Math 234A: Homework 2

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Problem 1

Definition (Cauchy Sequence). A sequence (z_n) is called a **Cauchy Sequence** if for all $\varepsilon > 0$, there exists a positive interger $\mathbb{N}$ such that for all $m, n \geq N$,

$$|z_m - z_n| < \varepsilon.$$

Show that a sequence (z_n) in $\mathbb{C}$ is convergent if and only if it is Cauchy.

Proof. ($\Longrightarrow$) Let $\varepsilon > 0$. If (z_n) is Cauchy, then we can find an $N \in \mathbb{N}$ such that for any $n, m \geq N$ such that

$$|z_n - z_m| < \varepsilon.$$

Then observe that for any $n, m \geq N$, we have

$$|\Re(z_n) - \Re(z_m)| = |\Re(z_n - z_m)| \leq |z_n - z_m| < \varepsilon$$

and similarly,

$$|\Im(z_n) - \Im(z_m)| = |\Im(z_n - z_m)| \leq |z_n - z_m| < \varepsilon.$$

Therefore, the real and imaginary parts of (z_n) are Cauchy. Since $\Re(z_n)$ and $\Im(z_n)$ are real-valued sequences, they must converge as well. But this holds if and only if (z_n) converges.

($\Longleftarrow$) Let $\varepsilon > 0$. Suppose (z_n) converges. This holds if and only if $\Re(z_n)$ and $\Im(z_n)$ converge. But this holds if and only if $\Re(z_n)$ and $\Im(z_n)$ are both Cauchy since they are real-valued sequences. Thus, we can find an $N \in \mathbb{N}$ such that for any $n, m \geq N$, we see that

$$|\Re(z_n) - \Re(z_m)| < \frac{\varepsilon}{2} \tag{1}$$

and

$$|\Im(z_n) - \Im(z_m)| < \frac{\varepsilon}{2}. \tag{2}$$

Using (1) and (2), we see that

$$\begin{aligned} |z_n - z_m| &= |\Re(z_n) + i\Im(z_n) - (\Re(z_m) + i\Im(z_m))| \\ &= |(\Re(z_n) - \Re(z_m)) + i(\Im(z_n) - \Im(z_m))| \\ &\leq |\Re(z_n) - \Re(z_m)| + |\Im(z_n) - \Im(z_m)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that (z_n) is Cauchy. ■

Problem 2

(i) Let $z_0 = x_0 + i\zeta_0 \in \mathbb{C}$: Define a sequence (z_n) by $z_{n+1} = \frac{1}{2}(z_n + \frac{1}{z_n})$, $n \geq 1$, and $z_1 = \frac{1}{2}(z_0 + \frac{1}{z_0})$. Show that

$$\lim_{n\to\infty} z_n = \begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0 \end{cases}$$

Proof. Let $z_0 = x_0 + i\zeta_0 \in \mathbb{C}$. Define a sequence (z_n) by

$$z_{n+1} = \frac{1}{2}\Big(z_n + \frac{1}{z_n}\Big),\ n \geq 1$$

and

$$z_1 = \frac{1}{2}\Big(z_0 + \frac{1}{z_0}\Big).$$

Our goal is to show that

$$\lim_{n\to\infty} z_n = \begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0. \end{cases}$$

Suppose $x_0 > 0$. First, we need to show that $z_n \neq 1$ for all $n \in \mathbb{N}$. We proceed via induction. Let our base case be $n = 1$. Then we want to show that $\Re(z_1) > 0$. Using the definition of z_0, we see that

$$\begin{aligned} z_1 = \frac{z_0^2+1}{2z_0} &= \frac{1}{2(x_0^2+\zeta_0^2)} \cdot [(x_0+i\zeta_0)^2(x_0-i\zeta_0) + (x_0 - i\delta_0)] \\ &= \frac{x_0^3 + x_0\zeta_0^2 + x_0}{2(x_0^2+\zeta_0^2)} + i\frac{\zeta_0 x_0^3 - \zeta_0}{2(x_0^2+\zeta_0^2)}. \end{aligned}$$

Since $z_1 = \Re(z_1) + i\Im(z_1)$ and $x_0 > 0$, we see that

$$\Re(z_1) = \frac{x_0^3 + x_0\zeta_0^2 + x_0}{2(x_0^2+\zeta_0^2)} > 0.$$

Now, suppose that the result holds for the nth case. We will show that the result holds for the $n+1$ case. Set $\alpha = \Re(z_n)$ and $\beta = \Im(z_n)$. Then by definition of z_{n+1}, we see that

$$\begin{aligned} z_{n+1} = \frac{z_n^2+1}{2z_n} &= \frac{1}{2(\alpha^2+\beta^2)} \cdot [(\alpha+i\beta)(\alpha-i\beta) + (\alpha - i\beta)] \\ &= \frac{\alpha^3 + \alpha\beta^3 + \alpha}{2(\alpha^2+\beta^2)} + i\frac{\beta\alpha^3 - \beta}{2(\alpha^2+\beta^2)}. \end{aligned}$$

By equating the real and imaginary parts of z_{n+1} with the right-hand side of the equation above and using the inductive hypothesis that $\alpha = \Re(z_n) > 0$, we see that

$$\Re(z_{n+1}) = \frac{\alpha^3 + \alpha\beta^3 + \alpha}{2(\alpha^2+\beta^2)} > 0.$$

Thus, we see that $\Re(z_n) > 0$ for all $n \in \mathbb{N}$. Consequently, this tells us that $\Re(z_n) + 1 > 1$ which further tells us that $|z_{n+1} + 1| > 1$. Now, define a new sequence (w_n) such that

$$w_{n+1} = \frac{z_{n+1} - 1}{z_{n+1} + 1}$$

and note that

$$z_{n+1} = \frac{1 + z_n^2}{2z_n}.$$

Our next step is to show that $w_{n+1} = w_n^2$. Thus, observe that

$$\begin{aligned} w_{n+1} = \frac{\frac{1+z_n^2}{2z_n} - 1}{\frac{1+z_n^2}{2z_n} + 1} = \frac{\frac{z_n^2 - 2z_n + 1}{2z_n}}{\frac{z_n^2+2z_n+1}{2z_n}} &= \frac{z_n^2 - 2z_n + 1}{z_n^2 + 2z_n + 1} \\ &= \frac{(z_n - 1)^2}{(z_n+1)^2} \\ &= \left(\frac{z_n - 1}{z_n + 1}\right)^2 \\ &= w_n^2. \end{aligned}$$

Now, we want to show, as a consequence of the result above, that (w_n) is a decreasing sequence; that is, we want to show that $0 < |w_{n+1}| < |w_n| < 1$. We will induct on $n \in \mathbb{N}$ to show this. Thus, let $n = 1$ be our base case. Observe that

$$\begin{aligned} 0 < |w_{1+1}| = |w_2| = |w_1^2| &= \left|\left(\frac{z_1 - 1}{z_1 + 1}\right)^2\right| \\ &< \left|\frac{z_1 - 1}{z_1 + 1}\right| \\ &< 1. \end{aligned}$$

Let us now assume that this result holds for the nth case. Thus, we see that

$$0 < |w_{n+2}| = |w_{n+1}^2| = \left|\left(\frac{z_{n+1} - 1}{z_{n+1} + 1}\right)^2\right| < |w_{n+1}| < |w_n| < 1.$$

Thus, we see that $0 < |w_{n+1}| < |w_n| < 1$ holds for all $n \in \mathbb{N}$. Clearly, we see that (w_n) is bounded and decreasing for all n. Hence, (w_n) must converge, and in this case, we claim that it converges to 0. Observe that

$$|w_n| = e^{\ln |w_n|}.$$

Now, since $0 < |w_n| < 1$, we must have $\ln |w_n| < 0$. By applying the limit as $n \to \infty$, we can see that

$$\lim_{n\to\infty} |w_n| = \lim_{n\to\infty} e^{\ln |w_n|} = 0.$$

Since $|z_{n+1} + 1| > 1$ for all $n \in \mathbb{N}$, we must have that

$$|z_{n+1} - 1| \to 0$$

as $n \to \infty$. Thus, we conclude that $\lim_{n\to\infty} z_n = 1$. Now, assume that $x_0 < 0$. Consider the sequence $(-z_n)$. Then observe that

$$\lim_{n\to\infty} -z_n = -\lim_{n\to\infty} z_n = -1.$$

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(ii) Discuss convergence and divergence of

$$z_n = 1 + i\frac{(-1)^n}{n^2} \text{ for } n = 1, 2, \dots \, .$$

Let $\Phi_n = \operatorname{Arg}(z_n)$. Show that $(\Phi_n) \to 0$.

Proof. Observe that $\Re(z_n) \to 1$ and $\Im(z_n) = \frac{(-1)^n}{n^2} \to 0$ as $n \to \infty$. Since the real part and imaginary part of z_n converge, we know that (z_n) must also converge. In fact, it converges to the following value

$$z_n \to 1 + 0i = 1.$$

By definition, we know that

$$\Phi_n = \operatorname{Arg}(z_n) = \tan^{-1}\left(\frac{\Im(z_n)}{\Re(z_n)}\right) \to \tan^{-1}(0) = 0 \text{ as } n \to \infty.$$

■

(iii) Assume that $0 < \gamma < 1$. Show that

$$\sum_{n=1}^{\infty} \gamma^n \cos n\theta = \frac{\gamma\cos\theta - \gamma^2}{1 - 2\gamma\cos\theta + \gamma^2}$$
$$\sum_{n=1}^{\infty} \gamma^n \sin n\theta = \frac{\gamma\sin\theta}{1 - 2\gamma\cos\theta + \gamma^2}.$$

Proof. Set $z = \gamma e^{i\theta}$. Consider the series

$$\sum_{n=1}^{\infty} z^n$$

which converges absolutely for $|z| < 1$ such that

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}. \tag{1}$$

By rearranging terms and using the polar representation of z, we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n &= \sum_{n=1}^{\infty} (\gamma e^{i\theta})^n \\
&= \sum_{n=1}^{\infty} [\gamma(\cos\theta + i\sin\theta)]^n \\
&= \sum_{n=1}^{\infty} \gamma^n(\cos\theta + i\sin\theta)^n \\
&= \sum_{n=1}^{\infty} \gamma^n(\cos n\theta + i\sin n\theta) && \text{(De Moivre's Theorem)} \\
&= \sum_{n=1}^{\infty} \gamma^n \cos n\theta + i\sum_{n=1}^{\infty} \gamma^n \sin n\theta.
\end{aligned}$$

Thus, we have

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} \gamma^n \cos n\theta + i\sum_{n=1}^{\infty} \gamma^n \sin n\theta. \tag{2}$$

Working with the sum in (1) and using the polar representation of z, we see that

$$\begin{aligned}
\frac{z}{1-z} = \frac{\gamma e^{i\theta}}{1-\gamma e^{i\theta}} &= \frac{(\gamma\cos\theta + i\gamma\sin\theta)}{(1-\gamma\cos\theta) - i\gamma\sin\theta} \\
&= \frac{(\gamma\cos\theta + i\gamma\sin\theta)}{(1-\gamma\cos\theta) - i\gamma\sin\theta} \cdot \frac{(1-\gamma\cos\theta) + i\gamma\sin\theta}{(1-\gamma\cos\theta) + i\gamma\sin\theta} \\
&= \frac{\gamma\cos\theta - \gamma^2(\sin^2\theta + \cos^2\theta) + i\gamma\sin\theta}{1 - 2\gamma\cos\theta + \gamma^2(\sin^2\theta + \cos^2\theta)} \\
&= \frac{\gamma\cos\theta - \gamma^2}{1 - 2\gamma\cos\theta + \gamma^2} + i\frac{\gamma\sin\theta}{1 - 2\gamma\cos\theta + \gamma^2}.
\end{aligned}$$

Thus, we see that

$$\frac{z}{1-z} = \frac{\gamma\cos\theta - \gamma^2}{1 - 2\gamma\cos\theta + \gamma^2} + i\frac{\gamma\sin\theta}{1-2\gamma\cos\theta + \gamma^2}. \tag{3}$$

Equating the real and imaginary parts of (2) and (3), we see that

$$\Re\Big(\sum_{n=1}^{\infty} z^n\Big) = \sum_{n=1}^{\infty}\gamma^n\cos n\theta = \frac{\gamma\cos\theta-\gamma^2}{1-2\gamma\cos\theta+\gamma^2},$$
$$\Im\Big(\sum_{n=1}^{\infty} z^n\Big) = \sum_{n=1}^{\infty}\gamma^n\sin n\theta = \frac{\gamma\sin\theta}{1-2\gamma\cos\theta+\gamma^2}.$$

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Problem 3

Let (z_n) be a sequence of non-zero complex numbers. Suppose that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L.$$

Show that $\lim_{n\to\infty}\sqrt[n]{|z_n|} = L$.

Proof. Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$||z_n|^{\frac{1}{n}} - L| < \varepsilon.$$

Indeed, we can use the fact that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L,$$

to find an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$\left|\frac{z_{n+1}}{z_n}\right| < L + \varepsilon.$$

Now, observe that

$$\begin{aligned}
|z_n|^{\frac{1}{n}} &= \left|\frac{z_n}{z_{n-1}}\cdot\frac{z_{n-1}}{z_{n-2}}\cdot\frac{z_{n-2}}{z_{n-3}}\cdots\frac{z_{N+1}}{z_N}\right|^{\frac{1}{n}} \\
&< \Big[(L+\varepsilon)^{n-N}\Big]^{\frac{1}{n}}|z_N|^{1/n} \\
&= \Big[(L+\varepsilon)\Big]^{1-\frac{N}{n}}|z_N|^{\frac{1}{n}}.
\end{aligned}$$

Taking the limit as $n \to \infty$ on the right-hand side, we can see that

$$|z_n|^{\frac{1}{n}} < L + \varepsilon \iff ||z_n|^{1/n} - L| < \varepsilon$$

which is our desired result.

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Problem 4

Determine all $z \in \mathbb{C}$ such that

(i) $\exp(z) = -2$

Solution. Note that $-2 = -2 + 0i$ which means that

$$\operatorname{Arg}(-2) = \tan^{-1}(-2) = \pi.$$

Thus, we see that

$$\begin{aligned} z = \operatorname{Log}(-2) &= \ln|-2| + i\operatorname{Arg}(-2) \\ &= \ln 2 + i\Big(\pi + 2\pi k\Big) \text{ for } k \in \mathbb{Z}. \end{aligned}$$

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(ii) $\exp(z) = -i$

Solution. Note that $-i = 0 - i$ which means that

$$\operatorname{Arg}(-i) = \frac{3\pi}{2}.$$

Thus, we see that

$$\begin{aligned} z = \operatorname{Log}(-i) &= \ln|-i| + i\operatorname{Arg}(-i) \\ &= \ln(1) + i\Big[\frac{3\pi}{2} + 2\pi k\Big] \text{ for } k \in \mathbb{Z} \\ &= i\Big[\frac{3\pi}{2} + 2\pi k\Big] \text{ for } k \in \mathbb{Z}. \end{aligned}$$

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(iii) $\sin z = 1 - i$

Solution. Using the fact that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

we see that

$$\begin{aligned} \frac{e^{iz} - e^{-iz}}{2i} = 1 - i &\implies e^{iz} - e^{-iz} = 2i + 2 \\ &\implies (e^{iz})^2 - 2(i+1)e^{iz} = 1 \\ &\implies w^2 - 2(i+1)w = 1 && (w = e^{iz}) \\ &\implies w^2 - 2(1+i)w + (1+i)^2 = 1 + (1+i)^2 \\ &\implies (w - (1+i))^2 = 1 + (1+i)^2 \\ &\implies (w - (1+i))^2 = 1 + 2i \\ &\implies w_{1,2} = \pm\sqrt{1+2i} + (1+i). \end{aligned}$$

Now, we will convert the first term on the right-hand side in terms of its respective polar representation. Thus, we have

$$\pm\sqrt{1+2i} = \pm 5^{\frac{1}{4}} e^{i\frac{\tan^{-1}(2)}{2}} = \pm 5^{1/4}\Big(\cos\Big(\frac{\tan^{-1}(2)}{2}\Big) + i\sin\Big(\frac{\tan^{-1}(2)}{2}\Big)\Big).$$

Now, set

$$\alpha = \Big(5^{1/4}\cos\Big(\frac{\tan^{-1}(2)}{2}\Big) + 1\Big) + i\Big(5^{1/4}\sin\Big(\frac{\tan^{-1}(2)}{2}\Big) + 1\Big).$$

and

$$\zeta = \Big(1 - 5^{1/4}\cos\Big(\frac{\tan^{-1}(2)}{2}\Big)\Big) + i\Big(1 - 5^{1/4}\sin\Big(\frac{\tan^{-1}(2)}{2}\Big)\Big).$$

$$\begin{aligned} e^{iz} = \alpha &\implies z = \frac{1}{i}\log(\alpha) \\ &\implies z = -i[\ln|\alpha| + i\operatorname{Arg}(\alpha)] \\ &\implies z = \operatorname{Arg}(\alpha) - i\ln|\alpha| \end{aligned}$$

and similarly

$$e^{iz} = \zeta \implies z = \{\operatorname{Arg}(\zeta) - i\ln|\zeta|\}.$$

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(iv) $\cos z = 3 + 4i$

Solution. Using the fact that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

we have

$$\begin{aligned} e^{iz} + e^{-iz} = 2(3+4i) &\implies (e^{iz})^2 + 1 = (6+8i)e^{iz} \\ &\implies w^2 + 1 = (6+8i)w \\ &\implies w^2 - (6+8i)w = -1 && (w = e^{iz}) \\ &\implies w^2 - (6+8i)w + (3+4i)^2 = (3+4i)^2 - 1 \\ &\implies (w - (3+4i))^2 = -1 + (3+4i)^2 \\ &\implies w = \pm\sqrt{8(3i-1)} + (3+4i) \end{aligned}$$

Now, let us convert the first term on the right-hand side into it's polar form which is

$$\pm\sqrt{8(3i-1)} = 8^{1/2}\cdot 10^{1/2} e^{i\frac{3\pi}{8}} = 8^{1/2}\cdot 10^{1/2}\Big(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\Big).$$

Let us set

$$\beta = \Big(3 + 8^{1/2}\cdot 10^{1/2}\cos\left(\frac{3\pi}{8}\right)\Big) + i\Big(4 + 8^{1/2}\cdot 10^{1/2}\sin\left(\frac{3\pi}{8}\right)\Big)$$

and

$$\omega = \Big(3 - 8^{1/2}\cdot 10^{1/2}\cos\left(\frac{3\pi}{8}\right)\Big) + i\Big(4 - 8^{1/2}\cdot 10^{1/2}\sin\left(\frac{3\pi}{8}\right)\Big)$$

Now, we have

$$\begin{aligned} e^{iz} = \beta &\implies z = \frac{1}{i}\log(\beta) \\ &\implies z = \{-i[\ln|\beta| + i\operatorname{Arg}(\beta)]\} \\ &\implies z = \{\operatorname{Arg}(\beta) - i\ln|\beta|\}. \end{aligned}$$

and

$$\begin{aligned} e^{iz} = \omega &\implies z = \frac{1}{i}\log(\omega) \\ &\implies z = \{-i[\ln|\omega| + i\operatorname{Arg}(\omega)]\} \\ &\implies z = \{\operatorname{Arg}(\omega) - i\ln|\omega|\}. \end{aligned}$$

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Problem 5

Let (z_n) and (w_n) be two sequences of complex numbers. Assume that $z_n = w_n - w_{n+1}$. Show that $\sum_{n=1}^{\infty} z_n$ converges if and only if (w_n) converges. If $\sum_{n=1}^{\infty} z_n$ converges, show that

$$\sum_{n=0}^{\infty} z_n = w_1 - \lim_{n\to\infty} w_{n+1}.$$

Use this to compute

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)}.$$

Proof. ($\Rightarrow$) Let (z_n) and w_n be two sequences of complex numbers. Our goal is to show that (w_n) converges. Assume that $z_n = w_n - w_{n+1}$. Since $\sum_{n=1}^{\infty} z_n$ converges , we know that (z_n) must converge to 0. But we must also have that

$$|z_n| = |w_n - w_{n+1}| \to 0$$

as $n \to \infty$. Hence, (w_n) must be a Cauchy sequence and so, it must converge by problem 1.

($\Leftarrow$) Suppose that (w_n) converges to some $w \in \mathbb{C}$. Let

$$s_n = \sum_{k=1}^{n} z_k.$$

Our goal is to show that $(s_n) \to 0$ as $n \to \infty$. Since $z_n = w_n - w_{n+1}$, we can see that

$$\begin{aligned}|s_n - s_m| = \Big|\sum_{k=m+1}^{n} z_k\Big| &= \Big|\sum_{k=m+1}^{n} w_k - w_{k+1}\Big| \\ &= |(w_{m+1} - w_{m+2}) + (w_{m+2} - w_{m+3}) + \cdots + (w_n - w_{n+1})| \\ &= |w_{m+1} - w_{n+1}|.\end{aligned}$$

Since (w_n) converges, we must also have that (w_n) is Cauchy. Thus, we see that

$$|s_n - s_m| = |w_{m+1} - w_{n+1}| \to 0 \text{ as } n \to \infty.$$

Thus, (s_n) must converge and hence the infinite series $\sum_{k=1}^{\infty} z_n$ converges. Using this result, we can now compute

$$\begin{aligned}s_n = \sum_{k=1}^{n} z_k &= \sum_{k=1}^{n} (w_k - w_{k+1}) \\ &= (w_1 - w_2) + (w_2 - w_3) + \cdots + (w_n - w_{n+1}) \\ &= w_1 - w_{n+1}\end{aligned}$$

whereby taking the limit gives us

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (w_1 - w_{n+1}) = w_1 - \lim_{n\to\infty} w_{n+1}.$$

Now, let us compute the following infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)}.$$

Using the formula we have just proved, we can see that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)} &= \frac{1}{(1+i)(2+i)} + \lim_{n\to\infty} \frac{1}{(n+i)((n+1)+i)} \\ &= \frac{1}{(1+i)(2+i)}.\end{aligned}$$

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Problem 6

For $z, w \in \mathbb{C}^{\cdot}$. Show that

$$\text{Log}(zw) = \text{Log}(z) + \text{Log}(w) + 2\pi i k$$

where

$$k = \begin{cases} 0 & \text{if } -\pi < \text{Arg}(z) + \text{Arg}(w) \leq \pi \\ 1 & \text{if } -2\pi < \text{Arg}(z) + \text{Arg}(w) \leq -\pi \\ -1 & \text{if } \pi < \text{Arg}\, z + \text{Arg}\, w \leq 2\pi. \end{cases}$$

Proof. Let $z, w \in \mathbb{C}$. We would like to consider three cases:

(i) $-\pi < \text{Arg}(z) + \text{Arg}(w) \leq \pi$

(ii) $-2\pi < \text{Arg}(z) + \text{Arg}(w) \leq -\pi$.

(iii) $\pi < \text{Arg}\, z + \text{Arg}\, w \leq 2\pi$.

Starting with case (i), we see that

$$\text{Log}(z) = \ln|z| + \text{Arg}(z)$$

and that $k = 0$. Thus, we see that we can see that

$$\begin{aligned} \text{Log}(zw) &= \ln|zw| + \text{Arg}(zw) \\ &= \ln|z||w| + (\text{Arg}(z) + \text{Arg}(w)) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w)) \\ &= \Big(\ln(z) + \text{Arg}(z)\Big) + \Big(\ln(w) + \text{Arg}(w)\Big) \\ &= \text{Log}(z) + \text{Log}(w). \end{aligned}$$

With case (ii), we have $k = 1$ so, we have

$$\begin{aligned} \text{Log}(zw) &= \ln|zw| + \text{Arg}(zw) \\ &= \ln|z||w| + (\text{Arg}(z) + \text{Arg}(w) + 2\pi i) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w) + 2\pi i) \\ &= \Big(\ln(z) + \text{Arg}(z)\Big) + \Big(\ln(w) + \text{Arg}(w)\Big) + 2\pi i \\ &= \text{Log}(z) + \text{Log}(w) + 2\pi i k. \end{aligned}$$

Lastly, with case (iii), we have $k = -1$ which implies that

$$\begin{aligned} \text{Log}(zw) &= \ln|zw| + \text{Arg}(zw) \\ &= \ln|z||w| + (\text{Arg}(z) + \text{Arg}(w) - 2\pi i) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w) - 2\pi i) \\ &= \Big(\ln(z) + \text{Arg}(z)\Big) + \Big(\ln(w) + \text{Arg}(w)\Big) - 2\pi i \\ &= \text{Log}(z) + \text{Log}(w) - 2\pi i. \end{aligned}$$

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Problem 7 (Hyperbolic Functions)

For $z \in \mathbb{C}$, define

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}$$

and

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}.$$

Show that

(i) $\sinh(z) = -i\sin(iz)$, $\cosh(z) = \cos(iz)$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Using the formulas above, we can see that

$$\begin{aligned}-i\sin(iz) &= -i\left[\frac{\exp(i^2z) - \exp(-i^2z)}{2i}\right] \\ &= \frac{\exp(z) - \exp(-z)}{2} \\ &= \sinh(z)\end{aligned}$$

and

$$\begin{aligned}\cos(iz) &= \frac{\exp(i^2z) + \exp(-i^2z)}{2} \\ &= \frac{\exp(-z) + \exp(z)}{2} \\ &= \cosh(z).\end{aligned}$$

■

(ii) $\sinh(z+w) = \sin(z)\cosh(w) + \cosh(z)\sinh(w)$.

Proof. We will show the formulas above by using the sum formulas for cosine and sine. By part (i), we see that

$$\begin{aligned}\sinh(z+w) &= -i\sin(i(z+w)) \\ &= -i\sin(iz+iw) \\ &= -i\Big[\sin(iz)\cos(iw) + \sin(iw)\cos(iz)\Big] \\ &= (-i\sin(iz))\cos(iw) + (-i\sin(iw))\cos(iz) \\ &= \sinh(z)\cosh(w) + \sinh(w)\cosh(z)\end{aligned}$$

and

$$\begin{aligned}\cosh(z+w) &= \cos(i(z+w)) \\ &= \cos(iz+iw) \\ &= \cos(iz)\cos(iw) - \sin(iz)\sin(iw) \\ &= \cosh(z)\cosh(w) + i^2\sin(iz)\sin(iw) \\ &= \cosh(z)\cosh(w) + (-i\sin(iz))(-i\sin(iw)) \\ &= \cosh(z)\cosh(w) + \sinh(z)\sinh(w).\end{aligned}$$

■

(iii) $\cosh^2(z) - \sinh^2(z) = 1$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Using part (i), we see that

$$\begin{aligned}\cosh^2(z) - \sinh^2(z) &= \cos^2(iz) - (-\sin^2(iz)) \\ &= \cos^2(iz) + \sin^2(iz) \\ &= 1.\end{aligned}$$

■

(iv) $\cosh(z+2\pi i) = \cosh(z)$ and $\sinh(z+2\pi i) = \sinh(z)$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Observe that

$$\sinh(2\pi i) = -i\sin(2\pi i^2) = -i\sin(-2\pi) = \sin(2\pi) = 0$$

and

$$\cosh(2\pi i) = \cos(2\pi i^2) = \cos(-2\pi) = \cos(2\pi) = 1.$$

Using part (ii), we can see that

$$\begin{aligned}\cosh(z + 2\pi i) &= \cosh(z)\cosh(2\pi i) + \sinh(z)\sinh(2\pi i)\\ &= \cosh(z)\end{aligned}$$

and

$$\begin{aligned}\sinh(z + 2\pi i) &= \sinh(z)\cosh(2\pi i) + \cosh(z)\sinh(2\pi i)\\ &= \sinh(z).\end{aligned}$$

∎

(v) $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

Proof. By using the series representation of cosine and sine, we can see that

$$\begin{aligned}\cosh(z) = \cos(iz) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(iz)^{2n}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(-1)^n z^{2n}\\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\end{aligned}$$

and

$$\begin{aligned}\sinh(z) = -i\sin(iz) &= -i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(iz)^{2n+1}\\ &= \sum_{n=0}^{\infty} \frac{(-i)(-1)^n}{(2n+1)!} i^{2n+1} z^{2n+1}\\ &= \sum_{n=0}^{\infty} \frac{(-i)(-1)^n}{(2n+1)!}(-1)^n i z^{2n+1}\\ &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

∎

Problem 8

(a) Find all possible $z \in \mathbb{C}$ so that

$$\sum_{n=1}^{\infty} n|z|^n$$

converges. Use it to discuss the convergence of the series $\sum_{n=1}^{\infty} nz^n$.

Solution. Using the ratio test, denote $a_n = n|z|^n$. Then observe that

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)|z|^{n+1}}{n|z|^n}\right| &= \frac{n+1}{n}\cdot|z| \\ &= \left(1+\frac{1}{n}\right)|z|.\end{aligned}$$

If we take the limit as $n\to\infty$, we see that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)|z| = |z|.$$

Note that by the ratio test, the series

$$\sum_{n=1}^{\infty} n|z|^n$$

converges if $|z|<1$. Thus, the series

$$\sum_{n=1}^{\infty} nz^n$$

converges for all $|z|<1$. ■

1. Show that the series $\sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$converges absolutely for all $z\in\mathbb{C}$.

Proof. Let $z\in\mathbb{C}$. We proceed via the ratio test to show that

$$\sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$$

converges absolutely. Thus, observe that

$$r_n = \frac{z^{4n+1}}{(2n)!}.$$

Hence, we have

$$\begin{aligned}\left|\frac{r_{n+1}}{r_n}\right| &= \left|\frac{z^{4n+5}}{(2n+2)!}\cdot\frac{(2n)!}{z^{4n+1}}\right| \\ &= \left|\frac{z}{(2n+2)(2n+1)}\right| \\ &= \frac{|z|}{(2n+2)(2n+1)}.\end{aligned}$$

Taking the limit as $n\to\infty$, we see that for any fixed $z\in\mathbb{C}$ that

$$\lim_{n\to\infty}\left|\frac{r_{n+1}}{r_n}\right| = \lim_{n\to\infty}\frac{|z|}{(2n+2)(2n+1)} = 0 < 1.$$

Thus, we see that the infinite series

$$\sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$$

converges. Now, observe that

$$\sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!} = z\sum_{n=0}^{\infty}\frac{(z^2)^{2n}}{(2n)!} = z\cosh(z^2)$$

and so we can find the sum of this infinite series granted that we pick a $z\in\mathbb{C}$. ■