Math 234 Final Exam

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Problem 1. Decide whether the following statements are true or false. You **do not need** to justify your answer.

(a) Let $f : \mathbb{C} \to \mathbb{R} \subset \mathbb{C}$ be defined by $f(z) = \cos(\overline{z})$. Then, f is complex differentiable at $z = 0$. **False.**

(b) Let $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Then, we can find a point $z \in A$ such that it is not an accumulation point of A. **False.**

(c) Consider the power series $\sum_{n=1}^{\infty} \frac{n^n(z-2i)^n}{n}$. Then, the radius of convergence of the power series is 1. **False.**

(d) Let $D \subseteq \mathbb{C}$ open, and $f : D \to \mathbb{C}$ be holomorphic. If $f'(z) = 0$ for all $z \in D$, then f is a constant function. **False.**

(e) Let $D \subseteq \mathbb{C}$ open, and $f : D \to \mathbb{C}$ be holomorphic. Then, all derivatives of f are also holomorphic functions. **True.**

Problem 2. Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{5}{((n+2)+i)((n+3)+i)}$ converges and compute its sum explicitly.

Proof. Consider the sequence (z_n) by

$$z_n = \frac{5}{((n+2)+i)((n+3)+i)}$$

observe through partial fraction decomposition that

$$\frac{5}{((n+2)+i)((n+3)+i)} = \frac{5}{n+(2+i)} - \frac{5}{(n+1)+(2+i)}.$$

Denote another sequence (w_n) by

$$w_n = \frac{5}{n+(2+i)}.$$

Note that $z_n = w_n - w_{n+1}$. To show that $\sum_{n=0}^{\infty} z_n$ converges, we will show that w_n converges (by problem 5 of homework 2). Clearly, we can see that as $n \to \infty$, we have $w_n \to 0$. So, we see that

$\sum_{n=0}^{\infty} z_n$ must converge. Now, we can see that

$$\begin{aligned}\sum_{n=0}^{\infty} z_n &= \sum_{n=0}^{\infty}(w_n - w_{n+1}) \\ &= w_0 - \lim_{n\to\infty} w_{n+1} \\ &= \frac{5}{2+i} - \lim_{n\to\infty} \frac{5}{(n+1)+(2+i)} \\ &= \frac{5}{2+i} - 0 \\ &= \frac{5}{2+i}.\end{aligned}$$

converges. ■

Problem 3 (Differential Equation Characterization of the exponential function)**.** Fix $c, w \in \mathbb{C}$ and consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = we^{cz}$. Then, f is holomorphic and it can be shown that f satisfies $f'(z) = cf(z)$ for all $z \in \mathbb{C}$ and $f(0) = w$. Assume that we have a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ that also satisfies $g'(z) = cg(z)$ for all $z \in \mathbb{C}$ and $g(0) = w$. Prove that $g(z) = f(z)$ for all $z \in \mathbb{C}$.

Proof. Note that $\mathbb{C}$ is both open and connected. Fix $c, w \in \mathbb{C}$ and define the function $f(z) = we^{cz}$. Our goal is to show that f is holomorphic and it can be shown that f satisfies $f'(z) = cf(z)$ for all $z \in \mathbb{C}$ and $f(0) = w$. As a consequence, we will show that if we have another holomorphic function $g : \mathbb{C} \to \mathbb{C}$ that also satisfies $g'(z) = cg(z)$ for all $z \in \mathbb{C}$ and $g(0) = w$, we will also show that $g(z) = f(z)$ for all $z \in \mathbb{C}$.

By definition of f, we see that f must be a holomorphic function (because e^{cz} is a holomorphic function and therefore complex differentiable). Now, we can see that

$$f'(z) = wce^{cz} = cwe^{cz} = cf(z).$$

Furthermore, we have

$$f(0) = we^{c\cdot 0} = we^0 = w.$$

Denote $h(z) = e^{-cz}g(z)$. We can see that

$$\begin{aligned}h'(z) &= -ce^{-cz}g(z) + e^{-cz}g'(z) \\ &= -ce^{-cz}g(z) + e^{-cz}(cg(z)) \\ &= 0.\end{aligned}$$

This tells us that h must be constant and so, for all $z \in \mathbb{C}$, we have $h(z) = k$ for some $k \in \mathbb{C}$.

$$h(z) = k \iff e^{-cz}g(z) = k \iff g(z) = ke^{cz}.$$

Since $g(0) = w$, we can see that $w = k$ and so, $f(z) = g(z)$ for all $z \in \mathbb{C}$. ■

Problem 4 (nth Root of Nowhere Vanishing Holomorphic Functions)**.** A nonempty open and connected set $D \subseteq \mathbb{C}$ is called an *elementary domain* if every holormorphic function on D has a primitive.

Let D be an elementary domain and $f : D \to \mathbb{C}$ be a nowhere vanishing holomorphic function. Here, nowhere vanishing means $f(z) \neq 0$ for all $z \in D$.

(a) Show that there exists a holormorphic function $g : D \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in D$.

(b) Let n be a positive integer. Use (a) to show that there is a holomorphic function $G : D \to \mathbb{C}$ such that $(G(z))^n = f(z)$ for all $z \in D$.

Proof. (a) Let $f : D \to \mathbb{C}$ be a holomorphic function. Let $G : D \to \mathbb{C}$ be defined by

$$G(z) = \frac{e^{F(z)}}{f(z)}$$

where $F : D \to \mathbb{C}$ is a primitive of the function f'/f. Indeed, since f defined on the elementary domain D, and f'/f must is holomorphic, we see that F must be the primitive of f'/f. Note that since e^z and $F(z)$ are holomorphic functions (e^z is also continuous), their composition $e^{F(z)}$ is also holomorphic. Since $f : D \to \mathbb{C}$ is nonwhere vanishing holomorphic function, we can see that $G = e^F/f$ must be a holomorphic function. Thus, observe that for all $z \in \mathbb{C}$

$$\begin{aligned} G'(z) &= \frac{F'(z)}{f(z)} e^{F(z)} - \frac{f'(z)}{(f(z))^2} e^{F(z)} \\ &= \frac{f'(z)}{(f(z))^2} e^{F(z)} - \frac{f'(z)}{(f(z))^2} e^{F(z)} \\ &= 0. \end{aligned}$$

This implies that $G(z) = k$ for some nonzero $k \in \mathbb{C}$. Hence, we have

$$G(z) = k \iff \frac{e^{F(z)}}{f(z)} = k \iff kf(z) = e^{F(z)}.$$

Since e^z is a surjective function $\mathbb{C}$ to $\mathbb{C}^\bullet$, we can find a $c \in \mathbb{C}$ such that $e^c = k$. Now, observe that

$$kf(z) = e^{F(z)} \iff e^c f(z) = e^{F(z)} \iff f(z) = e^{F(z)-c}.$$

Now, define $g : D \to \mathbb{C}$ by

$$g(z) = F(z) - c.$$

Clearly, F is a primitive which is holomorphic and $c \in \mathbb{C}$ implies that $g(z)$ is a holomorphic function which is our desired result.

(b) Let $n \in \mathbb{Z}^+$. Define $G : D \to \mathbb{C}$ in the following way:

$$G(z) = e^{\frac{1}{n} g(z)} = (e^{g(z)})^{\frac{1}{n}}.$$

By part (a), we see that $f(z) = e^{g(z)}$ is a nowhere vanishing holomorphic function defined on an elementary domain D where g is some holomorphic function from D to $\mathbb{C}$. Hence, $G(z)$ must be a holomorphic. Now, we see that

$$(G(z))^n = (e^{\frac{1}{n} g(z)})^n = e^{g(z)} = f(z)$$

as desired.

■

Problem 5 (Computation of Some Real Integrals using Complex Analysis-I). (a) Define $\alpha, \beta : [0,1] \to \mathbb{C}$ by $\alpha(t) = 3e^{2\pi i t}$ and $\beta(t) = 3\cos(2\pi t) + 4i\sin(2\pi t)$. Note that the trace of α is the circle $\{z \in \mathbb{C} : |z| = 3\}$ where as the trace of β is the ellipse whose equation is given by $x^2/9 + y^2/16 = 1$.

(i) Show that

$$\int_\alpha \frac{1}{z}\, dz = \int_\beta \frac{1}{z}\, dz.$$

(ii) Use (i) to show

$$\int_0^{2\pi} \frac{1}{9\cos^2 t + 16\sin^2 t}\, dt = \frac{\pi}{6}.$$

(b) Let $f, g : B(0, R) \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z} + \frac{2}{R-z}$, $g(z) = \frac{2}{R-z}$, where $R > 0$ and $B(0, R)$ is the open ball centered at $0 \in \mathbb{C}$ and radius R. Let $0 < r < R$.

   (i) Compute $\displaystyle\int_{\partial B(0,R)} f(z)\, dz$ and $\displaystyle\int_{\beta B(0,R)} g(z)\, dz$.

   (ii) Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\, dt = 1 \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\, dt = \frac{r}{R^2 - r^2}.$$

Solution (a). (i) Note that the circle defined by α is contained inside the ellipse β. Our goal is to show that

$$\int_\alpha \frac{1}{z}\, dz = \int_\beta \frac{1}{z}\, dz.$$

Suppose α has a negative orientation and β has a positive orientation. Now, we will introduce two new curves γ_1 and γ_2 by splitting β containing α in two regions down the middle. We define γ_1 to have a positive orientation and γ_2 to have a negative orientation. Note that γ_1 and γ_2 formed from this process are star-shaped regions. Since $\frac{1}{z}$ is holomorphic on γ_1 and γ_2, we can use Cauchy's Integral Theorem for star-shaped regions to state that

$$\int_{\gamma_1} \frac{1}{z}\, dz = 0 \quad \text{and} \quad \int_{\gamma_2} \frac{1}{z}\, dz = 0.$$

Adding these two integrals together, we get that

$$\int_{\gamma_1} \frac{1}{z}\, dz + \int_{\gamma_2} \frac{1}{z}\, dz = 0.$$

Note that this is equivalent to the composition of these two curves. Thus, we have

$$\int_{\gamma_1 \circ \gamma_2} \frac{1}{z}\, dz = 0.$$

But note that this is also equivalent to

$$-\int_\alpha \frac{1}{z}\, dz + \int_\beta \frac{1}{z}\, dz = 0.$$

Hence, we have

$$\int_\alpha \frac{1}{z}\, dz = \int_\beta \frac{1}{z}\, dz.$$

In particular, we see by Cauchy's Integral Theorem that

$$\int_\alpha \frac{1}{z}\, dz = 2\pi i.$$

(ii) From the parametrization $\beta(t) = 3\cos(2\pi t) + i4\sin(2\pi t)$ with $t \in [0, 1]$, we can rewrite $\beta(t)$ with bounds similar to the left-hand side of our desired integral. Thus, we have

$$\beta(t) = 2\cos t + 4i \sin t \quad \text{with } t \in [0, 2\pi].$$

Using this new parametrization, we can write

$$
\begin{aligned}
\int_\beta \frac{1}{z}\ dz &= \int_0^{2\pi} \frac{-3\sin t + 4i\cos t}{3\cos t 4i \sin t}\ dt \\
&= \int_0^{2\pi} \frac{(-3\sin t + 4i\cos t)(3\cos t - 4i\sin t)}{9\cos^2 t + 16\sin^2 t}\ dt \\
&= \int_0^{2\pi} \frac{12i - 7\sin t\cos t}{9\cos^2 t + 16\sin^2 t}\ dt \\
&= i\int_0^{2\pi} \frac{12}{9\cos^2 t + 16\sin^2 t}\ dt - \int_0^{2\pi} \frac{7\sin t\cos t}{9\cos^2 t + \sin^2 t}\ dt
\end{aligned}
$$

From part (i), we notice that

$$
2\pi i = \int_\alpha \frac{1}{z}\ dz = \int_\beta \frac{1}{z}\ dz.
$$

Equating imaginary parts, we can see that

$$
\int_0^{2\pi} \frac{12}{9\cos^2 t + 16\sin^2 t}\ dz = 2\pi.
$$

Dividing by 12 on both sides, we have

$$
\int_0^{2\pi} \frac{1}{9\cos^2 t + 16\sin^2 t}\ dz = \frac{\pi}{6}
$$

which is our desired integral.

■

Solution (b). (i) Since $0 \in B(0,R)$, we can use Cauchy's Integral Formula to write

$$
\int_{\partial B(0,R)} \frac{1}{z}\ dz = 2\pi i \cdot f(0) = 2\pi i.
$$

On the other hand, we see that $R \notin B(0,R)$, so we have

$$
\int_{\partial B(0,R)} g(z)\ dz = \int_{\partial B(0,R)} \frac{2}{R-z}\ dz = 0.
$$

By the linearity of the complex integral, we have that

$$
\begin{aligned}
\int_{\partial B(0,R)} f(z)\ dz &= \int_{\partial B(0,R)} \left[\frac{1}{z} + \frac{2}{R-z}\right]\ dz \\
&= \int_{\partial B(0,R)} \frac{1}{z}\ dz + \int_{\partial B(0,R)} \frac{2}{R-z}\ dz \\
&= 2\pi i + 0 \\
&= 2\pi i.
\end{aligned}
$$

(ii) To compute the first integral, we first notice that $f(z)$ can be written in the following way

$$
f(z) = \frac{R+z}{(R-z)z}.
$$

Now, parametrizing using $\alpha(t) = re^{it}$ with $t \in [0, 2\pi]$, we can write

$$\begin{aligned}
2\pi i = \oint_{\partial B(0,R)} f(z)\ dz &= \int_0^{2\pi} \frac{R + re^{it}}{(R - re^{it})re^{it}} rie^{it}\ dt \\
&= i \int_0^{2\pi} \frac{R + re^{it}}{R - re^{it}}\ dt \\
&= i \int_0^{2\pi} \frac{R + r(\cos t + i \sin t)}{R - r(\cos t + i \sin t)}\ dt \\
&= i \int_0^{2\pi} \frac{[(R + r\cos t) + ir \sin t][(R - r\cos t) + ir \sin t]}{(R - r\cos t)^2 + r^2 \sin^2 t}\ dt \\
&= \int_0^{2\pi} \frac{i(R^2 - r^2) - 2Rr \sin t}{R^2 - 2Rr \cos t + r^2}\ dt \\
&= i \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos t + r^2}\ dt - \int_0^{2\pi} \frac{Rr \sin t}{R^2 - 2Rr \cos t + r^2}\ dt.
\end{aligned}$$

By equating the imaginary part, we see that

$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos t + r^2}\ dt = 2\pi \implies \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos t + r^2}\ dt = 1$$

which establishes the first integral.

To get the second integral, we will consider the following function

$$h(z) = \frac{1}{R - z}.$$

Using the same parametrization to establish the first integral, we write

$$\begin{aligned}
\oint_{\partial B(0,R)} h(z)\ dz = \oint_{\partial B(0,R)} \frac{1}{R - z}\ dz &= \int_0^{2\pi} \frac{rie^{it}}{R - re^{it}}\ dt \\
&= \int_0^{2\pi} \frac{ri[\cos t + i \sin t]}{R - r[\cos t + i \sin t]}\ dt \\
&= ri \int_0^{2\pi} \frac{[\cos t + i \sin t][(R - r\cos t) + ir \sin t]}{(R - r\cos t)^2 + r^2 \sin^2 t}\ dt \\
&= ri \int_0^{2\pi} \frac{R\cos t + iR \sin t - r}{R^2 - 2Rr \cos t + r^2}\ dt \\
&= i \int_0^{2\pi} \frac{Rr\cos t - r^2}{R^2 - 2Rr \cos t + r^2}\ dt - \int_0^{2\pi} \frac{Rr \sin t}{R^2 - 2Rr \cos t + r^2}\ dt
\end{aligned}$$

By part (i), we see that

$$\oint_{\partial B(0,R)} h(z)\ dz = 0.$$

By equating the imaginary part, we see that

$$\int_0^{2\pi} \frac{Rr\cos t - r^2}{R^2 - 2Rr \cos t + r^2}\ dt = 0$$

By using the linearity of the complex integral, we see that

$$\int_0^{2\pi} \frac{Rr\cos t - r^2}{R^2 - 2Rr\cos t + r^2}\ dt = 0 \Longrightarrow \int_0^{2\pi} \frac{Rr\cos t}{R^2 - 2Rr\cos t + r^2}\ dt = \int_0^{2\pi} \frac{r^2}{R^2 - 2Rr\cos t + r^2}\ dt$$

Simplifying the right-hand side of the above further, we obtain

$$\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\ dt = \frac{r}{R^2 - r^2}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\ dt.$$

Multiplying by $\frac{1}{2\pi}$ on both sides of the above, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\ dt &= \frac{r}{R^2 - r^2}\cdot \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\ dt}_{\text{Apply the first integral}} \\ &= \frac{r}{R^2 - r^2}\cdot 1 \\ &= \frac{r}{R^2 - r^2}.\end{aligned}$$

Thus, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\ dt = 1 \quad\text{and}\quad \frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\ dt = \frac{r}{R^2 - r^2}.$$

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Problem 6 (Computation of Some Integrals Using Complex Analysis-II)**.** Consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^{iz^2}$.

(i) Let $R > 0$, and define $\alpha_R : [0, \pi/4] \to \mathbb{C}$ by $\alpha_R(t) = Rr^{it}$. Show that

$$\left|\int_{\alpha_R} f(z)\ dz\right| \le \frac{\pi(1 - e^{-R^2})}{4R}.$$

(ii) Use (i) to show $\lim_{R\to\infty}\int_{\alpha_R} f(z)\ dz = 0$.

(iii) consider the line segment L_R joining 0 to to R on the real axis and the line segment C_R joining $Re^{i\pi/4}$. Show that

$$\int_{C_R} f(z)\ dz = \int_{L_R} f(z)\ dz + \int_{\alpha_R} f(z)\ dz \quad\text{and}\quad \lim_{R\to\infty}\int_{C_R} f(z)\ dz = \lim_{R\to\infty}\int_{L_R} f(z)\ dz.$$

(iv) Show that

$$\lim_{R\to\infty}\int_{L_R} f(z)\ dz = \int_0^{\infty}\cos(t^2)\ dt + i\int_0^{\infty}\sin(t^2)\ dt$$

(v) Show that $\lim_{R\to\infty}\int_{C_R} f(z)\ dz = \frac{(1+i)\sqrt{2\pi}}{4}$ and use it to show $\int_0^{\infty}\cos(t^2)\ dt = \int_0^{\infty}\sin(t^2)\ dt = \frac{\sqrt{2\pi}}{4}$.

Solution. (i) Our goal is to show that

$$\Big|\int_{\alpha_R} f(z)\ dz\Big| \leq \frac{\pi(1-e^{-R^2})}{4R}.$$

We will first show that $|f(\alpha_R(t))| = e^{-R^2\sin 2t}$. Observe that

$$\begin{aligned} f(\alpha_R(t)) = e^{i(Re^{it})^2} &= e^{iR^2e^{2it}} \\ &= e^{R^2(i\cos 2t - \sin 2t)} \\ &= e^{iR^2\cos 2t}\cdot e^{-R^2\sin 2t}. \end{aligned}$$

Furthermore, we have

$$|f(\alpha_R(t))| = |e^{iR^2\cos 2t}|\cdot|e^{-R^2\sin 2t}| = e^{-R^2\sin 2t}. \tag{1}$$

Secondly, we will show that for $0 \leq t \leq \frac{\pi}{4}$, we see that

$$e^{-R^2\sin 2t} \leq e^{\frac{-4R^2}{\pi}t}.$$

Indeed, using the fact that $\sin 2t \geq \frac{4}{\pi}t$ for all $0 \leq t \leq \frac{\pi}{4}$, we have

$$\frac{1}{e^{R^2\sin 2t}} \leq \frac{1}{e^{\frac{4R^2}{\pi}t}} \implies e^{-R^2\sin 2t} \leq e^{-\frac{4R^2}{\pi}t}. \tag{2}$$

Lastly, we see that

$$\alpha_R'(t) = Rie^{it}$$

implies

$$|\alpha_R'(t)| = |Rie^{it}| = |Ri||e^{it}| = R. \tag{3}$$

Using (1), (2), and (3), we can see that

$$\begin{aligned} \Big|\int_{\alpha_R} f(z)\ dz\Big| &= \Big|\int_0^{\frac{\pi}{4}} f(\alpha_R(t))\alpha_R'(t)\ dt\Big| \\ C &\leq \int_0^{\frac{\pi}{4}} |f(\alpha_R(t))||\alpha_R'(t)|\ dt \\ &= \int_0^{\frac{\pi}{4}} Re^{-R^2\sin 2t}\ dt \\ &\leq \int_0^{\frac{\pi}{4}} Re^{-\frac{4R^2}{\pi}t}\ dt \\ &= \frac{-\pi}{4R}\int_0^{-\frac{4R^2}{\pi}} e^u\ du && (\text{Let } u = -\tfrac{4R^2}{\pi}t) \\ &= \frac{\pi(1-e^{-R^2})}{4R}. \end{aligned}$$

Thus, we can conclude that

$$\Big|\int_{\alpha_R} f(z)\ dz\Big| \leq \frac{\pi(1-e^{-R^2})}{4R}.$$

(ii) Notice that

$$\Big|\int_{\alpha_R} f(z)\ dz\Big| \leq \frac{\pi(1-e^{-R^2})}{4R} \iff -\frac{\pi(1-e^{-R^2})}{4R} \leq \int_{\alpha_R} f(z)\ dz \leq \frac{\pi(1-e^{-R^2})}{4R}.$$

Clearly, we see that as $R \to \infty$, we have

$$\frac{\pi(1-e^{-R^2})}{4R} \to 0 \quad \text{and} \quad -\frac{\pi(1-e^{-R^2})}{4R} \to 0.$$

Using the Squeeze Theorem, we can see that

$$\lim_{R\to\infty} \int_{\alpha_R} f(z)\ dz = 0.$$

(iii) Our goal is to show that

$$\int_{C_R} f(z)\ dz = \int_{L_R} f(z)\ dz + \int_{\alpha_R} f(z)\ dz.$$

Denote the closed piecewise curve Γ formed by C_R, L_R, and α_R. We define our orientation for both L_R and α_R to be positively oriented (counter-clockwise) and C_R to be negatively oriented (clockwise). Notice that Γ is also a star-shaped region and that $f(z) = e^{iz^2}$ is holomorphic on $\mathbb{C}$, and is therefore holomorphic on Γ. Taking into account our orientation for each curve making up Γ, we can apply Cauchy's Integral Formula for Star Domains to state that

$$\Big(\int_{L_R} f(z)\ dz + \int_{\alpha_R} f(z)\ dz\Big) - \int_{C_R} f(z)\ dz = 0$$

which implies that

$$\int_{C_R} f(z)\ dz = \int_{L_R} f(z)\ dz + \int_{\alpha_R} f(z)\ dz$$

which is our desired result.

From the first equation, we can see that

$$\begin{aligned}
\lim_{R\to\infty} \int_{C_R} f(z)\ dz &= \lim_{R\to\infty} \Big[\int_{L_R} f(z)\ dz + \int_{\alpha_R} f(z)\ dz\Big] \\
&= \lim_{R\to\infty} \int_{L_R} f(z)\ dz + \lim_{R\to\infty} \int_{\alpha_R} f(z)\ dz \\
&= \lim_{R\to\infty} \int_{L_R} f(z)\ dz + 0 && \text{(part(ii))} \\
&= \lim_{R\to\infty} \int_{L_R} f(z)\ dz.
\end{aligned}$$

(iv) We will start with the right-hand side of our desired result. Observe that

$$\begin{aligned}
\int_0^\infty \cos(t^2)\ dt + i\int_0^\infty \sin(t^2)\ dt &= \int_0^\infty [\cos(t^2) + i\sin(t^2)]\ dt \\
&= \lim_{R\to\infty} \int_0^R [\cos(t^2) + i\sin(t^2)]\ dt \\
&= \lim_{R\to\infty} \int_0^R e^{it^2}\ dt \\
&= \lim_{R\to\infty} \int_0^1 Re^{i(Ru)^2}\ du \\
&= \lim_{R\to\infty} \int_{L_R} f(z)\ dz.
\end{aligned}$$

Notice that in the second to last equality, we have the parametrization of the line L_R from 0 to R. Hence, we have

$$\lim_{R\to\infty}\int_{L_R} f(z)\ dz = \int_0^\infty \cos(t^2)\ dt + i\int_0^\infty \sin(t^2)\ dt$$

(v) Note that C_R is the line segment connecting 0 to the point $Re^{\frac{\pi}{4}i}$ can be parametrized by the following function

$$C_R(t) = (Re^{\frac{\pi}{4}i})t \quad \text{with } t \in [0,1].$$

Then observe that

$$\begin{aligned}
\lim_{R\to\infty}\int_{C_R} f(z)\ dz &= \lim_{R\to\infty}\int_0^1 f(C_R(t))C_R'(t)\ dt \\
&= \lim_{R\to\infty} e^{\frac{\pi}{4}i}\int_0^1 Re^{iR^2e^{i\frac{\pi}{2}}t^2}\ dt \\
&= \lim_{R\to\infty} e^{\frac{\pi}{4}i}\int_0^1 Re^{i^2R^2t^2}\ dt \\
&= \frac{\sqrt{2}}{2}(1+i)\lim_{R\to\infty}\int_0^1 Re^{-(Rt)^2}\ dt \\
&= \frac{\sqrt{2}}{2}(1+i)\lim_{R\to\infty}\int_0^R e^{-u^2}\ du && (\text{Let } u = Rt) \\
&= \frac{\sqrt{2}}{2}(1+i)\int_0^\infty e^{-u^2}\ du \\
&= \frac{\sqrt{2}}{2}(1+i)\cdot\frac{\sqrt{\pi}}{2} \\
&= \frac{(1+i)}{4}\sqrt{2\pi}.
\end{aligned}$$

Hence, we see that

$$\lim_{R\to\infty}\int_{C_R} f(z)\ dz = \frac{(1+i)\sqrt{2\pi}}{4}.$$

From part (iii), we see that

$$\lim_{R\to\infty}\int_{C_R} f(z)\ dz = \lim_{R\to\infty}\int_{L_R} f(z)\ dz$$

and

$$\lim_{R\to\infty}\int_{L_R} f(z)\ dz = \int_0^\infty \cos(t^2)\ dt + i\int_0^\infty \sin(t^2)\ dt. \tag{*}$$

As a consequence, we have

$$\lim_{R\to\infty}\int_{L_R} f(z)\ dz = \frac{(1+i)\sqrt{2\pi}}{4} = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}.$$

Equating real and imaginary parts with (*), we see that

$$\int_0^\infty \cos(t^2)\ dt = \frac{\sqrt{2\pi}}{4} \quad \text{and} \quad \int_0^\infty \sin(t^2)\ dt = \frac{\sqrt{2\pi}}{4}.$$

■

Problem 7 (Behavior of a non-constant holomorphic function on $\mathbb{C}$). Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function.

(i) Assume that the complement of $\overline{f(\mathbb{C})}$ is nonempty, where $\overline{f(\mathbb{C})}$ is the closure of $f(\mathbb{C})$. Let $w \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$, and define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \frac{1}{f(z)-w}$. Explain why g is holomorphic. Show that g is bounded; that is, there exists $M > 0$ such that $|g(z)| \leq M$ for all $z \in \mathbb{C}$. Using Liouville's Theorem, deduce that g is constant and from this deduce that f is constant.

(ii) Show that if f is non-constant, then $\overline{f(\mathbb{C})} = \mathbb{C}$.

Proof. (i) Let $w \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$ and let

$$g(z) = \frac{1}{f(z) - w}$$

where $g : \mathbb{C} \to \mathbb{C}$. Since $w \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$, it follows that $w \neq f(z)$ for all $z \in \mathbb{C}$. Since $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, we can see that g must also be holormorphic.

Now, we will show that g is bounded; that is, we will show that there exists an $M > 0$ such that $|g(z)| \leq M$ for all $z \in \mathbb{C}$. Since $w \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$, we know that

$$\exists\, \varepsilon > 0 \text{ such that } B(w, \varepsilon) \cap f(\mathbb{C}) = \emptyset.$$

This implies that for any $y \in f(\mathbb{C})$, we must have $y \notin B(w, \varepsilon)$. That is, $|y - w| \geq \varepsilon$. In particular, for any $z \in \mathbb{C}$, we have $|f(z) - w| \geq \varepsilon$; that is,

$$\frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon}.$$

Set $M = 1/\varepsilon$. By definition of g, we must have

$$|g(z)| = \frac{1}{|f(z) - w|} \leq M.$$

So, g must be bounded. By applying Liouville's Theorem, we can see that g must be a constant function. As a consequence, $g'(z) = 0$ for all $z \in \mathbb{C}$, and so

$$g'(z) = 0 \iff \frac{-f'(z)}{(f(z) - w)^2} = 0 \iff f'(z) = 0$$

since $f(z) \neq w$ for all $z \in \mathbb{C}$. Thus, f must be a constant function as desired.

(ii) We will proceed by proving the result via contrapositive. Suppose that $\overline{f(\mathbb{C})} \neq \mathbb{C}$; that is, $f(\mathbb{C})$ is NOT dense in $\mathbb{C}$. Our goal is to show that f is constant. Since $f(\mathbb{C})$ is not dense in $\mathbb{C}$, we know that there exists an open set V in $\mathbb{C}$ such that

$$V \cap f(\mathbb{C}) = \emptyset.$$

Hence, we have that for any $w \in V$, $w \notin \overline{f(\mathbb{C})}$; that is, $w \in \mathbb{C} \setminus f(\mathbb{C})$. Note that since f is holomorphic on $\mathbb{C}$, f must be holormorphic on $V \subseteq \mathbb{C}$. Now, we see that

$$g(z) = \frac{1}{f(z) - w}$$

must both be a holormorphic and bounded function on V (In fact, it is holormorphic and bounded on $\mathbb{C}$) by part (a). Hence, g must be constant and so f must be constant as a consequence.

■

Problem 8. Compute the following limits.

(i) $\displaystyle \lim_{z \to \infty} \frac{z^3 + iz + 1}{z^3 - z}$

(ii) $\lim_{z\to\infty} \frac{e^{-i|z|}}{z^2}$.

(iii) $\lim_{z\to\infty} z\sin(1/z)$.

Solution. (i) Notice that

$$\frac{z^3+iz+1}{z^3-z} = \frac{1+\frac{i}{z^2}+\frac{1}{z^3}}{1-\frac{1}{z^2}}.$$

Hence, we can see by the Algebraic Limit Theorem for functions that

$$\lim_{z\to\infty} \frac{z^3+iz+1}{z^3-z} = \lim_{z\to\infty} \frac{1+\frac{i}{z^2}+\frac{1}{z^3}}{1-\frac{1}{z^2}} = \frac{1+0+0}{1-0} = 1.$$

(ii) Set $f(z) = \frac{e^{-i|z|}}{z^2}$ and notice that

$$f(z) = \frac{e^{-i|z|}}{z^2} = \frac{1}{e^{i|z|}z^2}.$$

Clearly, we have that as $z \to \infty$, we can see that

$$e^{-i|z|} = \frac{1}{e^{i|z|}} \to 0$$

and similarly

$$\frac{1}{z^2} \to 0.$$

Thus, the Algebraic Limit Theorem for Functions, we can see that

$$f(z) \to 0$$

as $z \to \infty$.

(iii) Set $f(z) = z\sin(1/z)$ and notice that

$$f(1/z) = \frac{\sin(z)}{z}.$$

Furthermore, computing

$$\lim_{z\to\infty} z\sin(1/z)$$

is equivalent to computing the limit

$$\lim_{z\to 0} \frac{\sin(z)}{z}. \tag{*}$$

Using a fact from real analysis, we see that the limit of (*) is just 1. Hence, i

$$\lim_{z\to\infty} z\sin(1/z) = 1.$$

■

Problem 9. Examine whether the following functions $f : D \to \mathbb{C}$ are holomorphic. If you are using certain statements, make sure to check all the conditions.

(i) $D = \mathbb{C} \setminus \{0\}$ and $f(z) = e^{1/\overline{z}}$.

(ii) $D = \{z \in \mathbb{C} : |z-5i| < 1\}$ and $f(z) = \frac{\operatorname{Log} z}{z^2}$.

(iii) $D = \mathbb{C}$ with $f(x+iy) = (x^3 - 3xy^2 + 1) + i(3x^2y - y^3 - 1)$.

(iv) $D = \{z \in \mathbb{C} : |z| \neq 1\}$ and $f(z) = \dfrac{1}{1-|z|^2}$.

Solution. (i) Let $D = \mathbb{C} \setminus \{0\}$. Note that $f : D \to \mathbb{C}$ is a composition of $g(z) = e^{1/z}$ and $h(z) = \overline{z}$. But note that $h(z) = \overline{z}$ is not complex differentiable anywhere on $\mathbb{C}$ (By handout 11). Hence, the composition $f(z) = (g \circ h)(z) = e^{1/\overline{z}}$ fails to be holomorphic on D. Indeed, for any $z = x + iy \in \mathbb{C}$, we have $\overline{z} = x - iy$. Immediately, we see that the $h(z) = \overline{z}$ does not satisfy the Cauchy-Riemann equations; that is,

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}.$$

Therefore, $f(z)$ cannot be complex differentiable for all $z \in D$.

(ii) Let $D = \{z \in \mathbb{C} : |z - 5i| < 1\}$ and $f(z) = \frac{\mathrm{Log}(z)}{z^2}$. Since D does not include the negative real axis and 0, we know by problem 8 in Homework 4 that $\mathrm{Log}(z)$ is complex differentiable as well as $\frac{1}{z^2}$. Indeed, for any $z \in D$, we have that

$$\begin{aligned}
\frac{d}{dz}\left[\frac{1}{z^2}\right]\Bigg|_{z=c} = \lim_{z\to c} \frac{\frac{1}{z^2} - \frac{1}{c^2}}{z - c} &= \lim_{z \to c} \left[\frac{c^2 - z^2}{z^2c^2} \cdot \frac{1}{z - c}\right] \\
&= \lim_{z \to c} \frac{-(c+z)}{z^2c^2} \\
&= \frac{-2c}{c^4} \\
&= \frac{-2}{c^3}.
\end{aligned}$$

Hence, the product of two complex differentiable functions is complex differentiable by Problem 2 of Homework 4. Thus, we have

$$f(z) = \frac{\mathrm{Log}(z)}{z^2}$$

is complex differentiable and thus, holomorphic on D.

(iii) Let $D = \mathbb{C}$ and

$$f(x + iy) = (x^3 - 3xy^2 + 1) + i(3x^2y - y^3 - 1).$$

We will proceed by showing that the function f satisfies the Cauchy-Riemann Equations. Notice that real and imaginary parts of f are both differentiable and have continuous partial derivatives. Now, set

$$u(x, y) = x^3 - 3xy + 1 \quad \text{and} \quad v(x, y) = 3x^2y - y^3 - 1.$$

Observe that

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = -6xy \tag{1}$$

and

$$\frac{\partial v}{\partial x} = 6xy \quad \text{and} \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2. \tag{2}$$

Clearly, (1) and (2) imply that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, we conclude that f is complex differentiable on D and so f must be holomorphic on D.

(iv) Let $D = \{z \in \mathbb{C} : |z| \neq 1\}$ and $f(z) = \frac{1}{1-|z|^2}$. Notice that $z = x + iy$ implies that $\overline{z} = x^2 + y^2$. Now, we have

$$f(z) = \frac{1}{1 - |z|^2} = \frac{1}{1 - (x^2 + y^2)}.$$

Clearly, we can see that the real part

$$\Re(f(z)) = \frac{1}{1-(x^2-y^2)}$$

while the imaginary part

$$\Im(z) = 0.$$

If we label $u(x,y) = \Re(f(z))$ and $v(x,y) = \Im(f(z)) = 0$, we immediately have

$$\frac{\partial u}{\partial x} = 2x(1-(x^2+y^2))^{-2}$$

but

$$\frac{\partial v}{\partial y} = 0 \Longrightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

Similarly,

$$\frac{\partial u}{\partial y} = 2y(1-(x^2+y^2))^{-2}$$

but

$$\frac{\partial v}{\partial x} = 0 \Longrightarrow \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}.$$

Thus, f does not satisfy the Cauchy-Riemann Equations and so f cannot be complex differentiable on D. Therefore, f cannot be holomorphic on D as well.

$\blacksquare$