Math 230A: Homework 5

Lance Remigio

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Problem 1. Mark each statement True or False. Let (X, d) be a metric space and $K \subseteq X$.

1. If $Y \subseteq X$ and $\{G_\alpha\}$ is a collection of subsets of Y that are open relative to Y, then $\bigcup_\alpha G_\alpha$ is open relative to Y. **True.**
2. If there exists some open cover of K which has a finite subcover, then K is compact. **False.**
3. If K is compact, then $K' \subseteq K$. **True.**
4. If K is closed, then K is compact. **False.**
5. If K is compact and $E \subseteq K$, then E is compact. **False.**
6. Consider $E = [-10, 10] \subseteq \mathbb{R}$ and the open cover of E by $\Phi = \{(x-1, x+1) : x \in E\}$. Then the collection $\{(x - \frac{1}{2}, x + \frac{1}{2}) : x \in E\}$ is a subcover of Φ. **False.**
7. Let E and Φ be as above. Then the collection $\{(x-1, x+1) : x = -10, -9, -8, \ldots, 8, 9, 10\}$ is a subcover of Φ. **True.**
8. Let E and Φ be as above. Then the collection $\{(x-1, x+1) : x = -10, -8, -6, \ldots, 6, 8, 10\}$ is a subcover of Φ. **True.**

Problem 2. Show that compact implies bounded.

Proof. Let $E \subseteq X$. Suppose E is compact. Our goal is to show that E is bounded set in X; that is, there exists $p \in X$ and $\delta > 0$ such that $E \subseteq N_\delta(q)$. Let $x \in E$. To this end, we will show that there exists a $p \in X$ and $\delta > 0$ such that $E \subseteq N_\delta(x)$. Since E is compact, E is a closed set. Hence, we know that

$$E = \overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Choose $\delta = \varepsilon + 1$. Since E is closed, then there exists a $p \in N_\varepsilon(x) \cap E$ such that $p \in N_\varepsilon(x)$ and $p \in E$ (that is, $p \in X$). Since $p \in N_\varepsilon(x)$, we have

$$d(x, p) < \varepsilon < \varepsilon + 1 = \delta.$$

Hence, we see that $p \in N_\delta(x)$. Thus, $E \subseteq N_\delta(x)$ and so E is bounded. ∎

Problem 3. Show the union of finitely many compact sets is compact.

Proof. Let (X, d) be a metric space. Set $K = \bigcup_{i=1}^n K_i$ with K_i compact for all $1 \leq i \leq n$. Our goal is to show that K is compact. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X that forms an open cover for K. Since K_i is compact for all $1 \leq i \leq n$, we can find a finite subcover for each K_i. Thus, we have

$$K_i \subseteq \bigcup_{j=1}^{m_i} O_{i,j} \quad 1 \leq j \leq n_i.$$

Note that each $\bigcup_{j=1}^{m_i} O_{i,j}$ is open since each $O_{i,j}$ is open. Furthermore,

$$\sum m_i = n.$$

Denote each of these unions as

$$Q_{\alpha_i} = \bigcup_{j=1}^{m_i} O_{i,j}.$$

Thus, we see that

$$K = \bigcup_{i=1}^{n} K_i \subseteq \bigcup_{i=1}^{n} Q_{\alpha_i}$$

and so K is compact. ■

Problem 4. Show that an arbitrary intersection of compact sets is compact. (Hint. A closed subset of a compact set is compact)

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X and let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets in X. Define

$$K = \bigcap_{\alpha} K_\alpha. \tag{1}$$

Since each K_α is compact, we know that each K_α must be closed. Therefore, the arbitrary intersection above must be closed and so K is closed. But observe that

$$\bigcup_{\alpha} K_\alpha \subseteq K_\alpha$$

and that K_α is a closed set. Thus, K must be compact as well! ■

Problem 5. A metric space (X, d) is called **separable** if it contains a countable subset E which is dense in X. For example, $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable set which is dense in $\mathbb{R}$. Show that $\mathbb{R}^2$ is separable. (Hint: Consider the set of points which have only rational coordinates.)

Proof. Note that $\mathbb{R}^2$ is just $\mathbb{R} \times \mathbb{R}$. Recall that a finite product of countable sets is countable. Thus, we see that $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ is countable. All that is left to show is that $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there must exists $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ such that $x_1 < p < x_2$ and $y_1 < q < y_2$, respectively. Denote the points in $\mathbb{R}^2$ as $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ and $r = (p, q)$. Thus, we have that

$$p_1 < r < p_2.$$

Thus, $\mathbb{Q}^2$ must be dense in $\mathbb{R}^2$. ■

Problem 6. Let (X, d) be a separable metric space and $\emptyset \neq A \subseteq X$. Prove that the collection of the isolated points of A is at most countable.

Proof. Let (X, d) be a separable metric space and $\emptyset = A \subseteq X$. Denote the set of isolated points as

$$A_I = \{x \in X : \exists \varepsilon > 0 \text{ such that } N_\varepsilon(x) \cap A = \emptyset\}.$$

By assumption, X contains a subset E such that E is countable and $\overline{E} = X$. Our goal is to show that A_I is at most countable. It suffices to show that $A_I \subseteq E$. Let $x \in A_I$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \cap A = \emptyset$. Hence, $N_\varepsilon(x) \subseteq A^c$ for some $\varepsilon > 0$ and so A^c is an open set in X. By problem 19 of homework 4, we see that

$$A^c \cap E \neq \emptyset.$$

Thus, x must be contained in the intersection above. So, $A_I \subseteq E$. Since E is countable, A_I must be at most countable and we are done. ■

Problem 7. Let (X, d) be a metric space. A collection $\{V_\alpha\}$ of open subsets of X is said to be a **base** for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that the every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.)

Proof. Since (X, d) is a separable metric space, we know that X contains a countable dense subset E; that is, we have

$$X = \overline{E} = E = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Let $\{V_\alpha\}_{\alpha \in \Lambda}$. Let $x \in X$ and let $G \subseteq X$ be an open set such that $x \in G$. Our goal is to show that $V_\alpha \subseteq G$ for some α. Since $X = \overline{E}$, we must have $x \in \overline{E}$; that is, for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

To this end, let $\varepsilon = \frac{1}{n}$ and pick a point $y \in N_\varepsilon(x) \cap E$. Then

$$y \in N_{1/n}(x) \text{ and } y \in E.$$

Note that $N_{1/n}(x)$ is an open set in X. Thus, we can write this set in the following way

$$N_{1/n}(x) = \bigcup_\alpha V_\alpha.$$

Thus, y must be contained in the union above and so $y \in V_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since $y \in G$, we must also have $V_\alpha \subseteq G$ and we are done. ■

Problem 8. Let (X, d) be a metric space in which every infinite subset has a limit point. Prove that X is separable. (Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.)

Proof. Let (X, d) be a metric space in which every infinite subset has a limit point. Our goal is to show that X is separable. Fix $\delta > 0$, and pick $x_1 \in X$. Now, choose $x_1, \ldots, x_j \in X$ and then choose $x_{j+1} \in X$ such that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$.

We claim that if this leads to an infinite subset $E = \{x_1, x_2, \ldots\}$ of X that has the property that for any $x_i, x_j \in E$, $d(x_i, x_j) \geq \delta$, then E contains no limit points. Suppose for sake of contradiction that E does have a limit point. Denote this limit point of E as w. Thus, for any $\delta > 0$, we have

$$N_\delta(w) \cap (E \setminus \{w\}) \neq \emptyset. \tag{1}$$

Now, pick a point in this intersection, say, x such that

$$x \in N_\delta(w) \text{ and } x \in E.$$

Hence, we see that $d(x, w) < \delta$. Similarly, pick y in the intersection in (1) such that $y \in N_\delta(w)$. Then we have $d(y, w) < \delta$. Our goal is to find some $\varepsilon > 0$ such that $d(x, y) < \varepsilon$, contradicting our assumption that $d(x, y) \geq \delta$ for all $\delta > 0$. Choose $\varepsilon = 2\delta$. Using the triangle inequality, we see that

$$d(x, y) \leq d(x, w) + d(w, y) < \delta + \delta = 2\delta = \varepsilon$$

which produces desired contradiction. Hence, the process outlined in the first paragraph must terminate; that is, $E = \{x_1, x_2, \dots\}$ must be finite. From this, we denote this set as $E_n = \{x_1, x_2, \dots, x_n\}$.

Now, our goal is to find a subset of X that is both dense and countable. Consider the set

$$E_n = \{x_1, x_2, \dots, x_n\}.$$

Clearly, this set is at most countable. Now, consider the countable union

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Since each E_n is at most countable, we know that E must be at most countable.

Now, we will show that S is dense in X; that is, $\overline{S} = X$. Note that we can now cover X by finitely many neighborhoods of radius δ(By the process outlined in the hint). Fix $\delta = \frac{1}{n}$ and denote

$$X = \bigcup_{x \in E_n} N_{1/n}(x).$$

Hence, pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ by the Archimeadean Property. Consider the neighborhood $N_\varepsilon(x)$. If we let $y \in E_n$ with $x \in N_{1/n}(y)$, then we see that

$$d(x, y) < \frac{1}{n} < \varepsilon.$$

This implies that $y \in N_\varepsilon(x)$. Since y is also contained in S, we can conclude that

$$y \in N_\varepsilon(x) \cap S \implies N_\varepsilon(x) \cap S \neq \emptyset,$$

showing that S is dense in X. Hence, X is a separable metric space. ■

Problem 9 (Extra Credit). Let (X, d) be a metric space and Y be a nonempty subset of X. Let $E \subseteq Y$. Prove that

$$E \text{ is closed relative to } Y \iff E = A \cap Y \text{ for some closed set } A \subseteq X.$$

Proof. ($\implies$) Assume that E is closed relative to Y. Our goal is to show that $E = A \cap Y$; that is, we need to show two inclusions:

(1) $E \subseteq A \cap Y$,

(2) $A \cap Y \subseteq E$.

By assumption, we know that

$$E = \overline{E} = \{\forall x \in Y, \forall \varepsilon > 0 : N_\varepsilon^Y(x) \cap E \neq \emptyset\}.$$

Note that $N_\varepsilon(x) \subseteq \overline{N_\varepsilon(x)}$. Set

$$A = \bigcup_{i=1}^{n} \overline{N_{\varepsilon_i}(x)}$$

with radius $\varepsilon_i > 0$ for each N_{ε_i}. Notice that A must be a closed set in Y because each $\overline{N_{\varepsilon_i}(x)}$ is a closed set. Since

$$N_\varepsilon(x) \cap Y \subseteq \overline{N_\varepsilon(x)} \cap Y \subseteq E,$$

we must have

$$\begin{aligned} A \cap Y = \Big(\bigcup_{i=1}^{n} \overline{N_{\varepsilon_i}(x)}\Big) \cap Y &\subseteq \bigcup_{i=1}^{n} (\overline{N_{\varepsilon_i}(x)} \cap Y) \\ &\subseteq \bigcup_{i=1}^{n} E \\ &= E. \end{aligned}$$

Thus, we see that $A \cap Y \subseteq E$ which proves (1).

Now, we want to show (2). Let $p \in E$. Using our definition of A and the fact that $N_\varepsilon(p) \subseteq \overline{N_\varepsilon(p)}$,

$$\begin{aligned} E \subseteq N_\varepsilon(p) \cap Y \subseteq \overline{N_\varepsilon(p)} \cap Y &\subseteq \Big[\bigcup_{i=1}^{n} \overline{N_{\varepsilon_i}(x)}\Big] \cap Y \\ &= A \cap Y \end{aligned}$$

which shows (2). Hence, (1) and (2) imply that $E = A \cap Y$ for some closed set $A \subseteq Y$.

($\Longleftarrow$) Suppose $E = A \cap Y$ for some closed set $A \subseteq X$. Our goal is to show that E is closed relative to Y. Let p be a limit point of E. Since $E = A \cap Y$ and $p \in Y$, we see that

$$\begin{aligned} N_\varepsilon(p) \cap (E \setminus \{p\}) &\Longrightarrow N_\varepsilon(p) \cap ((A \cap Y) \setminus \{p\}) \neq \emptyset \\ &\Longrightarrow (N_\varepsilon(p) \cap Y) \cap (A \setminus \{p\}) \neq \emptyset \\ &\Longrightarrow N_\varepsilon^Y(p) \cap (A \setminus \{p\}) \neq \emptyset. \end{aligned}$$

This implies that p is a limit point of A. But A is a closed set in Y. Thus, p must be contained in A, showing that E is closed relative to Y. ∎

Problem 10 (Extra Credit)**.** Let (X, d) be a metric space. Let $E \subseteq X$. Prove that the following definitions of boundedness are equivalent:

- Rudin's Definition: There exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$.
- Anthony's Definition: There exists $R > 0$ such that for all x and y in E, we have $d(x, y) < R$.

Proof. ($\Longrightarrow$) Suppose there exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$. Let $x, y \in E$. Our goal is to show that there exists $R > 0$ such that for all $x, y \in E$, $d(x, y) < R$. To this end, let $x, y \in E$ be given. Choose $R = 2\varepsilon > 0$. Since $E \subseteq N_\varepsilon(q)$, we have that

$$d(x, y) \leq d(x, q) + d(q, y) < \varepsilon + \varepsilon = 2\varepsilon = R.$$

Thus, we see that $d(x, y) < R$ for any $x, y \in E$.

($\Longleftarrow$) Suppose there exists $R > 0$ such that for all $x, y \in E$, $d(x, y) < R$. We will show that there exists $q \in X$ and $\varepsilon > 0$ such that $E \subseteq N_\varepsilon(q)$. Let $x \in E$. Fix $y \in E$ such that $q = y$. Choose $\varepsilon = R + 1 > 0$. By assumption,

$$d(x, q) < R < R + 1 = \varepsilon.$$

Hence, $x \in N_\varepsilon(q)$. ∎