

Math 230A: Homework 5

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Problem 1. Mark each statement True or False. Let (X, d) be a metric space and $K \subseteq X$.

1. If $Y \subseteq X$ and $\{G_\alpha\}$ is a collection of subsets of Y that are open relative to Y , then $\bigcup_\alpha G_\alpha$ is open relative to Y .
2. If there exists some open cover of K which has a finite subcover, then K is compact.
3. If K is compact, then $K' \subseteq K$.
4. If K is closed, then K is compact.
5. If K is compact and $E \subseteq K$, then E is compact.
6. Consider $E = [-10, 10] \subseteq \mathbb{R}$ and the open cover of E by $\Phi = \{(x - 1, x + 1) : x \in E\}$. Then the collection $\{(x - \frac{1}{2}, x + \frac{1}{2}) : x \in E\}$ is a subcover of Φ .
7. Let E and Φ be as above. Then the collection $\{(x - 1, x + 1) : x = -10, -9, -8, \dots, 8, 9, 10\}$ is a subcover of Φ .
8. Let E and Φ be as above. Then the collection $\{(x - 1, x + 1) : x = -10, -8, -6, \dots, 6, 8, 10\}$ is a subcover of Φ .

Solution. 1. True.

2. False.
3. True.
4. False.
- 5.
- 6.
- 7.

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Problem 2. Show that compact implies bounded.

Proof. Let $E \subseteq X$. Suppose E is compact. Our goal is to show that E is bounded set in X ; that is, there exists $p \in X$ and $\delta > 0$ such that $E \subseteq N_\delta(p)$. Let $x \in E$. To this end, we will show that there exists a $p \in X$ and $\delta > 0$ such that $E \subseteq N_\delta(x)$. Since E is compact, E is a closed set. Hence, we know that

$$E = \overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Choose $\delta = \varepsilon + 1$. Since E is closed, then there exists a $p \in N_\varepsilon(x) \cap E$ such that $p \in N_\varepsilon(x)$ and $p \in E$ (that is, $p \in X$). Since $p \in N_\varepsilon(x)$, we have

$$d(x, p) < \varepsilon < \varepsilon + 1 = \delta.$$

Hence, we see that $p \in N_\delta(x)$. Thus, $E \subseteq N_\delta(x)$ and so E is bounded.

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Problem 3. Show the union of finitely many compact sets is compact.

Proof. Let (X, d) be a metric space. Set $K = \bigcup_{i=1}^n K_i$ with K_i compact for all $1 \leq i \leq n$. Our goal is to show that K is compact. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X that forms an open cover for K . Since K_i is compact for all $1 \leq i \leq n$, we can find a finite subcover for each K_i . Thus, we have

$$K_i \subseteq \bigcup_{j=1}^{m_i} O_{i,j} \quad 1 \leq i \leq n.$$

Note that each $\bigcup_{j=1}^{m_i} O_{i,j}$ is open since each $O_{i,j}$ is open. Furthermore,

$$\sum m_i = n.$$

Denote each of these unions as

$$Q_{\alpha_i} = \bigcup_{j=1}^{m_i} O_{i,j}.$$

Thus, we see that

$$K = \bigcup_{i=1}^n K_i \subseteq \bigcup_{i=1}^n Q_{\alpha_i}$$

and so K is compact. ■

Problem 4. Show that an arbitrary intersection of compact sets is compact. (Hint. A closed subset of a compact set is compact)

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X and let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets in X . Define

$$K = \bigcap_{\alpha} K_\alpha. \tag{1}$$

Since each K_α is compact, we know that each K_α must be closed. Therefore, the arbitrary intersection above must be closed and so K is closed. But observe that

$$\bigcup_{\alpha} K_\alpha \subseteq K_\alpha$$

and that K_α is a closed set. Thus, K must be compact as well! ■

Problem 5. A metric space (X, d) is called **separable** if it contains a countable subset E which is dense in X . For example, \mathbb{R} is separable because \mathbb{Q} is a countable set which is dense in \mathbb{R} . Show that \mathbb{R}^2 is separable. (Hint: Consider the set of points which have only rational coordinates.)

Proof. Note that \mathbb{R}^2 is just $\mathbb{R} \times \mathbb{R}$. Recall that a finite product of countable sets is countable. Thus, we see that $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ is countable. All that is left to show is that \mathbb{Q}^2 is dense in \mathbb{R}^2 . Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$. Since \mathbb{Q} is dense in \mathbb{R} , there must exist $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ such that $x_1 < p < x_2$ and $y_1 < q < y_2$, respectively. Denote the points in \mathbb{R}^2 as $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ and $r = (p, q)$. Thus, we have that

$$p_1 < r < p_2.$$

Thus, \mathbb{Q}^2 must be dense in \mathbb{R}^2 . ■

Problem 6. Let (X, d) be a separable metric space and $\emptyset \neq A \subseteq X$. Prove that the collection of the isolated points of A is at most countable.

Proof. Let (X, d) be a separable metric space and $\emptyset \neq A \subseteq X$. Denote the set of isolated points as

$$A_I = \{x \in X : \exists \varepsilon > 0 \text{ such that } N_\varepsilon(x) \cap A = \{x\}\}.$$

By assumption, X contains a subset E such that E is countable and $\overline{E} = X$. Our goal is to show that A_I is at most countable. It suffices to show that $A_I \subseteq E$. Let $x \in A_I$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \cap A = \{x\}$. Hence, $N_\varepsilon(x) \subseteq A^c$ for some $\varepsilon > 0$ and so A^c is an open set in X . By problem 19 of homework 4, we see that

$$A^c \cap E \neq \emptyset.$$

Thus, x must be contained in the intersection above. So, $A_I \subseteq E$. Since E is countable, A_I must be at most countable and we are done. ■

Problem 7. Let (X, d) be a metric space. A collection $\{V_\alpha\}$ of open subsets of X is said to be a **base** for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $V_\alpha \subseteq G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that the every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X .)

Proof. Since (X, d) is a separable metric space, we know that X contains a countable dense subset E ; that is, we have

$$X = \overline{E} = E = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Let $\{V_\alpha\}_{\alpha \in \Lambda}$. Let $x \in X$ and let $G \subseteq X$ be an open set such that $x \in G$. Our goal is to show that $V_\alpha \subseteq G$ for some α . Since $X = \overline{E}$, we must have $x \in \overline{E}$; that is, for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

To this end, let $\varepsilon = \frac{1}{n}$ and pick a point $y \in N_{1/n}(x) \cap E$. Then

$$y \in N_{1/n}(x) \text{ and } y \in E.$$

Note that $N_{1/n}(x)$ is an open set in X . Thus, we can write this set in the following way

$$N_{1/n}(x) = \bigcup_{\alpha} V_\alpha.$$

Thus, y must be contained in the union above and so $y \in V_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since $y \in G$, we must also have $V_{\alpha_0} \subseteq G$ and we are done. ■

Problem 8. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. (Hint: Fix $\delta > 0$, and pick $x_1 \in X$.) Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ . Take $\delta = 1/n$ ($n = 1, 2, 3, \dots$), and consider the centers of the corresponding neighborhoods.

Proof. ■

Problem 9 (Extra Credit). Let (X, d) be a metric space and Y be a nonempty subset of X . Let $E \subseteq Y$. Prove that

$$E \text{ is closed relative to } Y \iff E = A \cap Y \text{ for some closed set } A \subseteq X.$$

Proof. (\implies) Assume that E is closed relative to Y . Our goal is to show that $E = A \cap Y$; that is, we need to show two inclusions:

$$(1) E \subseteq A \cap Y,$$

$$(2) A \cap Y \subseteq E.$$

By assumption, we know that

$$E = \overline{E} = \{\forall x \in Y, \forall \varepsilon > 0 : N_\varepsilon^Y(x) \cap E \neq \emptyset\}.$$

Note that $N_\varepsilon(x) \subseteq \overline{N_\varepsilon(x)}$. Set

$$A = \bigcup_{i=1}^n \overline{N_{\varepsilon_i}(x)}$$

with radius $\varepsilon_i > 0$ for each N_{ε_i} . Notice that A must be a closed set in Y because each $\overline{N_{\varepsilon_i}(x)}$ is a closed set. Since

$$N_\varepsilon(x) \cap Y \subseteq \overline{N_\varepsilon(x)} \cap Y \subseteq E,$$

we must have

$$\begin{aligned} A \cap Y &= \left(\bigcup_{i=1}^n \overline{N_{\varepsilon_i}(x)} \right) \cap Y \subseteq \bigcup_{i=1}^n (\overline{N_{\varepsilon_i}(x)} \cap Y) \\ &\subseteq \bigcup_{i=1}^n E \\ &= E. \end{aligned}$$

Thus, we see that $A \cap Y \subseteq E$ which proves (1).

Now, we want to show (2). Let $p \in E$. Using our definition of A and the fact that $N_\varepsilon(p) \subseteq \overline{N_\varepsilon(p)}$,

$$\begin{aligned} E &\subseteq N_\varepsilon(p) \cap Y \subseteq \overline{N_\varepsilon(p)} \cap Y \subseteq \left[\bigcup_{i=1}^n \overline{N_{\varepsilon_i}(x)} \right] \cap Y \\ &= A \cap Y \end{aligned}$$

which shows (2). Hence, (1) and (2) imply that $E = A \cap Y$ for some closed set $A \subseteq Y$.

(\Leftarrow) Suppose $E = A \cap Y$ for some closed set $A \subseteq X$. Our goal is to show that E is closed relative to Y . Let p be a limit point of E .

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Problem 10 (Extra Credit). Let (X, d) be a metric space. Let $E \subseteq X$. Prove that the following definitions of boundedness are equivalent:

- Rudin's Definition: There exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$.
- Anthony's Definition: There exists $R > 0$ such that for all x and y in E , we have $d(x, y) < R$.

Proof.

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