

# Math 230A: Homework 4

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1. Consider  $\mathbb{R}$  with its standard metric. Let  $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ . Prove that  $E' = \{0\}$ .

**Proof.** Note that  $0 \notin E$ . To show that  $E' = \{0\}$ , it suffices to show that for  $\varepsilon > 0$

$$N_\varepsilon(0) \cap E \neq \emptyset$$

which is equivalent to

$$(-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let  $\varepsilon > 0$ . By the Archimedean Property of  $\mathbb{R}$ , there exists  $m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ . Thus,  $1/m \in (-\varepsilon, \varepsilon) \cap E$  and so  $0 \in E'$ . ■

2. Consider  $\mathbb{R}$  with its standard metric.

- (a) Prove that  $\mathbb{N}' = \emptyset$ . Is  $\mathbb{N}$  closed?

**Solution.** Suppose for sake of contradiction that  $\mathbb{N}' \neq \emptyset$ . Let  $n \in \mathbb{N}'$ . Let  $\varepsilon = 1$ . Then  $N_1(n) \cap \mathbb{N} \setminus \{n\} \neq \emptyset$  where  $N_1(n) = (n-1, n+1)$ . But note that  $(n-1, n+1)$  only contains one point, namely,  $n$  and nothing else. Hence,  $n$  must be an isolated point be an isolated point of  $\mathbb{N}$  which is a contradiction. Thus,  $\mathbb{N}' = \emptyset$ . Because  $\emptyset \subseteq \mathbb{N}$ ,  $\mathbb{N}$  must be a closed set. ■

- (b) Prove that  $\mathbb{Q}' = \mathbb{R}$ . Is  $\mathbb{Q}$  closed?

**Solution.** Let  $a \in \mathbb{R}$  and let  $\varepsilon > 0$ . Consider the open interval  $(a - \varepsilon, a + \varepsilon)$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we can find at least one  $x \in \mathbb{Q}$  such that  $x \in (a - \varepsilon, a + \varepsilon)$ ; that is, we have

$$N_\varepsilon(a) \cap (\mathbb{Q} \setminus \{a\}) \neq \emptyset.$$

So, any  $a \in \mathbb{R}$  must be a limit point of  $\mathbb{Q}$ . Therefore, we conclude that  $\mathbb{Q}' = \mathbb{R}$ . Consequently,  $\mathbb{Q}$  must not be closed since its limit points take are outside of  $\mathbb{Q}$ . ■

3. Consider  $\mathbb{R}^2$  with its standard metric. Let  $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ . Prove that  $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ .

**Proof.** Our goal is to show that  $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ ; that is, we will show that

$$E' = E \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}.$$

We proceed by considering two cases:

- (1) If  $(a, b)$  is such that  $a^2 + b^2 > 4$ , then we will prove that  $(a, b) \notin E'$ .
- (2) We will prove that if  $(a, b)$  is such that  $a^2 + b^2 \leq 4$ , then  $(a, b) \in E'$ ; that is,

$$\forall N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}) \neq \emptyset.$$

With (1), consider a point  $p = (a, b)$  that is outside of  $E$ . Let  $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$ . Then we

see that

$$0 < \frac{1}{2}(\sqrt{a^2 + b^2} - 2) \iff a^2 + b^2 > 4.$$

Hence, we must have that  $N_\delta(p) \cap (E \setminus \{p\}) = \emptyset$ .

With (2), consider the distance from the origin to any point  $(a, b)$  on the boundary of the circle  $E$ . Let  $\delta = \frac{1}{2} \min\{\varepsilon, 2 - \varepsilon\}$ . Suppose  $\varepsilon < 2$ . Let  $p$  be the points on the ray  $\overline{OQ}$  where the distance to  $q$  is  $\delta$ . Note that  $p \in N_\varepsilon((a, b))$  since the distance between  $p$  and  $(a, b)$  is  $\delta$  which is less than  $\varepsilon$ . Also,  $p \in E$  since  $p$  is on the ray  $\overline{OQ}$ . Thus, we have

$$p \in N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}).$$

On the other hand, if  $\varepsilon \geq 2$ , then let  $\delta = 1$ .  $E$  is clearly the intersection

$$N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\})$$

is nonempty with a similar argument shown when  $\varepsilon < 2$ .

Thus, we conclude that

$$E' = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}.$$

■

4. Let  $(X, d)$  be a metric space. Prove that  $X$  and  $\emptyset$  are both open sets.

**Proof.** We want to show the following:

- (i)  $X$  is open
- (ii)  $\emptyset$  is open

Let  $x \in X$ . For (i), we want to show that there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq X$ . Pick another point  $q \neq x$  such that  $q \in X$ . Choose  $\delta = d(x, q) > 0$ . But note that any open neighborhood is a subset of  $X$ . So,  $N_\delta(x) \subseteq X$  and so  $X$  is open.

Note that  $\emptyset^\circ = \emptyset$ . Hence,  $\emptyset$  is an open set.

■

5. Let  $(X, d)$  be a metric space. Prove that  $X$  and  $\emptyset$  are both closed sets.

**Proof.** We will proving the following statements:

- (i)  $X$  is closed.
- (ii)  $\emptyset$  is closed.

Let  $x$  be a limit point of  $X$ . Then for all  $\varepsilon > 0$ , we have  $N_\varepsilon(x) \cap (X \setminus \{x\}) \neq \emptyset$ . But note that  $N_\varepsilon(x)$  is an open set that is contained in  $X$ . Thus,  $x$  must be contained within  $X$  which proves (i).

For (ii), note that the complement of  $\emptyset$  is just  $X$  itself. Since  $X$  is open by problem 4, we see that  $X^c = \emptyset$  must be closed.

■

6. Consider  $\mathbb{R}$  equipped with the discrete metric. Let  $E = \{1, 2, 3\}$ . Prove that  $E$  is open, that is,  $E^\circ = E$ .

**Proof.** We will check that for every element  $x$  in  $E$  that there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq E$ . Choose  $\delta = 1$ , then we have  $d(1, x) = 0 < 1$ . Since this is the only point that satisfies  $\delta$ , we must have that  $N_1(1) = \{1\} \subseteq E$ . Similarly,  $N_1(2) = \{2\} \subseteq E$  and  $N_1(3) = \{3\} \subseteq E$ . Thus,  $E$  must be open.

■

7. Consider  $X = \mathbb{R}$  equipped with the standard metric. Let  $a < b$ .

- (a) Prove that the sets  $(a, b)$ ,  $(a, \infty)$ , and  $(-\infty, a)$  are open.

**Proof.** We will show that the following intervals in  $\mathbb{R}$  are open:

- (i)  $(a, b)$

(ii)  $(a, \infty)$ (iii)  $(-\infty, a)$ 

Starting with (i), we want to show that there exists a  $\delta > 0$  such that  $N_\delta(x) \subseteq (a, b)$  where  $N_\delta(x) = (x - \delta, x + \delta)$ . Let  $x \in (a, b)$ . Choose  $\delta = \frac{1}{2} \min\{|b - x|, |x - a|\}$ . Then we have  $N_\delta(x) \subseteq (a, b)$ . Thus, we conclude that  $(a, b)$  is open.

With (ii), fix  $x \in (a, \infty)$ . Observe that  $(a, x)$  is an open interval that is contained within  $(a, \infty)$ . Now, let  $y \in (a, x)$ . Then choose  $\delta = \frac{1}{2} \min\{|x - y|, |y - a|\} > 0$ . Hence, we have

$$N_\delta(y) \subseteq (a, x) \subseteq (a, \infty).$$

Thus, we have that  $(a, \infty)$  is an open set in  $\mathbb{R}$ .

Lastly, fix  $x \in (-\infty, b)$ . Consider the open interval  $(x, b)$  and let  $y \in (x, b)$ . Choose  $\delta = \frac{1}{2} \min\{|b - y|, |y - x|\} > 0$ . Then we have

$$N_\delta(y) \subseteq (x, b) \subseteq (-\infty, b)$$

which shows that  $N_\delta(y)$  is an open set. ■

(b) Prove that the sets  $[a, b]$ ,  $[a, \infty)$ , and  $(-\infty, b]$  are closed.

**Proof.** We will showing that the following intervals are closed in  $\mathbb{R}$ :

(i)  $[a, b]$ (ii)  $[a, \infty)$ (iii)  $(-\infty, b]$ 

Starting with (i), we want to show that every limit point  $x$  of  $[a, b]$  is contained in  $[a, b]$ . Thus, let  $x$  be a limit point of  $[a, b]$ . By definition, for all  $\delta > 0$ , we have  $N_\delta(x) \cap ([a, b] \setminus \{x\}) \neq \emptyset$ . So,  $N_\delta(x)$  contains a point  $q \neq x$  in this intersection such that  $q \in [a, b]$ . This means that  $a \leq q \leq b$ . Using the fact that  $q \in N_\delta(x)$ , we have

$$\begin{aligned} |x - q| < \delta &\iff q - \delta < x < q + \delta \\ &\implies a - \delta \leq x \leq b + \delta. \end{aligned}$$

Since  $\delta > 0$  is arbitrary, we conclude that  $a \leq x \leq b$  which means that  $x \in [a, b]$ . Thus,  $[a, b]$  is closed.

With (ii), we will employ the same process. Let  $x$  be a limit point of  $[a, \infty)$ . Then for all  $\varphi > 0$ , we see that  $N_\varphi(x) \cap ([a, \infty) \setminus \{x\}) \neq \emptyset$ . To this end, pick  $p \neq x$  in this intersection such that  $p \in [a, \infty)$ . Since  $p$  is also contained within  $N_\varphi(x)$ , we see that

$$\begin{aligned} |x - p| < \varphi &\iff p - \varphi < x < p + \varphi \\ &\iff a - \varphi \leq x < p + \varphi. \end{aligned}$$

This implies that  $a \leq x < p$  since  $\varphi > 0$  is arbitrary. Thus, we have  $x \in [a, \infty)$  and so  $[a, \infty)$  is closed.

The last case is similar to (ii). Let  $\varepsilon > 0$ . If  $x$  is a limit point of  $(-\infty, b]$ , then  $N_\varepsilon(x) \cap ((-\infty, b] \setminus \{x\}) \neq \emptyset$  implies that we can pick  $q \neq x$  in this intersection such that  $q \in (-\infty, b]$ . Since  $\varepsilon > 0$  is arbitrary, we have

$$q - \varepsilon < x < q + \varepsilon \leq b + \varepsilon$$

implies

$$q < x \leq b.$$

Hence,  $x \in (-\infty, b]$  and so  $(-\infty, b]$  is a closed set. ■

8. Let  $(X, d)$  be a metric space. Let  $p \in X$  and  $\delta > 0$ . Prove that the closed ball  $C_\delta(p) = \{x \in X :$

$d(x, p) \leq \delta\}$  is indeed closed.

**Proof.** Our goal is to show that  $C_\delta(p)$  is a closed set. To this end, we will show that  $[C_\delta(p)]^c$  is an open set. If this holds, then we can conclude that  $C_\delta(p)$  is a closed set. Let  $x \in [C_\delta(p)]^c$ . Then we have  $d(x, p) > \delta$ . Our goal is to find  $\varepsilon > 0$  such that  $N_\varepsilon(x) \subseteq [C_\delta(p)]^c$ . Since  $d(x, p) > \delta$  we can set  $\varepsilon = d(x, p) - \delta > 0$ . Using the radius  $\varepsilon > 0$ , we can construct the neighborhood of  $x$ ,  $N_\varepsilon(x)$ . Let  $y \in N_\varepsilon(x)$ . In order for  $y$  to be contained within  $[C_\delta(p)]^c$ , we have to show that  $d(y, p) > \delta$ . Using the triangle inequality, we have

$$\begin{aligned} d(x, p) &\leq d(x, y) + d(y, p) \implies d(y, p) \geq d(x, p) - d(x, y) \\ &> d(x, p) - \varepsilon & (y \in N_\varepsilon(x)) \\ &= \delta. \end{aligned}$$

Thus, we see that  $N_\varepsilon(x) \subseteq [C_\delta(p)]^c$  for some  $\varepsilon > 0$ . Hence,  $[C_\delta(p)]^c$  is open and so  $C_\delta(p)$  is closed. ■

9. Give an example that shows that the equality  $\overline{N_\delta(p)} = C_\delta(p)$  is not always true.

**Solution.** Consider the interval  $[0, 1]$  in  $\mathbb{R}$  with the discrete metric. Clearly, we see that  $1/2 \in [0, 1]$ . If we let  $\varepsilon = 1$ , then

$$N_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) < 1\} = \{1/2\}$$

since the only case when the inequality is satisfied is when  $x = 1/2$ . If we consider the closure of this neighborhood, we just get

$$\overline{N_1(1/2)} = \{1/2\}.$$

Now, consider the closed ball

$$C_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) \leq 1\}.$$

Observe that for any  $x \in \mathbb{R}$ , either  $x = 1/2$  or  $x \neq 1/2$  in  $[0, 1]$ , the inequality of the set above we always be satisfied; that is, the set will just be all elements contained in  $[0, 1]$ . Thus, we see that  $C_1(1/2) = [0, 1]$  and, in this case, that  $C_1(1/2) \neq \overline{N_1(1/2)}$ . ■

10. Let  $(X, d)$  be a metric space. Prove that an arbitrary intersection of closed sets is closed. Prove that a finite union of closed sets is closed.

**Proof.** Let  $(X, d)$  be a metric space. We will prove that

- (1) An arbitrary intersection of closed sets is closed.
- (2) A finite union of closed sets is closed.

To this end, we will proceed by showing the propositions above.

- (1) Let  $\{F_\alpha\}$  be an arbitrary collection of closed sets. We want to show that

$$\bigcap_{\alpha} F_{\alpha} \text{ is closed.}$$

It suffices to show that the complement of this set is open. Consider the complement

$$\left(\bigcap_{\alpha} F_{\alpha}\right)^c = \bigcup_{\alpha} F_{\alpha}^c. \quad (1)$$

Observe that each  $F_{\alpha}^c$  is open since each  $F_{\alpha}$  is closed. But note that the arbitrary union of open sets is open and so the union of the right-hand side of (1) is open. Thus, the

left-hand side of (1) is open. Hence, the complement

$$\left(\bigcap_{\alpha} F_{\alpha}\right)^c \text{ is open } \iff \bigcap_{\alpha} F_{\alpha} \text{ is closed}$$

and we are done.

- (2) Let  $\{F_i : 1 \leq i \leq n\}$  be a finite collection of closed sets  $F_i$ . We want to show that  $\bigcup_{i=1}^n F_i$  is a closed set. It suffices to show that the complement of this set, that is  $\left(\bigcup_{i=1}^n F_i\right)^c$  is open. Observe that

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c.$$

Note that each  $F_i^c$  is open if and only if each  $F_i$  is closed. Thus, the finite intersection of each open set  $F_i^c$  is open; that is,

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c \text{ is open.}$$

Hence, the finite union of closed sets

$$\bigcup_{i=1}^n F_i \text{ is closed.}$$

■

11. Let  $E$  be a subset of a metric space  $(X, d)$ . Show that  $\overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_{\varepsilon}(x) \cap E \neq \emptyset\}$ .

**Proof.** Let  $E$  be a subset of a metric space  $(X, d)$ . We want to show that

$$\overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_{\varepsilon}(x) \cap E \neq \emptyset\}.$$

Denote

$$A = \{x \in X : \forall \varepsilon > 0 \ N_{\varepsilon}(x) \cap E \neq \emptyset\}.$$

It suffices to show the following two inclusions:

$$(1) \ \overline{E} \subseteq A.$$

$$(2) \ A \subseteq \overline{E}.$$

Starting with (1), suppose  $x \in \overline{E}$ . Then either  $x \in E$  or  $x \in E'$  (or both). If  $x \in E'$ , then  $x$  is a limit point of  $E$ . Then for all  $\varepsilon > 0$ ,

$$N_{\varepsilon}(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Since  $E \setminus \{x\} \subseteq E$ , we see that

$$N_{\varepsilon}(x) \cap E \neq \emptyset$$

which implies that  $x \in A$ . Suppose  $x \in E$ . Then for all  $\varepsilon > 0$ , we see that

$$N_{\varepsilon}(x) \cap E \neq \emptyset.$$

If  $x$  is a limit point and  $x \in E$ , then we immediately have that

$$N_{\varepsilon}(x) \cap E \neq \emptyset.$$

by definition of limit point. Thus, we see that in the three cases outlined that  $\overline{E} \subseteq A$ .

With (2), suppose  $x \in A$ . We want to show that  $x \in \overline{E}$ ; that is,  $x$  is either a limit point of  $E$  or an element of  $E$ . Suppose that  $x$  is NOT an element of  $E$ . By definition of  $A$ , we see that for all  $\varepsilon > 0$ ,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

Note that  $x \notin E$  implies that  $E \setminus \{x\} = E$  and so we have

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus,  $x$  is a limit point of  $E$  and so  $x \in \overline{E}$ . Otherwise, when  $x$  is not a limit point of  $E$ , then the nonempty intersection

$$N_\varepsilon(x) \cap E$$

implies that  $x \in E$  and so  $x \in \overline{E}$ . Thus,  $A \subseteq \overline{E}$ . ■

12. Let  $(X, d)$  be a metric space. Show that the closure of a bounded set is bounded.

**Proof.** Let  $(X, d)$  be a metric space and suppose  $E \subseteq X$  is bounded; that is, there exists  $\varepsilon > 0$  and  $q \in X$  such that  $E \subseteq N_\varepsilon(q)$ . We claim that  $\overline{E} \subseteq N_{\varepsilon+1}(q)$  so that  $\overline{E}$  is bounded. Note that  $\overline{E} = E \cup E'$ . Let  $y \in \overline{E}$ . Then either  $y \in E$  or  $y \in E'$ . If  $y \in E$  and  $E$  is bounded, then

$$d(y, q) \leq \varepsilon < \varepsilon + 1.$$

Thus,  $y \in N_{\varepsilon+1}(q)$ . If  $y \in E'$ , then for all  $\delta > 0$ ,

$$N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset.$$

Let  $\delta = 1/2$ . Then pick a point  $x$  inside the intersection

$$N_{1/2}(y) \cap (E \setminus \{y\}).$$

Then  $x \in N_{1/2}(y)$  and  $x \in E$ . If  $x \in N_{1/2}(y)$ , then  $d(x, y) < 1/2$  and  $d(x, q) \leq \varepsilon$ , respectively. Using the triangle inequality, we see that

$$d(y, q) \leq d(y, x) + d(x, q) < \frac{1}{2} + \varepsilon < 1 + \varepsilon.$$

Thus,  $y \in N_{\varepsilon+1}(q)$ . Therefore, we see that  $\overline{E}$  is bounded in both cases. ■

13. Prove that a nonempty subset  $A$  of a metric space  $(X, d)$  is open if and only if it is a union of neighborhoods (open balls).

**Proof.** ( $\implies$ ) Suppose  $A$  is an open set in  $X$ . We will show that  $A$  can be written as a union of open balls; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x)$$

where  $N_{\varepsilon_x}(x)$  are open neighborhoods with radius  $\varepsilon_x$  ( $\varepsilon$  depends on  $x$ ) for each  $x \in A$ . To this end, we will show the following inclusions:

$$(1) \quad A \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$$

$$(2) \quad \bigcup_{x \in A} N_{\varepsilon_x}(x) \subseteq A$$

Starting with the inclusion in (1), let  $x \in A$ . Since  $A$  is open, we can find an  $\varepsilon_x > 0$  such that  $N_{\varepsilon_x}(x) \subseteq A$ . Note that  $N_{\varepsilon_x}(x) \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$ . So,  $x$  must be contained in  $\bigcup_{x \in A} N_{\varepsilon_x}(x)$  which satisfies (1).

Let  $p \in \bigcup_{x \in A} N_{\varepsilon_x}(x)$ . Then for some point  $x \in A$ ,  $p$  must be contained in some neighborhood  $N_{\varepsilon_x}(x)$  with radius  $\varepsilon_x > 0$ . Since  $x \in A$  and  $A$  is an open set, we must have  $N_{\varepsilon_x}(x) \subseteq A$ . Thus,  $p$  must be contained in  $A$  as well. Thus, the inclusion in (2) is satisfied.

( $\Leftarrow$ ) Suppose  $A$  is the union of open balls in  $A$ ; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x).$$

Since each  $N_{\varepsilon_x}(x)$  is open, the union of the right-hand side of the equation above is open. Thus,  $A$  must be open as well and we are done. ■

14. On page 9, Rudin implicitly defines a subset  $E \subseteq \mathbb{R}$  to be "dense" if it satisfies:

(i) For all  $x, y \in \mathbb{R}$  with  $x < y$ , there exists  $p \in E$  such that  $x < p < y$ .

On page 32, he defines a subset  $E$  of a metric space  $X$  to be "dense" if it satisfies:

(ii) Every point of  $X$  is either a limit point of  $E$  or a point of  $E$  (that is,  $\overline{E} = X$ ).

Prove that  $E \subseteq \mathbb{R}$  satisfies (i) if and only if it satisfies (ii).

**Proof.** ( $\Rightarrow$ ) Suppose  $E \subseteq \mathbb{R}$  satisfies the property (i). We want to show that every point in  $\mathbb{R}$  must either be a limit point of  $E$  or a point in  $E$ . To this end, let  $x \in \mathbb{R}$  be an arbitrary point. Suppose that  $x$  is not an element of  $E$ . Thus, we want to show that  $x$  is a limit point of  $E$ ; that is, we will show that

$$\forall \delta > 0 \quad N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Let  $\delta > 0$ . Then denote the open neighborhood of  $x$  as

$$N_\delta(x) = \{y \in \mathbb{R} : |x - y| < \delta\} = (x - \delta, x + \delta).$$

But note that  $x - \delta < x + \delta$  implies that there exists a  $p \neq x \in E$  such that  $x - \delta < p < x + \delta$ . Thus, we see that  $|x - p| < \delta$  and hence, we see that

$$N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Hence,  $x$  is a limit point of  $E$ . Now, if  $x$  is NOT a limit point, then  $x$  is an isolated point of  $E$  and that  $x \in E$ .

( $\Leftarrow$ ) Let  $x, y \in \mathbb{R}$  such that  $x < y$ . We will show that there exists  $p \in E$  such that  $x < p < y$ . Suppose that  $\mathbb{R} = \overline{E}$ . Since  $x \in \mathbb{R}$ , we must have  $x \in \overline{E}$ . That is, for all  $\varepsilon > 0$ , we have

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

This implies that  $N_\varepsilon(x)$  contains a point  $p \neq x$  such that  $p \in E$ . This means that  $p \in (x - \varepsilon, x + \varepsilon)$  and so

$$x - \varepsilon < p < x + \varepsilon.$$

Since  $x < y$ , we have that

$$x - \varepsilon < p < y + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we find that  $x < p < y$  for some  $p \in E$ . ■

15. Let  $A$  and  $B$  be subsets of a metric space  $(X, d)$ .

(a) If  $A \subseteq B$ , then  $A' \subseteq B'$ .

**Proof.** Let  $x \in A'$ . We want to show that  $x \in B'$ ; that is,  $x$  is a limit point of  $B$ . By definition,  $x \in A'$  implies that for all  $\varepsilon > 0$ ,  $N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$ . So, pick  $p \neq x$  in this intersection such that  $p \in A$ . But note that  $A \subseteq B$ . Thus,  $p \in B$  and we must have for all  $\varepsilon > 0$ ,

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset.$$

Thus,  $x$  is also a limit point of  $B$  (that is,  $x \in B'$ ) and we conclude that  $A' \subseteq B'$ . ■

(b) If  $A \subseteq B$ , then  $\overline{A} \subseteq \overline{B}$ .

**Proof.** Suppose  $A \subseteq B$ . By part (i), we see that  $A' \subseteq B'$ . Thus,

$$\overline{A} = A \cup A' \subseteq B \cup B' = \overline{B}.$$

Hence,  $\overline{A} \subseteq \overline{B}$ . ■

(c) Prove that  $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ .

**Proof.** Our goal is to show that  $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ . Let  $x \in \overline{A \cap B}$ . Then either  $x \in A \cap B$  or  $x \in (A \cap B)'$ . Suppose  $x \in A \cap B$ . Then both  $x \in A$  and  $x \in B$  and so  $x \in \overline{A}$  and  $x \in \overline{B}$  since  $A \subseteq \overline{A}$  and  $B \subseteq \overline{B}$ . Now, suppose  $x \in (A \cap B)'$ . Then  $x$  is a limit point of  $A \cap B$ ; that is, for all  $\varepsilon > 0$ ,  $N_\varepsilon(x) \cap ((A \cap B) \setminus \{x\}) \neq \emptyset$ . Since  $A \cap B \subseteq A$  and  $A \cap B \subseteq B$ , we must have that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \quad (1)$$

and

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset. \quad (2)$$

Thus, (1) and (2) imply that  $x \in A'$  and  $x \in B'$  meaning that  $x$  is both a limit point of  $A$  and  $B$ . Hence,  $x \in \overline{A}$  and  $x \in \overline{B}$ . Thus, we have  $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$  in both of these cases. ■

(a) Prove that  $A' \cup B' = (A \cup B)'$ .

**Proof.** Our goal is to show that  $A' \cup B' = (A \cup B)'$ ; that is, we need to show that

$$(1) \quad A' \cup B' \subseteq (A \cup B)'$$

$$(2) \quad (A \cup B)' \subseteq A' \cup B'.$$

Starting with (1), let  $x \in A' \cup B'$ . Then  $x \in A'$  or  $x \in B'$ . If  $x \in A'$ , then  $x$  is a limit point of  $A$ . Hence, for all  $\varepsilon > 0$  such that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset.$$

Then pick a point in this intersection, say,  $q \neq x$  such that  $q \in A$ . Since  $A \subseteq A \cup B$ , we have  $q \in A \cup B$ . Thus, we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Hence,  $x$  is a limit point of  $A \cup B$  and so  $x \in (A \cup B)'$ . If  $x \in B'$ , then for all  $\varepsilon > 0$  such that

$$N_\varepsilon(x) \cap B \setminus \{x\} \neq \emptyset.$$

Since  $B \subseteq A \cup B$ , we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Then  $x$  is a limit point of  $A \cup B$ ; that is,  $x \in (A \cup B)'$ . Hence, we have that  $A' \cup B' \subseteq (A \cup B)'$ .

Now, with (2), let  $x \in (A \cup B)'$ . Then  $x$  is a limit point of  $A \cup B$ . Thus, for all  $\varepsilon > 0$  we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Pick a point in this intersection, say,  $p \neq x$ . Note that  $p \in A \cup B$ . So, either  $p \in A$  or  $p \in B$ . If  $p \in A$ , then  $x$  is a limit point of  $A$ . Thus,  $x \in A'$  and so  $x \in A' \cup B'$ . If  $p \in B$ , then  $x$  is a limit point of  $B$  and so  $x \in B'$ . Thus,  $x \in A' \cup B'$ . Therefore, we have  $(A \cup B)' \subseteq A' \cup B'$  which proves (2).



With (1) and (2), we have

$$A' \cup B' = (A \cup B)'.$$

■

16. Let  $(X, d)$  be a metric space and let  $E \subseteq X$ .

(a) Prove that  $E'$  is closed.

**Proof.** Our goal is to show that  $E'$  is closed; that is, any limit point of  $E'$  is contained within  $E'$ . Let  $x$  be a limit point of  $E'$ . Then for all  $\varepsilon > 0$ ,  $N_\varepsilon(x) \cap (E' \setminus \{x\}) \neq \emptyset$ . To this end, pick  $y \neq x$  in this intersection such that  $y \in E'$ . But this tells us that  $y$  is a limit point of  $E$ . Thus, for all  $\delta > 0$ ,  $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$ . Thus, we can pick  $p \neq y$  such that  $p \in E$ . Note that  $p \neq y \neq x$  and  $p \in E$  imply that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus,  $x$  is a limit point of  $E$  and so  $x \in E'$ . Hence,  $E'$  is closed. ■

(b) Prove that  $E$  and  $\overline{E}$  have the same limit points.

**Proof.** To prove that  $E$  and  $\overline{E}$  have the same limit points, we need to show that  $E' = (\overline{E})'$ . We will prove the following two inclusions:

$$(1) E' \subseteq (\overline{E})'$$

$$(2) (\overline{E})' \subseteq E'.$$

Starting with (1), let  $x \in E'$ . Then  $x$  is a limit point of  $E$ . Thus, for all  $\varepsilon > 0$ ,  $N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset$ . Since  $E \subseteq \overline{E}$ , we must have that

$$N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset.$$

Thus,  $x \in \overline{E}'$  which proves (1).

Now with (2), let  $x \in \overline{E}'$ . Then for all  $\varepsilon > 0$ ,  $N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset$ . From this nonempty intersection, pick  $y \neq x$  such that  $y \in \overline{E}$ ; that is,  $y \in E$  or  $y \in E'$ . If  $y \in E$ , then  $x$  is a limit point of  $E$  and thus  $x \in E'$ . If  $y \in E'$ , then  $y$  is a limit point of  $E$ . Hence, for all  $\delta > 0$ ,  $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$ . From this, we can pick  $q \neq y \neq x$  such that  $q \in E$ . But this tells us that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset,$$

proving that  $x$  is a limit point of  $E$  and so  $x \in E'$ . ■

(c) Construct an example that shows  $E$  and  $E'$  do not necessarily have the same limit points?

**Proof.** Suppose we have the following set  $E = \{x < \frac{1}{n} : n \in \mathbb{N}, x \in \mathbb{R}\}$  and its set of limit points  $E' = \{\frac{1}{n} : n \in \mathbb{N}\}$ . By problem 1, we see that the set of limit points of  $E'$  must be  $\{0\}$ . Clearly,  $E$  and  $E'$  do not have the same limit points. ■

17. Construct a bounded set of real numbers with exactly three limit points.

**Solution.** Consider the sets  $A = \{1/n : n \in \mathbb{N}\}$ ,  $L = \{\pi + \frac{1}{n} : n \in \mathbb{N}\}$ ,  $I = \{3 + \frac{1}{n} : n \in \mathbb{N}\}$ . Note that each set is bounded. If we take the union of these sets, then we have the following set

$$A \cup L \cup I$$

which has three limit points which are 0,  $\pi$ , and 3, respectively. ■

18. Let  $(X, d)$  be a metric space and  $E \subseteq X$ .

(a) Prove that  $E^\circ$  is always open, that is  $(E^\circ)^\circ = E^\circ$ .

**Proof.** To show that  $E^\circ$  is open, we will show that  $(E^\circ)^\circ$ . To do this, we will need to show the following two inclusions:

$$(1) (E^\circ)^\circ \subseteq E^\circ$$

$$(2) E^\circ \subseteq (E^\circ)^\circ.$$

Starting with (1), let  $x \in (E^\circ)^\circ$ . By definition of interior points, there must exist a  $\delta > 0$  such that  $N_\delta(x) \subseteq E^\circ$ . Clearly, we have  $x \in E^\circ$  which proves (1).

Now, let  $x \in E^\circ$ . Then there exists  $\varepsilon > 0$  such that  $N_\varepsilon(x) \subseteq E$  where  $x \in E$ . Suppose for sake of contradiction that  $x$  is NOT in the interior of  $E^\circ$ . Then for all  $\delta > 0$ ,  $N_\delta(x) \not\subseteq E^\circ$ . That is,  $N_\delta(x) \cap (E^\circ)^c \neq \emptyset$ . Since  $E^\circ \subseteq E$ , it follows that  $x \in E^c$ . But this is a contradiction because we had assume that  $x \in E^\circ$  and so  $x \in E$ . Thus, we must have  $x$  lie in the interior of  $E^\circ$ . From (1) and (2), we conclude that  $E^\circ$  is open. ■

- (b) If  $G \subseteq E$  and  $G$  is open, prove that  $G \subseteq E^\circ$ .

**Proof.** Suppose  $G \subseteq E$  and  $G$  is open. We will show that  $G \subseteq E^\circ$ . Let  $x \in G$ . Since  $G$  is open, there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq G \subseteq E$ . Thus,  $x$  must be in the interior of  $E$ ; that is,  $x \in E^\circ$ . Hence,  $G \subseteq E^\circ$ . ■

- (c) Prove that  $E^\circ = \overline{E^c}^c$ .

**Proof.** We will show that  $E^\circ = (\overline{E^c})^c$ . We will show the following two inclusions:

$$(1) E^\circ \subseteq (\overline{E^c})^c,$$

$$(2) (\overline{E^c})^c \subseteq E^\circ.$$

Starting with (1), let  $x \in E^\circ$ . Then there exists a  $\delta > 0$  such that  $N_\delta(x) \subseteq E$ . This implies that  $x$  cannot be in  $E^c$ . Otherwise,  $x$  is not an interior point of  $E$ . Subsequently, we have that for all  $\delta > 0$ ,  $N_\delta(x) \cap E^c = \emptyset$ . Thus, neither  $x \in E^c$  nor  $x \in (E^c)'$ . Hence,  $x \in (\overline{E^c})^c$ .

With (2), let  $x \in (\overline{E^c})^c$ . Then  $x \notin \overline{E^c}$ . Since  $\overline{E^c} = E^c \cup (E^c)'$ , we have  $x \notin E^c$  and  $x \notin (E^c)'$ . Then there exists  $\delta > 0$  such that

$$N_\delta(x) \cap (E^c \setminus \{x\}) = \emptyset \iff N_\delta(x) \cap E^c = \emptyset.$$

Since  $x \notin E^c$ , we must have  $x \in E$ . Thus,  $N_\delta(x) \subseteq E$  for some  $\delta > 0$  and so  $x \in E^\circ$  which proves (2). ■

- (d) Give an example that shows that  $E$  and  $\overline{E}$  do not always have the same interiors.

**Solution.** Let  $E = \mathbb{Q}$  and note that  $\overline{E} = \mathbb{R}$  under the standard metric of  $\mathbb{R}$ . Since  $\mathbb{R}$  is open, we know that  $\mathbb{R}^\circ = \mathbb{R}$ . But the interior of  $\mathbb{Q}$  is empty, which is clearly not equal to  $\mathbb{R}^\circ = \overline{E}^\circ$ . ■

- (e) Give an example that shows that  $E$  and  $E^\circ$  do not always have the same closures.

**Proof.** Consider the set of rational numbers  $\mathbb{Q}$ . Then observe that  $\mathbb{Q}^\circ$  is empty since for all  $\varepsilon > 0$ ,  $N_\varepsilon(x)$  contains irrational numbers which are not contained in  $\mathbb{Q}$ . Hence,  $\overline{\mathbb{Q}^\circ} = \emptyset$ . But notice that the closure  $\overline{\mathbb{Q}} = \mathbb{R}$ . Clearly,  $\overline{\mathbb{Q}} \neq \overline{\mathbb{Q}^\circ}$ . ■

19. Let  $(X, d)$  be a metric space and let  $A \subseteq X$ . Prove that  $A$  is dense in  $X$  if and only if any nonempty open set in  $X$  has a nonempty intersection with  $A$ .

**Proof.** ( $\implies$ ) Let  $B$  be a nonempty open set in  $X$ . Let  $b \in B$ . We will show that  $A \cap B \neq \emptyset$ . Since  $b \in X$  and  $A$  is dense in  $X$ , either

- (1)  $b$  is a limit point of  $A$  or

(2)  $b \in A$ .

If  $b$  is a limit point of  $A$ , then for all  $\delta > 0$ ,

$$N_\delta(b) \cap (A \setminus \{b\}) \neq \emptyset.$$

Since  $B$  is an open set, we know that  $N_\delta(b) \subseteq B$ . Together with the fact that  $A \setminus \{b\} \subseteq A$ , we can conclude that

$$B \cap A \neq \emptyset.$$

On the other hand,  $b \in A$  immediately implies that  $B \cap A \neq \emptyset$ .

( $\Leftarrow$ ) Let  $p \in X$  and let  $A \subseteq X$ . We need to show that  $A$  is dense in  $X$ ; that is, we need to show that  $p$  is either a limit point of  $A$  or  $p \in A$ . Suppose that  $p \notin A$ . We will show that  $p$  is a limit point of  $A$ . Let  $\varepsilon > 0$  and consider the open neighborhood  $N_\varepsilon(p)$ . By assumption, the open ball  $N_\varepsilon(p)$  contains a nonempty intersection with  $A$ ; that is,

$$N_\varepsilon(p) \cap A \neq \emptyset.$$

Subsequently, we have

$$N_\varepsilon(p) \cap (A \setminus \{p\}) \neq \emptyset.$$

Thus,  $p$  is a limit point of  $A$ . On the other hand, if  $p$  is not a limit point of  $A$ , then  $p$  is an isolated point of  $A$ . Thus,  $p \in A$ . ■

**Proof.** ( $\Leftarrow$ ) Suppose  $A$  has a nonempty intersection with every nonempty open set in  $X$ . Thus, we have

$$\begin{aligned} (A^c)^\circ = \emptyset &\implies_{18-c) \left( \overline{(A^c)^c} \right)^c = \emptyset \\ &\implies \overline{(A^c)^c} = \emptyset \\ &\implies \overline{A} = X. \end{aligned}$$

Thus,  $A$  is dense in  $X$ .

( $\Rightarrow$ ) Suppose  $\overline{A} = X$ . By problem 11, we see that for any  $x \in X$  and for all  $\varepsilon > 0$  that

$$N_\varepsilon(x) \cap A \neq \emptyset. \quad (1)$$

If  $E$  is any nonempty open set, then let  $q \in E$ . Since  $q$  is an interior point of  $E$ , there exists  $\delta > 0$  such that

$$N_\delta(q) \subseteq E. \quad (2)$$

By (1) and (2), we must have that  $E \cap A \neq \emptyset$ . ■

20. Let  $(X, d)$  be a metric space and  $A_i \subseteq X$  for all  $i \in \mathbb{N}$ .

(a) Prove that for all  $n \in \mathbb{N}$ , we have  $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$ .

**Proof.** First, we will show by induction on  $n \in \mathbb{N}$  that

$$\left( \bigcup_{i=1}^n A_i \right)' = \bigcup_{i=1}^n A_i'.$$

Let  $n = 2$  be our base case, then we see that  $(A_1 \cup A_2)' = A_1' \cup A_2'$  by problem 15-(d).

Now, we will suppose that the result holds for the  $n$ th case. Thus, we have that

$$\begin{aligned} \left( \bigcup_{i=1}^{n+1} A_i \right)' &= \left( \bigcup_{i=1}^n A_i \cup A_{n+1} \right)' \\ &= \left( \bigcup_{i=1}^n A_i \right)' \cup A'_{n+1} \\ &= \left( \bigcup_{i=1}^n A'_i \right) \cup A'_{n+1} \\ &= \bigcup_{i=1}^{n+1} A'_i. \end{aligned}$$

Then for all  $i \in \mathbb{N}$ , we see that

$$\begin{aligned} \overline{\bigcup_{i=1}^n A_i} &= \left( \bigcup_{i=1}^n A_i \right) \cup \left( \bigcup_{i=1}^n A_i \right)' \\ &= \left( \bigcup_{i=1}^n A_i \right) \cup \left( \bigcup_{i=1}^n A'_i \right) \\ &= \bigcup_{i=1}^n A_i \cup A'_i \\ &= \bigcup_{i=1}^n \overline{A_i}. \end{aligned}$$

■

(b) Prove that  $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$ .

**Proof.** Our goal is to show that  $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$ . Let  $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$ . Then for some  $i \in \mathbb{N}$ , we see that  $x \in \overline{A_i}$ . Then either  $x \in A_i$  or  $x \in A'_i$ . If  $x \in A_i$  and  $A_i \subseteq \bigcup_{i=1}^{\infty} A_i$ , we see that  $x \in \bigcup_{i=1}^{\infty} A_i$ . Suppose  $x \in A'_i$ . Note that  $A'_i \subseteq \left( \bigcup_{i=1}^{\infty} A_i \right)'$ . So,  $x \in \left( \bigcup_{i=1}^{\infty} A_i \right)'$ . Thus,  $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$ . Hence, we have

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}.$$

■

(c) Show, by an example, that the inclusion in part (b) can be proper.

**Solution.** Consider the sequence of open intervals in  $\mathbb{R}$ ,  $A_i = (0, 1 + \frac{1}{i})$ . Observe that

$$\bigcup_{i=1}^{\infty} \overline{\left(0, 1 + \frac{1}{i}\right)} = \bigcup_{i=1}^{\infty} \left[0, 1 + \frac{1}{i}\right] = [0, 2).$$

But we have

$$\overline{\bigcup_{i=1}^{\infty} \left(0, 1 + \frac{1}{i}\right)} = \overline{(0, 2)} = [0, 2].$$

Clearly,  $[0, 2)$  is a proper subset of  $[0, 2]$ .

■

21. Let  $(X, d)$  be a metric space and  $A_i \subseteq X$ , for all  $i \in \mathbb{N}$ .

(a) Prove that for all  $n \in \mathbb{N}$ , we have  $\left( \bigcap_{i=1}^n A_i \right)^{\circ} = \bigcap_{i=1}^n A_i^{\circ}$ .

**Proof.** Our goal is to show that  $\left(\bigcap_{i=1}^n A_i\right)^\circ = \bigcap_{i=1}^n A_i^\circ$ ; that is, we need to show the following two inclusions:

$$(1) \left(\bigcap_{i=1}^n A_i\right)^\circ \subseteq \bigcap_{i=1}^n A_i^\circ.$$

$$(2) \bigcap_{i=1}^n A_i^\circ \subseteq \left(\bigcap_{i=1}^n A_i\right)^\circ.$$

Starting with (1), let  $x \in \left(\bigcap_{i=1}^n A_i\right)^\circ$ . Then there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq \bigcap_{i=1}^n A_i$ . Thus,  $x \in \bigcap_{i=1}^n A_i$  implies that  $x \in A_i$  for all  $1 \leq i \leq n$ . But this tells us that  $N_\varepsilon(x) \subseteq A_i$  for all  $1 \leq i \leq n$ , and so  $x \in A_i$ . Hence,

$$x \in \bigcap_{i=1}^n A_i^\circ.$$

Thus, we conclude that

$$\left(\bigcap_{i=1}^n A_i\right)^\circ \subseteq \bigcap_{i=1}^n A_i^\circ.$$

With (2), let  $x \in \bigcap_{i=1}^n A_i^\circ$ . Then for all  $1 \leq i \leq n$ , we have  $x \in A_i^\circ$ . By definition, there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq A_i$  for all  $1 \leq i \leq n$ . Thus,  $x \in A_i$  for all  $1 \leq i \leq n$  and so

$$x \in \bigcap_{i=1}^n A_i.$$

Subsequently,  $N_\delta(x) \subseteq \bigcap_{i=1}^n A_i$  and so  $x \in \left(\bigcap_{i=1}^n A_i\right)^\circ$ . Therefore, we have

$$\bigcap_{i=1}^n A_i^\circ \subseteq \left(\bigcap_{i=1}^n A_i\right)^\circ.$$

Hence, (1) and (2) gives us our desired result. ■

(b) Prove that  $\left(\bigcap_{i \in \mathbb{N}} A_i\right)^\circ \subseteq \bigcap_{i \in \mathbb{N}} A_i^\circ$ .

**Proof.** Let  $x \in \left(\bigcap_{i \in \mathbb{N}} A_i\right)^\circ$ . Then there exists  $\delta > 0$  such that  $N_\delta(x) \subseteq \bigcap_{i \in \mathbb{N}} A_i$ . Hence,  $x \in \bigcap_{i \in \mathbb{N}} A_i$  if and only if  $x \in A_i$  for all  $i \in \mathbb{N}$ . But this also tells us that  $N_\delta(x) \subseteq A_i$ . Hence,  $x \in A_i^\circ$  for all  $i \in \mathbb{N}$ . Thus,

$$\left(\bigcap_{i \in \mathbb{N}} A_i\right)^\circ \subseteq \bigcap_{i \in \mathbb{N}} A_i^\circ.$$

■

(c) Give an example where equality does not hold in (b).

**Solution.** Consider the sequence of closed intervals  $A_i$  in  $\mathbb{R}$  defined by

$$A_i = \left[a - \frac{1}{i}, b\right].$$

Then the interior of these closed intervals are

$$A_i^\circ = \left(a - \frac{1}{i}, b\right).$$

Now, observe that

$$\bigcup_{i \in \mathbb{N}} A_i^\circ = (a, b).$$

But

$$\left( \bigcup_{i \in \mathbb{N}} A_i \right)^\circ = [a, b]^\circ = (a, b).$$

Hence, we have  $(a, b) \subset [a, b]$ . ■

22. Let  $(X, d)$  be a metric space and  $A_i \subseteq X$ , for all  $i \in \mathbb{N}$ .

(a)  $\bigcup_{i \in \mathbb{N}} A_i^\circ \subseteq \left( \bigcup_{i \in \mathbb{N}} A_i \right)^\circ$ .

**Proof.** Let  $x \in \bigcup_{i \in \mathbb{N}} A_i^\circ$ . Then for some  $i \in \mathbb{N}$ , we have  $x \in A_i^\circ$ . Then there exists a  $\delta > 0$  such that  $N_\delta(x) \subseteq A_i$ . But note that

$$A_i \subseteq \bigcup_{i \in \mathbb{N}} A_i$$

and so

$$N_\delta(x) \subseteq \bigcup_{i \in \mathbb{N}} A_i.$$

This tells us that  $x \in \left( \bigcup_{i \in \mathbb{N}} A_i \right)^\circ$ . Therefore,

$$\bigcup_{i \in \mathbb{N}} A_i^\circ \subseteq \left( \bigcup_{i \in \mathbb{N}} A_i \right)^\circ.$$

■

(b) Give an example of a finite collection in which equality does not hold in (a).

**Solution.** Consider the two closed intervals  $A = [0, 1]$  and  $B = [1, 2]$ . Note that  $A^\circ = (0, 1)$  and  $B^\circ = (1, 2)$ . Thus,

$$A^\circ \cup B^\circ = (0, 1) \cup (1, 2)$$

whereas

$$(A \cup B)^\circ = ([0, 2])^\circ = (0, 2).$$

So,  $(0, 1) \cup (1, 2) \subsetneq (0, 2)$  since 1 is not included in  $(0, 1) \cup (1, 2)$ . ■