0.1 Lecture 4

0.1.1 Topics

(i) Continue discussion of convergence of sequence/series.

(ii) Discuss exponential, sine, and cosine function.

0.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

Definition (Convergence of Series)**.** If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example 0.1.1. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i}\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$

$$\frac{1}{2} is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider

$$\frac{1}{2} i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2} is_n = \frac{1}{2} is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow s_n = \frac{1}{i}\left[1 - \frac{1}{2^n}\right].$$

Thus, taking the limit as $n \to \infty$ gives us

$$\lim_{n\to\infty} s_n = \frac{1}{i}.$$

Example 0.1.2. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since $|z| < 1$, we know that

$$\sum_{n-1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n\to\infty} \sum_{k=1}^{n} z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as $n \to \infty$, we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series)**.** Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example 0.1.3. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses.

Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in $\mathbb{C}$.

0.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function)**.** For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series)**.** We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z+w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \text{ \textbf{Show this!}}.$$

0.2 Lecture 5

0.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

0.2.2 Sine and Cosine Functions

Recall that we defined

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C}$$
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}.$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2.$

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z + i \sin z.\end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x + iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.
- $\Im(\exp(z)) = e^x \sin y, \ |\exp(z)| = e^x$.
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and $w = u + iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that

$\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x + iy$. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ∎

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\cdot}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\cdot} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Proposition. Let $z \in \mathbb{C}^{\cdot}$. Find all possible solutions of $\exp(w) = z$.

Lemma. For $z \in \mathbb{C}^{\cdot}$, we have

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

0.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \mathrm{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \mathrm{Log}\, z + 2\pi ik$$

is a **multivalued function** and $\log z = \mathrm{Log}\, z$ if we want the solution to be in S.