# Math 230A Lecture Notes

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# Chapter 1

# Week 1

## 1.1 Lecture 1

### 1.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

### 1.1.2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

### 1.1.3 The First Defining Property

The set of real numbers is a field.

**Definition** (Fields). A field is a set $F$ with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

### 1.1.4 The Second Defining Property

**Definition** (Ordered Fields)**.** An **ordered field** is a field $F$ equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

**Remark.** We say that $x$ is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

**Definition** (Upper Bounds)**.** Suppose $F$ is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call $\beta$ an **upper bound of** $A$.

**Remark.** We call the collection of upper bounds of $A$ by $\text{UP}(A)$. If $\text{UP}(A) \neq \emptyset$, then we say that $A$ **is bounded above**.

Similarly, we define the lower bounds of a set.

**Definition** (Lower Bounds)**.** Suppose $F$ is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then $\alpha$ is called the **lower bound of** $A$.

**Remark.** Similarly, we denote the set of lower bounds of $A$ by $\text{LO}(A)$. We say that $A$ is bounded below if $\text{LO}(A) \neq \emptyset$.

**Example.** Suppose we have $A = [0, 1)$. We have

$$\mathrm{UP}(A) = [1, \infty)$$
$$\mathrm{LO}(A) = (-\infty, 0].$$

## 1.2 Lecture 2

### 1.2.1 Review of Least Upper Bound Property

**Definition** (Supremum)**.** Suppose $F$ is an ordered field, and $A \subseteq F$. Suppose there exists $\beta \in F$ such that

(i) $\beta \in \mathrm{UP}(A)$

(ii) If $\gamma \in F$ and $\gamma < \beta$, then $\gamma \notin \mathrm{UP}(A)$.

We call $\beta$ the **least upper bound** of $A$ or the **supremum** of $A$. We denote the supremum of $A$ as $\beta = \sup A$.

**Remark.** When we say THE supremum, we are implicitly stating that the supremum of $A$ is unique.

**Definition** (Infimum)**.** Suppose $F$ is an ordered field, and $A \subseteq F$. Suppose there exists $\alpha \in F$ such that

(i) $\alpha \in \mathrm{LO}(A)$

(ii) If $\gamma \in F$ and $\gamma > \alpha$, then $\gamma \notin \mathrm{LO}(A)$.

We call $\alpha$ the **greatest upper bound** of $A$ or the **infimum** of $A$, and write $\alpha = \inf A$.

**Definition** (Least Upper Bound Property)**.** An ordered field $F$ is said to have the **least-upper-bound property** if the following is true:

Every nonempty set $A$ in $F$ that is bounded above has a least upper bound in $F$.

That is, if $A \neq \emptyset$ and $\mathrm{UP}(A) \neq \emptyset$, then $\sup(A)$ exists.

**Theorem.** There is exactly one ordered field that has the least-upper-bound bound property. The set $\mathbb{R}$ is the unique ordered field that contains $\mathbb{Q}$ as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$ is dedekind complete
- $\mathbb{R}$ satisfies the Axiom of Completeness.

**Remark.** Note that $\mathbb{Q}$ being an ordered field does not immediately imply that $\mathbb{Q}$ has the LUBP.

**Definition** (Maximums and Infimums)**.** Let $A \subseteq \mathbb{R}$.

- If $\sup A \in A$, then we call, $\sup A$, the **maximum of** $A$ and we denote this by $\max A$.
- If $\inf A \in A$, we call, $\inf A$, the **minimum of** $A$ and we denote this by $\min A$.

**Lemma** (Useful Fact for Supremum)**.** Let $A \subseteq \mathbb{R}$. Then $\beta = \sup A$ if and only if

(i) $\beta \in \mathrm{UP}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a > \beta - \varepsilon$.

**Remark.** We can restate property (ii) above as "for all $\varepsilon > 0$, $\beta - \varepsilon \notin \mathrm{UP}(A)$".

**Lemma** (Useful Fact for Infimums). Let $A \subseteq \mathbb{R}$. Then $\alpha = \inf A$ if and only if

(i) $\alpha \in \mathrm{LO}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a < \alpha + \varepsilon$.

**Remark.** Similarly, we can restate property (ii) as "for all $\varepsilon > 0$, $\alpha + \varepsilon \notin \mathrm{LO}(A)$".

**Theorem** (Greatest Lower Bound Property of $\mathbb{R}$). Every nonempty subset $A$ of $\mathbb{R}$ that is bounded below has a **greatest upper bound in** $\mathbb{R}$.

Another way to say this is the following:

$$\text{If } A \neq \emptyset \text{ and } \mathrm{LO}(A) \neq \emptyset, \text{ then } \inf A \text{ exists in } \mathbb{R}.$$

### 1.2.2 Consequences of Least Upper Bound Property

**Theorem** (Archimedean Property). If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.

**Proof.** Let $A = \{nx : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ since $1 \cdot x \in A$. Suppose for sake of contradiction that for all $n \in \mathbb{Z}^+$, $nx \leq y$. This means that $y$ is an upper bound of $A$. Let $\beta = \sup A$. By the first useful fact, we have that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\beta - \varepsilon < nx$. Let $\varepsilon = x$. Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \implies \beta < x(n+1).$$

But this tells us that $x(n+1) \in A$ ($x \in A$ and $n+1 \in \mathbb{N}$) and that $\beta$ is NOT an upper bound which is a contradiction. Thus, it must be the case that $nx > y$ for some $n \in \mathbb{Z}^+$. ■

**Remark.** The well ordering property of $\mathbb{N}$ can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

**Corollary.** Let $A$ be a nonempty subset of $\mathbb{R}$ that consists of only integers.

(i) If $A$ is bounded above, then $\sup(A) \in A$.

(ii) If $A$ is bounded below, then $\inf(A) \in A$.

**Theorem** (Density of $\mathbb{Q}$ in $\mathbb{R}$). Let $x, y \in \mathbb{R}$ with $x < y$, there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

**Proof.** Our goal is to find a $p \in \mathbb{Q}$ such that

$$x < p < y$$

with $p = \frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; that is, find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$nx < m < ny.$$

First, notice that $x < y$. This implies that $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \tag{1}$$

Choose $m \in \mathbb{Z}$ such that $m$ to be the minimum element greater than $nx$; that is, choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m. \tag{2}$$

Let $A = \{k \in \mathbb{Z} : k > nx\}$ which is nonempty by the Archimedean Property. Furthermore, $nx$ is a lower bound for $A$. By the Well-ordering property, $A$ contains a minimum. Thus, $m = \min A$. Hence,we have

$$nx < m \implies x < \frac{m}{n}. \tag{3}$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \leq nx \implies m \leq nx + 1 &< n\Big(y - \frac{1}{n}\Big) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \tag{4}$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \iff x < p < y.$$

■

# Chapter 2

# Week 2

## 2.1 Lecture 3

### 2.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

### 2.1.2 Review, Existence of Roots

**Proposition.** There is no rational number whose square is 2.

**Theorem.** There is a unique positive real number $\alpha$ satisfying $\alpha^2 = 2$.

**Proof.** (i) **Uniqueness:** Suppose there are two of them $\alpha_1$ and $\alpha_2$. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that $A$ is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

■

**Remark.** A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number $\alpha$ such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

### 2.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

**Definition** (Usual Way of Defining Functions)**.** Let $A$ and $B$ be two sets. A **function** from $A$ to $B$ denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

**Definition** (The Correct Way of Defining Functions)**.** Let $A$ and $B$ be two sets. A function from $A$ to $B$ is a triple $(f, A, B)$ where $f$ is a **relation** form $A$ to $B$ satisfying

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, $A$ is called the **domain** of $f$ and $B$ is called the **codomain** of $f$)

**Example.** Let $A = \emptyset$ and $B$ be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to $B$ is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

**Definition** (Image, Range, Onto (Surjective))**.** Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of $f$ as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of $f$ as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say $f$ is **Onto (Surjective)**.

**Definition** (Preimage)**.** Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of $D$ under $f$ is denoted by

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

**Definition** (One-to-One (Injective))**.** Consider a function $f : A \to B$. We call $f$ one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of $A$.

### 2.1.4 Equivalent Sets

**Definition.** Let $A$ and $B$ be two sets. We say that $A$ and $B$ have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

**Remark.**

- An injective and surjective mapping is a bijective mapping.

- $A$ and $B$ have the same cardinal number

$$\begin{aligned} &= A \text{ and } B \text{ have the same cardinality} \\ &= A \text{ and } B \text{ can be put in the } \textbf{one-to-one correspondence} \\ &= \operatorname{card} A = \operatorname{card} B \\ &= A \text{ and } B \text{ are equivalent} \\ &= A \text{ and } B \text{ are equipotent} \end{aligned}$$

**Example.** Consider $\{1, 2, 3\} \sim \{a, b, c\}$. Indeed, the function $f : \{1, 2, 3\} \to \{a, b, c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

**Example.** $\mathbb{N} \sim \{2, 4, 6, \dots\}$. Indeed, the function $f : \mathbb{N} \to \{2, 4, 6, \dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

**Example.** $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

**Example.** $(-\infty, \infty) \sim (0, \infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty, \infty)$ and $(0, \infty)$.

**Example.** $(0, \infty) \sim (0, 1)$. Indeed, the function $f : (0, \infty) \to (0, 1)$ defined by

$$f(x) = \frac{x}{x + 1}$$

is a bijection.

**Example.** $[0, 1) \sim (0, 1)$. Indeed, the function $f : [0, 1) \to (0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

**Definition** (**$\sim$ is an equivalence relation**)**.** Let $A$ and $B$ be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}$$
$$\mathbb{N} = \{1, 2, 3, \dots\}$$

### 2.1.5 Finite, Infinite, Countable, At most countable

**Definition** (Finite, Infinite, Countable, At most countable)**.** Let $A$ be any set.

(a) We say that $A$ is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number $n$.

- (*) When $A \sim \mathbb{N}_n$, we say $A$ has $n$ elements and we write $\mathrm{card}(A) = n$.
- (*) Also, we set $\mathrm{card}(\emptyset) = 0$.

(b) The set $A$ is said to be **infinite** if it is not finite.

(c) The set $A$ is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set $A$ is said to be **uncountable** if it is neither countable or finite.

(e) The set $A$ is said to be **at most countable** if it is either finite or countable.

**Remark.** Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$A \text{ is finite} \iff B \text{ is finite}$$
$$A \text{ is countable} \iff B \text{ is countable}$$
$$A \text{ is uncountable} \iff B \text{ is uncountable}$$

(iii) The union of two finite sets is finite. If $A$ is infinite and $B$ is infinite, then $A \setminus B$ is infinite.

(iv) If $A$ **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.

# Chapter 3

# Week 3

## 3.1 Lecture 4

### 3.1.1 Topics

(1) Sequences

(2) Infinite subset of a countable set is countable.

### 3.1.2 Sequences

**Definition** (Sequence)**.** We call a **sequence**, we mean a function $f$ on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence $f$ by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in $A$.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

### 3.1.3 Infinite subset of a countable set is countable

**Theorem.** Every infinite subset of a countable set is countable.

**Proof.** Let $A$ be a countable set. Let $E \subseteq A$ and $E$ is infinite. Our goal is to show that $E$ is countable. Since $A$ is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let $n_1$ be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let $n_2$ be the smallest positive integer greater than $n_1$ such that $x_{n_2} \in E$.

$\vdots$

(k) Let $n_k$ be the smallest integer greater than $n_{k-1}$ such that $x_{n_k} \in E$.

Observe that the set

$$\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}$$

is both nonempty (since $E$ is infinite) and bounded below (by the well ordering property of natural numbers). Thus, we know that

$$n_k = \min\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}.$$

Now, define the function $f : \mathbb{N} \to E$ as follows:

$$f(k) = x_{n_k}.$$

We claim that this is a bijective map. If we can prove this fact, then surely $E$ is countable. We need to prove two things:

(1) $f$ is injective and

(2) $f$ is surjective.

Starting with (1), suppose $k_1 \neq k_2$. Then we have $n_{k_1} \neq n_{k_2}$ and thus, $x_{n_{k_1}} \neq x_{n_{k_2}}$. Hence, we see that $f(k_1) \neq f(k_2)$. Therefore, $f$ is injective which proves (1).

With (2), let $b \in E$. Since $E \subseteq A$, we must have $b \in A$ and thus there exists an $m \in \mathbb{N}$ such that $b = x_m$. Hence, there exists $1 \leq k \leq m$ such that $x_{n_k} = x_m = b$. Thus, $f(k) = b$ which shows that $f$ is surjective and so (2) is satisfied.

Therefore, $f$ must be a bijective map and we can now conclude that $E$ is countable. ■

**Remark.** Consider the contrapositive of the theorem above:

Let $E \subseteq A$. If $E$ is not countable, then $A$ is not countable.

When we say that a set is NOT countable, we do not necessarily mean that the set is uncountable. However, in the case that $E$ is an infinite set, we can say that $E$ being not countable is equivalent to saying that $E$ is uncountable. So, only in the case in which $E$ is infinite then the contrapositive is

Let $E \subseteq A$ be infinite. If $E$ is uncountable, then $A$ is uncountable. By the theorem we just proved, we see that $f(A) \subseteq S$ and $S$ being countable implies that $f(A)$ must be at most countable. Thus, $A$ must be at most countable.

**Corollary.** Let $A$ be any set and let $S$ be a countable set. If there exists an injective mapping $f : A \to S$, then $A$ is at most countable.

**Proof.** Let $A$ be any set and let $S$ be a countable set. Suppose there exists an injective mapping $f : A \to S$. We can restrict the codomain $S$ of $f$ to its range $f(A)$. Because $f$ is injective, the mapping $f : A \to f(A)$ will be bijective. Thus, we have that $A \sim f(A)$. ■

**Example** ($\mathbb{N} \times \mathbb{N}$ is countable). (1) The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(x, y) = 2^x 3^y$$

is injective. By the corollary we have just proved, we see that $\mathbb{N} \times \mathbb{N}$ is at most countable.

(2) Notice that $g : \mathbb{N} \to \{1\} \times \mathbb{N}$ is a bijection $g(a) = (1, a)$. Hence, $\{1\} \times \mathbb{N}$ is countable. Now, we have that

- (i) $\{1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$
- (ii) $\{1\} \times \mathbb{N}$ is countable implies that $\{1\} \times \mathbb{N}$ is infinite.

Thus, (i) and (ii) imply that $\mathbb{N} \times \mathbb{N}$ is countable.

Proving that a set $A$ is countable can be done by finding a subset $B$ that is countable which forces $A$ to be countable.

**Example** ($\mathbb{Q}$ is countable). (1) The function $f : \mathbb{Q} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2(2^p 3^q) & \text{if } x = \frac{p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 2(2^p 3^q) + 1 & \text{if } x = \frac{-p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is an injective mapping. So, $\mathbb{Q}$ is at most countable.

(2) Observe that $\mathbb{N} \subseteq \mathbb{Q}$ and $\mathbb{N}$ being a countable (and thus $\mathbb{N}$ is infinite) set implies that $\mathbb{Q}$ is infinite. Thus, $\mathbb{Q}$ is is countable.

### 3.1.4 Countable union of at most countable sets is at most countable

**Theorem.** Countable union of at most countable sets is at most countable.

**Proof.** Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of at most countable sets; that is, for each $n \in \mathbb{N}$, $A_n$ is at most countable. Our goal is to show that

$$K = \bigcup_{n \in \mathbb{N}} A_n \text{ is at most countable.}$$

To this end, it suffices to show that there exists an injective map $f : K \to \mathbb{N} \times \mathbb{N}$. Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\vdots \\ B_{n+1} &= A_{n+1} \setminus \Big( \bigcup_{k=1}^{n} A_k \Big). \end{aligned}$$

We leave as an exercise to show that

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \text{ are pairwise disjoint.}$$

Note that for all $n \in \mathbb{N}$, we have $B_n \subseteq A_n$ and $A_n$ is at most countable. Then for each $n \in \mathbb{N}$. $B_n$ must be at most countable. So, for each $n \in \mathbb{N}$, there exists an injective mapping $f_n : B_n \to \mathbb{N}$. Our goal is to show that

$$\bigcup_{n=1}^{\infty} B_n \text{ is at most countable.}$$

To this end, we define the function

$$f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N} \times \mathbb{N}$$

as follows:

For each $x \in \bigcup_{n=1}^{\infty} B_n$, there is exactly one $n \in \mathbb{N}$ such that $x \in B_n$; let us denote this $n$ by $n_x$.

Thus, we define

$$f(x) = (n_x, f_{n_x}(x)).$$

This function is injective because

$$\begin{aligned} f(x) = f(y) &\implies (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y)) \\ &\implies n_x = n_y \ \wedge \ f_{n_x}(x) = f_{n_y}(y) \\ &\implies f_{n_k}(x) = f_{n_x}(y) \\ &\implies x = y && \text{(Since } f_{n_x} \text{ is injective)} \end{aligned}$$

By the corollary, we see that the set

$$\bigcup_{n=1}^{\infty} B_n$$

is at most countable. ■

**Corollary.** A countable union of countable sets is countable.

**Proof.** Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. By the previous Theorem, we see that $\bigcup_{n\in\mathbb{N}} A_n$ is at most countable. Note that $A_1 \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and that $A_1$ is countable. Thus, $A_1$ must be infinite and so

$$\bigcup_{n\in\mathbb{N}} A_n \text{ is infinite.}$$

Thus, we see that

$$\bigcup_{n\in\mathbb{N}} A_n$$

must be countable. ■

**Corollary.** If $A$ and $B$ are at most countable, then $A \cup B$ at most countable.

**Proof.** Let $A_1 = A$ and $A_2 = B$ where $A_3 = A_4 = \cdots = A_n = \emptyset$. Then $A \cup B$ is at most countable by the previous Theorem. ■

**Theorem.** If $A$ is countable, then $A \times A$ is countable.

**Proof.** (1) Note that $A \times A = \bigcup_{b\in A}(\{b\} \times A)$ for each $b \in A$.

(2) The function $f : A \to \{b\} \times A$ defined by $f(x) = (b, x)$ is bijective. So, $A \sim \{b\} \times A$. Hence, $\{b\} \times A$ is countable.

Since a countable union of countable sets is countable, we see that $A \times A$ must be countable by (1) and (2). ■

## 3.2 Lecture 5

### 3.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks

- Inequalities

### 3.2.2 Summary of Last Lecture

1. How to prove $A$ is at most countable:
   - Show that $A \subseteq S$ where $S$ is countable.
   - Find an injective function $f : A \to S$ where $S$ is countable.
2. How to prove $A$ is infinite:
   - Prove that $A$ has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subsetneq A$ such that $A \sim B$.
3. How to prove that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

**Theorem (D).** Let $A$ be the set of all sequences whose terms are the digits 0 and 1 (that is $A$ is the collection of all binary sequences). This set $A$ is uncountable.

**Proof.** First, notice that $A$ is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

$$\text{for all } h(n) = \text{The binary sequence whose } n\text{th term is the digit 1 and all other terms are zero.}$$

Clearly, $h$ is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, $A$ is infinite

Suppose for sake of contradiction that $A$ is NOT uncountable. Since $A$ is infinite and not uncountable, this assumption tells us that $A$ must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned}
f(1) &= (a_1^1, a_2^1, \dots) \\
f(2) &= (a_1^2, a_2^2, \dots) \\
f(3) &= (a_1^3, a_2^3, \dots) \\
&\vdots \\
f(n) &= (a_1^n, a_2^n, \dots) \\
&\vdots
\end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that $A$ contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N} \; b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence $(b_n)$ is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

### 3.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of $a$ is defined to be $|a|$. But note that the **absolute value of** $a$ is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers $a$ and $b$ in $\mathbb{R}$, the **standard distance** between $a$ and $b$ is

$$\operatorname{dist}(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as $h$ gets close to zero...*.
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

**Theorem** (Basic Properties of Standard Size in $\mathbb{R}$)**.** Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

**Remark.** Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

**Definition.** Let $V$ be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on $V$ if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space $V$ equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

**Theorem** (Basic Properties of Standard Distance in $\mathbb{R}$)**.** For all $a, b \in \mathbb{R}$.

(i) $\text{dist}(a, b) \geq 0$

(ii) $\text{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\text{dist}(a, b) = \text{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

### 3.2.4 Metric Spaces

**Definition** (Metric Spaces)**.** Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set $X$ equipped with a metric $d$ is called a **metric space**. (Sometimes we write $(X, d)$ is a metric space)

**Remark.** $X$ does **NOT** need to be a vector space. Furthermore, $d$ is a more general notion of size than $\|\cdot\|$.

**Example.** $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)

# Chapter 4

# Week 4

## 4.1 Lecture 6

### 4.1.1 A few examples of Metrics

**Example.** Consider $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = \frac{|x-y|}{1+|x-y|}.$$

This is a metric on $(\mathbb{R}, d)$. **Prove this on homework!**

**Remark.** If $(X, D)$ is a metric space, then $(X, D)$ is also a metric space where

$$D(x, y) = \frac{d(x, y)}{1+d(x, y)}.$$

We can define a metric that will always be less than or equal to 1.

**Example** (Taxi Cab Metric). Consider $(\mathbb{R}^2, d)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ is defined by

$$d((a, b), (x, y)) = |a - x| + |b - y|$$

We want to show that this is a metric.

(i) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have $d((a, b), (x, y)) = |a-x|+|b-y| \geq 0$ by property of the absolute value $|\cdot|$.

(ii) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} d((a, b), (x, y)) = 0 &\iff |a - x| + |b - y| = 0 \\ &\iff |a - x| = 0 \text{ and } |b - y| = 0 \\ &\iff a - x = 0 \text{ and } b - y = 0 \\ &\iff (a, b) = (x, y). \end{aligned}$$

(iii) For all $(a, b), (x, y) \in \mathbb{R}^2$.

$$d((a, b), (x, y)) = |a - x| + |b - y| = |x - a| + |y - b| = d((x, y), (a, b)).$$

(iv) For all $(a, b), (x, y), (t, s) \in \mathbb{R}^2$, we want to show that

$$d((a, b), (x, y)) \leq d((a, b), (t, s)) + d((t, s), (x, y)).$$

We have

$$\begin{aligned} d((a, b), (t, s)) + d((t, s), (x, y)) &= |a - t| + |b - s| + |t - x| + |s - y| \\ &= (|a - t| + |t - x|) + (|b - s| + |s - y|) \\ &\geq |a - x| + |b - y| \\ &= d((a, b), (x, y)) \end{aligned}$$

**Example** (Discrete Metric)**.** Consider $X \to$ any nonempty set. Consider $d : X \times X \to [0, \infty)$,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

We want to show that this is a metric on $(X, \text{any non-empty set})$.

(i) For all $x, y \in X$, we have $d(x, y) \geq 0$ because either the $d(x, y) = 1$ or $d(x, y) = 0$.

(ii) Let $x, y \in X$. Clearly, we have $x = y$ if and only if $d(x, y) = 0$ by definition of $d(x, y)$.

(iii) Let $x, y \in X$. Clearly, we have $d(x, y) = d(y, x)$ by definition.

(iv) Let $x, y, z \in X$. We want to show that

$$d(x, y) \leq d(x, z) + d(z, y)$$

Let us consider two cases:

(1) $x = y$. Thus, both sides of the triangle inequality clearly hold since $d(x, y) = 0$.

(2) $x \neq y$. In this case, we have $d(x, y) = 1$. So, we need to show that

$$d(x, z) + d(z, y) \geq 1.$$

Since $x \neq y$, at least one of the statements $z \neq y$ or $z \neq x$ is true. If $z \neq x$, then $d(z, x) = 1$ and so

$$d(x, z) + d(z, y) = 1 + d(z, y) \geq 1.$$

If $z \neq y$, then $d(z, y) = 1$, and so

$$d(x, z) + d(z, y) = d(x, z) + 1 \geq 1.$$

**Example.** Consider $(V, \|\cdot\|) \to$ any normed vector space with the metric $d : V \times V \to [0, \infty)$ and $d(x, y) = \|x - y\|$. We wan to show that this is a metric on $V$.

(i) Let $x, y \in V$. By the property of the norm, we have $d(x, y) = \|x - y\| \geq 0$.

(ii) Let $x, y \in V$. By the 2nd property of the norm, we have

$$\begin{aligned} d(x, y) = 0 &\iff \|x - y\| = 0 \\ &\iff x - y = 0 \\ &\iff x = y. \end{aligned}$$

(iii) Let $x, y \in V$. We have

$$d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x).$$

(iv) Let $x, y, z \in V$. We want to show that

$$d(x,y) \leq d(x,z) + d(z,y).$$

We have,

$$\begin{aligned} d(x,z) + d(z,y) &= \|x-z\| + \|z-y\| \\ &\geq \|(x-z) + (z-y)\| \\ &= \|x-y\| \\ &= d(x,y). \end{aligned}$$

This example illustrates how every time we have a norm, we immediately have a metric space. But the other around is not necessarily true!

**Example** (Standard Distance in $\mathbb{R}^n$). Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we have

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

Note that if we define for all $x \in \mathbb{R}^n$, we have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

Then

$$d(x,y) = \|x-y\|_2.$$

Proving that the above is a norm is enough to show that the original in question is a metric.

Let $x, y \in \mathbb{R}^n$.

(i) We have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \geq 0.$$

(ii) We have

$$\begin{aligned} \|x_2\|_2 = 0 &\iff \sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0 \\ &\iff |x_1|^2 + \cdots + |x_n|^2 = 0 \\ &\iff |x_1| = 0, |x_2| = 0, \ldots, |x_n| = 0 \\ &\iff x_1 = 0, x_2 = 0, \ldots, x_n = 0. \end{aligned}$$

Thus, we have $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|\alpha x\|_2 &= \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2 (x_1^2 + \cdots + x_n^2)} \\ &= |\alpha| \sqrt{x_1^2 + \cdots + x_n^2} \\ &= |\alpha| \|x\|_2. \end{aligned}$$

(iv) Now, we want to show that

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

That is, we want to show that

$$\sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} \leq \sqrt{x_1^2+\cdots+x_n^2} + \sqrt{y_1^2+\cdots+y_n^2}.$$

**We will show this later!**

**Example.** Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$, we have

$$d_p(x,y) = \Big[|x_1-y_1|^p + \cdots + |x_n-y_n|^p\Big]^{\frac{1}{p}}.$$

### 4.1.2 Inequalities

In what follows we will list several key inequalities that can be used in proving that, for example, a given expression is a metric or a norm.

(1) The triangle inequality for the standard norm in $\mathbb{R}$. We have for all $x, y \in \mathbb{R}$, we have

$$|x+y| \leq |x| + |y|.$$

More generally, we have for all $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\Big|\sum_{i=1}^n x_i\Big| \leq \sum_{i=1}^n |x_i|.$$

(2) Reverse triangle inequality for the standard norm in $\mathbb{R}$; that is, we have for all $x, y \in \mathbb{R}$,

$$||x|-|y|| \leq |x-y|.$$

(3) For all $a, b \geq 0$ and for all $\rho > 0$, we have

$$ab \leq \frac{1}{2}\Big(\rho a^2 + \frac{1}{\rho}b^2\Big).$$

Note that we are dealing with real numbers here!

(4) Cauchy-Schwarz Inequality. For al $x, y \in \mathbb{R}^n$, we have

$$|x_1y_1+\cdots+x_ny_n| \leq \Big(\sqrt{x_1^2+\cdots+x_n^2}\Big)\Big(\sqrt{y_1^2+\cdots+y_n^2}\Big). \tag{*}$$

## 4.2 Lecture 7

### 4.2.1 Topics

- Review of Inequalities
- Neighborhood of a point
- Limit point of a set, isolated point of a set
- Closed set
- Interior point of a set

- Open set
- Bounded set
- Closure
- Dense

### 4.2.2 Inequalities

### 4.2.3 Minkowski

We wan to show the triangle inequality for $\|\cdot\|_2$ in $\mathbb{R}^n$; that is, we want to show that

**Proposition.** Let $\|\cdot\|_2$ be a norm in $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$, $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.

**Proof.** We see that $x \cdot y \leq |x \cdot y| \leq \|x\|_2 \|y\|_2$ by the Cauchy Cauchy-Schwarz Inequality. Thus, we have

$$\begin{aligned}
\|x+y\|_2^2 &= (x+y)\cdot(x+y) \\
&= x \cdot x + x \cdot y + y \cdot x + y \cdot y \\
&= \|x\|_2^2 + 2x \cdot y + \|y\|_2^2 \\
&\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
&= (\|x\|_2 + \|y\|_2)^2.
\end{aligned}$$

Hence, we have

$$\|x+y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2.$$

Therefore, we have

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

■

### 4.2.4 Minkowski for General p

For general $p$, we have Holder's Inequality which is

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

for any fixed real number $p \geq 1$ and for any $x, y \in \mathbb{R}^n$.

### 4.2.5 $(x+1)^n$

Recall that for all $a, b \in \mathbb{R}$, for all $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

So, in particular, if $x \geq 0$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\
&\geq 1 + nx.
\end{aligned}$$

Hence, we have for all $x \geq 0$ and for all $n \in \mathbb{N}$, we have

$$(x+1)^n \geq 1 + nx.$$

### 4.2.6 $p-$means

Let $x_1, \ldots, x_n$ be positive real numbers. Let $p \in \mathbb{N} \cup \{0\}$. By the $p-$mean of $x_1, \ldots, x_n$ denoted by $A_p(x_1, \ldots, x_n)$, we mean

$$A_p(x_1, \ldots, x_n) = \begin{cases} \frac{\sqrt[p]{x_1^p + \cdots + x_n s^p n}}{} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 \ldots x_n} & \text{if } p = 0. \end{cases}.$$

For example, if $p = 1$, we have

$$A_1(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

which is the Arithmetic Mean. If $p = 2$, we have

$$A_2(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

If $p = 0$, we have

$$A_0(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$$

which is the geometric mean. It can be shown that

$$A_0(x_1, \ldots, x_n) \leq A_1(x_1, \ldots, x_n) \leq A_2(x_1, \ldots, x_n) \leq \ldots \ .$$

In particular, we have $A_0 \leq A_1$; that is,

$$\sqrt[n]{x_1 \ldots x_n} \leq \frac{x_1 + \cdots + x_n}{n}. \tag{AM-GM Inequality}$$

We can prove this inequality using differentiation. Without differentiation, we can prove the same inequality using **Cauchy Induction**.

### 4.2.7 Jensen's Inequality

Suppose $f : (a, b) \to \mathbb{R}$ is a convex function ($f"(x) \geq 0$ for all $x \in (a, b)$). Let $x_1, \ldots, x_n$ be points in $(a, b)$. Let $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

We want to show this inequality holds for $n = 2$; that is,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

where $\lambda_1 + \lambda_2 = 1$. Note that

$$f((1 - \lambda_2)x_1 + \lambda_2 x_2) \leq (1 - \lambda_2) f(x_2) + \lambda_2 f(x_2).$$

**Remark.** If we want to have an expression that defines a function that gives us any number in between two points $e < h$, we can have

$$f(\lambda) = (1 - \lambda)e + \lambda h$$

for any $0 \leq \lambda \leq 1$.

### 4.2.8 Neighborhood of a point

**Definition** (Neighborhood)**.** Let $(X, d)$ be a metric space. Let $p \in X$. For any $\varepsilon > 0$, we call

$$N_\varepsilon(p) = \{x \in X : d(p, x) < \varepsilon\}.$$

the **neighborhood of** $p$ **of radius** $\varepsilon$.

**Example.** Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. The neighborhood of any $p \in \mathbb{R}$ with radius $\varepsilon > 0$ is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Note that $|x - p| < \varepsilon$ is the same thing as $p - \varepsilon < x < p + \varepsilon$ or that $x \in (p - \varepsilon, p + \varepsilon)$.

**Example.** Let $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$. Let $(a, b) \in \mathbb{R}^2$ with $\varepsilon > 0$. Then

$$\begin{aligned} N_\varepsilon((a, b)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y)(a, b)) < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \varepsilon^2.\} \end{aligned}$$

Thus, $N_\varepsilon((a, b))$ consists of the points inside the circle of radius $\varepsilon$ centered in $(a, b)$.

**Example.** $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = |a - x| + |b - y|$ and let $\varepsilon = 1$. We have

$$\begin{aligned} N_1((0, 0)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x - 0| + |y - 0| < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \end{aligned}$$

If we graph this out in $\mathbb{R}^2$, then the shape of the neighborhood will take on a rhombus.

**Example.** Let $(\mathbb{R}, d)$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $p \in \mathbb{R}$. Let $\varepsilon > 0$. Let us consider two cases:

(1) Let $\varepsilon \leq 1$. Note that if

$$d(x, p) < \varepsilon \leq 1,$$

then $d(x, p) < 1$, and so $d(x, p) = 0$. Hence, $x = p$. Then the neighborhood is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{p\}.$$

(2) Let $\varepsilon > 1$. Clearly, for all $x \in \mathbb{R}$, we have $d(x, p) \leq 1 < \varepsilon$. So,

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \mathbb{R}.$$

### 4.2.9 Limit Points

**Definition** (Limit Points, Isolated Points)**.** Let $(X, d)$ is a metric space with $E \subseteq X$. Then we call

(1) A point $p \in X$ is said to be a **limit point of** $E$ if for all $\varepsilon > 0$,

$$N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset.$$

(2) The **collection of all the limit points of** $E$ is denoted by $E'$; that is,

$$E' = \{p \in X : \text{for all } \varepsilon > 0, N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset\}.$$

(3) A point $p \in E$ is said to be an **isolated point of** $E$ if $p$ is NOT a limit point; that is, $p \in E$ but $p \notin E'$. Another way of saying this is $E \setminus E'$.

(4) If $p$ is NOT a limit point, we have $p \notin E'$ if and only if there exists $\varepsilon > 0$ such that

$$N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset.$$

**Remark.** The statement for all $\varepsilon > 0$, $N_\varepsilon(p)$ is equivalent to for all $N_\varepsilon(p)$.

**Example.** Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$. Note that $0 \notin E$. Also, recall that $0 \in E'$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(0) \cap (E \setminus \{0\}) \neq \emptyset$. If $0 \in E'$, we just need to show that for all $\varepsilon > 0$, $N_\varepsilon(0) \cap E \neq \emptyset$; that is, we need to show that

$$\text{for all } \varepsilon > 0,\ (-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Clearly, $1/m \in (-\varepsilon, \varepsilon) \cap E$.

**Example.** Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$ and

$$E = (1, 2) \cup \{5\}.$$

Prove that 5 is an isolated point. Since $5 \in E$, it is enough to show that 5 is not in $E'$. Recall that $5 \in E^{'}$ if and only if there exists an $\varepsilon > 0$ such that

$$N_\varepsilon(5) \cap (E \setminus \{5\}) = \emptyset.$$

Noticing that $E \setminus \{5\} = (1, 2)$, we can write $5 \notin E'$ if and only if there exists an $\varepsilon > 0$ such that $(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = \emptyset$. Clearly, $\varepsilon = 1$ does the job and we are done; that is,

$$(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = (4, 6) \cap (1, 2) = \emptyset.$$

**Example.** Let $(\mathbb{R}^2, d)$ and $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$ and

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

What is $E'$?

$$E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4.\}$$

For example, if $(a, b)$ is such that $a^2 + b^2 > 4$, then $(a, b) \notin E'$. Let $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$. Clearly, $N_\delta \cap (E \setminus \{p\}) = \emptyset$.

**Definition** (Closed Set)**.** Let $(X, d)$ be a metric space, $E \subseteq X$. We say that $E$ is **closed** if every limit point of $E$ is contained within $E$; that is,

$$E \text{ is closed} \iff E' \subseteq E.$$

**Example.** Let $(\mathbb{R}, d)$, $d(x, y) = |x - y|$ and $E = \{1, 2, 3\}$.

(i) What is $E'$?

**Claim:** $E' = \emptyset$. Let $p \in \mathbb{R}$. Our goal is to show that $p$ is not in $E'$. That is, we want to show that there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset$ where $N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon)$.

We may consider the following cases:

(1) If $p < 1$. Let $\varepsilon = \frac{1-p}{2}$ works.

(2) If $p > 3$, we have $\varepsilon = \frac{p-3}{2}$ works.

(3) If $p \in \{1, 2, 3\}$, then $\varepsilon = \frac{1}{4}$ works.

(4) If $1 < p < 2$, then let $\varepsilon = \frac{1}{2}\min\{p - 1, 2 - p\}$ works.

(5) If $2 < p < 3$, then $\varepsilon = \frac{1}{2}\{\min p - 2, 3 - p\}$ works.

(ii) Is $E$ closed? Since $E' = \emptyset$, we have $E' \subseteq E$ and so $E$ is closed.

**Remark.** Any finite set is closed!

**Definition** (Interior Point)**.** Let $(X, d)$ is a metric space and let $E \subseteq X$. We say that a point $p \in E$ is said to be an **interior point of** $E$ if there exists a neighborhood $N_\varepsilon(x)$ such that $N_\varepsilon(x) \subseteq E$. The collection of all interior points of $E$ is called the **interior of** $E$ and is denoted by the set:

$$E^\circ = \{x \in E : \exists\ N_\varepsilon(x) \subseteq E\}.$$

**Remark.** Note that by definition we know that the interior of $E$ is always contained within $E$; that is, $E^\circ \subseteq E$. Also,

$$p \in E^\circ \text{ if and only if there exists a neighborhood } N_\delta(p) \text{ such that } N_\delta(p) \subseteq E.$$

**Example.** Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$ with $E = (1, 3]$. What is $E^\circ =$?. We claim that $E^\circ = (1, 3)$.

Let $p \in (1, 3)$. We want to show that

(1) If $p \in (1, 3)$, then $p$ is an interior point.

(2) If $p = 3$, then $p \notin E^\circ$.

We proceed by showing each case above:

(1) It suffices to show that there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$. Clearly, choose $\delta = \frac{1}{2}\min\{p - 1, 3 - p\}$ and we are done.

(2) Suppose $p = 3$. It suffices to show that

$$\forall\ \varepsilon > 0,\ N_\varepsilon(3) \not\subseteq E.$$

That is, we want to show that

$$\forall\, \varepsilon > 0, (3-\varepsilon, 3+\varepsilon) \cap E^c \neq \emptyset.$$

Clearly, for all $\varepsilon > 0$, we have $3 + \frac{\varepsilon}{2} \in (3-\varepsilon, 3+\varepsilon)$. Thus, $3 + \frac{\varepsilon}{2} \in E^c$. Hence, we have

$$(3-\varepsilon, 3+\varepsilon) \cap E^c \neq \emptyset.$$

To show that boundary points are not interior points, it suffices to show that intersection with each neighborhood and the complement of the set is question is nonmepty.

**Example.** Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = \{1,2,3\}$. What is $E^\circ =?$. We claim that $E^\circ = \emptyset$. The reason is as follows: Let $p \in \{1,2,3\}$; that is, for all $\varepsilon > 0$, we have $N_\varepsilon(p) \not\subseteq E$ but $N_\varepsilon(p) = (p-\varepsilon, p+\varepsilon)$ has infinitely many points. We proved that if $p \in E$, then $p \notin E^\circ$. So, $E^\circ = \emptyset$.

**Definition** (Open Sets). Let $(X,d)$ be a metric space and $E \subseteq X$. We say that $E$ is **open** if every point of $E$ is an interior point of $E$; that is,

$$E \text{ is open} \iff E \subseteq E^\circ.$$

**Remark.** We know that, for any set, $E^\circ \subseteq E$. So, we can rewrite our definition as follows:

$$E \text{ is open} \iff E = E^\circ.$$

That is, the other inclusion holds!

**Example.** Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = \{1,2,3\}$. Is $E$ open? Note that $E^\circ = \emptyset$. So, $E^\circ \neq E$ and so $E$ is NOT open.

**Example.** Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = (1,4)$. Prove that $E$ is open. It suffices to show that every point $p \in E$ is an interior point. Let $p \in E$. That is, we want to show that there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$. If we choose $\delta = \frac{1}{2}\{p-1, 4-p\}$ does the job and we are done.

**Definition** (Bounded Sets). Let $(X,d)$ be a metric space and $E \subseteq X$. We say that $E$ is **bounded** if there exists $\varepsilon > 0$ and $q \in X$ such that $E \subseteq N_\varepsilon(q)$.

**Example.** Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = [0,\infty)$. Is $E$ bounded? **NO!** This is because for all $q \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$[0,\infty) \not\subseteq (q-\varepsilon, q+\varepsilon).$$

**Example.** Let $(\mathbb{R}, d)$ with the discrete metric and $E = [0,\infty)$. Is $E$ bounded? **YES!** For example, we have

$$E \subseteq N_{10}(0) = \mathbb{R}.$$

**Definition** (Closure). Let $(X,d)$ be a metric space and $E \subseteq X$. The **closure** of $E$, denoted by $\overline{E}$, is defined as follows:

$$\overline{E} = E \cup E'$$

that is, the closure of $E$ is the union of the isolated points and limit points.

**Example.** Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. What is $\overline{Q}$? Show within the homework that $\mathbb{Q}' = \mathbb{R}$. So,

**Definition** (Dense)**.** Let $(X, d)$ metric space and $E \subseteq X$. We say that $E$ is dense in $X$ if $\overline{E} = X$. (that is, every point of $X$ is either in $E$ or is a limit point of $E$).

**Example.** $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$.

# Chapter 5

# Week 5

### 5.0.1 Topics

(1) Every neigborhood is an open set.

(2) If $p$ is a limit point of a set $E$, then every neighborhood of $p$ contains infinitely many points of $E$.

(3) Theorem: $E$ is open $\iff E^C$ is closed.

(4) Theorem: arbitrary union of open sets is open, Finite intersection of open sets is open.

(5) Theorem: $\overline{E}$ is the smallest closed subset of $X$ that contains $E$.

**Theorem.** Let $(X, d)$ be a metric space and let $p \in X$ and $\varepsilon > 0$. Every neighborhood is an open set; that is, $N_\varepsilon(p)$ is an open set.

**Proof.** Our goal is to show that every point of $N_\varepsilon(p)$ is an interior point of $N_\varepsilon(p)$. Let $q \in N_\varepsilon(p)$. We need to show that there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$. Let $\delta = \frac{\varepsilon - d(p,q)}{2}$. We claim that $N_\delta(q)$ is a subset of $N_\varepsilon(p)$. Indeed, if $x \in N_\delta(q)$, then

$$d(q,x) < \delta \implies d(q,x) < \varepsilon - d(p,q)$$

and so

$$d(p,q) + d(q,x) < \varepsilon \iff d(p,x) < \varepsilon. \qquad \text{(triangle inequality)}$$

Thus, $x \in N_\varepsilon(p)$.

■

**Theorem.** Let $(X, d)$ be a metric space and $E \subseteq X$. If $p \in E'$, then every neighborhood of $p$ contains infinitely many points of $E$

**Proof.** Suppose for sake of contradiction that there exists a neighborhood of $p$ that contains finitely many points of $E$; that is,

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \cap E \text{ is a finite set.}$$

Since $N_\varepsilon(p) \cap (E \setminus \{p\}) \subseteq N_\varepsilon(p) \cap E$, we can immediately conclude that $N_\varepsilon(p) \cap (E \setminus \{p\})$ is finite also. Furthermore, $N_\varepsilon(p) \cap (E \setminus \{p\})$ is nonempty since $p \in E'$. Let us denote the elements of $N_\varepsilon(p) \cap (E \setminus \{p\})$ by $x_1, \ldots, x_n$. Our goal is to find a $\delta > 0$ such that the neighborhood with radius $\delta > 0$, we will not contain any of the finite points we introduced. For each $i \in \{1, \dots, n\}$, $d(p, x_i) > 0$,

$$\delta = \min\{d(p, x_i) : i \in 1, \dots, n\} > 0.$$

Clearly, $N_{\delta/2}(p) \cap (E \setminus \{p\}) = \emptyset$. But this contradicts our assumption that $p$ is NOT a limit point of $E$. ∎

**Corollary.** A finite set has no limit points; that is, if $E$ is finite, then $E' = \emptyset$.

**Proof.** This is just the contrapositive of the theorem above. ∎

**Theorem.** Let $(X, d)$ be a metric space and $E \subseteq X$. We have that $E$ is open if and only if $E^c$ closed.

**Proof.** ($\Longrightarrow$) Assume that $E$ is open. We want to show that $E^c$ is closed; that is, every limit point of $E^c$ is contained in $E^c$. Let $p$ be a limit point of $(E^c)$. Assume for sake of contradiction that $p \notin E^c$. Then $p \in E$. Since $E$ is open, $p$ is an interior point of $E$. Thus, there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$; that is, there exists $\delta > 0$ such that $N_\delta(p) \cap E^c = \emptyset$. Furthermore, we have that

$$N_\delta(p) \cap (E^c \setminus \{p\}) = \emptyset.$$

But this tells us that $p$ is not a limit point which is a contradiction.

($\Longleftarrow$) Assume that $E^c$ is closed. We want to show that $E$ is open; that is, every $x \in E$ is an interior point. Let $p \in E$. Assume for sake of contradiction that $p \notin E^\circ$. Then for all $\delta > 0$, $N_\delta(p) \not\subseteq E$. Hence, for all $N_\varepsilon(p) \cap E^c \neq \emptyset$. Therefore,

$$\forall \delta > 0 \ N_\delta(p) \cap (E^c \setminus \{p\}) \neq \emptyset.$$

That is, $p$ is a limit point of $E^c$. But by assumption, $E^c$ is closed. Thus, $p \in E^c$. But this contradicts the assumption that $p \in E$. ∎

**Theorem.** Let $(X, d)$ be a metric space. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets where $\Lambda$ is an index set (can be finite or infinite). Then

$$\bigcup_{\alpha \in \Lambda} A_\alpha$$

is an open set.

**Proof.** Our goal is to show that every point of $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ is an interior point. Let $p \in A$. Thus, there exists $\alpha \in \Lambda$ such that $p \in A_\alpha$. Since $A_\alpha$ is open, so there exists $\delta > 0$ such that $N_\delta(p) \subseteq A_\alpha$. But note that

$$A_{\alpha_0} \subseteq A.$$

Hence,

$$N_\delta(p) \subseteq A$$

and thus $p$ is an interior point of $A$.

∎

**Theorem.** Let $A_1, \dots, A_n$ be open sets in the metric space $(X, d)$. Then

$$\bigcap_{k=1}^{n} A_k \text{ is open.}$$

**Proof.** Our goal is to show that every point of $\bigcap_{k=1}^n A_k$ is an interior point of $\bigcap_{k=1}^n A_k$. Let $p \in \bigcap_{k=1}^n A_k$. Then for all $1 \leq k \leq n$ such that $p \in A_k$. Since $A_k$ is open for all $k$, we know that for all $1 \leq k \leq n$, there exists $\delta_k > 0$ such that $N_{\delta_k}(p) \subseteq A_k$. Let $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. Then we have that

$$N_\delta(p) \subseteq N_{\delta_k}(p) \subseteq A_k.$$

Consequently, we have

$$N_\delta(p) \subseteq \bigcap_{k=1}^n A_k.$$

Hence, $p$ is an interior point of the intersection $\bigcap_{k=1}^n A_k$. ■

**Theorem.** Let $(X, d)$ be a metric space and $E \subseteq X$.

(1) $\overline{E}$ is a closed set.

(2) $E$ is closed if and only if $E = \overline{E}$.

(3) If $E \subseteq F$ and $F$ is closed, then $\overline{E} \subseteq F$.

**Proof.** (1) Our goal is to show that $(\overline{E})^c$ is open. We need to show that every point of $(\overline{E})^c$ is an interior point of $(\overline{E})^c$. Let $p \in (\overline{E})^c$. We have

$$\begin{aligned} p \in (\overline{E})^c &\Longrightarrow p \notin \overline{E} \\ &\Longrightarrow p \notin (E \cup E') \\ &\Longrightarrow p \notin E \ \wedge \ p \notin E'. \end{aligned}$$

Note that

$$\begin{aligned} p \notin E' &\Longrightarrow \exists \varepsilon > 0 \ N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset \\ &\Longrightarrow \exists \varepsilon > 0 \ N_\varepsilon(p) \cap E = \emptyset. \end{aligned} \tag{1}$$

In what follows, we will show that $N_\varepsilon(p) \cap E' = \emptyset$. So, we have

$$\begin{aligned} & N_\varepsilon(p) \cap (E \cup E') = \emptyset \\ \Longrightarrow\ & N_\varepsilon(p) \cap \overline{E} = \emptyset \\ \Longrightarrow\ & N_\varepsilon(p) \subseteq (\overline{E})^c. \end{aligned}$$

Thus, we have that $p$ is an interior point of $(\overline{E})^c$. It remains to show that $N_\varepsilon(p) \cap E' = \emptyset$. Assume for sake of contradiction that $N_\varepsilon(p) \cap E' \neq \emptyset$. Let $q \in N_\varepsilon(p) \cap E'$. Then we have $q \in N_\varepsilon(p)$ and $q \in E'$. Because $N_\varepsilon(p)$ is an open set, there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$ and that $N_\delta(q) \cap (E \setminus \{q\}) \neq \emptyset$, respectively. But note that since $N_\delta(q) \subseteq N_\varepsilon(p)$ and $E \setminus \{q\} \subseteq E$ implies that

$$N_\varepsilon(p) \cap E \neq \emptyset$$

which contradicts (1).

■

**Definition** (Metric Subspace). Let $(X, d)$ be a metric space and let $Y$ be a nonempty subset of $X$. Then $(Y, d|_{Y \times Y})$ is called a **metric subspace** of $(X, d)$.

**Remark.** Consider the metric subspace $(Y, d|_{Y\times Y}$ of $(X, d)$. Let $p \in Y$ and $\varepsilon > 0$. Then we define

$$\begin{aligned} N_\varepsilon^Y(p) = \{y \in Y : d^Y(y,p) < \varepsilon\} &= \{y \in Y : d(y,p) < \varepsilon\} \\ &= N_\varepsilon(p) \cap Y. \end{aligned}$$

# Chapter 6

# Week 6

## 6.1 Lecture 10

**Theorem** ($E$ is open relative to $Y$)**.** Let $(X, d)$ be a metric space and $E \subseteq Y \subseteq X$ and $Y \neq \emptyset$. $E$ is open relative to $Y$ if and only if there exists an open set $G_0 \subseteq X$ such that $E = G \cap Y$.

**Proof.** ($\Longrightarrow$) Assume that $E$ is open relative to $Y$. Our goal is to show that there exists an open set $G \subseteq X$ such that $E = G \cap Y$. Since $E$ is open relative to $Y$, every $a \in E$, we have that $a$ is an interior point of $E$; that is, there exists $\varepsilon_0 > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$. Hence, for all $a \in E$, there exists $\varepsilon_a > 0$ such that $N_{\varepsilon_a}(a) \cap Y \subseteq E$. Let $G = \bigcup_{a \in E} N_{\varepsilon_a}(a)$. Clearly, $G$ is open in $X$ since

(1) For all $a \in E$, $N_{\varepsilon_a}(a)$ is a neighborhood and so it is open in $X$.

(2) A union of open sets is open in any metric space.

In what follows, we will prove that $E = G \cap Y$. Note that

$$G \cap Y = \Big( \bigcup_{a \in E} N_{\varepsilon_a}(a) \Big) \cap Y = \bigcup_{a \in E} \Big( N_{\varepsilon_a} \cap Y \Big) \subseteq \bigcup_{a \in E} E = E.$$

Suppose $b \in E$. We have $b \in N_{\varepsilon_b}(b)$ and thus $b \in G$. Furthermore, $b \in E$ implies $b \in Y$ since $E \subseteq Y$. Thus, we see that $E \subseteq G \cap Y$.

($\Longleftarrow$) Assume that there exists $G \subseteq X$ such that $E = G \cap Y$. We want to show that $E$ is open relative to $Y$. Our goal is to show that for all $a \in E$, there exists $\varepsilon > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$.

So, let $a \in E$ be given. Our goal is to find $\varepsilon > 0$ such that

$$N_\varepsilon(a) \cap Y \subseteq E.$$

By assumption, we have

$$\begin{aligned} a \in E = G \cap Y &\Longrightarrow a \in G \\ &\Longrightarrow_{G \text{ is open}} \exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \subseteq G. \end{aligned}$$

Hence, we see that

$$N_\varepsilon(a) \cap Y \subseteq G \cap Y = E.$$

■

**Theorem.** (i) If $E \subseteq \mathbb{R}$ is bounded above, then $\sup E \in \overline{E}$.

(ii) If $E \subseteq \mathbb{R}$ is bounded below, then $\inf E \in \overline{E}$.

**Proof.** Here we will prove (1) and the proof of (2) is completely analogous.

Since $E$ is bounded above, we have that $\sup E$ exists and is a real number by the least upper bound property of $\mathbb{R}$. Let $\alpha = \sup E$. Our goal is to show that $\alpha \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,

$$N_\varepsilon(\alpha) \cap E \neq \emptyset. \qquad \text{(Exercise 11 of HW4)}$$

Let $\varepsilon > 0$ be given. Since $\alpha = \sup E$, we know that there exists $x \in E$ such that $\alpha - \varepsilon < x$. Hence, there exists $x \in E$ such that

$$\alpha - \varepsilon < x \leq \alpha < \alpha + \varepsilon.$$

Hence, $\alpha - \varepsilon < x < \alpha + \varepsilon$; that is, $x \in N_\varepsilon(\alpha)$. Therefore, $N_\varepsilon(\alpha) \cap E \neq \emptyset$. ■

**Definition** (Open Covers)**.** Let $(X, d)$ be a metric space and $E \subseteq X$. A collection of sets $\{O_\alpha\}_{\alpha \in \Lambda}$ is said to be an **open cover** of $E$ if

(i) for every $\alpha \in \Lambda$, $O_\alpha$ is open in $X$.

(ii) $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$.

**Example.** Consider $(\mathbb{R}, |\bullet|)$ and the subset $E = [0, \infty)$. The collection $\{E_n\}_{n \in \mathbb{N}}$ defined by for all $n \in \mathbb{N}$, we have

$$E_n = \left(\frac{-1}{n}, n\right)$$

is an open cover of $E$. The reason is as follows:

(i) In homework 4, we proved that every open interval is an open set with the standard metric on $\mathbb{R}$.

(ii) $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Indeed, let $a \in E$. If $a = 0$, then $a$ belongs $E_n$ for all $n \in \mathbb{N}$. So, we have

$$a \in \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, if $a \neq 0$, then $\frac{1}{a} > 0$. By the Archimedean Property of $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a}$. So, $a < n$. Clearly,

$$\frac{-1}{n} < 0 < a < n.$$

Hence, $a \in E_n = \left(\frac{-1}{n}, n\right)$. Consequently,

$$a \in \bigcup_{m=1}^{\infty} E_m.$$

**Definition** (Compactness)**.** Let $(X, d)$ be a metric space and $K \subseteq X$. We say that $K$ is **compact** if every open cover of $K$ has a finite subcover; that is, for every open cover $\{O_\alpha\}_{\alpha \in \Lambda}$ of $K$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

**Example.** Let $(X, d)$ be a metric space and $E \subseteq X$. If $E$ is finite, then $E$ is compact. The reason is as follows:

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover. Our goal is to show that this open cover has a finite subcover; that is, we can choose $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

If $E = \emptyset$, there is nothing to prove. Otherwise, $E \neq \emptyset$, we can denote the elements of $E$ by $x_1, \ldots, x_n$. So,

$$E = \{x_1, \ldots, x_n\}.$$

We have

$$x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

and so continue this process until you have reached the $n$th element of $E$. Hence, we have

$$x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}.$$

Thus, we have that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$$

and so $E$ is compact.

**Example.** Let $(\mathbb{R}, |\cdot|)$ and

$$E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Prove that $E$ is compact (In general, if $a_n \to a$ in $\mathbb{R}$, then the set $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact).

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of $E$. Our goal is to show that this open cover has a finite subcover. Notice that $0 \in E$ and $E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha$ implies that

$$O \in \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_0 \in \Lambda \text{ such that } O \in O_{\alpha_0}.$$

So, if $O \in O_{\alpha_0}$ and $O_{\alpha_0}$ is open, we have

$$\exists \varepsilon > 0 \text{ such that } (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}.$$

By the Archimedean Property of $\mathbb{R}$, we see that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. So, for all $n \geq m$, we have $\frac{1}{n} < \varepsilon$. Hence, for all $n \geq m$, we have $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}$. Note that $1 \in E$ so there exists $\alpha_1 \in \Lambda$ such that $1 \in O_{\alpha_1}$. Similarly, $\frac{1}{2} \in E$ implies that there exists $\alpha_2 \in \Lambda$ such that $\frac{1}{2} \in O_{\alpha_2}$. Continue this process until, we have

$$\frac{1}{m-1} \in E \Longrightarrow \exists \alpha_m \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}}.$$

Thus, we conclude that

$$E \subseteq \bigcup_{n=0}^{\infty} O_{\alpha_n}$$

and so $E$ is compact.

**Remark.** If $X$ itself is compact, we say that $(X, d)$ is a compact metric space; that is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that

$$X = \bigcup_{\alpha\in\Lambda} O_\alpha$$

then there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$X = \bigcup_{i=1}^{n} O_{\alpha_i}.$$

**Theorem** (Compactness implies Closed). Let $(X, d)$ be a metric space and $E \subseteq K$ is closed and $K$ is compact. Then $E$ is compact.

**Proof.** It is enough to show that $K^c$ is open; that is, we need to find $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq K^c$ for every $x \in K^c$. That is, we want to show that

$$\exists\varepsilon > 0 \text{ such that } N_\varepsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \;\; d(x, a) > 0 \end{aligned}$$

For all $x \in K$, let $\varepsilon_x = \frac{1}{4}d(x, a)$. Clearly, we have

$$\forall x \in K \;\; N_{\varepsilon_x} \cap N_{\varepsilon_x}(a) = \emptyset.$$

Note that $\{N_{\varepsilon_x}(x)\}_{x\in K}$ is an open cover for $K$. Since $K$ is compact, there is a finite subcover. That is, there exists $x_1, \dots x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{x_i}}(x_i)$$

and of course

$$N_{\varepsilon_{x_1}}(x_1) \cap N_{\varepsilon_{x_1}}(a) = \emptyset.$$

If we continue this process until the $n$th step, we have that

$$N_{\varepsilon_{x_n}}(x) \cap N_{\varepsilon_{x_n}}(a) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon_{x_1}, \dots, \varepsilon_{x_n}\}$. Clearly, $N_\varepsilon(a) \subseteq N_{\varepsilon_{x_i}}(a) \forall 1 \le i \le n$. Hence, we have

$$N_{\varepsilon_{x_1}} \cap N_\varepsilon(a) = \emptyset$$

and similarly for the $n$th step, we have

$$N_{\varepsilon_{x_n}}(x) \cap N_\varepsilon(a) = \emptyset.$$

Therefore, we have

$$N_\varepsilon(a) \cap \left[N_{\varepsilon_{x_1}} \cup \cdots \cup N_{\varepsilon_{x_n}}\right] = \emptyset.$$

So,

$$N_\varepsilon(a) \cap K = \emptyset.$$

■

## 6.2 Lecture 11

### 6.2.1 Topics

- Metric subspace
- Theorem 2.35
- Theorem 2.33
- Theorem 2.37
- Theorem 2.36

**Theorem.** Let $(X, d)$ be a metric space and let $K \subseteq X$ be a compact set, and $E \subseteq K$ is closed. Then $E$ is compact.

**Proof.** Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $E$. Our goal is to show that this cover contains a finite subcover. Note that, $E$ is closed by assumption and so $E^c$ is open. Thus, we have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cup E^c.$$

Thus, $E^c$ together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set $K$. Since $K$ is compact, this open cover contains a finite subcover. So, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$K \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

Considering that $E \subseteq K$, we can write

$$E \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

So, we have $O_{\alpha_1}, \ldots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. ■

**Corollary.** If $F$ is closed and $K$ is compact, then $F \cap K$ is compact.

**Proof.** Note $F \cap K$ is a closed subset of the compact set $K$. Since $K$ is closed and $F$ is closed along with $F \cap K \subseteq K$. Thus, $K$ contains a finite subcover of $F \cap K$. Thus, $F \cap K$ is compact. ■

**Theorem.** Let $(X, d)$ be a metric space and Suppose $K \subseteq Y \subseteq X$. We have $K$ is compact relative to $X$

if and only if $K$ is compact relative to $Y$.

**Proof.** ($\Longleftarrow$) Assume that $K$ is compact relative to $Y$. Our goal is to show that $K$ is compact relative to $X$. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in $X$ that covers $K$. Our goal is to show that this cover has a finite subcover. Note that the set $K = K \cap Y$ since $K \subseteq Y$. Hence, we have

$$K = K \cap Y \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y = \bigcup_{\alpha\in\Lambda}(O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space $(Y, d^Y)$. So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in $(Y, d^Y)$ that covers $E$. Since $K$ is compact relative to $Y$, there exists a finite subcover; that is, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n}(O_{\alpha_i} \cap Y) \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

($\Longrightarrow$) Assume that $K$ is compact relative to $X$. Our goal is to show that $K$ is compact relative to $Y$. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in $(Y, d^Y)$ that covers $K$. Our goal is to show that this cover contains a finite subcover. It follows from Theorem 2.30 that for all $\alpha \in \Lambda$, there exists an open set $O_\alpha \subseteq X$ such that $G_\alpha = O_\alpha \cap Y$. We have

$$K \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda}(O_\alpha \cap Y) = \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$

Thus, $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for $K$ i n the metric space $(X, d)$. Since $K$ is compact, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Hence, we see that

$$\begin{aligned} K = K \cap Y \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y &= \bigcup_{i=1}^{n}(O_{\alpha_i} \cap Y) \\ &= \bigcup_{i=1}^{n} G_{\alpha_i}. \end{aligned}$$

That is, this is a finite subcover we were looking for and we are done. ■

Note that the property of compactness gets preserved in $X$, but when we are referring to open and closed sets these properties do not get preserved in $X$.

**Theorem.** Let $(X, d)$ be a metric space. If $E$ is an infinite subset of a compact set $K$, then $E$ has a limit point in $K$. (In fact, these two statements are equivalent, but we will only prove the forwards direction)

**Proof.** Let $(X, d)$ be a metric space. Assume that $K \subseteq X$ is compact and $E$ is infinite. We will show that $E$ has a limit point in $K$. Assume for sake of contradiction that $E$ has no limit point in $K$; that is, every limit point in $K$ will never be a limit point of $E$. In other words, for all $a \in E$, $a \notin E'$. Furthermore, for all $b \in K \setminus E$, we have $b \notin E'$. Therefore, for all $a \in E$, there exists $\varepsilon_a > 0$ such that

$$N_{\varepsilon_a} \cap (E \setminus \{a\}) = \emptyset. \tag{1}$$

Likewise, for all $b \in K \setminus E$, there exists $\delta_b > 0$ such that

$$N_{\delta_b} \cap (E \setminus \{b\}) = \emptyset \iff N_{\delta_b} \cap E = \emptyset. \tag{2}$$

Notice that (1) implies that

$$N_{\varepsilon_a} \cap E = \{a\}.$$

Clearly, $K$ is a subset of $\bigcup_{a \in E} N_{\varepsilon_a}(a) \cup \left( \bigcup_{b \in K \setminus E} N_{\delta_b}(b) \right)$. Since $K$ is compact, there exists $a_1, \dots, a_n \in E$ and $b_1, \dots, b_m \in K \setminus E$ such that

$$K \subseteq \left( \bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i) \right) \cup \left( \bigcup_{i=1}^{m} N_{\delta_{b_i}}(b_i) \right).$$

Since for all $b \in K \setminus E$ and the fact that $N_{\delta_b}(b) \cap E = \emptyset$, we have

$$E \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i).$$

Hence,

$$\begin{aligned} E &= E \cap \left[ \bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i) \right] \\ &= \bigcup_{i=1}^{n} (E \cap N_{\varepsilon_{a_i}}(a_i)) \\ &= \bigcup_{i=1}^{n} \{a_i\} \\ &= \{a_i : 1 \le i \le n\}. \end{aligned}$$

But this implies that $E$ is finite which is a contradiction of our assumption that $E$ is infinite. ∎

**Remark.** Let $(X, d)$ be a metric space and $K \subseteq X$. The following statements are equivalent:

(i) $K$ is compact.

(ii) Every infinite subset of $K$ has a limit point in $K$ (limit point compactness).

(iii) Every sequence in $K$ has a subsequence that converges to a point in $K$ (sequential compactness).

**Example.** Consider the set of intervals $A_i = [i, \infty)$, then

$$\bigcap_{i=1}^{\infty} A_i = \emptyset.$$

**Corollary.** If $K_1 \supseteq K_2 \supseteq \dots$ is a sequence of nonempty compact sets, and every finite intersection of these sets is nonempty, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

**Proof.** Suppose for sake of contradiction that the intersection

$$\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset.$$

Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \Big( \bigcap_{\alpha \in \Lambda} K_\alpha \Big) = \emptyset.$$

So,

$$K_{\alpha_0} \subseteq \Big( \bigcup_{\alpha \in \Lambda} K_\alpha \Big)^c \Longrightarrow K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} K_\alpha^c.$$

Note that $K_\alpha^c$ is open because $K_\alpha$ is closed. So, $\{K_\alpha^c\}$ is an open cover of $K_{\alpha_0}$. Since $K_{\alpha_0}$ is compact, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^n K_{\alpha_i}^c \subseteq \Big( \bigcap_{i=1}^n K_{\alpha_i} \Big)^c.$$

So, we have

$$K_{\alpha_0} \cap \Big( \bigcap_{i=1}^n K_{\alpha_i} \Big) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty.

■

**Theorem.** Let $\{K_\alpha\}$ a collection of compact subsets of $X$. If the intersection of any finite subcollection is nonempty, then

$$\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset.$$

**Proof.** ■

**Theorem** (Nested Interval Property)**.** If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^\infty I_n$ is nonempty.

**Proof.** ■

# Chapter 7

# Week 7

## 7.1 Lecture 12

### 7.1.1 Topics

- Definition of $K-$cell.
- Theorem: If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.
- Theorem: Every $k-$cell is compact.
- Theorem: Suppose $E \subseteq \mathbb{R}^k$. Then
- Connected sets

  $E$ is closed and bounded $\iff$ $E$ is compact $\iff$ Every infinite subset of $E$ has a limit point in $E$.
- Theorem: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

**Corollary.** If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of compact sets, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

**Theorem** (Nested Interval Property)**.** If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

**Definition** (K-cell)**.** The set $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is called a $k-$cell in $\mathbb{R}^k$.

**Example.** Let $I = [a_1, a_2] \times [a_2, b_2]$ is a $2-$cell in $\mathbb{R}^2$.

**Theorem** (Nested Cell Property)**.** If $I_1 \supseteq I_2 \supseteq I_3 \cdots$ is a nested sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

**Proof.** For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(n)}, b_1^{(n)}] \times \cdots \times [a_k^{(n)}, b_k^{(n)}]$$

Also, let

$$\forall n \in \mathbb{N} \text{ and } \forall 1 \le i \le k, \text{ we have } A_i^{(n)} = [a_i^{(n)}, b_i^{(n)}]$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$A_i^{(n)} \supseteq A_i^{(n+1)} \ \forall 1 \le i \le k.$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \cdots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \cdots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \cdots A_k^{(n)}. \end{aligned}$$

Hence, it follows from the nested interval property that there exists

$$\begin{aligned} &\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)} \\ &\exists x_2 \in \bigcap_{i=1}^{\infty} A_2^{(n)} \\ &\qquad \vdots \\ &\exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}. \end{aligned}$$

Thus, by a fact in set theory; that is,

$$(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D).$$

$$\begin{aligned} (x_1, \dots, x_k) &\in \Big[ \bigcup_{n=1}^{\infty} A_1^{(n)} \Big] \times \Big[ \bigcap_{n=1}^{\infty} A_2^{(n)} \Big] \times \cdots \times \Big[ \bigcap_{n=1}^{\infty} A_k^{(n)} \Big] \\ &\subseteq \bigcap_{n=1}^{\infty} [A_1^{(n)} \times \cdots \times A_k^{(n)}] \\ &= \bigcap_{n=1}^{\infty} I_n. \end{aligned}$$

Hence, we see that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

**Theorem.** Every $k-$cell in $\mathbb{R}^k$ is compact.

**Proof.** Here we will prove the claim for $2-$cells. The proof for a general $k-$cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a $2-$cell. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Let

$$\delta = d(a, b) = \|a - b\| = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$$

. Note that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in $I$, then

$$x_1, y_1 \in [a_1, b_2] \implies |x_1 - y_1| \le |b_1 - a_1|$$
$$x_2, y_2 \in [a_2, b_2] \implies |x_2 - y_2| \le |b_2 - a_2|$$

which implies that

$$\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \le \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta.$$

So, $d(x, y) \le \delta$. Let us assume for contradiction that $I$ is NOT compact. So, there exists an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $I$ that does NOT have a finite subcover; that is, $I \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$. For each $1 \le i \le 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2} \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i].$$

These subintervals determine 4 $2-$cells. There is at least one of these 4 $2-$cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let us call this $2-$cell as $I_1$. Notice that

$$\forall x, y \in I_1 \quad \|x - y\|_2 \le \frac{\delta}{2}.$$

Now, subdivide $I_1$ into 4 $2-$cells and continue this process inductively. In this manner, we will obtain a sequence of $2-$cells

$$I, I_1, I_2, I_3, \ldots$$

such that

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \tag{1}$$
$$\forall x, y \in I_n, \quad \|x - y\| \le \frac{\delta}{2} \tag{2}$$
$$\forall n \in \mathbb{N} \quad I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in I}$$

■

**Theorem** (Heine-Borel Theorem). Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

(a) $E$ is closed and bounded.

(b) $E$ is compact.

(c) Every infinite subset of $E$ has a limit point of $E$.

**Proof.** We will show that $(a) \implies (b) \implies (c) \implies (a)$.

($(a) \implies (b)$) Assume that $E$ is closed and bounded. Our goal is to show that $E$ is compact. Since $E$ is bounded, there exists a $K-$cell, $I$, such that $E \subseteq I$. Note that by Theorem 2.40, we see that $I$ is compact. By Theorem 2.3.5, $E$ is compact.

($(b) \implies (c)$) Assume that $E$ is compact. Our goal is to show that $E$ is limit point compact; that is, every infinite subset of $E$ has a limit point in $E$. See proof from last week.

($(c) \implies (a)$) Assume that every infinite subset of $E$ has a limit point in $E$. Our goal is to show that $E$ is closed and bounded. Suppose for sake of contradiction that $E$ is NOT bounded and NOT closed.

Suppose that $E$ is NOT bounded. In what follows, we will construct a sequence of points $x_1, x_2, \ldots$ in

$E$. Since $E$ is not bounded, we know that

$$\begin{aligned}
E \not\subseteq N_1(0) &\Longrightarrow \exists x_1 \in E \text{ such that } d(x_1, 0) = \|x_1\|_2 \geq 1.\\
E \not\subseteq N_2(0) &\Longrightarrow \exists x_2 \in E \text{ such that } d(x_2, 0) = \|x_2\|_2 \geq 2\\
E \not\subseteq N_3(0) &\Longrightarrow \exists x_3 \in E \text{ such that } d(x_3, 0) = \|x_3\|_2 \geq 3\\
&\vdots\\
E \not\subseteq N_n(0) &\Longrightarrow \exists x_n \in E \text{ such that } d(x_n, 0) = \|x_n\|_2 \geq n\\
&\vdots
\end{aligned}$$

That is, we have a sequence of points $S = \{x_n : n \in \mathbb{N}\}$ in $E$ with the property that

$$d(x_n, 0) = \|x_n\|_2 \geq n.$$

Note that $S$ is an infinite set; indeed, if $S$ were finite, then

$$S = \{a_1, \ldots, a_m\}.$$

Now, let

$$r = \max\{d(a_n, 0) : 1 \leq n \leq m\}$$

and let $n \in \mathbb{N}$ be such that $n > r + 1$ (By the Archimedean Property). Since $\|x_n\|_2 \geq n > r + 1$, we can conclude that none of the $a_1, \ldots, a_m$ is $x_n$ and this contradicts how $S$ was constructed.

Now, we will show that $E$ is closed. Assume for contradiction that $E$ is NOT closed; that is, $E' \not\subseteq E$. That is, there exists $y_0 \in \mathbb{R}^k$ such that $y_0 \in E'$ but $y_0 \in E$. We will construct a sequence of points $y_1, y_2, \ldots$ in $E$ as follows:

$$\begin{aligned}
y_0 \in E' &\Longrightarrow N_1(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_1 - y_0\| < 1\\
y_0 \in E' &\Longrightarrow N_{\frac{1}{2}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_2 - y_0\| < \frac{1}{2}\\
&\vdots\\
y_0 \in E' &\Longrightarrow N_{\frac{1}{m}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_m - y_0\| < \frac{1}{m}\\
&\vdots
\end{aligned}$$

Let $T = \{y_1, y_2, y_3, \ldots\}$ and note that $T \subseteq E$. Note $T$ is infinite (by construction $y_1, y_2, y_3, \ldots$ are distinct elements. We claim that if $z \neq y_0$, then $z \in T'$. To this end, we have for all $n \in \mathbb{N}$

$$d(y_0, z) = \|y_0 - z\|_2 \leq \|y_0 - y_n\|_2 + \|y_n - z\|_2$$

which implies further that

$$\|y_n - z\|_2 \geq \|y_0 - z\|_2 - \|y_n - y_0\|_2 > \|y_0 - z\|_2 - \frac{1}{n}.$$

Hence, for all $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{1}{2}\|y_0 - z\|_2$, we have

$$\begin{aligned} d(y_n, z) &> \|y_0 - z\|_2 - \frac{1}{n} \\ &> \|y_0 - z\|_2 - \frac{1}{2}\|y_0 - z\|_2 \\ &= \frac{1}{2}\|y_0 - z\|_2. \end{aligned}$$

So, for all but finitely many $n$, we see that

$$d(y_n, z) > \frac{1}{2}\|y_0 - z\|_2.$$

Hence, if we let $\varepsilon = \frac{1}{4}\|y_0 - z\|_2$, then $N_\varepsilon(z) \cap T$ is a finite set which proves $z \notin T'$. But this implies that the only possible limit point of $T$ is $y_0$, but $y_0 \notin E$. Hence, $T$ is an infinite subset of $E$ with not limit point in $E$. This contradicts our hypothesis that every infinite subset of $E$ has a limit point of $E$. ■

**Remark.** Note that in any general metric space, we have $(a) \implies (b)$ is not necessarily true.

**Theorem** (Bolzano-Weierstrass Theorem)**.** Let $E \subseteq \mathbb{R}^k$ and $E$ is an infinite set and bounded. Then $E' \neq \emptyset$.

**Proof.** Suppose that $E$ is bounded. Then there exists a $k-$cell $I$ such that $E \subseteq I$. By Theorem 2.40, we know that $I$ is a compact set. Furthermore, we know that $I$ is limit point compact by Theorem 2.41. So, every infinite set in $I$ has a limit point in $I$. In particular, $E$ has a limit point in $I$. So, $E \neq \emptyset$. ■

## 7.2 Lecture 13

### 7.2.1 Topics

- Separated sets, disconnected sets, connected set.
- Theorem: $E \subseteq \mathbb{R}$ is connected if and only if $x, y \in E$ and $z \in (x, y)$ implies $z \in E$.
- Perfect Sets
- Theorem: $P \subseteq \mathbb{R}^k$ is nonempty perfect implies $P$ is uncountable.
- The Cantor Set

**Definition** (Connected Sets, Disconnected, connected)**.** Let $(X, d)$ be a metric space.

(i) Two sets $A, B \subseteq X$ are aid to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(iii) A set $E \subseteq X$ is said to **disconnected** if it can be written as a union of tow nonempty separated sets $A$ and $B$; that is, $E = A \cup B$.

(iv) A set $E \subseteq X$ is said to be connected if it is NOT disconnected.

**Example** (ℝ with the standard metric). (*) If we have $A = (1, 2)$ and $B = (2, 5)$ are separated, then

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset$$
$$A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.$$

Hence, $E = A \cup B$ is disconnected.

(*) We have $C = (1, 2]$ and $D = (2, 5)$ are disjoint but not separated; that is, we have

$$C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\}$$
$$C \cup D = (1, 5) \text{ is indeed connected.}$$

**Theorem.** Let $E \subseteq \mathbb{R}$. $E$ is connected if and only if $E$ contains the following property

$$\text{If } x, y \in E \text{ and } x < z < y, \text{ then } z \in E.$$

**Proof.** ■

**Remark** (Proposition 3.3.5, "Differential Calculus on Normed Space", Cartan). Let $U$ be an open set in a normed (real) vector space. The following conditions are equivalent:

(a) $U$ is connected

(b) $U$ is path connected (any two points in $U$ can be connected by a path in $U$)

**Remark** (Chapter 4, "Introduction" to Topological Manifolds", John Lee). In any metric space,

$$\text{path connected} \implies \text{connected}.$$

However, the converse is not always true!

**Definition** (Perfect Set). Let $(X, d)$ be a metric space. Let $E \subseteq X$. The following are equivalent definitions:

(i) $E$ is said to be **perfect** if $E' = E$

(ii) $E$ is said to be **perfect** if $E' \subseteq E$ and $E \subseteq E'$.

(iii) $E$ is said to be perfect if $E$ is closed and every point of $E$ is a limit point of $E$.

(iv) $E$ is said to be perfect if $E$ is closed and $E$ does not have any isolated points.

**Example.**

- $E = [0, 1] \implies E' = [0, 1]$. Thus, we have $E = E'$ and so $E$ is perfect.
- $E = [0, 1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is NOT perfect.
- $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \implies E' = \{0\} \implies E \neq E'$, so $E$ is not perfect.
- Is $E'$ from the last example perfect? Indeed, $E' = \{0\} \implies (E')' = \emptyset$. Thus, $E' \neq (E')' \implies E'$ is NOT perfect.
- $E = \emptyset$ and $E' = \emptyset$. Thus, $E = E'$ so $E$ is perfect.

**Theorem.** Let $P$ be a nonempty perfect set in $\mathbb{R}^k$. Then $P$ is uncountable.

Our proof of the theorem above will use the following two lemmas:

**Lemma.** Let $(X, d)$ be a metric space. Let $E \subseteq X$ be perfect. If $V$ is any open set in $X$ such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

**Proof.** Let $q \in V \cap E$. Thus, $q \in V$ and $q \in E$. Then $q \in V$ implies that there exists $\delta > 0$ such that $N_\delta(q) \subseteq V$ and $q \in E$ implies $q \in E'$. As a consequence of these two results, we see that $N_\delta(q) \cap E$ is an infinite set. Thus, $V \cap E$ is an infinite set (Here, we are using the fact that an open set intersected with a perfect set is infinite whenever the intersection is nonempty). ■

**Lemma.** Let $q \in \mathbb{R}^k$. Let $r > 0$. Then $\overline{N_r(q)} = C_r(q)$.

**Proof.** Note that since $P' = P$ and $P \neq \emptyset$, we have $P' \neq \emptyset$. Thus, $P$ is infinite. Assume for contradiction that $P$ is countable. Let's denote the distinct elements of $P$ by $x_1, x_2, x_3, \ldots$; that is, we can denote

$$P = \{x_1, x_2, x_3, \dots\}.$$

In what follows, we will construct a sequence of neighborhoods $V_1, B_2, V_3, \ldots$ such that

(i) For all $n \in \mathbb{N}$, we have $\overline{V_{n+1}} \subseteq V_n$.

(ii) For all $n \in \mathbb{N}$, $x_n \notin \overline{V_{n+1}}$.

(iii) For all $n \in \mathbb{N}$, $V_n \cap P \neq \emptyset$.

Let's assume that we have constructed these neighborhoods (Construction of neighborhoods ). Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset.$$

Note that

(I) $\overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n}$ so $\overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P$ and so $K_{n+1} \subseteq K_n$ for each $n$.

(II) Since $\overline{V_n}$ is a closed and bounded set in $\mathbb{R}^k$, we have that $\overline{V_n}$ is compact. Furthermore, $P$ being a perfect set implies that $P$ is a closed set. As a consequence of these two facts, we can conclude that $K_n = \overline{V_n} \cap P$ is compact.

Using facts (I) and (II), we can conclude that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset \tag{*}$$

by Theorem 2.3.6. Recall that for all $n$, we have $K_n \subseteq P$, and so we have

$$\bigcap_{n=1}^{\infty} K_n \subseteq P.$$

In what follows, we will contradict (*). Let $b \in P$ be arbitrary. Then $b = x_m$ for some $m \in \mathbb{N}$. By property (ii), we see $x_m \notin \overline{V_{m+1}}$ and so $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$. This tells us that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

■

**Remark** (On the construction of neighborhoods $V_1, V_2, \ldots$)**.** Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$ (because $x_1 \in V_1$ and $c_1 \in P$). Our goal is to construct an open neighborhood $V_2$ such that

(i) $\overline{V_2} \subseteq V_1$,

(ii) $x_1 \notin \overline{V_2}$,

(iii) $V_2 \cap P \neq \emptyset$.

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$. By the first lemma found above, there exists $y_1 \in V_1 \cap P$ such that $y_1 \neq x_1$. Since $V_1$ is open and $y_1 \in V_1$, there exists $\delta_1 > 0$ such that $N_{\delta_1}(y_1) \subseteq V_1$.

$$\text{Let } r_2 = \tfrac{1}{2}\min\{d(x_1, y_1), \delta_1\}.$$

Let $V_2 = N_{r_2}(y_1)$. We claim $V_2$ has all the desired properties above. Indeed, we see that

(i) Observe that

$$\begin{aligned} \overline{V_2} = \overline{N_{r_2}(y_1)} &= \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\} \\ &\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1) \\ &\subseteq V_1. \end{aligned}$$

(ii) Notice that $d(x_1, y_1) > r_2$ implies that

$$x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}.$$

(iii) Since $y_1 \in V_2$ and $y_1 \in P$, we clearly have that $V_2 \cap P \neq \emptyset$.

We can construct $V_3, V_4, \ldots$ in a similar manner.

As a consequence of the theorem above, we have

**Corollary.** The interval $[0, 1]$ is uncountable.

### 7.2.2 The Cantor Set

The construction of the Cantor set is typically done in stages.

**Step 0:**

Let $E_0 = [0, 1]$.

**Step 1:**

Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, that is, remove the middle third, and define

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

The middle third, in this case, will be calculated by the following

$$\left(\frac{1}{3}, \frac{2}{3}\right) = \left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right).$$

**Step 2:**

Take each of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and remove the middle third of each of those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Likewise, we can remove the middle third move explicitly by calculating the following

$$\left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right) \text{ and } \left(\frac{3(2)+1}{3^2}, \frac{3(2)+2}{3^2}\right).$$

Continue in this manner until we obtain a sequence of compact sets:

$$E_1, E_2, E_3, E_4, \ldots$$

with the following properties

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \ \cdots$

(2) For each $n \in \mathbb{N}$, $E_n$ is the union of $2^n$ intervals of length $\frac{1}{3^n}$.

The set $P = \bigcap_{n=1}^{\infty} E_n$ is called the **Cantor Set**.

**Remark.** Notice that in order to obtain $E_n$, we remove intervals of the form $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ from $E_{n-1}$; that is, $k$ is such that $0 \leq k$ and $3k+2 < 3^n$.

**Theorem** (Properties of the Cantor Set). Let $P$ denote the Cantor set. Then

(1) $P$ is compact

(2) $P$ is nonempty

(3) $P$ contains no segment

(4) $P$ is perfect (and so it is uncountable)

(5) $P$ has measure zero.

**Proof.** (1) Note that $P$ is an intersection of compact sets. Hence, $P$ is compact (see hw5).

(2) It follows from Theorem 2.3.6 that the intersection of a sequence of nested nonempty compact sets is nonempty. (In fact, the endpoint of each interval that appears at any state belong to $P$)

(3) Our goal is to show that $P$ does NOT contain any set of the form $(\alpha, \beta)$ (where $0 \leq \alpha, \beta \leq 1$). Note that, by the construction of $P$, the intervals of the form:

$$I_{k,n} = \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \ n \in \mathbb{N}, 3k+2 < 3^n$$

have no intersection with $P$. However, $(\alpha, \beta)$ contains at least one of $I_{k,n}$'s. Indeed, $(\alpha, \beta)$ contains $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$. That is,

■

# Chapter 8

# Week 8

## 8.1 Lecture 14

### 8.1.1 Topics

- Convergence of a sequence
- Bounded Sequence
- Theorem: $x_n \to x \iff$ every neighborhood of $x$ contains $x_n$ for all but at most finitely many $n$.
- Theorem: A sequence cannot have more than one limit.
- Theorem: Convergent sequence $\implies$ Bounded sequence.
- If $x \in E'$, there is a sequence $(x_n)$ of distinct points in $E \setminus \{x\}$ such that $x_n \to x$.

**Definition** (Convergence of Sequence)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. We say that $(x_n)$ converges to a limit $x \in X$ if for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \varepsilon$.

**Definition** (Bounded Sequence)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. We say that $(x_n)$ is **bounded** if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space $X$; that is,

$$(x_n) \text{ is bounded if and only if there exists } q \in X, \text{ there exists } r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q).$$

**Theorem** (An Equivalent Characterization of Convergence)**.** Let $(X, d)$ be a metric space. We say that $(x_n) \to x$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains $x_n$ for all but at most finitely many $n$.

**Proof.** We have $(x_n) \to x$ if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $d(x_n, x) < \varepsilon$. This holds if and only if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \in N_\varepsilon(x)$. This holds if and only if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $N_\varepsilon(x)$ contains $x_n$ for all $n > N$. This is equivalent to saying that for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains $x_n$ for all but at most finitely many $n$. ∎

**Theorem** (Uniqueness of Limit)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. If $x_n \to x$ in $X$ and $x_n \to \tilde{x}$, then $x = \tilde{x}$.

**Proof.** In order to show that $\tilde{x} = d$, it suffices to show that $d(\tilde{x}, x) = 0$. That is, it suffices to show that for all $\varepsilon > 0$, $d(x, \tilde{x}) < \varepsilon$. Let $\varepsilon > 0$ be given. Since $x_n \to x$, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$d(x_n, x) < \frac{\varepsilon}{2}.$$

Similarly, $x_n \to \tilde{x}$ implies that there exists an $N_2 \in \mathbb{N}$ such that for any $n > N_2$, we have

$$d(x_n, \tilde{x}) < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. So, for any $n > N$, we have

$$d(x, \tilde{x}) \leq d(x, x_n) + d(x_n, \tilde{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we have $x = \tilde{x}$. ■

**Theorem** (Convergent $\Longrightarrow$ Bounded)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. If $x_n \to x$ in $X$, then $(x_n)$ is bounded.

**Proof.** By the definition of convergence with ($\varepsilon = 1$), we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \;\, x_n \in N_1(x).$$

Now, let

$$r = \max\{1, d(x_1, x), d(x_2, x), \ldots, d(x_N, x)\} + 1.$$

Then, clearly, we have

$$\forall n \in \mathbb{N} \;\, d(x_n, x) < r.$$

Hence,

$$\forall n \in \mathbb{N} \;\, x_n \in N_r(x).$$

■

**Corollary** (Contrapositive)**.** If $(x_n)$ is NOT bounded, then $(x_n)$ does NOT converge.

**Theorem** (Limit Point is a Limit of a Sequence)**.** Let $(X, d)$ be a metric space. Let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence $x_1, x_2, \ldots$ of distinct points in $E \setminus \{x\}$ that converges to $x$.

**Proof.** Since $x \in E'$, we have

$$\forall \varepsilon > 0 \;\; N_\varepsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular, we have

For $\varepsilon = 1$ $\exists x_1 \in E \setminus \{x\}$ such that $d(x_1, x) < 1$

For $\varepsilon = \dfrac{1}{2}$ $\exists x_2 \in E \setminus \{x\}$ such that $x_2 \neq x_1$ and $d(x_2, x) < \dfrac{1}{2}$

For $\varepsilon = \dfrac{1}{3}$ $\exists x_3 \in E \setminus \{x\}$ such that $x_3 \neq x_1, x_2$ and $d(x_3, x) < \dfrac{1}{3}$

$\vdots$

For $\varepsilon = \dfrac{1}{n}$ $\exists x_n \in E \setminus \{x\}$ such that $x_n \neq x_1, x_2, \ldots, x_{n-1}$ and $d(x_n, x) < \dfrac{1}{n}$

$\vdots$

In this way, we claim that the sequence $x_1, x_2, x_3, \ldots$ of distinct points in $E \setminus \{x\}$ converges to $x$.

Indeed, let $\varepsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that for any $n > N$, $d(x_n, x) < \varepsilon$. Choose $N$ such that $\frac{1}{N} < \varepsilon$. Then for all $n > N$, we have

$$d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

as desired. ■

## 8.2 Lecture 15

**Definition** (Subsequence)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$ and let

$$n_1 < n_2 < n_3 < \cdots$$

be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ is called a **subsequence** of $(x_1, x_2, x_3, \ldots)$ and is denoted by $(x_{n_k})$, where $k \in \mathbb{N}$ indexes the subsequence.

**Theorem.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. If $\lim\limits_{n\to\infty} x_n = x$, then every subsequence of $(x_n)$ converges to $x$.

**Proof.** Let $(x_{n_k})$ be a subsequence of $(x_n)$. Our goal is to show that $\lim\limits_{k\to\infty} x_{n_k} = x$. That is, we want to show that

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \text{ such that } \forall k > N \; d(x_{n_k}, x) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Our goal is to find an $N \in \mathbb{N}$ such that

$$\text{if } k > N \text{ then } d(x_{n_k}, x) < \varepsilon. \tag{I}$$

Since $x_n \to x$, we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N} \; d(x_n, x) < \varepsilon. \tag{II}$$

We claim that this $\hat{N}$ can be used as the same $N$ we were looking for. Indeed, if we let $N = \hat{N}$, then for any $k > N$, we can conclude that $n_k \geq k > N$ and so, by (II) we can write $d(x_{n_k}, x) < \varepsilon$. ■

**Corollary.** (1) If a subsequence of $(x_n)$ does NOT converge to $x$, then $(x_n)$ does NOT converge to $x$.

(2) If $(x_n)$ has a pair of subsequences converging to different limits, then $(x_n)$ does not converge.

**Theorem.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. The subsequential limits of $(x_n)$ form a closed set in $X$.

**Proof.** Let $E = \{b \in X : b \text{ is a limit of a subsequence of } (x_n)\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$, that is, we will show that there is a subsequence of $(x_n)$ that converges to some $a \in X$. We may considering two cases:

(1) $\forall n \in \mathbb{N}$, we have $x_n = a$. In this case, $(x_n)$ and any subsequence of $(x_n)$ converges to $a$. So, $a \in E$.

(2) $\exists n_1 \in \mathbb{N}$ such that $x_{n_1} \neq a$. In this case, let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, we have $N_{\frac{\delta}{2^2}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^3}.$$

Since $n \in E$, $b$ is a limit of a subsequence of $(x_n)$, so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

By the triangle inequality, we have that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b, a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of $(x_n)$ such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}.$$

From this construction, we can see that $x_{n_k} \to a$. Thus, $a \in E$.

■

**Theorem** (Compactness implies Sequential Compactness)**.** Let $(X, d)$ be a compact metric space. Then every sequence in $X$ has a convergent subsequence.

**Proof.** Let $(x_n)$ be a sequence is the compact metric space $X$. Let $E = \{x_1, x_2, \ldots\}$. If $E$ is finite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < \cdots$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \cdots = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ converges (it converges to $x$).

If $E$ is infinite, then since $X$ is compact by Theorem 2.3.7, $E$ has a limit point $x \in X$ (that is, there exists $x \in X$ such that $x \in E'$). Thus, we have

$$\forall \varepsilon > 0 \quad N_\varepsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned}
&\text{For } \varepsilon = 1, \ \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1 \\
&\text{For } \varepsilon = \frac{1}{2}, \ \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2} \\
&\text{For } \varepsilon = \frac{1}{3}, \ \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3} \\
&\vdots \\
&\text{For } \varepsilon = \frac{1}{m}, \ \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m} \\
&\vdots
\end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, \dots$ of $(x_n)$ that converges (it converges to $x$). ■

**Corollary.** Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

**Proof.** Let $(x_n)$ be a bounded sequence in $\mathbb{R}^k$. Then we have

$$\exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q).$$

Note that $N_r(q)$ is a bounded set and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_r(q)}$ is a compact subset of $\mathbb{R}^k$. So, we have $\overline{N_r(q)}$ is a compact metric space and $(x_n)$ is a sequence in $\overline{N_r(q)}$ implies, by the previous theorem, there exists a subsequence $(x_{n_k})$ of $(x_n)$ that converges in the metric space $\overline{N_r(q)}$. Since the distance $\overline{N_r(q)}$ is the same as distance function in $\mathbb{R}^k$, we can conclude that $(x_{n_k})$ converges in $\mathbb{R}^k$ as well. ■

**Definition** (Cauchy Sequence)**.** Let $(X, d)$ be a metric space. A sequence $(x_n)$ in $X$ is said to be a **Cauchy Sequence** if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N \ d(x_n, x_m) < \varepsilon.$$

**Theorem** (Convergent implies Cauchy)**.** Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. Then

$$(x_n) \text{ converges} \implies (x_n) \text{ is Cauchy}.$$

**Proof.** Suppose $(x_n) \to x$ for some $x \in X$. Our goal is to show that $(x_n)$ is a Cauchy sequence. Our goal is to show that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N \ d(x_n, x_m) < \varepsilon.$$

By assumption, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$, we have

$$d(x_n, x) < \frac{\varepsilon}{2}. \tag{1}$$

Similarly, for any $m > \hat{N}$

$$d(x_m, x) < \frac{\varepsilon}{2}. \tag{2}$$

We claim that $\hat{N}$ can be used as the $N$ we were looking for. Set $N = \hat{N}$ and suppose for any $n, m > N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

**Remark.** The converse of the theorem above does not necessarily hold. For example, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. In $\mathbb{Q}$, it is not true that every Cauchy sequence is convergent. For example, let $(q_n)$ be a sequence in $\mathbb{Q}$ such that $q_n \to \sqrt{2}$. That is,

$$\begin{aligned} q_n \to \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q}. \end{aligned}$$

However, we know that $(q_n)$ does not converge in $\mathbb{Q}$.

**Definition** (Complete Metric Space)**.** A metric space in which every Cauchy sequence is convergent is called a **complete metric space**.

# Chapter 9

# Week 9

## 9.1 Lecture 16

### 9.1.1 Topics

- Diameter of a set
- Theorem: $\operatorname{diam} \overline{E} = \operatorname{diam} E$
- Theorem: Nested sequence of nonempty compact sets $K_n$ and if $\operatorname{diam} K_n \to 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.
- Theorem: Every compact metric space is complete.
- Theorem: $\mathbb{R}^k$ is a complete metric space.

**Definition** (Diameter of a Set). Let $(X, d)$ be a metric space. Let $E$ be a nonmempty set in $X$. The diameter of $E$, denoted by $\operatorname{diam} E$, is defined as follows:

$$\operatorname{diam} E = \sup\{d(a, b) : a, d \in E\}.$$

**Remark.** Note that if $\emptyset \neq A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Thus, we have

$$\sup\{d(a, b) : a, b \in A\} \leq \sup\{d(a, b) : a, b \in B\}.$$

That is,

$$\operatorname{diam} A \leq \operatorname{diam} B.$$

**Theorem** (Interesting Observation). Let $(X, d)$ be a metric space. Let $(x_n)$ be a sequence in $X$. For all $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+1}, \dots\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n \to \infty} \operatorname{diam} E_n = 0.$$

**Proof.** ($\Longrightarrow$) Assume that $(x_n)$ is a Cauchy sequence. Note that

$$\begin{aligned}
E_1 &= \{x_2, x_3, x_4, x_5, \dots\} \\
E_2 &= \{x_3, x_4, x_5, \dots\} \\
E_3 &= \{x_4, x_5, \dots\} \\
E_4 &= \dots \\
&\vdots
\end{aligned}$$

Clearly, we have

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq \cdots.$$

So, we have

$$\operatorname{diam} E_1 \geq \operatorname{diam} E_2 \geq \operatorname{diam} E_3 \geq \cdots.$$

Our goal is to show that

$$\forall\, \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \ \underbrace{|\operatorname{diam} E_n - 0|}_{\operatorname{diam} E_n \geq 0} < \varepsilon. \tag{*}$$

So, it suffices to show that $\operatorname{diam} E_n < \varepsilon$. To this end, let $\varepsilon > 0$ be given. Since $(x_n)$ is Cauchy, there exists $\hat{N} \in \mathbb{N}$ such that

$$\forall n, m > \hat{N}\ \ d(x_n, x_m) < \frac{\varepsilon}{2}.$$

We claim that this $\hat{N}$ is the same $N$ we were looking for. Letting $N = \hat{N}$, we have

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}.$$

Hence, we have

$$\forall a, b \in E_{\hat{N}}\ \ d(a, b) < \frac{\varepsilon}{2}.$$

Thus, we have

$$\operatorname{diam} E_{\hat{N}} = \sup\{d(a, b) : a, b \in E_{\hat{N}}\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

If $n > \hat{N}$, then

$$\operatorname{diam} E_n \leq \operatorname{diam} E_{\hat{N}} < \varepsilon$$

as desired.

($\Longleftarrow$) Assume that $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$. Our goal is to show that $(x_n)$ is Cauchy; that is,

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N\ d(x_n, x_m) < \varepsilon. \tag{*}$$

Since $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$, for this given $\varepsilon$, there exists $\hat{N}$ such that

$$\forall n > \hat{N}\ \ \ \operatorname{diam} E_n < \varepsilon.$$

In particular, we have $\operatorname{diam} E_{\hat{N}+1} < \varepsilon$. Now, we claim that $N = \hat{N} + 1$ can be used as the same $N$ we were looking for. Indeed, if we let $N = \hat{N} = 1$, we have

$$\forall n, m > \hat{N} + 1,\ x_n, x_m \in E_{\hat{N}+1}$$

and thus

$$d(x_n, x_m) \leq \operatorname{diam} E_{\hat{N}+1} < \varepsilon.$$

■

**Theorem.** Let $(X, d)$ be a metric space. Let $E$ be a nonempty subset of $X$. Then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

**Proof.** Note that, since $E \subseteq \overline{E}$, we have $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$. In what follows, we will prove that $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$ by showing that

$$\forall \varepsilon > 0 \quad \operatorname{diam} \overline{E} \leq \operatorname{diam} E + \varepsilon.$$

Let $\varepsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \operatorname{diam} E + \varepsilon.$$

To this end, it suffices to show that $\operatorname{diam} E + \varepsilon$ is an upper bound for $\{d(a, b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$a \in \overline{E} \Longrightarrow N_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset \Longrightarrow \exists x \in E \text{ such that } d(x, a) < \frac{\varepsilon}{2}$$
$$b \in \overline{E} \Longrightarrow N_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset \Longrightarrow \exists y \in E \text{ such that } d(y, b) < \frac{\varepsilon}{2}.$$

Therefore, we have that

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\varepsilon}{2} + d(x, y) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \operatorname{diam} E + \frac{\varepsilon}{2} \\ &= \varepsilon + \operatorname{diam} E \end{aligned}$$

which is our desired result. ■

**Theorem.** Let $(X, d)$ be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a nested sequence of nonempty compact sets where $\lim_{n\to\infty} \operatorname{diam} K_n = 0$. Then

$$\bigcap_{n=1}^{\infty} K_n \text{ consists of exactly one point.}$$

**Proof.** Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.3.6, we know that $K \neq \emptyset$. In order to show that $K$ has only one element, we suppose that $a, b \in K$ and we will prove $a = b$. In order to show that $a = b$, we will prove that $d(a, b) = 0$. Showing this is equivalent to showing that

$$\forall \varepsilon > 0 \quad d(a, b) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n > N \quad \operatorname{diam} K_n < \varepsilon.$$

In particular, $\operatorname{diam} K_{N+1} < \varepsilon$. Now, we have

$$a \in \bigcap_{n=1}^{\infty} \Longrightarrow a \in K_{N+1}$$

and

$$b \in \bigcap_{n=1}^{\infty} K_n \implies b \in K_{N+1}$$

which implies further that

$$d(a,b) \leq \operatorname{diam} K_{N+1} < \varepsilon$$

which is our desired result. ■

**Theorem** (Compact Space $\implies$ Complete Space)**.** Any compact metric space is complete.

**Proof.** Let $(X,d)$ be a compact metric space. Let $(x_n)$ be a Cauchy sequence in $X$. Our goal is to show that $(x_n)$ converges in $X$. For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3} \dots\}$. We know that

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

(2) $(x_n)$ is Cauchy $\implies \lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$.

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \cdots \tag{†}$$

Since closed subsets of a compact metric space are compact, we know that (†) is a nested sequence of nonempty compact sets. Since $\operatorname{diam} E_n = \operatorname{diam} \overline{E_n}$, it follows from (2) that $\lim\limits_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$. Hence, we know by the previous theorem that $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let us denote this point as $a$ where

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}.$$

In what follows, we will prove that $\lim\limits_{n\to\infty} x_n = a$. To this end, it suffices to show that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ d(x_n, a) < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be give. Since $\lim\limits_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$, for this given $\varepsilon$, there exists $\hat{N}$ such that

$$\forall n > \hat{N} \quad \operatorname{diam} \overline{E_n} < \varepsilon.$$

We claim that $\hat{N}+1$ can be used as the $N$ that we were looking for. Indeed, if we let $N = \hat{N}+1$, then (*) holds. The reason is as follows: If $n > \hat{N}+1$, then

$$x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}}. \tag{I}$$

Furthermore,

$$a \in \bigcap_{n=1}^{\infty} \overline{E_n} \implies a \in \overline{E_{\hat{N}+1}}. \tag{II}$$

Thus, (I) and (II) imply that

$$d(x_n, a) \leq \operatorname{diam} \overline{E_{\hat{N}+1}} < \varepsilon.$$

■

**Theorem** ($\mathbb{R}^k$ is complete)**.** $\mathbb{R}^k$ is a complete metric space.

**Proof.** Let $(x_n)$ be a Cauchy sequence in $\mathbb{R}^k$. By homework 7, we see that $(x_n)$ must be bounded; that is,

$$\exists p \in \mathbb{R}^k, \varepsilon > 0 \text{ such that } \forall n \in \mathbb{N} \ x_n \in N_\varepsilon(p).$$

Note that $\overline{N_\varepsilon(p)}$ is a closed and bounded set in $\mathbb{R}^k$, and so it must be compact by the Heine-Borel Theorem. Thus, if $\overline{N_\varepsilon(p)}$ is a compact metric space and $(x_n)$ is a Cauchy sequence in $\overline{N_\varepsilon(p)}$, we have that $(x_n)$ converges to a point $x \in \overline{N_\varepsilon(p)}$ by the previous theorem. Since the metric in $\overline{N_\varepsilon(p)}$ is exactly the same as the metric in $\mathbb{R}^k$, we can conclude that $x_n \to x$ in $\mathbb{R}^k$.

■

## 9.2 Lecture 17

### 9.2.1 Topics

- Algebraic Limit Theorem (for sequences of numbers)
- Divergent sequences
- Monotone sequences
- Monotone Convergence Theorem
- Extended real numbers
- Properties of limit in $\overline{\mathbb{R}}$ (Extended Real Numbers)

**Theorem** (Algebraic Limit Theorem)**.** Suppose $(a_n)$ and $(b_n)$ are sequences of real numbers, and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$

(ii) n $\lim_{n\to\infty}(ca_n) = ca$ for any real number $c$.

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided that $b \neq 0$.

**Proof.** *Proof left to the reader.* ■

**Definition.** Consider $\mathbb{R}$ with its standard metric. Let $(x_n)$ be a sequence of real numbers. If $(x_n)$ does not converge, we say $(x_n)$ **diverges**.

Divergence can come in three different forms:

(i) $(x_n)$ becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n > M.$$

From this, we say that $x_n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$.

(ii) $(-x_n)$ becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ \underbrace{-x_n > M}_{x_n < -M}.$$

From this, we say that $x_n \to -\infty$ or $\lim_{n\to\infty} x_n = -\infty$.

(iii) $(x_n)$ is NOT convergent and it does NOT diverge to $\infty$ nor $-\infty$. For example,

$$(x_n) = ((-1)^n) = (-1, +1, -1, +1, \dots)$$

or

$$(x_n) = (n(-1)^n) = (-1, 2, -3, 4, -5, 6, \dots).$$

**Remark** (Strategies to Prove Divergence)**.** Let $(a_n)$ be a sequence of real numbers.

(1) If $(a_n)$ is unbounded, then $(a_n)$ diverges.

(2) If $(a_n)$ has a pair of subsequences converging to different limits, then $(a_n)$ diverges.

(3) Let $a \in \mathbb{R}$ be fixed but arbitrary. Show that the assumption $a_n \to a$ leads to a contradiction.

**Definition** (Increasing, Decreasing, Monotone)**.** Consider $\mathbb{R}$ with the standard metric.

(i) $(a_n)$ is said to be **increasing** if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$.

(ii) $(a_n)$ is said to be **decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

(iii) $(a_n)$ is said to be **monotone** if it is either increasing or decreasing or both.

(iv) $(a_n)$ is said to be **strictly increasing** if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

(v) $(a_n)$ is said to be **strictly decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} < a_n$.

**Example.** (i) $1, -1, 1, -1, 1, -1, \ldots$ is neither an increasing nor decreasing sequence.

(ii) $2, 4, 6, 8, 10, \ldots$ is strictly increasing.

(iii) $6, 6, 7, 7, 8, 8, \ldots$ is an increasing sequence but not strictly increasing.

(iv) $-2, -4, -6, -8, -10, -12, \ldots$ is a strictly decreasing sequence.

**Theorem** (Monotone Convergence Theorem)**.** Consider $\mathbb{R}$ with its standard metric.

(i) If $(a_n)$ is increasing and bounded, then $(a_n)$ converges to $\sup\{a_n : n \in \mathbb{N}\}$.

(ii) If $(a_n)$ is decreasing and bounded, then $(a_n)$ converges to $\inf\{a_n : n \in \mathbb{N}\}$.

(iii) If $(a_n)$ is increasing and unbounded, then $a_n \to \infty$.

(iv) If $(a_n)$ is decreasing and unbounded, then $a_n \to -\infty$.

**Proof.** Here will prove item (i). The rest of the parts follow analogously.

Suppose that $(a_n)$ is increasing ($\forall n\ a_n \leq a_{n+1}$) and $(a_n)$ is a bounded sequence. Our goal is to show that $a_n \to s$ where

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$

Notice that since $\{a_n : n \in \mathbb{N}\}$ is a bounded set, we know that $\{a_n : n \in \mathbb{N}\}$ must exists in $\mathbb{R}$. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ |a_n - s| < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. To show (*), it suffices to show that

$$\text{if } n > N, \text{ then } s - \varepsilon < a_n < s + \varepsilon.$$

Since $s = \sup\{a_n : n \in \mathbb{N}\}$, we see that $s - \varepsilon$ must NOT be an upper bound of $\{a_n : n \in \mathbb{N}\}$. Hence, there exists an element in $\{a_n : n \in \mathbb{N}\}$ that is larger than $s - \varepsilon$. That is, there exists $\hat{N} \in \mathbb{N}$ such that $a_{\hat{N}} > s - \varepsilon$. We claim that this $\hat{N}$ can be used as the $N$ that were looking for. Indeed, if we let $N = \hat{N}$, then

(1) if $n > \hat{N}$, then $a_n \geq a_{\hat{N}} > s - \varepsilon$

(2) if $n > \hat{N}$, then $a_n \leq \sup\{a_n : n \in \mathbb{N}\} = s < s + \varepsilon$.

Thus, (1) and (2) imply that if $n > \hat{N}$, then $s - \varepsilon < a_n < s + \varepsilon$ as desired. ∎

**Remark.** Consider $\mathbb{R}$ with its standard metric.

(*) Boundedness does not generally imply convergence. An example of such a bounded sequence that does not converge is $a_n = (-1)^n$.

(*) Monotone does not necessarily imply convergence. An example of such a sequence is $a_n = n$.

(*) Convergence does not generally imply being monotone; that is, consider $a_n = \frac{(-1)^n}{n}$ which converges to 0 but does not have a monotone behavior.

**Example.** Define the sequence $(a_n)$ recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Use induction to show that $a_n \leq 2$ for every $n$. We claim that $\forall n \geq 1$, $a_n \leq 2$. Let our base case be $n = 1$. Then $a_1 = 1 \leq 2$. Suppose for our induction hypothesis that the claim holds for $n = k (k \geq 1)$, that is, suppose that $a_k \leq 2$. Our goal is to show that the claim holds for $n = k+1$, that is, prove that $a_{k+1} \leq 2$. Hence, we have

$$a_{k+1} = \frac{1}{2}a_k + 1 \underbrace{\leq}_{a_k \leq 2} \frac{1}{2}(2) + 1 = 1 + 1 = 2.$$

(ii) Use induction to show that $(a_n)$ is an increasing sequence. Our claim that for all $n$, $a_n \leq a_{n+1}$. Let our base case be $n = 1$. Then $a_1 = 1$ and

$$a_2 = \frac{1}{2}a_1 + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2}.$$

Hence, this implies that $a_1 \leq a_2$. Suppose that the claim for $n = k + 1$, that is, prove that $a_{k+1} \leq a_{k+2}$. Now, we have

$$\begin{aligned} a_k \leq a_{k+1} &\implies \frac{1}{2}a_k \leq \frac{1}{2}a_{k+1} \\ &\implies \frac{1}{2}a_k + 1 \leq \frac{1}{2}a_{k+1} + 1 \\ &\implies a_{k+1} \leq a_{k+2}. \end{aligned}$$

(iii) We will show that (i) and (ii) will imply that $(a_n)$ converges. We have

(1) $\forall n$, $1 = a_1 \leq a_n \leq 2 \implies (a_n)$ is bounded,

(2) $(a_n)$ is increasing.

By the Monotone Convergence Theorem, we see that $(a_n)$ converges.

(iv) Now, we will show that $\lim_{n\to\infty} a_n = 2$. Let $A = \lim_{n\to\infty} a_n$. We have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[\frac{1}{2}a_n + 1\right] &\implies \lim_{n\to\infty} a_{n+1} = \frac{1}{2}(\lim_{n\to\infty} a_n) + 1 \\ &\implies A = \frac{1}{2}A + 1 \\ &\implies \frac{1}{2}A = 1 \\ &\implies A = 2. \end{aligned}$$

### 9.2.2 Extended Real Numbers

The set of extended real number, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols $+\infty, -\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

(*) $\overline{\mathbb{R}}$ is equipped with an order. We preserve the original order in $\mathbb{R}$ and we define

$$\forall x \in \mathbb{R} \quad -\infty < x < \infty.$$

(*) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{aligned}
&\forall x \in \overline{\mathbb{R}} \text{ with } x > 0 \ x \cdot (+\infty) = +\infty \quad x \cdot (-\infty) = -\infty \\
&\forall x \in \overline{\mathbb{R}} \text{ with } x < 0, x \cdot (+\infty) = -\infty \ x \cdot (-\infty) = +\infty \\
&\forall x \in \mathbb{R}, \ x + \infty = +\infty \\
&\forall x \in \mathbb{R}, \ x - \infty = -\infty.
\end{aligned}$$

As for the last convention, we define

$$+\infty + \infty = +\infty \quad -\infty - \infty = -\infty.$$

Furthermore, we have

$$\forall x \in \mathbb{R} \ \ \frac{x}{+\infty} = 0 \ \ \frac{x}{-\infty} = 0.$$

Please notice that we did <u>not</u> define the following

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \ldots, 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0.$$

(*) $\sup A = \infty \iff$ either $\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and $A$ is not bounded above in $\mathbb{R} \cup \{-\infty\}$.

(*) $\inf A = -\infty \iff -\infty$ or $A \subseteq \mathbb{R} \cup \{\infty\}$ and $A$ is not bounded below in $\mathbb{R} \cup \{\infty\}$.

(*) $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

**Remark.** Let $(a_n)$ bea sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

(i) $\lim\limits_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ |a_n - a| < \varepsilon$

(ii) $\lim\limits_{n\to\infty} a_n = \infty \iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ a_n > M$

(iii) $\lim\limits_{n\to\infty} a_n = -\infty \iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ -a_n > M$.

**Theorem** (Algebraic Limit Theorem for $\overline{\mathbb{R}}$)**.** Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $c \in \mathbb{R}$, then $ca_n \to ca$

(ii) $a_n + b_n \to a + b$ (provided that $\infty - \infty$ does not appear)

(iii) $a_n b_n \to ab$ (provided that $(\pm\infty) \cdot 0$ or $0 \cdot (\pm\infty)$ do not appear).

(iv) If $a = \infty$, then $\frac{1}{a_n} \to 0$. If $a = -\infty$, then $\frac{1}{a_n} \to 0$.

(v) If $a_n \to 0$ and $a_n > 0$, then $\frac{1}{a_n} \to \infty$. If $a_n \to 0$ and $a_n < 0$, then $\frac{1}{a_n} \to -\infty$.

**Theorem** (Order Limit Theorem for $\overline{\mathbb{R}}$). Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $a_n \leq b_n$, then $a \leq b$.

(ii) If $a_n \leq e_n$ and $a_n \to \infty$, then $e_n \to \infty$.

(iii) If $e_n \leq a_n$ and $a_n \to -\infty$, then $e_n \to -\infty$.

**Theorem** (Monotone Convergence Theorem in $\overline{\mathbb{R}}$). Let $(a_n)$ be a sequence in $\overline{\mathbb{R}}$.

(i) If $(a_n)$ is increasing, then $a_n \to \sup\{a_n : n \in \mathbb{N}\}$.

(ii) If $(a_n)$ is decreasing, then $a_n \to \inf\{a_n : n \in \mathbb{N}\}$.

Note that if $a_n \to \infty$, then $\frac{1}{a_n} \to 0$, however, $\frac{1}{a_n} \to 0$ does not imply that $a_n \to \infty$. For example, consider $(a_n) = (-n)$ or $(a_n) = ((-1)^n n)$.

**Remark.** (i) $\overline{\mathbb{R}}$ can be equipped with the following metric:

Let $f : \overline{\mathbb{R}} \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ be defined by

$$f(x) = \begin{cases} \frac{-\pi}{2} & x = \infty \\ \arctan(x) & -\infty < x < \infty \\ \frac{\pi}{2} & x = \infty \end{cases}$$

Define $\overline{d}(x, y) = |f(x) - f(y)| \; \forall x, y \in \overline{\mathbb{R}}$. The closure of $\mathbb{R}$ in $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(ii) One can show that if $(a_n)$ is a sequence in $\mathbb{R}$, then

$$a_n \to a \in \overline{\mathbb{R}} \iff \text{The sequence } a_n \text{ converges to } a \text{ in the metric space } (\overline{\mathbb{R}}, \overline{d}).$$

(iii) The closure of $\mathbb{R}$ in the metric space $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(iv) Every set in $(\overline{\mathbb{R}}, \overline{d})$ is bounded:

$$\forall x, y \in \overline{\mathbb{R}} \quad \overline{d}(x, y) \leq \pi.$$

# Chapter 10

# Week 10

## 10.1 Lecture 18-19

### 10.1.1 Topics

- lim sup and lim inf (Two equivalent characterizations)
- Theorem: $\liminf a_n \leq \limsup a_n$.
- Theorem: $\lim\limits_{n\to\infty} a_n$ exists in $\overline{\mathbb{R}}$ if and only if $\liminf a_n = \limsup a_n \in \overline{\mathbb{R}}$.
- Theorem: $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ provided that the right-hand side is not $\infty - \infty$.
- Some special sequences

### 10.1.2 First Characterization of Limsup and Liminf

Let $(x_n)$ be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \text{there exists a subsequence } (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \to x\}.$$

We define,

$$\limsup x_n = \sup S$$
$$\liminf x_n = \inf S.$$

### 10.1.3 Second Characterization of Limsup and Liminf

Let $(x_n)$ be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \geq n\}$. Clearly, we have

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \ .$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \cdots \qquad \text{(A decreasing sequence in } \overline{\mathbb{R}}\text{)}$$

and similarly, we have

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \dots \ . \qquad \text{(An increasng sequence in } \overline{\mathbb{R}}\text{)}$$

By the Monotone Convergence Theorem (in $\overline{\mathbb{R}}$), we know that $\lim\limits_{n\to\infty} \sup F_n$ and $\lim\limits_{n\to\infty} \inf F_n$ exists in $\overline{\mathbb{R}}$. We define

$$\limsup x_n = \lim_{n\to\infty} \sup F_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf F_n.$$

That is, we have

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf_n(\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup_n(\inf F_n).$$

Take note of the following notation:

$$\limsup x_n = \lim_{n\to\infty} \sup x_n = \overline{\lim} x_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf x_n = \underline{\lim} x_n.$$

**Example.** (i) $x_n = (-1)^n$

Notice that

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \sup\{-1, 1\} = \lim_{n\to\infty} 1 = 1$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 1\} = \lim_{n\to\infty} -1 = -1.$$

(ii) Consider $(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots)$

Then we have

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \{-1, 2, 3\} = \lim_{n\to\infty} 3 = 3$$
$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 2, 3\} = \lim_{n\to\infty} -1 = -1.$$

(iii) Consider $a_n = n$

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \sup\{n, n+1, n+2, \dots\}$$
$$= \lim_{n\to\infty} n = \infty.$$

and similarly, we have

$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{n, n+1, n+2, \dots\}$$
$$= \lim_{n\to\infty} n = \infty.$$

**Remark.** (i) $\liminf x_n = \sup_n \inf\{x_k : k \geq n\}$

(ii) $\limsup x_n = \inf_n \sup\{x_k : k \geq n\}$

**Theorem.** Let $(a_n)$ be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n.$$

**Proof.** Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}.$$

Since we already proved that the limits of both sides exists (in $\overline{\mathbb{R}}$), it follows from the order limit theorem (in $\overline{\mathbb{R}}$) that

$$\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

That is, we have

$$\liminf a_n \leq \limsup a_n.$$

■

**Theorem.** Let $(a_n)$ be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n \text{ exists in } \overline{\mathbb{R}} \text{ if and only if } \limsup a_n = \liminf a_n.$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.

**Proof.** ($\Longleftarrow$) Let $A = \limsup a_n = \liminf a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\lim a_n = A$. We may consider three cases; that is,

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), note that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}.$$

Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that $\lim_{n\to\infty} a_n = A$.

For (2) ($A = \infty$), we have for all $n \in \mathbb{N}$ that $\inf\{a_k : k \geq n\} \leq a_n$ and $\lim_{n\to\infty} \inf\{a_k : k \geq n\} = \infty$ implies $\lim_{n\to\infty} a_n = \infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

For (3) ($A = -\infty$), we know that for all $n \in \mathbb{N}$ that $a_n \leq \sup\{a_k : k \geq n\}$ and $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = -\infty$ implies that $\lim_{n\to\infty} a_n = -\infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

($\Longrightarrow$) Let $A = \lim_{n\to\infty} a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\limsup a_n = A$ and $\liminf a_n = A$. We may consider three cases:

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), suppose that $A \in \mathbb{R}$. Our goal is to show that

$$A \leq \liminf a_n \text{ and } \limsup a_n \leq A,$$

and so

$$A \leq \liminf a_n \leq \limsup a_n \leq A.$$

Thus, it suffices to show that for all $\varepsilon > 0$

$$A - \varepsilon \leq \liminf a_n \text{ and } \limsup a_n \leq A + \varepsilon.$$

To this end, let $\varepsilon > 0$ be given. Since $a_n \to A$, there exists an $N \in \mathbb{N}$ such that

$$\forall n > N \quad |a_n - A| < \varepsilon;$$

that is,

$$\forall n > N \quad A - \varepsilon < a_n < A + \varepsilon.$$

Now, observe that

$$\begin{aligned}
\forall n > N \quad a_n < A + \varepsilon &\Longrightarrow A + \varepsilon \text{ is an upper bound of } \{a_k : k \geq n\} \\
&\Longrightarrow \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \varepsilon \\
&\Longrightarrow \lim_{n\to\infty} \sup\{a_k : k \geq n\} \leq \lim_{n\to\infty} (A + \varepsilon) \qquad \text{(Order Limit Theorem)} \\
&\Longrightarrow \limsup a_n \leq A + \varepsilon
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
\forall n > N \quad A - \varepsilon < a_n &\Longrightarrow A - \varepsilon \text{ is a lower bound of} \{a_k : k \geq n\} \\
&\Longrightarrow \forall n > N \ \inf\{a_k : k \geq n\} \geq A - \varepsilon \\
&\Longrightarrow \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} A - \varepsilon \\
&\Longrightarrow \liminf a_n \geq A - \varepsilon.
\end{aligned}$$

Now, suppose (2). Our goal is to show that $\liminf a_n = \infty$ so that $\liminf a_n \leq \limsup a_n$ will imply that $\limsup a_n = \infty$. In order to show that $\liminf a_n = \infty$, it suffices to show that

$$\forall M > 0 \quad M \leq \liminf a_n.$$

To this end, let $M > 0$ be given. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
\forall n > N \quad a_n > M &\Longrightarrow \forall n > N \ \inf\{a_k : k \geq n\} \geq M \\
&\Longrightarrow \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} M \\
&\Longrightarrow \liminf a_n \geq M.
\end{aligned}$$

Note that an analogous process to the above is used to prove (3). ■

**Theorem.** Let $(a_n)$ and $(b_n)$ be the two sequences of real numbers. Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that the right-hand side is not of the form $\infty - \infty$ or $-\infty + \infty$.

**Proof.** First note that, by our assumption, $\limsup a_n + \liminf a_n$ is not of the form $\infty - \infty$ or $(-\infty + \infty)$, there exists $n_0$ such that

$$\forall n \geq n_0 \quad \sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\} \text{ is not of the form } \infty - \infty \text{ or } -\infty + \infty$$

For each $n \geq n_0$, we have

$$\begin{aligned}
\forall k \geq n \quad a_k \leq \sup\{a_\ell : \ell \geq n\} \\
\forall k \geq n \quad b_k \leq \sup\{b_m : m \geq n\}.
\end{aligned}$$

Thus, we have

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Therefore,

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Now, label $R_n = \sup\{a_k + b_k : k \geq n\}$, $L_n = \sup\{a_\ell : \ell \geq n\}$ and $S_n = \sup\{b_m : m \geq n\}$. From the above, we can see that $\lim\limits_{n\to\infty} R_n$, $\lim\limits_{n\to\infty} L_n$, and $\lim\limits_{n\to\infty} S_n$ all exists in $\overline{\mathbb{R}}$. Since $\lim\limits_{n\to\infty} L_n + \lim\limits_{n\to\infty} S_n$ is not of the form $\infty - \infty$, it follows form the Algebraic Limit Theorem that $\lim\limits_{n\to\infty}(L_n + S_n)$ exists and is equal to that of $\lim L_n + \lim S_n$. By the Order Limit Theorem, we see that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

■

**Theorem** ((e))**.** If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$.

**Proof.** Clearly, if x = 0, then the claim holds. So, let's assume $x \in (-1, 1)$ and $x \neq 0$. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ |x^n - 0| < \varepsilon.$$

That is, we need to show, given the setup above, that $|x^n| < \varepsilon$. Since $0 < |x| < 1$, there exists $y > 0$ such that $|x| = \frac{1}{1+y}$. Note that

$$|x|^n < \varepsilon \iff \frac{1}{(1+y)^n} < \varepsilon.$$

Using the Binomial Theorem ($(1 + y)^n \geq 1 + ny$), we can see that

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}.$$

Therefore, in order to ensure that $|x|^n < \varepsilon$, we just need to choose $n$ large enough so that $\frac{1}{ny} < \varepsilon$. To this end, it suffices to choose $n$ larger than $\frac{1}{\varepsilon y}$; that is, we can take $N = \frac{1}{\varepsilon y}$ and the result follows. ■

**Theorem** ((b))**.** If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.

**Proof.** If $p = 1$, the claim obviously holds. If $p \neq 1$, we may consider two cases.

For the first case, assume that $p > 1$. Then let $x_n = \sqrt[n]{p} - 1$. It suffices to show that $\lim\limits_{n\to\infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also, we have

$$\begin{aligned} \sqrt[n]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n}. \end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim\limits_{n\to\infty} x_n = 0$.

Now, suppose that $0 < p < 1$. Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by the previous case, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1 \iff \lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = 1.$$

■

**Theorem ((c)).** $\lim_{n\to\infty} \sqrt[n]{n} = 1$

**Proof.** Let $x_n = \sqrt[n]{n} - 1$. Observe that, by the binomial formula, we have for all $n \geq 2$,

$$\begin{aligned}
\sqrt[n]{n} = 1 + x_n &\implies n = (1+x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \\
&\implies \frac{2n}{n(n-1)} \geq x_n^2 \\
&\implies x_n \leq \sqrt{\frac{2}{n-1}}.
\end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \to 0$ and so $\sqrt[n]{n} \to 1$. ■

# Chapter 11

# Week 11

## 11.1 Lecture 20-21

### 11.1.1 Topics

- Infinite series (Basic definitions)
- Telescoping Series, Geometric Series
- Algebraic Limit Theorem for Series
- Divergence Test
- Cauchy Criterion for Series
- Absolute Convergence Test

Consider the following expression:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\ .$$

How can we make sense of the infinite sum above? More generally, let $(a_n)$ be a sequence of real numbers. Then what does the following expression mean?

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots\ ?$$

**Definition** (Infinite Series). Let $(X, \|\cdot\|)$ be a normed space. Let $(x_n)$ be a sequence in $X$.

(*) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1+x_2+x_3+\cdots$$

is called an **infinite series**.

(*) $x_1, x_2, \ldots$ are called the **terms** of this infinite series.

(*) The corresponding sequence of **partial sums** is defined by

$$\forall m \in \mathbb{N} \quad s_m = \text{(finite) sum of the first } m \text{ terms of the series;}$$

that is,

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + x_2 + \cdots + x_m \\ &\vdots \end{aligned}$$

1. We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty} x_n = L$) if $\lim_{n\to\infty} s_m = L$.
2. We say that the infinite series **diverges**, if $(s_m)$ diverges.
3. If $X = \mathbb{R}$ and $s_m \to \infty$, we write $\sum_{n=1}^{\infty} x_n = \infty$.

(*) If $X = \mathbb{R}$ and $s_m \to -\infty$, we write $\sum_{n=1}^{\infty} x_n = -\infty$.

**Remark (1).** Given an infinite series $\sum_{n=1}^{\infty} x_n$, it is important to keep a clear distinction between

(a) the sequence of terms: $(x_1, x_2, x_3, \dots)$

(b) the sequence of partial sums: $(s_1, s_2, s_3, \dots)$.

**Remark (2).** We may sometimes consider infinite series where the summation begins with $n = 0$ or $n = n_0$ for some integer $n_0$ different from 1.

As we shall see, some of our theorems apply specifically to series in $\mathbb{R}$ or to series with terms in $[0, \infty)$. Also, in our examples, we will primarily focus on series in $\mathbb{R}$; however, we will also consider encounter highly useful theorems that hold in more general normed spaces.

In most cases it is difficult (or even impossible) to find a simple formula for the partial sum $s_m$. However, there are two types of series for which we can easily find a simple formula for the partial sums. These two types are:

(1) Telescoping Series

(2) Geometric Series

### 11.1.2 Telescoping Series

**Example.** Consider the following series

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Notice that $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned}
s_1 &= 1 - \frac{1}{2} \\
s_2 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) = 1 - \frac{1}{3} \\
s_3 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) = 1 - \frac{1}{4} \\
&\vdots \\
s_m &= \sum_{n=1}^{m} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = \Big(\sum_{n=1}^{m} \frac{1}{n}\Big) - \Big(\sum_{n=1}^{m} \frac{1}{n+1}\Big) \\
&= 1 - \frac{1}{m+1}.
\end{aligned}$$

Clearly, we see that

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \Big[1 - \frac{1}{m+1}\Big] = 1.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

In general, a telescoping series is an infinite series where partial sums eventually have a finite number of terms after cancellation. For example, if $(y_n)$ is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n - y_{n+1})$ is a telescoping series; that is,

$$\begin{aligned}
s_m = \sum_{n=1}^{m}(y_n - y_{n+1}) = \Big(\sum_{n=1}^{m} y_n\Big) - \Big(\sum_{n=1}^{m} y_{n+1}\Big) &= [y_1 + y_2 + \cdots + y_m] - [y_2 + y_3 + \cdots + y_{m+1}] \\
&= y_1 - y_m.
\end{aligned}$$

### 11.1.3 Geometric Series

Let $k$ be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \cdots$ is called a **geometric series** with common ration "$r$". For example,

$$\sum_{n=1}^{\infty} \Big(\frac{1}{2}\Big)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ is a geometric series with common ratio } \frac{1}{2}.$$

Another example is that

$$\sum_{n=1}^{\infty} \frac{7^n}{29^n} \text{ is a geometric series with common ratio } \frac{7}{29}.$$

A non-example is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We can easily find a formula for the $m$th partial sum of $\sum_{n=k}^{\infty} r^k$ where

$$\begin{aligned}
s_1 &= r^k \\
s_2 &= r^k + r^{k+1} \\
s_3 &= r^k + r^{k+1} + r^{k+2} \\
&\vdots \\
s_m &= r^k + r^{k+1} + \cdots + r^{k+m-1} \qquad (*)
\end{aligned}$$

Now, if $r = 1$, we have
$$s_m = \underbrace{1 + 1 + \cdots + 1}_{m\text{summands}} = m.$$

If $r \neq 1$, then multiply both sides of (*) by $r$:
$$rs_m = r^{k+1} + r^{k+2} + \cdots + r^{k+m}. \tag{**}$$

Subtracting (**) from (*), we get
$$s_m - rs_m = r^k - r^{k+m}.$$

Since $r \neq 1$, we have
$$s_m = \frac{r^k - r^{k+m}}{1-r} = \frac{r^k(1-r^m)}{1-r}.$$

Note that

(i) If $|r| < 1$, then $\lim r^m = 0$.

(ii) If $|r| > 1$ or $r = -1$, then $\lim\limits_{n\to\infty} r^m$ does not exists.

Hence, we have
$$\lim_{m\to\infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{DNE} & \text{if} |r| \geq 1. \end{cases}$$

Thus,
$$\sum_{n=k}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{diverges} & \text{if} |r| \geq 1. \end{cases}$$

**Example.**

- $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

  Observe that
  $$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^1}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

- $\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n$

  Observe that
  $$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)} = \frac{1}{8}.$$

**Theorem.** Let $(X, \|\cdot\|)$ be a normed space. Let $(a_n)$ and $(b_n)$ be two sequence in $X$. Suppose that
$$\sum_{n=1}^{\infty} a_n = A \quad (A \in X), \quad \sum_{n=1}^{\infty} b_n = B \quad (B \in X).$$

Then

(i) For any scalar $\lambda$, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$.

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$.

**Proof.** Can easily be proven via the Algebraic Limit Theorem for Sequences. ■

**Theorem** (Divergence Test)**.** Let $(X, \|\cdot\|)$ be a normed space. Let $(x_n)$ be a sequence in $X$. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim\limits_{n\to\infty} x_n = 0$.

**Proof.** Let $s_n = x_1 + \cdots + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n\to\infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Note that $\lim s_n = L$ and $\lim s_{n-1} = L$. Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (s_n - s_{n-1}) = L - L = 0$$

by the Algebraic Limit Theorem for normed spaces. ■

**Remark.** Note that the divergence test is just the contrapositive of the above.

**Example.**

- $\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim\limits_{n\to\infty} (-1)^n$ does not exist.
- $\sum_{n=1}^{\infty} \frac{3n+1}{7n-4}$ diverges because $\lim\limits_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

From the above statements, we can now see make two key observations:

- If $\lim\limits_{n\to\infty} x_n = 0$, then $\sum_{n=1}^{\infty} x_n$ may or may not converge.
- If $\lim\limits_{n\to\infty} x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ diverges.

As for the first observation above, we see that $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^2}$ converges.

**Theorem** (Cauchy Criterion)**.** Let $(X, \|\cdot\|)$ be a complete normed space. Let $(x_n)$ be a sequence in $X$. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges } \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=1}^{n} x_k\| < \varepsilon.$$

**Proof.** Let $s_n = x_1 + \cdots + x_k$. Assuming that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\|$$

where $n > m$ and from the fact that

$$\begin{aligned} s_n - s_m &= (x_1 + \cdots + x_m + \cdots + x_n) - (x_1 + \cdots + x_m) \\ &= \sum_{k=m+1}^{n} s_k. \end{aligned}$$

Then we have

$$\begin{aligned}\sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges}\\ &\iff (s_k) \text{ is Cauchy}\\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \varepsilon\\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=m+1}^{n} x_k\| < \varepsilon\end{aligned}$$

as desired. ■

From here, we will refer to complete normed spaces as Banach spaces.

**Theorem** (Absolute Convergence Test)**.** Let $(X, \|\cdot\|)$ be a Banach Space. Let $(x_n)$ be a sequence in $X$. If $\underbrace{\sum_{n=1}^{\infty} \|x_n\|}_{\text{a sum in } \mathbb{R}}$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

**Proof.** By the Cauchy Criterion for series, it suffices to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \left\| \sum_{k=m+1}^{n} x_k \right\| < \varepsilon. \tag{*}$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for series that there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{n} \|x_k\| \right| < \varepsilon.$$

We claim that $\hat{N}$ is the same $N$ we were looking for. Hence, if $n > m > \hat{N}$, then we have

$$\left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| = \left| \sum_{k=m+1}^{n} \|x_k\| \right| < \varepsilon$$

as desired. ■

Please take note of the following observations:

(1) If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges (in Banach spaces).

(2) If $\sum_{n=1}^{\infty} \|x_n\|$ diverges, then $\sum_{n=1}^{\infty} x_n$ may converge or diverge.

From (2), we shall see (in the next lecture) that

(1) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ diverges but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

(2) $\sum_{n=1}^{\infty} |(-1)^n|$ diverges, also $\sum_{n=1}^{\infty} (-1)^n$ diverges (by the divergence test).

**Definition** (Absolute Convergence and Conditional Convergence)**.** We say that a series $\sum x_n$ **absolutely converges** if $\sum \|x_n\|$ converges and $\sum x_n$ converges. We say that $\sum x_n$ **conditionally converges** if $\sum \|x_n\|$ diverges but $\sum x_n$ converges.

**Example** (Conditionally Convergent). Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We see that this series is conditionally convergent since

$$\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \to 0$$

which tell us that the above series diverges by the divergence test. But the above series converges via the Leibniz Test (As we shall see in the next lecture).

## 11.2 Lecture 21-22

### 11.2.1 Topics

- Cauchy Condensation Test
- Comparison Test
- More on $\limsup$ and $\liminf$
- Root Test
- Ratio Test
- Dirichlet's Test

**Theorem** (Cauchy Condensation Test). Assume $a_n \geq 0$ for all $n$, and $(a_n)$ is a decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \text{ converges.}$$