Math-234-Final-Exam

Exam Rules: Please do not use any materials except the reference books and handouts, homework solutions (using outside resources is prohibited). In addition, please do not discuss the exam with your classmates. However, you can email me for any clarification. I will announce a few in-person or Zoom student hours on Canvas. Please submit your exam through Canvas by Friday 12/13/2024.

Name:

For all of these problems, i denotes the imaginary unit, $\overline{z}$ denotes the complex conjugate of the complex number z, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Question 1): (5 points) Decide whether the following statements are true or false. You **do not need** to justify your answer.

a) Let $f : \mathbb{C} \to \mathbb{R} \subset \mathbb{C}$ be defined by $f(z) = \cos(\bar{z})$ Then, f is complex differentiable at $z = 0$.

b) Let $A = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$. Then, we can find a point $z \in A$ such that it is not an accumulation point of A.

c) Consider the power series $\sum_{n=1}^{\infty} \dfrac{n^n (z-2i)^n}{n}$. Then, the radius of convergence of the power series is 1.

d) Let $D \subseteq \mathbb{C}$ open, and $f : D \to \mathbb{C}$ be holomorphic. If $f'(z) = 0$ for all $z \in D$, then f is a constant function.

e) Let $D \subseteq \mathbb{C}$ open, and $f : D \to \mathbb{C}$ be holomorphic. Then, all derivatives of f are also holomorphic functions.

Question 2) (15 points) Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{5}{(n+2+i)(n+3+i)}$ converges and compute its sum explicitly.

Question 3) (Differential Equation Characterization of the exponential function) (10 points) Fix $c, w \in \mathbb{C}$ and consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = we^{cz}$. Then, f is holomorphic and it can be shown that f satisfies $f'(z) = cf(z)$ for all $z \in \mathbb{C}$ and $f(0) = w$. Assume that we have a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ that also satisfies $g'(z) = cg(z)$ for all $z \in \mathbb{C}$ and $g(0) = w$. Prove that $g(z) = f(z)$ for all $z \in \mathbb{C}$. **Hint**: Think about using the function $h(z) = e^{-cz}g(z)$.

Question 4) (nth root of nowhere vanishing holomorphic function) (10 points) An non-empty open and connected set $D \subseteq C$ is called an *elementary domain* if every holomorphic function on D has a primitive.

Let D be an elementary domain and $f : D \to \mathbb{C}$ be a nowhere vanishing holomorphic function. Here, nowhere vanishing means $f(z) \neq 0$ for all $z \in D$.

(a) Show that there is a holomorphic function $g : D \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in D$. **Hint**: Think about using the function e^F/f where F is a primitive of f'/f (justify why F exists).

(b) Let n be a positive integer. Use (a) to show that there is a holomorphic function $G : D \to \mathbb{C}$ such that $(G(z))^n = f(z)$ for all $z \in D$.

Question 5) (Computation of some real integrals using complex analysis -I) (10 points)

(a) Define $\alpha, \beta : [0,1] \to \mathbb{C}$ by $\alpha(t) = 3e^{2\pi i t}$ and $\beta(t) = 3\cos(2\pi t) + 4i\sin(2\pi t)$. Note that the trace of α is the circle $\{z \in \mathbb{C} : \; |z| = 3\}$ where as the trace of β is the ellipse whose equation is given by $x^2/9 + y^2/16 = 1$.

(i) Show that

$$\int_{\alpha} \frac{1}{z}\, dz = \int_{\beta} \frac{1}{z}\, dz$$

Hint: Thinking about using the idea introduced in handout-18, pages 6-7.

(ii) Use (i) to show

$$\int_0^{2\pi} \frac{1}{9\cos^2 t + 16\sin^2 t}\, dt = \frac{\pi}{6}$$

(b) Let $f, g : B(0,R) \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \dfrac{1}{z} + \dfrac{2}{R-z}$ $g(z) = \dfrac{2}{R-z}$, where $R > 0$ and $B(0,R)$ is the open ball centered at $0 \in \mathbb{C}$ and radius R. Let $0 < r < R$.

(i) Compute $\displaystyle\oint_{\partial B(0,R)} f(z)\,dz$ and $\displaystyle\oint_{\partial B(0,R)} g(z)\,dz$

(ii) Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\,\mathrm{d}t = 1 \text{ and } \frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\,\mathrm{d}t = \frac{r}{R^2 - r^2}$$

Question 6) (Computation of some real integrals using complex analysis -II) (15 points) Consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^{iz^2}$.

(i) Let $R > 0$, and define $\alpha_R : [0, \pi/4] \to \mathbb{C}$ by $\alpha_R(t) = Re^{it}$. Show that

$$\left|\int_{\alpha_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi(1 - e^{-R^2})}{4R}$$

Hint: Show that $|f(\alpha(t))| = e^{-R^2 \sin 2t}$, then use the fact that $\sin 2t \geq \frac{4}{\pi} t$ for $0 \leq 0 \leq \pi/4$ to show $|f(\alpha(t))| \leq e^{-4R^2 t/\pi}$. Then, use the fact $\left| \int_{\alpha_R} f(z)\, \mathrm{d}z \right| \leq \int_0^{\pi/4} |f(\alpha(t))||\alpha'(t)|\, \mathrm{d}t$ to get the desired result. You do not have to establish these facts even though they do not require substantial work.

(ii) Use (i) to show $\lim_{R\to\infty} \int_{\alpha_R} f(z)\, \mathrm{d}z = 0$.

(iii) Consider the line segment L_R joining 0 to R on the real axis and the line segment C_R joining 0 to $Re^{i\pi/4}$. Show that

$$\int_{C_R} f(z)\, \mathrm{d}z = \int_{L_R} f(z)\, \mathrm{d}z + \int_{\alpha_R} f(z)\, \mathrm{d}z \quad \text{and} \quad \lim_{R\to\infty} \int_{C_R} f(z)\, \mathrm{d}z = \lim_{R\to\infty} \int_{L_R} f(z)\, \mathrm{d}z$$

Hint: Think about using Cauchy's Integral Theorem for Star domains.

(iv) Show that

$$\lim_{R\to\infty} \int_{L_R} f(z)\, \mathrm{d}z = \int_0^\infty \cos(t^2)\, dt + i \int_0^\infty \sin(t^2)\, dt$$

(v) Show that $\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = \dfrac{(1+i)\sqrt{2\pi}}{4}$ and use it to show $\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \dfrac{\sqrt{2\pi}}{4}$.

Hint: You may use the fact (without proof) $\displaystyle\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$.

Question 7) (Behavior of a non-constant holomophic function on $\mathbb{C}$) (10 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function.

(i) Assume that the complement of $\overline{f(\mathbb{C})}$ is non-empty, where $\overline{f(\mathbb{C})}$ is the closure of $f(\mathbb{C})$. Let $w \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$, and define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \dfrac{1}{f(z) - w}$. Explain why g is holomorphic. Show that g is bounded, i.e., there is $M > 0$ such that $|g(z)| \leq M$ for all $z \in \mathbb{C}$. Using Liouville's theorem, deduce that g is constant and from this deduce that f is constant.

(ii) Show that if f is non-constant, then $\overline{f(\mathbb{C})} = \mathbb{C}$. (This says that non-constant holomorphic functions with domain $\mathbb{C}$ are almost onto.)

Question 8) (15 points) Compute the following limits.

(i) $\displaystyle\lim_{z\to\infty} \frac{z^3 + iz + 1}{z^3 - z}$

(ii) $\displaystyle\lim_{z\to\infty} \frac{e^{-i|z|}}{z^2}$

(iii) $\displaystyle\lim_{z\to\infty} z\sin(1/z)$

Question 9) (10 points) Examine whether the following functions $f : D \to \mathbb{C}$ are holomorohic. If you are using certain statements, make sure to check all the conditions.

(i) $D = \mathbb{C} \setminus \{0\}$ and $f(z) = e^{1/\bar{z}}$.

(ii) $D = \{z \in \mathbb{C} : |z - 5i| < 1\}$ and $f(z) = \dfrac{\mathrm{Log} z}{z^2}$.

(iii) $D = \mathbb{C}$, $f(x + iy) = (x^3 - 3xy^2 + 1) + i(3x^2 y - y^3 - 1)$.

(iv) $D = \{z \in \mathbb{C} : |z| \neq 1\}$ and $f(z) = \dfrac{1}{1 - |z|^2}$.