# Math 230B: Homework 6

Lance Remigio

April 18, 2025

**Problem 1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

**Proof.** Our goal is to show that there exists an $M$ such that for all $n \geq 1$ and for all $x \in A$, we have

$$|f_n(x)| \leq M.$$

Since $f_n \to f$ uniformly, we know that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $m, n > N$ and for all $x \in A$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

In particular, if $\varepsilon = 1$, then

$$|f_n(x) - f_m(x)| < 1 \Longleftrightarrow |f_n(x)| < |f_m(x)| + 1 \quad \forall n, m > N. \tag{*}$$

Since each $f_n$ is bounded, it follows that there exists an $R_n$ such that

$$|f_n(x)| \leq R_n$$

for all $n \in \mathbb{N}$ and for all $x \in A$. Let $R = \max\{R_1, R_2, \dots, R_m\}$. Then from (*), we can see that

$$|f_n(x)| < |f_m(x)| + 1 \leq R_m + 1 \leq R + 1$$

for any $x \in A$ and for any $n \in \mathbb{N}$ where $M = R + 1$ is the desired $M$ we were looking for. Hence, $(f_n)$ is a uniformly bounded sequence of functions. ■

**Problem 2.** If $(f_n)$ and $(g_n)$ converge uniformly on a set $A$, prove that $(f_n + g_n)$ converges uniformly on $A$. Also,

**Proof.** Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in A$, we have

$$|(f_n + g_n)(x) - (f_m + g_m)(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $(f_n)$ converges uniformly on $A$, it follows from the Cauchy Criterion for uniform convergence that, with our given $\varepsilon$, there exists an $N_1 \in \mathbb{N}$ such that for any $n, m > N_1$ and for any $x \in A$, we have

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}. \tag{1}$$

Similarly, the uniform convergence of $(g_n)$ on $A$ implies that there exists an $N_2 \in \mathbb{N}$ such that for any $n, m > N_2$ and for any $x \in A$ that

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2}. \tag{2}$$

Then for any $n, m > \max\{N_1, N_2\} + 1$ and for any $x \in A$, we have

$$\begin{aligned} |(f_n + g_n)(x) - (f_m + g_m)(x)| &\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which is our desired result. ■

**Problem 3.** If $(f_n)$ and $(g_n)$ are two sequences of bounded functions that converge uniformly on a set $A$, prove that $(f_n g_n)$ converges uniformly on $A$.

**Proof.** Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in A$, we have

$$|(f_n g_n)(x) - (f_m g_m)(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. By problem 1, it follows from the uniform convergence of both bounded sequences $(f_n)$ and $(g_n)$ that there exists an $M_1, M_2 > 0$ such that for any $x \in A$ and for any $n \in \mathbb{N}$ that

$$|f_n(x)| \leq M_1 \quad \text{and} \quad |g_n(x)| \leq M_2,$$

respectively. Since $(f_n)$ converges uniformly on $A$, it follows that there exists an $N_1 \in \mathbb{N}$ such that for any $x \in A$ and for any $n, m > N_1$ that

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2M_2}. \tag{1}$$

Similarly, the uniform convergence of $(g_n)$ implies that there exists an $N_2 \in \mathbb{N}$ such that for any $x \in A$ and for any $n, m > N_2$, we have

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2M_1}. \tag{2}$$

Now, for any $x \in A$ and for any $n, m > \max\{N_1, N_2\} + 1$, we have

$$\begin{aligned}
|(f_n g_n)(x) - (f_m g_m)(x)| &\leq |f_n g_n(x) - f_m g_n(x) + f_m g_n(x) - f_m g_m(x)| \\
&\leq |f_n g_n(x) - f_m(x) g_n(x)| + |f_m(x) g_n(x) - f_m(x) g_m(x)| \\
&= |g_n(x)||f_n(x) - f_m(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
&\leq M_2 |f_n(x) - f_m 9x| + M_1 |g_n(x) - g_m(x)| \\
&< M_2 \cdot \frac{\varepsilon}{2M_2} + M_1 \cdot \frac{\varepsilon}{2M_1} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

■

**Problem 4.** Consider the sequences $(f_n : \mathbb{R} \to \mathbb{R})$ and $(g_n : \mathbb{R} \to \mathbb{R})$ defined as follows:

$$f_n(x) = 2 + \frac{5}{n} \quad g_n(x) = x + \frac{2}{n}$$

Prove that both $(f_n)$ and $(g_n)$ converge uniformly on the set $\mathbb{R}$, but $(f_n g_n)$ does not converge uniformly on $\mathbb{R}$.

**Proof.** Clearly, we see that $f_n \to f = 2$ uniformly. Also, it is not difficult to see that $g_n \to g$ where $g(x) = x$ for all $x \in \mathbb{R}$. We will show that this convergence is uniform. Let $\varepsilon > 0$ be given. Choose $N = \frac{5}{\varepsilon}$ and observe that for any $x \in \mathbb{R}$ and for any $n > N$, we have

$$|g_n(x) - g(x)| = \left|\left(x + \frac{5}{n}\right) - x\right| = \frac{5}{n} < \frac{5}{N} = \varepsilon.$$

Hence, we see that $g_n \to g$ uniformly on $\mathbb{R}$. Note that

$$\begin{aligned}
f_n g_n(x) &= \left(2 + \frac{5}{n}\right)\left(x + \frac{2}{n}\right) \\
&= \left(2 + \frac{5}{n}\right)x + \frac{4}{n} + \frac{10}{n^2}.
\end{aligned}$$

Since we see that $f_n g_n \to 2x$, we have

$$f_n g_n(x) - 2x = \frac{5}{n}x + \frac{6}{n}.$$

Define $b_n = f_n g_n(n) - 2n$. Then it follows that

$$b_n = 5 + \frac{6}{n}$$

which implies that $\lim_{n\to\infty} b_n = 5 > 0$ and that

$$\sup_{x\in\mathbb{R}} |f_n g_n(x) - 2x| \geq b_n.$$

Hence, $f_n g_n(x) \to 2x$ for any $x \in \mathbb{R}$ is NOT uniform. ■

**Problem 5.** Let $A \subseteq (X, d)$. Let $(f_n : A \to \mathbb{R})$ be a sequence of continuous functions which converges uniformly to a function $f$ on the set $A$. Let $(x_n)$ be a sequence in $A$ such that $x_n \to x \in A$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

**Proof.** Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, we have

$$|f_n(x_n) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly where each $f_n$ is a continuous function, we also have that $f$ is a continuous function by a theorem proven in class. Since $x_n \to x \in A$ and $f$ is continuous on $A$, it follows from the sequential criterion of continuity that $f(x_n) \to f(x)$ on $A$. With our given $\varepsilon$, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}. \tag{1}$$

Since $f_n \to f$ uniformly, there exists an $N_2 \in \mathbb{N}$ such that for any $n > N_2$ and any $x \in A$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}. \tag{2}$$

Choose $N = \max\{N_1, N_2\} + 1$. Because $x_n \in A$ for all $n \in \mathbb{N}$, it follows that for any $n > N_1$, we have

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$$

from (1). For any $n > N$, it follows that

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, we conclude that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

■

**Problem 6.** Let $A \subseteq (X, d)$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that if $(f_n : A \to \mathbb{R})_{n\geq 1}$ is a sequence of bounded functions that converges uniformly to $f : A \to \mathbb{R}$, then $(g \circ f_n)_{n\geq 1}$ converges uniformly to $g \circ f$.

**Proof.** Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$, we have

$$|(g \circ f_n)(x) - (g \circ f)(x)| < \varepsilon.$$

That is, we want to find an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$,

$$|g(f_n(x)) - g(f(x))| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Since $(f_n)$ is a sequence of bounded functions that converges uniformly to $f : A \to \mathbb{R}$, we have that $(f_n)$ is uniformly bounded by problem 1. Hence, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in A$. As a consequence, we see that $|f(x)| \leq M$ for all $x \in A$. Consider the compact interval $[-M, M]$ and $g|_{[-M,M]}$. Since $g$ is continuous and $[-M, M]$ is compact, it follows that $g$ is uniformly continuous. Hence, there exists (with our given $\varepsilon$) a $\delta > 0$ such that for all $s, t \in [-M, M]$ whenever $|s - t| < \delta$, we have

$$|g(s) - g(t)| < \varepsilon.$$

Since $f_n \to f$ uniformly, we can find an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$ and for any $x \in A$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

We claim that this $\hat{N}$ can be used as the same $N$ we were looking for. Indeed, if we take $\varepsilon = \delta$, then if $|f_n(x) - f(x)| < \delta$, then (*) will hold for any $n > \hat{N}$ and we are done. ■

**Problem 7.** For each $n \in \mathbb{N}$, let $f_n : (0, 1) \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{nx+1}$.

1. Explain in one line why $f_n \to f$ pointwise where $f \equiv 0$.
   **Proof.** Note that for all $x \in (0, 1)$ and for all $n \in \mathbb{N}$, we have
   $$0 \leq \frac{1}{nx+1} \leq \frac{1}{nx} \to 0.$$
   Thus, the Squeeze Theorem implies that $\frac{1}{nx+1} \to 0$ pointwise. ■

2. Explain in one line why each $f_n$ is continuous.
   **Proof.** Since 1 is a constant function and $nx+1$ is a polynomial function which are both continuous function where $nx + 1 \neq 0$, it follows from the Algebraic Continuity Theorem that each $f_n$ is continuous. ■

3. Explain why for each $n \in \mathbb{N}$, we have $f_{n+1} \leq f_n$.
   **Proof.** It immediately follows that for all $n \in \mathbb{N}$, $1 + nx$ is an increasing function. Define $\hat{f}_n(x) = 1 + nx$. Then from our observation $\hat{f}_n \leq \hat{f}_{n+1}$ for all $n \in \mathbb{N}$. Dividing we get
   $$\frac{1}{\hat{f}_{n+1}} \leq \frac{1}{\hat{f}_n} \implies f_{n+1} \leq f_n \quad \forall n \in \mathbb{N}.$$
   Thus, $f_n$ is a decreasing sequence of functions. ■

4. Explain why $f_n \to f$ is NOT uniform.
   **Solution.** Since $f_n$ is defined over a non-compact interval $(0, 1)$, it follows from Dini's Theorem that $f_n \to f$ is NOT uniform. ■

5. Explain why this example does not contradict the following theorem.
   **Solution.** This does not contradict the theorem because we still have pointwise convergence of $f_n \to f$. ■

**Problem 8.** (a) Prove that $(f_n : A \to \mathbb{R})_{n\geq1}$ converges uniformly to 0 if and only if $(|f_n|)_{n\geq1}$ converges uniformly to 0.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and assume that $f(1) = 0$. Prove that $(x^n f(x))$ converges uniformly on $[0, 1]$.

**Proof.** (a) ($\implies$) Our goal is to show that $(|f_n|)_{n\geq1}$ converges uniformly to 0. It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $m, n > N$ and for any $x \in A$, we have

$$\big||f_n(x)| - |f_m(x)|\big| < \varepsilon.$$

Let $\varepsilon > 0$. Since $(f_n)_{n\geq1}$ converges uniformly on $A$, it follows from our given $\varepsilon$ that there exists an $\hat{N} \in \mathbb{N}$ such that for any $n, m > \hat{N}$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

We claim that we can use $\hat{N}$ as the $N$ we were looking for. Indeed, we can see that for any $n, m > \hat{N}$ and $x \in A$ that

$$\big||f_n(x)| - |f_m(x)|\big| \leq |f_n(x) - f_m(x)| < \varepsilon$$

which is our desired result.

($\Longleftarrow$) Our goal is to show that $(f_n : A \to \mathbb{R})_{n \geq 1}$ converges uniformly to 0. It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$ and for any $x \in A$, we have

$$|f_n(x)| < \varepsilon.$$

Let $\varepsilon > 0$. Clearly, since $|f_n|$ converges uniformly to 0, then there exists an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$ and $x \in A$,

$$|f_n(x)| < \varepsilon$$

with $\hat{N}$ as the same $N$ we were looking for as desired. Hence, $f_n \to 0$ uniformly.

(b) We will show that $f_n$ defined by $f_n(x) = x^n f(x)$ for all $x \in [0, 1]$ converges to 0 uniformly using Dini's theorem. In what follows, we will show that each $f_n$ satisfies the following conditions:

(1) $[0, 1]$ is a compact set.

(2) For each $n \in \mathbb{N}$, $f_n : [0, 1] \to \mathbb{R}$ is continuous.

(3) $f_n \to 0$ pointwise on $K$ (Clearly, the zero function is continuous).

(4) For each $n \in \mathbb{N}$, we have $f_{n+1} \leq f_n$.

Clearly, (1) is satisfied by the Heine-Borel theorem on $\mathbb{R}$. Also, since $f$ is continuous on $[0, 1]$ and $x^n$ is a polynomial function which is clearly continuous on $[0, 1]$, it follows from the Algebraic Continuity Theorem that each $f_n$ is a continuous function and so (2) is satisfied. Next, notice that for $x = 0$, $f_n \to 0$ immediately. Similarly, if $x = 1$, then it immediately follows that $f_n \to 0$. On the other hand, if $x \in (0, 1)$, then $x^n \to 0$. Using the Algebraic Limit Theorem, it follows that $f(x)x^n \to 0$ for any $x \in (0, 1)$. Thus, we see that (3) is satisfied with the pointwise limit being clearly continuous on $[0, 1]$. Lastly, we see that for any $x \in [0, 1]$, $x^n$ is a decreasing function. Hence, $f_n(x) = f(x)x^n$ is a decreasing function and so (4) is satisfied.

By Dini's Theorem, we can conclude that $f_n \to 0$ uniformly on $[0, 1]$.

■

**Problem 9.** Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function, and for each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = f(x + \frac{1}{n})$. Prove that $(f_n)_{n \geq 1}$ converges uniformly to $f$ on $\mathbb{R}$.

**Proof.** It suffices to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in \mathbb{R}$, we have

$$|f_n(x) - f_m(x)| < \varepsilon. \qquad (*)$$

That is, we need to find an $N \in \mathbb{N}$ such that for any $n, m > N$ and for any $x \in \mathbb{R}$, we have

$$\left|f\Big(x + \frac{1}{n}\Big) - f\Big(x + \frac{1}{m}\Big)\right| < \varepsilon.$$

Since $f$ is uniformly continuous, it follows from our given $\varepsilon$ that there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$ satisfying $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Notice that $x_n = x + \frac{1}{n}$ converges to $x \in \mathbb{R}$. Hence, $(x_n)$ is a Cauchy sequence in $\mathbb{R}$. Thus, for any $\varepsilon > 0$, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n, m > \hat{N}$, we have

$$|x_n - x_m| < \varepsilon.$$

That is,

$$|x_n - x_m| = \left|\Big(x - \frac{1}{n}\Big) - \Big(x - \frac{1}{m}\Big)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$$

for any $n, m > \hat{N}$. Using $\varepsilon = \delta$, it follows from (*) that whenever $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \delta$,

$$\left| f\Big(x + \frac{1}{n}\Big) - f\Big(x + \frac{1}{m}\Big) \right| < \varepsilon \quad \forall n, m > \hat{N}.$$

That is,

$$|f_n(x) - f_m(x)| < \varepsilon$$

for any $n, m > \hat{N}$. Hence, $(f_n)$ converges to $f$ uniformly. ■

**Problem 10.** For each case, determine whether the given sequence of functions converges pointwise. If it does, determine whether the convergence is uniform.

10-1) $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = n^3 x^n$.
**Solution.** $f_n$ diverges since $n^3 \to \infty$ as $n \to \infty$. ■

10-2) $f_n : [0, \pi] \to \mathbb{R}$ defined by $f_n(x) = \sin^n(x)$.
**Solution.** Note that $|\sin x| \leq 1$ for all $x \in [0, \pi]$. Furthermore, $f_n$ is a sequence of continuous functions (since $\sin x$ is a continuous function and $x^n$ is a polynomial which is continuous so their composition is continuous) and decreasing. Also, note that the convergence of $f_n$ to 0 is pointwise and $[0, 1]$ is a compact set in $\mathbb{R}$. Dini's theorem implies that $f_n \to 0$ uniformly. ■

10-3) $f_n : (0, 1) \to \mathbb{R}$ defined by $f_n(x) = 2nxe^{-n^2x^2}$.
**Proof.** Converges pointwise to 0 but convergence is not uniform since $(0, \infty)$ is not a compact set (By Dini's Theorem). ■

10-4) $f_n : (0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{n^2 x}{(nx+1)^3}$.
**Proof.** Converges pointwise to 0 but convergence is not uniform since $(0, \infty)$ is not a compact set (By Dini's theorem). ■

10-5) $f_n : (0, 1) \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{nx+1}$.
**Proof.** Converges pointwise to 0 but convergence is not uniform since $(0, 1)$ is not a compact set (By Dini's theorem). ■

10-6) $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{nx^2+1}$.
**Proof.** Converges pointwise to 0 but convergence is not uniform since $\mathbb{R}$ is not a compact set (By Dini's theorem). ■

10-7) $f_n : [0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{nx}{n^3+x^3}$.
**Solution.** Note that $f_n$ is a sequence of continuous functions (since it is a ratio of continuous functions) which converges to 0 pointwise, but not uniform since $[0, \infty)$ is not a compact set by Dini's Theorem. ■

10-8) $f_n : [0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{nx^2}{n^3+x^3}$.
**Solution.** Note that $f_n$ is a sequence of continuous functions (since it is a ratio of continuous functions) which converges to 0 pointwise, but not uniform since $[0, \infty)$ is not a compact set by Dini's Theorem. ■

10-9) $f_n : [0, 1] \to \mathbb{R}$ with $g_n = f_n'$ where $f_n : [0, 1] \to \mathbb{R}$ is defined by $f_n(x) = \frac{\ln(1+nx)}{n}$.
**Solution.** Note that for all $x \in [0, 1]$, we have

$$g_n(x) = f_n'(x) = \frac{1}{nx + 1}$$

which is a sequence of continuous functions and that converges to 0 pointwise. Since $[0, 1]$ is compact and $g_n$ is a decreasing sequence of functions, it follows from Dini's Theorem that $g_n \to 0$ uniformly. Notice that for $x = 0$, the sequence $f_n(0) \to 0$. Hence, Exercise 12 from homework 5 implies that $f_n \to 0$ uniformly on $[0, 1]$. ■

**Problem 11.** For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

Prove that the Weierstrass M-Test cannot be applied to establish the uniform convergence of $\sum_{n=1}^{\infty} f_n$. Nevertheless, show that this series converges uniformly.

**Proof.** Note that we cannot use the Weierstrass M-Test because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series which diverges. Hence, we will show via the Cauchy Criterion that $\sum f_n$ converges uniformly on $[0,1]$.

Clearly, if $x \notin (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges. Suppose $x \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Then for all $n \in \mathbb{N}$, $f_n(x) = \frac{1}{n}$. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$|\sum_{k=m+1}^{n} f_n(x)| < \varepsilon.$$

Since at most one $f_n(x)$ is nonzero for any $x$, and $f_n(x) = \frac{1}{n}$, it follows that we can find an $N$ large enough so that $\frac{1}{N} < \varepsilon$. Then for all $x \in [0,1]$ and for all $n, m > N$, we have

$$\Big|\sum_{k=m+1}^{n} f_n(x)\Big| \leq \sup_{n\in\mathbb{N}} \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Hence, $f_n$ converges uniformly on $[0,1]$. ■

**Problem 12.** Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly on $[0,\infty)$.

**Proof.** Our goal is to show that the series $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly. It suffices to show that the series above satisfies the Weierstrass M-Test. Indeed, we see that for all $n \in \mathbb{N}$ and for all $x \in [0,\infty]$,

$$\Big|\frac{x}{1+n^4x^2}\Big| \leq \Big|\frac{x}{n^4x^2}\Big| = \Big|\frac{1}{n^4x}\Big| = \frac{1}{|n^4||x|} \leq \frac{1}{n^4}$$

where $M_n = \frac{1}{n^4}$. Clearly, the series $\sum_{n=1}^{\infty} M_n$ converges via the p-series test. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly. ■

**Problem 13.**
(a) Use Taylor's Theorem with Lagrange remainder to prove that for all $x > 0$, we have $e^x > \frac{x^2}{2}$.
(b) Prove that $\sum_{n=1}^{\infty} x^2e^{-nx}$ converges uniformly on $[0,\infty)$.

**Proof.**
(a) Clearly, we can see that $e^x$ is differentiable $n+1$ times and so by Taylor's Theorem with Lagrange Remainder, it follows that

$$e^x = \sum_{k=1}^{n} \frac{x^k}{k!} > \frac{x^2}{2!} = \frac{x^2}{2}$$

for all $x > 0$.

(b) We proceed via the Weierstrass M-Test to prove that $\sum_{n=1}^{\infty} x^2e^{-nx}$ converges uniformly on $[0,\infty)$. From part (a), it follows for all $n \in \mathbb{N}$ that

$$|x^2e^{-nx}| = x^2e^{-nx} < 2e^x \cdot e^{-nx} = 2e^{x(1-n)} = \frac{2}{e^{x(n-1)}} < 2e \cdot \Big(\frac{1}{e}\Big)^n.$$

Note that $|r| = \frac{1}{e} < 1$ and so the series

$$\sum_{n=1}^{\infty} 2e \cdot \Big(\frac{1}{e}\Big)^n.$$

is geometric which converges. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} x^2 e^{-nx}$ converges uniformly.

■

**Problem 14.** Let $a > 0$ be a fixed number. Prove that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a, \infty)$ and does not converge uniformly on $(0, \infty)$.

**Proof.** Our goal is to show that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a, \infty)$. We will do this via the Weierstrass M-Test. Note that for all $n \in \mathbb{N}$ and for all $x \in [a, \infty)$, we can use a result from homework 10 in Math 230A to write

$$\left|2^n \sin\left(\frac{1}{3^n x}\right)\right| \leq \frac{2^n}{|x| 3^n} = \frac{1}{|x|} \cdot \left(\frac{2}{3}\right)^n \leq \frac{1}{a} \cdot \left(\frac{2}{3}\right)^n. \tag{*}$$

Observe that $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent series because it is geometric series ($|r| = \frac{2}{3} < 1$). Thus, the Algebraic Theorem for Series implies that $\sum_{n=1}^{\infty} \frac{1}{a} \cdot (\frac{2}{3})^n$ converges. Using the Weierstrass M-Test, it follows that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ converges uniformly on $[a, \infty)$.

Now, let us consider the same series on the interval $(0, \infty)$. From our inequality in (*), we see that the series defined on the following sequence term

$$\frac{1}{|x|} \cdot \left(\frac{2}{3}\right)^n$$

depends on $x \in (0, \infty)$ and is not a constant sequence. Hence, it follows that $\sum_{n=1}^{\infty} 2^n \sin(\frac{1}{3^n x})$ does not converge uniformly via the Weierstrass M-Test. ■

**Problem 15.** Let $a > 0$ be a fixed number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{nx}{1 + n^4 x^2}$$

converges uniformly on $[a, \infty)$ and does not converge uniformly on $[0, \infty]$.

**Proof.** Consider $\sum_{n=1}^{\infty} \frac{nx}{1+n^4x^2}$ over the interval $[a, \infty)$ where $a > 0$ is fixed. For all $n \in \mathbb{N}$, it follows that

$$\left|\frac{nx}{1 + n^4 x^2}\right| \leq \frac{n|x|}{|1 + n^4 x^2|} \leq \frac{1}{n^3 |x|} \leq \frac{1}{an^3}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series (via the P-Series Test), we know by the Algebraic Limit Theorem for Series that $\sum_{n=1}^{\infty} \frac{1}{an^3}$ is also a convergent series. Hence, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} \frac{nx}{1+n^4x^2}$ converges uniformly on $[a, \infty)$

Now, consider the same series over $[0, \infty)$. Clearly, the series converges if $x = 0$. Performing a similar set of computations, we obtain the following inequality

$$\left|\frac{nx}{1 + n^4 x^2}\right| \leq \frac{n|x|}{|1 + n^4 x^2|} \leq \frac{n|x|}{2n^2 |x|} \leq \frac{1}{2n}$$

for any $x \in (0, \infty)$. Note that the series on the right-hand side of the above inequality diverges because it is a harmonic series. Hence, it follows from the Weierstrass M-Test that the series does NOT converge uniformly on the interval $(0, \infty)$. ■