Math 241 Homework 2

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Remark. In the first two problems of this homework, whenever I state $x_i^{(k)} \to x_i$, I mean whenever we let $k \to \infty$.

Problem 1. Prove that $(\mathbb{R}^n, d_\infty)$ is complete.

Proof. Let $(\vec{x_k})$ be a Cauchy sequence in $\mathbb{R}^n$. Note that $1 \leq i \leq n$ denotes the ith component of elements in $\mathbb{R}^n$ and $k \in \mathbb{N}$ is the index for each sequence in $\mathbb{R}^n$. By a result found in quiz 1, it follows that $(x_i^{(k)})$ for $1 \leq i \leq n$ is also Cauchy. Since $\mathbb{R}$ is a complete metric space with respect to the standard metric on $\mathbb{R}$, we find that each $x_i^{(k)}$ is also a convergent sequence. By another result in quiz 1, it follows that $(\vec{x_k})$ is a convergent sequence; that is, for each $1 \leq i \leq n$, $x_i^{(k)} \to x_i$ where $x_i \in \mathbb{R}$. Clearly, we have

$$\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

Hence, $\mathbb{R}^n$ with respect to the d_∞ metric is complete. ■

Problem 2.

(i) Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Prove that $\pi_i : \mathbb{R}^n \to \mathbb{R}$ defined by $\pi_i(\vec{x}) = x_i$ are continuous maps with respect to d_∞ on $\mathbb{R}^n$ and the standard metric on $\mathbb{R}$.

(ii) Prove that π_i in (i) are continuous maps with respect to d_{euclid} on $\mathbb{R}^n$ and the standard metric on $\mathbb{R}$.

Proof.

(i) Our goal is to show that π_i is a continuous map with respect to d_∞ on $\mathbb{R}^n$; we will do this via the sequential criterion of continuity. Suppose $\vec{x_k} \to \vec{x}$ for some $\vec{x}$ in $\mathbb{R}^n$. By a result found in quiz 1, we can see that $x_i^{(k)} \to x_i$ for $1 \leq i \leq n$. By definition of π_i, we find that as $k \to \infty$, we get

$$\pi_i(\vec{x_k}) = x_i^{(k)} \to x_i = \pi_i(\vec{x}).$$

Hence, we have that π_i is a continuous map with respect to d_∞ and the standard metric on $\mathbb{R}$.

(ii) Our goal is to show that $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to d_{euclid}. Let $(\vec{x_k})$ be a sequence in $\mathbb{R}^n$ such that $\vec{x_k} \to \vec{x}$. Since π_i is continuous on $\mathbb{R}^n$ with respect to the d_∞ metric, we have $d_\infty(\vec{x_k}, \vec{x}) \to 0$. Notice that

$$0 \leq d_{\text{euclid}}(\vec{x_k}, \vec{x}) \leq (n)^{1/2} d_\infty(\vec{x_k}, \vec{x}). \tag{1}$$

Now, $d_\infty(\vec{x_k}, \vec{x}) \to 0$ implies that $d_{\text{euclid}}(\vec{x_k}, \vec{x}) \to 0$ as $k \to \infty$ by applying the squeeze theorem to (1). Thus, we can see that

$$\begin{aligned}|\pi_i(\vec{x_k}) - \pi_i(\vec{x})| &= \Big(|\pi_i(\vec{x_k}) - \pi_i(\vec{x})|^2\Big)^{1/2} \\ &= \Big(|x_i^{(k)} - x_i|^2\Big)^{1/2} \\ &\le \Big(\sum_{i=1}^{n} |x_i^{(k)} - x_i|^2\Big)^{1/2} \\ &= d_\infty(\vec{x_k}, \vec{x}) \to 0.\end{aligned}$$

Hence, we conclude that

$$|\pi_i(\vec{x_k}) - \pi_i(\vec{x})| \to 0$$

and so π_i is continuous map with respect to d_∞ and the standard metric on $\mathbb{R}$.

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Problem 3. (i) Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |e^x - e^y|$. Prove that d is a metric on $\mathbb{R}$.

(ii) Prove or disprove: $(\mathbb{R}, d)$ is complete.

Proof. (i) (I) It follows immediately that $d(x, y) > 0$ by the way d is defined. Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned} d(x, y) = 0 &\iff |e^x - e^y| = 0 \\ &\iff e^x = e^y && \text{(standard metric on } \mathbb{R}) \\ &\iff \ln(e^x) = \ln(e^y) \\ &\iff x = y. \end{aligned}$$

Hence, property (i) is satisfied.

(II) We have

$$d(x, y) = |e^x - e^y| = |e^y - e^x| = d(y, x).$$

Hence, property (ii) is satisfied.

(III) Let $x, y, z \in \mathbb{R}$. Then we have

$$\begin{aligned}|e^x - e^y| &= |e^x - e^z + e^z - e^y| \\ &\le |e^x - e^z| + |e^z - e^y| \\ &= d(x, z) + d(z, y).\end{aligned}$$

Hence, we have $d(x, y) \le d(x, z) + d(z, y)$.

From the properties above, we conclude that $d(x, y) = |e^x - e^y|$ does indeed define a metric on $\mathbb{R}$.

(ii) We claim that the metric defined above does NOT make $\mathbb{R}$ complete. Define the sequence $x_n = \ln\left(\frac{1}{n}\right)$. It follows immediately with respect to d that (x_n) is a Cauchy sequence in $\mathbb{R}$. Indeed, let $\varepsilon > 0$. Since $1/n$ is a Cauchy sequence in $\mathbb{R}$ with respect to the standard metric, there exists an $N \in \mathbb{N}$ such that for any $n, m \ge N$, we have

$$d(x_n, x_m) = |e^{\ln(1/n)} - e^{\ln(1/m)}| = \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon.$$

Since $\mathbb{R}$ is complete with respect to the standard metric, we can see that $\frac{1}{n} \to 0$, but there does not exists an $x \in \mathbb{R}$ such that $x = \ln(0)$. Hence, $\mathbb{R}$ cannot be complete with the metric defined above.

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Problem 4. Let $X = \mathbb{N}$ be the set of positive integers.

(i) Let $d(m, n) = |m - n|$. Prove that (X, d) is complete.

(ii) Let $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$. Prove that (X, d) is not complete.

Proof. (i) Let (x_n) be a Cauchy sequence in $\mathbb{N}$. Let $\varepsilon > 0$. Our goal is to find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|x_n - x| < \varepsilon$$

where $x \in \mathbb{N}$. Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence in $\mathbb{N}$, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n, m \geq \hat{N}$

$$|x_n - x_m| < \varepsilon.$$

Note that since $x_n, x_m \in \mathbb{N}$, we can use $\hat{N}$ as the same N we were looking for. Indeed, if we fix $m > \hat{N}$ and suppose that for any $n \geq \hat{N}$, we have

$$|x_n - x_m| < \varepsilon.$$

Then clearly, $x_n \to x$ (where $x = x_m \in \mathbb{N}$, in this case). Hence, we conclude that $(x_n) \to x$.

We claim that (X, d) is not complete with respect to $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$. Consider the sequence $x_n = n$. With respect to the metric above, we can see that (x_n) is a Cauchy sequence in $\mathbb{N}$. Indeed, let $\varepsilon > 0$. Using the Archimedean Property, we can find an $\hat{N}$ such that

$$\frac{1}{\hat{N}} < \frac{\varepsilon}{2}.$$

If we let $n, m \geq \hat{N}$, we have

$$\frac{1}{n} \leq \frac{1}{\hat{N}} < \frac{\varepsilon}{2} \tag{1}$$

$$\frac{1}{m} \leq \frac{1}{\hat{N}} < \frac{\varepsilon}{2}. \tag{2}$$

Using (1) and (2) along with the triangle inequality, we can see that

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{2}{\hat{N}} < \varepsilon.$$

Hence, (x_n) is a Cauchy sequence, but $x_n \to 0$ (with respect to the metric above) where 0 is clearly not in $\mathbb{N}$. Hence, $\mathbb{N}$ cannot be complete with the above metric. ∎

Problem 5. Let $X = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$.

(i) Define $d(f, g) = \displaystyle\int_0^1 |f(t) - g(t)|\, dt$. Prove that d is a metric on X. Prove that d is a metric on X.

(ii) Prove that (X, d) is not complete.

Proof. (i) Our goal is to show that d is a metric on X.

(I) Let $f, g \in X$. Using the fact that the standard metric $|\cdot|$ is a nonnegative continuous function on $[0, 1]$ along with the fact given to us, it follows that

$$\int_0^1 |f(t) - g(t)|\, dt = 0 \Longleftrightarrow |f(t) - g(t)| = 0.$$

Thus, we have that

$$\begin{aligned}
d(f,g) = 0 &\iff \int_0^1 |f(t) - g(t)| \; dt \\
&\iff |f(t) - g(t)| = 0 \\
&\iff f(t) = g(t) && (|\cdot| \text{ is a metric on } \mathbb{R})
\end{aligned}$$

(II) Observe that for any $f, g \in X$, we see that

$$d(f,g) = \int_0^1 |f(t) - g(t)| \; dt = \int_0^1 |g(t) - f(t)| \; dt = d(g,f).$$

(III) Let $f, g, h \in X$. Then by the triangle inequality of the standard metric of $\mathbb{R}$ and the linearity of integral, we have

$$\begin{aligned}
d(f,g) &= \int_0^1 |f(t) - g(t)| \; dt \\
&\leq \int_0^1 \Big(|f(t) - h(t)| + |h(t) - g(t)| \Big) \; dt \\
&= \int_0^1 |f(t) - h(t)| \; dt + \int_0^1 |h(t) - g(t)| \; dt \\
&= d(f,h) + d(h,g).
\end{aligned}$$

Hence, property (III) is satisfied.

Thus, we conclude that d does indeed define a metric on X.

(ii) Our goal is to construct a sequence (x_n) that is Cauchy in X, but it does not converge in X. Based on the area of the triangle, which is represented by $d(x_m, x_n)$, found in figure 10 of the book, we can define $N = \frac{1}{\varepsilon}$ such that for any $m, n \geq N$, we have

$$d(x_n, x_m) < \varepsilon.$$

As a consequence, we can see that (x_n) is a Cauchy sequence. Now, we want to show that (x_n) does not converge in X. Suppose for sake of contradiction that (x_n) does converge in X. Suppose $x \in X$. Indeed, if we define

$$x_m(t) = 0 \text{ if } t \in [0, 1/2]$$

and

$$x_m(t) = 1 \text{ if } t \in [a_m, 1]$$

where $a_m = \frac{1}{2} + \frac{1}{m}$, we see that

$$\begin{aligned}
d(x_m, x) &= \int_0^1 |x_m(t) - x(t)| \; dt \\
&= \int_0^{1/2} |x(t)| \; dt + \int_{1/2}^{a_m} |x_m(t) - x(t)| \; dt + \int_{a_m}^1 |1 - x(t)| \; dt.
\end{aligned}$$

By our integration properties, we can see that

$$d(x_m, x) = \int_0^{1/2} |x(t)| \; dt + \int_{1/2}^{a_m} |x_m(t) - x(t)| \; dt + \int_{a_m}^1 |1 - x(t)| \; dt.$$

Since each corresponding integrand above is nonnegative, we can see that each integral on the right-hand side is nonnegative. Since $d(x_m, x) \to 0$ (by assumption), we can see that each integral on the right-hand side above approaches zero. Since $x(t)$ is a continuous function for all $t \in [0,1]$, we have that

$$x(t) = \begin{cases} 0 & \text{if } t \in [0, 1/2) \\ 1 & \text{if } t \in (1/2, 1]. \end{cases}$$

But note that $x(t)$ cannot be continuous; that is, $x \notin X$. The reason is as follows: if we take a sequence (t_n) in the interval $[0, 1/2)$ (that is, take a sequence from the left side), then we see that $x(t_n) \to 0$. However, if we take a sequence (r_n) in the interval $(1/2, 1]$ (that is, take the right-handed limit), then $x(r_n) \to 1$. By the sequential criterion of continuity, we see immediately that $x(t)$ cannot be continuous. Hence, we have $x_m(t)$ converges to a limit that does not belong to X. Hence, we conclude that X cannot be a complete metric space with the metric d defined above.

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