Homework 0: Extra Credit

Lance Remigio

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Problem 1. Let Ω be a nonempty open bounded set in $\mathbb{R}^n$. Prove (or disprove) that for all $x \in \Omega$, $\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, \Omega^c)$.

Proof. Let $x \in \Omega$. Our goal is to show that for any $\varepsilon > 0$

(i) $\operatorname{dist}(x, \Omega^c) \leq \operatorname{dist}(x, \partial\Omega) + \varepsilon$

(ii) $\operatorname{dist}(x, \partial\Omega) \leq \operatorname{dist}(x, \Omega^c) + \varepsilon$

Proving (i) and (ii), will give us our desired result. Let $\varepsilon > 0$ be given. Note that since $\operatorname{dist}(x, \partial\Omega)$ exists, there exists an $x_0 \in \partial\Omega$ such that

$$\|x - x_0\| < \operatorname{dist}(x, \partial\Omega) + \varepsilon. \tag{1}$$

Since $x_0 \in \partial\Omega$ and $\partial\Omega = \overline{\Omega} \cap \overline{\Omega^c}$, it follows that $x_0 \in \overline{\Omega^c}$ and $x_0 \in \overline{\Omega}$. Since Ω is open in $\mathbb{R}^n$, we get that Ω^c is closed and so, $\|x - x_0\| \geq \operatorname{dist}(x, \Omega^c)$. Taking the infimum over all elements on the left-hand side of (1), we have

$$\operatorname{dist}(x, \Omega^c) \leq \operatorname{dist}(x, \partial\Omega) + \varepsilon$$

which gives us (i).

Now, we will prove (ii). Since $\operatorname{dist}(x, \Omega^c)$ exists, there exists a $y \in \Omega^c$ such that

$$\|x - y\| < \operatorname{dist}(x, \Omega^c) + \varepsilon.$$

Note that Ω and Ω^c are separated sets in $\mathbb{R}^n$; that is, $\overline{\Omega} \cap \Omega^c = \emptyset$ and $\Omega \cap \overline{\Omega^c} = \emptyset$. Consider the line segment from $x \in \Omega$ to $y \in \Omega^c$ defined by the map $f(t) = x + t(y - x)$ for all $t \in [0,1]$. Observe that if $t = 0$, then $f(0) = x \in \Omega$ and $f(1) = y \in \Omega^c$. For some $t_0 \in (0,1)$, it follows that $z = f(t_0) = x + t_0(y - x)$ is in $\partial\Omega$. Hence, we have $\|x - z\| \geq \operatorname{dist}(x, \partial\Omega)$. Since $t \in (0,1)$, we get that

$$\|x - y\| > t\|x - y\| = \|t(x - y)\| = \|x - z\| \geq \operatorname{dist}(x, \partial\Omega).$$

Hence,

$$\operatorname{dist}(x, \partial\Omega) \leq \|x - y\| < \operatorname{dist}(x, \Omega^c) + \varepsilon$$

which gives us (ii). ■