Regular Surface Exercises

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Definition (Regular Surfaces). A subset $S \subseteq \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $x : U \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ such that

(1) $\mathbf{x}$ is differentiable; that is, the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders in U.

(2) $\mathbf{x}$ is a homeomorphism; that is, $\mathbf{x}$ contains an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous.

(3) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

Problem 1. Show that the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$ is a regular surface.

Proof. Define the following parametrization for C by $x = \cos(u)$, $y = \sin(u)$, and $z = v$ for every $(u, v) \in U$ where $U = (a, b) \times (c, d) \subseteq \mathbb{R}^2$ is an open set. We will show that this parametrization of C satisfies properties (1) through (3) of the definition above. Indeed, we have

(1) Clearly, we see that $\cos(u)$, $\sin(u)$, and v are functions that have derivatives of all order. Hence, $\mathbf{x}(u, v) = (\cos(u), \sin(u), v)$ contains derivatives of all order on $C \cap U$.

(2) We can see that the component functions contain their respective inverses on $C \cap U$; that is, $\cos^{-1}(x) = u$, $\sin^{-1}(y) = v$, and $z = v$. Thus, $\mathbf{x}^{-1}$ exists and is continuous (since their respective components are also continuous).

(3) Note that
$$\frac{\partial \mathbf{x}}{\partial u} = (\cos(u), \sin(u), v)$$
and
$$\frac{\partial \mathbf{x}}{\partial v} = (0, 0, 1).$$
Hence, we have
$$\mathrm{d}\mathbf{X}_{(u,v)} = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix}.$$
Notice that the matrix above has rank 2 and thus $\mathrm{d}\mathbf{x}_{(u,v)}$ must be an injective linear map.

Since properties (1)-(3) are satisfied, it follows that C is a regular surface. ■