1 Lecture 20-21

1.1 Topics

- Infinite series (Basic definitions)
- Telescoping Series, Geometric Series
- Algebraic Limit Theorem for Series
- Divergence Test
- Cauchy Criterion for Series
- Absolute Convergence Test

Consider the following expression:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots .$$

How can we make sense of the infinite sum above? More generally, let (a_n) be a sequence of real numbers. Then what does the following expression mean?

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots ?$$

Definition (Infinite Series). Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X.

(*) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1+x_2+x_3+\cdots$$

is called an **infinite series**.

(*) $x_1, x_2, \dots$ are called the **terms** of this infinite series.

(*) The corresponding sequence of **partial sums** is defined by

$$\forall m \in \mathbb{N} \quad s_m = \text{(finite) sum of the first } m \text{ terms of the series;}$$

that is,

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1+x_2 \\ s_3 &= x_1+x_2+x_3 \\ &\vdots \\ s_m &= x_1+x_2+\cdots+x_m \\ &\vdots \end{aligned}$$

1. We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty} x_n = L$) if $\lim_{n\to\infty} s_m = L$.
2. We say that the infinite series **diverges**, if (s_m) diverges.
3. If $X=\mathbb{R}$ and $s_m \to \infty$, we write $\sum_{n=1}^{\infty} x_n = \infty$.

(*) If $X=\mathbb{R}$ and $s_m \to -\infty$, we write $\sum_{n=1}^{\infty} x_n = -\infty$.

Remark (1). Given an infinite series $\sum_{n=1}^{\infty} x_n$, it is important to keep a clear distinction between

(a) the sequence of terms: $(x_1, x_2, x_3, \dots)$

(b) the sequence of partial sums: $(s_1, s_2, s_3, \dots)$.

Remark (2). We may sometimes consider infinite series where the summation begins with $n = 0$ or $n = n_0$ for some integer n_0 different from 1.

As we shall see, some of our theorems apply specifically to series in $\mathbb{R}$ or to series with terms in $[0, \infty)$. Also, in our examples, we will primarily focus on series in $\mathbb{R}$; however, we will also consider encounter highly useful theorems that hold in more general normed spaces.

In most cases it is difficult (or even impossible) to find a simple formula for the partial sum s_m. However, there are two types of series for which we can easily find a simple formula for the partial sums. These two types are:

(1) Telescoping Series

(2) Geometric Series

1.2 Telescoping Series

Example. Consider the following series

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Notice that $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned}
s_1 &= 1 - \frac{1}{2} \\
s_2 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) = 1 - \frac{1}{3} \\
s_3 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) = 1 - \frac{1}{4} \\
&\vdots \\
s_m &= \sum_{n=1}^{m} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = \Big(\sum_{n=1}^{m} \frac{1}{n}\Big) - \Big(\sum_{n=1}^{m} \frac{1}{n+1}\Big) \\
&= 1 - \frac{1}{m+1}.
\end{aligned}$$

Clearly, we see that

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \Big[1 - \frac{1}{m+1}\Big] = 1.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

In general, a telescoping series is an infinite series where partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n - y_{n+1})$ is a telescoping series; that is,

$$\begin{aligned}
s_m = \sum_{n=1}^{m}(y_n - y_{n+1}) = \Big(\sum_{n=1}^{m} y_n\Big) - \Big(\sum_{n=1}^{m} y_{n+1}\Big) &= [y_1 + y_2 + \cdots + y_m] - [y_2 + y_3 + \cdots + y_{m+1}] \\
&= y_1 - y_m.
\end{aligned}$$

1.3 Geometric Series

Let k be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \cdots$ is called a **geometric series** with common ration "r". For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ is a geometric series with common ratio } \frac{1}{2}.$$

Another example is that

$$\sum_{n=1}^{\infty} \frac{7^n}{29^n} \text{ is a geometric series with common ratio } \frac{7}{29}.$$

A non-example is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We can easily find a formula for the mth partial sum of $\sum_{n=k}^{\infty} r^k$ where

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k + r^{k+1} \\ s_3 &= r^k + r^{k+1} + r^{k+2} \\ &\vdots \\ s_m &= r^k + r^{k+1} + \cdots + r^{k+m-1} \end{aligned} \tag{*}$$

Now, if $r = 1$, we have

$$s_m = \underbrace{1 + 1 + \cdots + 1}_{m\text{summands}} = m.$$

If $r \neq 1$, then multiply both sides of (*) by r:

$$rs_m = r^{k+1} + r^{k+2} + \cdots + r^{k+m}. \tag{**}$$

Subtracting (**) from (*), we get

$$s_m - rs_m = r^k - r^{k+m}.$$

Since $r \neq 1$, we have

$$s_m = \frac{r^k - r^{k+m}}{1-r} = \frac{r^k(1-r^m)}{1-r}.$$

Note that

(i) If $|r| < 1$, then $\lim r^m = 0$.

(ii) If $|r| > 1$ or $r = -1$, then $\lim_{n\to\infty} r^m$ does not exists.

Hence, we have

$$\lim_{m\to\infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{DNE} & \text{if} |r| \geq 1. \end{cases}$$

Thus,

$$\sum_{n=k}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{diverges} & \text{if} |r| \geq 1. \end{cases}$$

Example. • $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Observe that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^1}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

- $\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n$

  Observe that

$$\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)} = \frac{1}{8}.$$

Theorem. Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) and (b_n) be two sequence in X. Suppose that

$$\sum_{n=1}^{\infty} a_n = A \quad (A \in X), \quad \sum_{n=1}^{\infty} b_n = B \quad (B \in X).$$

Then

(i) For any scalar λ, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$.

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$.

Proof. Can easily be proven via the Algebraic Limit Theorem for Sequences. ■

Theorem (Divergence Test)**.** Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $s_n = x_1 + \cdots + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \Longrightarrow \lim_{n\to\infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Note that $\lim s_n = L$ and $\lim s_{n-1} = L$. Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(s_n - s_{n-1}) = L - L = 0$$

by the Algebraic Limit Theorem for normed spaces. ■

Remark. Note that the divergence test is just the contrapositive of the above.

Example.

- $\sum_{n=1}^{\infty}(-1)^n$ diverges because $\lim_{n\to\infty}(-1)^n$ does not exist.
- $\sum_{n=1}^{\infty}\frac{3n+1}{7n-4}$ diverges because $\lim_{n\to\infty}\frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

From the above statements, we can now see make two key observations:

- If $\lim_{n\to\infty} x_n = 0$, then $\sum_{n=1}^{\infty} x_n$ may or may not converge.
- If $\lim_{n\to\infty} x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ diverges.

As for the first observation above, we see that $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^2}$ converges.

Theorem (Cauchy Criterion)**.** Let $(X, \|\cdot\|)$ be a complete normed space. Let (x_n) be a sequence in

X. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges } \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=1}^{n} x_k\| < \varepsilon.$$

Proof. Let $s_n = x_1 + \cdots + x_k$. Assuming that

$$\|s_n - s_m\| = \Big\| \sum_{k=m+1}^{n} x_k \Big\|$$

where $n > m$ and from the fact that

$$\begin{aligned} s_n - s_m &= (x_1 + \cdots + x_m + \cdots + x_n) - (x_1 + \cdots + x_m) \\ &= \sum_{k=m+1}^{n} s_k. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges} \\ &\iff (s_k) \text{ is Cauchy} \\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=m+1}^{n} x_k\| < \varepsilon \end{aligned}$$

as desired. ■

From here, we will refer to complete normed spaces as Banach spaces.

Theorem (Absolute Convergence Test)**.** Let $(X, \|\cdot\|)$ be a Banach Space. Let (x_n) be a sequence in X. If $\underbrace{\sum_{n=1}^{\infty} \|x_n\|}_{\text{a sum in } \mathbb{R}}$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for series, it suffices to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \Big\| \sum_{k=m+1}^{n} x_k \Big\| < \varepsilon. \tag{*}$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for series that there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \Big| \sum_{k=m+1}^{n} \|x_k\| \Big| < \varepsilon.$$

We claim that $\hat{N}$ is the same N we were looking for. Hence, if $n > m > \hat{N}$, then we have

$$\Big\| \sum_{k=m+1}^{n} x_k \Big\| \leq \sum_{k=m+1}^{n} \|x_k\| = \Big| \sum_{k=m+1}^{n} \|x_k\| \Big| < \varepsilon$$

as desired. ■

Please take note of the following observations:

(1) If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges (in Banach spaces).

(2) If $\sum_{n=1}^{\infty} \|x_n\|$ diverges, then $\sum_{n=1}^{\infty} x_n$ may converge or diverge.

From (2), we shall see (in the next lecture) that

(1) $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right|$ diverges but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

(2) $\sum_{n=1}^{\infty} |(-1)^n|$ diverges, also $\sum_{n=1}^{\infty}(-1)^n$ diverges (by the divergence test).

Definition (Absolute Convergence and Conditional Convergence)**.** We say that a series $\sum x_n$ **absolutely converges** if $\sum \|x_n\|$ converges and $\sum x_n$ converges. We say that $\sum x_n$ **conditionally converges** if $\sum \|x_n\|$ diverges but $\sum x_n$ converges.

Example (Conditionally Convergent)**.** Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We see that this series is conditionally convergent since

$$\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \to 0$$

which tell us that the above series diverges by the divergence test. But the above series converges via the Leibniz Test (As we shall see in the next lecture).

2 Lecture 21-22

2.1 Topics

- Cauchy Condensation Test
- Comparison Test
- More on lim sup and lim inf
- Root Test
- Ratio Test
- Dirichlet's Test