0.1 Lecture 4

0.1.1 Topics

(1) Sequences

(2) Infinite subset of a countable set is countable.

0.1.2 Sequences

Definition (Sequence). We call a **sequence**, we mean a function f on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence f by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in A.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

0.1.3 Infinite subset of a countable set is countable

Theorem. Every infinite subset of a countable set is countable.

Proof. Let A be a countable set. Let $E \subseteq A$ and E is infinite. Our goal is to show that E is countable. Since A is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let n_1 be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let n_2 be the smallest positive integer greater than n_1 such that $x_{n_2} \in E$.

$\vdots$

(k) Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Observe that the set

$$\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}$$

is both nonempty (since E is infinite) and bounded below (by the well ordering property of natural numbers). Thus, we know that

$$n_k = \min\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}.$$

Now, define the function $f : \mathbb{N} \to E$ as follows:

$$f(k) = x_{n_k}.$$

We claim that this is a bijective map. If we can prove this fact, then surely E is countable. We need to prove a two things:

(1) f is injective and

(2) f is surjective.

Starting with (1), suppose $k_1 \neq k_2$. Then we have $n_{k_1} \neq n_{k_2}$ and thus, $x_{n_{k_1}} \neq x_{n_{k_2}}$. Hence, we see that $f(k_1) \neq f(k_2)$. Therefore, f is injective which proves (1).

With (2), let $b \in E$. Since $E \subseteq A$, we must have $b \in A$ and thus there exists an $m \in \mathbb{N}$ such that $b = x_m$. Hence, there exists $1 \leq k \leq m$ such that $x_{n_k} = x_m = b$. Thus, $f(k) = b$ which shows that f is surjective and so (2) is satisfied.

Therefore, f must be a bijective map and we can now conclude that E is countable. ■

Remark. Consider the contrapositive of the theorem above:

Let $E \subseteq A$. If E is not countable, then A is not countable.

When we say that a set is NOT countable, we do not necessarily mean the set is uncountable. However, in the case in that E is an infinite set, we can say that E being not countable is equivalent to saying that E is uncountable. So, only in the case in which E is infinite then the contrapositive is

Let $E \subseteq A$ be infinite. If E is uncountable, then A is uncountable. By the theorem we just proved, we see that $f(A) \subseteq S$ and S being countable implies that $f(A)$ must be at most countable. Thus, A must be at most countable.

Corollary. Let A be any set and let S be a countable set. If there exists an injective mapping $f : A \to S$, then A is at most countable.

Proof. Let A be any set and let S be a countable set. Suppose there exists an injective mapping $f : A \to S$. We can restrict the codomain S of f to its range $f(A)$. Because f is injective, the mapping $f : A \to f(A)$ will be bijective. Thus, we have that $A \sim f(A)$. ∎

Example 0.1.1 ($\mathbb{N} \times \mathbb{N}$ is countable). (1) The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(x, y) = 2^x 3^y$$

is injective. By the corollary we have just proved, we see taht $\mathbb{N} \times \mathbb{N}$ is at most countable.

(2) Notice that $g : \mathbb{N} \to \{1\} \times \mathbb{N}$ is a bijection $g(a) = (1, a)$. Hence, $\{1\} \times \mathbb{N}$ is countable. Now, we have that

(i) $\{1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$

(ii) $\{1\} \times \mathbb{N}$ is countable implies that $\{1\} \times \mathbb{N}$ is infinite.

Thus, (i) and (ii) imply that $\mathbb{N} \times \mathbb{N}$ is countable.

Proving that a set A is countable can be done by finding a subset B that is countable which forces A to be countable.

Example 0.1.2 ($\mathbb{Q}$ is countable). (1) The function $f : \mathbb{Q} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2(2^p 3^q) & \text{if } x = \frac{p}{q} \; p \in \mathbb{N}, q \in \mathbb{N}, (p, q) = 1 \\ 2(2^p 3^q) + 1 & \text{if } x = \frac{-p}{q} \; p \in \mathbb{N}, q \in \mathbb{N}, (p, q) = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is an injective mapping. So, $\mathbb{Q}$ is at most countable.

(2) Observe that $\mathbb{N} \subseteq \mathbb{Q}$ and $\mathbb{N}$ being a countable (and thus $\mathbb{N}$ is infinite) set implies that $\mathbb{Q}$ is infinite. Thus, $\mathbb{Q}$ is is countable.

0.1.4 Countable union of at most countable sets is at most countable

Theorem. Countable union of at most countable sets is at most countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of at most countable sets; that is, for each $n \in \mathbb{N}$, A_n is at most countable. Our goal is to show that

$$K = \bigcup_{n \in \mathbb{N}} A_n \text{ is at most countable.}$$

To this end, it suffices to show that there exists an injective map $f : K \to \mathbb{N} \times \mathbb{N}$. Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\vdots \\ B_{n+1} &= A_{n+1} \setminus \Big(\bigcup_{k=1}^{n} A_k \Big). \end{aligned}$$

We leave as an exercise to show that

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \text{ are pairwise disjoint.}$$

Note that for all $n \in \mathbb{N}$, we have $B_n \subseteq A_n$ and A_n is at most countable. Then for each $n \in \mathbb{N}$. B_n must be at most countable. So, for each $n \in \mathbb{N}$, there exists an injective mapping $f_n : B_n \to \mathbb{N}$. Our goal is to show that

$$\bigcup_{n=1}^{\infty} B_n \text{ is at most countable.}$$

To this end, we define the function

$$f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N} \times \mathbb{N}$$

as follows:

For each $x \in \bigcup_{n=1}^{\infty} B_n$, there is exactly one $n \in \mathbb{N}$ such that $x \in B_n$; let us denote this n by n_x.

Thus, we define

$$f(x) = (n_x, f_{n_x}(x)).$$

This function is injective because

$$\begin{aligned} f(x) = f(y) &\implies (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y)) \\ &\implies n_x = n_y \ \wedge \ f_{n_x}(x) = f_{n_y}(y) \\ &\implies f_{n_k}(x) \wedge f_{n_x}(y) \\ &\implies x = y && \text{(Since } f_{n_x} \text{ is injective)} \end{aligned}$$

By the corollary, we see that the set

$$\bigcup_{n=1}^{\infty} B_n$$

is at most countable. ■

Corollary. A countable union of countable sets is countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. By the previous Theorem, we see that $\bigcup_{n \in \mathbb{N}} A_n$ is at most countable. Note that $A_1 \subseteq \bigcup_{n \in \mathbb{N}} A_n$ and that A_1 is countable. Thus, A_1 must be infinite and so

$$\bigcup_{n \in \mathbb{N}} A_n \text{ is infinite.}$$

Thus, we see that

$$\bigcup_{n \in \mathbb{N}} A_n$$

must be countable. ■

Corollary. If A and B are at most countable, then $A \cup B$ at most countable.

Proof. Let $A_1 = A$ and $A_2 = B$ where $A_2 = A_3 = \cdots = A_n = \emptyset$. Then $A \cup B$ is at most countable by the previous Theorem. ■

Theorem. If A is countable, then $A \times A$ is countable.

Proof. (1) Note that $A \times A = \bigcup_{b \in A} \{b\} \times A$ for each $b \in A$.

(2) The function $f : A \to \{b\} \times A$ defined by $f(x) = (b, x)$ is bijective. So, $A \sim \{b\} \times A$. Hence, $\{b\} \times A$ is countable.

Since a countable union of countable sets is countable, we see that $A \times A$ must be countable by (1) and (2). ■

0.2 Lecture 5

0.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks
- Inequalities

0.2.2 Summary of Last Lecture

1. How to prove A is at most countable:
   - Show that $A \subseteq S$ where S is countable.
   - Find an injective function $f : A \to S$ where S is countable.
2. How to prove A is infinite:
   - Prove that A has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subsetneq A$ such that $A \sim B$.
3. Find a proper subset $B \subsetneq A$ such that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

Theorem (D). Let A be the set of all sequences whose terms are the digits 0 and 1 (that is A is the collection of all binary sequences). This set A is uncountable.

Proof. First, notice that A is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

$$\text{for all } h(n) = \text{The binary sequence whose } n\text{th term is the digit 1 and all other terms are zero.}$$

Clearly, h is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, A is infinite

Suppose for sake of contradiction that A is NOT uncountable. Since A is infinite and not uncountable, this assumption tells us that A must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned} f(1) &= (a_1^1, a_2^1, \dots) \\ f(2) &= (a_1^2, a_2^2, \dots) \\ f(3) &= (a_1^3, a_2^3, \dots) \\ &\vdots \\ f(n) &= (a_1^n, a_2^n, \dots) \\ &\vdots \end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that A contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N} \; b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence (b_n) is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

0.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of a is defined to be $|a|$. But note that the **absolute value of** a is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers a and b in $\mathbb{R}$, the **standard distance** between a and b is

$$\text{dist}\,(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as h gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

Theorem (Basic Properties of Standard Size in $\mathbb{R}$). Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Remark. Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

Definition. Let V be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on V if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space V equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

Theorem (Basic Properties of Standard Distance in $\mathbb{R}$). For all $a, b \in \mathbb{R}$.

(i) $\text{dist}(a, b) \geq 0$

(ii) $\text{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\text{dist}(a, b) = \text{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

0.2.4 Metric Spaces

Definition (Metric Spaces). Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X equipped with a metric d is called a **metric space**. (Sometimes we write (X, d) is a metric space)

Remark. X does **NOT** need to be a vector space. Furthermore, d is a more general notion of size than $\|\cdot\|$.

Example 0.2.1. $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)