Math 230A: Homework 5

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Problem 1. Mark each statement True or False. Let (X, d) be a metric space and $K \subseteq X$.

1. If $Y \subseteq X$ and $\{G_\alpha\}$ is a collection of subsets of Y that are open relative to Y, then $\bigcup_\alpha G_\alpha$ is open relative to Y. **True.**
2. If there exists some open cover of K which has a finite subcover, then K is compact. **False.**
3. If K is compact, then $K' \subseteq K$. **True.**
4. If K is closed, then K is compact. **False.**
5. If K is compact and $E \subseteq K$, then E is compact. **False.**
6. Consider $E = [-10, 10] \subseteq \mathbb{R}$ and the open cover of E by $\Phi = \{(x-1, x+1) : x \in E\}$. Then the collection $\{(x - \frac{1}{2}, x + \frac{1}{2}) : x \in E\}$ is a subcover of Φ. **False.**
7. Let E and Φ be as above. Then the collection $\{(x-1, x+1) : x = -10, -9, -8, \ldots, 8, 9, 10\}$ is a subcover of Φ. **True.**
8. Let E and Φ be as above. Then the collection $\{(x-1, x+1) : x = -10, -8, -6, \ldots, 6, 8, 10\}$ is a subcover of Φ. **False.** (this is a subcollection of Φ but it is NOT a subcover)

Problem 2. Show that compact implies bounded.

Proof. Assume that K is compact. We want to show that K is bounded. Fix $\delta > 0$ and consider $K \subseteq \bigcup_{x \in k} N_\delta(x)$ where $\{N_\delta(x)\}_{x \in K}$ is an open cover of K. Since K is compact, we can find a finite subcover; that is, there exists $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_\delta(x_i). \tag{*}$$

Note that each $N_\delta(x_i)$ is bounded. Using the results from lemmas 1 and 2, we can conclude that the finite union in (*) must be bounded as well as its subset K. Thus, we can see that K must be bounded and we are done. ■

Problem 3. Show the union of finitely many compact sets is compact.

Proof. Let $E_1, \ldots, E_n$ be compact sets in the metric space (X, d). Our goal is to show that

$$K = \bigcup_{i=1}^{\infty} K_i$$

is a compact set in X. We proceed to show this result via induction on n. Let $n = 2$ be our base case. Our goal is to show that $E_1 \cup E_2$ is compact. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $E_1 \cup E_2$; that is,

$$E_1 \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \text{ and } E_2 \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha.$$

Since E_1 is compact, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$E_1 \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Likewise, E_2 being compact implies that there exists $\beta_1, \dots, \beta_m \in \Lambda$ such that

$$E_2 \subseteq \bigcup_{j=1}^{m} O_{\beta_j}.$$

Thus, we see that

$$E_1 \cup E_2 \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup \bigcup_{j=1}^{m} O_{\beta_j}.$$

Suppose the claim is true for $n = k \geq 1$. If $E_1, \dots, E_k$ are compact sets, then $\bigcup_{i=1}^{k} E_i$ is compact. Observe that

$$\bigcup_{i=1}^{k+1} E_i = \Big(\bigcup_{i=1}^{k} E_i\Big) \cup E_{k+1}. \qquad \text{(}\textstyle\bigcup_{i=1}^{k} E_i \text{ is compact)}$$

But by our base case $n = 2$, we know that the right-hand side of the equation above is compact and we are done. ■

Problem 4. Show that an arbitrary intersection of compact sets is compact. (Hint. A closed subset of a compact set is compact)

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X and let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a collection of compact sets in X. Define

$$K = \bigcap_{\alpha} K_\alpha. \tag{1}$$

Since each K_α is compact, we know that each K_α must be closed. Therefore, the arbitrary intersection above must be closed and so K is closed. But observe that

$$\bigcap_{\alpha} K_\alpha \subseteq K_\alpha$$

and that K_α is a closed set. Thus, K must be compact as well! ■

Problem 5. A metric space (X, d) is called **separable** if it contains a countable subset E which is dense in X. For example, $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable set which is dense in $\mathbb{R}$. Show that $\mathbb{R}^2$ is separable. (Hint: Consider the set of points which have only rational coordinates.)

Proof. Let

$$E = \mathbb{Q} \times \mathbb{Q} = \{(p, q) \in \mathbb{R}^2 : p, q \in \mathbb{Q}\}.$$

Since $\mathbb{Q}$ is a countable set and that a finite product of countable sets is countable, we must have that E is countable. All that is left to show is that E is dense in $\mathbb{R}^2$. Our goal is to show that $\overline{E} = \mathbb{R}^2$; that is,

$$\overline{E} = \{(x, y) \in \mathbb{R}^2 : \forall \varepsilon > 0,\ N_\varepsilon((x, y)) \cap E \neq \emptyset\}.$$

Hence, our goal is to show that for all $(x, y) \in \mathbb{R}^2$ and for all $\varepsilon > 0$ that

$$N_\varepsilon((x, y)) \cap E \neq \emptyset.$$

To this end, let $(x, y) \in \mathbb{R}^2$ and let $\varepsilon > 0$ be given. Since $x \in \mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that there exists $p \in \mathbb{Q}$ such that

$$p \in N_{\frac{\varepsilon}{\sqrt{2}}}(x) \iff |x-p| < \frac{\varepsilon}{\sqrt{2}}. \tag{1}$$

Likewise, $y \in \mathbb{R}$ implies that there exists $q \in \mathbb{Q}$ such that

$$q \in N_{\frac{\varepsilon}{\sqrt{2}}} \iff |y-q| < \frac{\varepsilon}{\sqrt{2}}. \tag{2}$$

We claim that $(p,q) \in N_\varepsilon((x,y)) \cap E$ so that

$$N_\varepsilon((x,y)) \cap E \neq \emptyset.$$

By using (1) and (2) as well as the triangle inequality, we see that

$$d((p,q),(x,y)) = \sqrt{|x-p|^2 + |y-q|^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon.$$

Hence, we see that $N_\varepsilon((x,y))$. Clearly, we see that $(p,q) \in E$ and so we conclude that

$$N_\varepsilon((x,y)) \cap E \neq \emptyset.$$

Thus, $E = \mathbb{Q}^2$ is dense in $\mathbb{R}^2$. ■

Problem 6. Let (X,d) be a separable metric space and $\emptyset \neq A \subseteq X$. Prove that the collection of the isolated points of A is at most countable.

Proof. Let (X,d) be a separable metric space and $\emptyset \neq A \subseteq X$. Denote the set A_I as the set of all isolated points of A. Since (X,d) is separable, we know that there must exists a subset of X denoted by E such that E is both countable and dense in X. Our goal is to construct an injective map $f : A_I \to E$ which proves that A_I is at most countable.

Observe that for every $p \in A_I$, we know that $p \in A \setminus A'$ if and only if there exists $\varepsilon_p > 0$ such that

$$N_{\varepsilon_p}(p) \cap A = \{p\}. \tag{1}$$

Furthermore, since E is dense in X, we know that there exists $\varepsilon_p > 0$ such that

$$N_{\varepsilon_p}(p) \cap E \neq \emptyset. \tag{2}$$

Hence, (1) and (2) imply that there exists $y_p \in X$ such that $y_p \in N_{\frac{\varepsilon_p}{2}} \cap E$. We define $f : A_I \to E$ by

$$f(p) = y_p.$$

Our goal is to show that f is injective. Let $p, q \in X$. Suppose $f(p) = f(q)$. From this, we will show that $p = q$. Since $f(p) = f(q)$, we know that

$$y_p = y_q.$$

Observe that

$$y_p \in N_{\frac{\varepsilon_p}{2}} \cap E \text{ and } y_q \in N_{\frac{\varepsilon_q}{2}}(q) \cap E,$$

Since $y_p = y_q$, we have

$$N_{\frac{\varepsilon_p}{2}}(p) \cap N_{\frac{\varepsilon_q}{2}}(q) \neq \emptyset.$$

Let $z \in N_{\frac{\varepsilon_p}{2}} \cap N_{\frac{\varepsilon_q}{2}}(q)$. Then we have

$$d(p,z) < \frac{\varepsilon_p}{2} \text{ and } d(z,q) < \frac{\varepsilon_q}{2}$$

which implies that

$$d(p,z) + d(z,q) < \frac{\varepsilon_p}{2} + \frac{\varepsilon_q}{2}.$$

Without loss of generality, assume that $\varepsilon_q < \varepsilon_p$. By the triangle inequality, we see that

$$d(p,q) \leq d(p,z) + d(z,q) < \frac{\varepsilon_p}{2} + \frac{\varepsilon_q}{2} < \varepsilon_p.$$

This implies that q must be contained in $N_{\varepsilon_p}(p) \cap A$. However, the only point contained in $N_{\varepsilon_p} \cap A$ is p. Hence, we must have $p = q$ and so f is injective. Thus, A_I is at most countable. ■

Problem 7. Let (X, d) be a metric space. A collection $\{V_\alpha\}$ of open subsets of X is said to be a **base** for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that the every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.)

Proof. Since (X, d) is a separable metric space, we know that X contains a countable dense subset E. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a collection of open subsets in X. Let $x \in X$ and let $G \subseteq X$ be an open set such that $x \in G$. Our goal is to show that $V_\alpha \subseteq G$ for some α. Since $X = \overline{E}$, we must have $x \in \overline{E}$; that is, for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

To this end, let $\varepsilon > 0$ be rational and pick a point $y \in N_\varepsilon(x) \cap E$. Then

$$y \in N_\varepsilon(x) \text{ and } y \in E.$$

Note that $N_\varepsilon(x)$ is an open set in X. Thus, we can write $N_\varepsilon(x)$ as a union of open sets V_α where $\alpha \in \Lambda$; that is,

$$N_\varepsilon(x) = \bigcup_\alpha V_\alpha.$$

Thus, y must be contained in the union above and so $y \in V_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since $y \in G$, we must also have $V_\alpha \subseteq G$ and we are done. ■

Problem 8. Let (X, d) be a metric space in which every infinite subset has a limit point. Prove that X is separable. (Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.)

Proof. Let (X, d) be a metric space in which every infinite subset has a limit point. Our goal is to show that X is separable. Fix $\delta > 0$, and pick $x_1 \in X$. Now, choose $x_1, \ldots, x_j \in X$ and then choose $x_{j+1} \in X$ such that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$.

By the lemma 3, the process outlined in the first paragraph must terminate; that is, we can consider finitely many neighborhoods of radius $\delta > 0$ that cover X.

Define

$$S = \{x_{n_j} : 1 \leq j \leq N_n : n = 1, 2, 3, \ldots\}$$

and fix $\delta = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ such that

$$X = \bigcup_{j=1}^{N_n} N_\delta(x_{n_j}) \quad n = 1, 2, 3, \ldots\ .$$

Clearly, S is countable. Hence, all that is left to show is that S is dense in X. That is, we would like to show that for every $x \in X$ and $\varepsilon > 0$ that

$$N_\varepsilon(x) \cap S \neq \emptyset.$$

To this end, pick $n \in \mathbb{N}$ such that $\delta = \frac{1}{n} < \varepsilon$ by the Archimeadean Property. Consider the neighborhood $N_\varepsilon(x)$. If we let $y \in S$ with $x \in N_\delta(y)$, then we see that

$$d(x, y) < \frac{1}{n} < \varepsilon.$$

This implies that $y \in N_\varepsilon(x)$. Since y is also contained in S, we can conclude that

$$y \in N_\varepsilon(x) \cap S \Longleftrightarrow N_\varepsilon(x) \cap S \neq \emptyset,$$

showing that S is dense in X. Hence, X is a separable metric space. ■

Problem 9 (Extra Credit). Let (X, d) be a metric space and Y be a nonempty subset of X. Let $E \subseteq Y$. Prove that

$$E \text{ is closed relative to } Y \Longleftrightarrow E = A \cap Y \text{ for some closed set } A \subseteq X.$$

Proof. ($\Longrightarrow$) Assume that E is closed relative to Y. Our goal is to show that $E = A \cap Y$. Since E is closed relative in Y, we know that E^c is open relative in Y; that is,

$$(Y \setminus E) = G \cap Y \text{ for some } G \subseteq Y.$$

Our goal is to show that $E = G^c \cap Y$ for some $G^c \subseteq Y$ (G is open $\Longleftrightarrow$ G^c is closed). Observe that

$$\begin{aligned} E = Y \setminus (Y \setminus E) &= Y \setminus (G \cap Y) \\ &= (Y \cap Y) \setminus (G \cap Y) \\ &= (Y \setminus G) \cap Y \\ &= G^c \cap Y && (\text{for some } G^c \subseteq Y) \end{aligned}$$

and we are done.

($\Longleftarrow$) Suppose $E = A \cap Y$ for some closed set $A \subseteq X$. Observe that

$$Y \setminus E = (X \setminus A) \cap Y \tag{1}$$

where $X \setminus A$ is open in X. By (1), we must have $Y \setminus E$ is open in Y. Hence, E is closed relative to Y. ■

Problem 10 (Extra Credit). Let (X, d) be a metric space. Let $E \subseteq X$. Prove that the following definitions of boundedness are equivalent:

- Rudin's Definition: There exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$.
- Anthony's Definition: There exists $R > 0$ such that for all x and y in E, we have $d(x, y) < R$.

Proof. ($\Longrightarrow$) Suppose there exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$. Let $x, y \in E$. Our goal is to show that there exists $R > 0$ such that for all $x, y \in E$, $d(x, y) < R$. To this end, let $x, y \in E$ be given. Choose $R = 2\varepsilon > 0$. Since $E \subseteq N_\varepsilon(q)$, we have that

$$d(x, y) \leq d(x, q) + d(q, y) < \varepsilon + \varepsilon = 2\varepsilon = R.$$

Thus, we see that $d(x, y) < R$ for any $x, y \in E$.

($\Longleftarrow$) Suppose there exists $R > 0$ such that for all $x, y \in E$, $d(x, y) < R$. We will show that there exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$. Let $x \in E$. Fix $y \in E$ such that $q = y$. Choose

$\varepsilon = R + 1 > 0$. By assumption,

$$d(x, q) < R < R + 1 = \varepsilon.$$

Hence, $x \in N_\varepsilon(q)$. ■

Lemma (1)**.** Let (X, d) be a metric space. A subset of a bounded set is bounded.

Proof. Since B is bounded, there exists $q \in X$ and $\varepsilon > 0$ such that $B \subseteq N_\varepsilon(q)$. Since $A \subseteq B$, we conclude that $A \subseteq N_\varepsilon(q)$. So, A is bounded. ■

Lemma (2)**.** Let (X, d) be a metric space. A finite union of bounded sets is bounded.

Proof. Let $E_1, \dots, E_n$ be a bounded sets in a metric space (X, d). We will show via induction on n that

$$\bigcup_{i=1}^{n} E_i$$

is bounded. Note that if $n = 1$, then E_1 is bounded and so the result is immediate. Thus, let our base case be $n = 2$. We will show that the union $E_1 \cup E_2$ is bounded. If E_1 is bounded, then we can find an $\varepsilon_1 > 0$ and $q_1 \in X$ such that

$$E_1 \subseteq N_{\varepsilon_1}(q_1). \tag{1}$$

Likewise, if E_2 is bounded, then we can find an $\varepsilon_2 > 0$ and $q_2 \in X$ such that

$$E_2 \subseteq N_{\varepsilon_2}(q_2). \tag{2}$$

Choose $\delta = 2[\varepsilon_1 + \varepsilon_2 + d(q_1, q_2)]$. From this, we will show that $E_1 \cup E_2 \subseteq N_\delta(q_1)$ so that $E_1 \cup E_2$ will be bounded. Let $x \in E_1 \cup E_2$. Then we either have $x \in E_1$ or $x \in E_2$. If $x \in E_1$, then

$$x \in N_{\varepsilon_1}(q) \subseteq N_\delta(q_1). \tag{$\varepsilon_1 < \delta$}$$

If $x \in E_2$, then

$$\begin{aligned} d(x, q_1) &\leq d(x, q_2) + d(q_2, q_1) \\ &\leq \varepsilon_2 + d(q_1, q_2) \\ &< \delta. \end{aligned}$$

This implies that $x \in N_\delta(q_1)$. Thus, (i) and (ii) imply that $E_1 \cup E_2 \subseteq N_\delta(q_1)$. Now, assume that the result holds for the nth case. We will now show that the result holds for $n = k + 1$. Thus, observe that

$$\bigcup_{n=1}^{k+1} E_n = \Big(\bigcup_{n=1}^{k} E_n\Big) \cup E_{k+1}. \tag{3}$$

By our inductive hypothesis, we know that

$$\bigcup_{n=1}^{k} E_n$$

is bounded. Clearly, with E_{k+1} being also bounded along with our base case $n = 2$, we can conclude that the union in (3) is also bounded which concludes our induction proof. ■

Lemma (3)**.** Let (X, d) be a metric space and $E \subseteq X$ be infinite. Suppose $d(x_n, x_m) > \varepsilon$ for all $n \neq m \in \mathbb{N}$ such that $x_n \neq x_m$. Then E has no limit points.

Proof. Suppose for sake of contradiction that E does have a limit point. Denote this limit point of E as x. Thus, for any $\delta > 0$, we have

$$N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset. \tag{1}$$

Since E is an infinite subset of X, pick $n \in \mathbb{N}$ such that

$$x_n \in N_\delta(x) \text{ and } x_n \in E.$$

Hence, we see that $d(x, x_n) < \delta$. Similarly, pick $m \in \mathbb{N}$ such that $n \neq m$ in the intersection in (1) such that $x_m \in N_\delta(w)$. Then we have $d(x_m, x) < \delta$. Our goal is to find some $\varepsilon > 0$ such that $d(x_n, x_m) < \varepsilon$, contradicting our assumption that $d(x_n, x_m) \geq \delta$ for all $\delta > 0$ and $n \neq m \in \mathbb{N}$. Choose $\varepsilon = 2\delta$. Using the triangle inequality, we see that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \delta + \delta = 2\delta = \varepsilon$$

which produces desired contradiction. ■