Understanding Analysis Notes

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Contents

Chapter 1

The Real Numbers

1.1 The Axiom of Completeness

Axiom 1: Axiom of Completeness

Every nonempty set of real numbers that is bounded above has a least upper bound.

Definition 1.1.1: Bounded sets

We call a set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Otherwise, a set is *bounded below* if there exists a $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for every $a \in A$.

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it satisfies the following criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We denote the least upper bound of a set A by calling it the *supremum* of A i.e $\sup(A)$. Similarly, we denote the greatest upper bound of set A by calling it the *infimum* of A i.e $\inf(A)$.

Note that a set can have many upper/lower bounds. But there can only exist one supremum and one infimum. In other words, these bounds are unique. Furthemore, the infimum and supremum need not be in the set.

Consider the following set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

This set is bounded above and below. In addition, we can see that $\sup(A) = 1$ and $\inf(A) = 0$ (this is because each subsequent number in the sequence gets smaller and smaller).

Definition 1.1.2: Maximum and Minimum of sets

We say that $a_0 \in \mathbb{R}$ is a *maximum* of the set A if $a_0 \in A$ and $a \leq a_0$ for all $a \in A$. Likewise, we say that $a_1 \in \mathbb{R}$ is a *minimum* of A if $a_1 \in A$ and $a \geq a_1$ for every $a \in A$.

If we have an open set $(0, 2)$ then the end points of this set are the infimum and supremum of the set respectively. Note that the maximum and the minimum do not exists because the infimum and the supremum are not in the set. If this set were to be closed, then the supremum and infimum would be in the set which implies that the max and min exists.

Now consider the Example

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

Notice that when we try and search for the supremum for this set, we cannot find one since we can always find a smaller number for an upper bound. One might say that $r = \sqrt{2}$ is the supremum of S but this is false since $r \notin \mathbb{Q}$ and is irrational.

Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and is bounded above. Let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Prove that $\sup(c + A) = c + \sup(A)$

Proof. We use defintion 0.2 to prove this proposition. First, we need to prove that this $\sup(c+A)$ is an upper bound. We have $\sup(A) = s$ for some $s \in A$ if $s \geq a$ for all $a \in A$. We find that adding $c \in \mathbb{R}$ gives us

$$c + s \geq c + a.$$

Hence, we have that $c + s$ is an upper bound for the set $c + A$.
Next, we prove that $\sup(c + A) = c + s$ is the *least upper bound*. We know that $c + s \geq c + a$ for all $a \in A$. Suppose we have another upper bound $b \in A$ such that $c + a \leq b$ for all $a \in A$. Another manipulation gives us $a \leq b - c$ for all $a \in A$. Since $\sup(A) = s$ is the least upper bound for A, it follows that $s \leq b - c$. Hence, we have

$$c + s \leq b \implies \sup(c + A) = c + \sup(A).$$

■

There is another way to restate part (ii) of defintion 0.2 i.e

Lemma 1.1.1: Alternative definition of least upper bound

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.

Proof. For the forward direction, suppose that $s = \sup A$ and consider $s - \epsilon$. Since s is an upper bound, we have that $s - \epsilon < s$. This means that $s - \epsilon$ is not an upper bound. Hence, we can find an element $a \in A$ such that $s - \epsilon < a$ because otherwise $s - \epsilon$ would be an upper bound. This concludes the forward direction.
For the backwards direction, assume s is an upper bound. We must satisfy part (ii) of defintion 0.2. Let $\epsilon > 0$, then $\epsilon = s - b$. But since any number smaller than s is not an upper bound, we have that $s \leq b$ if b is any other upper bound for S. Hence, $s = \sup A$. ■

1.2 Consequences of Completeness

The first application of the Axiom of Completeness is a result that says that the real line contains no gaps.

Theorem 1.2.1

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Our goal is to produce a real number x such that this element is in every closed interval I_n for every $n \in \mathbb{N}$. Using the Axiom of Completeness, we can denote the following sets

$$A = \{a_n : n \in \mathbb{N}\}$$
$$B = \{b_n : n \in \mathbb{N}\}$$

where A and B consists of the left-hand and right-hand endpoints respectively. Since every closed interval are nested, we know that every b_n serves as an upper bound for A. By the Axiom of completeness, we can say that a supremum exists for A and we can label this supremum as $x = \sup A$. By definition, this is an upper bound for A. Hence, we have that $a_n \leq x$. But since x is the least upper bound and every $b_n \in B$ is an upper bound for every $a_n \in A$, we have that $x \leq b_n$. Hence, we have that $a_n \leq x \leq b_n$ which means that $x \in I_n$ for all $n \in \mathbb{N}$. This precisely means that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. ∎

1.2.1 The Density of the Rationals

Theorem 1.2.2: Archimedean Property

- Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$
- Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$

Before we head on to the proof, it is important to notice that $\mathbb{N}$ is not bounded above and we shall not prove this fact since we are taking this property of the set to be a given just like all the properties that are contained in $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Proof. Assume for sake of contradiction that $\mathbb{N}$ is bounded above. Using the Axiom of Completeness, $\mathbb{N}$ contains a supremum, say, $\sup \mathbb{N} = \alpha$. Using lemma 1.3.8, we know that there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n. \qquad (\epsilon = 1)$$

This impplies that

$$\alpha < n + 1$$

but this shows that $n + 1 \in \mathbb{N}$ which is a contradiction because we assumed that $\alpha \geq n$ for all $n \in \mathbb{N}$ thereby rendering α to no longer be an upper bound for $\mathbb{N}$. Hence, we have that there exists an $n \in \mathbb{N}$ satisfying an $n \in \mathbb{N}$ satisfying $n > x$. The second part of this theorem follows immediately by setting $x = 1/y$. ∎

Theorem 1.2.3: Density of $\mathbb{Q}$ in $\mathbb{R}$

For every two $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Our goal is to choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b \tag{1}$$

The idea is to choose a denominator large enough so that when we increment by size $\frac{1}{n}$ that it will be too big to increment over the open interval (a, b). Using the (2) of the Archimedean Property, we choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a. \tag{2}$$

We now need to choose an $m \in \mathbb{Z}$ such that na is smaller than this chosen number. A diagram for choosing such a number is helpful. Hence,
Judging from our diagram, we can see that

$$m - 1 \leq na < m.$$

Focusing on the left part of the inequality, we can solve (2) for a and say that

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - 1/n) + 1 \\ &= nb \end{aligned}$$

This implies that $m < nb$ and consequently $na < m < nb$ which is equivalent to (1). ∎

1.3 The Existence of Square Roots

Theorem 1.3.1: The Existence of $\sqrt{2}$

There exists $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

Proof. Consider the set

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We need to show that $\alpha^2 = 2$. Hence, we need to show cases where $\alpha^2 < 2$ and $\alpha^2 > 2$. The idea behind these cases is to produce a contradiction that will show that having either one of these cases will violate the fact that $\alpha$is an upper bound for T and α is the least upper bound respectively.
Assume the first case, $\alpha^2 < 2$. We know that α is an upper bound for T. We need to construct an element that is larger than α. Hence, we construct

$$\alpha + \frac{1}{n} \in T \tag{1}$$

Squaring (1) we have that

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

We can use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

Rearranging we get that

$$\frac{2\alpha + 1}{n_0} < 2 - \alpha^2$$

and consequently

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2$$

But this means that $\alpha + 1/n_0 \in T$ showing that α is not an upper bound for T contradicting our assumption.
Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

1.4 Cardinality

1.4.1 Correspondence

Definition 1.4.1

A function $f : A \to B$ is *one-to-one* if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, there exists an element $a \in A$ for which $f(a) = b$.

An equivalent defintion for a function to be one-to-one is the following:

Definition 1.4.2

A function $f : A \to B$ is *one-to-one* if $f(a_1) \neq f(a_2)$ implies that $a_1 = a_2$.

A function that is both one-to-one and onto is said to be bijective. Meaning that we have a one-to-one correspodence between the sets A and B. Another way to explain a function being injective is to say that no two elements from A can map to the same element in B (think of the function x^2). And a function being onto can be explained as every element in A has to be mapped to an element in B.

From an algebraic perspective, we can denote a function being bijective to mean the same thing as two sets having the same cardinality i.e we can say that

Definition 1.4.3

Two sets A and B have the same cardinality if there exists $f : A \to B$ that is both one-to-one and onto. We can denote this symbolically as $A \sim B$

Some examples of bijective maps are

Example 1.4.1

1. Let the following map $f : \mathbb{N} \to \mathbf{E}$ be defined as $f(n) = 2n$. We can see that $\mathbb{N} \sim \mathbf{E}$. It's true that $\mathbf{E}$ is indeed a subset of $\mathbb{N}$, but do not conclude that it is a smaller set than $\mathbb{N}$ since they have the same cardinality or isomorphic to each other.

2. We can show this again. This time let us have a map $f : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ is odd.} \\ -n/2 \text{ if } n \text{ is even.} \end{cases}$$

   We have that $\mathbb{N} \sim \mathbb{Z}$ indeed.

1.4.2 Countable Sets

Definition 1.4.4

A set A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an *uncountable set.*

Theorem 1.4.1

Let $\mathbb{Q}, \mathbb{R}$. Then

- The set $\mathbb{Q}$ is countable.
- The set $\mathbb{R}$ is uncountable.

Proof. 1. Suppose we define A_n to be split into two sets. When $n = 1$, define A_n to be

$$A_1 = \{0\}$$

and define A_n when $n \geq 2$ as

$$A_n = \left\{ \pm \frac{p}{q} : \text{ where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

We can observe here that for every $n \in \mathbb{N}$ we can find every element of $\mathbb{Q}$ exactly once in the sets we have defined. So we can conclude that our map is onto. Since we designed our sets so that each rational numer appears once and the fact that for $n = 1$ and $n \geq 2$ produces two disjoint sets, we can see that our map is also one-to-one.

2. We can prove that second statement of theorem by contradiction. Assume for the sake of contradiction that there exists a *one-to-one* and *onto* function where $f : \mathbb{N} \to \mathbb{R}$. Letting $x_1 = f(1)$ and $x_2 = f(2)$ and so on, then we can enumerate each element of $\mathbb{R}$ i.e

$$\mathbb{R} = \{x_1, x_2, x_3, ...\}.$$

Using the Nested Interval Property, we will now produce a real number that is not in this set. Let I_n be a closed interval which does not contain x_n but contains x_{n+1}. Furthermore, I_{n+1} is contained within I_n. Note that within I_n there are two sets which are disjoint and x_{n+1} can be in either one of these sets. Now consider the following intersection $\cap_{n=1}^{\infty} I_n$. Using our construction that every $x_n \notin I_n$, then we can say that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

But this is a contradiction because the nested interval property asserts that this intersection is nonempty meaning that every $x \in \mathbb{R}$ is contained in the above set. Hence, we cannot emumerate every single element x_n of $\mathbb{R}$. Therefore, $\mathbb{R}$ is an *uncountable* set.

■

This gives us three insights:

1. The smallest type of infinite set is the countable set.
2. We can create another set by deleting or inserting elements into it.
3. Anything smaller than a countable set is either finite or countable.

We can create $\mathbb{R}$ by taking the union of $\mathbb{Q}$ and $\mathbb{I}$. Since $\mathbb{R}$ is not countable and $\mathbb{Q}$ is, this would mean that the set of irrational numbers $\mathbb{I}$ would be uncountable. This tells us that $\mathbb{I}$ is a bigger subset of $\mathbb{R}$ than $\mathbb{Q}$.

We can summarize these results in the follow two theorems:

Theorem 1.4.2

If $A \subseteq B$ and B is *countable*, then A is either countable or finite.

Theorem 1.4.3

1. If $A_1, A_2, ... A_n$ are each countable sets, then the union of

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

2. If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

Chapter 2

Sequences and Series

2.1 The Limit of a Sequence

Understanding infinite series depends on understanding sequences that make up sequences of partial sums.

Definition 2.1.1

A sequence is a function whose domain is $\mathbb{N}$.

A way we describe sequences is to assign each $n \in \mathbb{N}$, use a mapping rule, and then have an output for the nth term. Mathematically we can describe it as a map $f : \mathbb{N} \to \mathbb{R}$.

Example 2.1.1

Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$
2. $\{\frac{1+n}{n}\}_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots)$
3. (a_n), where $a_n = 2^n$ for each $n \in \mathbb{N}$,
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

It should not be confused that in some instances, the index n will start at $n = 0$ or $n = n_0$ for some other $n_0 > 1$. It is important to keep in mind that sequences are just infinite lists of real numbers. The main point of our analysis deals with what happens at the "tail" end of a given sequence.

Definition 2.1.2: Convergence of a Sequence

A sequence (a_n) *converges* to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Furthermore, the convergence of a sequence (a_n) to a is denoted by

$$\lim_{n \to \infty} a_n = a.$$

To understand the last part of this definition, namely, $|a_n - a| < \epsilon$, we can think of it as a neighborhood where a given value will be located in.

Definition 2.1.3

Given $a \in \mathbb{R}$ and $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the *ϵ-neighborhood of a.*

We can think of $V_\epsilon(a)$ as an interval where

$$a - \epsilon < a < a + \epsilon.$$

Another way is to think of it as a ball with radius $\epsilon > 0$ centered at a. we can also think about the convergence of a sequence to a point with the following definition.

Definition 2.1.4

A sequence (a_n) converges to a if, given any $\epsilon-$neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every $\epsilon-$neighborhood contains all but a finite number of the terms of (a_n).

The main idea here is that for some $n \in \mathbb{N}$ along a sequence (a_n), all the points of the sequence converge to some point within a certain $\epsilon-$neighborhood. Note that when increase the value of $n \in \mathbb{N}$, the smaller this $\epsilon-$neighborhood has to be and vice versa.

Example 2.1.2

Consider the sequence (a_n), where $a_n = \frac{1}{\sqrt{n}}$. From our regular understanding of calculus, one can see that the limit of this sequence goes to zero.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{e^2}.$$

We now proceed by verifying that this choice $N \in \mathbb{N}$ has the desired property that $a_n \to 0$. Let $n \geq N$ such that $n > \frac{1}{\epsilon^2}$. Hence, we have

$$\frac{1}{\sqrt{n}} < \epsilon.$$

But this implies that $|a_n - 0| < \epsilon$ and hence our sequence contains the desired property. ∎

The main idea of these convergence proofs is to find an $N \in \mathbb{N}$ such that the value we want can be "hit" within some range that we specify with any number $\epsilon > 0$.

Quantifiers

The phrase

"For all $\epsilon > 0$", there exists $N \in \mathbb{N}$ such that ..."

means that for every positive integer I give you, there exists some index or natural number that contains some property that allows the sequence to converge to some value that we desire and as long as we satisfy this rule, then we can say that the sequence converges to our desired value. The template for our subsequent covergence proof will follow the steps below:

- "Let $\epsilon > 0$" be arbitrary."
- Demonstrate that a specific choice of $N \in \mathbb{N}$ leads to the desired property. Note that finding this N often involves working backwards from $|a_n - a| < \epsilon$.
- Show that this N actually works.
- Now assume $n \geq N$.
- With this choice of $\mathbb{N}$, you can work towards the property that $|a_n - a| < \epsilon$

Example 2.1.3

Show

$$\lim \left(\frac{n+1}{n} \right) = 1.$$

In other words, show that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|a_n - 1| < \epsilon$$

where

$$a_n = \frac{n+1}{n}.$$

To obtain our choice of $N \in \mathbb{N}$, we must work backwards from our conclusion. Hence, we have

$$\begin{aligned} a_n - 1 &< \epsilon \\ \frac{n+1}{n} - \frac{n}{n} &< \epsilon \\ \iff \frac{1}{n} &< \epsilon \\ \iff \frac{1}{\epsilon} &< n. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ is $N = 1/\epsilon$.

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N = 1/\epsilon$ such that

$$N > \frac{1}{\epsilon}.$$

Let $n \geq N$. Then we proceed by showing that this choice of $N \in \mathbb{N}$ leads to the desired property. Hence,

$$\begin{aligned} n &> \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{n} \\ \epsilon &> \frac{n+1}{n} - \frac{n}{n} \\ \epsilon &> \frac{n+1}{n} - 1 \\ \epsilon &> |a_n - 1|. \end{aligned}$$

Hence, our choice of $N \in \mathbb{N}$ leads to $a_n \to 1$. We can now conclude that

$$\lim_{n\to\infty} a_n = 1.$$

■

Theorem 2.1.1: Uniqueness of Limits

The limit of a sequence, when it exists, must be unique.

Proof. Suppose we have $(a_n) \subseteq \mathbb{R}$. Suppose $a_n \to a$ and $a_n \to a'$. We want to show that

$$a = a'.$$

By definition, we have that

$$\begin{aligned} |a_n - a| &< \epsilon/2 \text{ for some } n_1 \in \mathbb{N} \\ |a_n - a'| &< \epsilon/2 \text{ for some } n_2 \in \mathbb{N}. \end{aligned}$$

We can show that $a = a'$ by showing that $|a - a'| < \epsilon$. Hence, choose $N = \min\{n_1, n_2\}$ such that

$$\begin{aligned} |a - a'| &< |a - a_n + a_n - a'| \\ &< |a - a_n| + |a_n - a'| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Hence, we have that $a = a'$ showing that our limit is unique. ■

2.1.1 Divergence

We can study the divergence of sequences by negating the definition we have above.

Example 2.1.4

Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}...\right)$$

We can prove that this sequence does not converge to zero. Why? When we choose an $\epsilon = 1/10$, there is none of the term of the sequence converge within the neighborhood $(-1/10, 1/10)$ since the sequence oscillates between $-1/5$ and $1/5$. There is no $N \in \mathbb{N}$, that satisfies $a_n \to 0$. We can also give a counter-example in which we disprove the claim that (a_n) converges to $1/5$. Choose $\epsilon = 1/10$. This produces the neighborhood $(1/10, 3/10)$. We can see that the sequence does in fact converge to $1/5$, but it does so in an oscillating fashion. Furthermore, the sequence does not stay within the neighbor we specified where we expect all the terms of the sequence to converge towards the value. Hence, there is no such $N \in \mathbb{N}$ where the property can be satisfied.

Definition 2.1.5

A sequence that does not converge is said to diverge.

2.2 The Algebraic and Order Limit Theorems

The goal of having a rigorous definition of convergence in Analysis is to prove statements about sequences in general like the notion of "boundedness" which we will define below.

Definition 2.2.1

A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n). This naturally leads us to the point that all convergent sequences are bounded i.e

Theorem 2.2.1

Every convergent sequence is bounded.

Proof. Assume (x_n) converges to a limit ℓ. This means that given $\epsilon = 1$, we can find an $N \in \mathbb{N}$ such that for every $n \geq N$, we can say that

$$\begin{aligned} &\implies |x_n - \ell| < 1 \\ &\iff -1 < x_n - \ell < 1 \\ &\iff \ell - 1 < x_n < \ell + 1. \end{aligned}$$

Note the terms of the sequence (x_n) can be found in the open interval $(\ell - 1, \ell + 1)$. Since $\ell \in \mathbb{R}$ can either be positive or negative, we can conclude that

$$|x_n| < |\ell| + 1$$

for all $n \geq N$ where

$$M = \max\{|x_1|, |x_2|, ..., |\ell| + 1\}.$$

Hence, it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. ∎

Theorem 2.2.2: Algebraic Limit Theorem

Let $\lim a_n = a$, and $\lim b_n = b$. Then,

(i) $\lim(ca_n) = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$ provided that $a \neq 0$.

Proof of (i). We begin by proving part (i). Suppose $a_n \to a$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n - a| < \epsilon/|c|. \tag{1}$$

In order to show (i), we need to show that

$$|ca_n - ca| < \epsilon.$$

Hence, observe that

$$\begin{aligned} |ca_n - ca| &= |c(a_n - a)| \\ &= |c||a_n - a| \\ &< |c|\frac{\epsilon}{|c|} \\ &= \epsilon. \end{aligned}$$

If $c = 0$, then our sequence (ca_n) reduces to the sequence $\{0, 0, 0, ..., 0\}$ which is clearly converging to $ca = 0$. Hence, we have attained our desired property that $\lim(ca_n) = ca$. The parts are left to you to prove. ∎

Proof of (ii). To show part (ii), it suffices to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n + b_n - (a+b)| < \epsilon.$$

Hence, we start with the left side of (ii). Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned} |a_n + b_n - (a+b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, it follows that $\lim(a_n + b_n) = a + b$ as required. ■

proof of (iii). To show part (iii), it suffices to show for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - ab| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$. We can choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we can say that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \\ &< M\frac{\epsilon}{2M} + |b|\frac{\epsilon}{2|b|} && (a_n \text{ is bounded}) \\ &< \epsilon \end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = ab$. ■

[]

Proof of (iv). To show part (iv), it suffices to show for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon.$$

Since $a_n \to a$ and $b_n \to b$ with $b \neq 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $n \geq N_1, N_2$, we can have

$$\begin{aligned}
|a_n - a| &< M\epsilon/2, \\
|b_n - b| &< \frac{|b|}{|a|} \cdot \frac{M\epsilon}{2}.
\end{aligned}$$

we can choose $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \left|\frac{a_n b - ab + ab - b_n a}{b_n b}\right| \\
&= \left|\frac{b(a_n - a) + (b - b_n)a}{b_n b}\right| \\
&\leq \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b|} \cdot \frac{|b_n - b|}{|b_n|} \\
&< \frac{M\epsilon}{2M} + \frac{|a|}{|b|} \cdot \frac{|b| M\epsilon}{|a| 2M} && (b_n \text{ bounded}) \\
&= \epsilon.
\end{aligned}$$

Hence, it follows that $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ provided that $b \neq 0$.

■

Theorem 2.2.3: Order Limit Theorem

ssume $\lim a_n = a$ and $\lim b_n = b$.

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iv) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$, for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Proof. (i) We proceed by contradiction by assuming that $a < 0$. Suppose $a_n \geq 0$ and $a_n \to a$. Let $\epsilon = |a|$ and suppose $n \geq N$. Then

$$|a_n - a| < |a| = -a.$$

But this means that $a_N < 0$ which is a contradiction since $a_N \geq 0$.

(ii) We can ensure that the sequence $b_n - a_n$ converges to $b - a$ by the Algebraic Limit Theorem. Since $b_n - a_n \geq 0$, we can use (i) to write $b - a \geq 0$. Hence, $a \leq b$.

(iii) Suppose there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$. Suppose $a_n = c$ then using (ii) yields $c \leq b$. Suppose $a_n \leq c$ for all $n \in \mathbb{N}$ then setting $b_n = c$ and using (ii) again yields $a \leq c$.

■

2.3 The Monotone Convergence Theorem

As we have seen in the last section, convergent sequences are bounded while the converse is not true. But if a sequence is monotone then surely it is convergent.

Definition 2.3.1: Monotone Sequences

A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.3.1: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be *monotone* and *bounded.* We need to show that (a_n) converges to some value s. Let our set of points a_n be defined as

$$A = \{a_n : \text{ for all } n \in \mathbb{N}\}$$

and because we have a bounded sequence, we must have an upper bound s which can be defined as out supremum i.e

$$s = \sup\{a_n : \text{ for all } n \in \mathbb{N}\}.$$

Let $\epsilon > 0$. We need to show that

$$|a_n - s| < \epsilon$$

Since $s - \epsilon$ is not an upper bound of A, there exists $N \in \mathbb{N}$ such that

$$s - \epsilon < a_N.$$

Let's assume that (a_n) is an increasing sequence. By assuming $n \geq N$, we can say that $a_n \geq a_N$. Since $s + \epsilon$ is an upper bound and s is the least upper bound, then we can say that

$$s - \epsilon < a_N \leq a_n < s \leq s + \epsilon$$

which imply that

$$\begin{aligned} s - \epsilon < a_n < s + \epsilon \\ \implies |a_n - s| < \epsilon. \end{aligned}$$

Hence, it follows that any *monotone* and *bounded* sequence converges. ■

The key takeaway from this theorem is that we don't actually need to specify a value for a limit in order to show that it converges. As long as we have a monotone sequence and that we know it is bounded then we know for sure that the sequence converges.

Definition 2.3.2: Infinite Series

Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots.$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + ... + b_m = \sum_{i=1}^{m} s_i,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges* to B if the sequence (s_m) converges to B. In this case, we write

$$\sum_{n=1}^{\infty} b_n = B.$$

Example 2.3.1

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Because the terms in the sum are all positive, the sequence of partial sums are given by

$$s_m = \sum_{k=1}^{m} \frac{1}{k^2}$$

is increasing. Our goal is to show that this sequence is convergent so that the series converges. We proceed by using the Monotone Convergence Theorem to do this. Since we already have a monotone sequence of partial sums, only we need to do now find an upper bound for s_m. Observe that

$$\begin{aligned}
s_m = 1 &= \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \ldots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \ldots + \frac{1}{m(m-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{(m-1)} - \frac{1}{m}\right) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2.
\end{aligned}$$

The third second equality is found by taking the partial fractions of the line before it. Thus, we find that 2 is an upper bound for the sequence of partial sums, so we can conclude that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Example 2.3.2: The Harmonic Series Diverges

Let's consider the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums is defined as follows

$$s_m = \sum_{k=1}^{m} \frac{1}{k}.$$

Like our last example, we expect these sequence of terms to be bounded by 2 but upon further inspection, we have

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

which is not true. Similarly, we find that $s_8 > 2\frac{1}{2}$, and we can see that in general we have that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^k} + ... + \frac{2}{2^k}\right) \\
&= 1 + ... + \left(2^{k-1}\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2} \\
&= 1 + k\frac{1}{2}.
\end{aligned}$$

This shows that our sequence is unbounded because we found $M = 1 + k\left(\frac{1}{2}\right) > 0$ such that $s_k > M$. Despite how slow the sequence of partial of sums may be at reaching this point, it does end up surpassing every number on the postive real line. Since we have an unbounded sequence of partial sums, we conclude that the Harmonic series as divergent.

Theorem 2.3.2: Cauchy Condensation Test

Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=0}^{\infty} b_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges.

Proof. For the forwards direction, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. This means that the sequence of partial sums

$$t_k = b_1 + 2b_2 + ... + 2^k b_{2k}$$

are bounded. Hence, there exists $M > 0$ such that $t_k \leq M$ for all $k \in \mathbb{N}$. Our goal is to show that the sequence of partial sums for the series

$$\sum_{n=0}^{\infty} b_n.$$

Since $b_n \geq 0$ and that for all $n \in \mathbb{N}$ b_n decreasing, we have that the partial sums t_k is monotone. Our goal is to show that

$$s_m = \sum_{k=0}^{m} b_k$$

is bounded. Hence, fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$ and hence $s_m \leq s_{2^{k+1}-1}$ which imply that

$$\begin{aligned}
s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}-1}) \\
&\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\
&= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\
&= t_k
\end{aligned}$$

Hence, we have $s_m \leq s_{2^{k+1}-1} < t_k \leq M$ which means that (s_m) is bounded. By the Monotone Convergence Theorem, it follows that the series $\sum_{n=1}^{\infty} b_n$ converges. For the forwards direction, we proceed with contrapostive. Hence, assume for sake of contradiction that the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

is a divergent series. We want to show that the series

$$\sum_{n=0}^{\infty} b_n$$

is also a divergent series. ■

2.4 Subsequences and Bolzano-Weierstrass

In the last section, we observed that the convergence of partial sums of a particular series can be determined by the behavior of a subsequence of the partial sums.

Definition 2.4.1: Subsequences

Let $(a_n) \subseteq \mathbb{R}$, and let $n_1 < n_2 < n_3 < n_4 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5} \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

A few remarks about subsequences:

Example 2.4.1

(a) The order of the subsequence is the same as in the original sequence. If we have the sequence

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

then the subsequences

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$$

and

$$\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots\right)$$

are permitted.

(b) Repetitions and swapping are not allowed. Like

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, \ldots\right)$$

and

$$\left(1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$$

Since subsequences have the same ordering as the original sequence, one can conjecture about them converging to the same limit.

Theorem 2.4.1

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \to a$ and let (a_{n_k}) be a subsequence for (a_n). We want to show (a_{n_k}) converges to a as well. Since $(a_n) \to a$, there exists an N such that for any $n \geq N$, we have $|a_n - a| < \epsilon$. We claim that $n_k \geq k$ for any $k \in \mathbb{N}$. Let us proceed by inducting on k. Let the base case be $k = 1$. Since n_k is an *increasing* sequence of natural numbers, we see that $n_1 \geq 1$. Now let us assume $n_{k-1} \geq k - 1$. Since (a_{n_k}) in *increasing*, we have $a_k \geq a_{k-1} \geq k - 1$ which implies that $n_k \geq k$.
Since any choice of $n \geq N$, we can say that $n_k \geq k \geq N$. Hence, we have

$$|a_{n_k} - a| < \epsilon$$

which is what we desired. ■

Example 2.4.2

Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > ... > 0,$$

the sequence (b^n) is *decreasing* and *bounded* below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some ℓ satisfying $0 \leq \ell < b$. To compute ℓ, notice that (b^{2n}) is a subsequence, so $b^{2n} \to \ell$ by Theorem 2.5.2. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $b^{2n} \to \ell \cdot \ell = \ell^2$. Because limits are unique (Theorem 2.2.7), $\ell^2 = \ell$, and thus $\ell = 0$.

Example 2.4.3

Suppose we have an oscillating sequence of numbers

$$\Big(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\Big)$$

Note that this sequence does not converge to any proposed limit yet if we take a subsequence of it, weget a sequence that converges! Observe, that the subsequence

$$\Big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, ...\Big)$$

and

$$\Big(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, ...\Big)$$

converge to $1/5$ and $-1/5$ respectively. Since we have two subsequences that converge to two different limits, we immediately conclude that the original sequence diverges.

This leads us to our next theorem that states that

Theorem 2.4.2: Bolzano-Weierstrass

Every bounded sequence contains a convergent subsequence.

Proof. Let (a_n) be a *bounded* sequence. Then there exists $M > 0$ such that $a_n \in [-M, M]$. Suppose we divide this interval in half for k times: that is, let the length of the intervals be defined by the sequence $M(1/2)^{k-1}$. We claim that a subsequence (a_{n_k}) lies in either one of these intervals: that is, let $n_k > n_{k-1}$ for all $k \in \mathbb{N}$ such that $a_{n_k} \in I_k$.
Let us induct on k. Then let our base case be $k = 1$. Since we have an increasing sequence of natural numbers n_k, we have that $n_2 > n_1$ which means that $a_{n_2} \in I_2$ as well as $a_{n_1} \in I_1$. Now let us assume that this holds for all $k \leq \ell - 1$. We want to show that this holds for $k < \ell$. By the monotonicity of n_k, we have that $n_\ell > n_{\ell-1} > n_k > n_1$ which implies that $a_{n_\ell} \in I_\ell$ for all $\ell \in \mathbb{N}$. Furthermore, the sets

$$I_1 \subseteq I_2 \subseteq I_3 ...$$

form a nested sequence of closed intervals.
By the *Nested Interval Property*, we can conclude that there exists an $x \in I_k$ for all $k \in \mathbb{N}$ such that $\bigcup_{k=1}^{\infty} I_k \neq \emptyset$. Let $\epsilon > 0$. Since $a_{n_k}, x \in I_k$ for all $k \in \mathbb{N}$ and $M(1/2)^{k-1} \to 0$ by the Algebraic Limit Theorem, we can choose an $N \in \mathbb{N}$ such that for any $k \geq n_k \geq N$, we have

$$|a_{n_k} - x| < \epsilon.$$

Hence, $(a_{n_k}) \to x$. ■

2.5 The Cauchy Criterion

Definition 2.5.1: Cauchy Sequence

A sequence (a_n) is called a *Cauchy Sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|a_n - a_m| < \epsilon.$$

In the regular convergence definition, we are given any $\epsilon > 0$ where there is a point in the sequence $N \in \mathbb{N}$ such that past this point, all of our terms fall within an ϵ range around some limit point. In the Cauchy Criterion defintion, we begin with the same conditions but this time, all the terms of the sequence are all tightly packed together wthin the $\epsilon > 0$ range we were given. It turns out, that these two definitions are equivalent: that is, *Cauchy sequences* are convergent sequences and convergent sequences are *Cauchy sequences*.

Theorem 2.5.1

Every convergent sequence is a Cauchy sequence.

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, there must existsa point $N \in \mathbb{N}$ after which we can conclude that

$$|x_n - x_m| < \epsilon.$$

Let $\epsilon > 0$. Since $(x_n) \to x$, we can choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|x_n - x| < \frac{\epsilon}{2}, \tag{1}$$

$$|x_m - x| < \frac{\epsilon}{2}. \tag{2}$$

Consider $|x_n - x_m|$. Then (1) and (2) imply that

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &< |x_n - x| + |x - x_n| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence.* ∎

We can prove the other direction, by using either the *Bolzano Weierstrass Theorem* or the *Monotone Convergence Theorem.* This is a little bit more difficult since we need to have a proposed limit for the sequence to converge to.

Lemma 2.5.1

Cauchy sequences are bounded.

Proof. Given $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < 1$ for amm $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$ (just substituted $m = N$ here). Hence, define

$$M = \max\{|x_1|, |x_1|, |x_1|, ..., |x_{N-1}|, |x_N| + 1\}.$$

Therefore, $|x_n| < M$ for all $n \in \mathbb{N}$ Hence, the *Cauchy sequence* (x_n) is *bounded.* ∎

Theorem 2.5.2: Cauchy Criterion in $\mathbb{R}$

A sequence converges if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) This direction is just Theorem 2.6.2 which we have proved above.
($\Leftarrow$) Suppose (x_n) is a *Cauchy sequence.* Let $\epsilon > 0$. Since (x_n) is a *bounded* sequence, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$ by the *Bolzano Weierstrass Theorem.* Let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, every $n_k \geq N$ has the property

$$|x_{n_k} - x| < \epsilon.$$

Our goal now is to show that $(x_n) \to x$. Hence, consider $|x_n - x|$. Then for every $n, n_k \geq N$, we have

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &< |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $(x_n) \to x$. ∎

2.5.1 Completeness Revisited

We can summarize all of our results thus far in the following way

$$\text{AOC} \begin{cases} \text{NIP} \implies \text{BW} \implies \text{CC} \\ \text{MCT} \end{cases}$$

where AOC is our defining axiom to base all our reults on and giving us the notion that an ordered field contains no holes. We could also take the MCT to be our defining axiom and gives us the notion of least upper bounds by proving NIP. In addition, we could also take NIP to be our starting point but we need to have an extra hypothesis; that is, the Archimedean Property to prove all our results above (This is unavoidable).

It could be possible to assume the Arcimedean property holds, suppose one of the results we have proven is true, and derive the others yet this is sort of limited since $\mathbb{Q}$ contains a set that is not complete.

Below is the least of implications we can prove based on which theorem we would like to select asour defining axiom. Hence, we have

$$\text{NIP} + \text{Archimedean Property} \implies \text{AOC}$$

and

$$\text{BW} \implies \text{MCT} \implies \text{Archimedean Property}$$

2.6 Properties of Infinite Series

We have learned the convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n) where

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim s_n = A.$$

We called (s_n) the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$. Just like the *Algebraic Limit Theorem* for sequences, we can also do the same thing for series.

Theorem 2.6.1: Properties of Infinite Series

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof. Suppose $\sum_{k=1}^{\infty} a_k = A$ and let $c \in \mathbb{R}$. Define the sequence of partial sums of $\sum_{k=1}^{\infty} ca_k$ as

$$t_k = cs_n = ca_1 + ca_2 + ca_3 + ... + ca_n.$$

By the *Algebraic Limit Theorem*, we know that $\lim cs_n = cA$. Hence,

$$\sum_{k=1}^{\infty} ca_k = cA.$$

To prove the addition rule, suppose $\sum_{k=1}^{\infty} b_k = B$. We want to show that

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B.$$

Define the sequence of partial sums for the two series as the following:

$$t_k = a_1 + a_2 + ... + a_n,$$
$$u_k = b_1 + b_2 + ... + b_n$$

Since $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, their sequence of partial sums also converges to the same value. Hence, let $\lim t_k = A$ and $\lim u_k = B$. By the *Algebraic Limit Theorem*, the sum of these two limits also converges i.e

$$\lim(t_k + u_k) = \lim t_k + \lim u_k = A + B.$$

Hence,

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B$$

■

We can summarize this theorem by keeping in mind that we can perform distribution over infinite addition and that we can add two infinite series together.

Theorem 2.6.2: Cauchy Criterion for Series

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Proof. Let $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + ... + a_n| < \epsilon.$$

Suppose $\sum_{k=1}^{\infty} a_k$ converges. This is true if and only if the sequence of partial sums (t_k) converges. This is true if and only if (s_k) is *Cauchy* by the *Cauchy Criterion.* Hence, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$

$$|s_n - s_m| < \epsilon.$$

Note that

$$\begin{aligned} |s_n - s_m| &= |\sum_{k=m+1}^{\infty} a_k - \sum_{k=m}^{m} a_k| \\ &= |\sum_{k=m+1}^{n} a_k| \\ &= |a_{m+1} + ... + a_n| < \epsilon \end{aligned}$$

■

This gives us the opportunity to prove some basic facts about series.

Theorem 2.6.3

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof. From the last theorem, we note that for every $\epsilon > 0$ such that whenever $n \geq m \geq N$, we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{\infty} a_k - 0 \right| < \epsilon$$

implies that $(a_n) \to 0$. ■

Keep in mind that the converse of this statement is not true! Just because (a_k) tends to 0 does not immediately imply that the series converges!

Theorem 2.6.4: Comparison Test

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then we have

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let us show part (i). Suppose $\sum_{k=1}^{\infty} b_k$ converges. We want to show that $\sum_{k=1}^{\infty} a_k$ converges. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n > m \geq N$ and the fact that $a_k \leq b_k$ for all $k \in \mathbb{N}$

$$\Big| \sum_{k=m+1}^{n} a_k \Big| \leq \Big| \sum_{k=m+1}^{n} b_k \Big|$$
$$< \epsilon.$$

Hence, a_k converges as well.
Note that part (ii) is just the contrapositive of part (i) which is also true. ■

Note that the convergence of sequences and series are relatively immutable when it comes to changes in some finite number of initial terms: that is, the behavior of sequences and series can be found past some choice of $N \in \mathbb{N}$. In order for the above test to be of any use to us, it is important to have a few examples under our belt i.e any $p > 1$ implies that

$\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

Example 2.6.1

A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots .$$

If $r = 1$ and $a \neq 0$, the series diverges. We can use the following algebraic identity, for $r \neq 1$, to write the following:

$$(1-r)(1+r+r^2 \ldots + r^{m-1}) = 1 - r^m$$

which allows us to rewrite the partial sum (s_m) of the above series to say that

$$s_m = a + ar + ar^2 + ar^3 + \ldots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

where $s_m = at_m$ where

$$t_m = 1 + r + r^2 + \ldots + r^{m-1}$$

is a convergent sequence. Using the *Algebraic Limit Theorem*, therefore, allows us to say that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

The next theorem is a modification of the *Comparison Test* to handle series that contain negative terms.

Theorem 2.6.5: Absolute Convergence Test

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof. Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. We want to show that $\sum_{n=1}^{\infty} a_n$ converges as well. Let $\epsilon > 0$. By the *Cauchy Criterion* for series, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$, we have

$$\begin{aligned} \left| \sum_{k=m+1}^{n} a_k \right| &\leq \sum_{k=m+1}^{n} |a_k| \\ &< \epsilon. \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges. ■

Note that the converse of the above statement is false as taking the absolute value of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

produces the regular harmonic series which *diverges*.

Theorem 2.6.6: Alternating Series Test

Let (a_n) be a sequence satisfying,

(i) $a_1 \geq a_2 \geq a_3 ... \geq a_n \geq a_{n+1} \geq ...$ and

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. See exercise 2.7.1 for proof ■

Definition 2.6.1: Absolute Convergence and Conditional Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.* If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

We can chart a few examples of some *conditionally convergent* series and *absolutely convergent* series.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \implies$ *conditionally convergent*
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \sum_{n=1}^{\infty} \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \implies$ *converges absolutely*

This tells us that any convergent series with positive terms must converge absolutely.

2.6.1 Rearrangements

We can obtain a rearrangement of an infinite series by permuting terms in the sum in some other order. In order for a sum to be a valid rearrangement, all the terms must appear and there should be no repeats.

Definition 2.6.2

Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a *bijective* function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

We can now explain the weird behavior for why the *harmonic series* converges to a different limit when rearranging the terms; that is, it is because the *harmonic series* is a *conditionally convergent* series which leads us to the next theorem.

Theorem 2.6.7: Rearrangement of Series

If a series converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k=1}^{\infty} a_k$ *converges absolutely* to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let us define the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ as

$$s_n = \sum_{k=1}^{n} a_k$$

and the sequence of partial sums for the rearranged series $\sum_{n=1}^{\infty} b_n$ as

$$t_m = \sum_{k=1}^{m} b_k.$$

Since $\sum_{n=1}^{\infty} a_n$ *converges absolutely*, let $\epsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|s_n - A| < \frac{\epsilon}{2}$$

as well some $N_2 \in \mathbb{N}$ such that whenever $n > m \geq N_2$, we have

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}.$$

All that is left to do is to set a point in the sequence of the rearranged series where our ultimate goal is to have $|t_m - A| < \epsilon$. Hence, define

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

Let $m \geq M$ such that, when using the *triangle inequality*, we get

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, we have that $\sum_{n=1}^{\infty} b_n$ converges to A. ■

2.7 Double Summations and Products

We discovered in an earlier section that given any doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbb{N}\}$, it can be an ambiguous task to define

$$\sum_{i,j=1}^{\infty} a_{ij}. \tag{1}$$

We also observed that performing *iterated summations*can lead to different summations. Of course, this can be avoided completely if we were to define the partial sum of (1) in the following way

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

for $m, n \in \mathbb{N}$. In order for the sum of (1) to converge we have to have the following hold:

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{mn}$$

Exercise 2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{mn}$. How does this value compare to the two iterated values for the sum already computed?

The double summation from section 2.1 is $a_{ij} = \frac{1}{2^{j-i}}$ where $\{a_{ij} : i, j \in \mathbb{N}\}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

Proof. To find $\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{mn}$, we first need to define the sequence of partial sums. We can fix j (the rows of the matrix) and define the sequence of partial sums for the series $\sum_{i,j=1}^{\infty} a_{ij}$ as

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}}\right) = -2 + \frac{1}{2^{n-1}}$$

which taking the limit leads to

$$\lim_{n\to\infty} \left(-2 + \frac{1}{2^{n-1}}\right) = -2.$$

■

The issue of rearrangements to an infinite series arises due to commutativity of addition in an infinite context. It was found that having an absolutely converging infinite series fixes this problem.

Exercise 2.8.2

Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some $b_i \in \mathbb{R}$, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \tag{1}$$

converges. This means that the (1) meets the *Cauchy Criterion*. Let $\epsilon > 0$. This implies that there exists $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| < \epsilon.$$

Consider $\left|\sum_{(i,j)\in A(m,n)} a_{ij}\right|$ where

$$A(m,n) = \{(i,j) : 1 \leq i \leq j \leq n\}.$$

Using the *Triangle Inequality*, we find that j

$$\begin{aligned}\Big|s_{mm} - s_{nn}\Big| &= \Big|\sum_{(i,j)\in A(m,n)} a_{ij}\Big| \\ &\leq \sum_{(i,j)\in A(m,n)} |a_{ij}| \qquad (2) \\ &< \epsilon.\end{aligned}$$

Since (2) meets the *Cauchy Criterion* for series, we know that $\sum_{m,n}^{\infty} a_{ij}$ must be *Cauchy* and thus must converge as well. ■

Another proof using the Comparison Test goes something like this

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges. This means that for each $i \in \mathbb{N}$ the infinite series

$$\sum_{j=1}^{\infty} a_{ij} = r_i$$

for some $r_i \in \mathbb{R}$. Hence, we have the infinite series

$$\sum_{i=1}^{\infty} r_i. \tag{1}$$

Our goal is to show that (1) converges. Suppose we look at the terms

$$|r_i| = \Big|\sum_{j=1}^{\infty} a_{ij}\Big|.$$

Note by the *Triangle Inequality* that

$$\sum_{i=1}^{\infty}|r_i| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|.$$

by assumption the infinite series to the right converges. Hence, the series to the left must also converge by the *Comparison Test*. Since $\sum|r_i|$ converges, then the series $\sum r_i$ converges by the *Absolute Convergence Test* . ■

Theorem 2.7.1

Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover, we have that

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1) , define

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$

■

Exercise 2.8.3

(a) Prove that (t_{nn}) converges.

Proof. From our definition of t_{nn} above we have

$$t_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

We want to show for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|t_{nn} - L| < \epsilon$. By assumption, we know that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \tag{1}$$

converges absolutely which implies that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

converges. Note that $t_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \to s_n = \sum_{i=1}^{\infty} r_i$ for some $r_i \in \mathbb{R}$. Furthermore, we have $s_n \to L$ since (1) converges. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have that

$$\begin{aligned} |t_{nn} - L| &= |t_{nn} - s_n + s_n - L| \\ &\leq |t_{nn} - s_n| + |s_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, the sequence of partial sums (t_{nn}) converges. ■

Another way we can prove this is to use the Monotone Convergence Theorem.

Proof. Our goal is to show that (t_{nn}) is converges to L. That is, our goal is to show that (t_{nn}) is bounded and monotone. We know that (t_{nn}) is monotone since all t_{nn} are non-negative terms and that $\sum_{n,m}^{\infty} |a_{ij}| = L$ where $L \geq 0$. To show that (t_{nn}) is bounded note that

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Hence, (t_{nn}) is a bounded sequence. By the Monotone Convergence Theorem, (t_{nn}) converges. ∎

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Proof. Suppose (t_{nn}) is a Cauchy Sequence. Then for some $N \in \mathbb{N}$ we have that for any $n \geq m > N$

$$|t_{nn} - t_{mm}| < \epsilon.$$

We can rewrite this in the following way to say that

$$|\sum_{n,m} t_{ij}| < \epsilon.$$

Our goal is to show that

$$|s_{nn} - s_{mm}| < \epsilon.$$

Hence, for any $n \geq m > N$, we have that

$$\begin{aligned} |s_{nn} - s_{mm}| &\leq |t_{nn} - t_{mm}| \\ &= \Big|\sum_{n,m} t_{ij}\Big| \\ &< \epsilon. \end{aligned}$$

Hence, (s_{nn}) converges. ∎

Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Exercise 2.8.4

(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

Proof. Since (t_{mn}) bounded, we can say that $t_{mn} \leq B$. Since the set

$$\{t_{mn} : m, n \in \mathbb{N}\}$$

is bounded above and non-empty, we also have that $B = \sup\{t_{mn} : m, n \in \mathbb{N}\}$ exists. Hence, for any $\epsilon > 0$, we have that $B - \frac{\epsilon}{2}$ is not an upper bound. Hence, there exists some $t_{n_0 m_0}$ such that $B - \frac{\epsilon}{2} < t_{m_0 n_0} \leq t_{mn}$. Furthermore, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq m > N_1$ since (t_{mn}) converges. Hence, we must have that $B - \frac{\epsilon}{2} < t_{mn} \leq B$ ∎

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Proof. Consider $|s_{mn} - S| < \epsilon$. Since $(s_{nn}) \to S$, let $\epsilon > 0$ such that for some $N_2 \in \mathbb{N}$ we have $n \geq m > N_2$, we have

$$|s_{nn} - S| < \frac{\epsilon}{2}.$$

Since (s_{nn}) meets the Cauchy Criterion, we have that there exists $N_2 \in \mathbb{N}$ such that for any $n \geq m > N$, we have

$$|s_{nn} - s_{mn}| < \frac{\epsilon}{2}.$$

Hence, observe that for any $n \geq m > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} - s_{nn} + s_{nn} - S| \\
&\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, we have that $(s_{mn}) \to S$. ∎

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 2.8.5

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + ... + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

Proof. By exercise 2.8.4, we know that $s_{mn} \to S$. Note that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} r_i \text{ for each } i.$$

Hence, we have

$$\lim_{m,n\to\infty} s_{mn} = \lim_{m\to\infty} \sum_{i=1}^{m} r_i = S$$

which is equivalent to saying that for all $m > N$ for some $N \in \mathbb{N}$ we have that

$$\left|\left(\sum_{i=1}^{m} r_i\right) - S\right| \leq \epsilon.$$

■

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} a_{ij}$ converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Proof. Using the same process above for summing up the columns of $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ leads to

$$\left|\left(\sum_{j=1}^{n} c_j\right) - S\right| \leq \epsilon.$$

Hence, we must have that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

■

Another way of computing double sums is to sum along the diagonals of a rectangular matrix. Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array where

$$d_2 = a_{11}, d_3 = a_{12} + a_{21}, d_4 = a_{13} + a_{22} + a_{31}$$

and in general

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$

Then, $\sum_{k=2}^{\infty} d_k$ represents another reasonable way of summing over every a_{ij} in the array.

Exercise 2.8.6

(a) Assuming the hypothesis and hence the conclusion of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

Proof. Our goal is to show that $\sum_{k=1}^{\infty} |d_k|$ converges. Since

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}$$

we can define the sequence of partial sums for $\sum_{k=1}^{\infty} |d_k|$ as

$$\sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} \Big|. \tag{1}$$

We know by Theorem 2.8.1 that the series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges. Hence, we can write

$$\sum_{k=2}^{n} |d_k| = \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} \Big| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

using the Triangle Inequality. By the Comparison Test, we must have that (1) converges as well. ■

2.7.1 Products of Series

We can take the product of two series by doing the following algebra below:

$$\begin{aligned} \Big(\sum_{i=1}^{\infty} a_i\Big)\Big(\sum_{j=1}^{\infty} b_j\Big) &= (a_1 + a_2 + a_3 + \dots)(b_1 + b_2 + b_3 \dots) \\ &= a_1b_1 + (a_1b_2 + a_2b_1) + (a_3b_1 + a_2b_2 + a_1b_3) + ... \\ &= \sum_{k=2}^{\infty} d_k \end{aligned}$$

where

$$d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-2} b_1.$$

Exercise 2.8.7

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

Proof. Our goal is to show that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j$ converges absolutely. Observe that

$$\begin{aligned}\sum_{i=1}^{\infty}\Big|\sum_{j=1}^{\infty} a_ib_j\Big| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j| && (1)\\ &= \sum_{i=1}^{\infty}|a_i|\Big(\sum_{j=1}^{\infty}|b_j|\Big)\\ &= \sum_{i=1}^{\infty}|a_i|\cdot|B| && (2)\end{aligned}$$

Since (2) converges absolutely by the Algebraic Series Theorem, we have that (1) converges. converges. ■

(b) Let $s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1$.

Proof. We can show that (t_{nn}) converges via the Monotone Convergence Theorem where

$$t_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_jj|.$$

Observe that all the terms of (t_{nn}) are positive and increasing. Now all we need to show is that (t_{nn}) is bounded. Since $\sum_{i=1}^{n}|a_i| \leq M$ and $\sum_{j=1}^{\infty}|b_i| \leq L$ for some $M, L \in \mathbb{R}$, we have that

$$\begin{aligned}t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_ib_j| &\leq \sum_{i=1}^{\infty}|a_j|\sum_{j=1}^{\infty}|b_j|\\ &\leq M\cdot L.\end{aligned}$$

Hence, (t_{nn}) is a bounded sequence of partial sums. Now by theorem 2.8.1, we can say that

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = AB.$$

■

Chapter 3

Basic Topology of The Real Numbers

3.1 Discussion: The Cantor Set

The following construction demonstrates that $\mathbb{R}$ is an uncountable set. Let C_0 be the closed interval $[0, 1]$, and define C_1 to be the set that results when we remove an open set in the middle third; that is,

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

We can construct the next iteration C_2 in a similar way above of each of the two sets unioned above. Hence, we have

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

or

$$C_n = [0, 1] \setminus \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{8}, \frac{8}{9}\right) \cup \dots\right]$$

If we continue this process inductively, then for each $n \in \mathbb{N}$, we get sets C_n consisting of 2^n closed intervals with each having a length of $1/3^n$. The Cantor set C is just the intersection of an infinite number of C_n; that is,

$$C = \bigcup_{n=0}^{\infty} C_n.$$

3.2 Open and Closed Sets

Recall that given any $\epsilon > 0$, the $\epsilon-$neighborhood of $a \in \mathbb{R}$ is the set

$$V_\epsilon = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

In other words, we have an open interval $(a - \epsilon, a + \epsilon)$ or $a - \epsilon < x < a + \epsilon$ centered at a with radius ϵ.

Definition 3.2.1: Open Sets

A set $A \subseteq \mathbb{R}$ is *open* if for all points $a \in A$ there exists an $\epsilon-$neighborhood $V_\epsilon(a) \subseteq A$.

Example 3.2.1

(i) The set $\mathbb{R}$ is an *open* set because for any $a \in \mathbb{R}$, we can pick a $\epsilon-$neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \subseteq \mathbb{R}$.

(ii) The empty set $\emptyset$ is an open subset of the real line. This statement is vacuously true due to the definition of an open *set* i.e this set has no interior points to consider so it is true by default.

(iii) Take any $c, d \in \mathbb{R}$ and create an open interval as such where

$$(c, d) = \{x \in \mathbb{R} : c < x < d\}.$$

To see why (c, d) is an *open* set, let $x \in (c, d)$ be an arbitrary point. Let $\epsilon = \min\{x - c, d - x\}$, then we can construct the following $\epsilon-$neighborhood where

$$V_\epsilon = \{x' \in \mathbb{R} : |x' - x| < \epsilon\}.$$

Theorem 3.2.1

(i) The union of an arbitrary collection open sets is open.

(ii) The intersection of a finite collection of open sets is open.

Proof. To prove (i), define $\{O_\lambda : \lambda \in A\}$ be a collection of open sets and let $O = \bigcup_{\lambda \in A} O_\lambda$. Let a be an arbitrary element of O. In order to show that O is *open*, we need to show that $V_\epsilon(a) \subseteq O$ where $V_\epsilon(a)$ is the $\epsilon-$neighborhood. Let $a \in O_\lambda$ be an arbitrary element. Since we have a collection of open sets

$$\{O_\lambda : \lambda \in A\}$$

we can create a $\epsilon-$neighborhood around $a \in O_\lambda$ for some $\lambda \in A$ such that $V_\epsilon(a) \subseteq O_\lambda$. But note that $O_\lambda \subseteq O$. Hence, we have that $V_\epsilon(a) \subseteq O = \bigcup_{\lambda \in A} O_\lambda$. Hence, O is an *open* set.
To prove (ii), suppose $O = \bigcap_{i=1}^{N} O_i$. Suppose $a \in O_i$ for all $1 \leq i \leq N$ where O_i is a collection of open sets. Hence, there exists an $\epsilon-$neighborhood for every O_i. We need only one value of ϵ to make this work so define $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, ...\epsilon_N\}$. This means that

$$V_{\epsilon_i}(a) \subseteq V_\epsilon(a) \subseteq O_i \subseteq O$$

Hence, we have

$$V_\epsilon(a) \subseteq \bigcap_{i=1}^{N} O_i.$$

■

3.2.1 Closed Sets

Definition 3.2.2: Limit Points

A point x is a *limit point* of a set A if every $\epsilon-$neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

In other words, we have the following intersection

$$x \notin V_\epsilon(x) \cap A.$$

This is another way of saying that a sequence approaches of values approaches the limit point x where $V_\epsilon(x)$ can be thought of as neighborhoods "clustering" around the point x.

Theorem 3.2.2

A point x is a limit point of a set A if and only of $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Proof. ($\Rightarrow$) Let $V_\epsilon(x)$ be an $\epsilon-$neighborhood around x. We want to show that $\lim a_n = x$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$. By definition,

$$V_\epsilon(x) = \{x \in \mathbb{R} : |a_n - x| < \epsilon\}.$$

Let $\epsilon = \frac{1}{n}$. Since x is a *limit point*, for each $n \in \mathbb{N}$, we can pick any point

$$a_n \in V_{1/n}(x) \cap A.$$

Then we have

$$|a_n - x| < \frac{1}{n}$$

which is equivalent to

$$x - \frac{1}{n} < a_n < x + \frac{1}{n}.$$

By the Algebraic limit theorem and Squeeze Theorem, we have that $(a_n) \to x$ where $a_n \neq x$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Suppose $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$. We want to show the converse. Let $\epsilon > 0$. Then By definition of $\lim a_n = x$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|a_n - x| < \epsilon.$$

But this is also the definition of an $\epsilon-$neighborhood. Hence, $a_n \neq x$ for all $n \in \mathbb{N}$ and $x \in A$ is a limit point implies

$$V_\epsilon(x) \cap A$$

for all $\epsilon-$neighborhoods. ∎

Keep in mind that $a \in A$ means that there is a sequence in A such that $a_n = a, a, a, ...$ which is uninteresting for the most part. We can distinguish *limit points* from *isolated points*.

Definition 3.2.3: Isolated Points

A point $a \in A$ is an *isolated point* of A if it is not a *limit point* of A.

Remember that an isolated point is always in the set A, but a limit point can be sometimes be outside of the set A. An example of this is the endpoint of an open interval. A sequence can approach the endpoint where $a_n \neq x$ for all $n \in \mathbb{N}$ but x is not in the set.

Definition 3.2.4: Closed Sets

A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

In other words, can say that a set A is closed if sequences contained in A converge to their limits that are within the set A.

Theorem 3.2.3

A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof. Suppose $F \subseteq \mathbb{R}$ is closed. Let $x \in F$ be a limit point. Let (x_n) be a Cauchy sequence contained in F. By the Cauchy Criterion, (x_n) converges to $x \in F$. ∎

Example 3.2.2

(i) Consider the set

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Let's show that each point of A is isolated. We can show that each point of A is isolated. Given $\frac{1}{n} \in A$. Choose $\epsilon = \frac{1}{n} - \frac{1}{(n+1)}$. Then,

$$V_\epsilon(1/n) \cap A = \left\{\frac{1}{n}\right\}.$$

It follows from Definition 3.2.4 that $\frac{1}{n}$ is not a limit point and so is isolated. Although all of the points of A are isolated, the set A does have only one limit point 0. The reason for this is can be explained by the very definition of A where $0 \notin A$. Since the limit of A is not contained in A, we can say that A is not closed. The set $F = A \cup \{0\}$ is an example of a closed set and is called the closure of A.

(ii) Let's prove that a closed interval

$$[c, d] = \{x \in \mathbb{R} : c \leq c \leq d\}$$

is a closed set using Definition 3.2.7. If x is a limit point of $[c, d]$, then by Theorem 3.2.5 there exists $(x_n) \subseteq [c, d]$ with $(x_n) \to x$. Since $(x_n) \to x$, we can use the Order Limit Theorem to say that

$$c \leq x_n \leq d \iff c \leq x \leq d.$$

This means $x \in [c, d]$ which proves that $[c, d]$ is a closed set.

(iii) Consider the set $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers. An interesting property of $\mathbb{Q}$ is that all of its limit points is actually all of $\mathbb{R}$. To see why this is so, let us have $y \in \mathbb{R}$ be arbitrary and construct $V_\epsilon(y)$ such that we have the open set $(y - \epsilon, y + \epsilon)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $x \in \mathbb{Q}$ where $x \neq y$ such that $x \in (y - \epsilon, y + \epsilon)$. Hence, y is a limit point of $\mathbb{Q}$.

We can actually restate the Density Property from the first chapter by saying the following:

Theorem 3.2.4

For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.

Proof. Let $y \in \mathbb{R}$ and let $\epsilon = \frac{1}{n}$. Create the following $\epsilon-$neighborhood $(y - \frac{1}{n}, y + \frac{1}{n})$. Since the end points of this $\epsilon-$neighborhood are real numbers, we can find a sequence of rational numbers $(x_n) \subseteq (y - \epsilon, y + \epsilon)$ by the Density of $\mathbb{Q}$ in $\mathbb{R}$ such that

$$y - \frac{1}{n} < x_n < y + \frac{1}{n}.$$

By the Squeeze Theorem, we can write that $(x_n) \to y$ where $x_n \neq y$ for all $n \in \mathbb{N}$. ■

3.2.2 Closure

Definition 3.2.5

Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be the $\bar{A} = A \cup L$.

Example 3.2.3

(i) Consider $A = \{1/n : n \in \mathbb{N}\}$, then the *closure of* A is just

$$\bar{A} = A \cup \{0\}.$$

(ii) In the last example, $y \notin (y - \epsilon, y + \epsilon)$ where $y \in \mathbb{R}$ gurantees that the closure of $\mathbb{Q}$ in $\mathbb{R}$; that is, $\bar{\mathbb{Q}} = \mathbb{R}$.

(iii) If A is an open interval (a, b), then the closure is just $\bar{A} = [a, b]$; that is, $\bar{A} = A \cup \{a, b\}$ where $a, b \in \mathbb{R}$ are the endpoints of the set (a, b).

(iv) If A is a closed interval then the closure is just $\bar{A} = A$. The obvious conclusion from this is that closed intervals are always closed sets.

Theorem 3.2.5

For any $A \subseteq \mathbb{R}$, the closure of $\bar{A}$ is a closed set and is the smallest closed set containing A.

Proof. Since L is the set of limit points of A, it follows immediately that $\bar{A}$ contains its limit points of A. The problem here is that taking the union of A and L could produce some new limit points.

The details are in exercise 3.2.7

Hence, any closed set containing A must contain L as well. Hence, we have $\bar{A} = A \cup L$ is the smallest closed set containing A. ■

3.2.3 Complements

The notions of open and closed imply that they are not antonyms of each other. Just because a set is not open, does not immediately imply that it is closed. We can see this in action by considering the half-open interval

$$(c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$$

as being neither open nor closed. Furthermore, $\mathbb{R}$ and $\emptyset$ are both simultaneously open and closed at the same time. Luckily, these are the only two sets that exhibit this confusing property. We do have a relationship between open and closed sets however.

Recall that the complement of a set $A \subseteq \mathbb{R}$ is defined to be the set

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

which describes all of the elements that are not in A.

Theorem 3.2.6

A set A is open if and only if A^c is closed. Likewise, a set B is closed if and only if B^c is open.

Proof. Suppose $A \subseteq \mathbb{R}$ is an open set. We want to show that A^c is a closed set. Let x be a limit point of A^c. Hence, there exists a sequence (x_n) such that $\lim x_n = x$ where $x_n \neq x$ for all $n \in \mathbb{N}$. By definition of $\lim x_n = x$, there is an $\epsilon-$neighborhood $V_\epsilon(x)$, but this means that $x \notin A$ and must be in A^c since every $\epsilon-$neighborhood of x intersects A at some point other than x. Hence, we have $x \in O^c$.
For the converse statement, we assume A^c is a closed set. We want to show that A is open. Hence, let $x \in A$. Since $x \in A$, x is not a limit point of A^c and A^c is a closed set, there must exist an $\epsilon-$neighborhood such that $x \notin V_\epsilon(x) \cap A^c$. This means $x \in A$ and so $V_\epsilon \subseteq A$. Hence, A is an open set. The second statement follows quickly when taking the complement of each going in each direction. ∎

Theorem 3.2.7

(i) The union of a finite collection of closed sets is closed.

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ it is true that

$$\Big(\bigcup_{\lambda \in \Lambda} E_\lambda\Big)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c \quad \text{and} \quad \Big(\bigcap_{\lambda \in \Lambda} E_\lambda\Big)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c.$$

∎

3.3 Compact Sets

3.3.1 Compactness

Definition 3.3.1

A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.

Example 3.3.1

Closed intervals are compact since all (a_n) in K are bounded and so we can always find a subsequence in K (By Bolzano-Weierstrass) that converges to a limit that is contained within the closed interval. We know the limit is contained in K since closed intervals are closed sets.

In the example above, we used the fact that bounded sequences contain subsequences that converge and the fact that their limits are contained since closed intervals are closed sets.

Definition 3.3.2: Bounded Sets

A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 3.3.1: Characterization of Compactness in $\mathbb{R}$

A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. Assume K is a compact set. Suppose for sake of contradiction that K is not a bounded set. Our goal is to construct a sequence (a_n) that diverges. Since K is not bounded, for all $M > 0$, there exists $a_n \in A$ such that $|a_n| > M$. But by assumption, K is compact so (a_n) must contain a subsequence (a_{n_k}) that converges to a limit in K. But since (a_{n_k}) is unbounded, we have a contradiction. Hence, K must be a bounded set.
Now we will show that K is closed. Since K has to be bounded, then $(a_n) \subseteq K$ must contain a subsequence (a_{n_k}) that converges to a limit in K. But this is the definition of a closed set. Hence, K is a closed set.
Assume that K is closed and bounded. Let (a_n) be an arbitrary sequence in K. Since K is bounded and hence (a_n) is bounded, (a_n) contains a subsequence (a_{n_k}) such that $(a_{n_k}) \to a$. Since K is closed, a is contained in K. Hence, we have that K is a compact set. ■

It is important remember that closed intervals are not all that is when considering compact sets. The structure is much more intricate and interesting. For example, we can say that the Cantor Set is compact. We can experiment with this new idea of compact sets with the Nested Interval Property from chapter 1.

Theorem 3.3.2: Nested Compact Set Property

If

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Proof. Assume $K_n \neq \emptyset$ compact for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, choose $x_n \in K_n$ where x_n is a sequence of points. Since we have a nested sequence of nonempty sets, it follows that $x_n \in K_1$. By definition of compactness, (x_n) contains a convergent subsequence (x_{n_k}) such that $\lim x_{n_k} = x$ where $x \in K_n$ for every $n \in \mathbb{N}$ and thus $x \in K_1$. Given $n_0 \in \mathbb{N}$, we have that the terms of the sequence x_n are contained within K_{n_0} given all $n \geq n_0$. We can ignore the finite number of terms for which $n_k < n_0$ so that (x_{n_k}) can be contained in K_{n_0}. Hence, we have that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, we have that $x \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. ■

3.3.2 Open Covers

In the last section, we proved that compact sets to be bounded and closed and vice versa. In other cases, we could also have defined compacts in this way and then proved that for every sequences that is bounded, there exists subsequences that converge to limits contained within the set. We can prove compactness in terms of open covers and finite subcovers.

Definition 3.3.3

Let $A \subseteq \mathbb{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.$$

Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Subcollection in this context is just a collection of sets that are subsets of the original collection of open sets.

Example 3.3.2

Consider the open interval $(0, 1)$. For each point $x \in (0, 1)$, let O_x be the open interval $(x/2, 1)$. Let the infinite collection of O_x be defined as

$$\{O_x : x \in (0, 1)\}$$

forms an open cover for the open interval $(0, 1)$; that is,

$$(0, 1) \subseteq \bigcup_{x \in (0,1)} O_x.$$

Note that it is impossible to find a finite subcover for the open set $(0, 1)$. Given any proposed finite subcollection

$$\{O_{x_1}, O_{x_2}, ..., O_{x_n}\},$$

let $x' = \min\{x_1, x_2, \ldots, x_n\}$ and observe that for any $y \in \mathbb{R}$ satisfying $0 < y \leq x'/2$ is not contained in the union $\bigcup_{i=1}^{n} O_{x_i}$.

Now consider a similar cover for the closed interval $[0, 1]$. For $x \in (0, 1)$, the sets $O_x = (x/2, 1)$

do relatively well to cover $(0, 1)$, but in order to have an open cover for the closed interval $[0, 1]$, we could let $\epsilon > 0$ so that we can have epsilon neighborhoods covering both endpoints. That is, we have $O_o = (-\epsilon, \epsilon)$ and $O_1 = (1 - \epsilon, 1 + \epsilon)$. Then the collection

$$\{O_0, O_1, O_x : x \in (0, 1)\}$$

is an open cover for $[0, 1]$ is a finite subcover for the closed interval $[0, 1]$.

Theorem 3.3.3: Heine-Borel Theorem

et K be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others.

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover for K has a finite subcover.

Proof. Notice that the proof of the equivalence of (i) and (ii) has already been proven in Theorem 3.3.1. All we need to show now is that (iii) implies (ii) and (iii) implies (i).
To show (ii), we must show that K is both bounded and closed. To show that K is bounded, let us construct an open cover for K by defining O_x to be an open interval of radius 1 ($\epsilon = 1$) for each $x \in K$. This means there exists a $\epsilon-$neighborhood for each $x \in K$; that is, $O_x = V_1(x)$. Since the open cover $\{O_x : x \in K\}$ contains a finite subcover for K, we have the K contained in the union of the collection of sets $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Hence, K itself is a bounded set.
Now let us show that K is closed. Let (y_n) be a Cauchy sequence contained in K with $\lim y_n = y$. To show that K is closed, we must show that $y \in K$. Suppose for sake of contradiction that $y \notin K$. By assumption we can construct an open cover by taking O_x to be an interval of radius $|x - y|/2$ around each point $x \in K$. Also, we are assuming that the open cover $\{O_x : x \in K\}$ for K contains a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. If $y \notin K$, then the distance from y to each $x_i \in K$ must be

$$\epsilon_0 := \min\left\{\frac{|x_i - y|}{2} : 1 \leq i \leq n\right\}.$$

Since (y_n) is a Cauchy sequence, so it must converge. Hence, for some $N \in \mathbb{N}$, we know that

$$|y_N - y| < \epsilon_0$$

we must have for every $n \geq N$, But note that since $y \notin K$, not all of the terms from the sequence (y_n) for every $n \geq N$ that is contained in K are not included in the finite subcover

$$\bigcup_{i=1}^{n} O_{x_i}.$$

Hence, our finite subcover does not actually cover all of K which is a contradiction and thus we must have $y \in K$. ■

3.4 Perfect Sets

Definition 3.4.1

A set $P \subseteq \mathbb{R}$ is *perfect* if it is closed and contains no isolated points.

An straightforward example of perfect sets are closed intervals and singleton sets.

Example 3.4.1

It is not too hard to see that the Cantor set from the very beginning of this chapter is perfect. We defined

$$C = \bigcap_{n=0}^{\infty} C_n$$

where each C_n is a finite union of closed intervals. We know by Theorem 3.2.14 that each C_n is closed, and as a result of using the same theorem that C is closed as well. Now all we need to show is that C contains no isolated points.

Let $x \in C$ be arbitrary. Let us construct a sequence (x_n) of points in C that are different from x such that $(x_n) \to x$. We know that C contains endpoints of each interval that make up each C_n. In exercise 3.4.3, we sketch the argument that these are all that is needed to construct such an (x_n).

An argument for uncountability of the Cantor set.

Theorem 3.4.1

A nonempty perfect set is uncountable.

Proof. Suppose P is a set that is perfect and nonempty. Hence, it must be the case that P is an infinite set because otherwise it would only consist of isolated points. Assume for sake of contradiction that P is countable. Thus, we can define P as the following:

$$P = \{x_1, x_2, x_3 \dots\},$$

where every element of P appears on this list. Our goal is to construct a sequence of nested compact sets K_n that is all contained within P with the property that $x_1 \notin K_2$, $x_2 \notin K_3$, $x_3 \notin K_4$ and so on. Before proceeding with our argument, we must be sure that, in fact, each K_n is nonempty. Hence, we use the nested Compact interval theorem to produce

$$x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$$

that cannot be on the list $\{x_1, x_2, x_3, \dots\}$.
Let I_1 be a closed interval such that $x_1 \in (I_0)^\circ$; that is, x_1 is not an endpoint of I_1. This produces an x_1 that is not isolated which means there exists some other point, say, $y_2 \in P$ such that $y_2 \in (I_1)^\circ$. Around y_2 we can construct a closed interval such that $I_2 \supseteq I_1$ with the condition that $x_1 \notin I_2$. Let $\epsilon > 0$, then if $I_1 = [a, b]$ we can define

$$\epsilon = \min\{y_2 - a, b - y_2, |x_1 - y_2|\}.$$

Then, the interval $I_2 = \{[y_2 - \epsilon/2, y_2 + \epsilon/2]\}$ has the desired properties. We can continue this process indefinitely.
Since $y_2 \in P$ is not isolated, there must exists another point $y_3 \in P$ in the interior of I_2 such that $y_3 \neq x_2$. Again, construct a closed interval centered on y_3 with an ϵ small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$ because this intersection contains at least y_3.
We find that when we carry out this construction inductively, we have a sequence of closed intervals I_n satisfying the following properties:

(i) $I_{n+1} \subseteq I_n$,

(ii) $x_n \notin I_{n+1}$, and

(iii) $I_n \cap P \neq \emptyset$.

To finish the proof, let $K_n = I_n \cap P$. For each $n \in \mathbb{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded since it is contained in the bounded sets I_n. Hence, K_n is compact. We can also see that K_n is nonempty and $K_{n+1} \subseteq K_n$. By employing the Nested Compact Set property, we can conclude that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

But we find that each $K_n \subseteq P$ where $x_n \notin i_{n+1}$ leads to the conclusion that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, which is a contradiction.

■

3.4.1 Connected Sets

Consider the two open intervals $(1,2)$ and $(2,5)$. Notice that these two intervals have the limit point $x=2$ in common. However, there is some space between them in the sense that 2 isn't contained in the other. Another way to say this is that $\overline{(1,2)} \cap (2,5) = \emptyset$ and likewise, $\overline{(2,5)} \cap (1,2) = \emptyset$. Notice that this same observation cannot be extended to the two sets $(1,2]$ and $(2,5)$ even though these two sets are disjoint.

Definition 3.4.2

(i) Two nonempty sets $A, B \subseteq \mathbb{R}$ are *separated* if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(ii) A set $E \subseteq \mathbb{R}$ is *disconnected* if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

(iii) A set that is not disconnected is called a *connected* set.

Example 3.4.2

(i) If we let $A = (1,2)$ and $B = (2,5)$, then it is not difficult to verify that $E = (1,2) \cup (2,5)$ is disconnected. Notice that the sets $C = (1,2]$ and $D = (2,5)$ are NOT separated because $C \cap \overline{D} = \{2\}$ is nonempty. We know that the interval $(1,5)$ is the union of sets C and D, but we cannot say they are disconnected. We will prove later that every interval is a connected subset of $\mathbb{R}$ and vice versa.

(ii) Consider the set of rational numbers

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \text{ and } B = \mathbb{Q} \cap (\sqrt{2}, \infty).$$

It turns out that these two sets are disconnected such that $\mathbb{Q} = A \cup B$. The fact that $A \subseteq (\infty, \sqrt{2})$ implies that any limit point of A will necessarily fall in $(-\infty, \sqrt{2}]$ by the Order Limit Theorem. Because this is disjoint from B, we get that $\overline{A} \cap B = \emptyset$. We can similarly show that $A \cap \overline{B} = \emptyset$, which implies that A and B are separated.

The definition of connected is stated as the negation of disconnected, but using the logical negation of the quantifiers in the definition above results in a positive characterization of connectedness.

A way to show that a set E is connected is to partition E into two nonempty disjoint sets where we can show at least one of the sets contains a limit point of the other.

Theorem 3.4.2

A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof. Exercise 3.4.6. ■

Theorem 3.4.3

A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $x \in E$ as well.

Proof. Assume E is connected, and let $a, b \in E$ and $a < c < b$. We can set A and B such that

$$A = (-\infty, c) \cap E \text{ and} B = (c, \infty) \cap E.$$

Because $a \in A$ and $b \in B$, neither set is empty and, just as in Example 3.4.5 (ii), neither set contains a limit point of the other. If $E = A \cup B$, then we have that E is disconnected. If $E = A \cup B$, then we would have that E is disconnected, which it is not. It must be the case that $A \cup B$ is missing some element of E, and c is the only possibility. Thus, $c \in E$.

Conversely, assume E is an interval in the sense that whenever $a, b \in E$ satisfy $a < c < b$ for some c, then $c \in E$. Our intent is to use the characterization of connected sets in Theorem 3.4.6, so let $E = A \cup B$, where A and B are nonempty and disjoint.

We need to show that one of these sets contains a limit point of the other. Pick $a_0 \in A$ and $b_0 \in B$, and suppose $a_0 < b_0$ for sake of argument. Since E is an interval, the interval $I_0 = [a_0, b_0]$ is contained in E. Now, let us bisect I_0 into two equal halves. The midpoint of I_0 must either be in A or B, and so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$. We can continue such a process inductively to get a sequence of nested intervals $I_n[a_n, b_n]$, where $a_n \in A$ and $b_n \in B$, and the length $(b_n - a_n) \to 0$. Hence, the following intersection

$$\bigcap_{n=0}^{\infty} I_n \neq \emptyset.$$

Since $(a_n - b_n) \to 0$, we have that the sequences of endpoints have the same limit point x. Since $x \in E$, it must be the case that x must belong to either A or B. Hence, E is a connected set. ■

Chapter 4

Functional Limits and Continuity

4.0.1 Towards a Formal Definition of Continuity

We want to define continuity at a point $c \in A$ to mean that if we have $x \in A$ that s chosen *near* c, then $f(x)$ will be near $f(c)$. We can define this notion mathematically to say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

This is a rational way to define continuity until we encounter a function like

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

where we haven't yet defined what it means for $\lim_{x \to 1/2} g(x)$. We can extend our notions of a limit from Chapter 2 to make sense of this expression. Hence, we can define a sequence (x_n) where $(x_n) \to 1/2$ and say that $\lim_{x \to 1/2} g(x)$ as the limit of $g(x_n)$. But the problem with this notion is that if (x_n) is defined as a sequence of rational points, then

$$\lim_{n \to \infty} g(x_n) = 1$$

but on the other hand, if (x_n) is irrational, then

$$\lim_{n \to \infty} g(x_n) = 0.$$

Very quickly, we can conclude that the limit of $g(x)$ as $x \to c$ does not exists. Furthermore, we can also see that $x = 1/2$ causes $g(x)$ to not be continuous. This is because both $\mathbb{Q}$ and $\mathbb{I}$ are both dense in the real line. Hence, it follows that for any $z \in \mathbb{R}$, we can find sequences $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{I}$ such that $\lim x_n = \lim y_n = z$. But because

$$\lim g(x_n) = \lim g(y_n),$$

we can say that the same line of reasoning applies to conclude that $g(x)$ is not continuous at z on $\mathbb{R}$. In other words, the Dirichlet function $g(x)$ is nowhere continuous on $\mathbb{R}$.

What happens when we adjust the definition of $g(x)$ slightly such that

$$h(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Suppose $h(x)$ is defined on $\mathbb{R}$. Then letting $c \in \mathbb{R}$ be different from 0,then we can construct sequences $(x_n) \to c$ of rationals and $(y_n) \to c$ of irrationals such that

$$\lim h(x_n) = c \text{ and } \lim h(y_n) = 0.$$

Thus, we have that h is not continuous at every point $c \neq 0$. If we set $c = 0$, then it turns out that these two functional limits are the same. This observation enables us to strive for a definition for functional limits by saying that

$$\lim_{x \to c} h(x) = L$$

if $h(z_n) \to L$ for all sequences $(z_n) \to c$. We can make sense of these two different limits by constructing $\epsilon-$neighborhoods around c and L respectively. Another example of a non-continuous function is

$$t(x) = \begin{cases} 1 \text{ if } x = 0 \\ 1/n \text{ if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ where } n > 0 \text{ and } (m,n) = 1 \\ 0 \text{ if } x \neq \mathbb{Q}. \end{cases}$$

If we let $c \in \mathbb{Q}$, then $t(c) > 0$. But since $\mathbb{I}$ is dense in $\mathbb{R}$, there exists a sequence of irrational numbers (y_n) in $\mathbb{R}$ such that $(y_n) \to c$. It immediately follows that $t(x)$ is not continuous at every point in $\mathbb{Q}$ since

$$\lim t(y_n) = 0 \neq t(c).$$

But if we let $c \in \mathbb{I}$, then we find that defining a sequence of rational numbers (y_n) such that $(y_n) \to c$ reveals that $\lim t(y_n) = 0$ which makes $t(x)$continuous at every irrational point in $\mathbb{R}$. The takeaway from this section is that the characteristics of a given set $A \subseteq \mathbb{R}$ greatly determines the continuity of a function.

4.1 Functional Limits

4.1.1 Defining the Functional Limit

Consider a function $f : A \to \mathbb{R}$. Suppose c of A is a limit point. From the last chapter, recall that the definition of a limit point is any $\epsilon-$neighborhood $V_\epsilon(c)$ intersects $A \setminus \{c\}$. In other words, c is a limit point of A if and only if $c = \lim x_n$ for some sequence $(x_n) \subseteq A$ with $x_n \neq c$ for all $n \in \mathbb{N}$. Furthermore, it is important to keep in mind that limit points of A do not necessarily belong to A unless it is closed.

If c is a limit point of our domain A, then, we can state that

$$\lim_{x \to c} f(x) = L$$

is intended to convey that the values of $f(x)$ gets arbitrarily close to L as x is chosen arbitrarily close to c. It is important to keep in mind that c need not be in the domain of A.

The structure of the definition of functional limits is as follows: Given a sequence (a_n), the assertion that $\lim a_n = L$ implies that for every $\epsilon-$neighborhood $V_\epsilon(L)$ centered at L, we can find a point in a sequence say a_N after which all the terms of a_n fall in $V_\epsilon(L)$. This is in response to an arbitrary choice of x in the domain where we have a $\delta-$neighborhood

Definition 4.1.1: Functional Limit

Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever

$$0 < |x - c| < \delta$$

(and $x \in A$) it follows that

$$|f(x) - L| < \epsilon.$$

This is often referred to as the epsilon-delta definition of a functional limit. The statement

$$|f(x) - L| < \epsilon$$

is equivalent to saying that $f(x) \in V_\epsilon(L)$. Likewise, the statement

$$|x - c| < \delta$$

is true if and only if $x \in V_\delta(c)$. Note that we imposed an additional restriction that $|x-c| > 0$. This is because we don't want $x = c$. We can recast the definition above in terms of $\epsilon-$neighborhoods to help get a more geometric view of what is happening with these functional limits.

Definition 4.1.2: Functional Limit In Terms of Neighborhoods

Let c be a limit point of the domain $f : A \to \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that for every $\epsilon-$neighborhood $V_\epsilon(L)$ of L, there exists a $\delta-$neighborhood $V_\epsilon(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

The reminder that we must have $x \in A$ ensures that every possible $x \in A$ must be a valid input for the function in question. Note that the appearance of $f(x)$ in our definitions carries an implicit assumption that x is always part of the domain of f. There is no use for considering isolated points outside of A. Hence, we can always expect that functional limits will have $x \in A$ that approach the limit point of A or $\text{dom}(f)$.

(i) Suppose we want to show that for $f(x) = 3x + 1$, we have

$$\lim_{x \to 2} f(x) = 7.$$

Proof. Let $\epsilon > 0$. Consider $|f(x) - 7|$. We want to show that whenever $|x - 2| < \delta$, that

$$|f(x) - 7| < \epsilon.$$

Hence, observe that

$$\begin{aligned}|f(x) - 7| &= |(3x + 1) - 7| \\ &= |3x - 6| \\ &= 3|x - 2| \\ &< 3\delta.\end{aligned}$$

We can choose $\delta = \epsilon/3$ such that

$$|f(x) - 7| < 3\delta = 3\frac{\epsilon}{3} = \epsilon.$$

Hence, we have that $\lim_{x\to 2} f(x) = 7$. ■

(ii) Let's show that

$$\lim_{x\to 2} g(x) = 4,$$

where $g(x) = x^2$.

Proof. Let $\epsilon > 0$. Suppose $0 < |x - 2| < \delta$. Then

$$\begin{aligned}|g(x) - 4| &= |x^2 - 4| \\ &= |(x - 2)(x + 2)| \\ &= |x - 2||x + 2|.\end{aligned}$$

Since $|x + 2| < \delta + 4$, observe that for $\delta = \min\{1, \epsilon/5\}$, we have

$$\begin{aligned}|g(x) - 4| &= |x - 2||x + 2| \\ &< \delta \cdot (\delta + 4) \\ &= \frac{\epsilon}{5} \cdot 5 \\ &= \epsilon.\end{aligned}$$

■

4.1.2 Sequential Criterion for Functional Limits

Theorem 4.1.1: Sequential Criterion for Function Limits

Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:

(i) $\lim_{x\to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Proof. (i) $\implies$ (ii) Suppose $\lim_{x\to c} f(x) = L$. Let $\epsilon > 0$. By assumption, we have a sequence $(x_n) \to c$. It immediately follows that for some $N \in \mathbb{N}$ such that for all $n \geq N$ that $x_n \in V_\delta(c)$. Hence, $f(x_n) \in V_\epsilon(L)$ by the Topological Definition of functional limits.
(ii) $\implies$ (i) Let $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$ such that $f(x_n) \to L$. Suppose for sake of contradiction that $\lim_{x\to c} f(x) \neq L$. Hence, there exists ϵ_0 such that for any $\delta > 0$ where $|x_n - c| < \delta$ that $|f(x_n) - L| \geq \epsilon_0$. Let $\delta = 1/n$ and suppose we pick $x_n \in V_\delta(c)$ such that $f(x_n) \notin V_\epsilon(L)$. But this implies that $f(x_n) \not\to L$ which contradicts our assumption that it is. Hence, it must be the case that (i) holds. ■

Corollary 4.1.1: Algebraic Limit Theorem for Function Limits

Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,

(i) $\lim_{x\to c} kf(x) = kL$ for all $k \in \mathbb{R}$,

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x\to c}[f(x)g(x)] = L \cdot M$, and

(iv) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$.

Proof. Exercise 4.2.1. ■

Corollary 4.1.2: Divergence Criterion for Functional Limits

Let f be a function defined on A and let c be a limit point of A. If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

Example 4.1.1

Suppose we wanted to show that $\lim_{x\to 0}\sin(1/x)$ does not exist. Set $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$, then
$$\lim x_n = \lim y_n = 0$$
but $\sin(1/x_n) = 0$ for all $n \in \mathbb{N}$ while $\sin(1/y_n) = 1$. Thus, we have

$$\lim \sin(1/x_n) \neq \lim \sin(1/y_n),$$

and thus we know that $\lim_{x\to 0}\sin(1/x)$ does not exist.

4.2 Continuous Functions

Definition 4.2.1: Continuity

A function $f : A \to \mathbb{R}$ is *continuous at a point* $c \in A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$. If f is continuous at every point in the domain A, then we say that f *continuous* on A.

The difference between this definition and the definition for functional limits is that we require the limit point c of A to be in the domain of f. The value of $f(c)$ is the value of $\lim_{x\to c} f(x)$. It is indeed possible to shorten this definition to say that f is continuous at $c \in A$ if

$$\lim_{x\to c} f(x) = f(c) \tag{1}$$

so as long as c is a limit point of A. The equation above gets undefined if c is an isolated point of A. But this does not stop f from being continuous at the point c. In fact, functions can still be continuous at their isolated points such as c.

We observed in the previous section that functional limits can be formulated using sequences from chapter 2. The same can be done for continuity.

Theorem 4.2.1: Characterizations of Continuity

Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:

(i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x-c| < \delta$ (and $x \in A$) implies $|f(x)-f(c)| < \epsilon$;

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(iii) For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.

If c is limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x\to c} f(x) = f(c)$.

Proof. (i) $\iff$ (ii) Let $V_\epsilon(f(c))$ and let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies

$$|f(x) - f(c)| < \epsilon.$$

We can rephrase in terms of ϵ and δ neighborhoods. Hence, the statement above is just equivalent to the statement that for all $V_\epsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c)$ which implies that $f(x) \in V_\epsilon(f(c))$.
(iii) $\iff$ (ii) First we show (ii) holds. Let $(x_n) \to c$ such that $f(x_n) \to f(c)$. In other words, $f(x_n) \in V_\epsilon(f(c))$. For sake of contradiction, that $f(x) \notin V_\epsilon(f(c))$. This means there exist ϵ_0 such that for all $\delta > 0$ with the property that $|x - c| < \delta$ that

$$|f(x) - f(c)| \geq \epsilon_0.$$

Let $\delta = 1/n$ and suppose we pick $x_n \in V_\delta(c)$ such that $f(x_n) \notin V_\epsilon(f(c))$. But this contradicts our assumption that $f(x_n) \in V_\epsilon(f(c))$. Hence, it must be the case that $f(x) \in V_\epsilon(f(c))$.
Now we want to show that (iii) holds. Let $(x_n) \subseteq A$. (with $x_n \in A$). Since $x_n \in A$ $x_n \neq c$ or $x_n = c$. If $x_n = c$, then it immediately follows that $f(x_n) \to f(c)$ given there exists $|x_n - c| < \delta$. Suppose $x_n \neq c$. Pick $x_n \in V_\delta(c)$ for some $\delta > 0$. Then by assumption, we have $f(x_n) \in V_\epsilon(f(c))$. Hence, $f(x_n) \to f(c)$.
To show (i) $\implies$ (iv), suppose c is a limit point of A. Let $x \in A$. Since $c \in A$, either $x \neq c$ or $x = c$. The conclusion follows immediately from the latter statement. Suppose $x_n \neq c$. By assumption, there exists $\delta > 0$ such that $|x_n - c| < \delta$. From this, it follows that

$$|f(x) - f(c)| < \epsilon.$$

And hence,

$$\lim_{x \to c} f(x) = f(c).$$

■

Corollary 4.2.1: Criterion for Discontinuity

Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n) \not\to f(c)$, we may conclude that f is not continuous at c.

This sequential characterization of continuity allows us to use all the results that we know of when it comes to sequences from Chapter 2.

Theorem 4.2.2: Algebraic Continuity Theorem

Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at point $c \in A$. Then,

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;

(ii) $f(x) + g(x)$ is continuous at c for all $k \in \mathbb{R}$;

(iii) $f(x)g(x)$ is continuous at c; and

(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Proof. All of these statements can be derived from the Characterizations of Continuity Theorem and the Algebraic Functional Limit Theorem. ■

Example 4.2.1

All polynomials are continuous on $\mathbb{R}$. In fact, rational functions (Quotients of polynomials) are continuous wherever they are defined. Consider the identity function $g(x) = x$. Since $|g(x) - g(c)| = |x - c|$, we can respond to a given $\epsilon > 0$ by choosing $\delta = \epsilon$ such that g is continuous on all of $\mathbb{R}$. Furthermore, this argument gets much simpler when we consider a constant function such as $f(x) = k$. Since any arbitrary polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

consists of sums and products of $g(x)$ with different constant functions, we can conclude that $p(x)$ is continuous. On the other hand, the Algebraic Continuity Theorem implies that quotients of polynomials are continuous as long as the denominator is not zero.

Example 4.2.2

In the sinuisodal example in the last section, we noticed that the oscillations of $\sin(1/x)$ are so rapid near the origin that $\lim_{x \to 0} \sin(1/x)$ does not exist. Consider the function,

$$g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose we want to observe the continuity of g at $c = 0$. We can do this by the following:

$$|g(x) - g(0)| = |x \sin(1/x) - 0| \leq |x|$$

since $|\sin(x)| \leq 1$. Given $\epsilon > 0$, choose $\delta = \epsilon$ such that whenever $|x| < \delta$ it follows that $|g(x) - g(0)| < \epsilon$. Thus, g is continuous at the origin.

Example 4.2.3

Consider the greatest integer function $h(x) = [[x]]$ which for each $x \in \mathbb{R}$ returns the largest integer $n \in \mathbb{Z}$ such that $n \leq x$. In previous math classes, this step function is observed to have discontinuous jumps at each integer value of its domain. We can show this more rigorously using the tools we have at our disposal. Given $m \in \mathbb{Z}$, define the sequence (x_n) by $x_n = m - 1/n$. It follows that $(x_n) \to m$, but not that

$$h(x_n) \to (m - 1),$$

which does not equal $m = h(m)$. By the Criterion for Discontinuity, we see that h fails to be continuous at each $m \in \mathbb{Z}$. Suppose we want to see why h is continuous at a point $c \notin \mathbb{Z}$. Given any $\epsilon > 0$, we must find a $\delta-$neighborhood $V_\delta(c)$ such that $x \in V_\delta(c)$ implies $h(x) \in V_\epsilon(h(c))$. We know that $c \in \mathbb{R}$ falls between consecutive integers $n < c < n + 1$ for some $n \in \mathbb{Z}$. Taking $\delta = \min\{c - n, (n + 1) - c\}$, then it follows from definition of h that $h(x) = h(c)$. Thus, we certainly have that $h(x) \in V_\epsilon(h(c))$ whenever $x \in V_\delta(c)$. This proof actually implies that our δ is not dependent on the value of $\epsilon > 0$.

Example 4.2.4

Consider $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbb{R} : x \geq 0\}$. Exercise 2.3.1 outlines a sequential proof that f is continuous on A. Show f is continuous below.

What about functions like $h(x) = \sqrt{3x^2 + 5}$ is continuous. Hence, a Compositions of Continuous functions type theorem is needed to show that $h(x)$ is continuous on its domain.

Theorem 4.2.3: Composition of Continuous Functions

Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Proof. Exercise 4.3.3. ■

4.3 Continuous Functions on Compact Sets

Given a function $f : A \to \mathbb{R}$ and a given subset $B \subseteq A$, the notation $f(B)$ can be defined as the range of f over the set B; in other words, we have that

$$f(B) = \{f(x) : x \in B\}.$$

We can describe properties such as subsets of $\mathbb{R}$ being open, closed, bounded, compact, perfect, and connected, but a more interesting analysis arises when see which ones are preserved when mapping B to $f(B)$ via a continuous function.

For example, if B is an open set and f is continuous, is the mapping $f(B)$ necessarily open? The answer to this is no.

Suppose $f(x) = x^2$ and $B = (-1, 1)$ is an open interval, then we have that the interval $[0, 1)$ is not open. What if B is closed? The same conjecture actually leads to the same conclusion that $f(B)$ is not closed as well. Consider the function

$$g(x) = \frac{1}{1 + x^2}$$

and the closed set $B = [0, \infty) = \{x : x \geq 0\}$. Because we have that $g(B) = (0, 1]$ is not closed, we must conclude that continuous functions do not generally map from closed sets to closed sets. However, if B is compact, then B gets mapped to closed and bounded subsets by continuous functions.

Theorem 4.3.1: Preservation of Compact Sets

Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.

Proof. Let $f : A \to \mathbb{R}$ and $K \subseteq A$ be a compact set. Let $(x_n) \subseteq K$ and $(y_n) \subseteq f(K)$. Since K is a compact set, there exists $(x_{n_k}) \to x$ such that x is contained in K. Suppose f is a continuous function. Define $f(x_n) = y_n$. Since (x_{n_k}) converges to x and f is a continuous function, we have that

$$f(x_{n_k}) = y_{n_k} \to f(x) = y.$$

This means our subsequence $(y_{n_k}) \subseteq f(K)$ converges to a limit y that is contained in $f(K)$. Thus, $f(K)$ is a compact set. ■

An extremely important result from this theorem deals with how compact sets are bounded and how they contain their supremums and infimums.

Theorem 4.3.2: Extreme Value Theorem

If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Proof. Since $f(K)$ is a compact set, we can set $\alpha = \sup f(K)$ and know that $\alpha \in f(K)$ from Exercise 3.3.1. It immediately follows that for some $x_1 \in K$, this element gets mapped to $\alpha = f(x_1)$ since f is a continuous function. Likewise, we have $\beta \in f(K)$ such that for some $x_0 \in K$, x_0 gets mapped to $\beta = f(x_0)$ by the same reasoning above. Hence, we have that for any $x \in K$,

$$f(x_0) \leq f(x) \leq f(x_1).$$

■

4.3.1 Uniform Continuity

We learned in the last section that polynomials are always continuous on $\mathbb{R}$. In this section, we show that these functions are actually uniform continuous on $\mathbb{R}$; that is, they are everywhere continuous.

Example 4.3.1

(i) Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x + 1$. We want to show that this is continuous for any point $c \in \mathbb{R}$. Let $\epsilon > 0$. Then choose $\delta = \epsilon/3$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |(3x+1) - (3c+1)| \\ &= 3|x - c| \\ &< 3 \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Hence, $\lim_{x \to c} f(x) = f(c)$. The key observation here is that our choice of δ is the same regardless of the point $c \in \mathbb{R}$, we are considering.

(ii) Suppose how the situation changes when we consider another function, say, $g(x) = x^2$ and see how the choice of δ changes with each point $c \in \mathbb{R}$. Given $c \in \mathbb{R}$, observe that

$$|g(x) - g(c)| = |x^2 - c^2| = |x - c||x + c|.$$

As was discussed in section 4.2, we need to upper bound $|x + c|$, which, in this case, can be obtained by letting our choice δ not exceed 1. This implies that all values of x under consideration will fall in the interval $(c-1, c+1)$. By using our assumption that $|x - c| < \delta$ and letting $\delta = 1$, we have that

$$|x + c| \leq |x| + |c| \leq (|c| + 1) + |c| = 2|c| + 1.$$

Now let $\epsilon > 0$. If we choose $\delta = \min\{1, \epsilon/(2|c|+1)\}$, then assume $|x - c| < \delta$ such that

$$|f(x) - f(c)| = |x - c||x + c| < \left(\frac{\epsilon}{2|c|+1}\right) \cdot (2|c| + 1) = \epsilon.$$

Notice how our choice of δ depended on our choice of $c \in \mathbb{R}$ where

$$\delta = \frac{\epsilon}{2|c|+1}.$$

This means that as our choice of $c \in \mathbb{R}$ gets bigger and bigger, our $\delta-$neighborhood must get smaller and smaller.

This leads us to our rigorous definition of what it means for a function to be uniform continuous.

Definition 4.3.1: Uniformly Continuous Functions

A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

The difference between regular continuity and uniform continuity is that regular implies continuity of a function at any point $c \in \mathbb{R}$ granted that our choice of δ is dependent on our choice of $c \in \mathbb{R}$ while uniform continuity implies that our choice of δ stays the same regardless of our choice of $c \in \mathbb{R}$.

On the other hand, saying that a function is not uniform continuous is to say that given some $\epsilon > 0$, there is not suitable choice of $\delta > 0$ that will be a valid response to our ϵ challenge. That is, every point $c \in \mathbb{R}$ has a unique $\delta > 0$.

Theorem 4.3.3: Sequential Criterion for Absence of Uniform Continuity

A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof. ($\Rightarrow$) We can negate the definition of uniform continuity to help us prove this direction. Hence, for some $\epsilon_0 > 0$, choose $\delta_n = 1/n$ such that whenever we have sequences (x_n) and (y_n) that satisfy

$$|x_n - y_n| < \frac{1}{n},$$

we have that

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

Clearly, we have $|x_n - y_n| \to 0$ by the Squeeze Theorem for sequences.
($\Leftarrow$) Since $|x_n - y_n| \to 0$ for any $n \geq N$ for some $N \in \mathbb{N}$, we can see that any choice of $\delta > 0$ will not be a suitable response to the ϵ_0 challenge; that is, $|x_n - y_n| \to 0$ implies that f fails to be uniformly continuous on A. ■

Example 4.3.2

Consider the function $h(x) = \sin(1/x)$. We can see that $h(x)$ is continuous at every point in the open interval $(0, 1)$ but is not uniformly continuous on this interval. We can prove this by defining the following sequence (x_n) and (y_n) to be

$$x_n = \frac{1}{\pi/2 + 2n\pi} \text{ and } y_n = \frac{1}{3\pi/2 + 2n\pi}.$$

Since both sequences tend to zero, we have that $|x_n - y_n| \to 0$ such that letting $\epsilon_0 = 2$ leads to

$$\begin{aligned}
|h(x_n) - h(y_n)| &= |\sin(\pi/2 + 2n\pi) - \sin(3\pi/2 + 2n\pi)| \\
&= |2\cos(n\pi) - (-2\cos(n\pi)| \\
&= 2|\cos(n\pi)| \\
&= 2.
\end{aligned}$$

Proof. Assume $f : K \to \mathbb{R}$ is continuous at every point of a compact set $K \subseteq \mathbb{R}$. Suppose for sake of contradiction that f is not uniformly continuous on K. Then by the Sequential Criterion for Absence of Uniform Continuity, we have that for some $\epsilon_0 > 0$, we have two sequences (x_n) and (y_n) that satisfy the following property

$$|x_n - y_n| \to 0$$

which implies that

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

Since K is a compact set, we can find a subsequence x_{n_k} such that $(x_{n_k}) \to x$ where $x \in K$. Suppose we want to show that $y_{n_k} \to x$. Hence, we can use the Algebraic Limit Theorem to say that

$$\lim(y_{n_k}) = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = \lim(y_{n_k} - x_{n_k}) + \lim x_{n_k} = 0 + x = x.$$

Since f is continuous on $x \in K$, we have that $f(x_{n_k}) = f(x)$ and $f(y_{n_k}) = f(x)$. But this means that

$$\lim(f(x_{n_k}) - f(y_{n_k})) = 0$$

that is, $|f(x_{n_k}) - f(y_{n_k})| < \epsilon$ for all $\epsilon > 0$ which contradicts our original assumption that

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbb{N}$. Hence, f must be uniformly continuous on K. ■

4.4 The Intermediate Value Theorem

In intuitive terms, the Intermediate Value Theorem is an observation that explains how a continuous function f on a closed interval $[a, b]$ attains every value that falls between two range values $f(a)$ and $f(b)$.

Theorem 4.4.1: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.

4.4.1 Preservation of Connected Sets

Before we prove the Intermediate Value Theorem, we should understand that it is a special case of continuous functions mapping connected sets to connected sets. In the last section, we saw how continuous functions on compact sets K produces range sets $f(K)$ that are also compact. This just so happened to also hold for connected sets.

Theorem 4.4.2: Preservation of Connected Sets

Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.

Proof. Our goal is to use the characterization of connected sets to prove this theorem. Let $f(E) = A \cup B$ where A and B are disjoint and nonempty. We want to produce a sequence that is contained in either A or B that converges to a limit contained in the other. Define the following sets:

$$C = \{x \in E : f(x) \in A\} \text{ and } D = \{x \in E : f(x) \in B\}.$$

These sets are the *preimages* of A and B respectively. We can see, from the properties of both A and B, that they are nonempty and disjoint such that they satisfy $E = C \cup D$. Now assume E is a connected set. This means there exists a convergent sequence (x_n) that is contained in either C or D with $\lim x_n = x$. Since f is continuous at x, we have that $f(x) = \lim f(x_n)$. Thus, it follows that $f(x_n)$ is a convergent sequence contained in either A or B while the limit $f(x)$is an element of the other. Hence, $f(E)$ is also connected. ■

In $\mathbb{R}$, a set is connected if and only if it is an interval (which is possibly unbounded). This in addition to the theorem above leads to a short proof of the Intermediate Value Theorem.

4.4.2 Completeness

A typical application of the Intermediate Value Theorem involves proving the existence of roots. Given a function $f(x) = x^2 - 2$, we can see that $f(1) = -1$ and $f(2) = 2$. Hence, there exists a point $c \in (1, 2)$ where $f(c) = 0$. In Chapter 1, we prove the existence of $\sqrt{2}$ using the Axiom of Completeness as our main assumption about the properties of $\mathbb{R}$. This fact sheds light on the relationship between the continuity of functions and the completeness of $\mathbb{R}$.

Proof of the Intermediate Value Theorem using the Axiom of Completeness.

Proof. First, let us consider a special case where f is a continuous function satisfying the property that $f(a) < 0 < f(b)$. We want to show that $f(c) = 0$ for some $c \in (a, b)$. Let us define the following set

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

This is the set of negative values that $f(x)$ takes on the interval $[a, b]$. Note that K is bounded above by b, and $a \in K$ so that K is nonempty. Hence, we are allowed to assert that $\sup K$ exists and that $\sup K = c$. We have three cases to consider:

$$f(c) > 0,\ f(c) < 0, \text{ and } f(c) = 0.$$

By the fact that c is the least upper bound of K rules out the first two cases. Hence, we reach our desired conclusion that $f(c) = 0$. The details are requested in Exercise 4.5.5(a). ■

Below is the second proof of the Intermediate Value Theorem using the Nested Interval Property.

Proof. Consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint $z = (a + b)/2$. If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. We have that, in either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right. This procedure can be inductively repeated such that the Nested Interval Property can be applied to gain the conclusion of the theorem. The remainder of the argument is left to the reader in Exercise 4.5.5(b). ■

4.4.3 The Intermediate Value Property

An interesting question we can ask is does the Intermediate Value Theorem have a converse that is true?

Definition 4.4.1

A function f has the *intermediate value property* on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

This is to say that every continuous function f on an interval $[a, b]$ must have the intermediate value property. We must be careful since this is not always true that a function that contains this property must necessarily be continuous. An example of this is the function

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at zero, but it does have the intermediate value property on $[0, 1]$.

4.5 Sets of Discontinuity

Definition 4.5.1: Set of Discontinuities

Given a function $f : \mathbb{R} \to \mathbb{R}$, we call the set $D_f \subseteq \mathbb{R}$ to be the set of points where the function f fails to be continuous.

Some examples of sets of discontinuous points are

(a) $D_g = \mathbb{R}$ in the case for Dirichlet's function,

(b) and $D_h = \mathbb{R} \setminus \{0\}$ in the case of the modified Dirichlet's function, and

(c) lastly, $D_t = \mathbb{Q}$ for Thomae's function $t(x)$.

We can always write the set of discontinuous points for a function D_f as a countable union of closed sets. For monotone functions, these closed sets can taken as single points.

4.5.1 Monotone Functions

Definition 4.5.2: Monotone Functions

A function $f : A \to \mathbb{R}$ is *increasing* on A if $f(x) \leq f(y)$ whenever $x < y$ and *decreasing* if $f(x) \geq f(y)$ whenever $x < y$ in A. A *monotone* function is one that is either increasing or decreasing.

The continuity of a function at a point c means that $\lim_{x \to c} f(x) = f(c)$. Discontinuities occur when right-hand limits do not equal the left-hand limits approaching c.

Definition 4.5.3: Right-Hand Limits

Given a limit point c of a set A and a function $f : A \to \mathbb{R}$, we write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$. Equivalently, in terms of sequences, $\lim_{x \to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n > c$ and $\lim(x_n) = c$.

Exercise 4.6.3

State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Solution. We say $\lim_{x \to c^-} f(x) = L$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |c - x| < \delta$. Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(y_n) = M$ for all sequences (y_n) satisfying $y_n < c$ and $\lim(y_n) = c$. ■

Theorem 4.5.1: Functional Limit

Given $f : A \to \mathbb{R}$ and a limit point c of A, $\lim_{x \to c} f(x) = L$ if and only if

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

Exercise 4.6.4

Supply a proof for this proposition.

Proof. Let $f : A \to \mathbb{R}$ and a limit point c of A. Assume $\lim_{x \to c} f(x) = L$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. To show that the right-hand limit equals L. Let $x > c$. Then immediately, we have that $0 < |x - c| < \delta$ implies $\lim_{x \to c^+} f(x) = L$ since $|f(x) - L| < \epsilon$. For the left-hand limit, suppose $x < c$. Then

$$0 < |x - c| < \delta \iff 0 < |c - x| < \delta$$

implies $|f(x) - L| < \epsilon$ holds. Hence, $\lim_{x \to c^-} f(x) = L$.
Conversely, the fact that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} = L$ implies for some $\delta > 0$ that for any $x > c$ or $x < c$ that $0 < |x - c| < \delta$ holds. Hence, we have that $|f(x) - L| < \epsilon$; that is, $\lim_{x \to c} f(x) = L$. ■

Generally speaking, discontinuities can be divided into three categories:

(i) If $\lim_{x \to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at c is called *removable*.

(ii) If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$, then f has a *jump* discontinuity.

(iii) If $\lim_{x \to c} f(x)$ does not exist for some other reason, then the discontinuity at c is called an *essential* discontinuity.

We now have the proper characteristics to describe our set of discontinuities D_f for an arbitrary monotone function f.

Exercise 4.6.5

Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Proof. Without loss of generality, let f be an increasing function. Then for all $x < y$, $f(y) \geq f(x)$. If $\lim_{x\to c} f(x) = L$, but $L \neq f(c)$, then either $f(c) < L$ or $f(c) > L$. In either case, there exists an $x \in A$ such that for all $y \in A$, we have $f(x) > f(y)$ or $f(x) < f(y)$. But this contradicts our assumption that f is increasing. Hence, the discontinuity cannot be a *removable* discontinuity. If there exist an *essential discontinuity* then f cannot be monotone since either the left-hand limit or the right-hand limit does not exists. Hence, the only type of discontinuity an increasing function can have is a jump discontinuity. ■

4.5.2 D_f for an Arbitrary Function

Review of infinite and finite closed or open sets:

(a) Recall that the intersection of an infinite collection of closed sets is closed and the union of a finite collection of closed sets is closed.

(b) On the other hand, the intersection of a finite collection of open sets must be open and the union of a infinite collection of open sets must be open.

Definition 4.5.4

A set that can be written as the countable union of closed sets is in the class F_σ.

Chapter 5

The Derivative

5.1 Are Derivatives Continuous?

The derivative of a function $g(x)$, namely $g'(x)$, can be defined as the slope of g at each point $x \in \mathrm{Dom}(f)$. As we have learned in our previous studies, the derivative is just the following limit

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

A couple questions we can ask about the relationship between continuity and differentiability of functions is that:

(i) Are they continuous?

(ii) Are continuous functions differentiable?

(iii) How nondifferentiable can a continuous function be?

In the last section, we identified the discontinuous points of a monotone function and expressed them in terms of countable closed sets. Some examples of such functions are of the form

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

When $n = 0$, we can see the oscillations of $\sin(1/x)$ prevent g from being continuous at $x = 0$. But when $n = 1$, the oscillations of g are sandwiched between $|x|$ and $-|x|$ which implies that g is continuous at $x = 0$. What can we say about $g_2'(0)$? Is it defined? Using our intuitive definition above, we have that

$$g_1'(0) = \lim_{x \to 0} \frac{g_1(x)}{x} = \lim_{x \to 0} \sin(1/x)$$

which, in this case, does not exist. Thus, we have that g_1 is not differentiable at zero. However, if we let $n = 2$, then we have the following

$$g_2'(0) = \lim_{x \to 0} x \sin(1/x) = 0.$$

At nonzero points in the domain of g, we can use rules of differentiation (that will be justified later) to conclude the g_2 is differentiable everywhere in $\mathbb{R}$ with

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

But if we now consider the limit

$$\lim_{x \to 0} g_2'(x)$$

we will find that it does not exist because for every $x \neq 0$, the $\cos(1/x)$ term is not preceded by a factor of x.

In summary, when $n = 2$, $g_2(x)$ is continuous and differentiable everywhere on $\mathbb{R}$, but the derivative function $g_2'(x)$ is defined everywhere but is not continuous at $x = 0$. The conclusion is that we don't the derivative of a function to be continuous in general.

The discontinuity we found from g_2' is an *essential* discontinuity; that is, the limit as $x \to 0$ does not exist as a one sided limit. What about a function with a simple jump discontinuity like

$$h'(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Notice that this function is actually the slopes of the absolute value function $|x|$ which is not differentiable at $x = 0$. How can we imply differentiability of h' at $x = 0$? Our main point here is that continuity is not a sufficient condition for derivatives to be possible.

5.2 Derivatives and the IVP

5.2.1 Definition of the Derivative

Definition 5.2.1: Differentiability

Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative* of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case, we say that g is *differentiable* at c. If g' exists for all points $c \in A$, we say that g is *differentible* on A.

Example 5.2.1

(i) Consider the function $f(x) = x^n$, where $n \in \mathbb{N}$, and let c be any arbitrary point in $\mathbb{R}$. Using the following identity,

$$x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

we can take the limit

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{x^n - c^n}{x - c} \\ &= \lim_{x\to c}(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}) \\ &= c^{n-1} + c^{n-1} + c^{n-1} + \cdots + c^{n-1} \\ &= nc^{n-1} \end{aligned}$$

(ii) If $g(x) = |x|$, then if we want to take the derivative at $c = 0$ produces the following limit

$$g'(0) = \lim_{x\to 0} \frac{|x|}{x}$$

which is 1 if we approach from the right and -1 if we approach from the left. Hence, we have that $g'(c) = 0$ does not exist.

This last example should remind us that continuity of a function does not necessarily imply that a function is differentiable. On the other hand, we can say that if g is differentiable at a point then g is continuous at that point.

Theorem 5.2.1: Differentiability Implies Continuity

If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.

Proof. Assume $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$. Hence, we have that the following limit exists

$$g'(c) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c}.$$

Using the Algebraic Limit Theorem for functional limits, we have that

$$\lim_{x\to c}(g(x) - g(c)) = \lim_{x\to c} \left(\frac{g(x) - g(c)}{x - c}\right)(x - c) = g'(c) \cdot 0 = 0.$$

Hence, it follows that $\lim_{x\to c} g(x) = g(c)$. ■

We can prove the same fact using the epsilon-delta definition for functional limits.

Proof. Assume $g : A \to \mathbb{R}$ is differentiable at at a point $c \in A$. Let $\epsilon > 0$. Then we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have that

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

With a few algebraic manipulations, we can manipulate the above to state that

$$|g(x) - g(c) - g(c)(x - c)| < |x - c| \tag{1}$$

with $\epsilon = 1$. Using the triangle inequality and choosing $\delta = \min\{1, \epsilon/(1 + |g(c)|)\}$

$$\begin{aligned}
|g(x) - g(c)| &= |g(x) - g(c)(x - c) + g(c)(x - c) - g(c)| \\
&\leq |g(x) - g(c)(x - c)| + |g(c)(x - c) - g(c)| \\
&< |x - c| + |g(c)||x - c| \\
&= |x - c|(1 + |g(c)|) \\
&< \delta \cdot (1 + |g(c)|) \\
&= \frac{\epsilon}{1 + |g(c)|} \cdot (1 + |g(c)|) \\
&= \epsilon.
\end{aligned}$$

Hence, g is continuous at $c \in A$. ■

5.2.2 Combinations of Differentiable Functions

We can use the Algebraic Limit Theorem for functional limits to prove some basic algebraic combinations of differentiable functions.

Theorem 5.2.2: Algebraic Differentiability Theorem

et f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,

(i) $(f + g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, for all $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, and

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ provided that $g(c) \neq 0$.

(i) $(f + g)'(c) = f'(c) + g'(c)$.

Proof. Assume f and g are functions that are both differentiable at some point $c \in A$. Since $(f+g)(x) = f(x)+g(x)$ and the Algebraic Function Limit Theorem, we have that

$$\begin{aligned}
(f+g)'(x) &= \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)+g(x) - (f(c)+g(c))}{x-c} \\
&= \lim_{x\to c} \frac{(f(x)-f(c)) + (g(x)-g(c))}{x-c} \\
&= \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}\right) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= f'(c) + g'(c).
\end{aligned}$$

■

(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

Proof. Since f is differentiable at $c \in A$, we have that

$$\begin{aligned}
(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x) - (kf)(c)}{x-c} \\
&= \lim_{x\to c} \frac{kf(x) - kf(c)}{x-c} \\
&= \lim_{x\to c} \frac{k(f(x) - f(c))}{x-c} \\
&= k \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= kf'(c).
\end{aligned}$$

■

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Proof. Let f and g be differentiable at some point $c \in A$. By using the Algebraic Function Limit Theorem, we have that

$$\begin{aligned}
(fg)'(c) &= \lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \Big(\frac{f(x)(g(x)-g(c))}{x-c} + \frac{g(c)(f(x)-f(c))}{x-c}\Big) \\
&= \lim_{x\to c} \frac{f(x)(g(x)-g(c))}{x-c} + \lim_{x\to c} \frac{g(c)(f(x)-f(c))}{x-c} && \text{(ALFT)} \\
&= \lim_{x\to c} f(x)\Big(\lim_{x\to c} \frac{g(x)-g(c)}{x-c}\Big) + g(c)\cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= f(c)g'(c) + g(c)f'(c).
\end{aligned}$$

Hence, we have that $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$. ■

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$.

Proof. Let f and g be differentiable functions where $g(x) \neq 0$ for all $x \in A$. Note that since f and g are differentiable, they are also continuous on A. Hence, $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then observe that

$$\begin{aligned}
(f/g)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c} \\
&= \lim_{x\to c} \Big[\frac{1}{g(x)g(c)} \cdot \frac{g(x)(f(x)-f(c)) - f(x)(g(x)-g(c))}{x-c}\Big] \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(\frac{g(x)(f(x)-f(c)) - f(x)(g(x)-g(c))}{x-c}\Big) \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \lim_{x\to c} \Big(g(x)\frac{f(x)-f(c)}{x-c} - f(x)\frac{g(x)-g(c)}{x-c}\Big) \\
&= \lim_{x\to c} \Big(\frac{1}{g(x)g(c)}\Big) \Big(\lim_{x\to c} g(x)\frac{f(x)-f(c)}{x-c} - \lim_{x\to c} f(x)\frac{g(x)-g(c)}{x-c}\Big) \\
&= \frac{1}{[g(c)]^2} \cdot (g(c)f'(c) - f(c)g'(c)).
\end{aligned}$$

■

We can also compose two differentiable functions together and still get a differentiable function.

This next fact is called the chain rule. A way to prove this fact is to use the following:

$$\begin{aligned}(g \circ f)'(c) &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\ &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\ &= g'(f(c)) \cdot f'(c).\end{aligned}$$

But an issue with this proof is that the expression $f(x) - f(c)$ might be zero in the denominator for arbitrarily small neighborhoods of c.

Theorem 5.2.3: Chain Rule

et $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Proof. Assume g is differentiable at $f(c)$. Then we have that

$$g'(f(c)) = \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)}.$$

We can rewrite the quotient in the limit above by setting $d(y)$ to be the following:

$$d(y) = \frac{g(y) - g(f(c))}{y - f(c)}.$$

This is equivalent to $\lim_{y \to f(c)} d(y) = g'(f(c))$. The issue at the moment is when we set $y = f(c)$, $d(y)$ becomes undefined. But this can be mitigated by rewriting $d(y)$ like

$$g(y) - g(f(c)) = d(y)(y - f(c)) \tag{1}$$

which hold for all $y \in B$ including $y = f(c)$. Hence, we are allowed to substitute $y = f(t)$ for any arbitrary $t \in A$ into $d(y)$. If $t \neq c$, we can divide (1) by $(t - c)$ to get

$$\frac{g(f(t) - g(f(c)))}{t - c} = d(f(t))\frac{f(t) - f(c)}{t - c}.$$

Hence, we have that

$$\begin{aligned}(g \circ f)'(c) &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\ &= \lim_{x \to c} \left(\frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}\right) \\ &= g'(f(c)) \cdot f'(c)\end{aligned}$$

■

5.2.3 Darboux's Theorem

A takeaway from our introduction to Derivatives is that differentiable functions need always be continuous and that our main example for this is for $n = 2$ with the function $g_2(x) = x^2 \sin(1/x)$

where $g_2(0)$. However, differentiable functions do posses the intermediate value property. This leads us to next theorem that says that functions attain their max and mins at points where the derivative is zero.

Theorem 5.2.4: Interior Extremum Theorem

Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$; that is, $f(c) \geq f(x)$ for all $x \in (a, b)$, then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.

Proof. Since c is in an open interval (a, b), we can construct two sequences $(x_n), (y_n) \subseteq (a, b)$ such that both sequences converge to $c \in (a, b)$ and satisfy $x_n < c < y_n$ for all $n \in \mathbb{N}$. Since f attains its maximum value at some point $c \in (a, b)$, we have that for all $n \in \mathbb{N}$, $f(y_n) \leq f(c)$. Hence, we can say that

$$f(y_n) - f(c) \leq 0.$$

Dividing by $(y_n - c)$ and using the Order Limit Theorem gives us the following

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0. \tag{1}$$

Now if f attains a minimum, we have $f(x_n) - f(c) \geq 0$ for all $n \in \mathbb{N}$ which implies further that

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

by the Order Limit Theorem. ■

This theorem leads to a very important result about differntiable functions containing the intermediate value property.

Theorem 5.2.5: Darboux's Theorem

If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.

Proof. We can define a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. Since g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \alpha$ and that $g'(a) < 0 < g'(b)$, we want to show that $g'(c) = 0$ for some $c \in (a, b)$. ■

5.3 The Mean Value Theorems

5.3.1 Mean Value Theorem

(a) We can find a point along some interval $[a, b]$ of a differentiable function f such that we will have a slope of f where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for at least one point $c \in (a, b)$.

(b) Used to prove L'hopital's rule for limits of quotients of differentiable functions.

(c) Used in the study of infinite series of differentiable functions.

(d) One of mechanisms needed to show Lagrange's Remainder Theorem and used to approximate the error between a Taylor polynomial.

Theorem 5.3.1: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.

Proof. Since f is continuous on a compact set, we know that f attains a maximum and a minimum. If f attains a maximum and minimum at the endpoints and the fact that $f(a) = f(b)$, we know that f must be a constant function. Hence, we can choose any $x \in [a, b]$ such that $f'(x) = 0$. If f attains a maximum or minimum in the interior of f then there exists $c \in (a, b)$ such that $f'(c) = 0$. ■

Theorem 5.3.2: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Notice that the Mean Value Theorem reduces to Rolle's Theorem in the case where $f(a) = f(b)$. Consider the equation of a line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \Big(\frac{f(b) - f(a)}{b - a}(x - a)\Big) + f(a).$$

Furthermore, we want to consider the difference between this line and the function $f(x)$. Define a new function d where

$$d(x) = f(x) - \Big[\Big(\frac{f(b) - f(a)}{b - a}\Big)(x - a) + f(a)\Big],$$

Observe that d is continuous on $[a, b]$ since f is continuous on $[a, b]$ and differentiable on (a, b) and satisfies $d(a) = 0 = d(b)$. By differentiating $d(x)$, we have that

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Now, using Rolle's Theorem, we can find a $c \in (a, b)$ such that $d'(c) = 0$. Hence,

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \iff f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

Now consider a constant function $f(x) = k$ for any k. Intuition suggests that for all $x \in A$, we have $f'(x) = 0$. Is there any way we can prove that $f(x)$ is constant given $f'(x) = 0$ for all $x \in A$? Indeed, we can using the Mean Value Theorem.

Corollary 5.3.1: I

$g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.

Proof. Take $x, y \in A$ and assume $x < y$. Applying the Mean Value Theorem to g on the interval $[a, b]$, we can see that

$$g'(c) = \frac{g(y) - g(x)}{y - x}.$$

Since $g(x) = 0$ for all $x \in A$, we have that

$$\frac{g(y) - g(x)}{y - x} = 0 \iff g(y) = g(x).$$

Set k equal to this common value. Since $x, y \in A$ are arbitrary, it follows that $g(x) = k$ for all $x \in A$. ■

Corollary 5.3.2

If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some interval $k \in \mathbb{R}$.

Proof. Suppose f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$. Let us define a new function $h(x) = f(x) - g(x)$. Differentiating this function gives us the following $h'(x) = f'(x) - g'(x)$. But since $g'(x) = f'(x)$, we have that $h'(x) = 0$. Since h is differentiable on an interval A, we know that $h(x) = k$. Hence, we have that

$$k = f(x) - g(x) \iff f(x) = g(x) + k.$$

■

We can build a more general form of the Mean Value Theorem which can be used to prove L'hopital's rules and the Lagrange Remainder Theorem.

Theorem 5.3.3: Generalized Mean Value Theorem

If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. This result follows by applying the Mean Value Theorem to the function

$$h(x) = [f(b) - f(a)]g(x) = [g(b) - g(a)]f(x).$$

The details are requested in Exercise 5.3.5. ∎

5.3.2 L'Hopital's Rules

The Algebraic Limit Theorem asserts that when taking a limit of a quotient of functions we can write

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

provided that the quotient is well-defined. What happens when the denominator has a limit that goes to zero while the numerator goes to a limit that is non-zero? Both zero? Both infinite?

Theorem 5.3.4: L'Hopital's Rule: 0/0 case

Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x\to a} \tfrac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x\to a} \tfrac{f(x)}{g(x)} = L.$$

Proof. Of course they would leave this to the reader to finish. It is requested in Exercise 5.3.11. ∎

Next is the case when we replace the assumption of the last theorem with the case that $\lim_{x\to a} g(x) = \infty$. We can define what it means to have an infinite limit.

Definition 5.3.1

Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x\to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$. We can define $\lim_{x\to c} g(x)$ in a similar way.

Next is the case of L'hopital's rule when applied to the case when both the numerator and the denominator go to infinity.

Theorem 5.3.5: L'Hopital's Rule: ∞/∞ case

ssume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\epsilon > 0$. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there exists $\delta_1 > 0$ such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2} \tag{1}$$

for all $a < x < a + \delta_1$. For convenience of notation, let $t = a + \delta_1$ and note that t is fixed for the remainder of the argument. Let our functions f and g be defined on the interval $[x, t]$ for any $x \in (a, t)$. We can use the Generalized Mean Value Theorem on the interval $[x, t]$ to get that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(t)}{g(x) - g(t)}$$

for some $c \in (x, t)$. Since we are considering $t = a + \delta_1$, we have that

$$L - \frac{\epsilon}{2} < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \frac{\epsilon}{2} \tag{2}$$

for all $x \in (a, t)$. Our goal is to isolate the fraction $f(x)/g(x)$ by multiplying (2) by $(g(x) - g(t))/g(x)$. We need to assume that $g(x) \geq g(t)$ so that the quantity we are multiplying by is positive (or else we will switch the order of the inequality which we don't want). Carrying our our plan results in the following inequality

$$L - \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} + \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}.$$

Since t is fixed and that $\lim_{x\to a} g(x) = \infty$, we can choose $\delta_2 > 0$ such that this our choice of multiplying by the above quantity will satisfy $g(x) \geq g(t)$ for all $a < x < a + \delta_2$. By the same fact, we can also choose δ_3 such that $a < c < a + \delta_3$ implies that $g(x)$ is large enough to ensure that both

$$\frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} \quad \text{and} \quad \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}$$

are less than $\epsilon/2$ in absolute value. Choosing $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ guarantees that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

for all $a < x < a + \delta$. ■

5.4 Continuous Nowhere-Differentiable Functions

5.4.1 The Sawtooth Function

In the last few sections, we have seen that continuity does not immediately imply that a function is differentiable. The prime example for this is the absolute value function $f(x) = |x|$ where differentiable at every point except $x = 0$. Can we construct functions that are continuous at every point but non-differentiable everywhere? The answer is yes and it is in the form of

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

where the values of a and b are carefully chosen. All of us may notice that this is just the Fourier Series. Define a function $h(x) = |x|$ that replaces the oscillations provided by $\cos(x)$ on the interval $[-1, 1]$ and extend h to be defined on all of $\mathbb{R}$ such that $h(x+2) = h(x)$. Instead of the smooth oscillations that we see with the $\cos(x)$ graph, we have a periodic "sawtooth" looking graph provided by $h(x) = |x|$.

Exercise 5.4.1

Sketch a graph of $(1/2)h(2x)$ on $[-2, 3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

as n gets larger.

Solution. The graph of $h_1(x)$ is just the Sawtooth function $h(x)$ but with a maximum height of $1/2$ and the length of the period is just 1. Notice that as n gets bigger and bigger our height for $h_n(x)$ is just $1/2^n$ while the period is $1/2^{n-1}$. The slopes of all the segments that make up $h_n(x)$ is just ± 1 for all $n \in \mathbb{N}$. ■

Let's define

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

which we claim as having the property that it is continuous for all $x \in \mathbb{R}$ but non-differentiable for all $x \in \mathbb{R}$.

5.4.2 Infinite Series of Functions and Continuity

As we can see, the way we defined $g(x)$ is very different from the way the functions we have been defining thus far where for each $x \in \mathbb{R}$, $g(x)$ is the value of an infinite series.

Exercise 5.4.2

Fix $x \in \mathbb{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges and thus $g(x)$ is properly defined.

Solution. Since the maximum height of $h_n(x)$ is $1/2^n$, it follows that for all $n \in \mathbb{N}$, the sequence of partial sums is bounded; that is

$$0 \leq \frac{1}{2^n} h_n(2^n x) \leq \frac{1}{2^n}.$$

Since $\frac{1}{2^n}$ produces a geometric series which converges, we know that $\frac{1}{2^n} h_n(2^n x)$ must also converge by the Comparison Test. This means the series that defines $g(x)$ converges and is, therefore, well defined. ■

We can ask a couple of questions about functions that are defined by infinite series.

(a) Certainly, finite sum of a continuous functions is continuous but is the infinite sum of continuous functions necessarily continuous? We will see that this is not always the case in the next chapter.

Exercise 5.4.4

As the graph in Figure 5.7 suggests, the structure of $g(x)$ is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.

(a) how do we know g attains a maximum value M on $[0, 2]$? What is the value?

> *Solution.* Since g is continuous on a compact set $[0, 2]$, we know that g must attain its maximum and minimum on the interval. Since g converges, we can use associativity to redefine g in the following way:
>
> $$f_n(x) = h_{2n}(x) + h_{2n+1}(x)$$
>
> for all $n \in \mathbb{N}$. A scaling argument is used to show that $g(x) \leq \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3}$. Hence, the max of $g(x) = \frac{4}{3}$. ∎

Nondifferentiability

Let us try to prove that g is not differentiable for all $x \in \mathbb{R}$. Looking at $x = 0$, we can see that g appears to not be differentiable here. Consider the sequence $x_m = 1/2^m$, where $m = 0, 1, 2, \ldots$.

Exercise 5.4.5

Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$

and use this to prove that $g'(0)$ does not exist.

Proof. For $g'(0)$ to exist, the sequential criterion for limits requires that

$$g'(0) = \lim_{m\to\infty} \frac{g(x_m) - g(0)}{x_m - 0}$$

exist for any sequence $(x_m) \to 0$. Let us fix $m \in \mathbb{N}$ and consider $x_m = 1/2^m$. Then plugging this in to g, we get that

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}).$$

Now we have to consider a few cases. Suppose $n > m$, then $h(2^{n-m}) = 0$ because the sawtooth function is zero for even n. If $n \leq m$, then we have $h(x) = x$ and we get that

$$\frac{1}{2^n} h(^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$$

by the definition of $h(x)$. This means we can represent $g(x_m)$ as a finite sum

$$g(x_n) = \sum_{n=0}^{m} \frac{1}{2^m}.$$

Using the difference quotient, we can see that

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{\sum_{n=0}^{m} 1/2^m}{1/2^m} = \sum_{n=0}^{m} 1 = m + 1.$$

Notice that the $g(x_m) = m+1$ is a sequence that diverges, we know that g is not differentiable at zero and therefore does not exist. ■

Although intuition might lead us astray by telling us that $g'(0) = \infty$, we need to remember that for $x_m = -(1/2^m)$ we can use the same argument above to produce a difference quotient that approaches $-\infty$. This is that "cusp" that we see at $x = 0$ in the graph of g.

Using the same argument above, we can prove g' does not exists for $x = 1$ and $x = 1/2$. In fact, $g'(x)$ does not exist for any $x \in \mathbb{Q}$ such that $x = p/2^k$ where $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. These values of $x \in \mathbb{Q}$ are called *dyadic* points. For every x that is *dyadic*, h_n contains a corner at x as long as $n \geq k$.

What if x is not *dyadic*? For fixed values of $m \in \mathbb{N} \cup \{0\}$, x can be between two dyadic points; that is,

$$\frac{p_m}{2^m} < x < \frac{p_m + 1}{2^m}.$$

Setting $x_m = p_m/2^m$ and $y_m = (p_m) + 1)/2^m$, we can repeat this for each m that produces two sequences (x_m) and (y_m) that satisfy the following:

(a) $\lim x_m = \lim y_m = x$,

(b) $x_m < x < y_m$.

We can prove the following lemma:

Lemma 5.4.1: L

t f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, then

$$f'(a) = \lim_{n \to \infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

Proof. ∎

We can use this lemma to show that $g'(x)$ does not exist.

Chapter 6

Sequences and Series of Functions

6.1 Discussion: Power Series

6.1.1 Geometric Series

Series that are easily summable are the Geometric series. One example of such a series is the following:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{1}$$

for all $|x| < 1$. A few more examples include the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{3}{4}.$$

We can take derivatives of both sides of (1) and get

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots. \tag{2}$$

A question we can ask ourselves is whether or not this formula is valid at least on the open interval $(-1, 1)$? It turns out that it is true for (2) to hold along $(-1, 1)$ (we will find out later why this is the case).

There is another surprising connection of (1) when we replace x term with an x^2 term and then take the integral of the left side. What we end up getting is a relationship between circles and infinite series; that is,

$$(\arctan(x))' = \frac{1}{1+x^2} \quad \text{and} \quad \arctan(0) = 0$$

where setting $x = 1$ gives us

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots.$$

Does treating the infinite series above like a polynomial really a valid way to produce a formula like the one above? It turns out we can treat these series as if we were just adding up an infinite amount of polynomials. Hence, which is why we have a whole section dedicated to *power series.*

What are some applications of power series? Well for one, we can express famous functions such as e^x, $\sqrt{1+x}$, or $\sin(x)$ in terms of an infinite series of polynomial terms. A notable example of power series is the generalization of the binomial formula. For any $n \in \mathbb{N}$, we have that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.$$

Say, we set $n = -1$, then our series is written as

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots,$$

which is equivalent (1). Setting $n = 1/2$ then our infinite series becomes

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{3}{2^3 3!}x^3 - \frac{3 \cdot 5}{2^4 3!}x^4 + \dots \ .$$

There are many more examples such as this that uses some sophisticated machinery that we do not quite know yet. One very important question we can ask ourselves is what properties of power series allows them to be manipulated in such a way that is so impervious to the infinite? We will explore this more in the upcoming sections.

6.2 Uniform Convergence of a Sequence of Functions

Just like our studies demonstrated in Chapter 2, we will first study the behaviors and properties of converging *sequences* of functions. The results that we have gathered about sequences and series so fat will be immediately applicable to our study of sequences of functions.

6.2.1 Pointwise Convergence

Definition 6.2.1: Pointwise Convergence

For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions *converges pointwise* on A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.
In this case, the following notations are all equivalent to each other

(i) $f_n \to f$

(ii) $\lim f_n = f$

(iii) $\lim_{n\to\infty} f_n(x) = f(x)$.

(iii) of the definition above is especially useful if there are any confusions the may arise as to whether or not x or n is the limiting variable.

Example 6.2.1

(i) Consider the sequence of functions f_n defined by

$$f_n(x) = \frac{x^2 + nx}{n}$$

on all of $\mathbb{R}$. We can compute the limit of f_n

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x.$$

Thus, we have that (f_n) converges *pointwise* to $f(x) = x$ on $\mathbb{R}$.

(ii) Let $g_n(x) = x^n$ on the set $[0, 1]$ where we consider the situation as $n \to \infty$. If $0 \le x < 1$, then we know that $x^n \to 0$. On the other hand, suppose $x = 1$, then we have that $x^n \to 1$. It follows that $g_n \to g$ converges pointwise on $[0, 1]$, where

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

We have a problem when considering continuity at $x = 1$.

(iii) Consider $h_n(x) = x^{1+\frac{1}{2n-1}}$ on the set $[-1, 1]$. For a fixed $x \in [-1, 1]$, we have

$$\lim_{n\to\infty} h_n(x) = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|.$$

Note that this function is not differentiable at $x = 0$.

6.2.2 Continuity of the Limit Function

We will begin this section by failing to prove that the pointwise limit of continuous functions is continuous. We will then find the holes of the subsequent argument so that we may understand why we need a stronger footing on the meaning of convergence for a sequence of functions.

Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and let us assume that (f_n) converges to a pointwise limit f. We will try to argue that the limit f is continuous. Let us fix $c \in A$, and let $\epsilon > 0$. Our objective is to find $\delta > 0$ such that whenever $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \epsilon.$$

We may use the Triangle Inequality to write

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|. \end{aligned}$$

Our impression is to make each term of the right hand side of this inequality small by using the fact that $f_n \to f$ and the continuity of f_n. Since $c \in A$ is fixed, let us choose $N \in \mathbb{N}$ such that

$$|f_N(c) - f(c)| < \frac{\epsilon}{3}.$$

Since N is chosen, the continuity of our particular choice f_N implies that there exists a $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

for all x whenever $|x - c| < \delta$. But here lies the problem of using the continuity of f_n; that is, we also need the following to hold:

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all x satisfying $|x - c| < \delta$. A few problems with this argument include

(i) Our choice of x depends on δ which also depends on our choice of N. This means for every choice of x along $(c-\delta, c+\delta)$, we will get a different N. We want our choice of δ to be uniform for any x.

(ii) The choice of x is not fixed the way c is on the interval $(c - \delta, c + \delta)$. This means that our choice x has to work along the interval.

This problem is apparent in our second example at the beginning of this section where the inequality

$$|g_n(1/2) - g(1/2)| < \frac{1}{3}$$

for $n \geq 2$ whereas

$$|g_n(9/10) - g(9/10)| < \frac{1}{3}$$

is true only after $n \geq 11$.

6.2.3 Uniform Convergence

To solve our the problems of pointwise convergence of functions, we introduce a stronger notion for convergence of functions.

Definition 6.2.2: Uniform Convergence

Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) *converges uniformly* on A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

whenever $n \geq N$ and $x \in A$.

Let us restate the definition of Pointwise convergence so that we are able to distinguish the key differences between the two.

Definition 6.2.3: Pointwise Convergence

et (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) *converges pointwise on* A to a limit f defined on A if, for every $\epsilon > 0$ and $x \in A$, there exists an $N \in \mathbb{N}$ (may depend on x) such that

$$|f_n(x) - f(x)| < \epsilon$$

whenever $n \geq N$.

Key Differences:

(i) In uniform convergence, notice that we only need

$$|f_n(x) - f(x)| < \epsilon$$

to hold for all $\epsilon > 0$; that is, our choice of x will not affect our choice of N. Another way to state this is $N \neq N(\epsilon, x)$

(ii) In pointwise convergence, not only do we need convergence to hold for all $\epsilon > 0$, we also need it to hold for all x.

Example 6.2.2

(i) Let

$$g_n(x) = \frac{1}{n(1+x^2)}.$$

For any fixed $x \in \mathbb{R}$, it is apparent that $\lim_{n\to\infty} g_n(x) = 0$ so that $g(x) = 0$ is the pointwise limit of the sequence (g_n) on $\mathbb{R}$. We want to know if (g_n) uniformly convergent. Since $1/(1+x^2) \leq 1$ for all $x \in \mathbb{R}$ implies that

$$|g_n(x) - g(x)| = \left|\frac{1}{n(1+x^2)} - 0\right| \leq \frac{1}{n}.$$

Hence, any given $\epsilon > 0$, we can choose $N > 1/\epsilon$ (which does not depend on x), we have that

$$n \geq N \text{ implies } |g_n(x) - g(x)| < \epsilon$$

for all $x \in \mathbb{R}$. Hence, $g_n \to 0$ uniformly on $\mathbb{R}$.

(ii) What about our first example from the very beginning of this section? Does it converge uniformly as well? Let $f_n(x) = (x^2 + nx)/n$. Since $(f_n) \to f$ pointwise where $f(x) = x$. It turns our that f_n is not uniformly convergent. To see why this is the case, we write that

$$|f_n(x) - f(x)| = \left|\frac{x^2 + nx}{n} - x\right| = \frac{x^2}{n}.$$

For $|f_n(x) - f(x)| < \epsilon$ to hold, we would need to create a choice of N such that

$$N > \frac{x^2}{\epsilon}.$$

While we certainly have convergence for every $x \in \mathbb{R}$, we still have our choice of N not uniform. Although not uniformly convergent on all of $\mathbb{R}$, we do end up having uniform convergence when we consider f_n over a closed interval $[-b, b]$. Hence, we have that

$$\frac{x^2}{n} \leq \frac{b^2}{n}.$$

Given any $\epsilon > 0$, we can choose $N > b^2/\epsilon$ that is not dependent on any $x \in [-b, b]$.

Graphically speaking, the uniform convergence of f_n to a limit f on a set A can be visualized by constructing an $\epsilon-$neighborhood around the limit f for which all of f_n is completely contained within the neighborhood for all $n \geq N$ for some point $N \in \mathbb{N}$.

6.2.4 Cauchy Criterion

Recall that the Cauchy Criterion states an equivalence between convergent sequences and Cauchy sequences without stating the limit of the sequence. The usefulness of such a theorem creates an opportunity for an analogous characterization of uniformly convergent sequences of functions.

Theorem 6.2.1: Cauchy Criterion for Uniform Convergence

A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

whenever $m, n \geq N$ and $x \in A$.

Proof. Exercise 6.2.5. ∎

6.2.5 Continuity Revisited

Let us now prove that the limit function of a sequence of continuous functions is continuous.

Theorem 6.2.2: Continuous Limit Theorem

Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f, If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Fix $c \in A$ and let $\epsilon > 0$. Since $(f_n) \to f$ is uniformly convergent on $\mathbb{R}$, we can choose an $N \in \mathbb{N}$ (that does not depend on x) such that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$. Since f_N is continuous, there exists $\delta > 0$ for which

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

is true whenever $|x - c| < \delta$. Just like our argument at the beginning of this section, we have that

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Hence, f is continuous at $c \in A$. ∎