Math 230A: Homework 6

Lance Remigio

October 17, 2024

Problem 1. In class we proved that every $2-$cell is a compact subset of $\mathbb{R}^2$. Use the same procedure to prove that every closed and bounded interval $[a, b]$ (that is, every $1-$cell) is a compact subset of $\mathbb{R}$.

Problem 2. Complete the proof of Theorem 2.41, that is, let $E \subseteq \mathbb{R}^k$ and prove that if every infinite subset of E has a limit point in E, then E is closed and bounded.

Problem 3. Give an example of a metric space (Y, d) and a set $E \subseteq Y$ where E is closed and bounded but not compact.

Problem 4. Prove that a discrete space is **totally disconnected**. That is, prove that in a metric space equipped with the discrete metric, the only subsets are singletons and $\emptyset$.

Problem 5. Let $E \subseteq \mathbb{R}$. Prove that E is connected if and only if it has the following property. If $x \in E$ and $y \in E$, and $x < z < y$, then $z \in E$.

Problem 6. Let (X, d) be a metric space. Let A and B be two connected subsets of X. Prove that if $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Problem 7. Recall that in class we mentioned that path-connectedness implies connectedness. Hence, every circle in $\mathbb{R}^2$ is a connected subset of $\mathbb{R}^2$. Use this fact to show that intersection of two connected sets in $\mathbb{R}^2$ is not necessarily connected.

Problem 8. (a) If A and B are disjoint closed sets in some metric space (X, d), prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable. (Hint: Use (c)).

Problem 9. Recall that in class we mentioned that implies connectedness. Hence, every closed disk in $\mathbb{R}^2$ is a connected subset of $\mathbb{R}^2$. Use this fact to show that the interior of a connected set in $\mathbb{R}^2$ is not necessarily connected.

Problem 10. Let (X, d) be a metric space. Suppose that $E \subseteq X$ is connected. Prove that $\overline{E}$ is connected.

Problem 11. Let (X, d) be a metric space. Prove that X is connected if and only if $\emptyset$ are the only subsets of X which are both open and closed.

Problem 12. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E compact? Is E perfect? (For this problem, you may use the fact that if $x = 0.x_1 x_2 \cdots x_n \cdots$ is an element of E, then $x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$).

Problem 13. Let (X, d) be a metric space.

(a) A subset of $A \subseteq X$ is called **nowhere dense in** X if the interior of the closure of A is empty, that is $(\overline{A})^\circ = \emptyset$. Prove that A is nowhere dense if and only if A^c contains a dense open set in X.

(b) Prove that if $\mathbb{R}^k = \bigcup_{n=1}^\infty F_n$ where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n is NOT nowhere dense.