Homework 7

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Problem 1. Let $(V, \|\cdot\|)$ be an infinite dimensional normed space.

(i) Assume that $(V, \|\cdot\|)$ is Banach.

Problem 2 (Extra Credit)**.** Let $(V, \|\cdot\|)$ be a normed space in which for any sequence (v_n) in V

$$\sum_{n=1}^{\infty} \|v_n\| < \infty \Longrightarrow \sum_{n=1}^{\infty} v_n \text{ converges in } V.$$

Prove that $(V, \|\cdot\|)$ is Banach.

Proof. Suppose that every absolutely convergent series is convergent. Our goal is to show that $(V, \|\cdot\|)$ is a Banach space. To do this, we will show that every Cauchy sequence in V converges. Let (v_n) be a Cauchy sequence in V. From here, our strategy is to find a subsequence (v_{n_k}) of (v_n) such that (v_{n_k}) converges in V (by the lemma). By definition, (v_n) being Cauchy implies that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$\|v_n - v_m\| < \varepsilon.$$

For $\varepsilon = 1$, there exists an $n_1 \in \mathbb{N}$ such that for any $n > m > n_1$, we have

$$\|v_n - v_m\| < 1.$$

Furthermore, if $\varepsilon = \frac{1}{2}$. So, there exists an $n_2 > n_1$ by the Archimedean Property such that for any $n > m > n_2$, we have

$$\|v_n - v_m\| < \frac{1}{2}.$$

In particular, if $\varepsilon = \frac{1}{2^{k-1}}$ for all $k \in \mathbb{N}$, then we can find an $n_k \in \mathbb{N}$ such that for any $n > m > n_k$, we have

$$\|v_n - v_m\| < \frac{1}{2^{k-1}}.$$

Moreover, by the Archimedean Property we can find an $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k > n_{k-1}$. Hence, it follows that (v_{n_k}) is a subsequence in V such that

$$0 \leq \|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^{k-1}}. \tag{*}$$

Note that since $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ is a geometric series it follows from the Comparison Test that

$$\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\|$$

converges to some $v \in V$. By assumption, this tells us that

$$\sum_{k=1}^{\infty} (v_{n_{k+1}} - v_{n_k})$$

converges to some v in V. Now, observe that

$$v_{n_1} + \sum_{j=1}^{k-1}(v_{n_{j+1}} - v_{n_j}) = v_{n_1} + (v_{n_2} - v_{n_1}) + (v_{n_3} - v_{n_2}) + \cdots + (v_{n_k} - v_{n_{k-1}})$$
$$= v_{n_k}.$$

Taking the limit on both sides of the above equality, we see that

$$\lim_{k\to\infty} v_{n_k} = \lim_{k\to\infty}\left[v_{n_k} + \sum_{j=1}^{k-1}(v_{n_{j-1}} - v_{n_j})\right]$$
$$= v_{n_1} + \lim_{k\to\infty}\sum_{j=1}^{k-1}(v_{n_{j+1}} - v_{n_j})$$
$$= v_{n_1} + v.$$

Thus, we now see that (v_{n_k}) converges in V which tells us that (v_n) is a converges in V. Hence, ■

Lemma. Let $(V, \|\cdot\|)$ be a normed space. Suppose (v_n) is a Cauchy sequence, and some subsequence (v_{n_k}) converges to a point v in V. Then (v_n) converges to v in V.

Proof. Let $n > m$. Since (v_n) is a Cauchy sequence in V, it follows that

$$\|v_n - v_m\| \to 0$$

as $n, m \to \infty$. Also, (v_{n_k}) converges to some $v \in V$. So, for $k \to \infty$, we have

$$\|v_{n_k} - v\| \to 0.$$

Using the triangle inequality, it follows that

$$0 \le \|v_n - v\| \le \|v_n - v_{n_k}\| + \|v_{n_k} - v\| \to 0.$$

Using the Squeeze Theorem, we have

$$\|v_n - v\| \to 0$$

as $n \to \infty$ and we are done. ■