Math 230B: Homework 1

Lance Remigio

February 14, 2025

Problem 1. Let $m, n \in \mathbb{N}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^m \sin \frac{1}{x^n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(i) Prove that f is differentiable at all $x \neq 0$.

(ii) Prove that $m > 1$, then f is differentiable at 0.

(iii) Prove that if $m > 1 + n$, then f' is continuous at 0.

(iv) Prove that if $m > 2 + n$, then f' is differentiable on $\mathbb{R}$.

(v) Prove that if $m > 2 + 2n$, then f'' is continuous at 0.

(vi) Prove that if $2 + n < m \leq 2 + 2n$, then f'' is not continuous at 0.

Proof. (i) Suppose $x \neq 0$. Notice that

(1) x^m is a polynomial that is differentiable for any $x \neq 0$.

(2) $\frac{1}{x^n}$ is differentiable for any $x \neq 0$.

(3) $\sin x$ is differentiable for any $x \neq 0$.

(ii) Suppose $m > 1$. Our goal is to show that f is differentiable at 0; that is, we will show that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^m \sin \frac{1}{x^n}}{x} = \lim_{x\to 0} x^{m-1} \sin \frac{1}{x^n}. \tag{*}$$

Since $|\sin \frac{1}{x^n}| \leq 1$, we can see that

$$0 \leq |x^{m-1} \sin \frac{1}{x^n}| = |x^{m-1}| \Big| \sin \frac{1}{x^n} \Big| \leq |x^{m-1}|. \tag{**}$$

Since $m - 1 > 0$ and x^{m-1} is a polynomial that is continuous everywhere on $\mathbb{R}$, we have $\lim_{x\to 0} x^{m-1} = 0$. As a consequence, we also have $\lim_{x \to 0} |x^{m-1}| = 0$. By applying the Squeeze Theorem for functions to the inequality in (**), we conclude that

$$\lim_{x\to 0} |x^{m-1} \sin \frac{1}{x^n}| = 0 \iff \lim_{x\to 0} x^{m-1} \sin \frac{1}{x^n} = 0.$$

But this implies that the limit in (*) exists and so f is differentiable at 0.

(iii) Computing f', we have

$$f'(x) = \begin{cases} mx^{m-1} \sin \frac{1}{x^n} - x^{m-n-1} \cos \frac{1}{x^n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

We wills show that both

$$mx^{m-1}\sin\frac{1}{x^n} \text{ and } x^{m-n-1}\cos\frac{1}{x^n}$$

approach 0 as $x \to 0$. Since $|\sin x| \le 1$ for all $x \in \mathbb{R}$, we have

$$0 \le |mx^{m-1}\sin\frac{1}{x^n}| \le |mx^{m-1}|. \tag{I}$$

Since $m-1 > n > 0$ and x^{m-1} is a polynomial which is continuous at 0, the Algebraic Continuity Theorem implies that

$$\lim_{x\to 0} mx^{m-1} = 0 \iff \lim_{x\to 0} |mx^{m-1}| = 0.$$

Using the Squeeze Theorem on (I), we conclude that

$$\lim_{x\to 0} |mx^{m-1}\sin\frac{1}{x^n}| = 0 \iff \lim_{x\to 0} mx^{m-1}\sin\frac{1}{x^n} = 0.$$

Using a similar argument, we can prove that

$$x^{m-n-1}\cos\frac{1}{x^n}$$

is continuous at 0. Indeed, we have

$$0 \le |x^{m-n-1}\cos\frac{1}{x^n}| \le |x^{m-n-1}|. \qquad (|\cos x| \le 1\ \forall x \in \mathbb{R})$$

Notice that $m-n-1 > 0$ and that x^{m-n-1} is a polynomial which is continuous everywhere on $\mathbb{R}$, we have $\lim\limits_{x\to 0} x^{m-n-1} = 0$. Hence,

$$\lim_{x\to 0} |x^{m-n-1}| = 0.$$

Applying the Squeeze Theorem, we have

$$\lim_{x\to 0} |x^{m-n-1}\cos\frac{1}{x^n}| = 0 \iff \lim_{x\to 0} x^{m-n-1}\cos\frac{1}{x^n} = 0.$$

Using the Algebraic Limit theorem for functions, we can conclude that as $x \to 0$

$$f'(x) = mx^{m-1}\sin\frac{1}{x^n} - x^{m-n-1}\cos\frac{1}{x^n} \to 0 = f'(0)$$

and so $f'(x)$ is at continuous at 0.

(iv) Our goal is to show that f' is differentiable on $\mathbb{R}$. Let $c \in \mathbb{R}$. Suppose $c = 0$. We will show that the limit

$$\begin{aligned}\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} &= \lim_{x\to 0}\frac{mx^{m-1}\sin\frac{1}{x^n} - x^{m-n-1}\cos\frac{1}{x^n}}{x}\\ &= \lim_{x\to 0}\left[mx^{m-2}\sin\frac{1}{x^n} - x^{m-n-2}\cos\frac{1}{x^n}\right]\\ &= 0.\end{aligned}$$

To prove this, we will use the same approach we used in part (iii); that is, we will show that each corresponding function of the above limit exists and equals 0. For te first function, notice that

$$0 \le |mx^{m-2}\sin\frac{1}{x^n}| \le |mx^{m-2}|. \tag{I}$$

Since x^{m-2} (note that $m-2>0$ by assumption) is a polynomial that continuous everywhere, we have $\lim_{x\to 0} x^{m-2} = 0$. This holds if and only if $\lim_{x\to 0} |x^{m-2}| = 0$. By applying the Squeeze Theorem to the inequality in (I), we have

$$\lim_{x\to 0} |mx^{m-2}\sin\frac{1}{x^n}| = 0 \Longleftrightarrow \lim_{x\to 0} mx^{m-2}\sin\frac{1}{x^n} = 0.$$

Now, we will show that

$$\lim_{x\to 0} x^{m-n-2}\cos\frac{1}{x^n} = 0.$$

Again, with a similar argument used in part (iii), we have

$$0 \le |x^{m-n-2}\cos\frac{1}{x^n}| \le |x^{m-n-2}|. \qquad (|\cos x| \le 1\ \forall x \in \mathbb{R})$$

Since x^{m-n-2} is a polynomial which is continuous everywhere on $\mathbb{R}$, we have $\lim_{x\to 0} x^{m-n-2} = 0$. This holds if and only if

$$\lim_{x\to 0} |x^{m-n-2}| = 0.$$

Using the Squeeze Theorem on the inequality above, we have

$$\lim_{x\to 0} |x^{m-n-2}\cos\frac{1}{x^n}| = 0 \Longleftrightarrow \lim_{x\to 0} x^{m-n-2}\cos\frac{1}{x^n} = 0.$$

Using the Algebraic limit theorem for functions, we can conclude that f' is differentiable for $c=0$. Suppose $c \neq 0$. From part (iii), we have

$$f'(x) = mx^{m-1}\sin\frac{1}{x^n} - x^{m-n-1}\cos\frac{1}{x^n}.$$

Note the following:

(1) x^{m-1} and x^{m-n-1} are polynomials which are differentiable everywhere on $\mathbb{R}$.

(2) $\sin x$ and $\cos x$ are trigonometric functions which are differentiable everywhere on $\mathbb{R}$.

(3) $\frac{1}{x^n}$ is a rational function are differentiable for every $x \neq 0$ in $\mathbb{R}$.

(4) $\sin(\frac{1}{x^n})$ is differentiable for every $x \neq 0$ in $\mathbb{R}$ by the chain rule.

From (1), (2), (3), and (4) we can use the Algebraic differentiability theorem to conclude that $f'(x)$ is indeed differentiable.

(v) We can use the same argument as the above to prove that $f''(x)$ is continuous at 0. By the Algebraic Continuity Theorem, we conclude that $f''(x)$ is continuous at 0.

(vi) Computing f'', we have

$$f''(x) = \begin{cases} m(m-1)x^{m-2}\sin\left(\frac{1}{x^n}\right) + n^2x^{m-2n-2}\sin\left(\frac{1}{x^n}\right) & \\ -n(2m-n-2)x^{m-n-2}\cos\left(\frac{1}{x^n}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Define $a_k = \frac{1}{\sqrt[n]{2\pi k + \frac{\pi}{2}}}$. We can see immediately that $a_k \to 0$ and that

$$\sin\frac{1}{(a_k)^n} = \sin\left(2\pi k + \frac{\pi}{2}\right) = 1,$$
$$\cos\frac{1}{(a_k)^n} = \cos\left(2\pi k + \frac{\pi}{2}\right) = 0$$

Now, for all $k \geq 1$, we can see that

$$\lim_{k\to\infty} f''(a_k) = \lim_{k\to\infty} \frac{1}{(a_k)^{(2+2n)-m}}.$$

Note that if $m = 2 + 2n$, we just have $\lim_{k\to+\infty} f''(a_k) = -n^2$. Otherwise, $\lim_{k\to+\infty} \frac{1}{(a_k)^{(2+2n)-m}} = \infty$. That is,

$$\lim_{k\to\infty} f''(a_k) \neq 0 = f''(0).$$

So, we conclude that f'' is not continuous at 0.

■

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t < 0 \\ 0 & \text{if } t \leq 0 \end{cases}.$$

Prove that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Proof. Our goal is to show that f is infinitely differentiable at 0. Note that for $t \leq 0$, one can immediately see that via induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. In what follows, we will show that $f^{(n)}(0) = 0$ for all $t > 0$ and for all $n \in \mathbb{N}$. First, we will show that for all $n \in \mathbb{N}$

$$f^{(n)}(t) = e^{-1/t} \frac{P_n(t)}{t^{2n}}.$$

Using a combination of the chain rule and product rule, we have that

$$f'(t) = \frac{e^{-1/t}}{t^2}.$$

Suppose that the result in (*) holds for $n = k$. We will show that it also holds for $n = k + 1$ case. Indeed, we see that

$$\begin{aligned} f^{(k+1)}(t) = \frac{d}{dt}[f^{(k)}(t)] &= \frac{d}{dt}\left[P_k(t) \cdot \frac{e^{-1/t}}{t^{2k+2}}\right] \\ \implies f^{(k+1)}(t) &= \frac{d}{dt}[P_k(t)]\frac{e^{-1/t}}{t^{2k}} + t^2 P_k(t)\frac{e^{-1/t}}{t^{2k}} - 2kP_k(t)\frac{e^{-1/t}}{t^{2k+1}} \\ &= \underbrace{\left[\frac{1}{t}\frac{d}{dt}[P_k(t)] + tP_k(t) - 2kP_k(t)\right]}_{\text{polynomial of degree at most} n+1} \frac{e^{-1/t}}{t^{2k+1}} \\ &= P_{k+1}(t)\frac{e^{-1/t}}{t^{2k+1}}. \end{aligned}$$

This implies that

$$f^{(k+1)}(t) = P_{k+1}(t)\frac{e^{-1/t}}{t^{2k+1}}.$$

Our goal is to show that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. We proceed via induction on $n \in \mathbb{N}$. We will start with proving that f is differentiable once. Indeed,

$$\begin{aligned}
\lim_{t\to 0} \frac{f(t)-f(0)}{t} &= \lim_{t\to 0} \frac{e^{\frac{-1}{t}}-0}{t} \\
&= \lim_{t\to 0} \frac{1}{t} e^{-\frac{1}{t}} \\
&= \lim_{t\to 0} \frac{1}{te^{1/t}} \\
&= 0.
\end{aligned}$$

Hence, $f'(0)$ exists and so $f'(0) = 0$ by the above. Suppose for our induction hypothesis that f is differentiable at 0 $k = n$ times. Our goal is to show that f is differentiable $n = k+1$ times. Note that where $P_n(t)$ is a polynomial of at most degree n. Since $f^{(k+1)}(t)$ exists, we have

$$\begin{aligned}
f^{(k+1)}(0) &= \lim_{t\to 0} \frac{f^{(k)}(t)-f^{(k)}(0)}{t-0} \\
&= \lim_{t\to 0} \frac{1}{t} \cdot \frac{e^{-1/t}P_k(t)}{t^{2k}} \\
&= \lim_{t\to 0} \left[P_k(t) \cdot \frac{e^{-1/t}}{t^{2k+1}}\right] \\
&= \lim_{t\to 0} P_k(t) \cdot \lim_{t\to 0} \frac{e^{-1/t}}{t^{2k+1}}
\end{aligned}$$

Note that $\lim\limits_{t\to 0} \frac{e^{-1/t}}{t^{2k+1}} = 0$ by L'Hopital's rule. Hence, we conclude that

$$f^{(k+1)}(0) = 0.$$

■

Problem 3. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Let $c \in I$. Recall that in class we proved that f is differentiable at c if and only if $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists. Use this result to prove that f is differentiable at c if and only if

$$\exists L \in \mathbb{R} \text{ such that } \lim_{h\to 0} \frac{f(c+h)-f(c)-Lh}{h} = 0.$$

Proof. Suppose that f is differentiable at c. Then

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists}$$

and so

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = L$$

for some $L \in \mathbb{R}$. Now, the right-hand side can be written in the following way:

$$L = \lim_{h\to 0} L = \lim_{h\to 0} \frac{Lh}{h}.$$

Note that the quantity $\frac{h}{h}$ holds because of the $\varepsilon - \delta$ definition of the derivative. Now, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = L &\iff \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{Lh}{h} \\
&\iff \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \lim_{h\to 0} \frac{Lh}{h} = 0.
\end{aligned}$$

By the Algebraic Limit Theorem for functions, we conclude that

$$\lim_{h\to 0} \frac{f(c+h) - f(c) - Lh}{h} = 0.$$

We can reverse this argument to get the prove the reverse direction. ∎

Problem 4. Let $g : A \to \mathbb{R}$ where A is a nonempty subset of $\mathbb{R}$. Suppose 0 is an interior point of A. Use the $\varepsilon - \delta$ definition of limit to prove that $\lim_{h\to 0} g(h) = L$, then $\lim_{h\to 0} g(-h) = L$.

Proof. Our goal is to show that $\lim_{h\to 0} g(-h) = L$; that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |h| < \delta$, we have

$$|g(-h) - L| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim_{h\to 0} g(h) = L$, we can find a $\hat{\delta} > 0$ such that whenever $0 < |h| < \hat{\delta}$ (note that here 0 is an interior point of A),

$$|g(h) - L| < \varepsilon. \qquad (*)$$

We claim that $\hat{\delta}$ is the same δ we were looking for. But observe that $|h| = |-(-h)| = |-h|$ and so, if $0 < |-h| < \delta$ and using our assumption in (*), we can write

$$|g(-h) - L| < \varepsilon$$

which is our desired result. ∎

Problem 5. Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Let c be an interior point of I. Assume f is differentiable at c.

(a) Recall that $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$. Use this and the result of Exercise 4 to show that

$$f'(c) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}.$$

(b) Use the result of (a) to prove that

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

Proof. (a) Define $g : I \to \mathbb{R}$ by

$$g(h) = \frac{f(c+h) - f(c)}{h}.$$

Notice that

$$g(-h) = \frac{f(c-h) - f(c)}{-h} = \frac{f(c) - f(c-h)}{h}.$$

By exercise 4, we can see that

$$f'(c) = \lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h)$$

Hence, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}.$$

(b) For h sufficiently small, we have

$$\begin{aligned}\frac{f(c+h)-f(c-h)}{2h} &= \frac{f(c+h)-f(c)}{2h} + \frac{f(c)-f(c-h)}{2h} \\ &= \frac{1}{2}\cdot\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\cdot\frac{f(c)-f(c-h)}{h}\end{aligned}$$

Now, taking the limit as $h \to 0$, we have by part (a) (and using the Algebraic Limit Theorem for functions) that

$$\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h} = \frac{1}{2}f'(c) + \frac{1}{2}f'(c) = f'(c).$$

■

Problem 6. Recall that in one of the homework assignments of Math 230A we proved that $\sin x$ and $\cos x$ are continuous functions on $\mathbb{R}$. We also proved that $\lim_{x\to 0}\frac{\sin x}{x} = 1$

(i) Use this result to show that

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

(ii) Use (i) to show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is differentiable at all points $c \in \mathbb{R}$ and $f'(c) = \cos c$ for all $c \in \mathbb{R}$.

Proof. (i) Suppose $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Then for for a sufficiently small neighborhood of zero, we may write

$$\begin{aligned}\frac{\cos h - 1}{h} &= \Big(\frac{\cos h - 1}{h}\Big)\Big(\frac{\cos h + 1}{\cos + 1}\Big) \\ &= \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \frac{-\sin^2 h}{h(\cos h + 1)} \\ &= \frac{\sin h}{h}\cdot\frac{-\sin h}{\cos h + 1}.\end{aligned}$$

Note that the first term of the product in the last equality above exists by assumption and the second term exists because

$$\lim_{h\to 0}\frac{-\sin h}{\cos h + 1} = 0.$$

Indeed, $\sin h$ and $\cos h$ are both continuous functions, and so $\lim_{h\to 0}(-\sin h) = -\sin 0 = 0$ and $\lim_{h\to 0}(\cos h + 1) = 2$ along with the Algebraic Continuity Theorem implies that the above limit holds. Now, using the Algebraic Limit Theorem for functions, we can write that

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\Big(\frac{\sin h}{h}\cdot\frac{-\sin h}{\cos h + 1}\Big) \\ &= \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{-\sin h}{\cos h + 1} \\ &= 1\cdot 0 \\ &= 0\end{aligned}$$

which is our desired result.

(ii) By the summation trigonometric identity

$$\begin{aligned}\frac{\sin(c+h)-\sin h}{h} &= \frac{[\sin c\cos h+\cos c\sin h]-\sin c}{h}\\ &= \frac{\sin c(1-\cos h)+\cos c\sin h}{h}\\ &= \sin c\cdot\frac{1-\cos h}{h}+\cos c\cdot\frac{\sin h}{h}.\end{aligned}$$

Using part (i) along with the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$, we have

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(c+h)-\sin h}{h} &= \lim_{h\to 0}\left(\sin c\cdot\frac{1-\cos h}{h}\right)+\lim_{h\to 0}\left(\cos c\cdot\frac{\sin h}{h}\right)\\ &= \sin c\cdot\lim_{h\to 0}\frac{1-\cos h}{h}+\cos c\cdot\lim_{h\to 0}\frac{\sin h}{h}\\ &= \sin c\cdot 0+\cos c\cdot 1\\ &= \cos c.\end{aligned}$$

Clearly, we can see that the limit above does exist. Now, we can conclude that

$$f'(c)=\cos c.$$

■

Problem 7. Prove the following theorem.

Theorem (Generalized Mean Value Theorem)**.** If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point $c\in(a,b)$ where

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c).$$

Proof. Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Our goal is to show that there exists a point $c\in(a,b)$ where

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)f'(c).]$$

To this end, define the function $h:[a,b]\to\mathbb{R}$ by

$$h(x)=[f(b)-f(a)]g(x)-[g(b)-g(a)]f(x).$$

Our goal is to show that h is continuous on $[a,b]$ and differentiable on the open interval (a,b). Indeed, knowing that f and g are continuous on $[a,b]$ implies, by the Algebraic Continuity Theorem, that $h(x)$ is continuous. Furthermore, f and g are differentiable on (a,b), and so $h(x)$ must also be differentiable by the Algebraic differentiability Theorem. Also, we have

$$\begin{aligned}h(b)-h(a) &= [f(b)-f(a)]g(b)-[g(b)-g(a)]f(b)\\ &\quad-\Big([f(b)-f(a)]g(a)-[g(b)-g(a)]f(b)\Big)\\ &= f(b)g(b)-f(a)g(b)-g(b)f(b)+g(a)f(b)\\ &\quad-f(b)g(a)+f(a)g(a)+g(b)f(b)-g(a)f(b)\\ &= 0\end{aligned}$$

Thus, we have $h(b)=h(a)$ and so, the Rolle's Theorem implies that there exists a $c\in(a,b)$ such that $h'(c)=0$. Hence, we have

$$h'(c)=[f(b)-f(a)]g'(c)-[g(b)-g(a)]f'(c)=0$$

and so

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

which is our desired result. ■

Problem 8. Prove the following theorem.

Theorem. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Prove that

$$\forall x \in I, f'(x) > 0 \implies f \text{ is strictly increasing on } I.$$

Proof. Suppose that for all $x \in I$, we have $f'(x) > 0$. Our goal is to show that f is strictly increasing on I; that is, for all $x_1, x_2 \in I$ with $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Since f is differentiable on I, we must also have that f is continuous on I. Consider the open interval (x_1, x_2) in I. Then f must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, we can see that $f'(c) > 0$. Since $x_2 - x_1 > 0$, we can see that

$$f(x_2) - f(x_1) > 0 \iff f(x_2) > f(x_1) \ \forall x_1, x_2 \in I.$$

■

Problem 9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $C > 0$.

(i) Suppose $|f(u) - f(v)| \leq C|u - v|$ for all $u, v \in \mathbb{R}$. Prove that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$.

(ii) Suppose $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. Prove that $|f(u) - f(v)| \leq C|u - v|$ for all $u, v \in \mathbb{R}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $C > 0$.

(i) Our goal is to show that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. To this end, let $x \in \mathbb{R}$. To show the result, we must show that

$$-C \leq \lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C. \tag{*}$$

By assumption, we can see that

$$\begin{aligned} |f(\hat{y}) - f(x)| \leq C|\hat{y} - x| &\iff \left| \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \right| \leq C \\ &\iff -C \leq \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C. \end{aligned}$$

Since f is differentiable on $\mathbb{R}$, we can see that

$$\lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \text{ exists.}$$

Applying the Order Limit Theorem for functions on the above inequality implies that

$$-C \leq \lim_{\hat{y} \to x} \frac{f(\hat{y}) - f(x)}{\hat{y} - x} \leq C$$

which tells us further that

$$|f'(x)| \leq C.$$

(ii) Suppose $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. Our goal is to show that

$$|f(u) - f(v)| \leq C|u - v| \ \forall u, v \in \mathbb{R}.$$

Let $u, v \in \mathbb{R}$. Consider the closed interval $[u, v] \subseteq \mathbb{R}$. Since f is continuous on $\mathbb{R}$, it follows immediately that f must also be continuous on $[u, v]$ (since f is differentiable on $\mathbb{R}$). Furthermore, f is differentiable on the open interval (u, v) since f is differentiable on $\mathbb{R}$. By the Mean Value Theorem, there exists a $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(u) - f(v)}{u - v}.$$

By assumption, we can see that $|f'(\xi)| \leq C$ and so

$$\Big|\frac{f(u) - f(v)}{u - v}\Big| = |f'(\xi)| \leq C.$$

Thus, we have

$$|f(u) - f(v)| \leq C|u - v|$$

which is our desired result.

■

Problem 10. Let $f : \mathbb{R} \to \mathbb{R}$ be given $f(x) = x^5 + x^3 - x^2 + 5x + 3$.

(i) Prove that there exists a solution to the equation $f(x) = 0$.

(ii) Prove that there cannot be more than one solution to the equation $f(x) = 0$.

Proof. (i) We proceed by using the Intermediate Value Theorem to show the result. Since f is continuous everywhere (because f is a polynomial), we can just consider a closed interval $[-1, 1]$. We will show that $f(-1) < 0$ and $f(1) > 0$. Indeed, we have

$$\begin{aligned} f(-1) &= (-1)^5 + (-1)^3 - (-1)^2 + 5(-1) + 3 \\ &= -5 < 0 \end{aligned}$$

and

$$\begin{aligned} f(1) &= (1)^5 + (1)^3 - (1)^2 + 5(1) + 3 \\ &> 9 > 0. \end{aligned}$$

Thus, the intermediate value theorem implies that there exists $\hat{c} \in [-1, 1]$ such that $f(\hat{c}) = 0$.

(ii) Suppose for sake of contradiction that there exists more than one solution $c_1, c_2 \in [-1, 1]$ such that $f(c_1) = 0$ and $f(c_2) = 0$. Thus, there exists $\tilde{c} \in (-1, 1)$ such that

$$f'(\tilde{c}) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = 0.$$

Since f is also differentiable everywhere on $\mathbb{R}$, we have

$$f'(x) = 5x^4 + 3x^2 - 2x + 5.$$

But note that $3x^2 - 2x + 5$ is a positive quadratic for all $x \in \mathbb{R}$. Hence, $f'(x) > 0$ for all $x \in \mathbb{R}$ which contradicts the fact that $f'(\tilde{c}) = 0$. ■

Problem 11. In class,we gave a proof of L'Hopital's Rule. If we add the following three assumptions to the hypotheses of the corresponding theorem, then we can give a shorter proof of H'opital's Rule:

(i) $f'(a)$ and $g'(a)$ exist.

(ii) $g'(a) \neq 0$.

(iii) f' and g' are continuous at a.

Here is the shorter proof:

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Solution. The first equality

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

holds because of (iii) and (ii). The third equality holds because of (i) and by definition of the derivative. Since are the referring to limits of functions, we can justify multiplying and dividing by $x - a$. The last equality holds because $f(a) = 0$ and $g(a) = 0$ from our original set of assumptions. ■

Problem 12. Let $n \in \mathbb{N}$ and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function for which the equation $f'(x) = 0$ has at most $n - 1$ solutions. Prove that the equation $f(x) = 0$ has at most n solutions.

Proof. Suppose for sake of contradiction that $f(x) = 0$ has at least n solutions. Denote the roots by

$$x_1 < x_2 < \cdots < x_n < x_{n+1}.$$

Now, notice that

$$f(x_1) = f(x_2) = \cdots = f(x_n) = f(x_{n+1}) = 0.$$

Since f is differentiable on $\mathbb{R}$, we can find an $c_i \in (x_i, x_i)$ for $1 \leq i \leq n$ such that $f'(c_i) = 0$ by the Mean Value Theorem. This implies that $f'(x)$ has n solutions which contradicts our assumption that $f'(x)$ has $n - 1$ solutions. ■

Problem 13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(i) Prove that f is differentiable at all $x \neq 0$.

(ii) Prove that $f'(0) = \frac{1}{2}$.

(iii) Prove that f is NOT increasing on any open interval containing 0.

Proof. (i) Note that x and x^2 are polynomials which is differentiable for all $x \neq 0$ in $\mathbb{R}$, $\sin x$ is differentiable for all $x \neq 0$ in $\mathbb{R}$, and $\frac{1}{x}$ is a rational function which is also differentiable for all $x \neq 0$ in $\mathbb{R}$. By the algebraic differentiability theorem, we have that $f(x)$ is a differentiable function for all $x \neq 0$.

(ii) Observe that

$$\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0} \left[\frac{1}{2} + x\sin\frac{1}{x}\right] \\
&= \lim_{x\to 0} \left[\frac{1}{2} + \frac{\sin\frac{1}{x}}{\frac{1}{x}}\right] \\
&= \frac{1}{2} + 0 && \text{(Algebraic Limit Theorem)} \\
&= \frac{1}{2}
\end{aligned}$$

where

$$\lim_{x\to 0} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 0.$$

Indeed, we have

$$0 \le |x\sin\frac{1}{x}| \le |x|.$$

Applying the squeeze theorem for functions to the inequality above, we have that

$$\lim_{x\to 0} \left|x\sin\frac{1}{x}\right| = 0$$

which further implies that

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

(iii) Define $a_n = \frac{1}{2\pi n} \to 0$. Our goal is to show that $f'(a_n) < 0$. Computing $f'(x)$ for $x \neq 0$, we have

$$f'(x) = \frac{1}{2} + \left[2x\sin\frac{1}{x} - \cos\frac{1}{x}\right].$$

Then we have

$$\begin{aligned}
f'(a_n) &= \frac{1}{2} + \left[\frac{1}{\pi}n \cdot \sin 2\pi n - \cos 2\pi n\right] \\
&= \frac{1}{2} + [0-1] \\
&= \frac{1}{2} - 1 \\
&= \frac{-1}{2} < 0.
\end{aligned}$$

Hence, $f'(a_n) < 0$ and so we conclude that f is NOT increasing on any open interval containing zero.

■

Problem 14. Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ be a differentiable function.

(a) Show that if there exists some $L \ge 0$ such that $|f'(x)| \le L$ for all $x \in I$, then f is uniformly continuous.

(b) Is the converse true? Prove it or give a counterexample.

Proof. (a) Suppose that there exists some $L \ge 0$ such that $|f'(x)| \le L$. Our goal is to show that f is uniformly continuous; that is, we need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-y|<\delta$, we have

$$|f(x)-f(y)|<\varepsilon.$$

Let $x,y\in\mathbb{R}$ and let $\varepsilon>0$. Suppose, without loss of generality, that $x<y$. Since f is differentiable on I and $(x,y)\subseteq I$, we can see that f is also differentiable on (x,y). Furthermore, f being differentiable on I implies that f is continuous on I and so f is continuous on $[x,y]$. By the Mean Value Theorem, we can find an $\ell\in(x,y)$ such that

$$f'(\ell)=\frac{f(x)-f(y)}{x-y}.$$

By assumption, we can see that for $L>0$ we have

$$\begin{aligned}|f'(\ell)|\leq L &\implies \left|\frac{f(x)-f(y)}{x-y}\right|\leq L\\ &\implies |f(x)-f(y)|\leq L|x-y|.\end{aligned}$$

Now, choose $\delta=\frac{\varepsilon}{L}$. Then whenever $|x-y|<\delta$, we can see that

$$|f(x)-f(y)|\leq L|x-y|<L\cdot\frac{\varepsilon}{L}=\varepsilon.$$

Hence, we conclude that f must be uniformly continuous on $\mathbb{R}$.

(b) Consider the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=\sqrt{x}$. We claim that this function is uniformly continuous on $\mathbb{R}$ but its derivative $f'(x)=\frac{1}{2}x^{-1/2}$ is not bounded above for some $L\geq 0$.

To show that f is uniformly continuous, let $x,y\in(0,\infty)$ and let $\varepsilon>0$. Choose $\delta=\sqrt{\varepsilon}$. Since $x,y\in(0,\infty)$, we have

$$|\sqrt{x}-\sqrt{y}|\leq|\sqrt{x}+\sqrt{y}|.$$

Observe that if $|x-y|<\delta$, then we have

$$|\sqrt{x}-\sqrt{y}|^2\leq|\sqrt{x}-\sqrt{y}||\sqrt{x}+\sqrt{y}|=|x-y|<\sqrt{\varepsilon}$$

which further implies that

$$|\sqrt{x}-\sqrt{y}|<\varepsilon.$$

Thus, $f(x)=\sqrt{x}$ is uniformly continuous on $(0,\infty)$.

Now, we want to show that the derivative of f is NOT bounded above. Note that

$$f'(x)=\frac{1}{2}x^{-1/2}.$$

We need to show that for all $M>0$ such that there exists $\hat{x}\in(0,\infty)$ such that $|f'(x)|>M$. To this end, let $\varepsilon>0$. Choose $\hat{x}=\frac{1}{64M^2}>0$. Then we have

$$\begin{aligned}|f'(\hat{x})|=\frac{1}{2\sqrt{\hat{x}}}&=\frac{1}{2}\cdot\frac{1}{\sqrt{1/64M^2}}\\ &=\frac{1}{2}\cdot\frac{1}{1/8M}\\ &=4M\\ &>M.\end{aligned}$$

Hence, f is not bounded above. ■

Problem 15. Let $f:(a,b)\to\mathbb{R}$ be differentiable on (a,b) with $|f'(x)|\leq M$ for $x\in(a,b)$ and $M\geq 0$.

Prove that

$$\lim_{x \to b^-} f(x)$$

exists.

Proof. Suppose $f : (a, b) \to \mathbb{R}$ is a differentiable function on (a, b) with $|f'(x)| \leq M$ for all $x \in (a, b)$. By Exercise 14, f must be uniformly continuous on (a, b). By Exercise 16 of Homework 10 from Math 230A, we can find a continuous function $F : [a, b] \to \mathbb{R}$ such that $F|_{(a,b)} = f$. As a consequence, we have $\lim_{x \to b} F(x) = F(b)$. Since we are only referring to the limit

$$\lim_{x \to b^-} f(x)$$

and $F|_{(a,b)} = f$, it follows that

$$\lim_{x \to b} f(x) = F(b)$$

exists. Clearly, if this holds, then

$$\lim_{x \to b^-} f(x) = F(b)$$

must also hold. ■

Problem 16. Let $f : (0, 1] \to \mathbb{R}$ be differentiable with $0 < f'(x) < 1$ for all $x \in (0, 1]$. Define a sequence (a_n):

$$a_n = f\left(\frac{1}{n}\right)$$

Prove that $\lim_{n \to \infty} a_n$ exists.

Proof. Our goal is to show that $\lim_{n \to \infty} a_n$ exists in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, it suffices to show that $a_n = f\left(\frac{1}{n}\right)$ is a Cauchy sequence; that is, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n, m > N$

$$|a_n - a_m| < \varepsilon.$$

To this end, let $\varepsilon > 0$ be given. Since $0 < f'(x) < 1$ for all $x \in (0, 1]$, we have

$$|a_n - a_m| = \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right) \right| \leq \left| \frac{1}{n} - \frac{1}{m} \right|$$

by exercise 9. Choose $\hat{N} = \frac{2}{\varepsilon}$. Then for any $n, m > N$, we have

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we see that $a_n = f\left(\frac{1}{n}\right)$ is a Cauchy sequence. ■

Problem 17. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Prove that, if $\lim_{x \to +\infty} f(x) = M \in \mathbb{R}$, then there exists a sequence (x_n) such that $|f'(x_n)| \to 0$.

Proof. Suppose $\lim_{x \to \infty} f(x) = M$. Our goal is to show that $|f'(x_n)| \to 0$. By assumption, let $\varepsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then there exists $\zeta_n > 0$ such that for any $x > \zeta_n$, we have

$$|f(x) - M| < \frac{1}{2n}.$$

In particular, we have $\zeta_n + 2 > \zeta_n + 1 > \zeta_n$. Since f is differentiable on $(0, \infty)$, f must also be continuous on $(0, \infty)$. Consider the open interval $(\zeta_n + 2, \zeta_n + 1)$ for all $n \in \mathbb{N}$. Since f is continuous on $(0, \infty)$, f is also continuous on $[\zeta_n + 1, \zeta_n + 2]$. By the Mean Value Theorem, we can find an

$x_n \in (\zeta_n + 1, \zeta_n + 2)$ such that

$$f'(x_n) = \frac{f(\zeta_n + 2) - f(\zeta_n + 1)}{\zeta_n + 2 - (\zeta_n + 1)} = f(\zeta_n + 2) - f(\zeta_n + 1).$$

By the triangle inequality, we have

$$0 < |f'(x_n)| \leq |f(\zeta_n + 2) - M| + |M - f(\zeta_n + 1)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

By using the Squeeze Theorem for sequential limits on the inequality above, we conclude that $|f'(x_n)| \to 0$.

■

Problem 18. Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Show that if $f'(x) \neq 1$ for all $x \in (0, 1)$, then there exists a unique $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

Proof. By exercise 16 of Homework 10 of 230A, we have that $f : [0, 1] \to [0, 1]$ being continuous on $[0, 1]$ implies that there exists a $c \in [0, 1]$ such that $f(c) = c$. Now, we want to show that this element $c \in [0, 1]$ is unique. Suppose for sake of contradiction that there exists $c_1, c_2 \in [0, 1]$ such that $f(c_1) = c_1$ and $f(c_2) = c_2$. Now, f must be differentiable on $(0, 1)$ implies that f is differentiable on (c_1, c_2). By the Mean Value Theorem, there exists $\hat{c} \in (0, 1)$ such that

$$f'(\hat{c}) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

But this contradicts the fact that $f(x) \neq 1$ for all $x \in [0, 1]$. Hence, the element $c \in [0, 1]$ must be unique. ■

Problem 19. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ with $f(0) = 0$. Assume that f is differentiable on $(0, \infty)$ with f' is increasing on $(0, \infty)$. Let $g : (0, \infty) \to \mathbb{R}$ be defined as $g(x) = \frac{f(x)}{x}$. Prove that g is increasing on $(0, \infty)$.

Proof. Our goal is to show that g is increasing on $(0, \infty)$; that is, for all $x, yin(0, \infty)$ with $x < y$, we have

$$g(x) < g(y).$$

Let $x, y \in (0, \infty)$. Since f and g are differentiable on $(0, \infty)$ and continuous on $[0, \infty)$, there exists $\hat{x} \in (0, x)$ and $\hat{y} \in (0, y)$ such that

$$f'(\hat{x}) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

and

$$f'(\hat{y}) = \frac{f(y) - f(0)}{y - 0} = \frac{f(y)}{y}.$$

Since f' is increasing on $(0, \infty)$ and $\hat{x} < \hat{y}$, we have

$$f'(\hat{x}) < f'(\hat{y}) \iff \frac{f(x)}{x} < \frac{f(y)}{y} \iff g(x) < g(y).$$

Hence, we can conclude that g is increasing. ■

Problem 20. Let $f : [0, +\infty) \to \mathbb{R}$ be continuous function, which is differentiable on $(0, +\infty)$.

(a) Prove that if $\lim\limits_{x \to +\infty} f'(x) = 0$, then f is uniformly continuous on $[0, \infty)$.

(b) Give an example of such a function with unbounded derivative.

Proof. (a) Suppose $\lim_{x \to +\infty} f'(x) = 0$. Our goal is to show that f is uniformly continuous on $[0, \infty)$; that is, we want to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Let $x, y \in [0, \infty)$ and $\varepsilon > 0$ be given. Since $\lim_{x \to \infty} f'(x) = 0$, there exists an $M > 0$ such that whenever $x > M$, we have

$$|f'(x)| < \frac{\varepsilon}{4}.$$

In particular, we can see that $M + 2 > M + 1 > M$. If we consider f on the interval $(M + 1, M + 2)$, we can use the Mean Value Theorem to write

$$\exists \tilde{x} \in (M + 1, M + 2) \text{ such that } f'(\tilde{x}) = f(M + 2) - f(M + 1).$$

One can immediately see that $\tilde{x} > M$ and so we have

$$|f'(\tilde{x})| < \frac{\varepsilon}{4}. \tag{1}$$

Using the differentiability of f at $\tilde{x} \in [0, +\infty)$, we can set $\delta_1 = 1$ so that whenever $0 < |x - \tilde{x}| < 1$, we have

$$|f(x) - f(\tilde{x})| < \Big(|f'(\tilde{x})| + \frac{\varepsilon}{4}\Big)|x - \tilde{x}| < |f'(\tilde{x})| + \frac{\varepsilon}{4}. \tag{2}$$

Likewise, whenever $0 < |y - \tilde{x}| < 1$, we have

$$|f(y) - f(\tilde{x})| < \Big(|f'(\tilde{x})| + \frac{\varepsilon}{4}\Big)|y - \tilde{x}| < |f'(\tilde{x})| + \frac{\varepsilon}{4}. \tag{3}$$

Now, choose $\delta = \frac{1}{2}\min\{M, 1\}$. Then whenever $|x - y| < \delta$ and using (1), (2), and (3), we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(\tilde{x})| + |f(\tilde{x} - f(y))| \\
&< \Big(|f'(\tilde{x})| + \frac{\varepsilon}{4}\Big) + \Big(|f'(\tilde{x})| + \frac{\varepsilon}{4}\Big) \\
&= 2|f'(\tilde{x})| + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we conclude that f must be continuous on the interval $[0, \infty)$.

(b) Consider the function

$$f(x) = \sqrt{x}\sin\frac{1}{x}.$$

We will show that f indeed satisfies the following properties:

(1) f is continuous on $[0, \infty)$

(2) f is differentiable on $(0, \infty)$

(3) f' is unbounded.

Starting with (1), we will consider two cases; that is, either $c = 0$ or $c \neq 0$. If $c = 0$, then observe that

$$0 \leq |\sqrt{x}\sin\frac{1}{x}| \leq |\sqrt{x}| = \sqrt{x}$$

where $|\sin x| \leq 1$ for all $x \in \mathbb{R}$. Using the squeeze theorem, we write

$$\lim_{x \to 0} |\sqrt{x}\sin\frac{1}{x}| = 0$$

which holds if and only if

$$\lim_{x\to 0} \sqrt{x}\sin\frac{1}{x} = 0.$$

Hence, f is continuous at $c = 0$. Suppose $c \neq 0$. Since $\sin x$ and $\frac{1}{x}$ are continuous functions at $c \neq 0$, their composition $\sin\frac{1}{x}$ is also continuous at $c \neq 0$. Furthermore, $\sqrt{x}$ is clearly continuous for $c \neq 0$. Thus, the Algebraic Continuity Theorem implies that

$$f(x) = \sqrt{x}\sin\frac{1}{x}$$

is continuous at $c \neq 0$.

Next, we show (2). Suppose $c \neq 0$. Since $\frac{1}{x}, \sin x$, are differentiable for every $x \neq 0$, we see that the chain rule implies that their composition is also differentiable. Since $\sqrt{x}$ is also differentiable for every $x \neq 0$, we can conclude via the Algebraic differentiability Theorem that f is indeed differentiable.

Finally we show that f' is unbounded as $x \to 0^+$. Computing f' using a combination of the chain rule and product rule, we have

$$f'(x) = \frac{1}{2x^{1/2}}\sin\frac{1}{x} - \frac{1}{x^{3/2}}\cos\frac{1}{x}.$$

Since $|\cos\frac{1}{x}| \leq 1$ for all $x \in \mathbb{R}$, we have that

$$f'(x) \geq \frac{1}{x^{1/2}}\Big(\frac{1}{2}\sin\frac{1}{x} - \frac{1}{x}\Big).$$

Since $\frac{1}{x^{1/2}}$ approaches ∞ as $x \to 0^+$, we can see that $|f'(x)| \to \infty$ as $x \to 0^+$. Thus, $f'(x)$ is unbounded, thereby satisfying (3). ■

Problem 21. Let f be differentiable on (a, b) and let $c \in (a, b)$. Suppose $f'(c) > 0$. Prove that there exists some $\delta > 0$ such that $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(x) > f(c)$ for $x \in (c, c + \delta)$.

Proof. Suppose that f is differentiable on (a, b). By definition, we have

$$\lim_{x\to c}\frac{f(x) - f(c)}{x - c} = f'(c) \text{ exists.}$$

Hence, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have

$$\Big|\frac{f(x) - f(c)}{x - c} - f'(c)\Big| < \varepsilon.$$

In particular, let $\varepsilon = \frac{f'(c)}{2}$. Then

$$\Big|\frac{f(x) - f(c)}{x - c} - f'(c)\Big| < 1 \tag{*}$$

which further implies that

$$\frac{f'(c)}{2} + f'(c) < \frac{f(x) - f(c)}{x - c} < \frac{f'(c)}{2} + f'(c) \Longrightarrow \frac{f'(c)}{2}(x - c) < f(x) - f(c) < \frac{3f'(c)}{2}(x - c).$$

Suppose $x \in (c - \delta, c)$. It immediately follows that $x < c$ and so $x - c < 0$. Then we have

$$f(x) - f(c) < \underbrace{\frac{3f'(c)}{2}}_{>0}\underbrace{(x - c)}_{<0} < 0.$$

Hence, we have $f(x) < f(c)$. Now, suppose $x \in (c, c+\delta)$. Similarly, we have $x > c$ which implies that $x - c > 0$. Then we have

$$f(x) - f(c) > \underbrace{\frac{f'(c)}{2}}_{>0} \underbrace{(x-c)}_{x>0} > 0.$$

Hence, we have $f(x) > f(c)$. ■