Math 230A Homework Solutions

Lance Remigio

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Contents

0.1 Week 1

1. Mark each statement True or False.

   1-1) If x and y are elements of an ordered field, then either $x \leq y$ or $y < x$. **True.**

   1-2) Every ordered field has the least upper bound property. **False.**

   1-3) If $E \subseteq \mathbb{R}$ is bounded above and $\alpha = \sup E$, then $\alpha \in E$. **False.**

   1-4) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, then $\alpha = \sup E$. **False.**

   1-5) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \in E$, then $\alpha = \sup E$. **True.**

   1-6) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \notin E$, then $\alpha \neq \sup(E)$. **False.**

2. Prove the following: Suppose α is an upper bound for $E \subset \mathbb{R}$. Then $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   Proof. Suppose α is an upper bound for $E \subseteq \mathbb{R}$. For the forwards direction, suppose $\alpha = \sup E$ and let $\varepsilon > 0$. Note that $\alpha - \varepsilon < \alpha$ implies that $\alpha - \varepsilon$ is NOT an upper bound of E. By definition, there must exist an element $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   For the backwards direction, let α be an upper bound of E. We need to show $\alpha = \sup E$; that is, we need to show that α is an upper bound of E and that if $\gamma < \alpha$, then γ is NOT an upper bound of E. Notice that the first property is satisfied by our assumption. Thus, all that is left to show is the second property.

   Suppose $\gamma < \alpha$. Then $\alpha - \gamma > 0$. Choose $\varepsilon = \alpha - \gamma$. Our goal is to find some element $x \in E$ such that $x > \gamma$. By assumption, there exists an element $x_0 \in E$ such that

   $$x_0 > \alpha - \varepsilon = \alpha - (\alpha - \gamma) = \gamma.$$

   Thus, $x_0 > \gamma$ for some $x_0 \in E$. Therefore, γ is NOT an upper bound of E and so we conclude that

   $$\alpha = \sup E.$$

   ■

3. Let $E = \{n/n + 1 : n \in \mathbb{N}\}$. Prove that $\sup E = 1$.

Proof. To show that 1 is the supremum of E, we need to show that 1 is an upper bound for E and that it satisfies the lemma found in Problem 2. First, we will show that 1 is an upper bound of E. Observe that for any $n \in \mathbb{N}$, we have

$$\frac{n}{n+1} < \frac{n}{n} = 1.$$

Thus, E is bounded above by 1.

Let $\varepsilon > 0$. By the Archimedean Property, there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} < \varepsilon.$$

Then observe that

$$\frac{1}{n+1} = \frac{(n+1)-n}{n+1} = 1 - \frac{n}{n+1}.$$

Thus,

$$1 - \frac{n}{n+1} < \varepsilon \iff \frac{n}{n+1} > 1 - \varepsilon.$$

But note that $\frac{n}{n+1} \in E$. Thus, we conclude that $\sup E = 1$ by the lemma found in Problem 2. ■

4. State the analogue of the Very Useful Theorem for infimums.

Solution. Suppose β is a lower bound for $E \subset \mathbb{R}$. Then $\beta = \inf E$ if and only if for all $\varepsilon > 0$, there exists $y_0 \in E$ such that $y_0 < \beta + \varepsilon$. ■

5. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Prove $a \leq b$.

Proof. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Suppose for sake of contradiction that $a > b$. Note that $a - b > 0$. Pick $\varepsilon = \frac{a-b}{2}$. Then

$$\begin{aligned} b + \varepsilon = b + \frac{a-b}{2} = \frac{2b + a - b}{2} &= \frac{a+b}{2} \\ &< \frac{a+a}{2} \\ &= \frac{2a}{2} \\ &= a. \end{aligned}$$

But this means that $a > b + \varepsilon$ which is a contradiction of our assumption that $a \leq b + \varepsilon$. Thus, it must be the case that $a \leq b$. ■

6. Complete the following proof.

Theorem (Greatest-lower-bound property of $\mathbb{R}$)**.** Every nonempty subset of A of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf A$ exists and is a real number.

Proof. Continuation of the proof presented in homework. Set $\beta = -\alpha$ where $\alpha = \sup(-A)$. Thus, there exists $a \in A$ such that $-\gamma < -a$. Multiplying by a negative on both sides of this inequality, we get $\gamma > a$ for some $a \in A$. If γ was a lower bound of A, then $\gamma \leq a$ for all $a \in A$. However, we have the negation of this, so we must have that γ is NOT a lower bound of A, which satisfies part (ii). Since $\beta \leq \alpha$ for all $a \in A$ and $\beta < \gamma$ implies γ is not a lower bound for A, we conclude that β must be the infimum of A. ■

7. (i) Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many elements of A.

Proof. Let $\varepsilon > 0$. Our goal is to show that $(\sup A - \varepsilon, \sup A)$ has infinitely many points. Note that as a consequence of the result found in Problem 2, there exists $x_0 \in A$ such that

$$\sup A - \varepsilon < x_0 \leq \sup A.$$

By assumption, we have that $\sup A \notin A$. Thus, we have

$$\sup A - \varepsilon < x_0 < \sup A$$

for some $x_0 \in A$. So, $(\sup A - \varepsilon, \sup A) \cap A$ is nonempty.

Now, we will show, that the set $(\sup A - \varepsilon, \sup A) \cap A$ contains infinitely many elements. Thus, suppose for sake of contradiction that the nonempty set $(\sup A - \varepsilon, \sup A) \cap A$ contains FINITELY many elements. Let us denote these elements by a_i for $1 \leq i \leq n$; that is,

$$(\sup A - \varepsilon, \sup A) \cap A = \{a_1, \dots, a_n : 1 \leq i \leq n\}.$$

For all $1 \leq k \leq n$, let the distance between each a_k and $\sup A$ be defined by

$$d_k = |\sup A - a_k|.$$

Since $a_k \in A$ and $\sup A \notin A$, we must have $|\sup A - a_k| > 0$. Let

$$d = \frac{\min\{d_1, \dots, d_n\}}{2}.$$

But this implies that none of the $a_1, \dots, a_n$ lies inside the interval $(\sup A - d, \sup A)$. That is,

$$A \cap (\sup A - d, \sup A) = \emptyset,$$

implies that $\sup A - d \in \mathrm{UP}(A)$ which is a contradiction because $\sup A - d$ is not an upper bound of A. Thus, $(\sup A - \varepsilon, \sup A) \cap A$ must contain infinitely many elements of A. ■

(ii) Let $B \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below. Show that if $\inf B \notin B$, then for all $\varepsilon > 0$ the open interval $(\inf B, \inf B + \varepsilon)$ contains infinitely many elements of B.

Proof. Let $\varepsilon > 0$. Note that as a consequence of Problem 3, there exists an $\beta \in B$ such that

$$\inf B \leq \beta < \inf B + \varepsilon.$$

Since $\inf B \notin B$, we must have

$$\inf B < \beta < \inf B + \varepsilon.$$

Thus, the set $(\inf B, \inf B + \varepsilon) \cap B$ is nonempty.

Now, we will show that $(\inf B, \inf B + \varepsilon) \cap B$ contains infinitely many points. Thus, suppose for sake of contradiction that the nonempty set $(\inf B, \inf B + \varepsilon) \cap B$ contains FINITELY many points. Denote these points by b_i for $1 \leq i \leq n$; that is

$$(\inf B, \inf B + \varepsilon) \cap B = \{b_i : 1 \leq i \leq n\}$$

For all $1 \le k \le n$, let the distance between each b_k and $\inf B$ be defined by

$$d_k = |\inf B - b_k|.$$

Since $b_k \in B$ and $\inf B \notin B$, we must have $|\inf B - b_k| > 0$. Let

$$d = \frac{\min\{d_1, d_2, \ldots, d_n\}}{2}.$$

Note that observe that none of the b_i for $1 \le i \le n$ lies within the open interval $(\inf B, \inf B + d)$. That is, the set

$$(\inf B, \inf B + d) \cap B = \emptyset$$

implies that $\inf B + d \in \mathrm{LO}(B)$ which is a contradiction because $\inf B + d$ is not a lower bound of B. Thus, the open interval $(\inf B, \inf B + \varepsilon)$ must have infinitely many points. ∎

8. Suppose the construction of $\mathbb{R}$ and the proof of the least-upper-bound property for $\mathbb{R}$ can be completed without directly using the well-ordering principle for $\mathbb{N}$. (Of course, statements equivalent to the well-ordering principle might have been used, but we are assuming that the nontrivial fact of their equivalence to the well-ordering principle was not utilized in the construction of $\mathbb{R}$ or in proving its Dedekind completeness.) Use the greatest-lower-bound property of $\mathbb{R}$ and the result of the previous exercises to prove the well-ordering principle for $\mathbb{N}$; that is, prove that every nonempty subset of $\mathbb{N}$ has a minimum.

Proof. Let $E \subseteq \mathbb{N}$ be a nonempty set. Note that E is bounded below by 0, and so $\mathrm{LO}(E) \neq \emptyset$. Since E is nonempty and bounded below, we know that $\inf E$ exists by the greatest-lower-bound property of $\mathbb{R}$.

To show that E contains a minimum, we suppose for contradiction that $\inf E \notin E$. Now, pick $\varepsilon = \frac{1}{5}$. By Part 2 of Problem 7, the open interval $(\inf E, \inf E + \frac{1}{5})$ intersects E at infinitely many points (these points being natural numbers). Suppose we choose an $m \in \mathbb{N}$ such that $m \in (\inf E, \inf E + \frac{1}{5})$. Note that m also lies interval $(m-1, m+1)$. Thus, $(\inf E, \inf E + \varepsilon) \subseteq (m-1, m+1)$. However, we see that $(\inf E, \inf E + \frac{1}{5})$ contains infinitely many natural numbers which cannot happen within the open interval $(m-1, m+1)$ since this interval contains at most only one element, namely m, which is a contradiction. Hence, it must be the case that $\inf E \in E$; that is, E contains a minimum. ∎

9. Let $A, B \subseteq \mathbb{R}$ be nonempty, bounded sets and let $c \in \mathbb{R}$. Define the following sets:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{ab : a \in A, b \in B\} \\ cA &= \{ca : a \in A\}. \end{aligned}$$

Prove that

9-1) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Our goal is to show that $A+B$ contains an infimum and that $\inf(A+B) = \inf A + \inf B$. Note that $A + B \neq \emptyset$ since $a + b \in A + B$ and A and B are nonempty sets. Since A and B are both bounded below, there exists $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that

$$a \ge x \text{ for all } a \in A \text{ and } b \ge y \text{ for all } b \in B.$$

Adding these two inequalities together, we see that $a + b \ge x + y$ for all $a + b \in A + B$. Thus, $x + y$ is a lower bound for $A + B$. Since $A + B$ is bounded below, it follows from greatest-lower-bound property that $\inf(A + B)$ exists.

Now, let us show $\inf(A+B) = \inf(A) + \inf(B)$. In order to show that $\inf(A+B) = \inf(A) + \inf(B)$, we need to show that

(i) The set $A + B$ is bounded below by $\inf(A) + \inf(B)$.

(ii) For every $\varepsilon > 0$, there exists $\varphi \in A + B$ such that

$$\varphi < \inf(A) + \inf(B) + \varepsilon.$$

Since A and B are nonempty and bounded below, we know that $\inf(A)$ and $\inf(B)$ exists by the greatest-lower-bound property. Thus, we can see that $a \geq \inf A$ for all $a \in A$ and $b \geq \inf B$ for all $b \in B$. Adding these two inequalities together results in

$$a + b \geq \inf(A) + \inf(B) \text{ for all } a + b \in A + B,$$

which shows (i). Now, let $\varepsilon > 0$. Using the lemma found in Problem 3, there exists an $\alpha \in A$ and $\beta \in \beta$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{2}$$

and

$$\beta < \inf(B) + \frac{\varepsilon}{2},$$

respectively. Adding these two inequalities, we get

$$\varphi = \alpha + \beta < \inf(A) + \inf(B) + \varepsilon \text{ for some } \varphi \in A + B,$$

showing (ii). Thus, we conclude that $\inf(A+B) = \inf(A) + \inf(B)$. ■

9-2) $\sup(A+B) = \sup(A) + \sup(B)$

Proof. Our goal is to show that $A+B$ contains a supremum and that $\sup(A+B) = \sup A + \sup B$. Note that $A + B \neq \emptyset$ since $a + b \in A + B$ and A and B are nonempty sets. Since A and B are bounded above, there exists $u \in \mathbb{R}$ and $v \in \mathbb{R}$ such that

$$a \leq u \text{ for all } a \in A \text{ and } b \leq v \text{ for all } b \in B, \text{ respectively.}$$

Adding these two inequalities together, we see that $a + b \leq u + v$ for all $a + b \in A + B$. Thus, $u + v$ is a upper bound for $A + B$. Since $A + B$ is bounded above, it follows from least-upper-bound property that $\sup(A + B)$ exists.

In order to show that $\sup(A+B) = \sup(A) + \sup(B)$, we need to show that

(i) $A + B$ is bounded above by $\sup(A) + \sup(B)$ and

(ii) For every $\varepsilon > 0$, there exists a $\lambda \in A + B$ such that

$$\lambda > \sup(A) + \sup(B) - \varepsilon.$$

Since A and B are nonempty and bounded above, we must have that $\sup(A)$ and $\sup(B)$ exists by the least-upper-bound property. Hence, we have that $a \leq \sup(A)$ for all $a \in A$ and $b \leq \sup(B)$ for all $b \in B$. Adding these two inequalities together, gives us

$$a + b \leq \sup(A) + \sup(B)$$

for all $a + b \in A + B$, which proves (i). Let $\varepsilon > 0$. Since $\sup(A)$ and $\sup(B)$ exists, we see that there exists $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{2}$$

and
$$\beta > \sup(B) - \frac{\varepsilon}{2}$$
by the lemma found in Problem 2. Adding these two inequalities results in
$$\lambda = \alpha + \beta > \sup(A) + \sup(B) - \varepsilon$$
for some $\lambda = \alpha + \beta \in A + B$, which shows (ii). Thus, we conclude that $\sup(A + B) = \sup(A) + \sup(B)$. ■

9-3) $\sup(-A) = -\inf(A)$

Proof. We will show that $-A$ contains a supremum and that $\sup(-A) = -\inf(A)$. Note that $-A$ is nonempty because $-a \in -A$ and A is nonempty. Since A is bounded below, we can find an $x \in \mathbb{R}$ such that $a \geq x$ for all $a \in A$. Multiplying by a negative, we see that $-a \leq -x$ for all $-a \in -A$. Thus, $-x$ is an upper bound for $-A$. By the least upper bound property, $\sup(-A)$ exists.

To show that $\sup(-A) = -\inf(A)$, we need to show that

(i) $-A$ is bounded above by $-\inf(A)$ and that

(ii) for all $\varepsilon > 0$, there exists $-\alpha \in -A$ such that
$$-\alpha > -\inf(A) - \varepsilon.$$

Since A is bounded below and nonempty, we know that $\inf(A)$ exists (by the greatest-lower-bound property) and that $a \geq \inf(A)$ for all $a \in A$. Multiplying this inequality by a negative, we get that
$$-a \leq -\inf(A) \quad \text{for all } -a \in -A.$$
Thus, $-A$ is bounded above by $-\inf(A)$, showing (i).

Let $\varepsilon > 0$. Since $\inf(A)$ exists, there exists $\alpha \in A$ such that
$$\alpha < \inf(A) + \varepsilon$$
by the lemma found in Problem 3. By multiplying by a negative on this inequality, we must have that
$$-\alpha > -\inf(A) - \varepsilon$$
for some $-\alpha \in -A$, showing (ii). Thus, we conclude that $\sup(-A) = -\inf(A)$. ■

9-4) $\inf(-A) = -\sup(A)$.

Proof. We will show that $-A$ contains an infimum and that $\inf(-A) = -\sup(A)$. Note that $-A$ is nonempty because $-a \in -A$ and A is nonempty. Since A is bounded above, we can find an $y \in \mathbb{R}$ such that $a \leq y$ for all $a \in A$. Multiplying by a negative, we see that $-a \geq -y$ for all $-a \in -A$. Thus, $-y$ is an lower bound for $-A$. By the greatest-lower-bound property, $\inf(-A)$ exists. Using 9-3), we see that
$$-\sup(A) = -\sup(-(-A)) = -(-\inf(-A)) = \inf(-A).$$
Thus, we conclude that $-\sup(A) = \inf(-A)$. ■

9-5) $\sup(A - B) = \sup(A) - \inf(B)$.

Proof. Observe that

$$\sup(A-B) = \sup(A+(-B)) = \sup(A)+\sup(-B) \quad (9\text{-}2))$$
$$= \sup(A)-\inf(B). \quad (9\text{-}3))$$

■

9-6) $\inf(A-B) = \inf(A)-\sup(B)$.

Proof. Observe that

$$\inf(A-B) = \inf(A+(-B)) = \inf(A)+\inf(-B) \quad (9\text{-}1))$$
$$= \inf(A)-\sup(B). \quad (9\text{-}4))$$

■

9-7) $\sup(cA) = c\sup(A)$ if $c>0$.

Proof. Let $c>0$. We will show that cA contains a supremum and that $\sup(cA) = c\sup(A)$. Note that cA is nonempty because $ca \in cA$ and $A \neq \emptyset$. Since A is bounded above, we can find an $x \in \mathbb{R}$ such that $a \leq x$ for all $a \in A$. Thus, $ca \leq cx$ for all $ca \in cA$. Thus, cx is an upper bound for cA. By the least-upper-bound property, $\sup(cA)$ exists. Now, we will show that $\sup(cA) = c\sup(A)$ by showing that

(i) The set cA is bounded above by $c\sup A$.

(ii) For all $\varepsilon>0$, there exists an $c\alpha \in cA$ such that

$$c\alpha > c\sup(A)-\varepsilon.$$

First, we show that cA is bounded above by $c\sup(A)$. Since A is nonempty and bounded above, we know that $\sup(A)$ exists (least-upper-bound property). Thus, we see that

$$a \leq \sup(A) \quad \text{for all } a \in A.$$

Multiplying by c, we must have

$$ca \leq c\sup(A) \quad \text{for all } ca \in cA.$$

Thus, cA is bounded above by $c\sup(A)$. Now, let $\varepsilon>0$. Since $\sup(A)$ exists, there exists $\alpha \in A$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{c}$$

by the result found in Problem 2. Multiplying by $c>0$, we must have

$$c\alpha > c\sup(A)-\varepsilon \quad \text{for some } c\alpha \in cA,$$

showing (ii). Thus, we conclude that $c\sup(A) = \sup(cA)$.

■

9-8) $\inf(cA) = c\inf(A)$ if $c>0$.

Proof. Let $c>0$. We will show that cA contains an infimum and that $\inf(cA) = c\inf(A)$. Note that cA is nonempty because $ca \in cA$ and $A \neq \emptyset$. Since A is bounded below, we can find a $y \in \mathbb{R}$ such that $a \geq y$ for all $a \in A$. Thus, $ca \geq cy$ for all $ca \in cA$. Thus, cy is an lower bound for cA. By the greatest-lower-bound property, $\inf(cA)$ exists.

To show that $\inf(cA) = c\inf(A)$, we need to show that

(i) $c\inf(A)$ is a lower bound of cA and

(ii) for all $\varepsilon > 0$, there exists a $\gamma \in cA$ such that

$$\gamma < c\inf(A) + \varepsilon.$$

Since $\inf(A)$ exists (because $A \neq \emptyset$ and A is bounded below), we have that $a \geq \inf(A)$ for all $a \in A$. Multiplying by $c > 0$, we have $ca \geq c\inf(A)$ for all $ca \in cA$. Thus, cA is bounded below by $c\inf(A)$. This proves (i).

Let $\varepsilon > 0$. Note the fact that $\inf(A)$ exists implies that there exists an $\alpha \in A$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{c}$$

by the lemma found in Problem 3. Multiplying by $c > 0$, we have

$$c\alpha < c\inf(A) + \varepsilon$$

for some $\gamma = c\alpha \in cA$, showing (ii). Thus, we conclude that

$$\inf(cA) = c\inf(A).$$

■

9-9) $\sup(cA) = c\inf(A)$ if $c < 0$.

Proof. Let $c < 0$. Observe that

$$\begin{aligned} c\inf(A) &= c\inf(-(-A)) \\ &= -c(\sup(-A)) && (9\text{-}4)) \\ &= \sup((-c)(-A)) && (-c > 0 \text{ and } 9\text{-}7)) \\ &= \sup(cA). \end{aligned}$$

Thus, we conclude that $c\inf(A) = \sup(cA)$ for $c < 0$. ■

9-10) $\inf(cA) = c\sup(A)$ if $c < 0$.

Proof. Let $c < 0$. Observe that

$$\begin{aligned} \inf(cA) &= \inf((-c)(-A)) && (-c > 0) \\ &= -c\inf(-A) && (9\text{-}8)) \\ &= -c(-\sup A) && (9\text{-}4)) \\ &= c\sup A. \end{aligned}$$

Thus, we conclude that

$$\inf(cA) = c\sup A$$

for $c < 0$. ■

9-11) Is it true that $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

Proof. This is false. Consider the sets $A = \{1, 2, 4, 8\}$ and $B = \{-5, -2, -3, -1\}$. Then $AB = \{-5, -4, -12, -8\}$. Observe that $\sup(A) \cdot \sup(B) = 8 \cdot -1 = -8$, but $\sup(AB) = -5 \neq -8 = \sup(A) \cdot \sup(B)$. ■

10. Recall that $\mathbb{Q}$ is a field; in particular, $\mathbb{Q}$ is closed under addition and multiplication.

10-1) Prove that if $p \in \mathbb{Q}$ and t is an irrational number, then $p + t$ is an irrational number.

Proof. Suppose $p \in \mathbb{Q}$ and t is an irrational number. Suppose for sake of contradiction that $p + t$ is an rational number. Then there exists $x, y \in \mathbb{Z}$ with $y \neq 0$ such that

$$p + t = \frac{x}{y}.$$

Subtracting p on both sides, we get

$$t = \frac{x}{y} - p.$$

Since $\mathbb{Q}$ is a field and $\frac{x}{y}, p \in \mathbb{Q}$, we get that $\frac{x}{y} - p \in \mathbb{Q}$. But this tells us that t is a rational number which is a contradiction. Thus, t must be irrational. ■

10-2) Complete the following proof.

Theorem. Given any two real numbers $x < y$, there exists an irrational number t satisfying $x < t < y$.

Proof. It follows from $x < y$ that $x - \sqrt{2} < y - \sqrt{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $p \in \mathbb{Q}$ such that $x - \sqrt{2} < p < y - \sqrt{2}$. Adding $\sqrt{2}$ on both sides gives us

$$x < p + \sqrt{2} < y.$$

Since $p \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, we get that $t = p + \sqrt{2}$ is irrational from 10-2). Thus, $x < t < y$ for some irrational number t.

■

11. Prove the following:

Theorem (Nested Interval Property)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, that is $\cap_n^\infty I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ because $a_1 \in A$. Since $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$ and that each I_{n+1} is contained within each I_n , we see that each $a_n \in A$ must be bounded above by b_n for all $n \in \mathbb{N}$. Since $A \neq \emptyset$ and bounded above, there must exists a number x such that $x = \sup(A)$. Since x is an upper bound of A, we have $a_n \leq x \leq b_n$. Thus, $x \in I_n$ for all $n \in \mathbb{N}$ which means that

$$x \in \bigcap_{n=1}^{\infty} I_n,$$

proving that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

12. Complete the following proof:

Prove that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.

Proof. Suppose for sake of contradiction $\bigcap_{n=1}^{\infty}(0,1/n) \neq \emptyset$. So there exists $x \in \bigcap_{n=1}^{\infty}(0,1/n)$. This implies that $x > 0$ and $x < 1/n$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < x.$$

But this contradicts our assumption that $x < 1/n$ holds for all $n \in \mathbb{N}$. Thus, the intersection

$$\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$$

must be empty. ■

0.2 Week 2

Problem 1

Mark each statement True or False.

1. If Y is a set and $\{G_\alpha : \alpha \in A\}$ is a family of subsets of Y, then $\bigcup_{\alpha \in A} G_\alpha$ is a subset of Y. **True.**
2. The range of every sequence is at most countable. **True.**
3. The set $\{q \in \mathbb{Q} : q^2 < 2\}$ is countable. **True.**
4. Every infinite subset of an uncountable set is uncountable. **False.**
5. The union of any collection of at most countable sets is at most countable. **False.**
6. If $A, B \subseteq \mathbb{R}$ and A is countable, then $A \cap B$ is at most countable. **True.**

Problem 2

Find a bijection from $\mathbb{N}$ to the set of odd integers.

Solution. Define the following bijection $f : \mathbb{N} \to \{\text{odd integers}\}$ as

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ -n+1 & \text{if } n \text{ is even.} \end{cases}$$

■

Problem 3

For the following problem, you may use your grade school knowledge that every real number has a decimal representation. Although there is a careful description of decimal representation on page 11 of Rudin, it will not be needed to solve this problem.

Recall that in class we proved that the set of all binary sequences is uncountable. Use that fact to show that interval $[0, 0.2)$ is uncountable. Explain how this result proves that $[0, 1]$ and $\mathbb{R}$ are uncountable as well.

Proof. Consider the set of binary sequences

$$A = \{n \in \mathbb{N} : (a_n)\}$$

Suppose we have the map

$$f : A \to [0, 0.2).$$

Note that f is injective since for any $(a_1, a_2, \dots), (b_1, b_2, \dots) \in A$, we have

$$\begin{aligned} f((a_1, a_2, \dots)) &= f((b_1, b_2, \dots)) \\ 0.a_1 a_2 \dots &= 0.b_1 b_2 \dots \ . \end{aligned}$$

Hence, $(a_1, a_2, \dots) = (b_1, b_2, \dots)$. Thus, f is injective. Consider the proper subset $f(A)$ of $[0, 0.2)$. Recall that A is uncountable and so, $f(A)$ is uncountable since $f : A \to [0, 0.2)$ is uncountable. Thus, $[0, 0.2)$ is also uncountable.

Note that $[0, 0, 2) \subseteq [0, 1]$. But $[0, 0, 2)$ is an uncountable set. Thus, $[0, 1]$ is uncountable. Similarly, we see that $[0, 1] \subseteq \mathbb{R}$. Since $[0, 1]$ is uncountable, we see that $\mathbb{R}$ is uncountable. ■

Problem 4

Is the set of all irrational numbers countable?

Solution. Suppose for sake of contradiction that the set of all irrational numbers $\mathbb{I}$ is countable. Note that $\mathbb{Q}$ is also countable and subsequently that the union $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$ is countable. However, $\mathbb{R}$ is uncountable from Problem 3 and so we have a contradiction. Thus, the set of irrational numbers must uncountable. ■

Problem 5

Show any collection of disjoint intervals of positive length is at most countable.

Proof. Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be a collection of disjoint intervals of positive length. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, for each $\alpha \in \Lambda$, there exists a $q_\alpha \in \mathbb{Q}$ such that $q_\alpha \in I_\alpha$. Define the map $f : \{I_\alpha\}_{\alpha \in \Lambda} \to \mathbb{Q}$ by $f(I_\alpha) = q_\alpha$. Since every interval I_α is disjoint, we must have that f is injective. If f is injective, then the map $f : \{I_\alpha\}_{\alpha \in \Lambda} \to f(\{I_\alpha\}_\Lambda)$ is bijective. Thus,

$$\{I_\alpha\}_{\alpha \in \Lambda} \sim f(\{I_\alpha\}_{\alpha \in \Lambda}).$$

Furthermore, we see that $f(\{I_\alpha\}_{\alpha \in \Lambda}) \subseteq \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we know that $f(\{I_\alpha\}_{\alpha \in \Lambda})$ is at most countable. Therefore, we conclude that $\{I_\alpha\}_{\alpha \in \Lambda}$ is at most countable. ■

Problem 6

A real number is said to be algebraic if it is a root of some nonzero polynomial with rational coefficients. Prove that the set of all algebraic numbers is countable. Define the map .

Proof. Let

$$A = \text{the set of all (real) algebraic numbers.}$$

For all $n \geq 1$, let

$$\hat{P}_n = \text{the set of all polynomials of degree exactly } n \text{ with rational coefficients}$$

and that

$$B_n = \{\alpha \in \mathbb{R} : \text{there exists } f \in \hat{P}_n \text{ with } f(\alpha) = 0\}$$

to be the collection of all polynomials in $\hat{P}_n$. Also, for any polynomial f, we let

$$R(f) = \{\alpha \in \mathbb{R} : f(\alpha) = 0\}$$

to be the collection of all roots of f. Here are some observations

(i) $A = \bigcup_{n=1}^{\infty} B_n$

(ii) For all $n \geq 1$ and for all $f \in \hat{P}_n$, we have that $R(f)$ is at most countable.

We will show the following statements:

(1) We will show that for each $n \geq 1$, $\hat{P}_n$ is countable.

(2) Using (1) and (ii), we will show that B_n is at most countable for each $n \geq 1$.

(3) Using (2) and (i), we will show that A is at most countable.

(4) Show that A is infinite.

First, we will show fact (1).

Let $n \geq 1$. Define the function $g : \hat{P}_n \to \mathbb{Q}^{n+1}$ by

$$g(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = (a_n, a_{n-1}, \ldots, a_0).$$

Clearly, this function is 1-1. So, $g : \hat{P}_n \to g(\hat{P}_n)$ is bijective. Thus, we have that $\hat{P}_n \sim g(\hat{P}_n)$ and $g(\hat{P}_n) \subseteq \mathbb{Q}^{n+1}$. Note that $\mathbb{Q}$ is countable implies that the product $\mathbb{Q}^{n+1}$ is countable. This implies that $g(\hat{P}_n)$ is at most countable. Now, we see that $j : \mathbb{N} \to \hat{P}_n$ defined by $h(m) = mx^n$ is a 1-1 function. Hence, we see that $h(\mathbb{N})$ is an infinite subset of $\hat{P}_n$. So, $\hat{P}_n$ must be infinite. Since $\hat{P}_n$ is at most countable and $\hat{P}_n$ is infinite, we must have that $\hat{P}_n$ is countable.

Next, we will show that for all $n \geq 1$, B_n is at most countable. Let $n \geq 1$. Since $\hat{P}_n$ is countable, we can enumerate its elements in the following way:

$$\hat{P}_n = \{f_1^n, f_2^n, \ldots\}.$$

We have

$$B_n = \bigcup_{f \in \hat{P}_n} R(f) = \bigcup_{m=1}^{\infty} R(f_m^n).$$

For each m, we see that $R(f_m^n)$ is finite (and thus, at most countable). Since a countable union of at most countable sets is at most countable, we can conclude that B_n is at most countable.

Next, we will show that A is at most countable. Since each B_n for all $n \geq 1$ is at most countable and the fact that

$$A = \bigcup_{n=1}^{\infty} B_n,$$

we have that A is at most countable (A countable union of at most countable sets is at most countable).

Lastly, we will show that A is infinite. For each $\alpha \in \mathbb{Q}$, clearly, α is a root of the polynomial $x - \alpha$ (which is a polynomial with rational coefficients). Thus, each rational number is algebraic. Since $\mathbb{Q} \subseteq A$ and $\mathbb{Q}$ is countable, we know that $\mathbb{Q}$ is infinite. Thus, A must be infinite, and thus countable. ■

Problem 7

Prove that not all real numbers are algebraic.

Proof. Suppose for sake of contradiction that all real numbers are algebraic. Let A be the set of all real numbers that are algebraic. By the last problem, this set is countable. However, $\mathbb{R}$ is an uncountable set by Problem 3 which is a contradiction. Thus, not all real numbers are algebraic. ■

Problem 8

Choose (specific) distinct real numbers a, b, c, d such that $a < b$ and $c < d$. Prove that card $[a, b] =$ card $[c, d]$. (Hint: Find a linear polynomial f such that $f(a) = c$ and $f(b) = d$.)

Proof. Our goal is to show that card $[a, b] =$ card $[c, d]$. It suffices to show that $[a, b] \sim [c, d]$. Consider the map $f : [a, b] \to [c, d]$ defined by

$$f(x) = \frac{d-c}{b-a}(x-a) + c.$$

We wan to show that f is a bijective function. Note that $f(a) = c$ and $f(b) = d$. First, we show that f is injective. Let $x, y \in [a, b]$. Suppose $f(x) = f(y)$. Then we have that

$$\begin{aligned}
\frac{d-c}{b-a}(x-a) + c &= \frac{d-c}{b-a}(y-a) + c \\
\frac{d-c}{b-a}(x-a) &= \frac{d-c}{b-a}(y-a) \\
x - a &= y - a \\
x &= y.
\end{aligned}$$

Thus, f is injective. Consider $x \in [c, d]$. Then

$$x = \frac{(y-c)(b-a)}{d-c} + a.$$

Then we have that

$$y = \frac{d-c}{b-a}(x-a) + c.$$

But this means that $f(x) = y$ for some $x \in [c, d]$. Hence, f is surjective. Since $f : [a, b] \sim [c, d]$ is bijective, we must have that card $[a, b] =$ card $[c, d]$. ■

Problem 9

Consider the function $f : (0, 1) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{n-1} & \text{if } x = \frac{1}{n}, \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}.$$

1. Prove that f is a bijection from $(0, 1)$ to $(0, 1]$.

   Proof. We will show that $f : (0, 1) \to (0, 1]$ where

   $$f(x) = \begin{cases} \frac{1}{n-1} & \text{if for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

   is a bijective map. First, we will show that f is injective. Suppose $x, y \in (0, 1)$. Then we have $x = 1/n$ and $y = 1/m$ for some $n, m \in \mathbb{N}$, respectively, such that $f(x) = \frac{1}{n-1}$ and $f(y) = \frac{1}{m-1}$.

Suppose $f(x) = f(y)$. Then observe that

$$\begin{aligned}\frac{1}{n-1} &= \frac{1}{m-1} \\ m-1 &= n-1 \\ m &= n.\end{aligned}$$

Thus, we see that $x = y$ and we conclude that f is injective.

Now, we will show that f is surjective. Note that for any $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < \frac{1}{n-1} < 1.$$

Then we can find some $x = \frac{1}{n} \in (0,1)$ such that

$$f(x) = \frac{1}{n-1}.$$

Thus, f must be surjective.

■

2. Find a bijection from $(0,1]$ to $[0,1)$.

Solution. Define the map

$$f(x) = \begin{cases} \frac{1}{n-1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

■

3. Find a bijection from $[0,1)$ to $[0,1]$.

Solution. Define the function

$$f(x) = \begin{cases} \frac{1}{n-1} & \text{if } x = \frac{1}{n} \text{ for some } n \geq 3 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1/2 \\ x & \text{otherwise} \end{cases}$$

■

0.3 Problem 10

Prove the following theorem.

Theorem. Let A and B be two nonempty sets. If $B \subseteq A$ and there exists a one-to-one function $f : A \to B$, then $A \sim B$.

Proof.

- If $x_1 \in C$ and $x_2 \in C$, then $h(x_1) = h(x_2)$ implies that

$$h(x_1) = h(x_2) \implies f(x_1) = f(x_2).$$

Since f is injective, we must have $x_1 = x_2$.

- To show that $h(A) \subseteq B$, we want to show that for all $a \in h(A)$, $a \in B$. Let $a \in h(A)$. By definition of h, we either have $a \in C$ or $a \in A \setminus C$. If $a \in C$, then we have $h(a) = f(a)$. Since $f : A \to B$, $h(a) \in B$. Thus, $h(A) \subseteq B$. If $a \in A \setminus C$, then $h(a) = a$ by definition of h.
- If $y \notin C$, then for all $n \geq 2$, $y \notin C_n$ which further implies that $y \in A \setminus C_n$. Since $C_n \subseteq C$, we see that $y \in A \setminus C$. By definition of h, we see that $h(y) = y$ and so, h is surjective.

■

0.4 Problem 11

Prove the following theorem.

Theorem (Schroder-Bernstein)**.** Let A and B be two nonempty sets. If there exists two one-to-one functions $f : A \to B$ and $g : B \to A$, then $B \sim A$.

Proof. Since $g(B) \subseteq A$ and the fact that the function $g \circ f : A \to g(B)$ is injective, we can use Theorem 1 to conclude that $A \sim g(B)$. Since $B \sim g(B)$ and $g(B) \sim A$, we conclude that $B \sim A$. ■

Problem 12

Prove the following theorem.

Theorem. Every infinite set contains a proper countable subset.

Proof. Continuing the proof presented on homework sheet. In this way, we have constructed a one-to-one function $f : \mathbb{N} \to S$. Note that $f : \mathbb{N} \to f(\mathbb{N})$ will be bijective. Since $\mathbb{N} \sim f(\mathbb{N})$ and $f(\mathbb{N})$ is a proper subset of S, we know that that $f(\mathbb{N})$ is countable. ■

Problem 13

Prove the following theorem.

Theorem. A set X is infinite if and only if there exists a proper subset $Y \subseteq X$ such that $X \sim Y$.

Proof. Assume that X is finite. Thus, there exists $n \in \mathbb{N}$ such that $X \sim \mathbb{N}_n$. Let $h : X \to \mathbb{N}_n$ be a bijective map. Assume for contradiction that there exists a proper subset $Y \subset X$ and a bijective map $f : X \to Y$. Thus, we have $Y \sim X$. Since $Y \sim X$ and $X \sim \mathbb{N}_n$, we have $Y \sim \mathbb{N}_n$. But by our lemma, we know that this cannot happen since there exists no one-to-one mapping of $\mathbb{N}_n$ onto $Y \subset \mathbb{N}_n$. Hence, we have a contradiction and we must conclude that X must be infinite. ■

Problem 14

Prove the following theorem.

Theorem. A set X is infinite if and only if there exists a one-to-one function $f : X \to X$ that is not onto.

Proof. By Theorem 4, it is enough to prove that there exists a proper subset $Y \subseteq X$ such that $X \sim Y$. Note that $f(X) \subseteq X$. Note that, by assumption, there exists a one-to-one function $f : X \to X$ that is NOT onto. Thus, we can restrict our codomain X by its proper subset $f(X)$ and produce the bijective function $f : X \to f(X)$. Thus, we see that $X \sim f(X)$ and $f(X) \subseteq X$. By Theorem 4, we conclude that X is infinite. ■

Problem 15

Prove the following theorem.

Theorem (Cantor's Theorem)**.** For any set A, the power set $P(A)$ has strictly greater cardinality than A. (Hint: If $f : A \to P(A)$ were bijective, consider $B = \{x \in A : x \notin f(x)\}$).

Proof. Let A be any set. Consider the map $f : A \to P(A)$. We will show that $\operatorname{card} A < \operatorname{card} P(A)$; that is, there exists no bijection $f : A \to P(A)$.

First, we note that $f : A \to P(A)$ is defined by

$$g(x) = \{x\}$$

which is a one-to-one function. Suppose for sake of contradiction that $f : A \to P(A)$ is a bijection. Thus, we see that f is surjective, by definition. Now, consider the set

$$B = \{x \in A : x \notin f(x)\}$$

note that $B \subseteq A$. Since f is surjective, there exists $y \in A$ such that $f(y) = B$. There are two cases to consider: either $y \in B$ or $y \notin B$. If $y \in B$, then $y \notin f(y)$. But note that $f(y) = B$, so $y \notin B$ which is a contradiction. On the other hand, if $y \notin B$, then $y \in f(y)$. Again, $f(y) = B$ implies y is also in B which is a contradiction. Thus, there exists no bijective map $f : A \to P(A)$ and so we conclude that

$$\operatorname{card} A < \operatorname{card} P(A).$$

■

Problem 16

Prove that A is infinite if and only if there exists a one-to-one function $f : \mathbb{N} \to A$.

Proof. ($\Rightarrow$) Suppose A is infinite. Our goal is to show that there exists a one-to-one function $f : \mathbb{N} \to A$. Since A is infinite, A is countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. By definition, this map is also injective.

($\Leftarrow$) Suppose there exists a one-to-one function $f : \mathbb{N} \to A$. Our goal is to show that A is infinite. Notice that $f : \mathbb{N} \to f(\mathbb{N})$ is a bijective map. Hence, $\mathbb{N} \sim f(\mathbb{N})$ implies that $f(\mathbb{N})$ infinite. Since $f(\mathbb{N}) \subseteq A$, we also have that A is infinite. ■

Problem 17

Prove that A is at most countable if and only if there exists a one-to-one function $f : A \to \mathbb{N}$.

Proof. ($\Rightarrow$) Suppose A is at most countable. By definition, A is either countable or finite. If A is finite, then there exists a bijective map such that $f : A \to \mathbb{N}_n$ for some $n \in \mathbb{N}$. By definition of bijection, we know that f is also injective. Suppose A is countable, then we have $A \sim \mathbb{N}$ and so there exists a bijective map $f : A \to \mathbb{N}$ which is also injective by definition.

($\Leftarrow$) Suppose there exists a one-to-one function $f : A \to \mathbb{N}$. Note that $f(A)$ is a proper subset of $\mathbb{N}$ and that $\mathbb{N}$ is countable. Since $f(A)$ is a proper subset of $\mathbb{N}$, $f(A)$ must be at most countable. Furthermore, $f : A \to f(A)$ is a bijective map and so, $A \sim f(A)$. Since $A \sim f(A)$ and $f(A)$ is at most countable, we must have that A is at most countable. ■

0.5 Week 3

1. Prove that for all $a, b \in \mathbb{R}$, we have $|ab| = |a||b|$.

Proof. Let $a, b \in \mathbb{R}$. Our goal is to show that $|ab| = |a||b|$. There are a few cases to consider to which we will list below:

(1) If $a = 0$ and $b = 0$.

(2) $a > 0$ and $b > 0$

(3) $a < 0$ and $b > 0$

(4) $a > 0$ and $b < 0$.

(5) $a < 0$ and $b < 0$.

Note that, by definition of absolute value, we have

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a \leq 0 \end{cases}$$
$$|b| = \begin{cases} b & \text{if } b > 0 \\ -b & \text{if } b \leq 0 \end{cases}$$
$$|ab| = \begin{cases} ab & \text{if } ab > 0 \\ -ab & \text{if } ab \leq 0 \end{cases}.$$

Now, we proceed with each case as follows:

(1) Suppose $a = 0$ and $b = 0$, then the result immediately follows.

(2) Suppose $a > 0$ and $b > 0$. Then $ab > 0$ which implies that

$$|ab| = a \cdot b = |a| \cdot |b|.$$

(3) Suppose $a < 0$ and $b > 0$, then we have $ab < 0$. Thus,

$$|ab| = -ab = (-a)(b) = |a||b|.$$

(4) Suppose $a > 0$ and $b < 0$. Then we have $ab < 0$ which tells us that

$$|ab| = -ab = (a)(-b) = |a||b|.$$

(5) Suppose $a < 0$ and $b < 0$. Then we have $ab > 0$ which tells us that

$$|ab| = ab = (-a)(-b) = |a||b|.$$

Thus, we conclude that for all $a, b \in \mathbb{R}$, we have $|ab| = |a||b|$. ■

2. Prove that for all $c \in \mathbb{R}$, we have $-|c| \leq c \leq |c|$.

Proof. Let $c \in \mathbb{R}$. We will show that $-|c| \leq c \leq |c|$. There are two cases to consider:

(1) $c = 0$,

(2) $c > 0$,

(2) $c < 0$.

We proceed with each case as follows:

(1) If $c = 0$, then the result immediately follows.

(2) Suppose $c > 0$. Then $|c| = c$ which implies that

$$c \leq c \Rightarrow c \leq |c|.$$

For the other inequality, note that $c \geq -c$. So, we have

$$c \geq -c = -|c|.$$

Thus, we have $-|c| \leq c \leq |c|$.

(3) Suppose $c < 0$. Note that $|c| = -c$. If $c < 0$, then we see that $-c < c$. Clearly, we have $c \leq c$, and so we have

$$-|c| = -(-c) = c \leq c \Rightarrow -|c| \leq c.$$

Since $c < -c$, we see that

$$c \leq -c = |c|.$$

Hence, we have $-|c| \leq c \leq |c|$.

■

3. Prove that for all $c, d \in \mathbb{R}$, we have

$$|c| \leq d \Longleftrightarrow -d \leq c \leq d.$$

Proof. Let $c, d \in \mathbb{R}$. We will show that $|c| \leq d$ if and only if $-d \leq c \leq d$.

($\Rightarrow$) Suppose $|c| \leq d$. Note that we have $|c| = c$ if $c > 0$ or $|c| = -c$ if $c < 0$ or $|c| = 0$ if $c = 0$. Clearly, if $c = 0$, then the result immediately follows. Thus, suppose $c > 0$. Then

$$|c| \leq d \Longrightarrow c \leq d$$

and

$$\begin{aligned} |c| \leq d &\Longrightarrow c \leq d \\ &\Longrightarrow -(-c) \leq d \\ &\Longrightarrow -c \geq -d. \end{aligned}$$

Note that $c > -c$, so $c \geq -d$. Thus, we see that

$$-d \leq c \leq d.$$

If $c < 0$, then $|c| = -c$. So,

$$\begin{aligned} |c| \leq d &\Rightarrow -c \leq d \\ &\Rightarrow c \geq -d. \end{aligned}$$

If $c < 0$, then $-c > 0$ implies $-c > c$. So, we have

$$\begin{aligned} c \geq -d &\Rightarrow -(-c) \geq -d \\ &\Rightarrow -c \leq d \\ &\Rightarrow c < -c \leq d \\ &\Rightarrow c \leq d. \end{aligned}$$

Thus, we see that

$$-d \leq c \leq d.$$

($\Leftarrow$) Suppose $-d \leq c \leq d$. Note that if $c = 0$, then the result immediately follows. Thus, we either have $c > 0$ or $c < 0$. If $c > 0$, then $|c| = c$. Then by assumption, we have

$$|c| = c \leq d.$$

If $c < 0$, then $|c| = -c$. By assumption, we see that

$$-d \leq c \Rightarrow -c \leq d.$$

Thus, we have

$$|c| = -c \leq d.$$

■

4. (i) Use induction to prove that $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ for any n real numbers $x_1, \ldots, x_n$.

Proof. We proceed with induction on n. Let $n = 1$, then we clearly we have $|x_1| \leq |x_1|$. If $n = 2$, then our claim is that

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

to which we will show that

$$(x_1 + x_2)^2 \leq (|x_1| + |x_2|)^2.$$

Using problem 1 and 2, we can see that

$$\begin{aligned} (x_1 + x_2)^2 &= x_1^2 + 2x_1x_2 + x_2^2 \\ &\leq |x_1|^2 + 2|x_1x_2| + |x_2|^2 \\ &= |x_1|^2 + 2|x_1||x_2| + |x_2|^2 \\ &= (|x_1| + |x_2|)^2. \end{aligned}$$

Then we have

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

by taking the square root of both sides. Thus, case $n = 2$ holds.

Now, suppose the result holds for the nth case. We will show that the $n+1$ case holds. Note that $p = (x_1 + x_2 + \cdots + x_n)$. So, applying the result for $n = 2$ yields the following result:

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &= |(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \\ &= |p + x_{n+1}| \\ &\leq |p| + |x_{n+1}|. \end{aligned}$$

By our induction hypothesis, we see that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Thus, we have

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |z_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

and we are done. ■

(ii) Prove $||x| - |y|| \leq |x - y|$ for any two real numbers x and y.

Proof. Let $x, y \in \mathbb{R}$. To show that $||x| - |y|| \leq |x - y|$, it is enough to show that

$$-(|x - y|) \leq |x| - |y| \leq |x - y|.$$

Observe that $|x| \leq |x|$. Then using the part (i), we see that

$$\begin{aligned} |x| \leq |x| &\iff |x| \leq |(x - y) + y| \leq |x - y| + |y| \\ &\iff |x| - |y| \leq |x - y|. \end{aligned}$$

Similarly, observe that $|y| \leq |y|$ implies that

$$\begin{aligned} |y| \leq |y| &\iff |y| \leq |(y - x) + x| \leq |y - x| + |x| \\ &\iff |y| - |x| \leq |y - x| \\ &\iff -(|x| - |y|) \leq |x - y|. \end{aligned}$$

Thus, we see that

$$-(|x - y|) \leq |x| - |y| \leq |x - y|$$

which is our desired result. ■

5. Recall that according to the AM-GM inequality (for the case $n = 2$) we have

$$\forall x, y > 0 \quad \frac{x + y}{2} \geq \sqrt{xy}.$$

Use the inequality above to show that for all real numbers $a > 0$ we have $a + \frac{1}{a} \geq 2$.

Proof. Let $a > 0$. Then by the $n = 2$ case of the AM-GM inequality, we have

$$\begin{aligned} a + \frac{1}{a} = \frac{2a^2}{2a} + \frac{2}{2a} &= \frac{2a^2/a}{2} + \frac{2/a}{2} \\ &\geq \sqrt{\frac{2a^2}{a} \cdot \frac{2}{a}} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

Thus, we conclude that

$$a + \frac{1}{a} \geq 2.$$

■

6. Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for all nonnegative real numbers a and b, we have
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \in \mathbb{R}$ be nonnegative. We have the following cases:

(1) $a = 0$ and $b = 0$

(2) $a = 0$ and $b > 0$

(3) $a > 0$ and $b = 0$

(4) $a > 0$ and $b > 0$.

We proceed with the proof of the result with the following cases.

(1) If $a = 0$ and $b = 0$, then the result is immediate.

(2) If $a = 0$ and $b > 0$, then we immediately have
$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{b^q}{q}.$$

(3) If $b = 0$ and $a > 0$, then we similarly have
$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{a^p}{p}.$$

(4) Suppose $a > 0$ and $b > 0$. By the property of logarithms, we see that
$$ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b}.$$
Also, we see that
$$\ln(a^p) = p \ln a \text{ and } \ln(b^q) = q \ln b.$$
Now, observe that
$$e^{\ln a + \ln b} = e^{\frac{p}{p} \ln a + \frac{q}{q} \ln b} = e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)}.$$
Notice that e^t, when differentiated twice, is a strictly positive function. Thus, e^t is convex for all $t \in \mathbb{R}$ our knowledge of calculus. Thus, we can use Jensen's inequality to conclude that
$$e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)} \leq \frac{1}{p} e^{\ln(a^p)} + \frac{1}{q} e^{\ln(b^q)} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

■

7. **(Holder's Inequality)** Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are two points in $\mathbb{R}^n$. Prove that
$$\sum_{i=1}^{n} |x_i y_i| \leq \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |y_i|^q \Big)^{\frac{1}{q}}$$
(Note that the Cauchy-Schwarz inequality is a special case of the inequality above where $p = q = 2$)

Proof. Let $a = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ and $b = \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}$.As a continuation of the proof presented in the homework sheet, we see that $a, b \neq 0$ implies

$$\begin{aligned}\sum_{i=1}^n \left|\left(\frac{x_i}{a}\right)\left(\frac{y_i}{b}\right)\right| \leq 1 &\implies \sum_{i=1}^n \left|\frac{x_i}{a}\right|\left|\frac{y_i}{b}\right| \leq 1 \\ &\implies \frac{1}{ab}\sum_{i=1}^n |x_i||y_i| \\ &\implies \sum_{i=1}^n |x_i y_i| \leq ab = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}.\end{aligned}$$

■

8. **(Minkowski's Inequality)** Let $p \geq 1$ be a real number. Suppose $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are two points in $\mathbb{R}^n$. Prove that

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

Proof. As a continuation of proof presented in the homework sheet, we have

$$\sum_{i=1}^n |x_i + y_i|^p \leq \left(\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}.$$

Dividing $\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}$ by both sides, we see that

$$\frac{\sum_{i=1}^n |x_i + y_i|^p}{\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}} \leq \left(\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right)$$

which can be re-written to

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

Now, observe that

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p}.$$

If we set

$$A = \sum_{i=1}^n |x_i + y_i|^p,$$

then we see that

$$A^{1-\frac{1}{q}} = A^{1-\left(1-\frac{1}{p}\right)} = A^{\frac{1}{p}}.$$

Thus, we see that

$$\left(\sum_{i=1}^n |x_i + y_i|\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}.$$

■

9. Let $n \geq 2$ where $n \in \mathbb{N}$. Prove that

$$y^n - x^n < n(y-x)y^{n-1}.$$

Proof. Using the identity that

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2} + \cdots + x^{n-1})$$

and the assumption that $x < y$, we can see that

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + \cdots + x^{n-1}) \\ &< (y-x)(y^{n-1} + y^{n-2} \cdot y + \cdots + y^{n-1}) \\ &= (y-x)(y^{n-1} + y^{n-1} + \cdots + y^{n-1}) \\ &= (y-x)ny^{n-1}. \end{aligned}$$

Thus, we conclude that

$$y^n - x^n < n(y-x)y^{n-1}.$$

■

10. **Every normed space is a metric space.** Let $(V, \|\cdot\|)$ be a normed space (in particular, V is a vector space). Prove that $d : V \times V \to \mathbb{R}$ defined by $d(x,y) = \|x - y\|$ is a metric space on V.

Proof. To show that $d(x,y)$ is a metric on V, we need to show the following properties:

(i) For all $x, y \in V$, $d(x,y) \geq 0$.

(ii) For all $x, y \in V$, $d(x,y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in V$, $d(x,y) = d(y,x)$.

(iv) For all $x, y, z \in V$, we have

$$d(x,y) \leq d(x,z) + d(z,y).$$

We proceed with the following proof of each property listed above.

(i) Let $x, y \in V$. Since $(V, \|\cdot\|)$ is a normed space, we know that $\|x - y\| \geq 0$ which satisfies property (i).

(ii) Let $x, y \in V$. Suppose $d(x,y) = 0$. By definition of $d(x,y)$ and by property (2) of norms, we have

$$\begin{aligned} d(x,y) = 0 &\implies \|x - y\| = 0 \\ &\implies x - y = 0 \\ &\implies x = y. \end{aligned}$$

This shows property (ii).

(iii) Let $x, y \in V$. Then by property (3) of norms, we see that

$$d(x,y) = \|x - y\| = \|-(y-x)\| = |-1|\|y - x\| = d(y,x).$$

Thus, property (iii) is satisfied.

(iv) Let $x, y, z \in V$. Then by the triangle inequality property of norms, we see that

$$\begin{aligned} d(x,y) = \|x-y\| &= \|(x-z)+(z-y)\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Thus, property (iv) is satisfied.

Since all four properties of a metric have been satisfied, we can now conclude that, indeed, $d(x,y) = \|x-y\|$ is a metric on V. ■

11. Let $p \geq 1$ be a real number. Define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \quad d_p(x,y) = \Big(\sum_{i=1}^n |x_i - y_i|^p\Big)^{\frac{1}{p}}.$$

Prove that d_p is a metric on $\mathbb{R}^n$. (Note that the special case $p = 2$ gives the standard metric on $\mathbb{R}^n$.)

Proof. Let us define the function $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ by $\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}}$. We will show that $\|\cdot\|_p$ is a norm on the vector space $\mathbb{R}^n$ and then define $d_p(x,y) = \|x-y\|_p$ and then show that this is a metric using the last exercise. Firstly, we proceed by showing that $\|\cdot\|_p$ satisfies the properties of norms. Thus, we have:

(i) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Note that $|x_i|^p \geq 0$ for all x_i's if and only if $|x_i| \geq 0$ which is true if and only if $x_i \geq 0$ for all $1 \leq i \leq n$. This tells us that

$$\sum_{i=1}^n |x_i|^p \geq 0.$$

Since $p \geq 1$ is a real number, we must have

$$\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} \geq 0.$$

Thus, the first property of norms are satisfied.

(ii) Let $x \in \mathbb{R}^n$. Then we see that

$$\begin{aligned} \|x\|_p = 0 &\iff \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} = 0 \\ &\iff \sum_{i=1}^n |x_i|^p = 0 \\ &\iff |x_i|^p = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff |x_i| = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff x_i = 0 && (\forall\ 1 \leq i \leq n) \\ &\iff x = 0. \end{aligned}$$

Hence, property (ii) is satisfied.

(iii) Let $\alpha \in \mathbb{R}$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then we see that

$$\begin{aligned}\|\alpha x\|_p &= \Big(\sum_{i=1}^{n} |\alpha x_i|^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{i=1}^{n} |\alpha|^p |x_i|^p\Big)^{\frac{1}{p}} \\ &= |\alpha| \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} \\ &= |\alpha| \|x\|_p.\end{aligned}$$

Thus, we see that property (iii) has been satisfied.

(iv) Let $x, y \in V$. By using Minkowski's inequality, we see that

$$\begin{aligned}\|x+y\|_p &= \Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p.\end{aligned}$$

Thus, the triangle inequality property of norms is satisfied.

This tells us that $\|\cdot\|_p$ is a norm on the vector space $\mathbb{R}^n$. Defining the function $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_p(x,y) = \|x - y\|_p$, we see that d_p must be a metric on $\mathbb{R}^n$ by Exercise 10. ■

12. Define $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \quad d_\infty = \max\{|x_i - y_i| : 1 \leq i \leq n\}.$$

Prove that d_∞ is a metric on $\mathbb{R}^n$.

Proof. Let us define the function $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ by

$$\|x\|_\infty = \max\{|x_i| : i = 1, \dots, n\} = \max_{1 \leq i \leq n} |x_i|.$$

We will prove that this is, indeed, a norm on the vector space $\mathbb{R}^n$ and thereby show that, by defining the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_\infty(x,y) = \|x - y\|_\infty$, it defines a metric on $\mathbb{R}^n$. In what follows, we will show the four properties of norms:

(i) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Since the absolute value $|\cdot|$ is always nonnegative, we can see that $|x_i| \geq 0$ for all $1 \leq i \leq n$. Thus, we have

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \geq 0.$$

Thus, the first property of norms is satisfied.

(ii) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Observe that

$$\begin{aligned}
\|x\|_\infty = 0 &\implies \max_{1 \leq i \leq n} |x_i| = 0 \\
&\implies |x_i| \leq 0 && (\forall 1 \leq i \leq n) \\
&\implies 0 \leq |x_i| \leq 0 && (\forall 1 \leq i \leq n) \\
&\implies |x_i| = 0 && (\forall 1 \leq i \leq n) \\
&\implies x_i = 0 && (\forall 1 \leq i \leq n) \\
&\implies x = 0.
\end{aligned}$$

Now, suppose $x = (x_1, x_2, \dots, x_n) = 0$. Then for all $1 \leq i \leq n$, we see that $|x_i| = 0$ and so $\max_{1 \leq i \leq n} |x_i| = 0$. By definition of $\| \cdot \|_\infty$, we see that $\|x\|_\infty = 0$.

(iii) Let $\alpha \in \mathbb{R}$ and let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then observe that by problem 9-7) of homework 1 that

$$\begin{aligned}
\|\alpha x\|_\infty &= \max_{1 \leq i \leq n} |\alpha x_i| \\
&= \max_{1 \leq i \leq n} |\alpha||x_i| && \text{(Problem 1)} \\
&= |\alpha| \max_{1 \leq i \leq n} |x_i| && \text{(Problem 9-7) of Homework 1)} \\
&= |\alpha|\|x\|_\infty.
\end{aligned}$$

Thus, we see that $\| \cdot \|_\infty$ satisfies the third property of norms.

(iv) Let $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. By the triangle inequality of the absolute value $| \cdot |$, we can see that

$$\begin{aligned}
\|x + y\|_\infty &= \max_{1 \leq i \leq n} |x_i + y_i| \\
&\leq \max_{1 \leq i \leq n} |x_i| + |y_i| \\
&= \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |y_i| \\
&= \|x\|_\infty + \|y\|_\infty.
\end{aligned}$$

Thus, property (iv) is satisfied.

Hence, we conclude that $\| \cdot \|_\infty$ defines a norm on $\mathbb{R}^n$. So, we can define the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d_\infty(x, y) = \|x - y\|_\infty$. Applying Exercise 10, we can see that this function defines a metric on $\mathbb{R}^n$. ■

Remark. Alternatively, we can show the triangle inequality above in the following way. By definition, we know that $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ and $\|y\|_\infty = \max_{1 \leq i \leq n} |y_i|$. Then by definition of maximums, we see that for all $1 \leq i \leq n$, we have

$$|y_i| \leq \|y\|_\infty \tag{1}$$

and

$$|x_i| \leq \|x\|_\infty. \tag{2}$$

Adding (1) and (2) together and then using the triangle inequality we see that

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_\infty + \|y\|_\infty.$$

This shows that that $\|x\|_\infty + \|y\|_\infty$ is an upper bound for the set

$$\{|x_i + y_i| : 1 \leq i \leq n\}.$$

Since the set above is clearly finite, bounded above, and nonempty, we see that the maximum for this set exists and thus

$$\max_{1 \leq i \leq n} |x_i + y_i| \leq \|x\|_\infty + \|y\|_\infty.$$

Hence, we have

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty.$$

13. Let $X = \mathbb{R}^2$. Is $d(x, y) = |x_1x_2 + y_1y_2|$ a metric on X?

Solution. No, $d(x, y) = |x_1x_2 + y_1y_2|$ does not define a metric on $\mathbb{R}^2$. Consider $(0, 1), (1, 0) \in \mathbb{R}^2$ with $x = (0, 1)$ and $(1, 0)$. Clearly, $(0, 1) \neq (1, 0)$ but

$$|x_1x_2 + y_1y_2| = |0 \cdot 1 + 1 \cdot 0| = |0 + 0| = |0| = 0.$$

■

14. Let $X = C([0, 1])$ be the set of continuous real-valued functions on $[0, 1]$. Is $d(f, g) = |f(1) - g(1)|$ a metric on X?

Solution. This is not a metric on X. Consider $f(x) = e^x$ and $g(x) = e$. We have $f \neq g$, but $d(f, g) = |f(1) - g(1)| = 0$. ■

15. Let $X = \{0\}$. Can you define a metric on X?

Solution. Yes, consider the function $d : X \times X \to X$ defined by $d(x, y) = 0$ for any $x, y \in X$. This function satisfies all the properties of a metric. ■

16. Let $X = C[0, 1]$. It can be proved that X is a vector space. Also, it can be shown that every real-valued continuous function on a closed and bounded interval attains a maximum. Define

$$\|f\|_\infty = \max\{|f(x)| : x \in [0, 1]\} = \max_{x \in [0,1]} |f(x)|.$$

Prove that $\| \cdot \|_\infty$ is a norm on X. What is the corresponding metric?

Proof. We will show that $\| \cdot \|_\infty$ defines a norm on X. We will do so by showing that $\| \cdot \|_\infty$ satisfies the properties of a norm.

(i) Let $f \in X$. Then for all $x \in [0, 1]$, we see that $|f(x)| \geq 0$ by the nonnegative property of the absolute value $| \cdot |$. Thus, we see that

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)| \geq 0.$$

(ii) Let $f \in X$. We have that

$$\begin{aligned}
\|f\|_\infty = 0 &\implies \max_{x\in[0,1]} |f(x)| = 0 \\
&\implies 0 \leq |f(x)| \leq 0 && (\forall x \in [0,1]) \\
&\implies |f(x)| = 0 && (\forall x \in [0,1]) \\
&\implies f(x) = 0 && (\forall x \in [0,1]) \\
&\implies f = 0.
\end{aligned}$$

Now, suppose $f = 0$. Then for all $x \in [0,1]$, we see that $f(x) = 0$. So, $|f(x)| = 0$ and thus $\max_{1\leq i\leq n} |x_i| = 0$. Hence, $\|x\|_\infty = 0$. Thus, property (ii) is satisfied.

(iii) Let $\alpha \in \mathbb{R}$ and $f \in X$. Then by exercise 9-7) of homework 1, we can see that

$$\begin{aligned}
\|\alpha f\|_\infty &= \max_{x\in[0,1]} |(\alpha f)(x)| \\
&= \max_{x\in[0,1]} |\alpha \cdot f(x)| \\
&= \max_{x\in[0,1]} |\alpha||f(x)| && \text{(Problem 1)} \\
&= |\alpha| \max_{x\in[0,1]} |f(x)| && \text{(Problem 9-7) of hw1)} \\
&= |\alpha|\|f\|_\infty.
\end{aligned}$$

Thus, we see that property (iii) is satisfied.

(iv) Let $f, g \in X$. Then observe that

$$\begin{aligned}
\|f+g\|_\infty &= \max_{x\in[0,1]} |(f+g)(x)| \\
&= \max_{x\in[0,1]} |f(x)+g(x)| \\
&\leq \max_{x\in[0,1]} |f(x)| + |g(x)| && \text{(Triangle Inequality)} \\
&= \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)| && \text{(Problem 9-2) of hw1)} \\
&= \|f\|_\infty + \|g\|_\infty.
\end{aligned}$$

Thus, the triangle inequality property is satisfied.

Since all four properties have been satisfied, we can now conclude that $\|\cdot\|_\infty$ is a norm on the vector space $X = C[0,1]$. The corresponding metric is the function $d_\infty : X \times X \to \mathbb{R}$ defined by $d_\infty(f,g) = \|f-g\|_\infty$. ■

Remark. We can appeal to the same methods in the remark of problem 12 to show the triangle inequality for the $\|\cdot\|_\infty$. By definition of maximums, we see that $|f(x)| \leq \|f\|_\infty$ and $|g(x)| \leq \|g\|_\infty$ for all $x \in [0,1]$. Adding these two inequalities together and using the triangle inequality of the $|\cdot|$, we see that

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

which holds for all $x \in [0,1]$. Thus, we see that the set

$$|f(x)+g(x)| : x \in [0,1]$$

is bounded above by $\|f\|_\infty + \|g\|_\infty$, and so we have

$$\|f+g\|_\infty \max_{1\leq i\leq n} |f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$

17. Let (X,d) be a metric space. Define $D(x,y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x,y \in X$. Prove that (X,D) is a metric space.

Proof. Let (X,d) be a metric space. We will show that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is a metric space.

(i) Let $x,y \in X$. Since (X,d) is a metric space, we know that $d(x,y) \geq 0$ for every $x,y \in X$ by property (i) of metric spaces. Thus, we see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0 \ \forall x,y \in X.$$

Hence, property (i) is satisfied.

(ii) Let $x,y \in X$. Since (X,d) is a metric space, we know that $d(x,y) = 0$ if and only if $x = y$. Thus, we see that

$$\begin{aligned} D(x,y) = 0 &\iff \frac{d(x,y)}{1+d(x,y)} = 0 \\ &\iff d(x,y) = 0 \\ &\iff x = y. \end{aligned}$$

Hence, property (ii) is satisfied.

(iii) Let $x,y \in X$. Since $d(x,y) = d(y,x)$ for all $x,y \in X$ (because (X,d) is a metric space), we see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = D(y,x).$$

Thus, property (iii) is satisfied.

(iv) To show that $D(x,y)$ satisfies the triangle inequality, we first need to show that $D(x,y)$ is a monotonically increasing function; that is, we need to show that for any $a < b$ where $a,b \in X$, that

$$\frac{a}{1+a} < \frac{b}{1+b}.$$

Now, let $a < b$. Then observe that

$$\begin{aligned} a < b &\iff ab + a < ab + b \\ &\iff a(b+1) < b(a+1) \\ &\iff \frac{a}{1+a} < \frac{b}{1+b}. \end{aligned}$$

Thus, we can see that $D(x,y)$ is a monotonically increasing function. Let $x,y,z \in X$. We need to consider a few cases when proving the triangle inequality:

(i) $d(x,y) \leq d(x,z)$.

(ii) $d(x,y) \leq d(z,y)$

(iii) $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$.

We proceed with each case as follows:

(i) If $d(x,y) \leq d(x,z)$, then by using the monotonicity of $D(x,y)$, we can see that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)}{1+d(x,z)} = D(x,z) \leq D(x,z) + D(z,y).$$

(ii) Similarly, if $d(x,y) \leq d(z,y)$, we have by using the monotonicity of $D(x,y)$ that

$$D(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(z,y)}{1+d(z,y)} = D(z,y) \leq D(x,z) + D(z,y).$$

(iii) If both $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$, then by the triangle inequality property of (X,d), we see that

$$\begin{aligned} D(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &\leq \frac{d(x,z)+d(z,y)}{1+d(x,y)} \\ &= \frac{d(x,z)}{1+d(x,y)} + \frac{d(z,y)}{1+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= D(x,z) + D(z,y). \end{aligned}$$

Thus, we can see that $D(x,y)$ satisfies the triangle inequality.

We can, therefore, conclude that $D(x,y)$ defines a metric on X. ■

Extra Credit Problems

1. Use the AM-GM inequality to prove that the sequence $(a_n)_{n\geq 1}$ given by $a_n = \left(1+\frac{1}{n}\right)^n$ is an increasing sequence. That is, show that for all $n \in \mathbb{N}$

$$\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Proof. Our goal is use the AM-GM inequality to show that the sequence $(a_n)_{n\geq 1}$ given by

$$a_n = \left(1+\frac{1}{n}\right)^n$$

is an increasing sequence; that is, we want to show that for all $n \in \mathbb{N}$ that $a_n \leq a_{n+1}$. From the AM-GM inequality, if we take $n+1$ points, then we see that

$$(x_1 \dots x_{n+1})^{\frac{1}{n+1}} \leq \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}$$

where

$$(xy^n)^{\frac{1}{n+1}} \leq \frac{x+ny}{n+1}$$

with $x = x_1$ and $y = x_2 = \cdots = x_{n+1}$. Now, taking $x = 1$ and $y = 1 + \frac{1}{n}$, we see that

$$\frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} = \frac{1 + (n+1)}{n+1} = \frac{1}{n+1} + 1$$

and so,

$$\begin{aligned}
(xy^n)^{\frac{1}{n+1}} \leq \frac{x + ny}{n+1} &\iff \left(\left(1 + \frac{1}{n}\right)^n\right)^{\frac{1}{n+1}} \leq \frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} \\
&\iff \left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}} \leq 1 + \frac{1}{n+1} \\
&\iff \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} \\
&\iff a_n \leq a_{n+1}.
\end{aligned}$$

Thus, we conclude that the sequence $(a_n)_{n\geq 1}$ is increasing. ■

2. Let (X, d) be a metric space. Define $D(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Prove that (X, D) is a metric space.

Proof. Let (X, d) be a metric space. Define $D(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Our goal is to show that (X, D) is a metric space.

(i) Let $x, y \in X$. Since $d(x, y) \geq 0$ by property (i) of metric spaces, we can see immediately that

$$D(x, y) = \min\{1, d(x, y)\} \geq 0.$$

Thus, the property (i) of metric spaces is satisfied.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}
D(x, y) = 0 &\iff \min\{1, d(x, y)\} = 0 \\
&\iff d(x, y) = 0 \\
&\iff x = y. && ((X, d) \text{ is a metric space})
\end{aligned}$$

Thus, property (ii) of metric spaces is satisfied.

(iii) Let $x, y \in X$. Then we see that

$$D(x, y) = \min\{1, d(x, y)\} = \min\{1, d(y, x)\} = D(y, x).$$

Thus, property (iii) of metric spaces is satisfied.

(iv) Let $x, y, z \in X$. Our goal is to show that

$$D(x, y) \leq D(x, z) + D(z, y). \tag{1}$$

We have some cases to consider; that is, either

(1) $d(x, z) = 0$, $d(z, y) = 0$, and $d(x, y) = 0$.

(2) $d(x, z) \geq 1$ or

(3) $d(z, y) \geq 1$ or

(4) both $d(x, z) \geq 1$ and $d(z, y) \geq 1$

(5) otherwise, $d(x, z) < 1$ and $d(z, y) < 1$.

Thus, we proceed by proving (1) with the following cases in mind:

(1) If $d(x,z)=0$, $d(z,y)=0$, and $d(x,y)=0$, then we get

$$D(x,y)=0\leq 0=0+0=D(x,z)+D(z,y).$$

(2) Suppose $d(x,z)\geq 1$. By definition of $D(\cdot,\cdot)$, we see that $D(x,z)=\min\{1,d(x,z)\}=1$. Then observe that

$$D(x,z)+D(z,y)=1+D(z,y)\geq 1\geq D(x,y).$$

(3) Suppose $d(z,y)\geq 1$. Then similarly, $D(z,y)=\min\{1,d(z,y)\}=1$. Thus, we see that

$$D(x,z)+D(z,y)=D(x,z)+1\geq 1\geq D(x,y).$$

(4) Suppose both $d(x,z)\geq 1$ and $d(z,y)\geq 1$. Then we have both $D(x,z)=\min\{1,d(x,z)\}=1$ and $D(z,y)=\min\{1,d(z,y)\}=1$. Thus, we have

$$D(x,z)+D(z,y)=1+1\geq 1\geq D(x,y).$$

(5) Now, suppose $d(x,z)<1$ and $d(z,y)<1$. Then by definition of $D(\cdot,\cdot)$, we see that $D(x,z)=\min\{1,d(x,z)\}=d(x,z)$ and $D(z,y)=\min\{1,d(z,y)\}=d(z,y)$. Since (X,d) is a metric space, we know that the triangle inequality for the metric $d(\cdot,\cdot)$ holds. Thus, we have that

$$D(x,z)+D(z,y)=d(x,z)+d(z,y)\geq d(x,y)\geq D(x,y).$$

Thus, we conclude that the $D(x,y)\leq D(x,z)+D(z,y)$ for all of these cases. ■