Math 230A: Homework 1

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1. Mark each statement True or False.

   1-1) If x and y are elements of an ordered field, then either $x \leq y$ or $y < x$. **False.**

   1-2) Every ordered field has the least upper bound property. **True.**

   1-3) If $E \subseteq \mathbb{R}$ is bounded above and $\alpha = \sup E$, then $\alpha \in E$. **False.**

   1-4) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, then $\alpha = \sup E$. **False.**

   1-5) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \in E$, then $\alpha = \sup E$. **True.**

2. Prove the following: Suppose α is an upper bound for $E \subset \mathbb{R}$. Then $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   Proof. Suppose α is an upper bound for $E \subseteq \mathbb{R}$. For the forwards direction, suppose $\alpha = \sup E$ and let $\varepsilon > 0$. Note that $\alpha - \varepsilon < \alpha$ implies that $\alpha - \varepsilon$ is NOT an upper bound of E. By definition, there must exist an element $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$. Thus, $\alpha < x_0 + \varepsilon$.

   For the backwards direction, let $\varepsilon > 0$ and let α be an upper bound of E. Suppose there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$. Let γ be any upper bound of E. Thus, $x_0 \leq \gamma$ by definition. By assumption, we see that

   $$\alpha - \varepsilon < x_0 \leq \gamma \Rightarrow \alpha \leq \gamma + \varepsilon.$$

   Since $\varepsilon > 0$ is arbitrary, we can use Exercise 5 to conclude that $\alpha \leq \gamma$, showing that α is the least upper bound. Thus, $\sup E = \alpha$. ■

3. Let $E = \{n/n + 1 : n \in \mathbb{N}\}$. Prove that $\sup E = 1$.

   Proof. ■

4. State the analogue of the Very Useful Theorem for infimums.

   Solution. Suppose β is a lower bound for $E \subseteq \mathbb{R}$. Then $\beta = \inf E$ if and only if for all $\varepsilon > 0$, there exists $y_0 \in E$ such that $y_0 < \beta + \varepsilon$. ■

5. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Prove $a \leq b$.

   Proof. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Suppose for sake of contradiction that $a > b$. Observe that

   $$b < a \leq b + \varepsilon \Rightarrow b \leq b + \varepsilon.$$

   Then subtracting b on both sides gives us $\varepsilon \geq 0$. But this is a contradiction because we had assumed that $\varepsilon > 0$. Thus, it must be the case that $a \leq b$. ■

6. Complete the following proof.

Theorem (Greatest-lower-bound property of $\mathbb{R}$)**.** Every nonempty subset of A of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf A$ exists and is a real number.

Proof. Continuation of the proof presented in homework. Set $\beta = -\alpha$ where $\alpha = \sup(-A)$. Thus, there exists $a \in A$ such that $-\gamma < -a$. Multiplying by a negative on both sides of this inequality, we get $\gamma > a$ for some $a \in A$. If γ was a lower bound of A, then $\gamma \leq a$ for all $a \in A$. However, we have the negation of this, so we must have that γ is NOT a lower bound of A, which satisfies part (ii). Since $\beta \leq \alpha$ for all $a \in A$ and $\beta < \gamma$ implies γ is not a lower bound for A, we conclude that β must be the infimum of A. ■

7. (i) Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many elements of A.

Proof. Since A is nonempty and bounded above, we know that A has the least-upper-bound property. Set $\sup A = \alpha$. Suppose $\sup A \notin A$. Suppose for sake of contradiction that there exists an $\varepsilon > 0$ such that the open interval $(\alpha - \varepsilon, \alpha)$ contains a finitely many elements of A. Thus, for every $x_i \in A$ with $1 \leq i \leq n$, we see that $x_i \in (\alpha - \varepsilon, \alpha)$ for some $\varepsilon > 0$. Then we see that

$$\alpha - \varepsilon < x_i < \alpha \Rightarrow \alpha < x_i < \alpha + \varepsilon.$$

But note that for all $x_i \in A$, we have $x_i > \alpha$. This tells us that α is NOT an upper bound of A, but instead that $\sup A \in A$ which is a contradiction. Thus, it must be the case that $(\alpha - \varepsilon, \alpha)$ must contain infinitely many points. ■

(ii) Let $B \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below. Show that if $\inf B \notin B$, then for all $\varepsilon > 0$ the open interval $(\inf B, \inf B + \varepsilon)$ contains infinitely many elements of B.

Proof. Since B is nonempty and bounded below, we know that B must contain the greatest upper bound property. Set this greatest upper bound to be $\beta = \inf B$. Suppose for sake of contradiction that there exists an $\varepsilon > 0$ such that the open interval $(\beta, \beta + \varepsilon)$ contains finitely many elements of B. Thus, for every $x_i \in B$ with $1 \leq i \leq n$, we see that $x_i \in (\beta, \beta + \varepsilon)$. Then we see that

$$\beta < x_i < \beta + \varepsilon \Rightarrow \beta - \varepsilon < x_i < \beta.$$

But note from this inequality that $x_i < \beta$, implying that β is not a lower bound of B. Which means that $\beta \in B$ which is a contradiction of our assumption that $\beta \notin B$. Hence, B must contain infinitely many points. ■

8. Let $A, B \subseteq \mathbb{R}$ be nonempty, bounded sets and let $c \in \mathbb{R}$. Define the following sets:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{ab : a \in A, b \in B\} \\ cA &= \{ca : a \in A\}. \end{aligned}$$

Prove that

9-1) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Since A and B are both nonempty and bounded below, we know that both $\inf(A)$ and $\inf(B)$ exists. Thus, $a \geq \inf(A)$ and $b \geq \inf(B)$ for all $a \in A$ and $b \in B$, respectively. Thus,

$$a + b \geq \inf(A) + \inf(B) \quad \text{for all } a + b \in A + B.$$

implies that $A + B$ is bounded below. Since $A + B \neq \emptyset$ (since A and B are both nonempty), we see that $\inf(A + B)$ exists.
Now, we will show that

$$\inf(A + B) = \inf(A) + \inf(B);$$

that is, we need to show that

$$\inf(A + B) \geq \inf(A) + \inf(B) \tag{1}$$

and

$$\inf(A + B) \leq \inf(A) + \inf(B). \tag{2}$$

To show (1), let $\varepsilon > 0$. Using the lemma found in Problem 2, there exists an $\alpha \in A$ and $\beta \in \beta$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{2}$$

and

$$\beta < \inf(B) + \frac{\varepsilon}{2},$$

respectively. Adding these two inequalities, we get

$$\alpha + \beta < \inf(A) + \inf(B) + \varepsilon.$$

Since $A + B$ is bounded below, we have $\inf(A + B) \leq \alpha + \beta$ such that

$$\inf(A + B) \leq \alpha + \beta < \inf(A) + \inf(B) + \varepsilon.$$

Using Problem 5, we conclude that

$$\inf(A + B) \leq \inf(A) + \inf(B).$$

To show (2), let $\varepsilon > 0$ again. Using the same lemma, there exiss $\varphi \in A + B$ with $\varphi = \gamma + \lambda$ where $\gamma \in A$ and $\lambda \in B$ such that

$$\gamma + \lambda = \varphi < \inf(A + B) + \varepsilon.$$

Since A and B are both bounded below, we can see that

$$\inf(A) + \inf(B) \leq \gamma + \lambda.$$

Thus, we have

$$\inf(A) + \inf(B) \leq \inf(A + B)$$

by Problem 5. ∎

9-2) $\sup(A + B) = \sup(A) + \sup(B)$

Proof. Since A and B are both nonempty, we can see that $A+B$ is also nonempty.

Furthermore, A and B are both bounded above, so $a \leq \sup(A)$ for all $a \in A$ and $b \leq \sup(B)$ for all $b \in B$ implies that

$$a + b \leq \sup(A) + \sup(B) \quad \text{for all } a \in A \text{ and } b \in B.$$

Thus, $A + B$ must be bounded above, and so by definition 1.10 in the textbook, we see that the $\sup(A + B)$ exists. Our goal is to show that

$$\sup(A + B) = \sup(A) + \sup(B);$$

that is, it suffices to show that both

$$\sup(A + B) \leq \sup(A) + \sup(B) \tag{1}$$

and

$$\sup(A + B) \geq \sup(A) + \sup(B). \tag{2}$$

To prove (1), let $\varepsilon > 0$. By the lemma found in Problem 2, there exists $\varphi \in A+B$ such that $\varphi > \sup(A+B) - \varepsilon$ with $\varphi = \alpha + \beta$ for some $\alpha \in A$ and $\beta \in B$. Thus, we have $\alpha \leq \sup(A)$ and $\beta \leq \sup(B)$, and so

$$\alpha + \beta \leq \sup(A) + \sup(B)$$

implies

$$\sup(A) + \sup(B) \geq \alpha + \beta > \sup(A + B) - \varepsilon.$$

Using Problem 5, we can see that

$$\sup(A) + \sup(B) \geq \sup(A + B)$$

which proves (1).
To prove (2), let $\varepsilon > 0$ again. Since $\sup(A)$ and $\sup(B)$ exists, we see that there exists $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{2}$$

and

$$\beta > \sup(B) - \frac{\varepsilon}{2}$$

by the lemma found in Problem 2. Adding these two inequalities results in

$$\alpha + \beta > \sup(A) + \sup(B) - \varepsilon.$$

Note that $\alpha + \beta \in A + B$ and $A + B$ is bounded above, we have that

$$\sup(A + B) \geq \alpha + \beta > \sup(A) + \sup(B) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can see that

$$\sup(A + B) \geq \sup(A) + \sup(B)$$

by Problem 5. Since (1) and (2) are satisfied, we can conclude that

$$\sup(A + B) = \sup(A) + \sup(B).$$

■

9-3) $\sup(-A) = -\inf(A)$

Proof. Since $A \neq \emptyset$ and A is bounded above, we can say that $-A$ is also nonempty and bounded above as well from Problem 6. Thus, we know that the supremum of $-A$ exists. Our goal is to show that both

$$\sup(-A) \leq -\inf(A) \tag{1}$$

and

$$\sup(-A) \geq -\inf(A). \tag{2}$$

Let $\varepsilon > 0$. To prove (1), we can use the lemma from Problem 2 to state that there exists an $-a \in -A$ such that

$$\sup(-A) < -a - \varepsilon.$$

Since A is bounded below, we know that $a \geq \inf(A)$. Multiplying through with a negative gives us

$$-a \leq -\inf(A)$$

which implies that

$$\sup(-A) < -a - \varepsilon \leq -\inf(A) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$-\inf(A) \geq \sup(-A)$$

which proves (1).

Now, to prove (2), we can use the lemma from problem 6, there exists a $\alpha \in A$ such that

$$\alpha < \inf(A) + \varepsilon.$$

Multiplying through by a negative on this inequality, we have

$$-\alpha > -\inf(A) - \varepsilon.$$

But A is bounded below, so we must have $-\alpha \leq \sup(-A)$. Thus, we have

$$\sup(-A) \geq -\alpha > -\inf(A) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\sup(-A) \geq -\inf(A)$$

by Problem 5, which proves (2).

■

9-4) $\inf(-A) = -\sup(A)$.

Proof. Our goal is to show that $\inf(-A) = -\sup(A)$; that is, we need to show that

$$\inf(-A) \leq -\sup(A) \tag{1}$$

and

$$\inf(-A) \geq -\sup(A). \tag{2}$$

First, we show (1). Let $\varepsilon > 0$. Using the lemma found in Problem 2, there exists $-\alpha - A$ such that

$$-\alpha < \inf(-A) + \varepsilon$$

. Since A is bounded above, $\alpha \leq \sup(A)$. Multiplying by a negative on this inequality produces $-\alpha \geq -\sup(A)$. Thus, we have

$$-\sup(A) \leq -\alpha < \inf(-A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$-\sup(A) \leq \inf(-A)$$

by Problem 5, showing (2). ∎

9-5) $\sup(A - B) = \sup(A) - \inf(B)$.

Proof. Observe that

$$\begin{aligned} \sup(A - B) = \sup(A + (-B)) &= \sup(A) + \sup(-B) && \text{(9-2))} \\ &= \sup(A) - \inf(B) && \text{(9-3))} \end{aligned}$$

∎

9-6) $\inf(A - B) = \inf(A) - \sup(B)$.

Proof. Observe that

$$\begin{aligned} \inf(A - B) = \inf(A + (-B)) &= \inf(A) + \inf(-B) && \text{(9-1))} \\ &= \inf(A) - \sup(A). && \text{(9-4))} \end{aligned}$$

∎

9-7) $\sup(cA) = c\sup(A)$ if $c > 0$.

9-8) $\inf(cA) = c\inf(A)$ if $c > 0$.

9-9) $\sup(cA) = c\inf(A)$ if $c < 0$.

9-10) $\inf(cA) = c\sup(A)$ if $c < 0$.

9-11) Is it true that $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

9. Recall that $\mathbb{Q}$ is a field; in particular, $\mathbb{Q}$ is closed under addition and multiplication.

10-1) Prove that if $p \in \mathbb{Q}$ and t is an irrational number, then $p + t$ is an irrational number.

Proof. ∎

10-2) Complete the following proof.

Theorem. Given any two real numbers $x < y$, there exists an irrational number t satisfying $x < t < y$.

Proof. ∎