq b$. This implies that $a \leq b/c$. Since $\alpha = \sup A$ is the least upper bound for A, we have that $\sup A \leq b/c$ which further implies that $c \sup A \leq b$ showing that it is the least upper bound in cA. Hence, we have that $\sup A(cA) = c \sup A$. ■

(b) Postulate a similar type of statement for $\sup(cA)$ for the cases $c < 0$.

Postulate. For cases $c < 0$, we have $\sup(cA) = c \inf A$. ■

Exercise 1.3.6

Suppose that $A, B \neq \emptyset$ and bounded above. Prove that
$$\sup(A + B) = \sup A + \sup B$$
We prove this proposition using two methods. One deals with direct application of the definition and the other deals with using lemma 1.3.8.

Proof. Our goal is to show that
$$\sup(A + B) = \sup A + \sup B$$
We know that since $A, B \neq \emptyset$ and bounded above, we have that $\sup A, \sup B$ exists. we denote the supremums by the following
$$\sup A = \alpha,$$
$$\sup B = \beta.$$

It suffices to show that following

$$\sup(A+B) \leq \sup A + \sup B \tag{1.1}$$

and

$$\sup(A+B) \geq \sup A + \sup B \tag{1.2}$$

We first show (2) first then we will show (1) next. Suppose we have abirtrary $x \in A$ and $y \in B$. Because $A, B \neq \emptyset$ and bounded above, we know that the set $A+B$ is also non-empty and bounded above which means its supremum $\sup(A+B)$ also exists. Hence, we know that

$$x + y \leq \sup(A+B)$$

Subtracting $y \in B$ to the other side of this inequality will yield

$$x \leq \sup(A+B) - y$$

But we know that since $x \in A$ and $\sup A \geq a$ for all $a \in A$, we have that

$$\sup A \leq \sup(A+B) - y.$$

Likewise, we isolate $y \in B$ to the other side and note that $b \leq \sup B$ for all $b \in B$. Then we get the following:

$$\begin{aligned} y &\leq \sup(A+B) - \sup A \\ \sup B &\leq \sup(A+B) - \sup A \end{aligned}$$

But this implies that

$$\sup A + \sup B \leq \sup(A+B)$$

Now we show (1). By lemma 1.3.8, we know that for all $\epsilon > 0$, we have that

$$\begin{aligned} \sup A - \frac{\epsilon}{2} &< a \\ \sup B - \frac{\epsilon}{2} &< b \end{aligned}$$

for some $a \in A$ and $b \in B$. Adding these two together we have that

$$\sup A + \sup B - \epsilon < a + b$$

But we also know that a and b are bounded above by their respective supremums so

$$\sup A + \sup B - \epsilon < a + b \leq \sup A + \sup B$$

Setting $\epsilon = \sup A + \sup B - \sup(A + B)$. Hence, we have that

$$\sup(A + B) \leq \sup A + \sup B.$$

Since we have (1) and (2), we see that

$$\sup(A + B) = \sup A + \sup B$$

■

Exercise 1.3.7

Prove that if a is an upper bound for A, and $a \in A$, then $\sup A = a$.

Proof. We want to show that $a \leq \sup A$ and $a \geq \sup A$. We start with the former. Since $A \neq \emptyset$ and bounded above, we have that the $\sup A$ exists. Label this supremum as $\sup A = \beta$. For every $\epsilon > 0$, we have that there exists $b \in A$ such that $\sup A - \epsilon \leq b$. Choose $\epsilon = 2\sup A - a - b$ such that

$$\begin{aligned} \sup A - 2\sup A + a + b &\leq b \\ -\sup A + a &\leq 0 \\ \implies a &\leq \sup A \end{aligned}$$

Now for the latter case, since $\sup A = \beta$ is the least upper bound of A and $a \in A$, it follows immediately that $a \leq \sup A$ for all $a \in A$. Hence, $\sup A = a$

■

Exercise 1.3.8

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Suppose $\sup A < \sup B$. Since we have $\sup B$, by lemma 1.3.8 we can say that for every $\epsilon > 0$, there exists $b \in B$ such that

$$\sup B - \epsilon < b \tag{1}$$

Choose $\epsilon = \sup B - \sup A$. We can do this because $\sup A < \sup B$. Hence, (1) implies

$$\begin{aligned} \sup B - \epsilon &< b \\ \sup B - (\sup B - \sup A) &< b \\ \sup A &< b. \end{aligned}$$

By definition, $\sup A$ is the least upper bound for A. Since $\sup A \geq a$ for all $a \in A$, it follows that from (1) that $a < b$ for all $a \in A$. Hence, for some $b \in B$, b is an upper bound for A.

■

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Exercise 1.3.10 (Cut Property)

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

Proof. Suppose A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$. By Axiom of Completeness, A and B are bounded above and below respectively. This implies that their supremum and infimums exists.

Firstly, we want to show that there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$. Since $a < b$ for all $a \in A$ and $b \in B$, every $b \in B$ is an upper bound for A. Denote B as the set of upper bounds for A. Hence, there must exist $c \in B$ such that c is the **least upper bound** for A due to the Axiom of Completeness. Furthermore, note that $\sup A \in B$ and not in A since $A \cap B = \emptyset$ which means $\sup A \in \mathbb{R}$. Hence, $\sup A \leq b$. But $x \in A$ so $x \leq \sup A$.

Now we want to show there exists $c \in \mathbb{R}$ such that $x \geq c$. Since every $a \in A$ is a lower bound for B and that $B \neq \emptyset$, there must exist an element in A such that it is the **greatest lower bound** for B. Denote this element as $c = \inf B$. Hence, $\inf B \geq a$ for all $a \in A$. Furthermore, $\inf B \in A$ and not in B since $A \cap B = \emptyset$ so $\inf B \in \mathbb{R}$ when we union A and B together. Since $x \in B$, we have that $\inf B \leq x$.

Furthermore, B is nonempty and bounded below and A is the set of lower bounds for B, we have that $\inf B = \sup A = c \in \mathbb{R}$. ∎

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove that $\sup E$ exists.

Proof. Assume $\mathbb{R}$ possesses the Cut Property and let $E \neq \emptyset$ that is bounded above. Suppose we have that $E \subseteq \mathbb{R}$. Since $\mathbb{R}$ possesses the cut property, we can find $c \in \mathbb{R}$ such that $x \leq c$ if $x \in E$. Since $A \cap B = \emptyset$, $c \in A \cup B = \mathbb{R}$. Hence, either $c \in A$ or $c \in B$. If $c \in A$, then c is not an upper bound for E since every $a \in A$ is less than every $b \in B$. Furthermore, if $c \in A$ and A is the set of lower bounds for B, then it would contradict that c is an upper bound for E. Thus, we must have $c \in B$. Since $c \in B$, B is the set of upper bounds for E, and $E \neq \emptyset$ and bounded above, $c \in B$ is the smallest element in B which makes it the **least upper bound** for E. Hence, $c = \sup E$ exists. ∎

1.2 Consequences of Completeness

The first application of the Axiom of Completeness is a result that says that the real line contains no gaps.

Theorem 1.2.1. *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Our goal is to produce a real number x such that this element is in every closed interval I_n for every $n \in \mathbb{N}$. Using the Axiom of Completeness, we can denote the following sets

$$A = \{a_n : n \in \mathbb{N}\}$$
$$B = \{b_n : n \in \mathbb{N}\}$$

where A and B consists of the left-hand and right-hand endpoints respectively. Since every closed interval are nested, we know that every b_n serves as an upper bound for A. By the Axiom of completeness, we can say that a supremum exists for A and we can label this supremum as $x = \sup A$. By definition, this is an upper bound for A. Hence, we have that $a_n \leq x$. But since x is the least upper bound and every $b_n \in B$ is an upper bound for every $a_n \in A$, we have that $x \leq b_n$. Hence, we have that $a_n \leq x \leq b_n$ which means that $x \in I_n$ for all $n \in \mathbb{N}$. This precisely means that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. ■

1.2.1 The Density of the Rationals

Theorem 1.2.2. *(Archimedean Property)*

- *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$*
- *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$*

Before we head on to the proof, it is important to notice that $\mathbb{N}$ is not bounded above and we shall not prove this fact since we are taking this property of the set to be a given just like all the properties that are contained in $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Proof. Assume for sake of contradiction that $\mathbb{N}$ is bounded above. Using the Axiom of Completeness, $\mathbb{N}$ contains a supremum, say, $\sup \mathbb{N} = \alpha$. Using lemma 1.3.8, we know that there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n. \qquad (\epsilon = 1)$$

This impplies that

$$\alpha < n + 1$$

but this shows that $n + 1 \in \mathbb{N}$ which is a contradiction because we assumed that $\alpha \geq n$ for all $n \in \mathbb{N}$ thereby rendering α to no longer be an upper bound for $\mathbb{N}$. Hence, we have that there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$. The second part of this theorem follows immediately by setting $x = 1/y$. ■

Theorem 1.2.3. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) For every two $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. Our goal is to choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b \qquad (1)$$

The idea is to choose a denominator large enough so that when we increment by size $\frac{1}{n}$ that it will be too big to increment over the open interval (a, b). Using the (2) of the Archimedean Property, we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a. \tag{2}$$

We now need to choose an $m \in \mathbb{Z}$ such that na is smaller than this chosen number. A diagram for choosing such a number is helpful. Hence,

Judging from our diagram, we can see that

$$m - 1 \leq na < m.$$

Focusing on the left part of the inequality, we can solve (2) for a and say that

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - 1/n) + 1 \\ &= nb \end{aligned}$$

This implies that $m < nb$ and consequently $na < m < nb$ which is equivalent to (1).

■

1.3 The Existence of Square Roots

Theorem 1.3.1. *There exists $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We need to show that $\alpha^2 = 2$. Hence, we need to show cases where $\alpha^2 < 2$ and $\alpha^2 > 2$. The idea behind these cases is to produce a contradiction that will show that having either one of these cases will violate the fact that $\alpha$is an upper bound for T and α is the least upper bound respectively.

Assume the first case, $\alpha^2 < 2$. We know that α is an upper bound for T. We need to construct an element that is larger than α. Hence, we construct

$$\alpha + \frac{1}{n} \in T \tag{1}$$

Squaring (1) we have that

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

We can use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

Rearranging we get that

$$\frac{2\alpha + 1}{n_0} < 2 - \alpha^2$$

and consequently

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2$$

But this means that $\alpha + 1/n_0 \in T$ showing that α is not an upper bound for T contradicting our assumption.

Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound.

■

1.3.1 Exercises

Exercise 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

Proof. Suppose $a, b \in \mathbb{Q}$. Then $p, q, m, n \in \mathbb{Z}$ such that $n, q \neq 0$. Hence, $a = \frac{p}{q}$ and $b = \frac{m}{n}$. Adding $a + b$ will give us

$$\begin{aligned} a + b &= \frac{p}{q} + \frac{m}{n} \\ &= \frac{pn + mq}{qn}. \end{aligned}$$

Since $pq + mn, qn \in \mathbb{Z}$ with $q, n \neq 0$, we have that $a + b \in \mathbb{Q}$. Now we multiply a and b together. Then we have

$$\begin{aligned} ab &= \frac{p}{q} \cdot \frac{m}{n} \\ &= \frac{pm}{qn}. \end{aligned}$$

Since $pm, qn \in \mathbb{Z}$ and $q, n \neq 0$, we have that $ab \in \mathbb{Q}$. ■

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Suppose for sake of contradiction that $at = r$ where $r \in \mathbb{Q}$. Solving for t, we have that $t = \frac{r}{a}$. But this tells us that $t \in \mathbb{Q}$ since $r, a \in \mathbb{Q}$ which is a contradics our assumption that $t \in \mathbb{I}$.

■

(c) Part (a) can be summarised by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. We can say that $s+t$ is an irrational number while st can either be rational or irrational depending if $s=t$ or $s \neq t$. If $s=t$, then st is rational and if $s \neq t$, then st is irrational. ■

Exercise 1.4.2

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. **Show that** $s = \sup A$.

Proof. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Since $s + \frac{1}{n}$ for all $n \in \mathbb{N}$ is an upper bound for A, we have that

$$\sup A \leq s + \frac{1}{n} \tag{1}$$

for all $n \in \mathbb{N}$. On the other hand, $s - \frac{1}{n}$ is a lower bound for A. Hence,

$$\sup A > s - \frac{1}{n} \tag{2}$$

for all $n \in \mathbb{N}$. We have (1) and (2) imply

$$s - \frac{1}{n} < \sup A \leq s + \frac{1}{n}. \tag{3}$$

This means that either $\sup A < s, \sup A > s$, or $\sup A = s,$. If $\sup A < s$, then $s - \sup A > 0$. Using the Archimedean Property, we can find an $n \in \mathbb{N}$ such that

$$s - \sup A > \frac{1}{n}$$

but this means that $\sup A < s - \frac{1}{n}$ which contradicts (3). On the other hand, if $\sup A > s$, then $\sup A - s > 0$. Using the Archimedean property again, we can find an $n \in \mathbb{N}$ such that

$$\sup A - s > \frac{1}{n}$$

but this means that $\sup A > s + \frac{1}{n}$ which is a contradiction since $\sup A < s + \frac{1}{n}$ from (3). Hence, it must be that $\sup A = s$. ■

Exercise 1.4.3

Prove that $\cap_{n=1}^{\infty}(0,1/n)=\emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion for the theorem to hold.

Proof. Suppose $x \in (0, \frac{1}{n})$, then $x > 0$. By the Archimedean Property, we can find an $N \in \mathbb{N}$ that is sufficiently large such that $x > \frac{1}{N}$. But this means that $x \in (0, 1/n)$ for all $n \in \mathbb{N}$. Hence, $x \notin \cap_{n=1}^{\infty}(0, \frac{1}{n})$ and then

$$\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset.$$

■

Exercise 1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup T = b$.

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Hence, we use lemma 1.3.8 and the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to state that for every $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $b - \epsilon < r < b$. But this means that $r \in T$ and $b - \epsilon$ is not an upper bound for T. Hence, $\sup T = b$.

■

Another proof for this:

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Since $a < b$ where $a, b \in \mathbb{R}$, we can find $x \in \mathbb{Q}$ such that $a < x < b$. Since $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ with $n \neq 0$, we have that $na < m < nb$. But note that nb is another upper bound for T for n sufficiently large and $nb > b$ implying that b is the least upper bound of T. Hence, $\sup T = b$. ■

Exercise 1.4.5

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. Consider the real numbers $a - \sqrt{p}$ and $b - \sqrt{p}$ where p is any prime number. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that

$$a - \sqrt{p} < r < b - \sqrt{p}$$

for some $r \in \mathbb{Q}$. Adding $\sqrt{p}$ to both sides, we have that

$$a < r + \sqrt{p} < b.$$

But know that $r + \sqrt{p} \in \mathbb{I}$ by (c) of Exercise 1.4.1. Hence, $t = r + \sqrt{p}$. We can follow the same procedure for trancendental numbers and make this conclusion. ■

Exercise 1.4.7

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Proof. Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound. ■

Exercise 1.4.6

Recall that a set B is dense in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set $\{r \in \mathbb{Q} : q \leq 10\}$

Solution. Yes, since $a < \frac{p}{10} < \frac{p}{q} < b$. ∎

(b) The set of all rationals p/q such that q is a power of 2.

Proof. Yes since $a < \frac{p}{2^n} < b$ for $n \in \mathbb{N}$. ∎

(c) The set of all rationals p/q with $10|p| \geq q$

Proof. ∎

1.4 Cardinality

1.4.1 Correspondence

Definition 1.4.1. A function $f : A \to B$ is *one-to-one* if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, there exists an element $a \in A$ for which $f(a) = b$.

An equivalent defintion for a function to be one-to-one is the following:

Definition 1.4.2. A function $f : A \to B$ is *one-to-one* if $f(a_1) \neq f(a_2)$ implies that $a_1 = a_2$.

A function that is both one-to-one and onto is said to be bijective. Meaning that we have a one-to-one correspodence between the sets A and B. Another way to explain a function being injective is to say that no two elements from A can map to the same element in B (think of the function x^2). And a function being onto can be explained as every element in A has to be mapped to an element in B.

From an algebraic perspective, we can denote a function being bijective to mean the same thing as two sets having the same cardinality i.e we can say that

Definition 1.4.3. Two sets A and B have the same cardinality if there exists $f : A \to B$ that is both one-to-one and onto. We can denote this symbolically as $A \sim B$

Example. Some examples of bijective maps are

1. Let the following map $f : \mathbb{N} \to \mathbf{E}$ be defined as $f(n) = 2n$. We can see that $\mathbb{N} \sim \mathbf{E}$. It's true that $\mathbf{E}$ is indeed a subset of $\mathbb{N}$, but do not conclude that it is a smaller set than $\mathbb{N}$ since they have the same cardinality or isomorphic to each other.

2. We can show this again. This time let us have a map $f : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ is odd.} \\ -n/2 \text{ if } n \text{ is even.} \end{cases}$$

   We have that $\mathbb{N} \sim \mathbb{Z}$ indeed.

1.4.2 Countable Sets

Definition 1.4.4. A set A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an *uncountable set*.

Theorem 1.4.1. *Let* $\mathbb{Q}, \mathbb{R}$. *Then*

- *The set* $\mathbb{Q}$ *is countable.*
- *The set* $\mathbb{R}$ *is uncountable.*

Proof. 1. Suppose we define A_n to be split into two sets. When $n = 1$, define A_n to be

$$A_1 = \{0\}$$

and define A_n when $n \geq 2$ as

$$A_n = \left\{ \pm \frac{p}{q} : \text{ where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

We can observe here that for every $n \in \mathbb{N}$ we can find every element of $\mathbb{Q}$ exactly once in the sets we have defined. So we can conclude that our map is onto. Since we designed our sets so that each rational numer appears once and the fact that for $n = 1$ and $n \geq 2$ produces two disjoint sets, we can see that our map is also one-to-one.

2. We can prove that second statement of theorem by contradiction. Assume for the sake of contradiction that there exists a *one-to-one* and *onto* function where $f : \mathbb{N} \to \mathbb{R}$. Letting $x_1 = f(1)$ and $x_2 = f(2)$ and so on, then we can enumerate each element of $\mathbb{R}$ i.e

$$\mathbb{R} = \{x_1, x_2, x_3, ...\}.$$

Using the Nested Interval Property, we will now produce a real number that is not in this set. Let I_n be a closed interval which does not contain x_n but contains x_{n+1}. Furthermore, I_{n+1} is contained within I_n. Note that within I_n there are two sets which are disjoint and x_{n+1} can be in either one of these sets. Now consider the following intersection $\cap_{n=1}^{\infty} I_n$. Using our construction that every $x_n \notin I_n$, then we can say that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

But this is a contradiction because the nested interval property asserts that this intersection is nonempty meaning that every $x \in \mathbb{R}$ is contained in the above set. Hence, we cannot emumerate every single element x_n of $\mathbb{R}$. Therefore, $\mathbb{R}$ is an *uncountable* set.

$\blacksquare$

This gives us three insights:

1. The smallest type of infinite set is the countable set.
2. We can create another set by deleting or inserting elements into it.
3. Anything smaller than a countable set is either finite or countable.

We can create $\mathbb{R}$ by taking the union of $\mathbb{Q}$ and $\mathbb{I}$. Since $\mathbb{R}$ is not countable and $\mathbb{Q}$ is, this would mean that the set of irrational numbers $\mathbb{I}$ would be uncountable. This tells us that $\mathbb{I}$ is a bigger subset of $\mathbb{R}$ than $\mathbb{Q}$.

We can summarize these results in the follow two theorems:

Theorem 1.4.2. *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Theorem 1.4.3. *1. If $A_1, A_2, ...A_n$ are each countable sets, then the union of*

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.4.3 Exercises

Exercise 1.5.1

Finish the following proof for Theorem 1.5.7.

Proof. Assume B is a countable set. So there exists a map $f : \mathbb{N} \to B$ such that f is surjective and injective. Let $A \subseteq B$ be an infinite subset of B. We want to show that A is countable. That is, A is both

1. injective
2. surjective.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Let $g : \mathbb{N} \to A$ be the map defined by

$$g(1) = f(n_1).$$

To show injectivity of g, we proceed via induction on the index $i \in \mathbb{N}$. Let the base case be $i = 2$. Then suppose $g(1) = g(2)$. By definition of g and injectivity of f, we have that

$$\begin{aligned} g(1) &= g(2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

But this means that $n_2 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Hence, g is injective. Now for the inductive step, assume this holds for every $1 \leq i \leq k-1$. We want to show that this holds for $i = k$. Suppose that

$$g(1) = g(k).$$

By defintion of g and injectivity of f, we have that

$$\begin{aligned} f(n_1) &= f(n_k) \\ n_1 &= n_k. \end{aligned}$$

But this also means that $n_k = \min\{n \in \mathbb{N} : f(n) \in \mathbb{N}\}$. Hence, g is injective.

Now we want to show that g is surjective. Note that we have

$$g(i) = A \cap \{f(n_1), f(n_2), f(n_3), ..., f(n_k)\}.$$

Then by definition of g, we have that $g(i) = f(n_i)$. Since f is surjective, there exists some $b \in B$ such that $f(n_i) = b$. But since $n_i = \{n_i \in \mathbb{N} : f(n_i) \in A\}$, we have that $f(n_i) \in A$ so g is surjective as well. Hence, we have that g is both injective and surjective which means that $\mathbb{N} \sim A$. Therefore, A is countable.

■

Exercise 1.5.2

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable.

The proposition is: $\mathbb{Q}$ is uncountable.

Proof. Assume for contradiction that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3\}$ and as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while the nested interval property implies that this intersection is nonempty. This contradiction implies $\mathbb{Q}$ must therefore be uncountable. ■

Thoughts. I think the main issue with this proof is when the author assumed that the set of rationals are closed. Since $\mathbb{Q}$ contains irrantional numbers within each subset of the $\mathbb{Q}$ as well as real numbers, $\mathbb{Q}$ cannot be closed. Hence, we cannot apply the nested interval property here. ■

Exercise 1.5.3

Prove theorem 1.5.8

Theorem 1.4.4. *1. If $A_1, A_2, ...A_n$ are each countable sets, then the union of*

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1. First, prove statement (i) for two countable sets, A_1 and A_2.

   Proof. Suppose A_1 and A_2 are countable sets. Then $\mathbb{N} \sim A_1$ and $\mathbb{N} \sim A_2$. Furthermore, we have that the maps $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to A_2$ are bijective. Our goal is to show the union $A_1 \cup A_2$ is also countable i.e we need to show that the map $h : \mathbb{N} \to A_1 \cup A_2$ is bijective. Before we proceed, let us replace A_2 with the following set B_2 defined as

   $$B_2 = A_2 \setminus A_1 = \{h(n) \in A_2 : h(n) \notin A_1\}.$$

   Now our following map is $h : \mathbb{N} \to A_1 \cup B_2$ (this is equivalent to $A_1 \cup A_2$)and define it as follows

   $$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k + 1 \in B_2 \end{cases}$$

   Suppose we have $n_1, n_2 \in \mathbb{N}$ and $h(n_1) = h(n_2)$. Since f and g are injective, we have

   $$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

   This shows that h is injective (the same process can be applied to g when $h \in B_2$). Note that $A_1 \cap B_2 = \emptyset$ because otherwise h would not be well defined. Now we need to show that h is surjective. Since f and g are surjective, there exists either $x \in A_1$ or $x \in B_2$ such that $h(n) = f(n) = x$ or $h(n) = g(n) = x$. Hence, we have that h is surjective. Since h is a bijective map, we now have that $\mathbb{N} \sim A_1 \cup B_2$.

   Suppose we use induction on the index $i \in \mathbb{N}$. Since we have already proven the base case for two countable sets, let us assume $A_1, A_2, ..., A_k$ are all countable sets such that for $i \leq k - 1$, the union $A_1 \cup A_2 ... \cup A_{k-1}$ is countable. Let's set $A' = A_1 \cup A_2 ... \cup A_{k-1}$. Our goal is to show that the union $A' \cup A_k$ is countable. Let's define the map $h : \mathbb{N} \to A' \cup B'$ such that

   $$B' = A_k \setminus A' = \{h(n) \in A_k : x \notin A'\}.$$

   and

   $$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k + 1 \end{cases}$$

Let $n_1, n_2 \in \mathbb{N}$. Since A' and A_k are countable sets, we have that

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

Hence, h is injective. Now we want to show that h is surjective. If either $h(n) \in A'$ or $h(n) \in A_k$, then since $f : \mathbb{N} \to A'$ and $g : \mathbb{N} \to A_k$ are surjective functions, we have that there exists $x \in A_k$ or $x \in A'$ such that $h(n) = x$. Hence, h is surjective as well. Since h is now a bijective function, we conclude that the union $A_1 \cup A_2 ... \cup A_k$ is countable.

■

2. Explain why induction cannot be used to prove part (1) of Theorem 1.5.8 from part (2)

   Solution. We cannot use induction on part (2) of theorem 1.5.8 because the index itself $n \in \mathbb{N}$ is infinite and induction only works only finite n. ■

Proof of the second part of theorem

Proof. Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of countable sets. Define the union

$$S = \bigcup_{n \in \mathbb{N}} S_n.$$

for all $n \in \mathbb{N}$. Assume each S_n is disjoint. Otherwise, let S_1 such that for each $n \geq 1$, define

$$S'_{n+1} = S_{n+1} \setminus S_n = \{x \in S_{n+1} : x \notin S_n\}.$$

This is to ensure that our following map is well-defined. let F_n denote the set of all injections from $S_n \to \mathbb{N}$ Let $\varphi : S \to \mathbb{N} \times \mathbb{N}$ be the map that is defined by

$$\varphi(x) = (n, f_n(x))$$

where $n \in \mathbb{N}$ smallest guranteed by the Well-Ordering Principle. Since each f_n is an injection, it follows that φ is also an injection. Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists an injection $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Composing the two functions φ and α, we have that $\alpha \circ \varphi : S \to \mathbb{N}$ is an injection. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, we know that α is also surjective. Hence, the composition $\alpha \circ \varphi$ is also surjective. Therefore, we have that S is countable.

■

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Proof. Let (a, b) any interval. Then define the function $f : (a, b) \to \mathbb{R}$ as

$$f(x) = x^2$$

Our objective is to show that f is injective and surjective. To show that f is injective, we need to let $x_1, x_2 \in (a, b)$. Then suppose

$$f(x_1) = f(x_2).$$

Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^2 &= x_2^2 \\ x_1 &= x_2. \end{aligned}$$

This shows that f is injective. Now we want to show that f is surjective. Then there exists $\sqrt{y} \in (a, b)$. Let

$$x = \sqrt{y}.$$

Then we have that

$$\begin{aligned} x^2 &= y \\ f(x) &= y. \end{aligned}$$

Hence, f is surjective. Since $f : (a, b) \to \mathbb{R}$ is a bijective function, we have that $(a, b) \sim \mathbb{R}$. ■

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof. Let $(0, \infty) = \{x : x > 0\}$. Our goal is to show that $(a, \infty) \sim \mathbb{R}$. To show this, we need to show the map $f : (a, \infty) \to \mathbb{R}$ is bijective. Define $f$as the following:

$$f(x) = \ln(x).$$

Then suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in (a, \infty)$. Then

$$\begin{aligned} \ln(x_1) &= \ln(x_2) \\ x_1 &= x_2 \end{aligned}$$

Hence, we have that f is an injective function. Now we want to show that f is surjective. Then let $e^y = x \in (0, \infty)$. Then taking the natural log of both sides, we have that $\ln(x) = y$. Hence, we have that f is a surjective function. Since f is a bijective function, we know that $(0, \infty) \sim \mathbb{R}$. ■

(c) Using open intervals makes it more covenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Proof. We want to show that $[0, 1) \sim (0, 1)$. Define the map $f : [0, 1) \to (0, 1)$ as

$$f(x) = \frac{1}{x - 1}$$

Our goal is to show that this map is bijective. Hence, we need to show that this map is both injective and surjective.

To show that f is injective. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in [0, 1)$. Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{1}{x_1 - 1} &= \frac{1}{x_2 - 1} \\ x_1 - 1 &= x_2 - 1 \\ x_1 &= x_2. \end{aligned}$$

Hence, f is injective.

To show that f is surjective, suppose we have $x - 1 = \frac{1}{y}$. Then

$$y = \frac{1}{x - 1}.$$

But we have that $f(x) = \frac{1}{x-1}$ so we have

$$f(x) = \frac{1}{x - 1} = y.$$

Hence, f is surjective.

Since f is bijective, we have that $[0, 1) \sim (0, 1)$. ■

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

Solution. $A \sim A$ because A is a bijection onto itself (same elements map to the same elements of the same set). ■

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Solution. If $A \sim B$, then the map $f : A \to B$ is a bijection. Meaning we can map unique eleemnts from A to unique elements to B. Since there is unique mapping of elements from $A \to B$ thenn we would expect to see the same thing when we map the same elements from $B \to A$. ■

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Proof. Suppose we have three sets A, B, and C. Suppose $A \sim B$ and $B \sim C$ then we have two maps $f : A \to B$ and $g : B \to C$ that are bijective. Composing the two functions we get $g \circ f : A \to C$. We want to show that this mapping is also bijective. Let $x_1, x_2 \in A$ then suppose $g \circ f(x_1) = g \circ f(x_2)$. By definition of composition, we have

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) \\ f(x_1) &= f(x_2) && (\ g \text{ is injective}) \\ x_1 &= x_2. && (\ f \text{ is injective}) \end{aligned}$$

Hence, $g \circ f$ is an injective function. Now we want to show that $g \circ f$ is a surjective mapping. Since f is surjective, there exists a $y \in B$ such that $f(x) = y$. Since g is also surjective, there exists a $z \in C$ such that $g(y) = c$. Hence, we have that $g(f(x)) = c$ which means $g \circ f$ is a surjective mapping. Therefore, $A \sim C$. ■

Exercise 1.5.11

[Shroder-Bernstein Theorem] Assume there exists an injective function $f : X \to Y$ and another injective function $g : Y \to X$. Show that $X \sim Y$. The strategy is to partition X and Y into components

$$\begin{aligned} X &= A \cup A' \\ Y &= B \cup B' \end{aligned}$$

with $A \cup A' = \emptyset$ and $B \cup B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

Solution. Taking disjoint sets prevents the problem of an element from either map mapping to two elements onto it's image. Thus, allowing us to have a well-defined function. Having two injective maps also would lead to the $X \sim Y$ because composing these two functions would allow us take a unique mapping from one element from each other. ■

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is similar collection in Y.

Proof. We set proceed by induction on $n \in \mathbb{N}$ and let $P(n)$ be the statement that $\{A_n : n \in \mathbb{N}\}$ and $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint. Define

$$A_{n+1} = g(f(A_n))$$

and for each $n \in \mathbb{N}$. Note that $g(f(A_{n+1})) = A_{n+2}$.

Let our base case be $n = 1$. Then $A_2 = g(f(A_1))$. By definition of A_1, we have that $x \in X$ but not in $g(Y)$. If $x \notin g(Y)$ then $x \notin g(f(A_1))$ as well. Hence, we have that $A_1 \cap A_2 = \emptyset$ Now assume $P(n)$ holds for $n \leq k - 1$. Define

$$A_{n+1} = A_{n+1} \setminus g(f(A_{n+1})) = \{x \in A_{n+1} : x \notin g(f(A_{n+1}))\}$$

Since A_n is pairwise disjoint for each $n \leq k - 1$, we also have that

$$A_{k-1} = A_{k-1} \setminus g(f(A_{k-1})) = \{x \in A_{k-1} : x \notin g(f(A_{k-1}))\}.$$

But notice that $g(f(A_{k-1})) = A_k$ which tells us that $A_{k-1} \cap A_k$ are also pairwise disjoint. Thus, A_n for each $n \in \mathbb{N}$ is pairwise disjoint.

■

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

Proof. ■

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'

Proof. ■

Chapter 2

Sequences and Series

2.1 The Limit of a Sequence

Understanding infinite series depends on understanding sequences that make up sequences of partial sums.

Definition 2.1.1. A sequence is a function whose domain is $\mathbb{N}$.

A way we describe sequences is to assign each $n \in \mathbb{N}$, use a mapping rule, and then have an output for the nth term. Mathematically we can describe it as a map $f : \mathbb{N} \to \mathbb{R}$.

Example. Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$
2. $\{\frac{1+n}{n}\}_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ...)$
3. (a_n), where $a_n = 2^n$ for each $n \in \mathbb{N}$,
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

It should not be confused that in some instances, the index n will start at $n = 0$ or $n = n_0$ for some other $n_0 > 1$. It is important to keep in mind that sequences are just infinite lists of real numbers. The main point of our analysis deals with what happens at the "tail" end of a given sequence.

Definition 2.1.2 (Convergence of a Sequence)**.** A sequence (a_n) *converges* to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Furthermore, the convergence of a sequence (a_n) to a is denoted by

$$\lim_{n\to\infty} a_n = a.$$

To understand the last part of this definition, namely, $|a_n - a| < \epsilon$, we can think of it as a neighborhood where a given value will be located in.

Definition 2.1.3. Given $a \in \mathbb{R}$ and $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the *ϵ-neighborhood of a.*

We can think of $V_\epsilon(a)$ as an interval where

$$a - \epsilon < a < a + \epsilon.$$

Another way is to think of it as a ball with radius $\epsilon > 0$ centered at a. we can also think about the convergence of a sequence to a point with the following definition.

Definition 2.1.4. A sequence (a_n) converges to a if, given any $\epsilon-$neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every $\epsilon-$neighborhood contains all but a finite number of the terms of (a_n).

The main idea here is that for some $n \in \mathbb{N}$ along a sequence (a_n), all the points of the sequence converge to some point within a certain $\epsilon-$neighborhood. Note that when increase the value of $n \in \mathbb{N}$, the smaller this $\epsilon-$neighborhood has to be and vice versa.

Example. Consider the sequence (a_n), where $a_n = \frac{1}{\sqrt{n}}$. From our regular understanding of calculus, one can see that the limit of this sequence goes to zero.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{e^2}.$$

We now proceed by verifying that this choice $N \in \mathbb{N}$ has the desired property that $a_n \to 0$. Let $n \geq N$ such that $n > \frac{1}{\epsilon^2}$. Hence, we have

$$\frac{1}{\sqrt{n}} < \epsilon.$$

But this implies that $|a_n - 0| < \epsilon$ and hence our sequence contains the desired property. ■

The main idea of these convergence proofs is to find an $N \in \mathbb{N}$ such that the value we want can be "hit" within some range that we specify with any number $\epsilon > 0$.

Quantifiers

The phrase

"For all $\epsilon > 0$", there exists $N \in \mathbb{N}$ such that ..."

means that for every positive integer I give you, there exists some index or natural number that contains some property that allows the sequence to converge to some value that we desire and as long as we satisfy this rule, then we can say that the sequence converges to our desired value. The template for our subsequent covergence proof will follow the steps below:

- "Let $\epsilon > 0$" be arbitrary."
- Demonstrate that a specific choice of $N \in \mathbb{N}$ leads to the desired property. Note that finding this N often involves working backwards from $|a_n - a| < \epsilon$.
- Show that this N actually works.
- Now assume $n \geq N$.
- With this choice of $\mathbb{N}$, you can work towards the property that $|a_n - a| < \epsilon$

Example. Show

$$\lim \left(\frac{n+1}{n}\right) = 1.$$

In other words, show that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|a_n - 1| < \epsilon$$

where

$$a_n = \frac{n+1}{n}.$$

To obtain our choice of $N \in \mathbb{N}$, we must work backwards from our conclusion. Hence, we have

$$\begin{aligned} a_n - 1 &< \epsilon \\ \frac{n+1}{n} - \frac{n}{n} &< \epsilon \\ \iff \frac{1}{n} &< \epsilon \\ \iff \frac{1}{\epsilon} &< n. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ is $N = 1/\epsilon$. Now for the actual proof.

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N = 1/\epsilon$ such that

$$N > \frac{1}{\epsilon}.$$

Let $n \geq N$. Then we proceed by showing that this choice of $N \in \mathbb{N}$ leads to the desired property. Hence,

$$\begin{aligned} n &> \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{n} \\ \epsilon &> \frac{n+1}{n} - \frac{n}{n} \\ \epsilon &> \frac{n+1}{n} - 1 \\ \epsilon &> |a_n - 1|. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ leads to $a_n \to 1$. We can now conclude that

$$\lim_{n \to \infty} a_n = 1.$$

■

Theorem 2.1.1 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Proof. Suppose we have $(a_n) \subseteq \mathbb{R}$. Suppose $a_n \to a$ and $a_n \to a'$. We want to show that

$$a = a'.$$

By definition, we have that

$$|a_n - a| < \epsilon/2 \text{ for some } n_1 \in \mathbb{N}$$
$$|a_n - a'| < \epsilon/2 \text{ for some } n_2 \in \mathbb{N}.$$

We can show that $a = a'$ by showing that $|a - a'| < \epsilon$. Hence, choose $N = \min\{n_1, n_2\}$ such that

$$\begin{aligned} |a - a'| &< |a - a_n + a_n - a'| \\ &< |a - a_n| + |a_n - a'| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Hence, we have that $a = a'$ showing that our limit is unique. ■

2.1.1 Divergence

We can study the divergence of sequences by negating the definition we have above.

Example. Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5} ...\right)$$

We can prove that this sequence does not converge to zero. Why? When we choose an $\epsilon = 1/10$, there is none of the term of the sequence converge within the neighborhood $(-1/10, 1/10)$ since the sequence oscillates between $-1/5$ and $1/5$. There is no $N \in \mathbb{N}$, that satisfies $a_n \to 0$. We can also give a counter-example in which we disprove the claim that (a_n) converges to $1/5$. Choose $\epsilon = 1/10$. This produces the neighborhood $(1/10, 3/10)$. We can see that the sequence does in fact converge to $1/5$, but it does so in an oscillating fashion. Furthermore, the sequence does not stay within the neighbor we specified where we expect all the terms of the sequence to converge towards the value. Hence, there is no such $N \in \mathbb{N}$ where the property can be satisfied.

Definition 2.1.5. A sequence that does not converge is said to diverge.

2.1.2 Exercises

Exercise 2.2.1

What happens if we reverse the order of the quantifiers in our convergence definition?

> **Definition 2.1.6** (Reversed). A sequence x_n converges to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ we have for $n \geq N$ such that
>
> $$|x_n - x| < \epsilon.$$

Give an example of a convergent sequence. Is there an example of a convergent sequence that is divergent? Can a sequence converge to two different values? What exactly is being described in this strange definition.

(a) When we reverse the quantifiers, the definition now requires us to construct such an ϵ such that any choice of $N \in \mathbb{N}$ will satisfy the property.

(b) An example of a convergent sequence is $x_n = 1/n$. It can be easily shown that $x_n \to 0$.

(c) Based on our definition and the fact that we can choose any $N \in \mathbb{N}$ suggest that we can have two different values for which the sequence can converge to.

(d) There is a specific contruction of an ϵ such that all x_n clusters converges towards a point determined by any choice of $N \in \mathbb{N}$.

Exercise 2.2.2

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1. $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$ Let $x_n = \frac{2n+1}{5n+4}$. We want to work backwards from our conclusion

   $$|x_n - \frac{2}{5}| < \epsilon$$

   to find our choice of $N \in \mathbb{N}$. Hence,

   $$\begin{aligned} \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &< \epsilon \\ \frac{3}{5(5n+4)} &< \epsilon. \end{aligned}$$

Solving for n, we get that
$$n > \frac{3/\epsilon - 20}{25}.$$
This only holds for all $0 < \epsilon < 3/20$. Hence, our choice of $N \in \mathbb{N}$ is
$$N = \frac{3/\epsilon - 20}{25}.$$

Proof. Let $0 < \epsilon < 3/20$. Choose $N = \frac{3/\epsilon - 20}{25}$ such that $N > \frac{3/\epsilon - 20}{25}$. Suppose $n \geq N$. We want to show that
$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon.$$
So we have the folowing manipulations
$$n > \frac{3/\epsilon - 20}{25}$$
$$25n\epsilon > 3 - 20\epsilon$$
so we have
$$\epsilon(25n + 20) > 3.$$

Hence, we have
$$\epsilon > \frac{3}{25n + 20}$$
which satisfies our given property that
$$\lim x_n = 2/5.$$

■

2. $\lim \frac{2n^2}{n^3+3} = 0$ Let $x_n = \frac{2n^2}{n^3+3}$. We want to produce an $N \in \mathbb{N}$ from
$$|x_n - 0| < \epsilon.$$
Observe that
$$\frac{2n^2}{n^3+3} < \epsilon$$

Notice that it is somewhat difficult to solve for n so we need to upper bound and lower bound the numerator and the denominator separately. Furthermore, we notice that (x_n) is bounded by $\frac{2n^2}{n^3} = \frac{2}{n}$. Then we lower bound the denominator. Observe that $n^3 + 3 \geq n^3$. Hence, we can estimate x_n to have the following form:

$$\frac{2n^2}{n^3+3} \leq \frac{2}{n} < \epsilon$$

which implies that

$$n > \frac{2}{\epsilon}$$

for $n > 2$.

Proof. Let $\epsilon > 0$. Choose $N = \min\{2, \frac{2}{\epsilon}\}$ and suppose $n \geq N$. Then observe that

$$\epsilon > \frac{2}{n} \geq \frac{2n^2}{n^3+3}.$$

Hence, we have

$$\frac{2n^2}{n^3+3} < \epsilon$$

and our property is satisfied. ∎

3. $\lim \frac{\sin(n^2)}{n^{1/3}} = 0$

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N = 1/\epsilon^3 \in \mathbb{N}$ and assume $n > N$. Then observe that

$$\frac{\sin(n^2)}{n^{1/3}} \leq \frac{1}{n^{1/3}} < \epsilon$$

since $\sin(n^2) \leq 1$. Hence, we have that

$$\left|\frac{\sin(n^2)}{n^{1/3}} - 0\right| < \epsilon.$$

Hence, the property is satisfied. ∎

Definition 2.1.7 (Greatest Integer)**.** For all $x \in \mathbb{R}$, if for all $k \in \mathbb{Z}$, $r \in \mathbb{Z}$ where $k > r$ such that $k \leq x < k+1$ and $r \leq x < r+1$ then we say that $\max(k, r)$ is the greatest integer less than or equal to x and denote it as

$$k = [[x]].$$

Exercise 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. Find $\lim a_n$ and supply proofs for each conlusion if

(a) $a_n = [[1/n]]$,

Proof. We claim that the limit of $a_n = [[1/n]]$ is equal to zero. We want to show that for all $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that for every $n \geq N$

$$|a_n - 0| < \epsilon.$$

We proceed by choosing $N > 1$. Suppose $n \geq N$. Our goal is to show that following property above. Since for every $N > 1$ such that $a_n = 0$, we have $n \geq N$

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Hence, our $N \in \mathbb{N}$ shows that $\lim a_n = 0$. ■

(b) $a_n = [[(10 + n)/2n]]$.

Proof. We claim that $\lim a_n = 0$. Our goal is to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - 0| < \epsilon$$

Choose $N > 10$. Suppose $n \geq N$ then we have

$$|a_n - 0| = |0 - 0| < \epsilon.$$

Hence, we have $\lim a_n = 0$. ■

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the $\epsilon-$neighborhood, the larger N may have to be."

Exercise 2.2.6

Prove the uniqueness of limits theorem. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Proof. Suppose $a_n \to a$ and $a_n \to b$. Then for every $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for every $n \geq N_1$ and $n \geq N_2$

$$|a_n - a| < \epsilon/2,$$
$$|a_n - b| < \epsilon/2.$$

Choose $N = \min\{N_1, N_2\}$ and assume $n \geq N$. We want to show that $a = b$ by showing that

$$|a - b| < \epsilon.$$

Hence, we have

$$\begin{aligned} |a - b| &< |a - a_n + a_n - b| \\ &< |a - a_n| + |a_n - n| && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\ a_n \to a,\ a_n \to b) \\ &= \epsilon. \end{aligned}$$

Therefore, $|a - b| < \epsilon$ and thus $a = b$.

■

Exercise 2.2.7

Here are two useful definitions

Definition 2.1.8. A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

and

Definition 2.1.9. A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

Solution. The sequence $(-1)^n$ is frequently in the set $\{1\}$ since for every $n > 0$, the sequence oscillates between two values in the set $\{-1, 1\}$. ■

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

Solution. The first definition is stronger because it implies that any sequence (x_n) will eventually converge to a point in some set $A \subseteq \mathbb{R}$ whereas the second definition explains how a point is constantly being "hit" but not letting all the terms of x_n settle within $A \subseteq \mathbb{R}$ past some $N \in \mathbb{N}$. ■

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

Solution. We can rephrase definition 2.2.3B (Convergence of a Sequence: Topological Version) by replacing every instance of the word *converge* with the phrase "eventually settling into" and rephrasing the $\epsilon-$neighborhood as a set $A \subseteq \mathbb{R}$ that a sequence x_n "eventually settles into to".

■

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. Since $(x_n) = 2$ for all $n \in \mathbb{N}$, x_n is frequently in the interval $(1.9, 2.1)$.
■

2.2 The Algebraic and Order Limit Theorems

The goal of having a rigorous definition of convergence in Analysis is to prove statements about sequences in general like the notion of "boundedness" which we will define below.

Definition 2.2.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n). This naturally leads us to the point that all convergent sequences are bounded i.e

Theorem 2.2.1. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit ℓ. This means that given $\epsilon = 1$, we can find an $N \in \mathbb{N}$ such that for every $n \geq N$, we can say that

$$\begin{aligned} &\implies |x_n - \ell| < 1 \\ &\iff -1 < x_n - \ell < 1 \\ &\iff \ell - 1 < x_n < \ell + 1. \end{aligned}$$

Note the terms of the sequence (x_n) can be found in the open interval $(\ell - 1, \ell + 1)$. Since $\ell \in \mathbb{R}$ can either be positive or negative, we can conclude that

$$|x_n| < |\ell| + 1$$

for all $n \geq N$ where

$$M = \max\{|x_1|, |x_2|, ..., |\ell| + 1\}.$$

Hence, it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. ∎

Theorem 2.2.2 (Algebraic Limit Theorem). *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$ *for all* $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$ *provided that* $a \neq 0$.

Proof of (i). We begin by proving part (i). Suppose $a_n \to a$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - a| < \epsilon/|c|. \tag{1}$$

In order to show (i), we need to show that

$$|ca_n - ca| < \epsilon.$$

Hence, observe that

$$\begin{aligned} |ca_n - ca| &< |c(a_n - a)| \\ &< |c||a_n - a| \\ &< |c|\frac{\epsilon}{|c|} \\ &= \epsilon. \end{aligned}$$

If $c = 0$, then our sequence (ca_n) reduces to the sequence $\{0, 0, 0, ..., 0\}$ which is clearly converging to $ca = 0$. Hence, we have attained our desired property that $\lim(ca_n) = ca$. The parts are left to you to prove. ∎

Proof of (ii). To show part (ii), it suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n + b_n - (a + b)| < \epsilon.$$

Hence, we start with the left side of (ii). Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}
|a_n + b_n - (a + b)| &< |(a_n - a) + (b_n - b)| \\
&< |a_n - a| + |b_n - b| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(a_n + b_n) = a + b$ as required.

■

proof of (iii). To show part (iii), it suffices to show for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - ab| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned}
|a_n b_n - ab| &< |a_n b_n - a_n b + a_n b - ab| \\
&< |a_n(b_n - b) + b(a_n - a)| \\
&< |a_n(b_n - b)| + |b(a_n - a)| \\
&< |a_n||b_n - b| + |b||a_n - a| \\
&< M\frac{\epsilon}{2M} + |b|\frac{\epsilon}{2|b|} && (\ a_n \text{ is bounded}) \\
&< \epsilon
\end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = ab$. ■

Proof of (iv). To show part (iv), it suffices to show for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$ with $b \neq 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $n \geq N_1, N_2$, we can have

$$|a_n - a| < M\epsilon/2,$$
$$|b_n - b| < \frac{|b|}{|a|} \cdot \frac{M\epsilon}{2}.$$

we can choose $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - ab + ab - b_n a}{b_n b}\right| \\
&= \left|\frac{b(a_n - a) + (b - b_n)a}{b_n b}\right| \\
&< \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b|} \cdot \frac{|b_n - b|}{|b_n|} \\
&< \frac{M\epsilon}{2M} + \frac{|a|}{|b|} \cdot \frac{|b|M\epsilon}{|a|2M} && (\ b_n \text{ bounded}) \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ provided that $b \neq 0$.

■

Theorem 2.2.3 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a \geq 0$.

(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.

(iv) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$, *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

(i) *Proof.* We proceed by contradiction by assuming that $a < 0$. Suppose $a_n \geq 0$ and $a_n \to a$. Let $\epsilon = |a|$ and suppose $n \geq N$. Then

$$|a_n - a| < |a| = -a.$$

But this means that $a_N < 0$ which is a contradiction since $a_N \geq 0$. ■

(ii) *Proof.* We can ensure that the sequence $b_n - a_n$ converges to $b - a$ by the Algebraic Limit Theorem. Since $b_n - a_n \geq 0$, we can use (i) to write $b - a \geq 0$. Hence, $a \leq b$. ■

(iii) *Proof.* Suppose there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$. Suppose $a_n = c$ then using (ii) yields $c \leq b$. Suppose $a_n \leq c$ for all $n \in \mathbb{N}$ then setting $b_n = c$ and using (ii) again yields $a \leq c$. ■

2.2.1 Exercises

Exercise 2.3.1

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

Proof. Suppose $x_n \geq 0$ and $x_n \to 0$. In order to show that $\sqrt{x_n} \to 0$, it suffices to show that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|\sqrt{x_n} - 0| < \epsilon.$$

Choose $N \in \mathbb{N}$. Suppose $x_n = 0$ for all $n \in \mathbb{N}$, then $(\sqrt{x_n}) = 0$ for all $n \geq N$ which means that $(\sqrt{x_n}) \to 0$. Suppose $x_n > 0$ for all $n \in \mathbb{N}$, then observe that since $(x_n) \to 0$ and (x_n) bounded, we have

$$\begin{aligned}
|\sqrt{x_n} - 0| &= |\sqrt{x_n}| \\
&= \left|\frac{x_n}{\sqrt{x_n}}\right| \\
&= \left|\frac{x_n - 0}{\sqrt{x_n}}\right| \\
&= \frac{|x_n - 0|}{\sqrt{x_n}} \\
&< \sqrt{M}\frac{\epsilon}{\sqrt{M}} \\
&= \epsilon
\end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to 0$. ■

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $(x_n) \to x$. We want to show that $(\sqrt{x_n}) \to x$. Suppose $x_n = 0$ and suppose $N \in \mathbb{N}$ such that for every

$n \geq N$, then we have the first case above where $x = 0$ and $(\sqrt{x_n}) \to 0$. Now suppose $x_n > 0$ and choose $N \in \mathbb{N}$ such that for every $n \geq N$, then observe that since $(x_n) \to x$ and (x_n) is bounded by an integer $M > 0$, we have that

$$\begin{aligned}
|\sqrt{x_n} - \sqrt{x}| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\
&= \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} \\
&< (\sqrt{M} + \sqrt{x})\frac{\epsilon}{(\sqrt{M} + \sqrt{x})} \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to \sqrt{x}$. ■

Exercise 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

Proof. Suppose $(x_n) \to 2$. Our goal is to show that property above. It suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{2x_n - 1}{3} - 1\right| < \epsilon.$$

Choose $N \in \mathbb{N}$ and suppose $n \geq N$

$$\begin{aligned}
\left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 4}{3}\right| \\
&= \left|\frac{2}{3}(x_n - 2)\right| \\
&= \left|\frac{2}{3}\right||x_n - 2| \\
&< \frac{2}{3} \cdot \frac{3\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that

$$\left(\frac{2x_n - 1}{3}\right) \to 1.$$

■

(b) $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$.

Proof. We want to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have
$$\left|\frac{1}{x_n} - \frac{1}{2}\right| < \epsilon.$$
Choose $N \in \mathbb{N}$ and assume $n \geq N$. Since $(x_n) \to 2$, we can write
$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &= \frac{|x_n - 2|}{2|x_n|}. \end{aligned} \tag{1}$$

Since $(x_n) \to 2$, we can set $\epsilon = 1$ so that we can lower bound the denominator of (1) using
$$2 - \epsilon < |x_n| \implies 1 < |x_n|.$$
Then we can set $N = \max\{1, \epsilon/2\}$ so that
$$\frac{|x_n - 2|}{2|x_n|} < \frac{2\epsilon}{2} = \epsilon$$
which satisifes our desired property. ■

Exercise 2.3.3

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots x_n, y_n, \ldots$ Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Theorem 2.2.4 (Squeeze Theorem)**.** *If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = \ell$, then $\lim y_n = \ell$.*

Proof. Suppose $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and suppose $\lim x_n = \lim z_n = \ell$. We want to show that $\lim y_n = \ell$. By the Order Limit Theorem, we have $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ implies that $\ell \leq y_n \leq \ell$ for all $n \in \mathbb{N}$. But this means that $y_n = \ell$ for all $n \in \mathbb{N}$. Hence, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$
$$|y_n - \ell| = |\ell - \ell| = 0 < \epsilon.$$

Hence, it follows that $\lim y_n = \ell$. ■

Exercise 2.3.4

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined).

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\
&= \frac{\lim 1+\lim(2a_n)}{\lim 1+\lim(3a_n)-\lim(4a_n^2)} \\
&= \frac{1+2\cdot 0}{1+3\cdot 0+4\cdot 0^2} \\
&= 1.
\end{aligned}$$

■

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n}{a_n}\right) \\
&= \lim\Big(a_n+4\Big) \\
&= \lim a_n + \lim 4 \\
&= 0+4 \\
&= 4.
\end{aligned}$$

■

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution. Let $(a_n) \to 0$. Then

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right)\\ &= \frac{\lim 2 + \lim(3a_n)}{\lim 1 + \lim(5a_n)}\\ &= \frac{2+3\cdot 0}{1+5\cdot 0}\\ &= 2.\end{aligned}$$

■

Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence

$$(x_1, y_1, x_2, y_2, ..., x_n, y_n).$$

For the forwards direction, assume (z_n) is a convergent sequence. We want to show that $\lim x_n = \lim y_n$. It suffices to show that given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|x_n - y_n| < \epsilon.$$

Suppose $(x_n) \to x$ and $(y_n) \to y$, then we can write

$$\begin{aligned}|x_n - y_n| &= |x_n - z_n + z_n - y_n|\\ &< |x_n - z_n| + |z_n - y_n|\\ &= |x_n - z + z - z_n| + |z_n - z + z - y_n|\\ &< |x_n - x| + |x - z_n| + |z_n - y| + |y - y_n|. \qquad (1)\end{aligned}$$

By definition, (z_n) is a shuffled sequence and convergent. Hence, $z_n \to x$ and $z_n \to y$. But by the uniqueness of limits, $x = y$ so we have that

$$|x_n - y_n| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

which means $\lim(x_n - y_n) = \lim x_n - \lim y_n = 0$.

Now for the backwards direction, assume $\lim x_n = \lim y_n$. We want to show (z_n) converges i.e for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|z_n - z| < \epsilon.$$

Exercise 2.3.6

Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Proof. Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Assume $(1/n) \to 0$ and $\sqrt{x_n} \to \sqrt{x}$. Then taking the limit of b_n, we have

$$\begin{aligned}
\lim b_n &= \lim(n - \sqrt{n^2+2n}) \\
&= \lim \frac{-2n}{n + \sqrt{n^2+2n}} \\
&= \lim \frac{-2}{1 + \sqrt{1+2/n}} \\
&= \frac{\lim(-2)}{\lim(1+\sqrt{1+2/n})} \\
&= \frac{\lim(-2)}{\lim(1) + \lim(\sqrt{1+2/n})} \\
&= \frac{-2}{1+1+0} && ((1/n) \to 0,\ (\sqrt{x_n}) \to \sqrt{x}) \\
&= -1.
\end{aligned}$$

Hence, we have $\lim b_n = -1$. Now we can show that b_n does reach this limit.

Let $\epsilon > 0$. Then choose

$$N = \frac{2}{\sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1}.$$

Then assumme $n \geq N$. Our goal is to show that

$$|b_n + 1| < \epsilon.$$

Then

$$\begin{aligned}
n &> \frac{2}{\sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1} \\
\implies \sqrt{\frac{1+\epsilon}{1-\epsilon}} - 1 &> \frac{2}{n}
\end{aligned}$$

Then we have

$$\begin{aligned} \sqrt{1+2/n} &< \frac{1+\epsilon}{1-\epsilon} \\ (1-\epsilon)\sqrt{1-2/n} &< 1+\epsilon \\ (1-\epsilon)\sqrt{1-2/n} - 1 &< \epsilon. \end{aligned}$$

Then we get

$$-1+\sqrt{1+2/n} < \epsilon(1+\sqrt{1+2/n})$$

and then

$$\begin{aligned} \frac{-1+\sqrt{1+2/n}}{1+\sqrt{1+2/n}} &< \epsilon \\ \frac{-2n}{n+\sqrt{n^2+2n}} + \frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}} &< \epsilon \\ n-\sqrt{n^2+2n}+1 &< \epsilon. \end{aligned}$$

Hence, it follows that $|b_n+1| < \epsilon$.

■

Exercise 2.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

Proof. Let $(x_n) \to x$ and let $p(x)$ be a polynomial. Let

$$p(x) = \sum_{i=0}^{m} a_i x^i$$

and

$$p(x_n) = \sum_{i=0}^{m} a_i x_n^i.$$

Our goal is to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|p(x_n) - p(x)| < \epsilon.$$

Then by part (i) of the Algebraic Limit Theorem, we have

$$\begin{aligned}
|p(x_n) - p(x)| &= \Big| \sum_{i=0}^{m} a_i x_n^i - \sum_{i=0}^{m} a_i x^i \Big| \\
&= \Big| \sum_{i=0}^{m} a_i (x_n^i - x^i) \Big| \\
&< \sum_{i=0}^{m} |a_i x_n^i - a_i x^i| && \text{(T.I)} \\
&< \sum_{i=0}^{m} \frac{\epsilon}{m} && (x_n \to x) \\
&= \frac{\epsilon}{m} \cdot m \\
&= \epsilon.
\end{aligned}$$

Hence, we have $p(x_n) \to p(x)$.

■

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Exercise 2.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

Proof. Let (a_n) be a bounded but not necessarily convergent sequence, and assume $\lim b_n = 0$. We want to show that $\lim(a_n b_n) = 0$. It suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - 0| < \epsilon. \tag{1}$$

Since (a_n) bounded, there exists an $M > 0$ such that $|a_n| < M$. Starting with the left side of (1), choose $N \in \mathbb{N}$ such that for every $n \geq N$

$$\begin{aligned}
|a_n b_n - 0| &= |a_n||b_n| \\
&< M \cdot \frac{\epsilon}{M} && (b_n \to 0) \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(a_n b_n) \to 0$. We cannot use the Algebraic Limit Theorem here because (a_n) does not necessarily have a defined limit even though it is bounded. ■

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

Solution. It would simply not converge. ■

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Proof. Suppose $a_n \to a$ where $a = 0$ and $b_n \to b$. Our goal is to show that $\lim(a_n b_n) = 0$. Let $\epsilon > 0$, then choose $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\begin{aligned} |a_n b_n - 0| &< |a_n||b_n| \\ &< \frac{\epsilon}{M} \cdot M && (a_n \to 0, b_n \to b) \\ &< \epsilon \end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = 0$. ■

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

Counterexample. Suppose $a_n = \frac{n}{2n+1}$ and $b_n = \frac{n}{2n+5}$. We have $\lim a_n = \lim b_n$ but $\lim(a_n - b_n) \neq 0$. ■

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

Proof. Let $\epsilon > 0$. Consider $||b_n| - |b||$. Assume $n \geq N$ then

$$||b_n| - |b|| < |b_n - b| < \epsilon$$

by reverse triangle inequality and $(b_n) \to b$. ■

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

Proof. Assume $(a_n) \to a$ and $(b_n - a_n) \to 0$. Let $\epsilon > 0$. By assumption,

$$\begin{aligned} |a_n - a| &< \frac{\epsilon}{2}, \ \ n \geq N_1 \\ |b_n - a_n| &< \frac{\epsilon}{2}, \ \ n \geq N_2. \end{aligned}$$

Hence, choose $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned}
|b_n - a| &= |b_n - a_n + a_n - a| \\
&< |b_n - a_n| + |a_n - a| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, $(b_n) \to a$. ■

(d) If $(a_n) \to a$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Proof. Let $\epsilon > 0$. Choose N so that $a_n \to 0$. Then consider $|b_n - b|$ and observe that

$$|b_n - b| \leq a_n < \epsilon,$$

Hence, it follows that $(b_n) \to b$. ■

Exercise 2.3.13(Iterated Limits).

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* Limits

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn} \right) \text{ and } \lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn} \right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

Proof. Let $a_{mn} = m/(m+n)$. We can compute the *iterated* limits

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} \frac{m}{m+n} \right) \tag{1}$$

and

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \frac{m}{m+n} \right). \tag{2}$$

We start with (1). Hence, we have

$$\begin{aligned}
\lim_{n\to\infty} \left(\lim_{m\to\infty} \frac{m}{m+n} \right) &= \lim_{n\to\infty} (1) \\
&= 1.
\end{aligned}$$

With (2), we have

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right) = \lim_{m\to\infty}(0)$$
$$= 0.$$

■

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exists? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

Proof. Let us try and compute the limit of a_{mn} the same way we did above. Hence, we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{1}{m+n}\right) = \lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{1}{1+n/m}\right) = \lim_{n\to\infty}(1) = 1 \tag{1}$$

and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{1}{m+n}\right) = 1. \tag{2}$$

We claim that for $\lim_{m,n\to\infty} a_{mn}$ to exists, we have to have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

hold. Since (1) and (2) are the equal to each other, we claim that $\lim_{m,n\to\infty} a_{mn}$ exists.

We can try to do the same process for the sequence $a_{mn} = mn/(m^2+n^2)$. Hence, we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{mn}{m^2+n^2}\right) = \lim_{n\to\infty}(0) = 0 \tag{3}$$

and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty}(0) = 0 \tag{4}$$

Since (2) and (3) are equal, we have that $\lim_{m,n\to\infty} a_{mn}$ exists. ■

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but neither iterated limit can be computed.

Solution. Consider

$$a_{m,n} = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}$$

where the iterated limits cannot be computed. ■

1. Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

   Proof. Suppose $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. We want to show that $\lim_{m\to\infty} b_m = a$. Consider $|b_m - a|$. Then fix $m \in \mathbb{N}$ such that for any $m, n \geq N$, we have that

$$\begin{aligned} |b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |b_m - a_{mn}| + |a_{mn} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

   Hence, $(b_m) \to a$. ■

2.3 The Monotone Convergence Theorem

As we have seen in the last section, convergent sequences are bounded while the converse is not true. But if a sequence is monotone then surely it is convergent.

Definition 2.3.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.3.1 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be *monotone* and *bounded*. We need to show that (a_n) converges to some value s. Let our set of points a_n be defined as

$$A = \{a_n : \text{ for all } n \in \mathbb{N}\}$$

and because we have a bounded sequence, we must have an upper bound s which can be defined as out supremum i.e

$$s = \sup\{a_n : \text{ for all } n \in \mathbb{N}\}.$$

Let $\epsilon > 0$. We need to show that

$$|a_n - s| < \epsilon$$

Since $s - \epsilon$ is not an upper bound of A, there exists $N \in \mathbb{N}$ such that

$$s - \epsilon < a_N.$$

Let's assume that (a_n) is an increasing sequence. By assuming $n \geq N$, we can say that $a_n \geq a_N$. Since $s + \epsilon$ is an upper bound and s is the least upper bound, then we can say that

$$s - \epsilon < a_N \leq a_n < s \leq s + \epsilon$$

which imply that

$$\begin{aligned} s - \epsilon < a_n < s + \epsilon \\ \implies |a_n - s| < \epsilon. \end{aligned}$$

Hence, it follows that any *monotone* and *bounded* sequence converges. ■

The key takeaway from this theorem is that we don't actually need to specify a value for a limit in order to show that it converges. As long as we have a monotone sequence and that we know it is bounded then we know for sure that the sequence converges.

Definition 2.3.2 (Convergence of a Series)**.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots.$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + ... + b_m = \sum_{i=1}^{m} s_i,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges* to B if the sequence (s_m) converges to B. In this case, we write

$$\sum_{n=1}^{\infty} b_n = B.$$

Example. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Because the terms in the sum are all positive, the sequence of partial sums are given by

$$s_m = \sum_{k=1}^{m} \frac{1}{k^2}$$

is increasing. Our goal is to show that this sequence is convergent so that the series converges. We proceed by using the Monotone Convergence Theorem to do this. Since we already have a monotone sequence of partial sums, only we need to do now find an upper bound for s_m. Observe that

$$\begin{aligned}
s_m &= 1 = \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + ... + \frac{1}{m^2} \\
&< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + ... + \frac{1}{m(m-1)} \\
&= 1 + \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + ... + \Big(\frac{1}{(m-1)} - \frac{1}{m}\Big) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2.
\end{aligned}$$

The third second equality is found by taking the partial fractions of the line before it. Thus, we find that 2 is an upper bound for the sequence of partial sums, so we can conclude that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Example (Harmonic Series)**.** Let's consider the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums is defined as follows

$$s_m = \sum_{k=1}^{m} \frac{1}{k}.$$

Like our last example, we expect these sequence of terms to be bounded by 2 but upon further inspection, we have

$$s_4 = 1 + \frac{1}{2} + \Big(\frac{1}{3} + \frac{1}{4}\Big) > 1 + \frac{1}{2} + \Big(\frac{1}{4} + \frac{1}{4}\Big) = 2$$

which is not true. Similarly, we find that $s_8 > 2\frac{1}{2}$, and we can see that in general we have that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^k} + ... + \frac{2}{2^k}\right) \\
&= 1 + ... + \left(2^{k-1}\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2} \\
&= 1 + k\frac{1}{2}.
\end{aligned}$$

This shows that our sequence is unbounded because we found $M = 1+k\left(\frac{1}{2}\right) > 0$ such that $s_k > M$. Despite how slow the sequence of partial of sums may be at reaching this point, it does end up surpassing every number on the postive real line. Since we have an unbounded sequence of partial sums, we conclude that the Harmonic series as divergent.

Theorem 2.3.2 (Cauchy Condensation Test)**.** *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series*

$$\sum_{n=0}^{\infty} b_n$$

converges if and only if

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

converges.

Proof. For the forwards direction, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. This means that the sequence of partial sums

$$t_k = b_1 + 2b_2 + ... + 2^k b_{2k}$$

are bounded. Hence, there exists $M > 0$ such that $t_k \leq M$ for all $k \in \mathbb{N}$. Our goal is to show that the sequence of partial sums for the series

$$\sum_{n=0}^{\infty} b_n.$$

Since $b_n \geq 0$ and that for all $n \in \mathbb{N}$ b_n decreasing, we have that the partial sums t_k is monotone. Our goal is to show that

$$s_m = \sum_{k=0}^{m} b_k$$

is bounded. Hence, fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$ and hence $s_m \leq s_{2^{k+1}-1}$ which imply that

$$\begin{aligned}
s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}-1}) \\
&\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\
&= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\
&= t_k
\end{aligned}$$

Hence, we have $s_m \leq s_{2^{k+1}-1} < t_k \leq M$ which means that (s_m) is bounded. By the Monotone Convergence Theorem, it follows that the series $\sum_{n=1}^{\infty} b_n$ converges. For the forwards direction, we proceed with contrapostive. Hence, assume for sake of contradiction that the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

is a divergent series. We want to show that the series

$$\sum_{n=0}^{\infty} b_n$$

is also a divergent series. ■

2.3.1 Exercises

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$ converges.

Proof. Let (x_n) be the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$. Our goal is to show that (x_n) is convergent. It is sufficient to show that (x_n) is both *monotone* and *bounded*. We first show that (x_n) is *monotone*. We claim that (x_n) is a *decreasing* sequence. Hence, we will show that for all $n \in \mathbb{N}$, we have $x_n > x_{n+1}$. We proceed by inducting on n. Let the base case be $n = 1$. Then we have that

$$x_1 = 3 > x_2 = \frac{1}{4 - 3} = 1.$$

Hence, we have $x_1 > x_2$. Now we assume that (x_n) is decreasing for all $1 < n \leq k - 1$. We want to show that $x_n > x_k$ for all $n < k$. Since $n \leq k - 1$, we have $x_{k-1} \leq x_n$ by inductive hypothesis. Consider x_k. By definition, we have that $x_k = 1/(4 - x_{k-1})$. Since $x_{k-1} \leq x_n$, then for all $n \in \mathbb{N}$ we have

$$x_k = \frac{1}{4 - x_{k-1}} < \frac{1}{4 - x_n}.$$

Hence, $x_k < x_n$ for all $n \in \mathbb{N}$. This is equivalent to showing $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, (x_n) is a *monotone* sequence. Now we show that (x_n) is *bounded*. Since $3 = x_1 \geq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_{n+1} = 1/(4 - x_n) > 0$, we have that

$$0 < x_n \leq 3.$$

Hence, (x_n) is bounded. Since (x_n) is *monotone* and *bounded*, we have that (x_n) is a convergent sequence by the Monotone Convergence theorem. ■

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Solution. Since (x_n) is *monotone* and *bounded*, then (x_{n+1}) is also *monotone* and *bounded*. By the Monotone Convergence Theorem, we have that (x_{n+1}) is also convergent. Hence, $\lim x_{n+1}$ also exists. ■

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution. Since $\lim x_n = \lim x_{n+1}$, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim \frac{1}{4 - x_n} \\ &= \frac{\lim 1}{\lim(4 - x_n)} \\ &= \frac{1}{\lim(4) - \lim x_n} \\ &= \frac{1}{4 - x}. && (\lim x_n = x) \end{aligned}$$

Then we have

$$x = \frac{1}{4 - x}$$

and then

$$x^2 - 4x + 1 = 0$$

which we can solve via the quadratic formula. Hence, we have $x = 2 + \sqrt{3}$. ■

Exercise 2.4.3

Following the model of Exercise 2.4.2, show that the sequence defined by $y_1 = 1$ and $y_{n+1} = 2 - \frac{1}{y_n}$ converges and find the limit.

Proof. Let (y_n) be the sequence defined by $y_1 = 1$ and

$$y_{n+1} = 4 - \frac{1}{y_n}.$$

for all $n \in \mathbb{N}$. We want to show that (y_n) converges. Hence, our goal is to show that (y_n) is *monotone* and *bounded*. We claim that (y_n) is increasing. Hence, we show this by inducting on $n \in \mathbb{N}$. Our goal is to show that $y_n \leq y_{n+1}$ for all $n \in \mathbb{N}$. Let the base case be $n = 1$. Then observe that

$$y_1 = 1 < y_2 = 4 - \frac{1}{1} = 3$$

Hence, we have $y_1 < y_2$.

Now assume that (y_n) is increasing for all $1 \leq n \leq k-1$. Hence, $y_n \leq y_{k-1}$. Our goal now is to show that $y_n \leq y_k$ for all $n \in \mathbb{N}$. Let's consider y_k. Then by definition of (y_n), we have

$$y_k = 4 - \frac{1}{y_{k-1}}.$$

Since $y_n \leq y_{k-1}$, we have

$$y_k = 4 - \frac{1}{y_{k-1}} \geq 4 - \frac{1}{y_n}$$

This shows that $y_k \geq y_n$ for any $n \in \mathbb{N}$. Hence, it follows that y_n is an increasing sequence and, therefore, *monotone*. Now

Now we want to show that (y_n) is *bounded*. Observe that $1 < y_n$ for all $n \in \mathbb{N}$ which means (y_n) contains a lower bound. Furthermore, for each $n \in \mathbb{N}$ we also have that $y_{n+1} = 4 - 1/y_n < 4$ which means that (y_n) also contains an uppper bound. Hence, it follows that

$$1 < y_n < 4$$

for all $n \in \mathbb{N}$. Hence, we have (y_n) is *bounded*. By the Monotone Convergence Theorem, it follows that (y_n) is a convergent sequence.

By last exercise, we know that $\lim y_n = \lim y_{n+1}$. Let's assume $(y_n) \to y$. Our goal is to compute $\lim y_n$. By the Algebraic Limit Theorem, we have

$$\begin{aligned}
y = \lim y_n &= \lim\left(4 - \frac{1}{y_n}\right) \\
&= \lim(4) - \lim\left(\frac{1}{y_n}\right) \\
&= 4 - \frac{\lim(1)}{\lim y_n} \\
&= 4 - \frac{1}{y}.
\end{aligned}$$

Hence, we have

$$y = 4 - \frac{1}{y}$$

which yields the following quadratic equation set to zero

$$y^2 - 4y + 1 = 0.$$

Solving for y using the quadratic formula yields $y = 2 + \sqrt{3}$ ■

Exercise 2.4.5 (Calculating Square Roots)

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big).$$

(a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

Proof. Our first goal is to show that $x_n^2 > 2$ for all $n \in \mathbb{N}$. We proceed by inducting on $n \in \mathbb{N}$. Let our base case be $n = 1$. Then

$$\begin{aligned} x_1 = 2 < x_1^2 \\ &= 4 \\ &< \frac{9}{4} \\ &= \frac{1}{4}\Big(x_1^2 + \frac{4}{x_1^2} + 4\Big) \\ &= x_2^2 \end{aligned}$$

which implies that $2 < x_1^2 < x_2^2$. Now suppose $x_{k-1}^2 > 2$ for all $n \leq k - 1$. We want to show that $x_k^2 > 2$ for all $n \in [1, k)$. Consider x_k and then by definition, we have

$$\begin{aligned} x_k^2 &= \frac{1}{4}\Big(x_{k-1}^2 + \frac{4}{x_{k-1}^2} + 4\Big) \\ &> \frac{1}{4}(2 + 2 + 4) \\ &= \frac{8}{4} \\ &= 2. \end{aligned}$$

Hence, $x_k^2 > 2$ for all $n \in \mathbb{N}$. Now we want to show that $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Consider $x_n - x_{n+1}$ then observe that since $x_n^2 > 2$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{x_n^2 - 2}{2x_n} \\ &> \frac{2-2}{2\sqrt{2}} \\ &= 0. \end{aligned}$$

Furthermore, when $x^2 = 2$ we get that $x_n - x_{n+1} = 0$. Hence, we have $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, we get that (x_n) is a convergent sequence. Since $\lim x_n = \lim x_{n+1}$, we can show that $\lim x_n = \sqrt{2}$. By the Algebraic Limit Theorem, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim \Big(\frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big)\Big) \\ &= \frac{1}{2}\lim\Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(\lim x_n + \lim \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(x + \frac{2}{x}\Big) \\ &= \frac{1}{2}x + \frac{1}{x} \\ &= \frac{x^2+2}{2x} \end{aligned}$$

which implies that

$$\begin{aligned} & x^2 = 2 \\ \implies & x = \sqrt{2}. \end{aligned}$$

Hence, we have $\lim x_n = \sqrt{2}$ ■

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution. Let the sequence (x_n) be defined recursively as $x_1 = c$ and

$$x_{n+1} = \frac{1}{c}\Big(x_n + \frac{c}{x_n}\Big).$$

Assume $x_n^2 > c$ for all $n \in \mathbb{N}$ and $x_n - x_{n+1} \geq 0$, then we have $\lim x_n = \sqrt{c}$. ■

Exercise 2.4.6 (Limit Superior.)

Let (a_n) be a *bounded* sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. Let (a_n) be a *bounded* sequence. Let (y_n) be defined as

$$y_n = \sup\{a_k : k \geq n\}$$

and further denote

$$A_n = \{a_k : k \geq n\}.$$

Our goal is to show that (y_n) converges. Thus, our goal is to show that (y_n) is both *monotone* and *bounded*.

We first show the former. We claim that (y_n) is a *decreasing* sequence. Hence, we need to show that $y_n \geq y_{n+1}$ for all $n \in \mathbb{N}$. We know that for each term y_n, by definition, is the *least upper bound* of the set $\{a_k : k \geq n\}$. By exercise 1.3.4, we know that since $A_{n+1} \subseteq A_n$, we have that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$. Hence, we have that y_n is a *decreasing* sequence.

Now we want to show that (y_n) is *bounded*. Since (a_n) is a *bounded* sequence, there exists an $M > 0$ such that $M \leq a_n \leq y_n$ for all $n \in \mathbb{N}$. Hence, (y_n) is a *bounded* sequence. In this case, it is enough to have (y_n) be bounded below.

Since (y_n) is *monotone* and *bounded*, we have that (y_n) is a convergent sequence.

■

(b) The *limit superior* of (a_n), or $\limsup a_n$ is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Solution. Let a_n be a *bounded* sequence. Then define

$$w_n = \inf\{a_k : k \geq n\}$$

so we can have

$$\lim w_n = \lim \inf\{a_k : k \geq n\}.$$

This limit exists because the terms of (w_n) are *increasing* and that (w_n) is *bounded* since there exists an $L > 0$ such that $w_n \leq a_n \leq L$.

■

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every *bounded* sequence, and give an example of a sequence for which the inequality is strict.

Proof. Suppose that (a_n) is a *bounded* sequence. Since $\liminf a_n$ and $\limsup a_n$ exists, we have that $\liminf a_n \leq a_n \leq \limsup a_n$. Hence,

$$\liminf a_n \leq \limsup a_n.$$

■

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We start by assuming that $\liminf a_n = \limsup a_n$. We want to show that the $\lim a_n$ exists. Define the following:

$$w = \liminf a_n = \lim w_n,$$
$$y = \limsup a_n = \lim y_n.$$

Since w_n and y_n both bound a_n, it follows that $w_n \leq a_n \leq y_n$. Since the $\liminf a_n$ and $\limsup a_n$ both exists such that $\lim w_n = \lim y_n = \ell$, it follows that $\lim a_n = \ell$ by the *Squeeze Theorem*. Hence, $\lim a_n$ exists.

Now assume the converse. Hence, our goal now is to show that $\liminf a_n = \limsup a_n$. Since (a_n) is a convergent sequence, let $\epsilon > 0$ such that there exists an $N \in \mathbb{N}$ where for every $n \geq N$, we have

$$|a_n - \ell| < \epsilon.$$

This is equivalent to saying

$$\ell - \epsilon \leq a_n \leq \ell + \epsilon.$$

Since $\epsilon > 0$ is abitrary, we can conclude that $\liminf a_n = \limsup a_n$.

■

Exercise 2.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Proof. Our goal is to show that $\sum_{n=1}^{\infty} b_n$ diverges. Since $\sum_{n=0}^{\infty} b_{2^n}$ diverges, the sequence of partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2^k}$$

diverges. This implies that t_k is *unbounded*: that is, there exists $M > 0$ such that for some $K \in \mathbb{N}$, we have for all $k \geq K$ we have $t_k > M$. Since $b_n \geq 0$, it suffices to show that the partial sums of $\sum_{n=1}^{\infty} b_n$ are *unbounded.* Let

$$s_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + b_2 + ... + b_m.$$

Let us fix m such that we choose k sufficiently large so that $m \geq 2^{k+1} + 1$. Hence, we have that $s_m \geq s_{2^{k-1}+1}$. Observe that

$$\begin{aligned} s_{2^{k+1}+1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}+1}) \\ &\geq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\ &= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\ &= t_k. \end{aligned}$$

This implies that $s_m \geq s_{2^{k+1}+1} \geq t_k > M$ for all $k \geq K$. Hence, we conclude t_k is *unbounded.* Thus, the series

$$\sum_{n=1}^{\infty} b_n.$$

diverges. ■

Exercise 2.4.10 (Infinite Products)

A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \dots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \dots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\dots \text{ where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$) for positive x will be useful in one direction.)

Exercise 2.4.6 (Limit Superior.)

Let (a_n) be a *bounded* sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. Let (a_n) be a *bounded* sequence. Let (y_n) be defined as

$$y_n = \sup\{a_k : k \geq n\}$$

and further denote

$$A_n = \{a_k : k \geq n\}.$$

Our goal is to show that (y_n) converges. Thus, our goal is to show that (y_n) is both *monotone* and *bounded*.

We first show the former. We claim that (y_n) is a *decreasing* sequence. Hence, we need to show that $y_n \geq y_{n+1}$ for all $n \in \mathbb{N}$. We know that for each term y_n, by definition, is the *least upper bound* of the set $\{a_k : k \geq n\}$. By exercise 1.3.4, we know that since $A_{n+1} \subseteq A_n$, we have that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$. Hence, we have that y_n is a *decreasing* sequence.

Now we want to show that (y_n) is *bounded*. Since (a_n) is a *bounded* sequence, there exists an $M > 0$ such that $M \leq a_n \leq y_n$ for all $n \in \mathbb{N}$. Hence, (y_n) is a *bounded* sequence. In this case, it is enough to have (y_n) be bounded below.

Since (y_n) is *monotone* and *bounded*, we have that (y_n) is a convergent sequence.

■

(b) The *limit superior* of (a_n), or $\limsup a_n$ is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Solution. Let a_n be a *bounded* sequence. Then define

$$w_n = \inf\{a_k : k \geq n\}$$

so we can have

$$\lim w_n = \lim \inf\{a_k : k \geq n\}.$$

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■

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every *bounded* sequence, and give an example of a sequence for which the inequality is strict.

Proof. Suppose that (a_n) is a *bounded* sequence. Since $\liminf a_n$ and $\limsup a_n$ exists, we have that $\liminf a_n \leq a_n \leq \limsup a_n$. Hence,

$$\liminf a_n \leq \limsup a_n.$$

■

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We start by assuming that $\liminf a_n = \limsup a_n$. We want to show that the $\lim a_n$ exists. Define the following:

$$\begin{aligned} w &= \liminf a_n = \lim w_n, \\ y &= \limsup a_n = \lim y_n. \end{aligned}$$

Since w_n and y_n both bound a_n, it follows that $w_n \leq a_n \leq y_n$. Since the $\liminf a_n$ and $\limsup a_n$ both exists such that $\lim w_n = \lim y_n = \ell$, it follows that $\lim a_n = \ell$ by the *Squeeze Theorem*. Hence, $\lim a_n$ exists.

Now assume the converse. Hence, our goal now is to show that $\liminf a_n = \limsup a_n$. Since (a_n) is a convergent sequence, let $\epsilon > 0$ such that there exists an $N \in \mathbb{N}$ where for every $n \geq N$, we have

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Since $\epsilon > 0$ is abitrary, we can conclude that $\liminf a_n = \limsup a_n$.

■

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Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

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$$t_k = b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2^k}$$

diverges. This implies that t_k is *unbounded*: that is, there exists $M > 0$ such that for some $K \in \mathbb{N}$, we have for all $k \geq K$ we have $t_k > M$. Since $b_n \geq 0$, it suffices to show that the partial sums of $\sum_{n=1}^{\infty} b_n$ are *unbounded*. Let

$$s_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + b_2 + ... + b_m.$$

Let us fix m such that we choose k sufficiently large so that $m \geq 2^{k+1} + 1$. Hence, we have that $s_m \geq s_{2^{k-1}+1}$. Observe that

$$\begin{aligned} s_{2^{k+1}+1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}+1}) \\ &\geq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\ &= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\ &= t_k. \end{aligned}$$

This implies that $s_m \geq s_{2^{k+1}+1} \geq t_k > M$ for all $k \geq K$. Hence, we conclude t_k is *unbounded*. Thus, the series

$$\sum_{n=1}^{\infty} b_n.$$

diverges. ■

Exercise 2.4.10 (Infinite Products)

A close relative of infinite series is the *infinite product*

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which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \dots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\dots \text{ where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$) for positive x will be useful in one direction.)

2.4 Subsequences and Bolzano-Weierstrass

In the last section, we observed that the convergence of partial sums of a particular series can be determined by the behavior of a subsequence of the partial sums.

Definition 2.4.1. Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \dots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

A few remarks about subsequences:

(a) The order of the subsequence is the same as in the original sequence.

Example. If we have the sequence

$$(a_n) = \Big(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\Big)$$

then the subsequences

$$\Big(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\Big)$$

and

$$\Big(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, ...\Big)$$

are permitted.

(b) Repetitions and swapping are not allowed.

Example. Like

$$\Big(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, ...\Big)$$

and

$$\Big(1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, ...\Big)$$

Since subsequences have the same ordering as the original sequence, one can conjecture about them converging to the same limit.

Theorem 2.4.1. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Let $(a_n) \to a$ and let (a_{n_k}) be a subsequence for (a_n). We want to show (a_{n_k}) converges to a as well. Since $(a_n) \to a$, there exists an N such that for any $n \geq N$, we have $|a_n - a| < \epsilon$.

We claim that $n_k \geq k$ for any $k \in \mathbb{N}$. Let us proceed by inducting on k. Let the base case be $k = 1$. Since n_k is an *increasing* sequence of natural numbers, we see that $n_1 \geq 1$. Now let us assume $n_{k-1} \geq k - 1$. Since (a_{n_k}) in *increasing*, we have $a_k \geq a_{k-1} \geq k - 1$ which implies that $n_k \geq k$.

Since any choice of $n \geq N$, we can say that $n_k \geq k \geq N$. Hence, we have

$$|a_{n_k} - a| < \epsilon$$

which is what we desired. ■

Example. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > ... > 0,$$

the sequence (b^n) is *decreasing* and *bounded* below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some ℓ satisfying $0 \leq \ell < b$. To compute ℓ, notice that (b^{2n}) is a subsequence, so $b^{2n} \to \ell$ by Theorem 2.5.2. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $b^{2n} \to \ell \cdot \ell = \ell^2$. Because limits are unique (Theorem 2.2.7), $\ell^2 = \ell$, and thus $\ell = 0$.

Example. Suppose we have an oscillating sequence of numbers

$$\Big(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\Big)$$

Note that this sequence does not converge to any proposed limit yet if we take a subsequence of it, weget a sequence that converges! Observe, that the subsequence

$$\Big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, ...\Big)$$

and

$$\Big(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, ...\Big)$$

converge to $1/5$ and $-1/5$ respectively. Since we have two subsequences that converge to two different limits, we immediately conclude that the original sequence diverges.

This leads us to our next theorem that states that

Theorem 2.4.2 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a *bounded* sequence. Then there exists $M > 0$ such that $a_n \in [-M, M]$. Suppose we divide this interval in half for k times: that is, let the length of the intervals be defined by the sequence $M(1/2)^{k-1}$. We claim that a subsequence (a_{n_k}) lies in either one of these intervals: that is, let $n_k > n_{k-1}$ for all $k \in \mathbb{N}$ such that $a_{n_k} \in I_k$.

Let us induct on k. Then let our base case be $k = 1$. Since we have an increasing sequence of natural numbers n_k, we have that $n_2 > n_1$ which means that $a_{n_2} \in I_2$ as well as $a_{n_1} \in I_1$. Now let us assume that this holds for all $k \leq \ell - 1$. We want to show that this holds for $k < \ell$. By the monotonicity of n_k, we have that $n_\ell > n_{\ell-1} > n_k > n_1$ which implies that $a_{n_\ell} \in I_\ell$ for all $\ell \in \mathbb{N}$. Furthermore, the sets

$$I_1 \subseteq I_2 \subseteq I_3 ...$$

form a nested sequence of closed intervals.

By the *Nested Interval Property*, we can conclude that there exists an $x \in I_k$ for all $k \in \mathbb{N}$ such that $\bigcup_{k=1}^{\infty} I_k \neq \emptyset$. Let $\epsilon > 0$. Since $a_{n_k}, x \in I_k$ for all $k \in \mathbb{N}$ and $M(1/2)^{k-1} \to 0$ by the Algebraic Limit Theorem, we can choose an $N \in \mathbb{N}$ such that for any $k \geq n_k \geq N$, we have

$$|a_{n_k} - x| < \epsilon.$$

Hence, $(a_{n_k}) \to x$. ■

Definitions

Definition 2.4.2. Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \dots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.4.3. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Theorem 2.4.4 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

2.4.1 Exercises

Exercise 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Solution. The subsubsequence of the bounded subsequence must converge by the *Bolzano-Weierstrass* theorem. ■

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Solution. Let (a_n) be a sequence defined by

$$a_n = \begin{cases} \frac{1}{2n} \text{ if } n = 2k \\ \frac{1}{2n} + 1 \text{ if } n = 2k+1. \end{cases}$$

Note that $1, 0 \notin (a_n)$ but if we take the subsequences $(a_{2k}) = 1/4k$ and $(a_{2k+1}) = 1/(4k+2)+1$, and take their limit, then we end up with the former converging to 0 and the latter converging to 1.

■

(c) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

Solution. Let's define the infinite set

$$A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

and define a subsequence such that we can make a subsequence for each $n \in \mathbb{N}$ where (a_n) hits every value of A_n. ■

(d) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\},$$

and no subsequences converging to points outside of this set.

Solution. This is not possible. There exists such a subsequence that does go to 0 but it is not within the infinite set. ■

Exercise 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

Solution. If every proper subsequence of (x_n) converges to x, then $(x_2, x_3, x_4, ...)$ also converges to x. Hence, $(x_n) \to x$ by the *uniqueness of limits.* ■

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

Solution. This is just the contrapositve of the statement:

"If (x_n) converges then every subsequence of (x_n) converges as well."

■

(c) If (x_n) contains a divergent subsequence, then there exists two subsequences of (x_n) that converge to different limits.

Solution. This is false. we can find an (x_n) that is not bounded such that we cannot find a subsequence that converges to a limit. ■

(d) If (x_n) is *monotone* and contains a convergent subsequence, then (x_n) converges.

Proof. Assume (x_n) is *monotone* and contains a convergent *subsequence* (x_{n_k}). It suffices to show that (x_n) is *bounded*. Since (x_{n_k}) is convergent, it is also *bounded*. Hence, there exists $M > 0$ such that for all $n_k \in \mathbb{N}$, we have $|x_{n_k}| \leq M$. Since (x_n) *monotone* then either $n \leq n_k$ or $n \geq n_k$ for all $n \in \mathbb{N}$. Hence, we can write either $-M \leq x_n$ or $x_{n_k} \leq M$. But this means that (x_n) is also *bounded*. Since (x_n) both *bounded* and *monotone*, (x_n) is convergent by the *Monotone Convergence Theorem*. Also, (x_n) and (x_{n_k}) converge to the same limit by the *Uniqueness of Limits*. ■

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative propeprty holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + ...$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + ... + a_{n_1}) + (a_1 + a_2 + ... + a_{n_2}) + (a_1 + a_2 + ... + a_{n_3}) + ...$$

leads to a series that also converges to L.

Proof. Our goal is to show that the associative property for a converging infinite series holds. Let us define the terms of the subsequence

$$\begin{aligned}
b_1 &= a_1 + a_2 + a_3 + ... + a_{n_1} \\
b_2 &= a_{n_1+1} + a_{n_1+2} + a_{n_1+3} + \cdots + a_{n_2} \\
&\vdots \\
b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}.
\end{aligned}$$

Our goal is to show that the subsequence (b_m) converges to L as well. Suppose $\lim s_n = L$. Let the partial sums (t_m) be regrouped in terms of the subsequence above

$$\begin{aligned}
t_m &= b_1 + b_2 + \cdots + b_m \\
&= (a_1 + a_2 + a_3 + ... + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \ldots \\
&+ (a_{n_{m-1}+1} + \cdots + a_{n_m}).
\end{aligned}$$

Since $\lim s_n = L$, its sequence of partial sums also converge to L. But this means every subsequence of (t_k) also converges to L. Hence, (b_m) converges to L as well. ■

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to associative. Why doesn't our proof in (a) apply to this example?

Solution. We cannot have infinite series be associative if the sequence of partial sums diverges. This means we cannot regroup the terms of our partial sums into a subsequence that converges, since there are divergent subsequences. ■

Exercise 2.5.5

Assume (a_n) is *bounded* sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose for sake of contradiction that $(a_n) \not\to a$. Then there exists $\epsilon_0 > 0$ such that $|a_n - a| \geq \epsilon_0$ for all $N \in \mathbb{N}$. Since (a_n) is *bounded*, we can find a subsequence (a_{n_k}) that converges to some $\ell \in \mathbb{R}$. Since $(a_n) \not\to a$, then $(a_{n_k}) \to \ell$ where $\ell \neq a$. Yet we assumed every convergent subsequence of (a_n) converges to the same limit a but $a \neq \ell$ which is a contradiction. Hence, it must be the case that $\lim a_n = a$. ■

Exercise 2.5.6

Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed)

Proof. Let $b \geq 0$. Our goal is to show that $\lim b^{1/n}$ exists. We observe that

$$b > b^{1/2} > b^{1/3} > b^{1/4} > ... > b^{1/n} \geq 0$$

and conclude by induction that $b^{1/n}$ is a *decreasing* sequence. Since $0 \leq b^{1/n} < b$, we can also conclude that $(b^{1/n})$ is a *bounded* sequence. Hence, $(b^{1/n})$ is a convergent sequence. But note that $(b^{1/n}) \to 0$ for all $b^{1/n} \geq 0$ by exercise 2.3.1. Hence, $\lim b^{1/n} = 0$. ■

Exercise 2.5.7

Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof. Suppose $\lim b^n = 0$. Let $\epsilon = 1$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $-1 < b^n < 1$. Then we have $-1 < b < 1$. Hence, $|b| < 1$.

Now let us show the converse. Assume $|b| < 1$; that is, $-1 < b < 1$. Since $0 \leq b < 1$ holds by Example 2.5.3, we can write $\lim b^n = 0$. Suppose $-1 < b < 0$. We observe that

$$b < b^2 < b^3 < b^4 < ... < 0$$

impying that b^n is an *increasing* sequence for all $n \in \mathbb{N}$ for $b^n \in (-1, 0)$. Furthermore, (b^n) is *bounded* since $1 < b^n < 0$. Hence, (b^n) is a convergent sequence by the *Monotone Convergence Theorem*. Hence, $(b^{2n}) \to \ell$ satisfying $b < \ell \leq 0$. Suppose $\lim(b^n) = \ell$. Let (b^{2n}) be a subsequence, then (b^{2n}) also converges to the same limit b. Hence, we have

$$\begin{aligned} \ell = \lim b^n &= \lim b^{2n} \\ &= \lim(b^n \cdot b^n) \\ &= \lim b^n \cdot \lim b^n \\ &= \ell^2 \end{aligned}$$

Then by the same process in Example 2.5.3, we have $\lim(b^n) = 0$.

■

Exercise 2.5.6

Let (a_n) be a *bounded* sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Proof. Let (a_n) be a *bounded* sequence. We observe that $S \neq \emptyset$ and S *bounded above* since there exists $M > 0$ such that $|a_n| \leq M$ where $x < a_n \leq M$. By the *Axiom of Completeness*, $s = \sup S$ exists. Then by lemma 1.3.8, let $\epsilon = 1/n_k$ such that for some $a_{n_k} \in S$, we have

$$s - \frac{1}{n_k} \leq a_{n_k} \leq s$$

Note that we can write $\lim(s - 1/n_k) = s$ by the *Algebraic Limit Theorem*. By the *Squeeze Theorem*, it follows that $a_{n_k} \to s = \sup S$. ■

2.5 The Cauchy Criterion

Definition 2.5.1. A sequence (a_n) is called a *Cauchy Sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|a_n - a_m| < \epsilon.$$

In the regular convergence defintion, we are given any $\epsilon > 0$ where there is a point in the sequence $N \in \mathbb{N}$ such that past this point, all of our terms fall within an ϵ range around some limit point. In the Cauchy Criterion defintion, we begin with the same conditions but this time, all the terms of the sequence are all tightly packed together wthin the $\epsilon > 0$ range we were given. It turns out, that these two definitions are equivalent: that is, *Cauchy sequences* are convergent sequences and convergent sequences are *Cauchy sequences*.

Theorem 2.5.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, there must existsa point $N \in \mathbb{N}$ after which we can conclude that

$$|x_n - x_m| < \epsilon.$$

Let $\epsilon > 0$. Since $(x_n) \to x$, we can choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|x_n - x| < \frac{\epsilon}{2}, \tag{1}$$

$$|x_m - x| < \frac{\epsilon}{2}. \tag{2}$$

Consider $|x_n - x_m|$. Then (1) and (2) imply that

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &< |x_n - x| + |x - x_n| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence.* ■

We can prove the other direction, by using either the *Bolzano Weierstrass Theorem* or the *Monotone Convergence Theorem.* This is a little bit more difficult since we need to have a proposed limit for the sequence to converge to.

Lemma 2.5.1. *Cauchy sequences are bounded.*

Proof. Given $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < 1$ for amm $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$ (just substituted $m = N$ here). Hence, define

$$M = \max\{|x_1|, |x_1|, |x_1|, ..., |x_{N-1}|, |x_N| + 1\}.$$

Therefore, $|x_n| < M$ for all $n \in \mathbb{N}$ Hence, the *Cauchy sequence* (x_n) is *bounded.* ■

Theorem 2.5.2. *A sequence converges if and only if it is a Cauchy sequence.*

Proof. ($\Rightarrow$) This direction is just Theorem 2.6.2 which we have proved above.

($\Leftarrow$) Suppose (x_n) is a *Cauchy sequence.* Let $\epsilon > 0$. Since (x_n) is a *bounded* sequence, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$ by the *Bolzano Weierstrass Theorem.* Let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, every $n_k \geq N$ has the property

$$|x_{n_k} - x| < \epsilon.$$

■

Our goal now is to show that $(x_n) \to x$. Hence, consider $|x_n - x|$. Then for every $n, n_k \geq N$, we have

$$\begin{aligned}
|x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\
&< |x_n - x_{n_k}| + |x_{n_k} - x| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, $(x_n) \to x$.

2.5.1 Completeness Revisited

We can summarize all of our results thus far in the following way

$$\text{AOC} \begin{cases} \text{NIP} \implies \text{BW} \implies \text{CC} \\ \text{MCT} \end{cases}$$

where AOC is our defining axiom to base all our reults on and giving us the notion that an ordered field contains no holes. We could also take the MCT to be our defining axiom and gives us the notion of least upper bounds by proving NIP. In addition, we could also take NIP to be our starting point but we need to have an extra hypothesis; that is, the Archimedean Property to prove all our results above (This is unavoidable).

It could be possible to assume the Arcimedean property holds, suppose one of the results we have proven is true, and derive the others yet this is sort of limited since $\mathbb{Q}$ contains a set that is not complete.

Below is the least of implications we can prove based on which theorem we would like to select asour defining axiom. Hence, we have

$$\text{NIP} + \text{Archimedean Property} \implies \text{AOC}$$

and

$$\text{BW} \implies \text{MCT} \implies \text{Archimedean Property}$$

2.5.2 Exercises

Exercise 2.6.1

Prove that every convergent sequence is *Cauchy*.

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, we must have for every $\epsilon > 0$, there must exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we must have

$$|x_n - x_m| < \epsilon.$$

Let $\epsilon > 0$. Since $(x_n) \to x$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we must have

$$|x_n - x| < \frac{\epsilon}{2}.$$

Since (x_n) converges to x, every subsequence (x_{n_k}) of (x_n) converges to x. This means for $n_k \geq N$, we also have

$$|x_{n_k} - x| < \frac{\epsilon}{2}.$$

Now consider $|x_n - x_{n_k}|$ and assume $n, n_k \geq N$. Then by the *triangle inequality*, we can write

$$\begin{aligned} |x_n - x_{n_k}| &= |x_n - x + x - x_{n_k}| \\ &< |x_n - x| + |x - x_{n_k}| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence.* ■

Exercise 2.6.2

Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not *monotone.*

Solution. This is possible. Suppose (x_n) is defined such that

$$x_n = \frac{(-1)^n}{n}.$$

We have (x_n) is Cauchy and thus convergent but it is not monotonic. ■

Takeaway: Just because a sequence is convergent does not generally imply that it is monotonic.

(b) A cauchy sequence with an unbounded subsequence.

Solution. This is not possible since cauchy sequences must be convergent and convergent sequences are bounded which means every subsequence is bounded as well. ■

(c) A divergent monotone sequence with a Cauchy subsequence.

Solution. This is not possible. A divergent monotone sequence must contain divergent subsequences. Thus, these subsequences cannot be Cauchy by the Cauchy Criterion. ■

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. This is possible. Let's define the following sequence

$$(1, 2, 1, 4, 1, 6, 1, 8, ...)$$

where

$$x_n = \begin{cases} 1 \text{ if } n \text{ odd} \\ \text{even if } n \text{ even} \end{cases}.$$

is an *unbounded* sequence. As we can see, if we take the subsequence (x_{2k+1}), then we find the subsequence

$$(1, 1, 1, 1, 1, ...)$$

converges to 1. ■

Exercise 2.6.3

If (x_n) and (y_n) are *Cauchy* sequences, then one easy way to prove that $(x_n + y_n)$ is *Cauchy* is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the *Algebraic Limit Theorem* then implies $(x_n + y_n)$ is convergent and hence *Cauchy*.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algbraic Limit Theorem.

Proof. Suppose (x_n) and (y_n) are *Cauchy Sequences*. Our goal is to show that $(x_n + y_n)$ is also a *Cauchy* sequence. Since (x_n) is *Cauchy*, let $\epsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ for every $m, n \geq N$, we have

$$|x_n - x_m| < \frac{\epsilon}{2}.$$

Likewise, there exists $N_2 \in \mathbb{N}$ such that for every $m, n \geq N$, we have

$$|y_n - y_m| < \frac{\epsilon}{2}.$$

Our goal is to show that

$$|(x_n + y_n) - (x_m + y_m)| < \epsilon$$

Now choose $N = \max\{N_1, N_2\}$ such that $n, m \geq N$ and using the triangle inequality, we write

$$\begin{aligned}|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &< |x_n - x_m| + |y_n - y_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence, we have $(x_n + y_n)$ is also *Cauchy*. ■

(b) Do the same for the product $(x_n y_n)$.

Proof. Suppose (x_n) and (y_n) are both *Cauchy*. This implies (x_n) and (y_n) are *bounded* as well as their subsequences (x_{n_k}) and (y_{b_k}). Hence, there exists $N_1, N_2 \in \mathbb{N}$ such that for every $n_k \geq N_1, N_2$, Since (x_{n_k}) converges and (y_n) are Cauchy sequences, it follows that they are also *bounded*. Hence, there exists $M_1, M_2 > 0$ such that $|y_n| < M_1$ and $|x_{n_k}| < M_2$ for all n, n_k. Our goal is to show

$$|x_n y_n - x_{n_k} y_{n_k}| < \epsilon.$$

Choose $N = \max\{N_1, N_2\}$ such that for every $n, n_k \geq N$

$$\begin{aligned}|x_n y_n - x_{n_k} y_{n_k}| &= |x_n y_n - x_{n_k} y_n + x_{n_k} y_n - x_{n_k} y_{n_k}| \\ &= |y_n(x_n - x_{n_k}) + x_{n_k}(y_n - y_{n_k})| \\ &< |y_n||x_n - x_{n_k}| + |x_{n_k}||y_n - y_{n_k}| \\ &< M_1 \cdot \frac{\epsilon}{2M_1} + M_2 \cdot \frac{\epsilon}{2M_2} \\ &= \epsilon.\end{aligned}$$

Hence, $(x_n y_n)$ is a *Cauchy sequence*. ■

Exercise 2.6.4

Let (a_n) and (b_n) be *Cauchy sequences*. Decide whether each of the following sequences is a *Cauchy sequence*, justifying each conclusion.

(a) $c_n = |a_n - b_n|$.

Solution. We claim that (c_n) is a *Cauchy* sequence. Let $\epsilon > 0$. We want to show that given some $N \in \mathbb{N}$, if $n, m \geq N$, then we have

$$|c_n - c_m| < \epsilon.$$

Then, by the *Reverse Triagle Inequality*

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)| \\
&= |(a_n - a_m) + (b_n - b_m)| \\
&< |a_n - a_m| + |b_n - b_m| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, (c_n) is *Cauchy sequence.* ■

(b) $c_n = (-1)^n a_n$.

Solution. This is false. Consider the *Cauchy sequence*

$$a_n = (1, 1, 1, , 1, ...)$$

If we take $c_n = (-1)^n a_n$, then (c_n) is not *Cauchy sequence* since

$$(1, -1, 1, -1, 1, ...)$$

is not. ■

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution. This is false. Consider (a_n) defined by the alternating sequence

$$a_n = \frac{(-1)^n}{n}.$$

This sequence is *Cauchy* but (c_n) is not because we have for all $n \in \mathbb{N}$

$$c_n = \left[\left[\frac{(-1)^n}{n}\right]\right] = \begin{cases} 0 \text{ if } n = 2k \\ -1 \text{ if } n = 2k+1. \end{cases}$$

which diverges. ■

Exercise 2.6.5

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

Solution. False. Take the sequence $a_n = n$ and note that $|a_{n+1} - a_n| < \epsilon$ given any $\epsilon > 0$, but a_n is *unbounded*. ■

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution. Suppose (x_n) and (y_n) are *pseudo-Cauchy*. We want to show that $(x_n + y_n)$ is also *pseudo-Cauchy*. Let $\epsilon > 0$. Choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we have

$$\begin{aligned} |(x_{n+1} + y_{n+1}) - (x_n + y_n)| &= |(x_{n+1} - x_n) + (y_{n+1} - y_n)| \\ &< |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

by the *Triangle Inequality*. Hence, $(x_n + y_n)$ are *pseudo-Cauchy*. ■

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness ans the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without manking any appeal to the Archimedean Property. This shows that BW, AOC, and MCT are all equivalent.

Proof. Suppose (x_n) is a *bounded* and *monotone* sequence. Our goal is to show that $(x_n) \to x$. By assumption, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for every $n_k \geq N$, we have

$$|x_{n_k} - x| < \epsilon.$$

Since (x_n) is *monotone*, then either $n_k \geq n$ or $n \geq n_k$. If $n \geq n_k \geq N$ for all $n \in \mathbb{N}$, then $|x_n - x| < \epsilon$. If $n_k \geq n$, then for any choice of $n \geq N$, we observe that

$$|x_n - x| \leq |x_{n_k} - x| < \epsilon.$$

Hence, we conclude that (x_n) is a convergent sequence. ■

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

Proof. Assume the *Cauchy Criterion* holds. We want to show that there exists (x_{n_k}) such that $(x_{n_k}) \to x$. Since (x_n) *bounded above* and *non-empty*, $x = \sup(x_n)$ exists. Furthermore, $(x_n) \to x$ since (x_n) is *Cauchy*. Since n_k is an *increasing* set of natural numbers and (x_n) is *bounded above*, we have that

$$x_n - \frac{1}{n_k} \leq x_{n_k} \leq x.$$

By the *Squeeze Theorem*, we have $(x_{n_k}) \to x$. ■

2.6 Properties of Infinite Series

We have learned the convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n) where

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim s_n = A.$$

We called (s_n) the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$. Just like the *Algebraic Limit Theorem* for sequences, we can also do the same thing for series.

Theorem 2.6.1 (Algebraic Limit Theorem for Series). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof. Suppose $\sum_{k=1}^{\infty} a_k = A$ and let $c \in \mathbb{R}$. Define the sequence of partial sums of $\sum_{k=1}^{\infty} ca_k$ as

$$t_k = cs_n = ca_1 + ca_2 + ca_3 + ... + ca_n.$$

By the *Algebraic Limit Theorem*, we know that $\lim cs_n = cA$. Hence,

$$\sum_{k=1}^{\infty} ca_k = cA.$$

To prove the addition rule, suppose $\sum_{k=1}^{\infty} b_k = B$. We want to show that

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B.$$

Define the sequence of partial sums for the two series as the following:

$$t_k = a_1 + a_2 + ... + a_n,$$
$$u_k = b_1 + b_2 + ... + b_n$$

Since $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, their sequence of partial sums also converges to the same value. Hence, let $\lim t_k = A$ and $\lim u_k = B$. By the *Algebraic Limit Theorem*, the sum of these two limits also converges i.e

$$\lim(t_k + u_k) = \lim t_k + \lim u_k = A + B.$$

Hence,

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B$$

■

We can summarize this theorem by keeping in mind that we can perform distribution over infinite addition and that we can add two infinite series together.

Theorem 2.6.2 (Cauchy Criterion for Series)**.** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Proof. Let $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Suppose $\sum_{k=1}^{\infty} a_k$ converges. This is true if and only if the sequence of partial sums (t_k) converges. This is true if and only if (s_k) is *Cauchy* by the *Cauchy Criterion.* Hence, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$

$$|s_n - s_m| < \epsilon.$$

Note that

$$\begin{aligned}
|s_n - s_m| &= |\sum_{k=m+1}^{\infty} a_k - \sum_{k=m}^{m} a_k| \\
&= |\sum_{k=m+1}^{n} a_k| \\
&= |a_{m+1} + ... + a_n| < \epsilon
\end{aligned}$$

■

This gives us the opportunity to prove some basic facts about series.

Theorem 2.6.3. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Proof. From the last theorem, we note that for every $\epsilon > 0$ such that whenever $n \geq m \geq N$, we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{\infty} a_k - 0 \right| < \epsilon$$

implies that $(a_n) \to 0$. ■

Keep in mind that the converse of this statement is not true! Just because (a_k) tends to 0 does not immediately imply that the series converges!

Theorem 2.6.4 (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then we have*

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let us show part (i). Suppose $\sum_{k=1}^{\infty} b_k$ converges. We want to show that $\sum_{k=1}^{\infty} a_k$ converges. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n > m \geq N$ and the fact that $a_k \leq b_k$ for all $k \in \mathbb{N}$

$$\begin{aligned} \Big| \sum_{k=m+1}^{n} a_k \Big| &\leq \Big| \sum_{k=m+1}^{n} b_k \Big| \\ &< \epsilon. \end{aligned}$$

Hence, a_k converges as well.

Note that part (ii) is just the contrapositive of part (i) which is also true. ■

Note that the convergence of sequences and series are relatively immutable when it comes to changes in some finite number of initial terms: that is, the behavior of sequences and series can be found past some choice of $N \in \mathbb{N}$. In order for the above test to be of any use to us, it is important to have a few examples under our belt i.e any $p > 1$ implies that

$$\sum_{n=1}^{\infty} 1/n^p \text{ converges if and only if } p > 1.$$

Example. A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots .$$

If $r = 1$ and $a \neq 0$, the series diverges. We can use the following algebraic identity, for $r \neq 1$, to write the following:

$$(1-r)(1 + r + r^2 \dots + r^{m-1}) = 1 - r^m$$

which allows us to rewrite the partial sum (s_m) of the above series to say that

$$s_m = a + ar + ar^2 + ar^3 + \dots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

where $s_m = at_m$ where

$$t_m = 1 + r + r^2 + ... + r^{m-1}$$

is a convergent sequence. Using the *Algebraic Limit Theorem*, therefore, allows us to say that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

The next theorem is a modification of the *Comparison Test* to handle series that contain negative terms.

Theorem 2.6.5 (Absolute Convergence Test)**.** *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Proof. Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. We want to show that $\sum_{n=1}^{\infty} a_n$ converges as well. Let $\epsilon > 0$. By the *Cauchy Criterion* for series, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have

$$\begin{aligned} \Big|\sum_{k=m+1}^{n} a_k\Big| &\leq \sum_{k=m+1}^{n} |a_k| \\ &< \epsilon. \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges. ■

Note that the converse of the above statement is false as taking the absolute value of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$$

produces the regular harmonic series which *diverges*.

Theorem 2.6.6 (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 ... \geq a_n \geq a_{n+1} \geq ...$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof. See exercise 2.7.1 for proof ■

Definition 2.6.1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

We can chart a few examples of some *conditionally convergent* series and *absolutely convergent* series.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \implies$ *conditionally convergent*
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \sum_{n=1}^{\infty} \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \implies$ *converges absolutely*

This tells us that any convergent series with positive terms must converge absolutely.

2.6.1 Rearrangements

We can obtain a rearrangement of an infinite series by permuting terms in the sum in some other order. In order for a sum to be a valid rearrangement, all the terms must appear and there should be no repeats.

Definition 2.6.2. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

We can now explain the weird behavior for why the *harmonic series* converges to a different limit when rearranging the terms; that is, it is because the *harmonic series* is a *conditionally convergent* series which leads us to the next theorem.

Theorem 2.6.7. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ *converges absolutely* to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let us define the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ as

$$s_n = \sum_{k=1}^{n} a_k$$

and the sequence of partial sums for the rearranged series $\sum_{n=1}^{\infty} b_n$ as

$$t_m = \sum_{k=1}^{m} b_k.$$

Since $\sum_{n=1}^{\infty} a_n$ *converges absolutely*, let $\epsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|s_n - A| < \frac{\epsilon}{2}$$

as well some $N_2 \in \mathbb{N}$ such that whenever $n > m \geq N_2$, we have

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}.$$

All that is left to do is to set a point in the sequence of the rearranged series where our ultimate goal is to have $|t_m - A| < \epsilon$. Hence, define

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

Let $m \geq M$ such that, when using the *triangle inequality*, we get

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have that $\sum_{n=1}^{\infty} b_n$ converges to A. ■

2.6.2 Definitions

Theorem 2.6.8 (Algebraic Limit Theorem for Series). *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Theorem 2.6.9 (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Theorem 2.6.10. *If the series* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Theorem 2.6.11 (Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$. *Then we have*

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Theorem 2.6.12 (Absolute Convergence Test)**.** *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges as well.*

Definition 2.6.3. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

Definition 2.6.4. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Theorem 2.6.13. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Theorem 2.6.14 (Alternating Series Test)**.** *Let* (a_n) *be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 ... \geq a_n \geq a_{n+1} \geq ...$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

2.6.3 Exercises

Exercise 2.7.1

Proving the *Alternating Series Test* amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - ... \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n). Different characterizations of completeness lead to different proofs.

(a) Prove the *Alternating Series Test* by showing that (s_n) is a *Cauchy Sequence*.

Proof. Let (a_n) be a *decreasing sequence* and suppose $(a_n) \to 0$. We want to show that the *Alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ meets the *Cauchy Criterion*.

We first need to show that for every $n > m$, we have the property

$$0 \leq |a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n| \leq |a_{m+1}|$$

Hence, we proceed by induction on k. Note that

$$\sum_{k=m+1}^{n}(-1)^{k+1}a_k = a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n.$$

Let our base case be $P(1)$. Then $a_{m+1} \geq 0$. For $P(2)$, we have $a_{m+1} \geq a_{m+2}$ for all m since (a_n) is a *decreasing sequence*. Suppose this holds for all $m \leq k-1$. We want to show that this holds for $P(k)$. Since (a_n) is *decreasing*, we have that $a_{k-1} \geq a_k$. Hence, $a_{k-1} - a_k \geq 0$. Since $P(k-1)$ holds where

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_{k-1} \leq a_{m+1}.$$

But this means that every term leading up to a_k is bounded by a_{m+1}. Hence,

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_k \leq a_{m+1}.$$

Let $\epsilon > 0$. All is left to show is that

$$\Big|\sum_{k=1}^{n}(-1)^{k+1}a_k\Big| < \epsilon.$$

Hence, for some $N \in \mathbb{N}$, let $n > m \geq N$ and $(a_n) \to 0$ such that

$$\begin{aligned}\Big|\sum_{k=1}^{\infty}(-1)^{k+1}a_k\Big| &\leq |a_{m+1}| \\ &< \epsilon.\end{aligned}$$

Hence, the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ meets the *Cauchy Criterion.* ■

(b) Supply another proof for this result using the Nested Interval Property.

Proof. Suppose (a_n) is *decreasing* sequence and $(a_n) \to 0$. Our goal is to show the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Since (a_n) is *decreasing*, we can use the *Nested Interval Property* to construct closed intervals $I_n = [s_n, s_{n+1}]$ such that the length of these intervals is $|s_n - s_{n+1}| \leq a_n$. The *Nested Interval Property* gurantees the following property that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

where $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Hence, $S \in \mathbb{R}$ can be our candidate limit since $S \in I_n$ for all n. Let $\epsilon > 0$. Since $(a_n) \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$

$$|s_n - S| \leq a_n < \epsilon.$$

■

Hence, $(s_n) \to S$.

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the *Monotone Convergence Theorem* leads to a third proof for the *Alternating Series Test.*

Proof. Define the subsequence of partial sums (s_{2n}) as

$$\sum_{k=1}^{n}(-1)^{2k}a_{2k}.$$

Since (a_n) is a *decreasing sequence*, we have that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}s_1 &= a_2 \geq 0 \\ s_2 &= a_2 + a_4 \geq s_1 \\ s_3 &= a_2 + a_4 + a_6 \geq s_2 \\ &\vdots \\ s_n &= a_2 + a_4 + a_6 + ... + a_{2n}.\end{aligned}$$

We can see that s_{2n} is an *increasing sequence*. Also, $|s_{2n}| < M$ since (a_n) is a *bounded sequence*. Hence, we can conclude that the subsequence of partial sums (s_{2n}) is converges to some $S \in \mathbb{R}$.

We can show that (s_{2n+1}) converges to S as well. Since $s_{2n+1} = s_{2n} + a_{2n+1}$, we can use the *Algebraic Limit Theorem* to say that

$$\begin{aligned}\lim(s_{2n+1}) &= \lim(s_{2n} + a_{2n+1}) \\ &= \lim(s_{2n}) + \lim(a_{2n+1}) \\ &= S + 0 \\ &= S.\end{aligned}$$

Since $(s_{2n}) \to S$ and $(s_{2n+1}) \to S$, we have $(s_n) \to S$ as well. ■

Exercise 2.7.4

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the *Cauchy Criterion* for Series.

Proof. Suppose (a_k) and (b_k) are sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Assume $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to show that $\sum_{k=1}^{\infty} a_k$ converges. Define the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ as

$$t_n = \sum_{k=1}^{n} a_k.$$

Let $\epsilon > 0$. Since $a_k \leq b_k$ and $\sum_{n=1}^{\infty} b_n$ converges, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, we have

$$\begin{aligned}|t_n - t_m| &= \Big|\sum_{k=m+1}^{n} a_k\Big| \\ &\leq \Big|\sum_{k=m+1}^{n} b_k\Big| \\ &< \epsilon.\end{aligned}$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ converges. Note that part (ii) is just the contrapositive of part (i). Hence, it is also true. ■

(b) Give another proof for the *Comparison Test*, this time using the *Monotone Convergence Theorem.*

Proof. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to use the *Monotone Convergence Theorem* to show that $\sum_{n=1}^{\infty} a_n$ converges i.e our goal is to show that the sequence of partial sums $t_n = \sum_{k=1}^{n} a_n$ is *bounded* and *monotone.*

Since the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$ are *bounded* and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, it follows that we have $|t_n| \leq M$ as well.

Now we want to show that (t_n) is a *decreasing sequence.* Since $\sum_{n=1}^{\infty} b_n$ is convergent, we know that $b_n \to 0$. Since $a_n \geq 0$ and $(b_n) \to 0$, the terms (t_n) must also be *decreasing*. Hence, $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$.

Since (t_n) is both *decreasing* and *bounded*, it follows that $\sum_{n=1}^{\infty} a_n$ is a convergent ■

Exercise 2.7.4

Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

Solution. Take $\sum x_n = (-1)^n$ and $\sum y_n = 1/n$. These two series diverge but $\sum x_n y_n = (-1)^n/n$ converges. ■

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

Solution. Take the convergent series $\sum 1/n^2$ and the bounded sequence $y_n = \sin(n)$. We have $\sum x_n y_n = \sum \sin(n)/n^2$ is divergent by the comparison test. ■

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converges but $\sum y_n$ diverges.

Solution. This is impossible. By the Algebraic Series Theorem, we cannot have $\sum (x_n + y_n)$ converge without $\sum y_n$ converging as well. ■

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution. By the comparison test, $\sum (-1)^n x_n$ diverges. ■

Exercise 2.7.5

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Corollary 2.6.14.1. *The series* $\sum_{n=1}^{\infty} 1/n^p$ *converges if and only if* $p > 1$.

Proof. We start with the backwards direction. Suppose $p > 1$. Our goal is to show that $\sum_{n=1}^{\infty} 1/n^p$ converges. Notice that $b_n = 1/n^p$ where $b_n \geq 0$ and b_n *decreasing*. By the *Cauchy Condensation Test*, we can prove that

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^p}\right)^n.$$

converges. Since $p > 1$, we have that

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^{p-1} = \sum_{n=0}^{\infty} \left(\frac{1}{2^p}\right)^n.$$

Since $|r| = |1/2^p| < 1$, we know that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ is a *Geometric Series*. By the *Cauchy Condensation Test*, we can say that $\sum_{n=1}^{\infty} b_n$ converges.

For the forwards direction, since $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges, the only reasonable choice of p is when $p > 1$ or else it is *Harmonic Series* which diverges. ■

Exercise 2.7.6

Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about *subvergent* series:

(a) If (a_n) is *bounded*, then $\sum a_n$ *subverges.*

Solution. This is a valid proposition since the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ are bounded which implies that the sequence of partial sums contains a subsequence partial sums that is convergent. Hence, we can say that $\sum a_n$ is a *subvergent* series. ■

(b) All convergent series are *subvergent.*

Solution. This is valid since the sequence of partial sums for a convergent series converges and hence all of the possible subsequence of partial sums for the series converges to the same limit. ■

(c) If $\sum |a_n|$ *subverges*, then $\sum a_n$ *subverges* as well.

Solution. This is not valid. ■

(d) If $\sum a_n$ *subverges*, then (a_n) has a convergent subsequence.

Solution. This is not valid. ■

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

Proof. Suppose for sake of contradiction that $\sum a_n$ converges. Hence, $(a_n) \to 0$. This means that $\lim(na_n) = 0$ but this contradicts our assumption that $\lim(na_n) = l \neq 0$. Hence, the series $\sum a_n$ must diverge. ■

Another why is to use the limit assumption directly.

Proof. Suppose $a_n > 0$ and $\lim(na_n) = l$. We want to show that $\sum a_n$ diverges. Since $\lim(na_n) = l \neq 0$, let $\epsilon = 1$ such that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ for all n, we have

$$|na_n - l| < 1 \iff a_n < \frac{1+l}{n}.$$

This implies that

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n}.$$

Note that $\sum \frac{1+l}{n}$ is not a *p-series* since n^p where $p = 1$. Hence, the series $\sum \frac{1+l}{n}$ diverges. Hence, we have that $\sum a_n$ is also a divergent series by the comparison test. ■

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Proof. Suppose $a_n > 0$ and $\lim(n^2 a_n)$ exists. Suppose $\lim(n^2 a_n) = L$ for some $L \in \mathbb{R}$. Let $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|n^2 a_n - L| < \epsilon.$$

Hence, we have

$$n^2 a_n - L < 1 \iff a_n < \frac{1+l}{n^2} \tag{1}$$

Our goal is to show via *comparison test* that the series $\sum a_n$ converges. From (1), we have

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n^2}.$$

Observe that the series $\sum \frac{1+l}{n^2}$ is a *p-series* test which converges. Hence, the series $\sum a_n$ converges by the *Comparison test.* ■

Exercise 2.7.8

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ *converges absolutely*, then $\sum a_n^2$ also *converges absolutely.*

Proof. Since $\sum a_n$ converges absolutely, then we have the series $\sum |a_n|$ converges. In order for $\sum a_n^2$ to converge absolutely, we need to show that $\sum |a_n^2|$ converges. Furthermore, (a_n) is a *bounded* sequence. Hence, there exists $M > 0$ such that $|a_n| \leq M$. Since there exists $N \in \mathbb{N}$, for any $n \geq N$, we can write

$$\begin{aligned} \sum |a_n^2| &= \sum |a_n \cdot a_n| \\ &= \sum |a_n| \cdot |a_n| \\ &\leq \sum M \cdot |a_n| \\ &= M \sum |a_n| \end{aligned}$$

We know by the *Algebraic Limit Theorem* for series that $M \sum |a_n|$ converges. Hence, the series $\sum a_n^2$ converges absolutely by the *Comparison Test.* ■

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

Proof. Since (b_n) converges, we have that (b_n) is also *bounded.* Hence, there exists $M > 0$ such that for all n we have $b_n \leq M$. Hence, we have

$$\sum a_n b_n \leq M \sum a_n.$$

By the *Algebraic Limit Theorem* for series, we have that $M\sum a_n$ converges. Since $a_nb_n \leq Ma_n$, we have that the series $\sum a_nb_n$ also converges by the *Comparison test.* ■

(c) If $\sum a_n$ *converges conditionally*, then $\sum n^2a_n$ diverges.

Solution. This is false. Consider the series $\sum \frac{(-1)^n}{n^2}$ which *converges conditionally* but note that $\sum n^2\frac{(-1)^n}{n^2} = \sum(-1)^n$ diverges. ■

Exercise 2.7.9 (Ratio Test).

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the *Ratio Test* states that if (a_n) satisfies

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n|r'$.

Proof. There exists $N \in \mathbb{N}$ such that $n \geq N$ because $\lim|\frac{a_{n+1}}{a_n}| = r$. This means that $|\frac{a_{n+1}}{a_n}|$ is *bounded*. Hence, we have that $|\frac{a_{n+1}}{a_n}| \leq r'$ which means that $|a_{n+1}| \leq r'|a_n|$. ■

(b) Why does $|a_N|\sum(r')^n$ converge?

Proof. The series $|a_N|\sum(r')^n$ converges because $|r'| < 1$ which means that $|a_N|\sum(r')^n$ is a *geometric series* which converges. ■

(c) Now, show that $\sum|a_n|$ converges, and conclude that $\sum a_n$ converges.

Proof. Consider the series $\sum|a_n|$ and the fact that

$$\sum|a_n| \leq |a_N|\sum(r')^n$$

for all $n \geq N$. Since the right hand series is *geometric* which converges, we can conclude that $\sum|a_n|$ also converges by the comparison test. Hence, the series $\sum a_n$ converges *absolutely* and thus the series $\sum a_n$ converges.

■

2.7 Double Summations and Products

We discovered in an earlier section that given any doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbb{N}\}$, it can be an ambiguous task to define

$$\sum_{i,j=1}^{\infty} a_{ij}. \tag{1}$$

We also observed that performing *iterated summations*can lead to different summations. Of course, this can be avoided completely if we were to define the partial sum of (1) in the following way

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

for $m, n \in \mathbb{N}$. In order for the sum of (1) to converge we have to have the following hold:

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n \to \infty} s_{mn}$$

Exercise 2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{mn}$. How does this value compare to the two iterated values for the sum already computed?

The double summation from section 2.1 is $a_{ij} = \frac{1}{2^{j-i}}$ where $\{a_{ij} : i, j \in \mathbb{N}\}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

Proof. To find $\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{mn}$, we first need to define the sequence of partial sums. We can fix j (the rows of the matrix) and define the sequence of partial sums for the series $\sum_{i,j=1}^{\infty} a_{ij}$ as

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}}\right) = -2 + \frac{1}{2^{n-1}}$$

which taking the limit leads to

$$\lim_{n\to\infty} \left(-2 + \frac{1}{2^{n-1}}\right) = -2.$$

∎

The issue of rearrangements to an infinite series arises due to commutativity of addition in an infinite context. It was found that having an absolutely converging infinite series fixes this problem.

Exercise 2.8.2

Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some $b_i \in \mathbb{R}$, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges.

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \tag{1}$$

converges. This means that the (1) meets the *Cauchy Criterion*. Let $\epsilon > 0$. This implies that there exists $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| < \epsilon.$$

Consider $\left|\sum_{(i,j)\in A(m,n)} a_{ij}\right|$ where

$$A(m,n) = \{(i,j) : 1 \leq i \leq j \leq n\}.$$

Using the *Triangle Inequality*, we find that j

$$\begin{aligned}\Big|s_{mm} - s_{nn}\Big| &= \Big|\sum_{(i,j)\in A(m,n)} a_{ij}\Big| \\ &\leq \sum_{(i,j)\in A(m,n)} |a_{ij}| \qquad (2) \\ &< \epsilon.\end{aligned}$$

Since (2) meets the *Cauchy Criterion* for series, we know that $\sum_{m,n}^{\infty} a_{ij}$ must be *Cauchy* and thus must converge as well. ■

Another proof using the Comparison Test goes something like this

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges. This means that for each $i \in \mathbb{N}$ the infinite series

$$\sum_{j=1}^{\infty} a_{ij} = r_i$$

for some $r_i \in \mathbb{R}$. Hence, we have the infinite series

$$\sum_{i=1}^{\infty} r_i. \tag{1}$$

Our goal is to show that (1) converges. Suppose we look at the terms

$$|r_i| = \Big|\sum_{j=1}^{\infty} a_{ij}\Big|.$$

Note by the *Triangle Inequality* that

$$\sum_{i=1}^{\infty}|r_i| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|.$$

by assumption the infinite series to the right converges. Hence, the series to the left must also converge by the *Comparison Test*. Since $\sum |r_i|$ converges, then the series $\sum r_i$ converges by the *Absolute Convergence Test* . ■

Theorem 2.7.1. *Let* $\{a_{ij} : i, j \in \mathbb{N}\}$ *be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ *and* $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ *converge to the same value. Moreover, we have that*

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1) , define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|.$$

■

Exercise 2.8.3

(a) Prove that (t_{nn}) converges.

Proof. From our definition of t_{nn} above we have

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|.$$

We want to show for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|t_{nn} - L| < \epsilon$. By assumption, we know that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \tag{1}$$

converges absolutely which implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges. Note that $t_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \to s_n = \sum_{i=1}^{\infty} r_i$ for some $r_i \in \mathbb{R}$. Furthermore, we have $s_n \to L$ since (1) converges. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have that

$$\begin{aligned} |t_{nn} - L| &= |t_{nn} - s_n + s_n - L| \\ &\leq |t_{nn} - s_n| + |s_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, the sequence of partial sums (t_{nn}) converges. ■

Another way we can prove this is to use the Monotone Convergence Theorem.

Proof. Our goal is to show that (t_{nn}) is converges to L. That is, our goal is to show that (t_{nn}) is bounded and monotone. We know that (t_{nn}) is monotone since all t_{nn} are non-negative terms and that $\sum_{n,m}^{\infty} |a_{ij}| = L$ where $L \geq 0$. To show that (t_{nn}) is bounded note that

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m}\sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Hence, (t_{nn}) is a bounded sequence. By the Monotone Convergence Theorem, (t_{nn}) converges. ■

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Proof. Suppose (t_{nn}) is a Cauchy Sequence. Then for some $N \in \mathbb{N}$ we have that for any $n \geq m > N$

$$|t_{nn} - t_{mm}| < \epsilon.$$

We can rewrite this in the following way to say that

$$|\sum_{n,m} t_{ij}| < \epsilon.$$

Our goal is to show that

$$|s_{nn} - s_{mm}| < \epsilon.$$

Hence, for any $n \geq m > N$, we have that

$$\begin{aligned} |s_{nn} - s_{mm}| &\leq |t_{nn} - t_{mm}| \\ &= \Big| \sum_{n,m} t_{ij} \Big| \\ &< \epsilon. \end{aligned}$$

Hence, (s_{nn}) converges.

■

Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Exercise 2.8.4

(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

Proof. Since (t_{mn}) bounded, we can say that $t_{mn} \leq B$. Since the set

$$\{t_{mn} : m, n \in \mathbb{N}\}$$

is bounded above and non-empty, we also have that $B = \sup\{t_{mn} : m, n \in \mathbb{N}\}$ exists. Hence, for any $\epsilon > 0$, we have that $B - \frac{\epsilon}{2}$ is not an upper bound. Hence, there exists some $t_{n_0 m_0}$ such that $B - \frac{\epsilon}{2} < t_{m_0 n_0} \leq t_{mn}$. Furthermore, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq m > N_1$ since (t_{mn}) converges. Hence, we must have that $B - \frac{\epsilon}{2} < t_{mn} \leq B$ ■

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Proof. Consider $|s_{mn} - S| < \epsilon$. Since $(s_{nn}) \to S$, let $\epsilon > 0$ such that for some $N_2 \in \mathbb{N}$ we have $n \geq m > N_2$, we have

$$|s_{nn} - S| < \frac{\epsilon}{2}.$$

Since (s_{nn}) meets the Cauchy Criterion, we have that there exists $N_2 \in \mathbb{N}$ such that for any $n \geq m > N$, we have

$$|s_{nn} - s_{mn}| < \frac{\epsilon}{2}.$$

Hence, observe that for any $n \geq m > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} - s_{nn} + s_{nn} - S| \\
&\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, we have that $(s_{mn}) \to S$. ■

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 2.8.5

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + ... + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

Proof. By exercise 2.8.4, we know that $s_{mn} \to S$. Note that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} r_i \text{ for each } i.$$

Hence, we have

$$\lim_{m,n\to\infty} s_{mn} = \lim_{m\to\infty} \sum_{i=1}^{m} r_i = S$$

which is equivalent to saying that for all $m > N$ for some $N \in \mathbb{N}$ we have that

$$\Big|\Big(\sum_{i=1}^{m} r_i\Big) - S\Big| \leq \epsilon.$$

■

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} a_{ij}$ converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Proof. Using the same process above for summing up the columns of $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ leads to

$$\Big|\Big(\sum_{j=1}^{n} c_j\Big) - S\Big| \leq \epsilon.$$

Hence, we must have that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

■

Another way of computing double sums is to sum along the diagonals of a rectangular matrix.

Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array where

$$d_2 = a_{11}, d_3 = a_{12} + a_{21}, d_4 = a_{13} + a_{22} + a_{31}$$

and in general

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$

Then, $\sum_{k=2}^{\infty} d_k$ represents another reasonable way of summing over every a_{ij} in the array.

Exercise 2.8.6

(a) Assuming the hypothesis and hence the conclusion of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

Proof. Our goal is to show that $\sum_{k=1}^{\infty} |d_k|$ converges. Since

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}$$

we can define the sequence of partial sums for $\sum_{k=1}^{\infty} |d_k|$ as

$$\sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} \Big|. \tag{1}$$

We know by Theorem 2.8.1 that the series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges. Hence, we can write

$$\sum_{k=2}^{n} |d_k| = \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} \Big| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

using the Triangle Inequality. By the Comparison Test, we must have that (1) converges as well. ■

2.7.1 Products of Series

We can take the product of two series by doing the following algebra below:

$$\begin{aligned}
\Big(\sum_{i=1}^{\infty} a_i\Big)\Big(\sum_{j=1}^{\infty} b_j\Big) &= (a_1 + a_2 + a_3 + \dots)(b_1 + b_2 + b_3 \dots) \\
&= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_3 b_1 + a_2 b_2 + a_1 b_3) + ... \\
&= \sum_{k=2}^{\infty} d_k
\end{aligned}$$

where

$$d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-2} b_1.$$

Exercise 2.8.7

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges so that we may apply Theorem 2.8.1.

Proof. Our goal is to show that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j$ converges absolutely. Observe that

$$\begin{aligned}\sum_{i=1}^{\infty}\Big|\sum_{j=1}^{\infty}a_ib_j\Big| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j| && (1)\\ &= \sum_{i=1}^{\infty}|a_i|\Big(\sum_{j=1}^{\infty}|b_j|\Big)\\ &= \sum_{i=1}^{\infty}|a_i|\cdot|B| && (2)\end{aligned}$$

Since (2) converges absolutely by the Algebraic Series Theorem, we have that (1) converges. converges. ■

(b) Let $s_{nn}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j$, and prove that $\lim_{n\to\infty}s_{nn}=AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_ib_j=\sum_{k=2}^{\infty}d_k=AB,$$

where, as before, $d_k=a_1b_{k-1}+a_2b_{k-2}+\cdots+a_{k-1}b_1$.

Proof. We can show that (t_{nn}) converges via the Monotone Convergence Theorem where

$$t_{nn}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_jj|.$$

Observe that all the terms of (t_{nn}) are positive and increasing. Now all we need to show is that (t_{nn}) is bounded. Since $\sum_{i=1}^{n}|a_i|\leq M$ and $\sum_{j=1}^{\infty}|b_i|\leq L$ for some $M,L\in\mathbb{R}$, we have that

$$\begin{aligned}t_{nn}=\sum_{i=1}^{n}\sum_{j=1}^{n}|a_ib_j| &\leq \sum_{i=1}^{\infty}|a_j|\sum_{j=1}^{\infty}|b_j|\\ &\leq M\cdot L.\end{aligned}$$

Hence, (t_{nn}) is a bounded sequence of partial sums. Now by theorem 2.8.1, we can say that

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB.$$

■

Chapter 3

Basic Topology of The Real Numbers

3.1 Discussion: The Cantor Set

The following construction demonstrates that $\mathbb{R}$ is an uncountable set. Let C_0 be the closed interval $[0,1]$, and define C_1 to be the set that results when we remove an open set in the middle third; that is,

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

We can construct the next iteration C_2 in a similar way above of each of the two sets unioned above. Hence, we have

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

or

$$C_n = [0,1] \setminus \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{8}, \frac{8}{9}\right) \cup \dots\right]$$

If we continue this process inductively, then for each $n \in \mathbb{N}$, we get sets C_n consisting of 2^n closed intervals with each having a length of $1/3^n$. The Cantor set C is just the intersection of an infinite number of C_n; that is,

$$C = \bigcup_{n=0}^{\infty} C_n.$$

3.2 Open and Closed Sets

Recall that given any $\epsilon > 0$, the $\epsilon-$neighborhood of $a \in \mathbb{R}$ is the set

$$V_\epsilon = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

In other words, we have an open interval $(a - \epsilon, a + \epsilon)$ or $a - \epsilon < x < a + \epsilon$ centered at a with radius ϵ.

Definition 3.2.1. A set $A \subseteq \mathbb{R}$ is *open* if for all points $a \in A$ there exists an $\epsilon-$neighborhood $V_\epsilon(a) \subseteq A$.

Example. (i) The set $\mathbb{R}$ is an *open* set because for any $a \in \mathbb{R}$, we can pick a $\epsilon-$neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \subseteq \mathbb{R}$.

(ii) The empty set $\emptyset$ is an open subset of the real line. This statement is vacuously true due to the definition of an open *set* i.e this set has no interior points to consider so it is true by default.

(iii) Take any $c, d \in \mathbb{R}$ and create an open interval as such where

$$(c, d) = \{x \in \mathbb{R} : c < x < d\}.$$

To see why (c, d) is an *open* set, let $x \in (c, d)$ be an arbitrary point. Let $\epsilon = \min\{x - c, d - x\}$, then we can construct the following $\epsilon-$neighborhood where

$$V_\epsilon = \{x' \in \mathbb{R} : |x' - x| < \epsilon\}.$$

Theorem 3.2.1. *(i) The union of an arbitrary collection open sets is open.*
(ii) The intersection of a finite collection of open sets is open.

Proof. To prove (i), define $\{O_\lambda : \lambda \in A\}$ be a collection of open sets and let $O = \bigcup_{\lambda \in A} O_\lambda$. Let a be an arbitrary element of O. In order to show that O is *open*, we need to show that $V_\epsilon(a) \subseteq O$ where $V_\epsilon(a)$ is the $\epsilon-$neighborhood. Let $a \in O_\lambda$ be an arbitrary element. Since we have a collection of open sets

$$\{O_\lambda : \lambda \in A\}$$

we can create a $\epsilon-$neighborhood around $a \in O_\lambda$ for some $\lambda \in A$ such that $V_\epsilon(a) \subseteq O_\lambda$. But note that $O_\lambda \subseteq O$. Hence, we have that $V_\epsilon(a) \subseteq O = \bigcup_{\lambda \in A} O_\lambda$. Hence, O is an *open* set.

To prove (ii), suppose $O = \bigcap_{i=1}^{N} O_i$. Suppose $a \in O_i$ for all $1 \leq i \leq N$ where O_i is a collection of open sets. Hence, there exists an $\epsilon-$neighborhood for every O_i. We need only one value of ϵ to make this work so define $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, ...\epsilon_N\}$. This means that

$$V_{\epsilon_i}(a) \subseteq V_\epsilon(a) \subseteq O_i \subseteq O$$

Hence, we have

$$V_\epsilon(a) \subseteq \bigcap_{i=1}^{N} O_i.$$

■

3.2.1 Closed Sets

Definition 3.2.2. A point x is a *limit point* of a set A if every $\epsilon-$neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

In other words, we have the following intersection

$$x \notin V_\epsilon(x) \cap A.$$

This is another way of saying that a sequence approaches of values approaches the limit point x where $V_\epsilon(x)$ can be thought of as neighborhoods "clustering" around the point x.

Theorem 3.2.2. *A point x is a limit point of a set A if and only of $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Let $V_\epsilon(x)$ be an $\epsilon-$neighborhood around x. We want to show that $\lim a_n = x$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$. By definition,

$$V_\epsilon(x) = \{x \in \mathbb{R} : |a_n - x| < \epsilon\}.$$

Let $\epsilon = \frac{1}{n}$. Since x is a *limit point*, for each $n \in \mathbb{N}$, we can pick any point

$$a_n \in V_{1/n}(x) \cap A.$$

Then we have

$$|a_n - x| < \frac{1}{n}$$

which is equivalent to
$$x - \frac{1}{n} < a_n < x + \frac{1}{n}.$$
By the Algebraic limit theorem and Squeeze Theorem, we have that $(a_n) \to x$ where $a_n \neq x$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Suppose $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$. We want to show the converse. Let $\epsilon > 0$. Then By definition of $\lim a_n = x$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have
$$|a_n - x| < \epsilon.$$
But this is also the definition of an $\epsilon-$neighborhood. Hence, $a_n \neq x$ for all $n \in \mathbb{N}$ and $x \in A$ is a limit point implies
$$V_\epsilon(x) \cap A$$
for all $\epsilon-$neighborhoods. ■

Keep in mind that $a \in A$ means that there is a sequence in A such that $a_n = a, a, a, ...$ which is uninteresting for the most part. We can distinguish *limit points* from *isolated points*.

Definition 3.2.3. A point $a \in A$ is an *isolated point* of A if it is not a *limit point* of A.

Remember that an isolated point is always in the set A, but a limit point can be sometimes be outside of the set A. An example of this is the endpoint of an open interval. A sequence can approach the endpoint where $a_n \neq x$ for all $n \in \mathbb{N}$ but x is not in the set.

Definition 3.2.4. A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

In other words, can say that a set A is closed if sequences contained in A converge to their limits that are within the set A.

Theorem 3.2.3. *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Proof. Suppose $F \subseteq \mathbb{R}$ is closed. Let $x \in F$ be a limit point. Let (x_n) be a Cauchy sequence contained in F. By the Cauchy Criterion, (x_n) converges to $x \in F$. ∎

Example. (i) Consider the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Let's show that each point of A is isolated. We can show that each point of A is isolated. Given $\frac{1}{n} \in A$. Choose $\epsilon = \frac{1}{n} - \frac{1}{(n+1)}$. Then,

$$V_\epsilon(1/n) \cap A = \left\{ \frac{1}{n} \right\}.$$

It follows from Definition 3.2.4 that $\frac{1}{n}$ is not a limit point and so is isolated. Although all of the points of A are isolated, the set A does have only one limit point 0. The reason for this is can be explained by the very definition of A where $0 \notin A$. Since the limit of A is not contained in A, we can say that A is not closed. The set $F = A \cup \{0\}$ is an example of a closed set and is called the closure of A.

(ii) Let's prove that a closed interval

$$[c, d] = \{x \in \mathbb{R} : c \leq c \leq d\}$$

is a closed set using Definition 3.2.7. If x is a limit point of $[c, d]$, then by Theorem 3.2.5 there exists $(x_n) \subseteq [c, d]$ with $(x_n) \to x$. Since $(x_n) \to x$, we can use the Order Limit Theorem to say that

$$c \leq x_n \leq d \iff c \leq x \leq d.$$

This means $x \in [c, d]$ which proves that $[c, d]$ is a closed set.

(iii) Consider the set $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers. An interesting property of $\mathbb{Q}$ is that all of its limit points is actually all of $\mathbb{R}$. To see why this is so, let us have $y \in \mathbb{R}$ be arbitrary and construct $V_\epsilon(y)$ such that we have the open set $(y - \epsilon, y + \epsilon)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $x \in \mathbb{Q}$ where $x \neq y$ such that $x \in (y - \epsilon, y + \epsilon)$. Hence, y is a limit point of $\mathbb{Q}$.

We can actually restate the Density Property from the first chapter by saying the following:

Theorem 3.2.4. *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Proof. Let $y \in \mathbb{R}$ and let $\epsilon = \frac{1}{n}$. Create the following $\epsilon-$neighborhood $(y - \frac{1}{n}, y + \frac{1}{n})$. Since the end points of this $\epsilon-$neighborhood are real numbers, we can find a sequence of rational numbers $(x_n) \subseteq (y - \epsilon, y + \epsilon)$ by the Density of $\mathbb{Q}$ in $\mathbb{R}$ such that

$$y - \frac{1}{n} < x_n < y + \frac{1}{n}.$$

By the Squeeze Theorem, we can write that $(x_n) \to y$ where $x_n \neq y$ for all $n \in \mathbb{N}$. ■

3.2.2 Closure

Definition 3.2.5. Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be the $\bar{A} = A \cup L$.

Example. (i) Consider $A = \{1/n : n \in \mathbb{N}\}$, then the *closure of* A is just

$$\bar{A} = A \cup \{0\}.$$

(ii) In the last example, $y \notin (y - \epsilon, y + \epsilon)$ where $y \in \mathbb{R}$ gurantees that the closure of $\mathbb{Q}$ in $\mathbb{R}$; that is, $\bar{\mathbb{Q}} = \mathbb{R}$.

(iii) If A is an open interval (a, b), then the closure is just $\bar{A} = [a, b]$; that is, $\bar{A} = A \cup \{a, b\}$ where $a, b \in \mathbb{R}$ are the endpoints of the set (a, b).

(iv) If A is a closed interval then the closure is just $\bar{A} = A$. The obvious conclusion from this is that closed intervals are always closed sets.

Theorem 3.2.5. *For any $A \subseteq \mathbb{R}$, the closure of $\bar{A}$ is a closed set and is the smallest closed set containing A.*

Proof. Since L is the set of limit points of A, it follows immediately that $\bar{A}$ contains its limit points of A. The problem here is that taking the union of A and L could produce some new limit points.

The details are in exercise 3.2.7

Hence, any closed set containing A must contain L as well. Hence, we have $\bar{A} = A \cup L$ is the smallest closed set containing A. ■

3.2.3 Complements

The notions of open and closed imply that they are not antonyms of each other. Just because a set is not open, does not immediately imply that it is closed. We can see this in action by considering the half-open interval

$$(c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$$

as being neither open nor closed. Furthermore, $\mathbb{R}$ and $\emptyset$ are both simultaneously open and closed at the same time. Luckily, these are the only two sets that exhibit this confusing property. We do have a relationship between open and closed sets however.

Recall that the complement of a set $A \subseteq \mathbb{R}$ is defined to be the set

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

which describes all of the elements that are not in A.

Theorem 3.2.6. *A set A is open if and only if A^c is closed. Likewise, a set B is closed if and only if B^c is open.*

Proof. Suppose $A \subseteq \mathbb{R}$ is an open set. We want to show that A^c is a closed set. Let x be a limit point of A^c. Hence, there exists a sequence (x_n) such that $\lim x_n = x$ where $x_n \neq x$ for all $n \in \mathbb{N}$. By definition of $\lim x_n = x$, there is an $\epsilon-$neighborhood $V_\epsilon(x)$, but this means that $x \notin A$ and must be in A^c since every $\epsilon-$neighborhood of x intersects A at some point other than x. Hence, we have $x \in O^c$.

For the converse statement, we assume A^c is a closed set. We want to show that A is open. Hence, let $x \in A$. Since $x \in A$, x is not a limit point of A^c and A^c is a closed set, there must exist an $\epsilon-$neighborhood such that $x \notin V_\epsilon(x) \cap A^c$. This means $x \in A$ and so $V_\epsilon \subseteq A$. Hence, A is an open set. The second statement follows quickly when taking the complement of each going in each direction. ■

Theorem 3.2.7. *(i) The union of a finite collection of closed sets is closed.*
(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ it is true that

$$\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \quad \text{and} \quad \Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

■

3.2.4 Definitions

Definition 3.2.6. A set $A \subseteq \mathbb{R}$ is *open* if for all points $a \in A$ there exists an $\epsilon-$neighborhood $V_\epsilon(a) \subseteq A$.

Theorem 3.2.8. *(i) The union of an arbitrary collection open sets is open.*
(ii) The intersection of a finite collection of open sets is open.

Definition 3.2.7. A point x is a *limit point* of a set A if every $\epsilon-$neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Theorem 3.2.9. *A point x is a limit point of a set A if and only of $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition 3.2.8. A point $a \in A$ is an *isolated point* of A if it is not a *limit point* of A.

Definition 3.2.9. A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

Theorem 3.2.10. *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Theorem 3.2.11. *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Definition 3.2.10. Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be the $\bar{A} = A \cup L$.

Theorem 3.2.12. *For any $A \subseteq \mathbb{R}$, the closure of $\bar{A}$ is a closed set and is the smallest closed set containing A.*

Theorem 3.2.13. *A set A is open if and only if A^c is closed. Likewise, a set B is closed if and only if B^c is open.*

3.2.5 Exercises

Exercise 3.2.2

Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, ...\right\}$$

and

$$B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$

Answer the following questions for each set:

(a) What are the limit points?

Solution. The limit points of A is $L = \{-1, 1\}$ and the limit points of B is $L = \{0, 1\}$. ■

(b) Is the set open? Closed?

Solution. The sets A and B are not closed since their limit points are not contained and open since we can create $V_\epsilon(x) \subseteq A$ while B is not open since $V_\epsilon(x) \nsubseteq B$ for every $x \in \mathbb{Q}$ however small ϵ is. ■

(c) Does the set contain any isolated points?

Solution. From part (b), since we cannot find any points near each $x \in \mathbb{Q}$ in B, we have that all the points of B are isolated points. ■

(d) Find the closure of the set.

Solution. The closure of sets A and B are $\overline{A} = A \cup \{-1, 1\}$ and $\overline{B} = B \cup \{0, 1\}$. ■

Exercise 1.2.13

Show De Morgan's Laws where $\{A_i : 1 \leq i \leq n\}$ is a collection of sets such that

$$\Big(\bigcup_{i=1}^{n} A_i\Big)^c = \bigcap_{i=1}^{n} A_i^c \tag{1}$$

$$\Big(\bigcap_{i=1}^{n} A_i\Big)^c = \bigcup_{i=1}^{n} A_i^c \tag{2}$$

for any finite $n \in \mathbb{N}$.

Proof. Our goal is to show that both inclusions hold for (1) and (2). Our first step is to induct on $n \in \mathbb{N}$ to show that

$$\Big(\bigcup_{i=1}^{n} A_i\Big)^c \subseteq \bigcap_{i=1}^{n} A_i^c. \tag{1}$$

Let $n = 1$ be the base case. It follows immediately that $A_1^c \subseteq A_1^c$. Let $n = 2$, then it follows that $(A_1 \cup A_2)^c \subseteq A_1^c \cap A_2^c$ by exercise 1.2.5. For the other inclusion, we also have $A_1^c \cap A_2^c \subseteq (A_1 \cup A_2)^c$. Now suppose (1) holds for $1 \leq n \leq k-1$. We want to show that (1) holds for k. Let

$$A' = \bigcup_{n=1}^{k-1} A_n$$

then consider the following

$$\Big(\bigcup_{n=1}^{k} A_n\Big)^c = \Big(A_k \cup \Big[\bigcup_{n=1}^{k-1} A_n\Big]\Big)^c = (A_k \cup A')^c$$

Let $x \in (A_k \cup A')^c$, then we know that $x \notin (A_k \cup A')$. This means that $x \notin A_k$ and

$x \notin A'$. Hence, we have $x \in A_k^c$ and $x \in (A')^c$; that is,

$$\begin{aligned}(A_k \cup A')^c &\subseteq A_k^c \cap (A')^c \\ &= A_k^c \cap \Big(\bigcup_{n=1}^{k-1} A_n\Big)^c. \\ &\subseteq A_k^c \cap \Big(\bigcap_{n=1}^{k-1} A_n^c\Big) \\ &= A_k \cap (A_{k-1} \cap ... \cap A_1) \\ &= \bigcap_{n=1}^{k} A_n^c.\end{aligned}$$

Hence, we have

$$\Big(\bigcup_{i=1}^{n} A_i\Big)^c \subseteq \bigcap_{i=1}^{n} A_i^c.$$

For the other inclusion, suppose the containment

$$\bigcap_{n=1}^{k-1} A_k^c \subseteq \Big(\bigcup_{n=1}^{k-1} A_k\Big)^c \tag{2}$$

holds for $1 \leq n \leq k-1$. We want to show that (2) holds for k. Consider the finite intersection

$$\bigcap_{n=1}^{k} A_n^c = A_k^c \cap \Big(\bigcap_{n=1}^{k-1} A_n^c\Big).$$

If we know that $x \notin \bigcap_{n=1}^{k-1} A_n$ and $x \notin A_k$ then $x \notin \Big(A_k \cup \Big(\bigcap_{n=1}^{k-1} A_n\Big)\Big)$. Hence, using our inductive hypothesis, we have

$$\begin{aligned}\bigcap_{n=1}^{k} A_n^c &= A_k^c \cap \Big(\bigcap_{n=1}^{k-1} A_n^c\Big) \\ &\subseteq A_k^c \cup \Big(\bigcap_{n=1}^{k-1} A_n\Big)^c \\ &\subseteq A_k^c \cup \Big(\bigcup_{n=1}^{k-1} A_n^c\Big) \\ &= \Big(\bigcup_{n=1}^{k} A_n\Big)^c\end{aligned}$$

Since both containments hold, we must have

$$\Big(\bigcup_{n=1}^{k} A_n\Big)^c = \bigcap_{n=1}^{k} A_n^c.$$

The proof to the other equation is similar. ■

Exercise 3.2.4

Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

Proof. Let $A \neq \emptyset$ and bounded above. Since $s = \sup A$ exists we can let $\epsilon > 0$ such that for some $\alpha \in A$, we have $s - \epsilon < \alpha$. Our goal is to show that $s \in \overline{A}$. Let (a_n) be a sequence in A such that $a_n \neq s$ for all $n \in \mathbb{N}$. Let $\epsilon = 1/n$ such that

$$s - \frac{1}{n} < \alpha \leq a_n \leq s.$$

By the Squeeze Theorem, we have $\lim a_n = s = \sup A$. This means $s = \sup A$ is a limit point where $L = \{s\}$ such that $\overline{A} = A \cup L$. Hence, $s \in \overline{A}$.

■

(b) Can an open set contain its supremum?

Solution. An open set A cannot contain its supremum, which is a limit point in part (a), since otherwise A would be a closed set. ■

Exercise 3.2.5

Prove Theorem 3.2.8: Show that a set $F \subseteq \mathbb{R}$ is closed if and only if if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof. ($\Rightarrow$) Let $F \subseteq \mathbb{R}$ be a closed set. Let x be a limit point and let (x_n) be a Cauchy sequence be arbitrary. Since F is a closed set, the limit point $x \in F$; that is, $\lim x_n = x \in F$ where $x_n \neq x$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Let $F \subseteq \mathbb{R}$. We want to show that F is closed. Let (x_n) be a Cauchy sequence contained in F such that $\lim x_n = x \in F$. Note that $x_n \neq x$ for all $n \in \mathbb{N}$. Since all the limit points of F are contained in F, then F must be a closed set. ■

Exercise 3.2.7

Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

Proof. Let L be the set of limit points of A, and suppose that x is a limit point of L. Our goal is to show that x is a limit point of A. Let $V_\epsilon(x)$ be arbitrary. Let $\epsilon > 0$, then we know that $V_\epsilon(x)$ intersects L at a point $\ell \in L$ where $\ell \neq x$. Choose $\epsilon' > 0$ small enough so that $V_{\epsilon'}(\ell) \subseteq V_\epsilon(x)$ and $x \notin V_{\epsilon'}(\ell)$. Since $\ell \in L$, we know that ℓ is a limit point of Am and therefore x is a limit point of A and thus an element of L. ■

(b) Argue that if x is a limit point $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Proof. Suppose x is a limit point of $\overline{A} = A \cup L$. By definition, we can construct $V_\epsilon(x)$ such that $V_\epsilon(x)$ intersects $a \in \overline{A}$ where $a \neq x$. This means $x \in A$ or $x \in L$. If $x \in A$, then $V_\epsilon(x)$ intersects every point $a \in A$ where $x \neq a$. Hence, x is a limit point of A. If $x \in L$, then we can use the same argument from above to construct an $\epsilon' > 0$ small enough so that $V_{\epsilon'}(\ell) \subseteq V_\epsilon(x)$ where $x \notin V_{\epsilon'}(\ell)$. Since $\ell \in L$ is a limit point, this means that that $V_{\epsilon'}(\ell)$ intersects A. But since $x \notin V_{\epsilon'}(\ell)$ this means that $V_\epsilon(x)$ intersects A at every point of A that is not x. Hence, x is a limit point of A. ■

Exercise 3.2.9

A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \text{ and } \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

Proof. Suppose we show the first equation. Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$ then for all E_λ where $\lambda \in \Lambda$, we must have $x \notin E_\lambda$. But this is equivalent to saying that $x \in E_\lambda^c$ for all $\lambda \in \Lambda$ where $E_\lambda^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Hence, we have

$$x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

Now let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then for all $\lambda \in \Lambda$, we have $x \in E_\lambda^c$. This is true if and only if $x \notin E_\lambda$ for all $\lambda \in \Lambda$. Since $E_\lambda \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda$, we also have that $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. Hence, we have

$$x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c.$$

Hence,

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

The other equation can be shown through a similar process above. ■

(b) Now, provide the details for the proof of Theorem 3.2.14.

Proof. To prove part (i), suppose we have a finite collection of open sets where

$$\{E_i : 1 \leq i \leq N\}.$$

Since E_i closed, their complements E_i^c is open. Since the finite intersection of open sets is open, we have that

$$\left(\bigcup_{i=1}^N E_i\right)^c = \bigcap_{i=1}^N E_i^c$$

is open. But this means that

$$\bigcup_{i=1}^N E_i$$

is closed.

To prove part (ii), suppose we have an arbitrary collection of closed sets

$$\{E_\lambda : \lambda \in \Lambda\}.$$

Since E_λ is closed, we have that their complement E_λ^c is open. But this means that the union

$$\bigcup_{\lambda\in\Lambda} E_\lambda^c = \Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big)^c \tag{1}$$

is also open. But since the complement of the intersection of (1) is open, we have

$$\bigcap_{\lambda\in\Lambda} E_\lambda$$

is closed. ■

Exercise 3.2.11

(a) Prove that $\overline{A\cup B} = \overline{A}\cup\overline{B}$

Proof. We want to show the following containments

$$\overline{A\cup B} \subseteq \overline{A}\cup\overline{B},$$
$$\overline{A\cup B} \supseteq \overline{A}\cup\overline{B}$$

Suppose $x \in \overline{A\cup B}$. Then x is a limit point of $A \cup B$. Hence, either $x \in A$ or $x \in B$. But x is a limit point so there exists $V_\epsilon(x)$ that either intersects $a \neq x \in A$ or $b \neq x \in B$. But this means that x is a limit point of A or B. Hence, $x \in \overline{A}\cup\overline{B}$. Suppose $x \in \overline{A}\cup\overline{B}$. Then either $x \in \overline{A}$ or $x \in \overline{B}$. But this means that x is a limit point of A or B which imply that $V_\epsilon(x)$ intersects elements of both A or B that is not x. Hence, x must be a limit point of either A or B. Hence, $x \in \overline{A\cup B}$. Since both containments are true, we have that $\overline{A\cup B} = \overline{A}\cup\overline{B}$. ■

(b) Does this result about closures extend to infinite unions of sets?

Solution. No this result does not extend to infinite unions. Consider the counter-example where we have a closed set $H_n = [1/n, 1] \subseteq \mathbb{R}$ where

$$\bigcup_{i=1}^{\infty} \overline{H_n} = (0,1] \tag{1}$$

$$\overline{\bigcup_{i=1}^{\infty} H_n} = [0,1] \tag{2}$$

It is clear that (1) and (2) are not the same sets. ■

Exercise 3.2.14

A dual notion to the closure of a set is the interior of a set. The *interior* of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \exists V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors posses a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$. First we show that first statement.

Proof. ($\Rightarrow$) Suppose E is closed. We want to show that $\overline{E} = E$; that is, we want to show that $\overline{E} \subseteq E$ and $\overline{E} \supseteq E$. Note that the first containment follows immediately since $\overline{E}$ is the smallest set containing E. Now we want to show that $\overline{E} \supseteq E$. Let $x \in E$ be a limit point. Since x is a limit point and E is a closed set, we know that x must be contained in E. This means that set of limit points L of E must where $x \in L$ implies that $x \in \overline{E}$. Hence, $\overline{E} = E$.

($\Leftarrow$) It follows that E is closed since $\overline{E}$ contains its limit points and that $\overline{E} = E$. ■

Now we show the second statement

Proof. ($\Rightarrow$) Suppose E is an open set. We must show the following two containments: $E^\circ \subseteq E$ and $E^\circ \supseteq E$. We show the first containment. Let $x \in E^\circ$ be arbitrary. Then there exists $V_\epsilon(x)$ such that $V_\epsilon(x) \subseteq E$. Hence, $x \in E$ so we have $E^\circ \subseteq E$. Now we show that second containment. Since E is an open set, let $x \in E$ be arbitrary such that there exists $V_\epsilon(x) \subseteq E$. But this is by definition the interior of E so we must have $x \in E^\circ$.

($\Leftarrow$) Suppose $E = E^\circ$. We want to show that E is an open set. Let $x \in E$ be arbitrary. Since $E = E^\circ$, there exists $V_\epsilon(x)$ such that $V_\epsilon(x) \subseteq E$. But this means E is an open set by definition. ■

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Show that $\overline{E}^c = (E^c)^\circ$

Proof. We want to show that first equation; that is, we want to show the following two containments $\overline{E}^c \subseteq (E^c)^\circ$ and $\overline{E}^c \supseteq (E^c)^\circ$. First we show the former containment. Let $x \in \overline{E}^c$ be arbitrary. If $x \notin \overline{E}$, then x is not a limit point of E and $x \notin E$. But this means that $x \in (E^c)^\circ$ and hence, $\overline{E}^c \subseteq (E^c)^\circ$. Now we show the second containment. Let $x \in (E^c)^\circ$ be arbitrary. There exists $V_\epsilon(x) \subseteq E^c$. We can be sure that x is not a limit point of $\overline{E}$ since $\overline{E}$ contains all its limit points. Hence, we must have $x \in \overline{E}^c$. Hence, we have $\overline{E}^c = (E^c)^\circ$. ■

Now we show $(E^\circ)^c = \overline{E^c}$

Proof. We want to show the following two containments; namely, $(E^\circ)^c \subseteq \overline{E^c}$ and $\overline{E^c} \subseteq (E^\circ)^c$.

We start with the first containment. Let $x \in (E^\circ)^c$ be arbitrary. This means $x \notin E^\circ$ and hence for all $\epsilon-$neighborhoods of x, we have $V_\epsilon(x) \nsubseteq E$. Our goal is to show that $x \in \overline{E^c}$. If x is not a limit point of E^c, then we just have $x \in E^c$ and hence $x \in \overline{E^c}$. Otherwise, we can prove x is a limit point of E^c. Suppose L is the set of limit points of E^c. Let $\epsilon' > 0$ be as small as possible and $\ell \in L$ such that $V_{\epsilon'}(\ell) \subseteq V_\epsilon(x)$ where $x \notin V_{\epsilon'}(\ell)$. Since ℓ is a limit point of E^c, $V_{\epsilon'}(\ell)$ intersects E^c. But this also means $V_\epsilon(x)$ intersects points of E^c that is not x. Hence, x is a limit point of E^c and thus $x \in \overline{E^c}$.

Now let $x \in \overline{E^c}$ be arbitrary. Then either $x \in E^c$ or $x \in L$ where L denotes the set of limit points of E^c. If $x \in E^c$, then surely $x \notin E^\circ$. Hence, $x \in (E^\circ)^c$. If $x \in L$ and $\overline{E}$ is a closed set, then x cannot be in E°. Hence, x must be in $(E^\circ)^c$. Hence $\overline{E^c} \subseteq (E^\circ)^c$

■

3.3 Compact Sets

3.3.1 Compactness

Definition 3.3.1. A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.

Example. (i) Closed intervals are compact since all (a_n) in K are bounded and so we can always find a subsequence in K (By Bolzano-Weierstrass) that converges to a limit that is contained within the closed interval. We know the limit is contained in K since closed intervals are closed sets.

In the example above, we used the fact that bounded sequences contain subsequences that converge and the fact that their limits are contained since closed intervals are closed sets.

Definition 3.3.2. A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 3.3.1 (Characterization of Compactness in $\mathbb{R}$). *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. Assume K is a compact set. Suppose for sake of contradiction that K is not a bounded set. Our goal is to construct a sequence (a_n) that diverges. Since K is not bounded, for all $M > 0$, there exists $a_n \in A$ such that $|a_n| > M$. But by assumption, K is compact so (a_n) must contain a subsequence (a_{n_k}) that converges to a limit in K. But since (a_{n_k}) is unbounded, we have a contradiction. Hence, K must be a bounded set.

Now we will show that K is closed. Since K has to be bounded, then $(a_n) \subseteq K$ must contain a subsequence (a_{n_k}) that converges to a limit in K. But this is the definition of a closed set. Hence, K is a closed set.

Assume that K is closed and bounded. Let (a_n) be an arbitrary sequence in K. Since K is bounded and hence (a_n) is bounded, (a_n) contains a subsequence (a_{n_k}) such that $(a_{n_k}) \to a$. Since K is closed, a is contained in K. Hence, we have that K is a compact set. ■

It is important remember that closed intervals are not all that is when considering compact sets. The structure is much more intricate and interesting. For example, we can say that the Cantor Set is compact. We can experiment with this new idea of compact sets with the Nested Interval Property from chapter 1.

Theorem 3.3.2 (Nested Compact Set Property)**.** *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ *is not empty.*

Proof. Assume $K_n \neq \emptyset$ compact for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, choose $x_n \in K_n$ where x_n is a sequence of points. Since we have a nested sequence of nonempty sets, it follows that $x_n \in K_1$. By definition of compactness, (x_n) contains a convergent subsequence (x_{n_k}) such that $\lim x_{n_k} = x$ where $x \in K_n$ for every $n \in \mathbb{N}$ and thus $x \in K_1$. Given $n_0 \in \mathbb{N}$, we have that the terms of the sequence x_n are contained within K_{n_0} given all $n \geq n_0$. We can ignore the finite number of terms for which $n_k < n_0$ so that (x_{n_k}) can be contained in K_{n_0}. Hence, we have that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, we have that $x \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. ■

3.3.2 Open Covers

In the last section, we proved that compact sets to be bounded and closed and vice versa. In other cases, we could also have defined compacts in this way and then proved that for every sequences that is bounded, there exists subsequences that converge to limits contained within the set. We can prove compactness in terms of open covers and finite subcovers.

Definition 3.3.3. Let $A \subseteq \mathbb{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.$$

Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Subcollection in this context is just a collection of sets that are subsets of the original collection of open sets.

Example. Consider the open interval $(0, 1)$. For each point $x \in (0, 1)$, let O_x be the open interval $(x/2, 1)$. Let the infinite collection of O_x be defined as

$$\{O_x : x \in (0,1)\}$$

forms an open cover for the open interval $(0,1)$; that is,

$$(0,1) \subseteq \bigcup_{x\in(0,1)} O_x.$$

Note that it is impossible to find a finite subcover for the open set $(0,1)$. Given any proposed finite subcollection

$$\{O_{x_1}, O_{x_2}, ..., O_{x_n}\},$$

let $x' = \min\{x_1, x_2, \dots, x_n\}$ and observe that for any $y \in \mathbb{R}$ satisfying $0 < y \leq x'/2$ is not contained in the union $\bigcup_{i=1}^{n} O_{x_i}$.

Now consider a similar cover for the closed interval $[0,1]$. For $x \in (0,1)$, the sets $O_x = (x/2, 1)$ do relatively well to cover $(0,1)$, but in order to have an open cover for the closed interval $[0,1]$, we could let $\epsilon > 0$ so that we can have epsilon neighborhoods covering both endpoints. That is, we have $O_o = (-\epsilon, \epsilon)$ and $O_1 = (1-\epsilon, 1+\epsilon)$. Then the collection

$$\{O_0, O_1, O_x : x \in (0,1)\}$$

is an open cover for $[0,1]$ is a finite subcover for the closed interval $[0,1]$.

Theorem 3.3.3 (Heine-Borel Theorem)**.** *Let K be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others.*

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover for K has a finite subcover.

Proof. Notice that the proof of the equivalence of (i) and (ii) has already been proven in Theorem 3.3.1. All we need to show now is that (iii) implies (ii) and (iii) implies (i).

To show (ii), we must show that K is both bounded and closed. To show that K is bounded, let us construct an open cover for K by defining O_x to be an open interval of radius 1 ($\epsilon = 1$) for each $x \in K$. This means there exists a $\epsilon-$neighborhood for each $x \in K$; that is, $O_x = V_1(x)$. Since the open cover $\{O_x : x \in K\}$ contains a finite subcover for K, we have the K contained in the union of the collection of sets $\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$. Hence, K itself is a bounded set.

Now let us show that K is closed. Let (y_n) be a Cauchy sequence contained in K with $\lim y_n = y$. To show that K is closed, we must show that $y \in K$. Suppose for sake of contradiction that $y \notin K$. By assumption we can construct an open cover by taking O_x to be an interval of radius $|x - y|/2$ around each point $x \in K$. Also, we are assuming that the open cover $\{O_x : x \in K\}$ for K contains a finite subcover $\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$. If $y \notin K$, then the distance from y to each $x_i \in K$ must be

$$\epsilon_0 := \min\left\{\frac{|x_i - y|}{2} : 1 \leq i \leq n\right\}.$$

Since (y_n) is a Cauchy sequence, so it must converge. Hence, for some $N \in \mathbb{N}$, we know that

$$|y_N - y| < \epsilon_0$$

we must have for every $n \geq N$, But note that since $y \notin K$, not all of the terms from the sequence (y_n) for every $n \geq N$ that is contained in K are not included in the finite subcover

$$\bigcup_{i=1}^{n} O_{x_i}.$$

Hence, our finite subcover does not actually cover all of K which is a contradiction and thus we must have $y \in K$.

■

3.3.3 Exercises

Exercises 3.3.1

Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exists and are contained in K.

Proof. Suppose K is compact. By the equivalence theorem, K is bounded. Since K is nonempty and bounded above and below, $\sup K$ exists and $\inf K$ exists by the Axiom of Completeness. We begin by constructing a sequence (x_n) that is contained in K such that we want to show that $(x_n) \to \sup K$. Let $\epsilon > 0$. Since $\sup K$ is the least upper bound, we can have $\sup K - \epsilon$ be not an upper bound. But since $(x_n) \subseteq K$, we must have $x_n \leq \sup K$. Together we have

$$\sup K - \epsilon \leq x_n \leq \sup K.$$

Since $\epsilon > 0$ is arbitrary, the sequence $(x_n) \to \sup K$. Since K is closed $\sup K$ must be contained in K. By a similar reasoning, we can generate (x_n) so that $(x_n) \to \inf K$ which is also contained in K. Hence, $\sup K$ and $\inf K$ contained in K. ■

Exercise 3.3.2

Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbb{N}$.

Solution. Not compact. We can see that for every $n \in \mathbb{N}$, the sequence $a_n = n$ diverges to infinity and so does its subsequences; that is, if we define the odd and even subsequences (a_n) where both sequences also converge to infinity. ■

(b) $\mathbb{Q} \cap [0, 1]$.

Solution. This set is compact because it is bounded and closed. ■

(c) The Cantor set.

Solution. The cantor set is closed. ■

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

Solution. This set is not compact since the limit point of this set ($L = \{0\}$) is not contained in the set. ■

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \dots\}$.

Solution. This set is compact since its limit point $L = \{1\}$ is contained in the set and the fact that it is bounded by 1 and 0. ■

Exercise 3.3.3

Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Proof. Assume that K is closed and bounded. Our goal is to show that K is compact. Let (a_n) be a sequence contained in K. Since K is bounded, we know that (a_n) is also bounded. Hence, there exists a subsequence (a_{n_k}) such that $(a_{n_k}) \to a$. But since K

is closed and a is a limit point of K, we have that $\lim a_{n_k} = a$ is also contained in K. ■

Exercise 3.3.4

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both or neither.

(a) $K \cap F$.

Solution. Compact. ■

(b) $\overline{F^c \cup K^c}$.

Solution. Closed. ■

(c) $K \setminus F = \{x \in K : x \notin F\}$

Solution. Definitely compact. ■

(d) $\overline{K \cap F^c}$.

Solution. Definitely compact. ■

Exercise 3.3.8

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}.$$

This turns out to a reasonable definition between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

Proof. Suppose K and L are disjoint, nonempty compact sets. Our goal is to show that $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$. Since K and L are nonempty compact sets, we also have that K and L are bounded sets. Hence, $\inf A$ exists where

$$A = \{|x - y| : x \in K \text{ and } y \in L\}$$

by the Axiom of Completeness. Then by lemma 1.3.8, we have that for some $x_0 \in K$ and $y_0 \in L$, consider the distance $d = |x_0 - y_0| \in A$ such that we have

$$d = |x_0 - y_0| > \inf A + \epsilon$$

for all $\epsilon > 0$. Since K and L are disjoint and $\epsilon > 0$ is arbitrary, we have $d = |x_0 - y_0| > 0$. Hence, we have that $d > 0$. ∎

Technically, the proof above is not valid yet since we haven't gone over functions and how compactness imply continuity.

Proof. Suppose K and L are non-empty compact sets that are disjoint. Suppose for sake of contradiction that $d = 0$. Since K and L are both compact, let (x_n) and (y_n) contain subsequences (x_{n_k}) and (y_{n_k}) that converge to x_0 and y_0 respectively. We want to show that $\lim |x_{n_k} - y_{n_k}| = |x_0 - y_0|$. Let $\epsilon > 0$. Then for every $n_k > N$ for some $N \in \mathbb{N}$, we have that

$$\begin{aligned}
||x_{n_k} - y_{n_k}| - |x_0 - y_0|| &\leq |(x_{n_k} - y_{n_k}) - (x_0 - y_0)| \\
&= |(x_{n_k} - x_0) + (y_0 - y_{n_k})| \\
&\leq |x_{n_k} - x_0| + |y_{n_k} - y_0| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Since $d = |x_0 - y_0| = 0$, we have that $\lim |x_{n_k} - y_{n_k}| = d = 0$. But this means that $K \cap L \neq \emptyset$ since K and L are closed sets which is a contradiction. Hence, we must have $d > 0$. ∎

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Proof. Basically the argument above but taking away the fact that K and L are compact sets. ∎

Exercise 3.3.9

Follow these steps to prove the final implication in Theorem 3.3.8. Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

Proof. Let I_1 be a half of I_0 whose intersection with K does not have a finite subcover, so that $I_1 \cap K$ cannot be finitely covered and $I_1 \subseteq I_0$. Then bisect I_1 into two closed intervals A_2 and B_2, and again let $A_2 = I_2$ such that $A_2 \cap K$

does not have a finite subcover. Otherwise, $B_2 = I_2$ such that $B_2 \cap K$ does not contain a finite subcover. Hence, $I_2 \subseteq I_1 \subseteq I_0$. We can repeat this process of bisecting each interval I_n and determining which closed set does not have a finite subcovers until we have our desired sequence I_n such that $\lim I_n = 0$. ∎

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

Proof. Since K is compact, $K \cap I_n$ is also compact for each $n \in \mathbb{N}$. Hence, we know that $\bigcap_{n=1}^{\infty} I_n \cap K$ is non-empty, and there exists $x \in K \cap I_n$ for all n. ∎

(c) Because $x \in K$, there must exists an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Proof. Let $x \in K$ and let O_{λ_0} be an open set that contains x. Since O_{λ_0} is an open set, we can construct an $\epsilon-$neighborhood such that $V_{\epsilon_0}(x) \subseteq O_{\lambda_0}$. Now choose n_0 such that for every $n \geq n_0$, $|I_{n_0}| < \epsilon_0$. Hence, I_{n_0} is contained within a single open set O_{λ_0} which means it has a finite subcover. This contradiction tells us that K must have had a finite subcover. ∎

Exercise 3.3.10

Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine-Borel Theorem. Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

Proof. Let S be defined as the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$. Since $[a, b]$ is a closed interval, we can define a sequence of points (x_n) that converges to $x \in [a, b]$. Since $\{O_\lambda : \lambda \in \Lambda\}$ is an open cover for $[a, b]$, we know that there must exists a finite cover for $[a, x]$. Hence, S is nonempty and S is bounded since $a \leq x \leq b$. By the Axiom of Completeness, $s = \sup S$ exists. ∎

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

Proof. Since $[a, b]$ is a closed and bounded interval, it follows that $[a, b]$ is a compact set. By exercise 3.3.1, $[a, b]$ must contain its supremum. Hence, $\sup[a, b] \in S$ and hence, $[a, b]$ must have a finite subcover. ∎

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Proof. Let K be a closed and bounded set. From (a) and (b), K must have finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$. ∎

Other solutions (not mine).

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

Proof. Suppose for contradiction that $s < b$, letting $s \in O_{\lambda_0}$ implies $[a, s]$ is finitely coverable since we can take the finite cover of an $x \in O_{\lambda_0}$ with $x < s$. This causes a contradiction however since there exists points $y > s$ with $y \in O_{\lambda_0}$ meaning $[a, y]$ is also finitely coverable. Therefore, the only option is $s = b$, since any $s < b$ fails. ∎

(c) We must also consider the case where y does not exists; that is, there exists a "gap". Let $y = [s, b] \cap K$ and suppose $y \neq s$. Since $y \in [s, b] \cap K$ we know

$$[a, y] \cap K = ([a, s] \cap K) \cup ([s, y] \cap K) = [a, s] \cap K \cup y.$$

Therefore if $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$ covered $[a, s]$ then letting $y \in O_{\lambda_{n+1}}$ would give the finite cover $\{O_{\lambda_1}, \ldots, O_{\lambda_{n+1}}\}$ contradicting the assumption that $s < b$, therefore $s = b$ is the only option, and so K can be finitely covered.

3.4 Perfect Sets

Definition 3.4.1. A set $P \subseteq \mathbb{R}$ is *perfect* if it is closed and contains no isolated points.

An straightforward example of perfect sets are closed intervals and singleton sets.

Example. It is not too hard to see that the Cantor set from the very beginning of this chapter is perfect. We defined

$$C = \bigcap_{n=0}^{\infty} C_n$$

where each C_n is a finite union of closed intervals. We know by Theorem 3.2.14 that each C_n is closed, and as a result of using the same theorem that C is closed as well. Now all we need to show is that C contains no isolated points.

Let $x \in C$ be arbitrary. Let us construct a sequence (x_n) of points in C that are different from x such that $(x_n) \to x$. We know that C contains endpoints of each interval that make up each C_n. In exercise 3.4.3, we sketch the argument that these are all that is needed to construct such an (x_n).

An argument for uncountability of the Cantor set.

Theorem 3.4.1. *A nonempty perfect set is uncountable.*

Proof. Suppose P is a set that is perfect and nonempty. Hence, it must be the case that P is an infinite set because otherwise it would only consist of isolated points. Assume for sake of contradiction that P is countable. Thus, we can define P as the following:

$$P = \{x_1, x_2, x_3 \dots\},$$

where every element of P appears on this list. Our goal is to construct a sequence of nested compact sets K_n that is all contained within P with the property that $x_1 \notin K_2$, $x_2 \notin K_3$, $x_3 \notin K_4$ and so on. Before proceeding with our argument, we must be sure that, in fact, each K_n is nonempty. Hence, we use the nested Compact interval theorem to produce

$$x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$$

that cannot be on the list $\{x_1, x_2, x_3, \dots\}$.

Let I_1 be a closed interval such that $x_1 \in (I_0)^\circ$; that is, x_1 is not an endpoint of I_1. This produces an x_1 that is not isolated which means there exists some other point, say, $y_2 \in P$ such that $y_2 \in (I_1)^\circ$. Around y_2 we can construct a closed interval such that $I_2 \supseteq I_1$ with the condition that $x_1 \notin I_2$. Let $\epsilon > 0$, then if $I_1 = [a, b]$ we can define

$$\epsilon = \min\{y_2 - a, b - y_2, |x_1 - y_2|\}.$$

Then, the interval $I_2 = \{[y_2 - \epsilon/2, y_2 + \epsilon/2]\}$ has the desired properties. We can continue this process indefinitely.

Since $y_2 \in P$ is not isolated, there must exists another point $y_3 \in P$ in the interior of I_2 such that $y_3 \neq x_2$. Again, construct a closed interval centered on y_3 with an ϵ small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$ because this intersection contains at least y_3.

We find that when we carry out this construction inductively, we have a sequence of closed intervals I_n satisfying the following properties:

(i) $I_{n+1} \subseteq I_n$,

(ii) $x_n \notin I_{n+1}$, and

(iii) $I_n \cap P \neq \emptyset$.

To finish the proof, let $K_n = I_n \cap P$. For each $n \in \mathbb{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded since it is contained in the bounded sets I_n. Hence, K_n is compact. We can also see that K_n is nonempty and $K_{n+1} \subseteq K_n$. By employing the Nested Compact Set property, we can conclude that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

But we find that each $K_n \subseteq P$ where $x_n \notin i_{n+1}$ leads to the conclusion that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, which is a contradiction.

■

3.4.1 Connected Sets

Consider the two open intervals $(1, 2)$ and $(2, 5)$. Notice that these two intervals have the limit point $x = 2$ in common. However, there is some space between them in the sense that 2 isn't contained in the other. Another way to say this is that $\overline{(1, 2)} \cap (2, 5) = \emptyset$ and likewise, $\overline{(2, 5)} \cap (1, 2) = \emptyset$. Notice that this same observation cannot be extended to the two sets $(1, 2]$ and $(2, 5)$ even though these two sets are disjoint.

Definition 3.4.2. (i) Two nonempty sets $A, B \subseteq \mathbb{R}$ are *separated* if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(ii) A set $E \subseteq \mathbb{R}$ is *disconnected* if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

(iii) A set that is not disconnected is called a *connected* set.

Example. (i) If we let $A = (1, 2)$ and $B = (2, 5)$, then it is not difficult to verify that $E = (1, 2) \cup (2, 5)$ is disconnected. Notice that the sets $C = (1, 2]$ and $D = (2, 5)$ are NOT separated because $C \cap \overline{D} = \{2\}$ is nonempty. We know that the interval $(1, 5)$ is the union of sets C and D, but we cannot say they are disconnected. We will prove later that every interval is a connected subset of $\mathbb{R}$ and vice versa.

(ii) Consider the set of rational numbers

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \text{ and } B = \mathbb{Q} \cap (\sqrt{2}, \infty).$$

It turns out that these two sets are disconnected such that $\mathbb{Q} = A \cup B$. The fact that $A \subseteq (\infty, \sqrt{2})$ implies that any limit point of A will necessarily fall in $(-\infty, \sqrt{2}]$ by the Order Limit Theorem. Because this is disjoint from B, we get that $\overline{A} \cap B = \emptyset$. We can similarly show that $A \cap \overline{B} = \emptyset$, which implies that A and B are separated.

The definition of connected is stated as the negation of disconnected, but using the logical negation of the quantifiers in the definition above results in a positive characterization of connectedness.

A way to show that a set E is connected is to partition E into two nonempty disjoint sets where we can show at least one of the sets contains a limit point of the other.

Theorem 3.4.2. *A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.*

Proof. Exercise 3.4.6. ■

Theorem 3.4.3. *A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $x \in E$ as well.*

Proof. Assume E is connected, and let $a, b \in E$ and $a < c < b$. We can set A and B such that

$$A = (-\infty, c) \cap E \text{ and} B = (c, \infty) \cap E.$$

Because $a \in A$ and $b \in B$, neither set is empty and, just as in Example 3.4.5 (ii), neither set contains a limit point of the other. If $E = A \cup B$, then we have that E is disconnected. If $E = A \cup B$, then we would have that E is disconnected, which it is not. It must be the case that $A \cup B$ is missing some element of E, and c is the only possibility. Thus, $c \in E$.

Conversely, assume E is an interval in the sense that whenever $a, b \in E$ satisfy $a < c < b$ for some c, then $c \in E$. Our intent is to use the characterization of connected sets in Theorem 3.4.6, so let $E = A \cup B$, where A and B are nonempty and disjoint.

We need to show that one of these sets contains a limit point of the other. Pick $a_0 \in A$ and $b_0 \in B$, and suppose $a_0 < b_0$ for sake of argument. Since E is an interval, the interval $I_0 = [a_0, b_0]$ is contained in E. Now, let us bisect I_0 into two equal halves. The midpoint of I_0 must either be in A or B, and so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$. We can continue such a process inductively to get a sequence of nested intervals $I_n[a_n, b_n]$, where $a_n \in A$ and $b_n \in B$, and the length $(b_n - a_n) \to 0$. Hence, the following intersection

$$\bigcap_{n=0}^{\infty} I_n \neq \emptyset.$$

Since $(a_n - b_n) \to 0$, we have that the sequences of endpoints have the same limit point x. Since $x \in E$, it must be the case that x must belong to either A or B. Hence, E is a connected set. ∎

3.4.2 Definition

Definition 3.4.3. A set $P \subseteq \mathbb{R}$ is *perfect* if it is closed and contains no isolated points.

Theorem 3.4.4. *A nonempty perfect set is uncountable.*

Definition 3.4.4. (i) Two nonempty sets $A, B \subseteq \mathbb{R}$ are *separated* if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(ii) A set $E \subseteq \mathbb{R}$ is *disconnected* if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

(iii) A set that is not disconnected is called a *connected* set.

Theorem 3.4.5. *A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.*

Theorem 3.4.6. *A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.*

3.4.3 Exercises

Exercise 3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution. $P \cap K$ always compact but not always perfect. This is because $P \cap K$ is always a closed and bounded set. ■

Exercise 3.4.5

Let A and B be nonempty subsets of $\mathbb{R}$. Show that if there exists disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Proof. Let A and B be nonempty subsets of $\mathbb{R}$. Suppose there exists sets U and V such that $U \cap V = \emptyset$. Let $x \in U$. Since U is an open set, let $x \in U^{\circ}$. Hence, there exists $V_{\epsilon}(x)$ such that $V_{\epsilon}(x) \subseteq U$. Let x be a limit point of A such that there exists $(x_n) \to x$ where $x_n \neq x$ for all $n \in \mathbb{N}$. Doing the same process for the set V suppose there exists (y_n) is a sequence contained in B such that $(y_n) \to y$ where $y \in V$ is an interior point such that there exist $V_{\epsilon}(y) \subseteq V$.

Since $U \cap V = \emptyset$ and x and y are interior points of U and V respectively, it follows that $V_{\epsilon}(x) \cap V_{\epsilon}(y) = \emptyset$. But this means neither limit point of A nor B is contained in the other. Hence, A and B must be separated sets. ■

Exercise 3.4.6

Prove Theorem 3.4.6.

Proof. ($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected and suppose A and B are disjoint. Since E is connected, we have $\overline{A} \cap B \neq \emptyset$ and $\overline{B} \cap A \neq \emptyset$. Without loss of generality, let $x \in \overline{A} \cap B$. Since A and B are disjoint, we must not have $x \in A$. Hence, $x \in B$ else A and B would not be disjoint. Hence, x is a limit point of A. Hence, there exists $(x_n) \subseteq A$ such that $(x_n) \to x$ where $x \neq x_n$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Our goal now is to show the converse; that is, show that A and B are not separated i.e $\overline{A} \cap B \neq \emptyset$ and $\overline{B} \cap A \neq \emptyset$. Suppose there exists $(x_n) \subseteq A$. By assumption $(x_n) \to x$ such that $x \in B$. Since $\overline{A}$ is a closed set, we have that $x \in \overline{A}$. But this means that $\overline{A} \cap B \neq \emptyset$. The argument is similar when $(x_n) \subseteq B$. Hence, $\overline{B} \cap A \neq \emptyset$. But this means that $E = A \cup B$ is connected. ■

Exercise 3.4.7

A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exists separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

Proof. Since the rational numbers are dense in $\mathbb{I}$, we can construct the following interval $x < c < y$ where $c \in \mathbb{I}$. We can set A and B such that

$$A = (-\infty, c) \cap \mathbb{Q} \quad \text{and} \quad B = \mathbb{Q} \cap (c, +\infty)$$

Let $x \in A$ and $y \in B$, then neither set is empty and neither set contains a limit point of the other. Since $A \cap B = \mathbb{Q}$, we must have $\mathbb{Q}$ as a totally disconnected set unless $c \in \mathbb{Q}$ which is not. ■

(b) Is the set of irrational numbers totally disconnected?

Proof. The set of irrational numbers is totally disconnected because we can always find $x \in \mathbb{Q}$ such that for every $z, y \in \mathbb{I}$, we have $z < x < y$. Thus, we can follow the same argument above to produce two sets that are separated. ■

Chapter 4

Functional Limits and Continuity

4.0.1 Towards a Formal Definition of Continuity

We want to define continuity at a point $c \in A$ to mean that if we have $x \in A$ that s chosen *near* c, then $f(x)$ will be near $f(c)$. We can define this notion mathematically to say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

This is a rational way to define continuity until we encounter a function like

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

where we haven't yet defined what it means for $\lim_{x \to 1/2} g(x)$. We can extend our notions of a limit from Chapter 2 to make sense of this expression. Hence, we can define a sequence (x_n) where $(x_n) \to 1/2$ and say that $\lim_{x \to 1/2} g(x)$ as the limit of $g(x_n)$. But the problem with this notion is that if (x_n) is defined as a sequence of rational points, then

$$\lim_{n \to \infty} g(x_n) = 1$$

but on the other hand, if (x_n) is irrational, then

$$\lim_{n \to \infty} g(x_n) = 0.$$

Very quickly, we can conclude that the limit of $g(x)$ as $x \to c$ does not exists. Furthermore, we can also see that $x = 1/2$ causes $g(x)$ to not be continuous. This is because both $\mathbb{Q}$ and $\mathbb{I}$ are both dense in the real line. Hence, it follows that for any $z \in \mathbb{R}$, we can find sequences $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{I}$ such that $\lim x_n = \lim y_n = z$. But because

$$\lim g(x_n) = \lim g(y_n),$$

we can say that the same line of reasoning applies to conclude that $g(x)$ is not continuous at z on $\mathbb{R}$. In other words, the Dirichlet function $g(x)$ is nowhere continuous on $\mathbb{R}$.

What happens when we adjust the definition of $g(x)$ slightly such that

$$h(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Suppose $h(x)$ is defined on $\mathbb{R}$. Then letting $c \in \mathbb{R}$ be different from 0,then we can construct sequences $(x_n) \to c$ of rationals and $(y_n) \to c$ of irrationals such that

$$\lim h(x_n) = c \text{ and } \lim h(y_n) = 0.$$

Thus, we have that h is not continuous at every point $c \neq 0$. If we set $c = 0$, then it turns out that these two functional limits are the same. This observation enables us to strive for a definition for functional limits by saying that

$$\lim_{x \to c} h(x) = L$$

if $h(z_n) \to L$ for all sequences $(z_n) \to c$. We can make sense of these two different limits by constructing $\epsilon-$neighborhoods around c and L respectively. Another example of a non-continuous function is

$$t(x) = \begin{cases} 1 \text{ if } x = 0 \\ 1/n \text{ if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ where } n > 0 \text{ and } (m, n) = 1 \\ 0 \text{ if } x \neq \mathbb{Q}. \end{cases}$$

If we let $c \in \mathbb{Q}$, then $t(c) > 0$. But since $\mathbb{I}$ is dense in $\mathbb{R}$, there exists a sequence of irrational numbers (y_n) in $\mathbb{R}$ such that $(y_n) \to c$. It immediately follows that $t(x)$ is not continuous at every point in $\mathbb{Q}$ since

$$\lim t(y_n) = 0 \neq t(c).$$

But if we let $c \in \mathbb{I}$, then we find that defining a sequence of rational numbers (y_n) such that $(y_n) \to c$ reveals that $\lim t(y_n) = 0$ which makes $t(x)$continuous at every irrational point in $\mathbb{R}$. The takeaway from this section is that the characteristics of a given set $A \subseteq \mathbb{R}$ greatly determines the continuity of a function.

4.1 Functional Limits

4.1.1 Defining the Functional Limit

Consider a function $f : A \to \mathbb{R}$. Suppose c of A is a limit point. From the last chapter, recall that the definition of a limit point is any $\epsilon-$neighborhood $V_\epsilon(c)$ intersects $A \setminus \{c\}$. In other words, c is a limit point of A if and only if $c = \lim x_n$ for some sequence $(x_n) \subseteq A$ with $x_n \neq c$ for all $n \in \mathbb{N}$. Furthermore, it is important to keep in mind that limit points of A do not necessarily belong to A unless it is closed.

If c is a limit point of our domain A, then, we can state that

$$\lim_{x \to c} f(x) = L$$

is intended to convey that the values of $f(x)$ gets arbitrarily close to L as x is chosen arbitrarily close to c. It is important to keep in mind that c need not be in the domain of A.

The structure of the definition of functional limits is as follows: Given a sequence (a_n), the assertion that $\lim a_n = L$ implies that for every $\epsilon-$neighborhood $V_\epsilon(L)$ centered at L, we can find a point in a sequence say a_N after which all the terms of a_n fall in $V_\epsilon(L)$. This is in response to an arbitrary choice of x in the domain where we have a $\delta-$neighborhood

Definition 4.1.1 (Functional Limit)**.** Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever

$$0 < |x - c| < \delta$$

(and $x \in A$) it follows that

$$|f(x) - L| < \epsilon.$$

This is often referred to as the epsilon-delta definition of a functional limit. The statement

$$|f(x) - L| < \epsilon$$

is equivalent to saying that $f(x) \in V_\epsilon(L)$. Likewise, the statement

$$|x - c| < \delta$$

is true if and only if $x \in V_\delta(c)$. Note that we imposed an additional restriction that $|x-c| > 0$. This is because we don't want $x = c$. We can recast the definition above in terms of $\epsilon-$neighborhoods to help get a more geometric view of what is happening with these functional limits.

Definition 4.1.2. Let c be a limit point of the domain $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that for every $\epsilon-$neighborhood $V_\epsilon(L)$ of L, there exists a $\delta-$neighborhood $V_\epsilon(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

The reminder that we must have $x \in A$ ensures that every possible $x \in A$ must be a valid input for the function in question. Note that the appearance of $f(x)$ in our definitions carries an implicit assumption that x is always part of the domain of f. There is no use for considering isolated points outside of A. Hence, we can always expect that functional limits will have $x \in A$ that approach the limit point of A or $\text{dom}(f)$.

Example. (i) Suppose we want to show that for $f(x) = 3x+1$, we have

$$\lim_{x\to 2} f(x) = 7.$$

Proof. Let $\epsilon > 0$. Consider $|f(x) - 7|$. We want to show that whenever $|x-2| < \delta$, that

$$|f(x) - 7| < \epsilon.$$

Hence, observe that

$$\begin{aligned}
|f(x) - 7| &= |(3x+1) - 7| \\
&= |3x - 6| \\
&= 3|x-2| \\
&< 3\delta.
\end{aligned}$$

We can choose $\delta = \epsilon/3$ such that

$$|f(x) - 7| < 3\delta = 3\frac{\epsilon}{3} = \epsilon.$$

Hence, we have that $\lim_{x\to 2} f(x) = 7$. ∎

(ii) Let's show that
$$\lim_{x\to 2} g(x) = 4,$$
where $g(x) = x^2$.

Proof. Let $\epsilon > 0$. Suppose $0 < |x-2| < \delta$. Then
$$\begin{aligned} |g(x)-4| &= |x^2-4| \\ &= |(x-2)(x+2)| \\ &= |x-2||x+2|. \end{aligned}$$

Since $|x+2| < \delta + 4$, observe that for $\delta = \min\{1, \epsilon/5\}$, we have
$$\begin{aligned} |g(x)-4| &= |x-2||x+2| \\ &< \delta \cdot (\delta+4) \\ &= \frac{\epsilon}{5} \cdot 5 \\ &= \epsilon. \end{aligned}$$
∎

4.1.2 Sequential Criterion for Functional Limits

Theorem 4.1.1 (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x\to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Proof. (i) $\implies$ (ii) Suppose $\lim_{x\to c} f(x) = L$. Let $\epsilon > 0$. By assumption, we have a sequence $(x_n) \to c$. It immediately follows that for some $N \in \mathbb{N}$ such that for all $n \geq N$ that $x_n \in V_\delta(c)$. Hence, $f(x_n) \in V_\epsilon(L)$ by the Topological Definition of functional limits.

(ii) $\implies$ (i) Let $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$ such that $f(x_n) \to L$. Suppose for sake of contradiction that $\lim_{x\to c} f(x) \neq L$. Hence, there exists ϵ_0 such that for any $\delta > 0$ where $|x_n - c| < \delta$ that $|f(x_n) - L| \geq \epsilon_0$. Let $\delta = 1/n$ and suppose we pick $x_n \in V_\delta(c)$ such that $f(x_n) \notin V_\epsilon(L)$. But this implies that $f(x_n) \not\to L$ which contradicts our assumption that it is. Hence, it must be the case that (i) holds. ∎

Corollary 4.1.1.1 (Algebraic Limit Theorem for Function Limits)**.** *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x\to c}[f(x)g(x)] = L \cdot M$, *and*

(iv) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided that* $M \neq 0$.

Proof. Exercise 4.2.1. ■

Corollary 4.1.1.2 (Divergence Criterion for Functional Limits)**.** *Let f be a function defined on A and let c be a limit point of A. If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$*

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

Example. Suppose we wanted to show that $\lim_{x\to 0} \sin(1/x)$ does not exist. Set $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$, then

$$\lim x_n = \lim y_n = 0$$

but $\sin(1/x_n) = 0$ for all $n \in \mathbb{N}$ while $\sin(1/y_n) = 1$. Thus, we have

$$\lim \sin(1/x_n) \neq \lim \sin(1/y_n),$$

and thus we know that $\lim_{x\to 0} \sin(1/x)$ does not exist.

4.1.3 Definitions

Definition 4.1.3 (Functional Limit)**.** Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever

$$0 < |x - c| < \delta$$

(and $x \in A$) it follows that

$$|f(x) - L| < \epsilon.$$

Definition 4.1.4. Let c be a limit point of the domain $f : A \to \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that for every $\epsilon-$neighborhood $V_\epsilon(L)$ of L, there exists a $\delta-$neighborhood $V_\epsilon(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

Theorem 4.1.2 (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Theorem 4.1.3 (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Corollary 4.1.3.1 (Algebraic Limit Theorem for Function Limits). *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x \to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,

(ii) $\lim_{x \to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x \to c}[f(x)g(x)] = L \cdot M$, *and*

(iv) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided that* $M \neq 0$.

Corollary 4.1.3.2 (Divergence Criterion for Functional Limits). *Let f be a function defined on A and let c be a limit point of A. If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$*

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

4.1.4 Exercises

Exercise 4.2.1

(a) Supply the details for how Corollary 4.1.1.1 follows from the Sequential Criterion for Functional Limits in Theorem 4.1.1 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

Proof. Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$. Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. By the Sequential Criterion for functional limits, let $(x_n), (y_n) \subseteq A$ such that $(x_n) \to c$ and $(y_n) \to c$ where $x_n, y_n \neq c$ implying that $f(x_n) \to L$ and $g(y_n) \to M$. By the Algebraic Limit Theorem, we can state that

$$\lim[f(x_n) + g(y_n)] \lim f(x_n) + \lim g(y_n) = L + M.$$

Hence, we have that

$$\lim_{x \to c}[f(x) + g(x)] = L + M$$

by the Sequential Criterion for Functional Limits. ■

(b) Now, write another proof of Corollary 4.1.1.1 part (ii) directly from Definition for functional limits without using the sequential criterion in Theorem 4.1.1.

Proof. We can also prove the addition rule for Functional Limits directly from the definition. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then for some $\delta > 0$, suppose c is a limit point of A such that $0 < |x - c| < \delta$. We want to show that for any arbitrary $\epsilon > 0$ that

$$|(f(x) + g(x)) - (L + M)| < \epsilon.$$

Hence, choose $\delta = \min\{\delta_1, \delta_2\}$

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have

$$\lim_{x \to c} [f(x) + g(x)] = L + M.$$

■

(c) Repeat (a) and (b) for Corollary 4.1.1.1 part (iii).

Proof. Let $(x_n), (y_n) \subseteq A$ such that $(x_n) \to c$ and $(y_n) \to c$ for some limit point c of A where we have $\lim f(x_n) = L$ and $\lim g(y_n) = M$. By the Algebraic Limit Theorem, we have that

$$\lim [f(x_n) g(y_n)] = \lim f(x_n) \cdot \lim g(y_n) = L \cdot M.$$

By the Sequential Criterion, this also means that

$$\lim_{x \to c} [f(x) g(x)] = L \cdot M.$$

Now we can prove the same fact but this time we use only the Definition of functional limit. Let $f : A \to \mathbb{R}$. Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$ where c is a limit point of A. Let $\epsilon > 0$. Our goal is to show that

$$|f(x)g(x) - LM| < \epsilon.$$

Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - g(x)L + g(x)L - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + L|g(x) - M| \\ &< M + 1 \cdot \frac{\epsilon}{2(M+1)} + L \cdot \frac{\epsilon}{2L} \\ &= \epsilon.\end{aligned}$$

Hence, we have that

$$\lim_{x\to c} f(x)g(x) = LM.$$

■

Exercise 4.2.2

For each stated limit, find the largest possible $\delta-$neighborhood that is a proper response to the given ϵ challenge.

(a) $\lim_{x\to 3}(5x - 6) = 9$.

Solution. Since $\epsilon = 1$, we know that

$$|(5x - 6) - 9| < 1.$$

To solve for δ, we do the following

$$\begin{aligned}|(5x - 6) - 9| &= |5(x - 3)| < 1 \\ &= 5|x - 3| < 1\end{aligned}$$

which implies that

$$|x - 3| < \frac{1}{5} = \delta.$$

Hence, the largest possible $\delta-$neighborhood that is a proper response to the given ϵ challenge. ■

(b) $\lim_{x\to 4} \sqrt{x} = 2$, where $\epsilon = 1$.

Solution. Since $\epsilon = 1$, we know that

$$|\sqrt{x} - 2| < 1.$$

To get our δ, we just do the following manipulations

$$\begin{aligned} \sqrt{x} - 2 &< 1 \\ \sqrt{x} &< 3 \\ x &< 9 \\ |x-4| &< 5 = \delta. \end{aligned}$$

Hence, our ϵ response is $\delta = 5$. ■

(c) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)

Solution. Let $\epsilon = 1$. We want to generate a $\delta > 0$ that satisfies the ϵ challenge. Since $[[\pi]] = 3$, our max $\delta-$neighborhood can only have $\delta = \pi - 3$ ■

(d) We have (c) but this time with $\epsilon = .01$.

Solution. It would still remain as $\delta = \pi - 3$. ■

Exercise 4.2.5

Use Definition 4.1.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

Proof. Suppose there exists a $\delta > 0$ such that $0 < |x-2| < \delta$. Let $\epsilon > 0$. Then observe that by using definition 4.1.1 that

$$\begin{aligned} |f(x) - 10| &= |(3x+4) - 10| \\ &= 3|x-2| \\ &< 3\delta \\ &= 3 \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

■

(b) $\lim_{x\to 0} x^3 = 0$.

Proof. Suppose there exists $\delta > 0$ such that $0 < |x| < \delta$. By applying the definition of functional limits, choose $\delta = (\epsilon)^{1/3}$

$$\begin{aligned}|f(x) - 0| &= |x^3| \\ &< \delta^3 \\ &= \epsilon.\end{aligned}$$

Hence, we have $|f(x) - 0| < \epsilon$ which implies that

$$\lim_{x \to 0} x^3 = 0.$$

■

(c) $\lim_{x\to 2}(x^2 + x - 1) = 5$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\begin{aligned}|f(x) - 5| &= |(x^2 + x - 1) - 5| \\ &= |x^2 + x - 6| \\ &= |x + 3||x - 2| \\ &< |x + 3|\delta \\ &< 3 \cdot \frac{\epsilon}{3}. && = \epsilon.\end{aligned}$$

■

(d) $\lim_{x\to 3} 1/x = 1/3$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\begin{aligned}|f(x) - \frac{1}{3}| &= |\frac{1}{x} - \frac{1}{3}| \\ &= \left|\frac{x-3}{3x}\right| \\ &= \frac{|x-3|}{3|x|} \\ &< 12\epsilon \cdot \frac{1}{12} \\ &= \epsilon.\end{aligned}$$

■

Exercise 4.2.6

Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.

Solution. This is true since δ that has been constructed is the largest possible neighborhood that one can produce as a response to the ϵ challenge. ∎

(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $f(a) = L$.

Solution. This is false. I have trouble thinking of an example. Will get back to this one soon. ∎

(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.

Solution. This is true based on the Algebraic Functional Limit Theorem. ∎

(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Solution. This is not true in general since we can easily produce $g(x)$ that is unbounded. For this to work, we would need $g(x)$ to be bounded. ∎

Exercise 4.2.7

Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Proof. Let $g : A \to \mathbb{R}$. Suppose $\lim_{x\to c} g(x) = 0$ and suppose $f(x)$ is bounded; that is, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Using the Functional Limit Definition, suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Hence, we have

$$\begin{aligned}|f(x)g(x) - 0| &= |f(x)g(x)| \\ &= |f(x)||g(x)| \\ &\leq M|g(x)| \\ &< M \cdot \frac{\epsilon}{M} \\ &= \epsilon.\end{aligned}$$

It follows from this that $\lim_{x\to c} f(x)g(x) = 0$. ■

Exercise 4.2.9 (Infinite Limits).

The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge response style of Definition 4.1.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\epsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition 4.1.5. We say $\lim_{x\to c} f(x) = \infty$ to mean that for all $M > 0$, we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

Proof. Let $M > 0$ and $f(x) = 1/x^2$. Choose $\delta = 1/\sqrt{M}$. Since $0 < |x| < \delta$, we have

$$x^2 < \delta^2 \iff \frac{1}{x^2} > \frac{1}{\delta^2}.$$

But this means that

$$f(x) = \frac{1}{x^2} > \frac{1}{\delta^2} = \frac{1}{1/M} = M.$$

Hence, we have

$$\lim_{x\to 0} f(x) = \infty.$$

■

(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show that $\lim_{x\to\infty} 1/x = 0$.

Definition 4.1.6. Let $f : A \to \mathbb{R}$. We say $\lim_{x\to\infty} f(x) = L$ to mean for any $\epsilon > 0$, there exists $x_0 \in A$ where $A \subseteq \mathbb{R}$ such that for any $x \geq x_0$, we have

$$|f(x) - L| < \epsilon.$$

Solution. We start with some sketch work for what x_0 might be. Observe that

$$\begin{aligned}|f(x) - L| &= |\frac{1}{x} - 0| \\ &= \frac{1}{x} \\ &< \epsilon.\end{aligned}$$

Solving for x, we get that

$$x > \frac{1}{\epsilon}.$$

Hence, $x_0 = 1/\epsilon$.

Let $\epsilon > 0$ and let $f(x) = 1/x$. Choose $x_0 = 1/\epsilon$. Suppose $x > x_0 = 1/\epsilon$. Then we have that

$$x > \frac{1}{\epsilon} \iff \frac{1}{x} < \epsilon.$$

Hence, we have that

$$|f(x) - 0| < \epsilon.$$

This means $\lim_{x\to\infty} f(x) = 0$. ■

Exercise 4.2.10

Introductory calculus courses typically refer to the *right-hand limit* of a function as the limit obtained by "letting x approach a from the right-hand side"

(a) Give a proper definition in the style of Definition 4.1.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = M.$$

Definition 4.1.7. We say that

$$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = M$$

if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ and $0 < |c - x| < \delta$ we have

$$|f(x) - L| < \epsilon,$$
$$|f(x) - M| < \epsilon$$

respectively.

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal to L.

Proof. Suppose $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have

$$|f(x) - L| < \epsilon. \tag{1}$$

This denotes our right-hand limit. Note that this also works if we flip the order of our δ assumption. Hence, we have $0 < |a - x| < \delta$ such that (1) holds. ■

But this means our right-hand and left-hand limits are equal to each other.

Now we show the converse. Suppose the right-hand and left-hand limits are equal to each other; that is,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

We want to show that $\lim_{x\to a} f(x) = L$. Since both $|x - a| < \delta$ and $|a - x| < \delta$ hold for both limits, we immediately have that

$$|f(x) - L| < \epsilon.$$

Hence, we have $\lim_{x\to a} f(x) = L$.

Exercise 4.2.11(Squeeze Theorem).

Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Suppose $f(x) \leq g(x) \leq h(x)$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show

$$\lim_{x\to c} g(x) = L$$

as well.

Proof. Let $(x_n), (y_n), (z_n) \subseteq A$. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A. Using the Sequential Criterion for Functional Limits, let $f(x_n) \to L$ and $h(z_n) \to L$. There exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$f(x_n) \leq g(y_n) \leq h(z_n) \iff L \leq g(y_n) \leq L.$$

Hence, we have $g(y_n) \to L$ by the Squeeze Theorem for Sequences. By the Sequential Criterion, we must have $\lim_{x\to c} g(x) = L$ as well. ■

Another proof using the definition of Functional Limits directly.

Proof. Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, we know that

$$\lim_{x\to c}[h(x) - f(x)] = 0.$$

Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Since $f(x) \leq g(x) \leq h(x)$, we have

$$\begin{aligned}
|g(x) - L| &\leq |h(x) - L| \\
&= |h(x) - f(x) + f(x) - L| \\
&\leq |h(x) - f(x)| + |f(x) - L| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

But, this means that $\lim_{x\to a} g(x) = L$. ■

Another proof

Proof. Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Choose $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Since $f(x) \leq g(x) \leq h(x)$, we have

$$\begin{aligned}
|g(x) - L| &= |g(x) - h(x) + h(x) - L| \\
&\leq |g(x) - f(x) + f(x) - h(x)| + |h(x) - L| \\
&\leq |g(x) - f(x)| + |f(x) - h(x)| + |h(x) - L| \\
&\leq |h(x) - f(x)| + |f(x) - L| + |L - h(x)| + |h(x) - L| \\
&\leq |h(x) - L| + |L - f(x)| + 2|h(x) - L| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{2\epsilon}{4} \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that

$$\lim_{x \to a} g(x) = L.$$

■

4.2 Continuous Functions

Definition 4.2.1 (Continuity)**.** A function $f : A \to \mathbb{R}$ is *continuous at a point* $c \in A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$. If f is continuous at every point in the domain A, then we say that f *continuous* on A.

The difference between this definition and the definition for functional limits is that we require the limit point c of A to be in the domain of f. The value of $f(c)$ is the value of $\lim_{x \to c} f(x)$. It is indeed possible to shorten this definition to say that f is continuous at $c \in A$ if

$$\lim_{x \to c} f(x) = f(c) \tag{1}$$

so as long as c is a limit point of A. The equation above gets undefined if c is an isolated point of A. But this does not stop f from being continuous at the point c. In fact, functions can still be continuous at their isolated points such as c.

We observed in the previous section that functional limits can be formulated using sequences from chapter 2. The same can be done for continuity.

Theorem 4.2.1 (Characterizations of Continuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(iii) For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.

If c is limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x \to c} f(x) = f(c)$.

Proof. (i) $\iff$ (ii) Let $V_\epsilon(f(c))$ and let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies

$$|f(x) - f(c)| < \epsilon.$$

We can rephrase in terms of ϵ and δ neighborhoods. Hence, the statement above is just equivalent to the statement that for all $V_\epsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c)$ which implies that $f(x) \in V_\epsilon(f(c))$.

(iii) $\iff$ (ii) First we show (ii) holds. Let $(x_n) \to c$ such that $f(x_n) \to f(c)$. In other words, $f(x_n) \in V_\epsilon(f(c))$. For sake of contradiction, that $f(x) \notin V_\epsilon(f(c))$. This means there exist ϵ_0 such that for all $\delta > 0$ with the property that $|x - c| < \delta$ that

$$|f(x) - f(c)| \geq \epsilon_0.$$

Let $\delta = 1/n$ and suppose we pick $x_n \in V_\delta(c)$ such that $f(x_n) \notin V_\epsilon(f(c))$. But this contradicts our assumption that $f(x_n) \in V_\epsilon(f(c))$. Hence, it must be the case that $f(x) \in V_\epsilon(f(c))$.

Now we want to show that (*iii*) holds. Let $(x_n) \subseteq A$. (with $x_n \in A$). Since $x_n \in A$ $x_n \neq c$ or $x_n = c$. If $x_n = c$, then it immediately follows that $f(x_n) \to f(c)$ given there exists $|x_n - c| < \delta$. Suppose $x_n \neq c$. Pick $x_n \in V_\delta(c)$ for some $\delta > 0$. Then by assumption, we have $f(x_n) \in V_\epsilon(f(c))$. Hence, $f(x_n) \to f(c)$.

To show (i) $\implies$ (iv), suppose c is a limit point of A. Let $x \in A$. Since $c \in A$, either $x \neq c$ or $x = c$. The conclusion follows immediately from the latter statement. Suppose $x_n \neq c$. By assumption, there exists $\delta > 0$ such that $|x_n - c| < \delta$. From this, it follows that

$$|f(x) - f(c)| < \epsilon.$$

And hence,

$$\lim_{x \to c} f(x) = f(c).$$

■

Corollary 4.2.1.1 (Criterion for Discontinuity). *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n) \not\to f(c)$, we may conclude that f is not continuous at c.*

This sequential characterization of continuity allows us to use all the results that we know of when it comes to sequences from Chapter 2.

Theorem 4.2.2 (Algebraic Continuity Theorem). *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at point $c \in A$. Then,*

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;
(ii) $f(x) + g(x)$ is continuous at c for all $k \in \mathbb{R}$;
(iii) $f(x)g(x)$ is continuous at c; and
(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Proof. All of these statements can be derived from the Characterizations of Continuity Theorem and the Algebraic Functional Limit Theorem. ■

Example. All polynomials are continuous on $\mathbb{R}$. In fact, rational functions (Quotients of polynomials) are continuous wherever they are defined. Consider the identity function $g(x) = x$. Since $|g(x) - g(c)| = |x - c|$, we can respond to a given $\epsilon > 0$ by choosing $\delta = \epsilon$ such that g is continuous on all of $\mathbb{R}$. Furthermore, this argument gets much simpler when we consider a constant function such as $f(x) = k$. Since any arbitrary polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

consists of sums and products of $g(x)$ with different constant functions, we can conclude that $p(x)$ is continuous. On the other hand, the Algebraic Continuity Theorem implies that quotients of polynomials are continuous as long as the denominator is not zero.

Example. In the sinuisodal example in the last section, we noticed that the oscillations of $\sin(1/x)$ are so rapid near the origin that $\lim_{x \to 0} \sin(1/x)$ does not exist.

Consider the function,

$$g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose we want to observe the continuity of g at $c = 0$. We can do this by the following:

$$|g(x) - g(0)| = |x\sin(1/x) - 0| \leq |x|$$

since $|\sin(x)| \leq 1$. Given $\epsilon > 0$, choose $\delta = \epsilon$ such that whenever $|x| < \delta$ it follows that $|g(x) - g(0)| < \epsilon$. Thus, g is continuous at the origin.

Example. Consider the greatest integer function $h(x) = [[x]]$ which for each $x \in \mathbb{R}$ returns the largest integer $n \in \mathbb{Z}$ such that $n \leq x$. In previous math classes, this step function is observed to have discontinuous jumps at each integer value of its domain. We can show this more rigorously using the tools we have at our disposal. Given $m \in \mathbb{Z}$, define the sequence (x_n) by $x_n = m - 1/n$. It follows that $(x_n) \to m$, but not that

$$h(x_n) \to (m-1),$$

which does not equal $m = h(m)$. By the Criterion for Discontinuity, we see that h fails to be continuous at each $m \in \mathbb{Z}$. Suppose we want to see why h is continuous at a point $c \notin \mathbb{Z}$. Given any $\epsilon > 0$, we must find a $\delta-$neighborhood $V_\delta(c)$ such that $x \in V_\delta(c)$ implies $h(x) \in V_\epsilon(h(c))$. We know that $c \in \mathbb{R}$ falls between consecutive integers $n < c < n+1$ for some $n \in \mathbb{Z}$. Taking $\delta = \min\{c - n, (n+1) - c\}$, then it follows from definition of h that $h(x) = h(c)$. Thus, we certainly have that $h(x) \in V_\epsilon(h(c))$ whenever $x \in V_\delta(c)$. This proof actually implies that our δ is not dependent on the value of $\epsilon > 0$.

Example. Consider $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbb{R} : x \geq 0\}$. Exercise 2.3.1 outlines a sequential proof that f is continuous on A. Show f is continuous below.

What about functions like $h(x) = \sqrt{3x^2 + 5}$ is continuous. Hence, a Compositions of Continuous functions type theorem is needed to show that $h(x)$ is continuous on its domain.

Theorem 4.2.3 (Composition of Continuous Functions)**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.*

Proof. Exercise 4.3.3. ■

4.2.1 Definitions

Definition 4.2.2 (Continuity)**.** A function $f : A \to \mathbb{R}$ is *continuous at a point* $c \in A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$. If f is continuous at every point in the domain A, then we say that f *continuous* on A.

Theorem 4.2.4 (Characterizations of Continuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(iii) For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.

If c is limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x \to c} f(x) = f(c)$.

Corollary 4.2.4.1 (Criterion for Discontinuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n) \not\to f(c)$, we may conclude that f is not continuous at c.*

Theorem 4.2.5 (Algebraic Continuity Theorem)**.** *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at point $c \in A$. Then,*

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;

(ii) $f(x) + g(x)$ is continuous at c for all $k \in \mathbb{R}$;

(iii) $f(x)g(x)$ is continuous at c; and

(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Theorem 4.2.6 (Algebraic Continuity Theorem). *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at point $c \in A$. Then,*

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;

(ii) $f(x) + g(x)$ is continuous at c for all $k \in \mathbb{R}$;

(iii) $f(x)g(x)$ is continuous at c; and

(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Theorem 4.2.7 (Composition of Continuous Functions). *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.*

4.2.2 Exercises

Exercise 4.3.1

Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

Proof. Let $\epsilon > 0$. Suppose $c = 0$. Let $c \in A$ and $g(x) = \sqrt[3]{x}$. Since $|x| < \delta$, we can say that $|\sqrt[3]{x}| < \sqrt[3]{\delta}$. Then choose $\delta = \epsilon^3$ such that

$$\begin{aligned}
|g(x) - g(c)| &= |\sqrt[3]{x} - 0| \\
&= |\sqrt[3]{x}| \\
&< \sqrt[3]{\epsilon^3} \\
&= \epsilon.
\end{aligned}$$

Hence, we have that

$$\lim_{x \to 0} \sqrt[3]{x} = 0.$$

■

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Proof. Now let $c \neq 0$. Let $\epsilon > 0$ and suppose $|x - c| < \delta$. By the given identity

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we can write

$$\begin{aligned} |g(x) - g(c)| &= |\sqrt[3]{x} - \sqrt[3]{c}| \\ &= \left| x^{3/9} - c^{3/9} \right| \\ &= |x^{1/9} - c^{1/9}||x^{2/9} + x^{2/9}c^{2/9} + c^{2/9}|. \end{aligned}$$

Since $|x - c| < \delta$, we have

$$x^{1/9} - c^{1/9} < (\delta + c)^{1/9} - c^{1/9} \tag{1}$$
$$x^{2/9} + x^{1/9}c^{1/9} + c^{2/9} < (\delta + c)^{2/9} + x^{1/9}c^{1/9} + c^{1/9} \tag{2}$$
$$x^{1/9}c^{1/9} < (\delta + c)^{1/9}c^{1/9}. \tag{3}$$

By using the identity again, we write

$$\begin{aligned} |g(x) - g(c)| &= |x^{1/9} - c^{1/9}||x^{2/9} + x^{2/9}c^{2/9} + c^{2/9}| \\ &< [(\delta + c)^{1/9} - c^{1/9}][(\delta + c)^{2/9} + (\delta + c)^{1/9}c^{1/9} + c^{1/9}] \\ &= (\delta + c)^{3/9} - c^{3/9} \\ &= (\delta + c)^{1/3} - c^{1/3}. \end{aligned} \tag{4}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$ such that (4) implies

$$(\delta + c)^{1/3} - c^{1/3} = (\delta^3 - c + c)^{1/3} - c^{1/3} = \epsilon + c^{1/3} - c^{1/3} = \epsilon.$$

Hence, we have $|g(x) - g(c)| < \epsilon$ implying that

$$\lim_{x \to c} g(x) = g(c).$$

■

Exercise 4.3.3

(a) Supply a proof for Theorem 4.2.3 using the $\epsilon - \delta$ characterization of continuity.

Proof. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Assume that the range

$$f(A) = \{f(x) : x \in A\}$$

is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. Suppose f is continuous at $c \in A$ and g is continuous at $f(c) \in B$.

Let $\epsilon > 0$. Since f is continuous at $c \in A$, we can construct $\delta > 0$ such that $|x - c| < \delta$ where
$$|f(x) - f(c)| < \epsilon.$$
Let $\epsilon = \delta$. Then assume
$$|f(x) - f(c)| < \delta$$
since g is continuous at $f(c) \in B$. Then we immediately have that
$$|g \circ f(x) - g \circ f(c)| = |g(f(x)) - g(f(c))| < \epsilon.$$
Hence, we have
$$\lim_{x \to c} g \circ f(x) = g \circ f(c).$$
∎

(b) Give another proof of this theorem using the sequential characterization of continuity.

Proof. Let $f(x_n) \subseteq f(A)$ and $(x_n) \subseteq A$ where the image
$$f(A) = \{f(x) : x \in A\}.$$
Let $\epsilon > 0$. Since $(x_n) \to c$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have
$$|f(x_n) - f(c)| < \epsilon.$$
But $g(f(x_n 0) \to g(f(c))$ since g is continuous at $f(c) \in B$ so we end up having
$$|g \circ f(x_n) - g \circ f(c)| = |g(f(x_n)) - g(f(c))| < \epsilon.$$
Hence, we have
$$\lim g \circ f(x_n) = g \circ f(c).$$
Note that this fact follows immediately if $f(x_n) = f(c)$ and $x_n = c$. ∎

Exercise 4.3.5

Show using definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Since $c \in A$ and c is an isolated point of $A \subseteq \mathbb{R}$, we must have $x = c$ and $f(x) = f(c)$ where $x \in A$ is an arbitrary point. This follows from the fact that there exists an $\epsilon-$neighborhood such that $V_\epsilon(x) \cap A = \emptyset$. Let $\epsilon > 0$. Then choose $\delta = \epsilon$ such that whenever $|x - c| < \delta$, we have that

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon.$$

■

Exercise 4.3.6

Using the $\epsilon - \delta$ characterization of continuity (and thus using no previous results about sequences), show that the linear function $f(x) = ax + b$ is continuous at every point $x \in \mathbb{R}$.

Proof. Let $\epsilon > 0$ and let $c \in \mathbb{R}$ be arbitrary. Choose $\delta = \epsilon/a$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |(ax + b) - (ac + b)| \\ &= |a(x - c)| \\ &= a|x - c| \\ &< a \cdot \frac{\epsilon}{a} \\ &= \epsilon. \end{aligned}$$

Hence, every linear function $f(x) = ax + b$ is continuous at any point $c \in \mathbb{R}$. ■

Exercise 4.3.7

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show K is a closed set.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Let $c \in K$. Since h is continuous on $\mathbb{R}$ we can use the Sequential Criterion. Let $x_n \in K$. There exists $(x_n) \subseteq K$ such that $(x_n) \to c$ implying that $h(x_n) \to h(c)$. Since every $x \in K$ has the property that $h(x) = 0$ and $c \in K$, we have that $h(c) = 0$. Hence, $h(x_n) \to h(c)$ is contained in K. Thus, K is closed. ■

Exercise 4.3.10

Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|].$$

(a) Show that if $f_1, f_2, \dots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \dots f_n(x)\}$$

is a continuous function.

Proof. Suppose $f_1, f_2, \dots, f_n$ are continuous functions. We proceed by inducting on n. Let our base case be $n = 2$. Then we have our continuous functions f_1, f_2. By using the $\epsilon - \delta$ definition for continuity, we have that

$$\begin{aligned}
|g(x) - g(c)| &= \Big|\frac{1}{2}\Big[(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|\Big] \\
&\quad - \frac{1}{2}\Big[(f_1(c) + f_2(c)) + |f_1(c) - f_2(c)|\Big]\Big| \\
&= \Big|\frac{1}{2}\Big[(f_1(x) - f_1(c)) + (f_2(x) - f_2(c))\Big] \\
&\quad + \frac{1}{2}\Big[|f_1(x) - f_2(x)| + |f_1(c) - f_2(c)|\Big]\Big| \\
&\leq \frac{1}{2}\Big[|f_1(x) - f_1(c)| + |f_2(x) - f_2(c)| \\
&\quad + |(f_1(x) - f_2(x)) - (f_1(c) - f_2(c))|\Big]
\end{aligned}$$

■

Exercise 4.3.11 (Contraction Mapping Theorem).

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and let $x, c \in \mathbb{R}$ where c is a limit point of A. We want to show that f is continuous on $\mathbb{R}$; that is, we want to show that

$$\lim_{x \to c} f(x) = f(c).$$

Let $\epsilon > 0$. Choose $\delta = \epsilon/c$ such that whenever $|x - c| < \delta$ we have

$$|f(x) - f(c)| \leq k|x - c| < k \cdot \frac{\epsilon}{k} = \epsilon.$$

Hence, we have

$$\lim_{x \to c} f(x) = f(c).$$

∎

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), ...).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence, we may let $y = \lim y_n$.

Proof. Observe that for any fixed $n \in \mathbb{N}$,

$$|y_{m+1} - y_{m+1}| = |f(y_m) - f(y_{m+1})| \leq c|y_m - y_{m+1}|.$$

We can show this via induction that

$$\begin{aligned} |y_{m+1} - y_{m+2}| &\leq c|y_m - y_{m+1}| \\ &\leq c^2|y_{m-1} - y_m| \\ &\vdots \\ &\leq c^m|y - 1 - y_2|. \end{aligned}$$

The fact that $0 < c < 1$ means that $\sum_{n=1}^{\infty} c^n$ converges (because it is a geometric series) enables us to conclude that (y_n) is Cauchy sequence. To see this, observe that

$$\begin{aligned} |y_m - y_n| &= |y_m - y_{m+1} + y_{m+1} + \cdots + y_{n-1} - y_n| \\ &\leq c^{m-1}|y_1 - y_2| + c^m|y_1 - y_2| + \ldots c^{n-2}|y_1 - y_2| \\ &= c^{m-1}|y_1 - y_2|(1 + c + \cdots + c^{n-m-r}) \\ &< c^{m-1}|y_1 - y_2|\left(\frac{1}{1-c}\right). \end{aligned}$$

Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ large enough so that $c^{N-1} < \epsilon 1 - c/|y_1 - y_2|$. Then the previous calculation shows that $n > m \geq N$ implies $|y_m - y_n| < \epsilon$; that is, we have

$$\begin{aligned} |y_{m+1} - y_{n+1}| &\leq |y_m - y_n| \\ &< c^{N-1}|y_1 - y_2|\left(\frac{1}{1-c}\right) \\ &< \frac{\epsilon(1-c)}{|y_1 - y_2|} \cdot \frac{|y_1 - y_2|}{1-c} \\ &= \epsilon. \end{aligned}$$

Hence, (y_n) is Cauchy Sequence. ■

(c) Prove that y is a fixed point of f; that is, $f(y) = y$ and that it is unique in this regard.

Proof. Since f is continuous on $\mathbb{R}$ and $(y_n) \subseteq \mathbb{R}$ is a Cauchy sequence and that $\lim y_n = y$ for all $n \in \mathbb{N}$, we have that

$$f(y_n) = y_{n+1} \to f(y) = y.$$

Hence, y is a fixed point of f. ■

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$ then the sequence $(x, f(x), f(f(x)), ...)$ converges to y defined in (b).

Proof. Let us fix $x \in \mathbb{R}$ where $(x_n) \subseteq \mathbb{R}$ is a sequence of points. Since the f is defined like $f(x_n) = x_{n+1}$. Since $x \in \mathbb{R}$ is fixed, we know that $f(x_n) \to f(x)$ and that $(x_{n+1}) \to x$. Hence, $f(x) = x$ as defined in (b). ■

Exercise 4.3.12

Let $F \subseteq \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Proof. Let $F \subseteq \mathbb{R}$ be a nonempty closed set. Let $x \in F$. Since F is a closed, let (x_n) be a Cauchy sequence such that $(x_n) \to x$ that is contained in F. But since $g : F \to \mathbb{R}$, we must have

$$|g(x_n) - g(x)| < \epsilon$$

by the Sequential Criterion of Continuity. This means g is continuous on all of $\mathbb{R}$.

Let $x \notin F$. Suppose for sake of contradiction that $g(x) = 0$. Let $(x_n) \subseteq F$ where $x_n \in F$. Since F is closed, (x_n) is a Cauchy sequence such that $(x_n) \to x$. But since g is continuous x has to be contained in F which is a contradiction. Hence, it must be that $g(x) \neq 0$. ■

Exercise 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Proof. By using the linearity property of f, we have $f(0+0) = f(0)+f(0)$ which implies $f(0) = 0$. For the inverse property, suppose $x \in \mathbb{R}$ such that $x+(-x) = 0$. Then using the linearity property of f, we have $f(x)+(-x)) = f(x)+f(-x)$. Since $f(0) = 0$, we have that $f(x) + f(-x)$ implies $f(-x) = -f(x)$. ■

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

Proof. Let $k = f(1)$. We proceed to show $f(n) = kn$ by inducting on $n \in \mathbb{N}$. Let our base case be $n = 1$. If $n = 1$, then $f(1) = k$. Now suppose $f(n)$ holds for $1 \leq n \leq \ell - 1$. Then

$$\begin{aligned} f(\ell) &= k\ell \\ &= k(\ell + 1 - 1) \\ &= k(\ell - 1) + k \\ &= f(\ell - 1) + f(1). \end{aligned}$$

Hence, $f(n) = kn$ for all $n \in \mathbb{N}$. To prove $f(z) = kz$ for all $z \in \ Z$ we can just prove it for all negative integers and zero. This is easy to see when $z = 0$. It's also easy to see that f holds for $z \in \mathbb{Z}^-$. Since $f(-x) = -f(x)$, we have that $f(-\ell) = -f(\ell)$. Since $f(n)$ holds for all $n \in \mathbb{Z}^+$. Hence, $f(z) = nz$ for all $z \in \mathbb{Z}$. Before proving $f(r) = kr$ for any rational number r, let us consider $1/n$ where $n \in \mathbb{N}$. Note that

$$\begin{aligned} k &= f(1) \\ &= f\Big(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\Big) \\ &= nf\Big(\frac{1}{n}\Big). \end{aligned}$$

Applying this for any given $r \in \mathbb{Q}$, we can see that

$$\begin{aligned} f(m/n) &= f\Big(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\Big) \\ &= mf\Big(\frac{1}{n}\Big) \\ &= k\Big(\frac{m}{n}\Big). \end{aligned}$$

We can prove that this holds for any rational number $r < 0$ by using a similar strategy to the used to prove the negative integers case above. ■

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Proof. Assume f is continuous at $x = 0$. Let $\epsilon > 0$. Choose $\delta = \epsilon/k$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |kx - kc| \\ &= k|x - c| \\ &< k \cdot \frac{\epsilon}{k} \\ &= \epsilon. \end{aligned}$$

Hence, f is continuous for $c \neq 0$. But f is also continuous at $x = 0$. Hence, f is continuous at every point in $\mathbb{R}$ and thus $f(x) = kx$ for all $x \in \mathbb{R}$. ■

4.3 Continuous Functions on Compact Sets

Given a function $f : A \to \mathbb{R}$ and a given subset $B \subseteq A$, the notation $f(B)$ can be defined as the range of f over the set B; in other words, we have that

$$f(B) = \{f(x) : x \in B\}.$$

We can describe properties such as subsets of $\mathbb{R}$ being open, closed, bounded, compact, perfect, and connected, but a more interesting analysis arises when see which ones are preserved when mapping B to $f(B)$ via a continuous function.

For example, if B is an open set and f is continuous, is the mapping $f(B)$ necessarily open? The answer to this is no.

Suppose $f(x) = x^2$ and $B = (-1, 1)$ is an open interval, then we have that the interval $[0, 1)$ is not open. What if B is closed? The same conjecture actually leads to the same conclusion that $f(B)$ is not closed as well. Consider the function

$$g(x) = \frac{1}{1 + x^2}$$

and the closed set $B = [0, \infty) = \{x : x \geq 0\}$. Because we have that $g(B) = (0, 1]$ is not closed, we must conclude that continuous functions do not generally map from closed sets to closed sets. However, if B is compact, then B gets mapped to closed and bounded subsets by continuous functions.

Theorem 4.3.1 (Preservation of Compact Sets)**.** *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Proof. Let $f : A \to \mathbb{R}$ and $K \subseteq A$ be a compact set. Let $(x_n) \subseteq K$ and $(y_n) \subseteq f(K)$. Since K is a compact set, there exists $(x_{n_k}) \to x$ such that x is contained in K. Suppose f is a continuous function. Define $f(x_n) = y_n$. Since (x_{n_k}) converges to x and f is a continuous function, we have that

$$f(x_{n_k}) = y_{n_k} \to f(x) = y.$$

This means our subsequence $(y_{n_k}) \subseteq f(K)$ converges to a limit y that is contained in $f(K)$. Thus, $f(K)$ is a compact set. ■

An extremely important result from this theorem deals with how compact sets are bounded and how they contain their supremums and infimums.

Theorem 4.3.2 (Extreme Value Theorem)**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Proof. Since $f(K)$ is a compact set, we can set $\alpha = \sup f(K)$ and know that $\alpha \in f(K)$ from Exercise 3.3.1. It immediately follows that for some $x_1 \in K$, this element gets mapped to $\alpha = f(x_1)$ since f is a continuous function. Likewise, we have $\beta \in f(K)$ such that for some $x_0 \in K$, x_0 gets mapped to $\beta = f(x_0)$ by the same reasoning above. Hence, we have that for any $x \in K$,

$$f(x_0) \leq f(x) \leq f(x_1).$$

■

4.3.1 Uniform Continuity

We learned in the last section that polynomials are always continuous on $\mathbb{R}$. In this section, we show that these functions are actually uniform continuous on $\mathbb{R}$; that is, they are everywhere continuous.

Example. (i) Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x + 1$. We want to show that this is continuous for any point $c \in \mathbb{R}$. Let $\epsilon > 0$. Then choose $\delta = \epsilon/3$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |(3x + 1) - (3c + 1)| \\ &= 3|x - c| \\ &< 3 \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Hence, $\lim_{x \to c} f(x) = f(c)$. The key observation here is that our choice of δ is the same regardless of the point $c \in \mathbb{R}$, we are considering.

(ii) Suppose how the situation changes when we consider another function, say, $g(x) = x^2$ and see how the choice of δ changes with each point $c \in \mathbb{R}$. Given $c \in \mathbb{R}$, observe that

$$|g(x) - g(c)| = |x^2 - c^2| = |x - c||x + c|.$$

As was discussed in section 4.2, we need to upper bound $|x + c|$, which, in this case, can be obtained by letting our choice δ not exceed 1. This implies that all values of x under consideration will fall in the interval $(c - 1, c + 1)$. By using our assumption that $|x - c| < \delta$ and letting $\delta = 1$, we have that

$$|x + c| \leq |x| + |c| \leq (|c| + 1) + |c| = 2|c| + 1.$$

Now let $\epsilon > 0$. If we choose $\delta = \min\{1, \epsilon/(2|c| + 1)\}$, then assume $|x - c| < \delta$ such that

$$|f(x) - f(c)| = |x - c||x + c| < \left(\frac{\epsilon}{2|c| + 1}\right) \cdot (2|c| + 1) = \epsilon.$$

Notice how our choice of δ depended on our choice of $c \in \mathbb{R}$ where

$$\delta = \frac{\epsilon}{2|c| + 1}.$$

This means that as our choice of $c \in \mathbb{R}$ gets bigger and bigger, our $\delta-$neighborhood must get smaller and smaller.

This leads us to our rigorous definition of what it means for a function to be uniform continuous.

Definition 4.3.1. A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

The difference between regular continuity and uniform continuity is that regular implies continuity of a function at any point $c \in \mathbb{R}$ granted that our choice of δ is dependent on our choice of $c \in \mathbb{R}$ while uniform continuity implies that our choice of δ stays the same regardless of our choice of $c \in \mathbb{R}$.

On the other hand, saying that a function is not uniform continuous is to say that given some $\epsilon > 0$, there is not suitable choice of $\delta > 0$ that will be a valid response to our ϵ challenge. That is, every point $c \in \mathbb{R}$ has a unique $\delta > 0$.

Theorem 4.3.3 (Sequential Criterion for Absence of Uniform Continuity)**.** *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof. ($\Rightarrow$) We can negate the definition of uniform continuity to help us prove this direction. Hence, for some $\epsilon_0 > 0$, choose $\delta_n = 1/n$ such that whenever we have sequences (x_n) and (y_n) that satisfy

$$|x_n - y_n| < \frac{1}{n},$$

we have that

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

Clearly, we have $|x_n - y_n| \to 0$ by the Squeeze Theorem for sequences.

($\Leftarrow$) Since $|x_n - y_n| \to 0$ for any $n \geq N$ for some $N \in \mathbb{N}$, we can see that any choice of $\delta > 0$ will not be a suitable response to the ϵ_0 challenge; that is, $|x_n - y_n| \to 0$ implies that f fails to be uniformly continuous on A. ∎

Example. Consider the function $h(x) = \sin(1/x)$. We can see that $h(x)$ is continuous at every point in the open interval $(0, 1)$ but is not uniformly continuous on this interval. We can prove this by defining the following sequence (x_n) and (y_n) to be

$$x_n = \frac{1}{\pi/2 + 2n\pi} \text{ and } y_n = \frac{1}{3\pi/2 + 2n\pi}.$$

Since both sequences tend to zero, we have that $|x_n - y_n| \to 0$ such that letting $\epsilon_0 = 2$ leads to

$$\begin{aligned} |h(x_n) - h(y_n)| &= |\sin(\pi/2 + 2n\pi) - \sin(3\pi/2 + 2n\pi)| \\ &= |2\cos(n\pi) - (-2\cos(n\pi)| \\ &= 2|\cos(n\pi)| \\ &= 2. \end{aligned}$$

Theorem 4.3.4 (Uniform Continuity on Compact Sets). *A function that is continuous on a compact set K is uniformly continuous on K.*

Proof. Assume $f : K \to \mathbb{R}$ is continuous at every point of a compact set $K \subseteq \mathbb{R}$. Suppose for sake of contradiction that f is not uniformly continuous on K. Then by the Sequential Criterion for Absence of Uniform Continuity, we have that for some $\epsilon_0 > 0$, we have two sequences (x_n) and (y_n) that satisfy the following property

$$|x_n - y_n| \to 0$$

which implies that

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

Since K is a compact set, we can find a subsequence x_{n_k} such that $(x_{n_k}) \to x$ where $x \in K$. Suppose we want to show that $y_{n_k} \to x$. Hence, we can use the Algebraic Limit Theorem to say that

$$\lim(y_{n_k}) = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = \lim(y_{n_k} - x_{n_k}) + \lim x_{n_k} = 0 + x = x.$$

Since f is continuous on $x \in K$, we have that $f(x_{n_k}) = f(x)$ and $f(y_{n_k}) = f(x)$. But this means that

$$\lim(f(x_{n_k}) - f(y_{n_k})) = 0$$

that is, $|f(x_{n_k}) - f(y_{n_k})| < \epsilon$ for all $\epsilon > 0$ which contradicts our original assumption that

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbb{N}$. Hence, f must be uniformly continuous on K. ■

4.3.2 Definitions

Theorem 4.3.5 (Preservation of Compact Sets). *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Theorem 4.3.6 (Extreme Value Theorem). *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Definition 4.3.2. A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Theorem 4.3.7 (Sequential Criterion for Absence of Uniform Continuity). *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Theorem 4.3.8 (Uniform Continuity on Compact Sets). *A function that is continuous on a compact set K is uniformly continuous on K.*

4.3.3 Exercises

Exercise 4.4.1

(a) Show that $f(x) = x^3$ is continuous on all $c \in \mathbb{R}$.

Proof. Let $\epsilon > 0$ and let $c \in \mathbb{R}$. Choose $\delta = \min\{1, \epsilon/|(c+1)^2 + (c+1)c + c^2|\}$. Let $f(x) = x^3$. Then whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |x^3 - c^3| \\ &= |x - c||x^2 + xc + c^2| \\ &< \frac{\epsilon}{|(c+1)^2 + (c+1)c + c^2|} \cdot |(c+1)^2 + (c+1)c + c^2| \\ &= \epsilon. \end{aligned}$$

Hence, we have that $\lim_{x\to c} f(x) = f(c)$ for any $c \in \mathbb{R}$. ■

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbb{R}$.

Proof. Let $(x_n) = n$ and $(y_n) = n + 1/n$, then

$$|x_n - y_n| = \left|\frac{1}{n}\right| \to 0.$$

Then we have that

$$\begin{aligned}
|f(x_n) - f(y_n)| &= \left|n^3 - \left(n + \frac{1}{n}\right)^3\right| \\
&= \left| -\left(3n + \frac{3}{n} + \frac{1}{n^3}\right)\right| \\
&= 3n + \frac{3}{n} + \frac{1}{n^3} \\
&\geq 3.
\end{aligned}$$

Hence, there exists $\epsilon_0 = 3$ such that $|f(x_n) - f(y_n)| \geq \epsilon_0$. which implies $f(x) = x^3$ is not uniform continuous. ■

(c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

Proof. Let (a, b) be a bounded subset of $\mathbb{R}$. Then let $\epsilon > 0$. Let $x, y \in [a, b]$. Choose $\delta =$. Hence, whenever $|x - y| < \delta$ we have that

$$\begin{aligned}
|f(x) - f(y)| &= |x^3 - y^3| \\
&= |x - y||x^2 + xy + y^2| \\
&< \delta \cdot 3b^2 \\
&= \frac{\epsilon}{3b^2} \cdot 3b^2 \\
&= \epsilon.
\end{aligned}$$

Hence, we have that f is uniformly continuous on the bounded set $(a, b) \subseteq \mathbb{R}$. ■

Exercise 4.4.2

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

Solution. No it is not. Let $(x_n), (y_n)$ be two sequences such that $x_n = 1/2n$ and $y_n = 1/(2n + 1)$. Observe that $|x_n - y_n| \to 0$ since both sequences tend to zero. Now we have that

$$|f(x_n) - f(y_n)| = |2n - 2n - 1| = 1 = \epsilon_0.$$

Hence, $f(x)$ is not uniformly continuous on $(0,1)$. ■

(b) Is $g(x) = \sqrt{x^2+1}$ uniformly continuous on (0,1)?

Proof. Let $\epsilon > 0$ and choose $\delta = \epsilon$. Assume $f(x)$ is defined on $(0,1)$. Then whenever $|x-y| < \delta$, we have that

$$\begin{aligned}
|g(x) - g(y)| &= |\sqrt{x^2+1} - \sqrt{y^2+1}| \\
&= \frac{|x^2 - y^2|}{|\sqrt{x^2+1} + \sqrt{y^2+1}|} \\
&= \frac{|x-y||x+y|}{|\sqrt{x^2+1} + \sqrt{y^2+1}|} \\
&< \delta \cdot \frac{|x+y|}{|\sqrt{x^2+1} + \sqrt{y^2+1}|} \\
&< \delta \cdot \frac{2}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, we have that $g(x)$ is uniform continuous on $(0,1) \subseteq \mathbb{R}$. ■

Exercise 4.4.3

Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1,\infty)$ but not on the set $(0,1]$.

Proof. Let $\epsilon > 0$. Choose $\delta = \delta/2$ such that whenever $|x-y| < \delta$ for any $x, y \in [1,\infty)$

we have that

$$
\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \frac{|y^2 - x^2|}{|x^2y^2|} \\
&= \frac{|x-y||x+y|}{|x^2||y^2|} \\
&\leq \frac{|x-y|}{|x^2||y^2|}(|x| + |y|) && \text{(Triangle Inequality)} \\
&< \delta \cdot \frac{1}{|x^2y^2|}(|x| + |y|) && (|x - y| < \delta) \\
&= \delta \cdot \left(\frac{1}{xy^2} + \frac{1}{x^2y}\right) \\
&< \frac{\epsilon}{2} \cdot 2 && (x \geq 1 \text{ and } \delta = \epsilon/2) \\
&= \epsilon.
\end{aligned}
$$

Hence, $f(x) = 1/x^2$ is a continuous function on $[1, \infty)$. Now we want to show that $f(x)$ is not uniformly continuous on $(0, 1]$. Hence, let $(x_n), (y_n)$ be two sequences that are contained in $(0, 1]$. Defined these two sequences as follows:

$$
\begin{aligned}
x_n &= \frac{1}{\sqrt{2n}}, \\
y_n &= \frac{1}{\sqrt{2n+1}}.
\end{aligned}
$$

Since (x_n) and (y_n) both tend towards zero, we have that $|x_n - y_n| \to 0$. Now consider $|f(x_n) - f(y_n)|$. We want to construct an ϵ_0 such that $|f(x_n) - f(y_n)| \geq \epsilon_0$. Hence, observe that

$$
\begin{aligned}
|f(x_n) - f(y_n)| &= \left|\left(\frac{1}{1/\sqrt{2n}}\right)^2 - \left(\frac{1}{1/\sqrt{2n+1}}\right)^2\right| \\
&= |2n - 2n - 1| \\
&= 1 \\
&= \epsilon_0.
\end{aligned}
$$

Hence, we have that $f(x)$ cannot be uniformly continuous on the interval $(0, 1]$. ■

Exercise 4.4.4

Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

Solution. Let f be a continuous function $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$. Since $[a, b]$ are compact sets, and f is continuous on $[a, b]$, we have $f(a) \leq f(x) \leq f(b)$. Since $f > 0$ for all $a \leq x \leq b$, we have that

$$\frac{1}{f(b)} \leq \frac{1}{f(x)} \leq \frac{1}{f(a)}$$

which means $1/f$ is a bounded function on $[a, b]$. ■

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

Solution. Let f be a uniformly continuous function on a bounded set A. Hence, let $\epsilon = 1$. There exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have that

$$|f(x) - f(y)| < \epsilon.$$

Since A is bounded, there exists a subsequence $(x_n) \to x$ where x is a limit point of A. Suppose for sake of contradiction that $f(A)$ is unbounded. Let $\delta = 1$. Since f is unbounded, we have that whenever $|x_n - x_m| < 1$ where $n \neq m$, we have

$$|f(x_n) - f(x_m)| > 1.$$

But this means that our function is not uniformly continuous on A which is a contradiction. Thus, f must have a bounded range.

■

(c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

Solution. This is false. Suppose we have the function f defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{I}. \end{cases}$$

We have that for any compact set K, we have that every element of x gets mapped to $f(K)$ but $f(x)$ is not a continuous function. ■

Exercise 4.4.5

Assume that g is defined on an open interval (a, c) a nd it is known to be uniformly continuous on $(a, b]$ and $[b, c)$. Let $x \in (a, b]$ and $y \in [b, c)$. Note that $(a, c) = (a, b] \cup [b, c)$. Since f is uniformly continuous on $(a, b]$, we have that for any $x, b \in (a, b]$, there exists $\delta_1 > 0$ such that whenever $|x - b| < \delta_2$, we have that

$$|f(x) - f(b)| < \frac{\epsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that whenever $|b - y| < \delta_2$ such that whenever

$$|f(b) - f(y)| < \frac{\epsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$ such that whenever $|x - y| < \delta$, we have that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(b) + f(b) - f(y)| \\ &= |f(x) - f(b)| + |f(b) - f(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $f(x)$ is uniformly continuous on (a, c).

Exercise 4.4.7

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. Let $\epsilon > 0$ and let $x, y \in [0, \infty)$. If $x, y = 0$, then it immediately follows that f is uniformly continuous since choosing $\delta = \epsilon$ implies that whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |\sqrt{0} - \sqrt{0}| = 0 < \epsilon.$$

Suppose $x, y \neq 0$. Then choose $\delta = \epsilon \cdot 2\sqrt{c}$ for any $c \in [0, \infty)$ such that whenever

$|x-y|<\delta$ we have that

$$\begin{aligned}|f(x)-f(y)| &= |\sqrt{x}-\sqrt{y}| \\ &= \Big|\frac{x-y}{\sqrt{x}+\sqrt{y}}\Big| \\ &= \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \\ &\leq \frac{|x-y|}{2\sqrt{c}} \\ &< \frac{\delta}{2\sqrt{c}} \\ &= \epsilon.\end{aligned}$$

Hence, we have that $f(x)=\sqrt{x}$ is continuous on $[0,\infty)$. ■

Exercise 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0,1]$ with range $(0,1)$.

Solution. Let $f(x)=1/x(x+1)$. The range of $f(x)$ is $(0,1)$. ■

(b) A continuous function defined on $(0,1)$ with range $[0,1]$.

Solution. Let the following function $f(x)$ be defined as follows:

$$f(x)=\begin{cases}0 & \text{if } x\in(0,\frac{1}{4}) \\ 2x-\frac{1}{2} & \text{if } x\in[\frac{1}{4},\frac{3}{4}] \\ 1 & \text{if } x\in(3/4,1)\end{cases}$$

■

(c) A continuous function defined on $(0,1]$ with range $(0,1)$.

Solution. ■

Exercise 4.4.9

Definition 4.3.3 (Lipschitz Functions)**.** A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$$

for all $x \neq y \in A$.

Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

Proof. Suppose $f : A \to \mathbb{R}$ is Lipschitz and let $x \neq y \in A$. Let $\epsilon > 0$ and choose $\delta = \epsilon/M$. Assume $|x-y| < \delta$. Using the fact that f is Lipschitz, we have

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq M.$$

But this is equivalent to

$$|f(x)-f(y)| \leq M|x-y| < M \cdot \delta = M \cdot \frac{\epsilon}{M} = \epsilon.$$

Hence, we have that $|f(x)-f(y)| < \epsilon$ for all $x \neq y \in A$ implying that $f : A \to \mathbb{R}$ is a uniformly continuous function. ∎

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. No the converse statement is not true. Take the function $f(x) = \sqrt{x}$. This function contains a slope that gets arbitrarly steep as the values of x tend toward infinity. Another example is the function $g(x) = 1/x^2$. We can see that if $g(x)$ is defined on $(0,1)$ then the slope of the $g(x)$ gets arbitrary big as $x \to 0$. ∎

Exercise 4.4.10

Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x)+g(x),\ f(x)g(x),\ \frac{f(x)}{g(x)},\ g\circ f(x).$$

(a) $f(x)+g(x)$.

Proof. The addition of two uniformly continuous functions $f(x), g(x)$ defined on the common domain A is continuous. Let $\epsilon > 0$. Then define $\delta = \min\{\delta_1, \delta_2\}$ such that whenever

$$\begin{aligned} |x-y| &< \delta_1, \\ |x-y| &< \delta_2 \end{aligned}$$

we have that

$$\begin{aligned} |f(x)+g(x)-(f(y)-g(y))| &= |(f(x)-f(y))+(g(x)-g(y))| \\ &\leq |f(x)-f(y)|+|g(x)-g(y)| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, the sum of two uniformly continuous functions is uniformly continuous. ■

(b) $f(x)g(x)$.

Solution. Not necessarily true unless both of the functions are both bounded. ■

(c) $f(x)/g(x)$ where $g(x) > 0$.

Solution. Like the product of two uniformly continuous functions, the quotient is also not necessarily true unless both $g(x)$ and $f(x)$ are bounded below and above respectively. ■

(d) $f(g(x))$.

Proof. Given $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ uniformly continuous, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in A so that composition is defined on A. Let $\epsilon > 0$. Let $x \neq y \in A$. Since $f(x)$ is uniformly continuous on A, choose $\epsilon = \delta$ such that whenever $|f(x)-f(y)| < \delta$ and $g : A \to \mathbb{R}$ being uniformly continuous on A, we have that

$$\begin{aligned}|g \circ f(x) - g \circ f(y)| &= |g(f(x)) - g(f(y))| \\ &< \epsilon.\end{aligned}$$

■

Exercise 4.4.11 (Topological Characterization of Continuity).

Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Proof. ($\Leftarrow$) Let us construct an $\epsilon-$neighborhood around $g(c)$; that is, $V_\epsilon(g(c))$. Suppose $V_\delta(c)$ with $x \in V_\delta(c)$. Since $g^{-1}(O)$ is an open set, we have that $V_\delta(c) \subseteq g^{-1}(O)$. But this means that $x \in g^{-1}(O)$ implying that $g(x) \in V_\epsilon(g(c))$ since $g(x) \in O$.

($\Rightarrow$) Let $V_\epsilon(g(c))$ where c is a limit point of O. Since g is a continuous function, there exists $V_\delta(c)$ such that whenever $x \in V_\delta(c)$, we have that $g(x) \in V_\epsilon(g(c))$. Since O is an open set, we have that $V_\epsilon(g(c)) \subseteq O$. But this means that $g(x) \in O$ as well implying that x is also an element of $g^{-1}(O)$. Hence, we have that $V_{\delta(c)} \subseteq g^{-1}(O)$ and thus $g^{-1}(O)$ is an open set. ■

4.4 The Intermediate Value Theorem

In intuitive terms, the Intermediate Value Theorem is an observation that explains how a continuous function f on a closed interval $[a, b]$ attains every value that falls between two range values $f(a)$ and $f(b)$.

Theorem 4.4.1 (Intermediate Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

4.4.1 Preservation of Connected Sets

Before we prove the Intermediate Value Theorem, we should understand that it is a special case of continuous functions mapping connected sets to connected sets. In the last section, we saw how continuous functions on compact sets K produces range sets $f(K)$ that are also compact. This just so happened to also hold for connected sets.

Theorem 4.4.2 (Preservation of Connected Sets)**.** *Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.*

Proof. Our goal is to use the characterization of connected sets to prove this theorem. Let $f(E) = A \cup B$ where A and B are disjoint and nonempty. We want to produce a sequence that is contained in either A or B that converges to a limit contained in the other. Define the following sets:

$$C = \{x \in E : f(x) \in A\} \text{ and } D = \{x \in E : f(x) \in B\}.$$

These sets are the *preimages* of A and B respectively. We can see, from the properties of both A and B, that they are nonempty and disjoint such that they satisfy $E = C \cup D$. Now assume E is a connected set. This means there exists a convergent sequence (x_n) that is contained in either C or D with $\lim x_n = x$. Since f is continuous at x, we have that $f(x) = \lim f(x_n)$. Thus, it follows that $f(x_n)$ is a convergent sequence contained in either A or B while the limit $f(x)$is an element of the other. Hence, $f(E)$ is also connected. ■

In $\mathbb{R}$, a set is connected if and only if it is an interval (which is possibly unbounded). This in addition to the theorem above leads to a short proof of the Intermediate Value Theorem.

4.4.2 Completeness

A typical application of the Intermediate Value Theorem involves proving the existence of roots. Given a function $f(x) = x^2 - 2$, we can see that $f(1) = -1$ and $f(2) = 2$. Hence, there exists a point $c \in (1, 2)$ where $f(c) = 0$. In Chapter 1, we prove the existence of $\sqrt{2}$ using the Axiom of Completeness as our main assumption about the properties of $\mathbb{R}$. This fact sheds light on the relationship between the continuity of functions and the completeness of $\mathbb{R}$.

Proof of the Intermediate Value Theorem using the Axiom of Completeness.

Proof. First, let us consider a special case where f is a continuous function satisfying the property that $f(a) < 0 < f(b)$. We want to show that $f(c) = 0$ for some $c \in (a, b)$. Let us define the following set

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

This is the set of negative values that $f(x)$ takes on the interval $[a, b]$. Note that K is bounded above by b, and $a \in K$ so that K is nonempty. Hence, we are allowed to assert that $\sup K$ exists and that $\sup K = c$. We have three cases to consider:

$$f(c) > 0, \; f(c) < 0, \text{ and } f(c) = 0.$$

By the fact that c is the least upper bound of K rules out the first two cases. Hence, we reach our desired conclusion that $f(c) = 0$. The details are requested in Exercise 4.5.5(a). ■

Below is the second proof of the Intermediate Value Theorem using the Nested Interval Property.

Proof. Consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint $z = (a + b)/2$. If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. We have that, in either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right. This procedure can be inductively repeated such that the Nested Interval Property can be applied to gain the conclusion of the theorem. The remainder of the argument is left to the reader in Exercise 4.5.5(b). ■

4.4.3 The Intermediate Value Property

An interesting question we can ask is does the Intermediate Value Theorem have a converse that is true?

Definition 4.4.1. A function f has the *intermediate value property* on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

This is to say that every continuous function f on an interval $[a, b]$ must have the intermediate value property. We must be careful since this is not always true that a function that contains this property must necessarily be continuous. An example of this is the function

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at zero, but it does have the intermediate value property on $[0, 1]$.

4.4.4 Definitions

Theorem 4.4.3 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

Theorem 4.4.4 (Preservation of Connected Sets)**.** *Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.*

Definition 4.4.2. A function f has the *intermediate value property* on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

4.4.5 Exercises

Exercise 4.5.3

Definition 4.4.3. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A.

Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property, then f is continuous on $[a, b]$.

Proof. Let f be an increasing function on $[a, b]$. Since f satisfies the intermediate value property, we know that $f(a) \leq f(c) \leq f(b)$ for some $c \in (x, y)$. Since we know that $f(a) \leq f(c)$, let us suppose two cases; that is, let us suppose either $f(c) - \epsilon/2 < f(a)$ or $f(a) \leq f(c) - \epsilon/2$. Then the former implies that we can set $x_1 = a$ and the latter implies that we can set $f(c) - \epsilon/2 = f(x_1)$. Taking the latter case, we can set $f(x_1) = f(c) - \epsilon/2$ and let $x \in (x_1, c]$ because f satisfies the intermediate value property. Hence, we have that

$$f(c) - \epsilon/2 \leq f(x) \leq f(c). \tag{1}$$

Likewise, we know that $f(c) \leq f(b)$. Let us suppose two cases again; either $f(b) < f(c) + \epsilon/2$ or $f(b) \geq f(c) + \epsilon/2$. The former we can set $b = x_2$ and the latter we can set $f(x_2) = f(c) + \epsilon/2$ because f satisfies the intermediate value theorem. If we let $x \in [c, x_2)$, then we have that

$$f(c) \leq f(x) \leq f(c) + \epsilon/2 = f(x_2). \tag{2}$$

Taking (1) and (2) together then choose $\delta = \min\{x_1 - c, x_2 - c\}$, we have that

$$\epsilon/2 \leq f(x) - f(c) \leq \epsilon/2$$

which is equivalent to

$$|f(x) - f(c)| \leq \epsilon/2.$$

■

Exercise 4.5.5

(a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

Proof. Considering the special case where f is a continuous function satisfying the property that $f(a) < 0 < f(b)$. Our goal is to show that $f(c)$ for some $c \in (a, b)$. Define the set

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

Note that $f(b)$ is an upper bound for K and we know $a \in K$ since $f(a) < 0$. Hence, we have that $\sup K$ exists by the Axiom of Completeness. Our goal is to show that $\sup K = f(c) = 0$. Consider the cases where $f(c) > 0, f(c) < 0$, and $f(c) = 0$. We will show that the first two cases contradict our notion that $\sup K$ is the least upper bound. Assume $f(c) < 0$ for some $c \in (a, b)$. But this means that $f(c)$ would not be an upper bound of K since there exists some α such that $\sup K < \alpha$where α is not an upper bound of K. Assume $f(c) > 0$ for some $c \in (a, b)$. This implies that $f(c) > 0$ is an upper bound of K but is not the least upper bound of K which is a contradiction. Hence, it must be that $f(c) = 0$. ■

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Proof. Consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$ and consider the midpoint $z = (a + b)/2$. If $f(z) \geq 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and positive at the right. We can extend this inductively to produce a sequence of closed intervals $I_n = [a_n, b_n]$ with the property described above to make a nested sequence

$$I_0 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots .$$

Since the intersection of these nested intervals I_n for all n is nonempty, we can find a $c \in \bigcup_{n=0}^{\infty} I_n$ such that $f(c) = 0$ since $f(x)$ is a continuous function. ∎

Exercise 4.5.6

Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x-y| = 1/2$ and $f(x) = f(y)$.

Proof. Define $g(x) = f(x)-f(x+1/2)$. Note that g is continuous over $[0, 1/2]$. Hence, we have

$$\begin{aligned} g(0) &= f(0) - f(1/2) && (1) \\ g(1/2) &= f(1/2) - f(1) \\ &= -g(0) && (2) \end{aligned}$$

By the Intermediate Value Theorem, we can see that there must exists $c \in [0, 1/2]$ such that $g(c) = 0$. Hence, we must have $g(c) = f(c) - f(c + 1/2) = 0$ which implies $f(c) = f(c + 1/2)$. ∎

(b) Show that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

Initial Attempt at Solution. Define $g(x_n) = f(x_n) - f(x_n + 1/n)$. Note that $g(x_n)$ is a continuous function over $[0, 1]$. Since g is continuous, let $(x_n) \to 0$ and observe that $g(0) = f(0) - f(0) = 0$ and let $(x_n) \to 1$ such that $g(1) = f(1) - f(1) = 0$. Hence, we can see that $f(0) = f(1) = 0$. Hence, we can use the Intermediate Value Theorem to state that there exists $c \in [0, 1]$ such that $g(c) = 0$. Hence, we have

$$g(c) = 0 \iff f(x_n) = f(x_n + 1/n).$$

■

Corrected Solution. For fixed $n \in \mathbb{N}$, define $g(x_n) = f(x_n) - f(x_n + 1/n)$ such that $|x_n - y_n| = 1/n$ where $y_n = x_n + 1/n$. Note that g is continuous over $[0, (n-1)/n]$. We want to show that there exists a root in the interval $[0, (n-1)/n]$. Consider the following

$$\begin{aligned}\sum_{k=0}^{n-1} g\Big(\frac{k}{n}\Big) &= \sum_{k=0}^{n-1} f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\\ &= f(1) - f(0)\\ &= 0\end{aligned}$$

and note that if there exists $0 \leq k \leq n-1$ such that $g(k/n) = 0$ then we are done. Otherwise, if $g(k/n) \neq 0$ for $0 \leq k \leq n-1$, then there must exist $0 \leq k_1 \leq n-1$ and $0 \leq k_2 \leq n-1$ such that $g(k_1/n)$ and $g(k_2/n)$ have opposite sign. Hence, we can use the intermediate value theorem to posit the existence of $c \in [0, (n-1)/n]$ such that $g(c) = 0$. Hence, we have

$$f(x_n) = f\Big(x_n + \frac{1}{n}\Big).$$

■

Exercise 4.5.7

Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Proof. Let g be defined by $g(x) = x - f(x)$. We can see that g is continuous over $[0, 1]$ since $f(x)$ is continuous over $[0, 1]$. Now observe that

$$g(0) = -f(x), \tag{1}$$
$$g(1) = 1 - f(x). \tag{2}$$

Since $g(1) = 1 + g(0)$, we have that $g(1) \geq g(0)$. Since g is continuous over $[0, 1]$, we can use the Intermediate Value Theorem to find a $c \in [0, 1]$ such that $g(c) = 0$. Hence, we must have

$$g(c) = c - f(c) = 0 \iff f(c) = c.$$

■

Exercise 4.5.8 (Inverse Functions).

If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous on an interval $[a, b]$ and injective, then f^{-1} is also continuous.

Proof. Assume f is continuous on an interval $[a, b]$ and injective. Let $\epsilon > 0$. Assume f is monotone so that we can use the intermediate value theorem to posit the existence of an y_0 such that it is between $f(x)$ and $f(y)$. Choose f to be an increasing function such that there exists some $x_1 \in [a, b]$ such that $f(x_1) < f(c)$. Hence, there exists y_1 in the image of f such that $x_1 = f^{-1}(y_1) = c - \epsilon$ where $x_1 = c - \epsilon < c$. Let $x \in (x_1, c]$ such that

$$x_1 = c - \epsilon < x \leq c. \tag{4.1}$$

Likewise, there exists some $x_2 \in [a, b]$ such that $f(x_2) > f(c)$. Define $x_2 = c + \epsilon$ such that $x_2 > c$. Let $x \in [c, x_2)$, we have that

$$c \leq x < x_2 = c + \epsilon. \tag{4.2}$$

Since f is injective, we have $f^{-1}(x') = x$ for all $x' \in \text{im}(f)$ and $f^{-1}(c') = c$ for some $c' \in \text{im}(f)$. Hence, we have that

$$|f^{-1}(x') - f^{-1}(c')| < \epsilon$$

whenever $|f(x) - f(c)| < \delta$. ∎

4.5 Sets of Discontinuity

Definition 4.5.1. Given a function $f : \mathbb{R} \to \mathbb{R}$, we call the set $D_f \subseteq \mathbb{R}$ to be the set of points where the function f fails to be continuous.

Some examples of sets of discontinuous points are

(a) $D_g = \mathbb{R}$ in the case for Dirichlet's function,

(b) and $D_h = \mathbb{R} \setminus \{0\}$ in the case of the modified Dirichlet's function, and

(c) lastly, $D_t = \mathbb{Q}$ for Thomae's function $t(x)$.

We can always write the set of discontinuous points for a function D_f as a countable union of closed sets. For monotone functions, these closed sets can taken as single points.

4.5.1 Monotone Functions

Definition 4.5.2. A function $f : A \to \mathbb{R}$ is *increasing* on A if $f(x) \leq f(y)$ whenever $x < y$ and *decreasing* if $f(x) \geq f(y)$ whenever $x < y$ in A. A *monotone* function is one that is either increasing or decreasing.

The continuity of a function at a point c means that $\lim_{x \to c} f(x) = f(c)$. Discontinuities occur when right-hand limits do not equal the left-hand limits approaching c.

Definition 4.5.3. Given a limit point c of a set A and a function $f : A \to \mathbb{R}$, we write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$. Equivalently, in terms of sequences, $\lim_{x \to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n > c$ and $\lim(x_n) = c$.

Exercise 4.6.3

State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Solution. We say $\lim_{x \to c^-} f(x) = L$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |c - x| < \delta$. Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(y_n) = M$ for all sequences (y_n) satisfying $y_n < c$ and $\lim(y_n) = c$. ■

Theorem 4.5.1. *Given $f : A \to \mathbb{R}$ and a limit point c of A, $\lim_{x \to c} f(x) = L$ if and only if*

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

Exercise 4.6.4

Supply a proof for this proposition.

Proof. Let $f : A \to \mathbb{R}$ and a limit point c of A. Assume $\lim_{x\to c} f(x) = L$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. To show that the right-hand limit equals L. Let $x > c$. Then immediately, we have that $0 < |x-c| < \delta$ implies $\lim_{x\to c^+} f(x) = L$ since $|f(x)-L| < \epsilon$. For the left-hand limit, suppose $x < c$. Then

$$0 < |x - c| < \delta \iff 0 < |c - x| < \delta$$

implies $|f(x) - L| < \epsilon$ holds. Hence, $\lim_{x\to c^-} f(x) = L$.

Conversely, the fact that $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} = L$ implies for some $\delta > 0$ that for any $x > c$ or $x < c$ that $0 < |x-c| < \delta$ holds. Hence, we have that $|f(x)-L| < \epsilon$; that is, $\lim_{x\to c} f(x) = L$. ■

Generally speaking, discontinuities can be divided into three categories:

(i) If $\lim_{x\to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at c is called *removable*.

(ii) If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then f has a *jump* discontinuity.

(iii) If $\lim_{x\to c} f(x)$ does not exist for some other reason, then the discontinuity at c is called an *essential* discontinuity.

We now have the proper characteristics to describe our set of discontinuities D_f for an arbitrary monotone function f.

Exercise 4.6.5

Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Proof. Without loss of generality, let f be an increasing function. Then for all $x < y$, $f(y) \geq f(x)$. If $\lim_{x\to c} f(x) = L$, but $L \neq f(c)$, then either $f(c) < L$ or $f(c) > L$. In either case, there exists an $x \in A$ such that for all $y \in A$, we have $f(x) > f(y)$ or $f(x) < f(y)$. But this contradicts our assumption that f is increasing. Hence, the discontinuity cannot be a *removable* discontinuity. If there exist an *essential discontinuity* then f cannot be monotone since either the left-hand limit or the right-hand limit does not exists. Hence, the only type of discontinuity an increasing function can have is a jump discontinuity. ■

4.5.2 D_f for an Arbitrary Function

Review of infinite and finite closed or open sets:

(a) Recall that the intersection of an infinite collection of closed sets is closed and the union of a finite collection of closed sets is closed.

(b) On the other hand, the intersection of a finite collection of open sets must be open and the union of a infinite collection of open sets must be open.

Definition 4.5.4. A set that can be written as the countable union of closed sets is in the class F_σ.

Chapter 5

The Derivative

5.1 Are Derivatives Continuous?

The derivative of a function $g(x)$, namely $g'(x)$, can be defined as the slope of g at each point $x \in \text{Dom}(f)$. As we have learned in our previous studies, the derivative is just the following limit

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

A couple questions we can ask about the relationship between continuity and differentiability of functions is that:

(i) Are they continuous?

(ii) Are continuous functions differentiable?

(iii) How nondifferentiable can a continuous function be?

In the last section, we identified the discontinuous points of a monotone function and expressed them in terms of countable closed sets. Some examples of such functions are of the form

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

When $n = 0$, we can see the oscillations of $\sin(1/x)$ prevent g from being continuous at $x = 0$. But when $n = 1$, the oscillations of g are sandwiched between $|x|$ and $-|x|$ which implies that g is continuous at $x = 0$. What can we say about $g_2'(0)$? Is it defined? Using our intuitive definition above, we have that

$$g_1'(0) = \lim_{x \to 0} \frac{g_1(x)}{x} = \lim_{x \to 0} \sin(1/x)$$

which, in this case, does not exist. Thus, we have that g_1 is not differentiable at zero. However, if we let $n = 2$, then we have the following

$$g_2'(0) = \lim_{x \to 0} x \sin(1/x) = 0.$$

At nonzero points in the domain of g, we can use rules of differentiation (that will be justified later) to conclude the g_2 is differentiable everywhere in $\mathbb{R}$ with

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

But if we now consider the limit

$$\lim_{x \to 0} g_2'(x)$$

we will find that it does not exist because for every $x \neq 0$, the $\cos(1/x)$ term is not preceded by a factor of x.

In summary, when $n = 2$, $g_2(x)$ is continuous and differentiable everywhere on $\mathbb{R}$, but the derivative function $g_2'(x)$ is defined everywhere but is not continuous at $x = 0$. The conclusion is that we don't the derivative of a function to be continuous in general.

The discontinuity we found from g_2' is an *essential* discontinuity; that is, the limit as $x \to 0$ does not exist as a one sided limit. What about a function with a simple jump discontinuity like

$$h'(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Notice that this function is actually the slopes of the absolute value function $|x|$ which is not differentiable at $x = 0$. How can we imply differentiability of h' at $x = 0$? Our main point here is that continuity is not a sufficient condition for derivatives to be possible.

5.2 Derivatives and the IVP

5.2.1 Definition of the Derivative

Definition 5.2.1 (Differentiability)**.** Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative* of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case, we say that g is *differentiable* at c. If g' exists for all points $c \in A$, we say that g is *differentible* on A.

Example. (i) Consider the function $f(x) = x^n$, where $n \in \mathbb{N}$, and let c be any arbitrary point in $\mathbb{R}$. Using the following identity,

$$x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

we can take the limit

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{x^n - c^n}{x - c} \\ &= \lim_{x \to c}(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}) \\ &= c^{n-1} + c^{n-1} + c^{n-1} + \cdots + c^{n-1} \\ &= nc^{n-1} \end{aligned}$$

(ii) If $g(x) = |x|$, then if we want to take the derivative at $c = 0$ produces the following limit

$$g'(0) = \lim_{x \to 0} \frac{|x|}{x}$$

which is 1 if we approach from the right and -1 if we approach from the left. Hence, we have that $g'(c) = 0$ does not exist.

This last example should remind us that continuity of a function does not necessarily imply that a function is differentiable. On the other hand, we can say that if g is differentiable at a point then g is continuous at that point.

Theorem 5.2.1. *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof. Assume $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$. Hence, we have that the following limit exists

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

Using the Algebraic Limit Theorem for functional limits, we have that

$$\lim_{x \to c}(g(x) - g(c)) = \lim_{x \to c} \left(\frac{g(x) - g(c)}{x - c} \right)(x - c) = g'(c) \cdot 0 = 0.$$

Hence, it follows that $\lim_{x \to c} g(x) = g(c)$. ∎

We can prove the same fact using the epsilon-delta definition for functional limits.

Proof. Assume $g : A \to \mathbb{R}$ is differentiable at at a point $c \in A$. Let $\epsilon > 0$. Then we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have that

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

With a few algebraic manipulations, we can manipulate the above to state that

$$|g(x) - g(c) - g(c)(x - c)| < |x - c| \tag{1}$$

with $\epsilon = 1$. Using the triangle inequality and choosing $\delta = \min\{1, \epsilon/(1 + |g(c)|)\}$

$$\begin{aligned}
|g(x) - g(c)| &= |g(x) - g(c)(x - c) + g(c)(x - c) - g(c)| \\
&\leq |g(x) - g(c)(x - c)| + |g(c)(x - c) - g(c)| \\
&< |x - c| + |g(c)||x - c| \\
&= |x - c|(1 + |g(c)|) \\
&< \delta \cdot (1 + |g(c)|) \\
&= \frac{\epsilon}{1 + |g(c)|} \cdot (1 + |g(c)|) \\
&= \epsilon.
\end{aligned}$$

Hence, g is continuous at $c \in A$. ∎

5.2.2 Combinations of Differentiable Functions

We can use the Algebraic Limit Theorem for functional limits to prove some basic algebraic combinations of differentiable functions.

Theorem 5.2.2 (Algebraic Differentiability Theorem)**.** *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ *provided that* $g(c) \neq 0$.

(i) $(f+g)'(c) = f'(c) + g'(c)$.

Proof. Assume f and g are functions that are both differentiable at some point $c \in A$. Since $(f+g)(x) = f(x) + g(x)$ and the Algebraic Function Limit Theorem, we have that

$$\begin{aligned}
(f+g)'(x) &= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} \\
&= \lim_{x \to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x-c} \\
&= \lim_{x \to c} \frac{(f(x) - f(c)) + (g(x) - g(c))}{x-c} \\
&= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x-c} + \frac{g(x) - g(c)}{x-c} \right) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x-c} + \lim_{x \to c} \frac{g(x) - g(c)}{x-c} \\
&= f'(c) + g'(c).
\end{aligned}$$

■

(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

Proof. Since f is differentiable at $c \in A$, we have that

$$\begin{aligned}
(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x) - (kf)(c)}{x - c} \\
&= \lim_{x\to c} \frac{kf(x) - kf(c)}{x - c} \\
&= \lim_{x\to c} \frac{k(f(x) - f(c))}{x - c} \\
&= k \cdot \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
&= kf'(c).
\end{aligned}$$

■

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Proof. Let f and g be differentiable at some point $c \in A$. By using the Algebraic Function Limit Theorem, we have that

$$\begin{aligned}
(fg)'(c) &= \lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x - c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} \\
&= \lim_{x\to c} \Big(\frac{f(x)(g(x) - g(c))}{x - c} + \frac{g(c)(f(x) - f(c))}{x - c}\Big) \\
&= \lim_{x\to c} \frac{f(x)(g(x) - g(c))}{x - c} + \lim_{x\to c} \frac{g(c)(f(x) - f(c))}{x - c} \qquad \text{(ALFT)} \\
&= \lim_{x\to c} f(x)\Big(\lim_{x\to c} \frac{g(x) - g(c)}{x - c}\Big) + g(c) \cdot \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
&= f(c)g'(c) + g(c)f'(c).
\end{aligned}$$

Hence, we have that $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$. ■

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$.

Proof. Let f and g be differentiable functions where $g(x) \neq 0$ for all $x \in A$. Note that since f and g are differentiable, they are also continuous on A. Hence,

$\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then observe that

$$\begin{aligned}
(f/g)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c} \\
&= \lim_{x\to c} \Big[\frac{1}{g(x)g(c)} \cdot \frac{g(x)(f(x)-f(c)) - f(x)(g(x)-g(c))}{x-c}\Big] \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(\frac{g(x)(f(x)-f(c)) - f(x)(g(x)-g(c))}{x-c}\Big) \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(g(x)\frac{f(x)-f(c)}{x-c} - f(x)\frac{g(x)-g(c)}{x-c}\Big) \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \Big(\lim_{x\to c} g(x)\frac{f(x)-f(c)}{x-c} - \lim_{x\to c} f(x)\frac{g(x)-g(c)}{x-c}\Big) \\
&= \frac{1}{[g(c)]^2} \cdot (g(c)f'(c) - f(c)g'(c)).
\end{aligned}$$

■

We can also compose two differentiable functions together and still get a differentiable function. This next fact is called the chain rule. A way to prove this fact is to use the following:

$$\begin{aligned}
(g \circ f)'(c) &= \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} \\
&= \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} \\
&= g'(f(c)) \cdot f'(c).
\end{aligned}$$

But an issue with this proof is that the expression $f(x) - f(c)$ might be zero in the denominator for arbitrarily small neighborhoods of c.

Theorem 5.2.3 (Chain Rule)**.** *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof. Assume g is differentiable at $f(c)$. Then we have that

$$g'(f(c)) = \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)}.$$

We can rewrite the quotient in the limit above by setting $d(y)$ to be the following:

$$d(y) = \frac{g(y) - g(f(c))}{y - f(c)}.$$

This is equivalent to $\lim_{y \to f(c)} d(y) = g'(f(c))$. The issue at the moment is when we set $y = f(c)$, $d(y)$ becomes undefined. But this can be mitigated by rewriting $d(y)$ like

$$g(y) - g(f(c)) = d(y)(y - f(c)) \tag{1}$$

which hold for all $y \in B$ including $y = f(c)$. Hence, we are allowed to substitute $y = f(t)$ for any arbitrary $t \in A$ into $d(y)$. If $t \neq c$, we can divide (1) by $(t - c)$ to get

$$\frac{g(f(t) - g(f(c)))}{t - c} = d(f(t)) \frac{f(t) - f(c)}{t - c}.$$

Hence, we have that

$$\begin{aligned}
(g \circ f)'(c) &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\
&= \lim_{x \to c} \Big(\frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \Big) \\
&= g'(f(c)) \cdot f'(c)
\end{aligned}$$

■

5.2.3 Darboux's Theorem

A takeaway from our introduction to Derivatives is that differentiable functions need always be continuous and that our main example for this is for $n = 2$ with the function $g_2(x) = x^2 \sin(1/x)$ where $g_2(0)$. However, differentiable functions do posses the intermediate value property. This leads us to next theorem that says that functions attain their max and mins at points where the derivative is zero.

Theorem 5.2.4 (Interior Extremum Theorem)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$; that is, $f(c) \geq f(x)$ for all $x \in (a, b)$, then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof. Since c is in an open interval (a, b), we can construct two sequences $(x_n), (y_n) \subseteq (a, b)$ such that both sequences converge to $c \in (a, b)$ and satisfy $x_n < c < y_n$ for all $n \in \mathbb{N}$. Since f attains its maximum value at some point $c \in (a, b)$, we have that for all $n \in \mathbb{N}$, $f(y_n) \leq f(c)$. Hence, we can say that

$$f(y_n) - f(c) \leq 0.$$

Dividing by $(y_n - c)$ and using the Order Limit Theorem gives us the following

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0. \tag{1}$$

Now if f attains a minimum, we have $f(x_n) - f(c) \geq 0$ for all $n \in \mathbb{N}$ which implies further that

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

by the Order Limit Theorem. ■

This theorem leads to a very important result about differntiable functions containing the intermediate value property.

Theorem 5.2.5 (Darboux's Theorem). *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof. We can define a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. Since g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \alpha$ and that $g'(a) < 0 < g'(b)$, we want to show that $g'(c) = 0$ for some $c \in (a, b)$. ■

5.2.4 Definitions and Theorems

Definition 5.2.2 (Differentiability). Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative* of g at c is defined by

$$g'(c) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case, we say that g is *differentiable* at c. If g' exists for all points $c \in A$, we say that g is *differentible* on A.

Theorem 5.2.6. *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Theorem 5.2.7 (Algebraic Differentiability Theorem)**.** *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ *provided that* $g(c) \neq 0$.

Theorem 5.2.8 (Chain Rule)**.** *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Theorem 5.2.9 (Interior Extremum Theorem)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$; that is, $f(c) \geq f(x)$ for all $x \in (a, b)$, then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Theorem 5.2.10 (Darboux's Theorem)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

5.2.5 Exercises

Exercise 5.2.1

Supply proofs for parts (i) and (ii) of Theorem 5.2.4.

(i) $(f+g)'(c) = f'(c) + g'(c)$.

Proof. Assume f and g are functions that are both differentiable at some point $c \in A$. Since $(f+g)(x) = f(x) + g(x)$ and the Algebraic Function Limit

Theorem, we have that

$$\begin{aligned}(f+g)'(x) &= \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)+g(x)-(f(c)+g(c))}{x-c} \\ &= \lim_{x\to c} \frac{(f(x)-f(c))+(g(x)-g(c))}{x-c} \\ &= \lim_{x\to c} \Big(\frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}\Big) \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= f'(c)+g'(c).\end{aligned}$$

■

(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

Proof. Since f is differentiable at $c \in A$, we have that

$$\begin{aligned}(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x)-(kf)(c)}{x-c} \\ &= \lim_{x\to c} \frac{kf(x)-kf(c)}{x-c} \\ &= \lim_{x\to c} \frac{k(f(x)-f(c))}{x-c} \\ &= k \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\ &= kf'(c).\end{aligned}$$

■

Exercise 5.2.2

Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

Solution. Take $g(x) = |x|$ and $f(x) = \frac{1}{|x|}$ where $(fg)(x) = 1$ is just a constant function that is differentiable everywhere (including zero). ■

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

Solution. Take $f(x) = \frac{1}{x}$ and $g(x) = x^2$. Their product $(fg)(x) = x$ is differentiable at $x = 0$. ■

(c) A function f not differentiable at zero and a function g differentiable at zero when $f + g$ is differentiable at zero.

Solution. This is impossible since $f = (f + g) - f$ would be differentiable at zero. ■

(d) A function f is differentiable at zero but not differentiable at any other point.

Solution. Let $f(x)$ be defined by the following:

$$f(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

■

Exercise 5.2.3

(a) Use the definition of the derivative to produce the proper formula for the derivative of $h(x) = 1/x$.

Proof. Let $h : A \to \mathbb{R}$ be a function defined on an interval A. Let $c \neq 0$ in A be arbitrary. Using the definition of the derivative, we have that

$$\begin{aligned} h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\ &= \lim_{x \to c} \frac{1/x - 1/c}{x - c} \\ &= \lim_{x \to c} \frac{-(x - c)}{xc} \cdot \frac{1}{x - c} \\ &= \lim_{x \to c} -\frac{1}{xc} \\ &= \frac{-1}{c^2}. \end{aligned}$$

Hence, we have that $h'(c) = -1/c^2$ for any $c \neq 0$ in A. ■

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

Proof. Define $h = (g)^{-1}$. By the result in part (a) and the Chain Rule, we have that $h'(c) = \frac{-1}{[g(c)]^2} \cdot g'(c)$. Then by the Product Rule, we have that

$$\begin{aligned}(f \cdot h)'(c) &= f'(c)h(c) + h'(c)f(c) \\ &= \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{[g(c)]^2} \\ &= \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.\end{aligned}$$

■

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (ii).

Proof. Let f and g be differentiable functions where $g(x) \neq 0$ for all $x \in A$. Note that since f and g are differentiable, they are also continuous on A. Hence, $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then observe that

$$\begin{aligned}(f/g)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\ &= \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x - c} \\ &= \lim_{x\to c} \Big[\frac{1}{g(x)g(c)} \cdot \frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c}\Big] \\ &= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(\frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c}\Big) \\ &= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(g(x)\frac{f(x) - f(c)}{x - c} - f(x)\frac{g(x) - g(c)}{x - c}\Big) \\ &= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \Big(\lim_{x\to c} g(x)\frac{f(x) - f(c)}{x - c} - \lim_{x\to c} f(x)\frac{g(x) - g(c)}{x - c}\Big) \\ &= \frac{1}{[g(c)]^2} \cdot (g(c)f'(c) - f(c)g'(c)).\end{aligned}$$

■

Exercise 5.2.4

Follow these steps to provide a slightly modified proof of the Chain Rule.

(a) Show that a function $h : A \to \mathbb{R}$ is differentiable at $a \in A$ if and only if there exists a function $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = \ell(x)(x - a)$$

for all $x \in A$.

Proof. ($\Rightarrow$) Suppose $h : A \to \mathbb{R}$ is differentiable at $a \in A$. We want to show that there exists $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies the following:

$$h(x) - h(a) = \ell(x)(x - a).$$

Since h is differentiable at $a \in A$, we know that the following

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

exists. But we can define a function $\ell(x)$ inside the limit as

$$\ell(x) = \frac{h(x) - h(a)}{x - a} \iff h(x) - h(a) = \ell(x)(x - a).$$

But we know that h is continuous since h is differentiable so we have that $\ell(x)$ is also continuous.

($\Leftarrow$) Now we want to show the converse. Suppose there exists $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = \ell(x)(x - a)$$

for all $a \in A$. Since ℓ is continuous, we now for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - a| < \delta$, we have that

$$|\ell(x) - \ell(a)| < \epsilon. \tag{1}$$

But since

$$\ell(x) = \frac{h(x) - h(a)}{x - a}$$

we know that the limit

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

exists. Hence, h is differentiable at a. ∎

(b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5.

Proof. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. Let f be a differentiable function at $c \in A$ and let $g : B \to \mathbb{R}$ be a differentiable function at $f(c) \in B$.

Since g is differentiable at $f(c) \in B$, we can construct the function ℓ such that

$$\ell(f(x)) = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}. \tag{1}$$

By part (a), we know that $\ell(f(x))$ is a continuous function and that it satisfies the following property

$$g(f(x)) - g(f(c)) = \ell(f(x))(f(x) - f(c)) \tag{2}$$

for all $f(c) \in B$. Dividing (2) by $(x - c)$, we get that

$$(g \circ f)'(c) = \frac{g(f(x)) - g(f(c))}{x - c} = \ell(f(x))\frac{f(x) - f(c)}{x - c}. \tag{3}$$

Assuming our result in part (a) and the fact that f and g are both differentiable at $c \in A$ and $f(c) \in B$ respectively, taking the limit of (3) leads to the conclusion that $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$. ■

Exercise 5.2.5

Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

Solution. The function f is continuous for all $a > 0$. ■

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

Solution. The function f is differentiable at zero for all $a > 1$ and yes the derivative function is continuous in the same conditions. ■

(c) For which values of a is f twice-differentiable?

Solution. The function f is twice-differentiable whenever $a > 2$. ■

Exercise 5.2.6

Let g be defined on an interval A, and let $c \in A$.

(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

Solution. Since g is differentiable at $c \in A$ and A is an interval, we can always define the difference quotient between two points, in this case $c+h$ and c, and still have the limit as $h \to 0$ and still have $g'(c)$. ■

(b) Assume A is open. If g is differentiable at $c \in A$, show that

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

Proof. Let $c \in A$ and let $\epsilon = h$. Since g is defined on an interval A and A is open, there exists $V_h(c) \subseteq A$. Then let $x \in (c-h, c+h)$. Hence, our difference quotient can be written as follows

$$d(h) = \frac{g(c+h) - g(c-h)}{c+h-(c-h)} = \frac{g(c+h) - g(c-h)}{2h}$$

which taking the limit as $h \to 0$ will give us $g'(c)$ since g is differentiable at c. Hence,

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

■

Exercise 5.2.7

Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0,1]$.

Exercise 5.2.8

Review the definition of uniform continuity. Given a differentiable function $f : A \to \mathbb{R}$, let's say that f is *uniformly differentiable* on A if, given $\epsilon > 0$ there exists $\delta > 0$ such that
$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \epsilon$$
whenever $0 < |x-y| < \delta$.

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? How about $g(x) = x^3$?

Solution. $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$ but $g(x) = x^3$ is not. ■

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

Proof. Let $f : A \to \mathbb{R}$ be uniformly differentiable on an interval A. Let $\epsilon > 0$ and let $c \in A$. We want to show that f' is continuous on A; that is, whenever $|x-c| < \delta$ for some $\delta > 0$, we have that
$$|f'(x) - f'(c)| < \epsilon.$$

By using the fact that f is uniformly differentiable on an interval A, we have
$$\begin{aligned} |f'(x) - f'(c)| &= \left|f'(x) - \frac{f(x)-f(c)}{x-c} + \frac{f(x)-f(c)}{x-c} - f'(c)\right| \\ &\leq \left|f'(x) - \frac{f(x)-f(c)}{x-c}\right| + \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have that f is continuous on A. ■

(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Solution. Not necessarily. It is differentiable on $[0, 1]$, but not uniformly differentiable since the δ response gets progressively smaller as we compute $g_2'(x)$ as $x \to 0$. ■

Exercise 5.2.11

Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$ and a point $y \in (a, b)$ where $g(y) < g(b)$.

Proof. To prove $g(a) > g(x)$ for some $x \in (a, b)$, let us construct the sequence $(x_n) \subseteq (a, b)$ such that $x_n \to x$. We can do this because g being differentiable on $[a, b]$ immediately implies that g is continuous on $[a, b]$. Since $g'(a) < 0$, we know that

$$\lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} < 0. \tag{1}$$

Since g is continuous on $[a, b]$, we have that (1) implies

$$\frac{g(x) - g(a)}{x - a} < 0 \iff g(x) < g(a).$$

Likewise, we can construct the sequence $(y_n) \subseteq (a, b)$ such that $(y_n) \to y$. Since g is continuous we know that

$$\lim_{n\to\infty} \frac{g(y_n) - g(b)}{y_n - b} > 0$$

implies that

$$\frac{g(y) - g(b)}{y - b} > 0 \iff g(y) > g(b).$$

■

(b) Now complete the proof of Darboux's Theorem started earlier.

Proof. Without loss of generality, suppose $g(x) < g(y)$. Since g is a continuous function on (a, b), there exists L such that $g(x) < L < g(y)$ where $g(c) = L$ for some $c \in (a, b)$. Taking the derivative of both sides leads to $g'(x) < 0 < g'(y)$. But by definition, we have that $g'(x) = f'(x) - \alpha$ and thus we have $f'(c) = \alpha$. ■

Exercise 5.2.12 (Inverse functions).

If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8, we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for all $x \in [a,b]$. Show f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

where $y = f(x)$.

Proof. Since $f : [a,b] \to \mathbb{R}$ is a one-to-one function, there exists an inverse function f^{-1} which is defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. Since f is differentiable on $[a,b]$ with $f(x) \neq 0$ for all $x \in [a,b]$, we know that

$$\begin{aligned}
(f^{-1})'(y) &= \lim_{y \to f(c)} \frac{f^{-1}(y) - f^{-1}(f(c))}{y - f(c)} \\
&= \lim_{y \to f(c)} \frac{x - c}{y - f(c)} \\
&= \lim_{x \to c} \frac{1}{\frac{f(x)-f(c)}{x-c}} \qquad (1) \\
&= \frac{1}{f'(c)}.
\end{aligned}$$

The reason why we can exchange the limits on (1) is due to the fact that f^{-1} is continuous on the range of f.

■

5.3 The Mean Value Theorems

5.3.1 Mean Value Theorem

(a) We can find a point along some interval $[a,b]$ of a differentiable function f such that we will have a slope of f where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for at least one point $c \in (a,b)$.

(b) Used to prove L'hopital's rule for limits of quotients of differentiable functions.

(c) Used in the study of infinite series of differentiable functions.

(d) One of mechanisms needed to show Lagrange's Remainder Theorem and used to approximate the error between a Taylor polynomial.

Theorem 5.3.1 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Proof. Since f is continuous on a compact set, we know that f attains a maximum and a minimum. If f attains a maximum and minimum at the endpoints and the fact that $f(a) = f(b)$, we know that f must be a constant function. Hence, we can choose any $x \in [a, b]$ such that $f'(x) = 0$. If f attains a maximum or minimum in the interior of f then there exists $c \in (a, b)$ such that $f'(c) = 0$. ■

Theorem 5.3.2 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Notice that the Mean Value Theorem reduces to Rolle's Theorem in the case where $f(a) = f(b)$. Consider the equation of a line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \Big(\frac{f(b) - f(a)}{b - a}(x - a)\Big) + f(a).$$

Furthermore, we want to consider the difference between this line and the function $f(x)$. Define a new function d where

$$d(x) = f(x) - \Big[\Big(\frac{f(b) - f(a)}{b - a}\Big)(x - a) + f(a)\Big],$$

Observe that d is continuous on $[a, b]$ since f is continuous on $[a, b]$ and differentiable on (a, b) and satisfies $d(a) = 0 = d(b)$. By differentiating $d(x)$, we have that

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Now, using Rolle's Theorem, we can find a $c \in (a, b)$ such that $d'(c) = 0$. Hence,

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \iff f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

Now consider a constant function $f(x) = k$ for any k. Intuition suggests that for all $x \in A$, we have $f'(x) = 0$. Is there any way we can prove that $f(x)$ is constant given $f'(x) = 0$ for all $x \in A$? Indeed, we can using the Mean Value Theorem.

Corollary 5.3.2.1. *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Proof. Take $x, y \in A$ and assume $x < y$. Applying the Mean Value Theorem to g on the interval $[a, b]$, we can see that

$$g'(c) = \frac{g(y) - g(x)}{y - x}.$$

Since $g(x) = 0$ for all $x \in A$, we have that

$$\frac{g(y) - g(x)}{y - x} = 0 \iff g(y) = g(x).$$

Set k equal to this common value. Since $x, y \in A$ are arbitrary, it follows that $g(x) = k$ for all $x \in A$. ■

Corollary 5.3.2.2. *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some interval $k \in \mathbb{R}$.*

Proof. Suppose f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$. Let us define a new function $h(x) = f(x) - g(x)$. Differentiating this function gives us the following $h'(x) = f'(x) - g'(x)$. But since $g'(x) = f'(x)$, we have that $h'(x) = 0$. Since h is differentiable on an interval A, we know that $h(x) = k$. Hence, we have that

$$k = f(x) - g(x) \iff f(x) = g(x) + k.$$

■

We can build a more general form of the Mean Value Theorem which can be used to prove L'hopital's rules and the Lagrange Remainder Theorem.

Theorem 5.3.3 (Generalized Mean Value Theorem)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. This result follows by applying the Mean Value Theorem to the function

$$h(x) = [f(b) - f(a)]g(x) = [g(b) - g(a)]f(x).$$

The details are requested in Exercise 5.3.5. ■

5.3.2 L'Hopital's Rules

The Algebraic Limit Theorem asserts that when taking a limit of a quotient of functions we can write

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

provided that the quotient is well-defined. What happens when the denominator has a limit that goes to zero while the numerator goes to a limit that is non-zero? Both zero? Both infinite?

Theorem 5.3.4 (L'Hopital's Rule: 0/0 case)**.** *Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Of course they would leave this to the reader to finish. It is requested in Exercise 5.3.11. ■

Next is the case when we replace the assumption of the last theorem with the case that $\lim_{x\to a} g(x) = \infty$. We can define what it means to have an infinite limit.

Definition 5.3.1. Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x\to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$. We can define $\lim_{x\to c} g(x)$ in a similar way.

Next is the case of L'hopital's rule when applied to the case when both the numerator and the denominator go to infinity.

Theorem 5.3.5 (L'Hopital's Rule: ∞/∞ case)**.** *Assume f and g are differentiable on (a,b) and that $g'(x) \neq 0$ for all $x \in (a,b)$. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\epsilon > 0$. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there exists $\delta_1 > 0$ such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2} \tag{1}$$

for all $a < x < a + \delta_1$. For convenience of notation, let $t = a + \delta_1$ and note that t is fixed for the remainder of the argument. Let our functions f and g be defined on the interval $[x,t]$ for any $x \in (a,t)$. We can use the Generalized Mean Value Theorem on the interval $[x,t]$ to get that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(t)}{g(x) - g(t)}$$

for some $c \in (x,t)$. Since we are considering $t = a + \delta_1$, we have that

$$L - \frac{\epsilon}{2} < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \frac{\epsilon}{2} \tag{2}$$

for all $x \in (a,t)$. Our goal is to isolate the fraction $f(x)/g(x)$ by multiplying (2) by $(g(x) - g(t))/g(x)$. We need to assume that $g(x) \geq g(t)$ so that the quantity we are multiplying by is positive (or else we will switch the order of the inequality which we don't want). Carrying our our plan results in the following inequality

$$L - \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} + \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}.$$

Since t is fixed and that $\lim_{x \to a} g(x) = \infty$, we can choose $\delta_2 > 0$ such that this our choice of multiplying by the above quantity will satisfy $g(x) \geq g(t)$ for all $a < x < a + \delta_2$. By the same fact, we can also choose δ_3 such that $a < c < a + \delta_3$ implies that $g(x)$ is large enough to ensure that both

$$\frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} \quad \text{and} \quad \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}$$

are less than $\epsilon/2$ in absolute value. Choosing $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ guarantees that

$$\left| \frac{f(x)}{g(x)} - L \right| < \epsilon$$

for all $a < x < a + \delta$. ■

5.3.3 Definitions and Theorems

Theorem 5.3.6 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Theorem 5.3.7 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 5.3.7.1. *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Corollary 5.3.7.2. *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some interval $k \in \mathbb{R}$.*

Corollary 5.3.7.3. *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Definition 5.3.2. Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x\to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$. We can define $\lim_{x\to c} g(x)$ in a similar way.

Theorem 5.3.8 (Generalized Mean Value Theorem)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 5.3.9 (L'Hopital's Rule: 0/0 case)**.** *Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x\to a} \tfrac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x\to a} \tfrac{f(x)}{g(x)} = L.$$

Theorem 5.3.10 (L'Hopital's Rule: ∞/∞ case)**.** *Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

5.3.4 Exercises

Exercise 5.3.1

Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\Big|\frac{f(x) - f(y)}{x - y} \leq M\Big|$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Proof. Let f be differentiable on a closed interval $[a, b]$ and f' continuous on $[a, b]$. Let us define our closed interval as A. Let $x \neq y \in [a, b]$. By the Mean Value Theorem, we can find a $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Since f' is continuous on a compact set $[a, b]$, the range of f' is also bounded. Hence, there exist $M > 0$ such that $|f'(c)| \leq M$. Hence, we have that

$$\Big|\frac{f(x) - f(y)}{x - y}\Big| \leq M.$$

■

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution. Yes. Suppose $|f'(x)| < 1$ on $[a, b]$ and let $x \neq y$ in $[a, b]$ as before. By using the fact that $f'(x)$ is continuous on $[a, b]$, f' differentiable on $[a, b]$ and the Mean Value Theorem, we get that for some $c \in (x, y)$, we have that

$$|f'(c)| = \Big|\frac{f(x) - f(y)}{x - y}\Big| < |f'(x)| + 1 \iff \Big|\frac{f(x) - f(y)}{x - y}\Big| < 2$$

which implies that

$$|f(x) - f(y)| < 2|x - y|.$$

Hence, f is a contractive function. ■

Exercise 5.3.2

Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is injective on A. Provide an example that the converse statement need not be true.

Proof. Let f be differentiable on an interval A. Let $x \neq y \in A$. By the Mean Value Theorem, we can find a $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}. \tag{1}$$

Since $f'(c) \neq 1$ on A and $x \neq y$, (1) implies that $f(x) \neq f(y)$. Hence, f is an injective function. ∎

Exercise 5.3.3

Let h be differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

Proof. Since h is a differentiable function that is defined on $[0, 3]$, we can find a $d \in (1, 3)$ such that

$$h'(d) = \frac{h(3) - h(1)}{3 - 1} = \frac{2 - 2}{3 - 1} = 0.$$

Since $h'(d) = 0$, we know that $h(d)$ must be a constant. Hence, $h(d) = d$ for some $d \in [0, 3]$. ∎

(b) Argue that at some point c we have $h'(c) = 1/3$.

Proof. Since h is a differentiable function defined on $[0, 3]$, we can use the Mean Value Theorem to state that there exists a $c \in (0, 3)$ such that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{1}{3}.$$

∎

(c) Argue that $h'(x) = 1/4$ at some point in the domain.

Proof. ∎

Exercise 5.3.4

Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(a) If $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(0) = 0$ and $f'(0) = 0$.

Proof. Suppose $f(x_n) = 0$ for all $n \in \mathbb{N}$. Since f is differentiable on the interval A, we know that f is also continuous on A. Since $(x_n) \to 0$ for $x_n \neq 0$, we know that $f(x_n) = 0$ implies $f(0) = 0$. Since f is differentiable, $f(0) = 0$, and $x_n \neq 0$, we have that

$$\begin{aligned} f'(0) &= \lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n} \\ &= 0. \end{aligned}$$

■

(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Proof. Suppose f is twice-differentiable at zero. This means $f'(x_n) = 0$ as well. Since $x_n \neq 0$ and $f'(0) = 0$, we have

$$f''(0) = \lim_{n\to\infty} \frac{f'(x_n) - f'(0)}{x_n} = 0.$$

■

Exercise 5.3.5

(a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem.

Proof. Suppose f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). By the Mean Value Theorem, we can find a $c \in (a, b)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a}, \\ g'(c) &= \frac{g(b) - g(a)}{b - a}. \end{aligned}$$

Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Note that h is also continuous on $[a, b]$ and differentiable on the open interval (a, b) since f and g are both continuous on $[a,b]$ and differentiable on (a,b). Using the Mean Value Theorem again, we can find a $c \in (a,b)$ such that

$$h'(c) = \frac{h(b)-h(a)}{b-a}. \tag{1}$$

Using algebraic manipulations, we end up having $h(b)-h(a) = 0$ implying that $h'(c) = 0$. Expanding the right side of (1), we have that

$$\begin{aligned}\frac{h(b)-h(a)}{b-a} &= (f(b)-f(a))\cdot\frac{g(b)-g(a)}{b-a} - (g(b)-g(a))\cdot\frac{f(b)-f(a)}{b-a}\\ &= (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c).\end{aligned}$$

Since $h'(c) = 0$, we arrive at

$$[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c).$$

If $g' \neq 0$ for all $x \in (a,b)$, then our conclusion can be stated as follows:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

■

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve).

Solution. Graphically, we can find a tangent through $(f(a), g(a))$ and $(f(b), g(b))$ such that it, meaning $g's$ slope is parallel to the tangent of f. ■

Exercise 5.3.6

(a) Let $g : [0,a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0,a]$. Show $|g(x)| \leq Mx$ for all $x \in [0,a]$.

Proof. Suppose g is a differentiable function defined on $[0,a]$. Since g is continuous on $[0,a]$, differentiable on (a,b) and $g(0) = 0$, we use the Mean Value Theorem to state that there exists $c \in (a,b)$ such that

$$g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}.$$

Since $|g'(x)| \leq M$, we know that

$$|g'(c)| = \left|\frac{g(x)}{x}\right| \leq M \iff |g(x)| \leq Mx.$$

■

(b) Let $h : [0, a] \to \mathbb{R}$ be twice-differentiable, $h'(0) = h(0) = 0$ and $|h"(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

Proof. Let h be twice-differentiable with $h'(0) = h(0) = 0$ as well as $|h"(x)| \leq M$ for all $x \in [0, a]$. Since h is twice-differentiable, we can find a $c \in (a, b)$ such that

$$h"(c) = \frac{h'(x) - f(0)}{x - 0} = \frac{h'(x)}{x}.$$

Since we are also given $|h'(x)| \leq M$, we know that $|h(x)| \leq Mx$. Since $|h"(x)| \leq M$ for all $x \in [0, a]$, we have that

$$\begin{aligned} |h"(c)| \leq M &\iff \left|\frac{h'(x)}{x}\right| \leq \frac{M}{2} \\ &\iff |h'(x)| \leq \frac{M}{2}x \\ &\iff |h(x)| \leq \frac{M}{2}x^2. \end{aligned}$$

■

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Proof. Let h be differentiable three times. Suppose $h"(0) = h'(0) = h(0) = 0$. It is a similar process as above. ■

Exercise 5.3.7

A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof. Suppose for sake of contradiction that f contains more than one fixed point. Let $a, b \in A$ be fixed points of f. Then we have $f(a) = a$ and $f(b) = b$. Suppose $f'(x) \neq 1$ for any $x \in A$. Since f is differentiable on A, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.$$

But this tell us that $f'(c) = 1$ which contradicts our assumption that $f'(x) \neq 1$ on A. Hence, a and b must be the same fixed point. ■

Exercise 5.3.8

Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Proof. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. We want to show that $f'(0)$ exists and is equal to L. Define $h(x) = f(x) - f(0)$ and $g(x) = x$. If we take the following limit

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{h(x)}{g(x)}$$

we notice that $f'(0) = 0/0$ which prompts us to use L'Hopital's rule for limits. Hence, if we take differentiate $h(x)$ and $g(x)$ and then take the limit of their quotient $h'(x)/g'(x)$, we get that $h'(x) = f'(x)$ and $g'(x) = 1$. But note that $h'(x)/g'(x) = f'(x)$. Since $\lim_{x\to 0} f'(x) = L$, we know that $\lim_{x\to 0} \frac{h'(x)}{g'(x)}$ must also equal to L. But this means that by L'hopital's rule that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = L.$$

■

Exercise 5.3.9

Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the $0/0$ case of L'Hopital's Rule under this stronger Hypothesis.

Proof. Since f' and g' are continuous at a, we know that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)} = L.$$

But we know f and g are also differentiable at a which means f and g are also continuous at a. Since $x \neq a$ for all $x \in A$ where A is an interval, we know that

$$\begin{aligned} L &= \frac{f'(a)}{g'(a)} \\ &= \frac{\lim_{x\to a} f(x) - f(a)}{\lim_{x\to a} g(x) - g(a)} \\ &= \lim_{x\to a} \frac{f(x)}{g(x)} && (f(a) = g(a) = 0) \end{aligned}$$

Hence, we have that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$. ■

Exercise 5.3.11

(a) Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hopital's Rule (Theorem 5.3.6).

Proof. Assume f and g continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. Suppose $f(a) = g(a) = 0$, $g'(x) \neq 0$ for all $x \neq a$, and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < |x - a| < \delta$. Let $x \in (a, a+\delta)$. Since f is differentiable on A, there exists $c \in (x, a)$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $\lim_{x\to a} f'(x)/g'(x) = L$, we have that

$$\left|\frac{f(x) - f(a)}{g(x) - g(a)} - L\right| < \epsilon.$$

Since $g(a) = f(a) = 0$, we have that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon.$$

Hence, we have that $\lim_{x\to a} f(x)/g(x) = L$. ■

Exercise 5.3.12

If f is twice differentiable on an open interval containing a and f" is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f\text{"}(a).$$

Proof. Since f is twice-differentiable on an open interval A where $a \in A$, we can write

$$f'(a+h) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

and likewise

$$f'(a-h) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$$

by Exercise 5.2.6. Hence, we can write

$$\begin{aligned} f\text{"}(a) &= \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \left[\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - \frac{f(a) - f(a-h)}{h} \right] \\ &= \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a+h)}{h^2}. \end{aligned}$$

■