0.1 Lecture 4

0.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

0.1.2 Class Exercises

(i) Show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z.

(ii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ converges for all z.

(iii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all z.

Definition (Exponential, Cosine, and Sine). We define

$$\begin{aligned}
\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Our main goal for this lecture is to show that $\exp(z+w) = \exp(z)\exp(w)$.

0.1.3 Cauchy Multiplication Theorem

Theorem (Cauchy Multiplication Theorem). Assume that $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converges absolutely. Then

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} z_k w_{n-k} \Big) = \Big(\sum_{n=0}^{\infty} z_n \Big) \Big(\sum_{n=0}^{\infty} w_n \Big) \tag{1}$$

where the series on the left-hand side of (1) converges absolutely.

Proof. We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is bounded. Observe that

$$\begin{aligned}
\sum_{n=0}^{N} \Big| \sum_{i+j=n} z_i w_j \Big| &\leq \sum_{n=0}^{N} \sum_{i+j=n} |z_i w_j| \\
&= \sum_{0 \leq i+j \leq N} |z_i w_j| \\
&\leq \sum_{0 \leq i,j \leq N} |z_i w_j| \\
&= \sum_{i=0}^{N} |z_i| \sum_{j=0}^{N} |w_j|.
\end{aligned}$$

By our assumption, we can see that the sequence of partial sums of $\sum_{n=0}^{\infty}$ and $\sum_{n=0}^{\infty} w_n$ are bounded, and thus the left side of the equation above is bounded. Thus, we see that

$$\sum_{n=0}^{N}\Big|\sum_{i+j=n} z_i w_j\Big|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \Big|\sum_{n=0}^{2N}\sum_{i+j=n} z_i w_j - \sum_{i=0}^{N} z_i \sum_{j=0} w_j\Big| \to 0$$

as $N \to \infty$. Let us define the following sets

$$\begin{aligned} T_N &= \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : i \geq 0, j \geq 0, 0 \leq i+j \leq N\} \\ t_n &= \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : 0 \leq i \leq N, 0 \leq j \leq N\}. \end{aligned}$$

Observe that $T_N \subseteq t_n$ and $t_N \subseteq T_{2N} \subseteq t_{2N}$. Thus,

$$\begin{aligned} \alpha_N &= \Big|\sum_{(i,j)\in T_{2N}\setminus t_N} z_i w_j\Big| \leq \sum_{(i,j)\in T_{2N}\setminus t_N} |z_i w_j| \\ &\leq \sum_{(i,j)\in t_{2N}\setminus t_N} |z_i w_j| \\ &= \sum_{i=0}^{2N}\sum_{j=0}^{2N} |z_i w_j| - \sum_{i=0}^{N}\sum_{j=0}^{N} |z_i w_j| \\ &= \sum_{i=0}^{2N} |z_i| \sum_{j=0}^{2N} |w_j| - \sum_{i=0}^{N} |z_i| \sum_{j=0}^{N} |w_j| \to 0 \text{ as } N \to \infty. \end{aligned}$$

■

Corollary. For any $z, w \in \mathbb{C}$, we have $\exp(z)\cdot\exp(w) = \exp(z+w)$.

Proof. Let

$$C_n = \sum_{k=0}^{n} \frac{z^k}{k!}\cdot\frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by $n!$ on both sides of the equation above, we see that

$$n!C_n = \sum_{k=0}^{n}\binom{n}{k} z^k w^{n-k} = (z+w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^{N} C_n = \sum_{n=0}^{N}\frac{n!C_n}{n!} = \sum_{n=0}^{N}\frac{(z+w)^n}{n!}.$$

Let $n \to \infty$. Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} = \exp(z+w).$$

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0.2 Lecture 5

0.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

0.2.2 Sine and Cosine Functions

Recall that we defined

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C}$$
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}.$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2.$

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z + i \sin z. \end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x + iy) = e^x(\cos y + i \sin y).$
- $\Re(\exp(z)) = e^x \cos y.$
- $\Im(\exp(z)) = e^x \sin y, \ |\exp(z)| = e^x.$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w.$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w.$

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and

$w = u + iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x + iy$. Let $z = \gamma(\cos\varphi + i \sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ■

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\bullet}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\bullet} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Problem 1. Let $z \in \mathbb{C}^{\bullet}$. Find all possible solutions of $\exp(w) = z$.

Solution. Taking the log of both sides of $\exp(w) = z$, we see that

$$w = \log(z) = \ln|z| + i\Big(\mathrm{Arg}(z) + 2\pi k\Big), \ k \in \mathbb{Z}.$$

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Lemma. For $z \in \mathbb{C}^{\bullet}$, we have

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

0.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \mathrm{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \mathrm{Log}\, z + 2\pi ik$$

is a **multivalued function** and $\log z = \mathrm{Log}\, z$ if we want the solution to be in S.