0.0.1 Topics

(1) Every neigborhood is an open set.

(2) If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

(3) Theorem: E is open $\iff E^C$ is closed.

(4) Theorem: arbitrary union of open sets is open, Finite intersection of open sets is open.

(5) Theorem: $\overline{E}$ is the smallest closed subset of X that contains E.

Theorem. Let (X, d) be a metric space and let $p \in X$ and $\varepsilon > 0$. Every neighborhood is an open set; that is, $N_\varepsilon(p)$ is an open set.

Proof. Our goal is to show that every point of $N_\varepsilon(p)$ is an interior point of $N_\varepsilon(p)$. Let $q \in N_\varepsilon(p)$. We need to show that there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$. Let $\delta = \frac{\varepsilon - d(p,q)}{2}$. We claim that $N_\delta(q)$ is a subset of $N_\varepsilon(p)$. Indeed, if $x \in N_\delta(q)$, then

$$d(q, x) < \delta \implies d(q, x) < \varepsilon - d(p, q)$$

and so

$$d(p, q) + d(q, x) < \varepsilon \iff d(p, x) < \varepsilon. \qquad \text{(triangle inequality)}$$

Thus, $x \in N_\varepsilon(p)$.

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Theorem. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary. A finite set has no limit points.

Theorem.

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