0.1 Lecture 3

0.1.1 Topics

- Polar Representation of complex numbers.
- Convergence of sequences in $\mathbb{C}$.

0.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^{\cdot} = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_{+} = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos\varphi, \sin\varphi)$$

with $\tan\varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos\psi, \sin\psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_{+} \times \mathbb{R} \to \mathbb{C}^{\cdot}$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^{\cdot}$, then

$$z = \gamma(\cos\varphi + i\sin\varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument)**.** Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\operatorname{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\ &= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\ &= \gamma\gamma'(\cos(\varphi+\varphi') + i\sin(\varphi+\varphi')). \end{aligned}$$

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The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^{\cdot}$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi). \end{aligned}$$

Corollary (De Moivre's Theorem)**.** Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example 0.1.1 (An example of De Moivre's Theorem)**.** Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i\sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

0.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have
$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$
Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have
$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$
Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have
$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
Thus, we can conclude that
$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that
$$|z_n w_n - zw| < \varepsilon.$$
Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have
$$|z_n - z| < \frac{\varepsilon}{2M}$$
where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have
$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\
&\leq |w_n||z_n - z| + |z||w_n - w| \\
&\leq M|z_n - z| + |z||w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we conclude that

$$z_n w_n \to zw.$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon|z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned}
|z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\
&< \frac{\varepsilon|z|^2}{2} \cdot \frac{2}{|z|^2} \\
&= \varepsilon.
\end{aligned}$$

Thus, we conclude that $z_n^{-1} \to z^{-1}$.

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