Homework 5

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Problem 1. Let A be a nonempty set of $\mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, $f_n : A \to \mathbb{R}$ is a *uniformly continuous* function on A. Prove that if (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is *uniformly continuous* on A.

Proof. Suppose that $f_n \to f : A \to \mathbb{R}$ uniformly. Our goal is to show that f is uniformly continuous; that is, we want to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x, y \in A$. Since $f_n \to f$ uniformly, there exists an $N \in \mathbb{N}$ such that for any $x \in A$ and for any $n > N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}. \tag{1}$$

Since f_n is uniformly continuous on A for all $n \in \mathbb{N}$. In particular, f_n is uniformly if $n = N + 1$; that is, there exists a$\hat{\delta} > 0$ such that for any $|x - y| < \hat{\delta}$, we have

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}. \tag{2}$$

We claim that $\hat{\delta}$ can be used as the same δ we were looking for. Indeed, whenever $|x - y| < \hat{\delta}$, (1) and (2) imply that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

as our desired result. ∎

Problem 2. Let A be a nonempty set and suppose $(f_n : A \to \mathbb{R})_{n \geq 1}$ is a sequence of functions. Suppose $f : A \to \mathbb{R}$ is a function. Prove that the following statement are equivalent:

(i) (f_n) converges uniformly to $f : A \to \mathbb{R}$.

(ii) $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ $\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon$.

(iii) $\lim\limits_{n \to \infty} (\sup_{x \in A} |f_n(x) - f(x)|) = 0$.

Proof. ($(i) \implies (ii)$) Suppose that $f_n \to f$ uniformly. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$

$$\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly, there exists an $\tilde{N} \in \mathbb{N}$ such that for all $x \in A$, for all

$n > \tilde{N}$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Note that $\varepsilon/2$ is an upper bound for the set

$$\{|f_n(x) - f(x)| : \forall x \in A \;\; \forall n > \tilde{N}\}.$$

We claim that $\tilde{N}$ is the same N we were looking for. Taking the supremum of the inequality above, we have

$$\sup_{x\in A} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

for any $n > \tilde{N}$ which is our desired result.

($(ii) \implies (iii)$) Suppose that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon.$$

Our goal is to show that $\lim\limits_{n\to\infty} \Big(\sup_{x\in A} |f_n(x) - f(x)|\Big) = 0$. By assumption, we can let $\varepsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$ such that there exists an $\kappa_n \in \mathbb{N}$ such that for any $n > \kappa_n$, we have k

$$0 \leq \sup_{x\in A} |f_n(x) - f(x)| < \frac{1}{n}.$$

Clearly, we see that $\frac{1}{n} \to 0$ as $n \to \infty$. Applying the squeeze theorem to the inequality above as $n \to \infty$, we have that

$$\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

($(iii) \implies (i)$) Suppose that $\lim\limits_{n\to\infty} \Big(\sup_{x\in A} |f_n(x) - f(x)|\Big) = 0$. Our goal is to show that $f_n \to f : A \to \mathbb{R}$ uniformly; that is, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $x \in A$, for any $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \Big(\sup_{x\in A} |f_n(x) - f(x)|\Big) = 0$, (with our given ε) there exists an $\tilde{N} \in \mathbb{N}$ such that for any $n > \hat{N}$

$$|\sup_{x\in A} |f_n - f(x)|| < \varepsilon;$$

that is, for any $n > \hat{N}$

$$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon.$$

Note that $|f_n(x) - f(x)| \leq \sup_{x\in A} |f_n(x) - f(x)|$ for all $x \in A$. We claim that $\hat{N}$ is the same N we were looking for. Hence, for any $n > \hat{N}$, we have

$$|f_n(x) - f(x)| < \varepsilon$$

Hence, $f_n \to f$ uniformly. ■

Problem 3. Suppose (a_n) and (b_n) are two sequences of real numbers and $a_n \geq b_n$ for all $n \in \mathbb{N}$. Suppose $\lim\limits_{n\to\infty} b_n > 0$. Explain in one line why it follows from the order limit theorem that $\lim\limits_{n\to\infty} a_n$ cannot be zero.

Proof. There exists $n_o \in \mathbb{N}$ such that $a_n \geq b_n$ for all $n \geq n_0$, $\lim\limits_{n\to\infty} a_n \geq \lim\limits_{n\to\infty} b_n > 0$. ■

Problem 4 (4-1)**.** For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^2}{n^4+x^2}$.

Proof. Clearly, f_n converges to f pointwise. Indeed, for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{x^2}{n^4 + x^2} \leq \frac{x^2}{n^4}.$$

Consider the right-hand side of the above inequality, we have $\lim_{n\to\infty} \frac{x^2}{n^4} = x^2 \lim_{n\to\infty} \frac{1}{n^4} = x^2 \cdot 0 = 0$. By applying the Squeeze theorem as $n \to \infty$ to the inequality above, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{n^4 + x^2} = 0$$

pointwise.

Now, we want to show that $f_n \to 0$ is NOT uniform. Immediately, we see that

$$\sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^4 + x^2} \right| \geq |f_n(x)|$$

for all $x \in \mathbb{R}$. In particular, if we let $x = n^2$, then we have

$$\sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^4 + x^2} \right| \geq |f_n(n^2)| = \frac{1}{2}.$$

Clearly, if we define $b_n = f(n^2)$, we have

$$\lim_{n\to\infty} f(n^2) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} > 0.$$

■

Problem 5 (4-2)**.** For each $n \in \mathbb{N}$, let $f_n : [0, 1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$. It is easy to show that $f_n \to 0$ pointwise. Prove that the convergence is NOT uniform.

Proof. We can easily show that $f_n \to 0$ pointwise. We will show that the convergence is NOT uniform. Define $b_n = \left(1 - \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Indeed, we see that

$$\sup_{x\in[0,1)} |f_n(x)| = \sup_{x\in[0,1)} |x^n| = \sup_{x\in[0,1)} \geq \left(1 - \frac{1}{n}\right)^n \tag{1}$$

and

$$\lim_{n\to\infty} \left(1 - \frac{1}{kn}\right)^n = \frac{1}{\varepsilon} > 0. \tag{2}$$

Thus, (1) and (2) imply that $f_n \to 0$ does NOT converge uniformly. ■

Problem 6. Suppose that $A = G \cup H$ where G and H are nonempty sets. Prove that if (f_n) converges uniformly to f on both G and H, then (f_n) converges uniformly to f on A.

Proof. Our goal is to show that for any $\varepsilon > 0$ be given and for any $x \in A$, for any $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $A = G \cup H$, we either have $x \in G$ or $x \in H$. If $x \in G$, then we can use the fact that $f_n \to f$ uniformly on G, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

On the other hand, if $x \in H$, then using the fact that $f_n \to f$ uniformly on H, there exists an

$N_2 \in \mathbb{N}$, with our given ε, such that for any $n > N_2$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Hence, in both cases $f_n \to f$ on A uniformly. ■

Problem 7. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and suppose that $a_n \to a$ in $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. For each $n \in \mathbb{N}$,define $f_n : A \to \mathbb{R}$ by $f_n(x) = f(x) + a_n$. Prove that (f_n) converges uniformly to the function $f + a$ on the set A.

Proof. Our goal is to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in A$ and for all $n > N$, we have

$$|f_n(x) - (f(x) + a)| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $a_n \to a$, there exists $\tilde{N} \in \mathbb{N}$ such that for any $n > \tilde{N}$, we have

$$|a_n - a| < \varepsilon.$$

We claim that $\tilde{N}$ is the same N we were looking for. Hence, for any $n > \tilde{N}$, we have

$$\begin{aligned}|f_n - (f(x) + a)| &= |(f(x) + a_n) - (f(x) + a)| \\ &= |a_n - a| \\ &< \varepsilon.\end{aligned}$$

Hence, $f_n \to f + a$ uniformly. ■

Problem 8. Suppose that (g_k) converges uniformly to g on the nonempty st A. Use the Cauchy Criterion for uniform convergence of sequences to prove that the sequence (h_k) where $h_k = g_{k+1} - g_k$ converges uniformly to zero on A.

Proof. Our goal is to show that $h_k \to 0$ uniformly on A; that is, we want to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $k > N$, we have

$$|h_k| < \varepsilon.$$

Let $\varepsilon > 0$ be given and let $x \in A$. Since $g_n \to g$ converges uniformly on A, the Cauchy Criterion implies that there exists an $\tilde{N} \in \mathbb{N}$ such that for any $m > n > \tilde{N}$, we have

$$|g_n(x) - g_m(x)| < \varepsilon. \tag{*}$$

We claim that $\tilde{N}$ can be used as the same N we were looking for. Indeed, for any $k + 1 > k > \tilde{N}$, (*) implies that

$$|g_{k+1}(x) - g_k(x)| < \varepsilon.$$

But we have $h_k = g_{k+1} - g_k$ for all $k \in \mathbb{N}$, we have

$$|h_k| = |g_{k+1} - g_k| < \varepsilon$$

as desired. ■

Problem 9. Complete the following the proof presented.

Proof. Because $|\sin(n_{\hat{i}}x_0) - \sin(n_{N+1}x_0)| \geq 1$ and $b_n = \sin(n_{\hat{i}}x_0) > 0$ for all $n \in \mathbb{N}$, we have found a subsequence of $(\sin(nx))_{n\geq 1}$ that does not converge uniformly on the interval $[0, 2\pi]$ which is a contradiction. ■

Problem 10. For all $n \geq 1$ define $f_n : [0,1] \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} 1 & \text{if } n!x \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $f_n \to f$ pointwise where $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{I} \\ 1 & \text{if} x \in \mathbb{Q}. \end{cases}$$

Proof. (1) If $x \in [0,1]$ is irrational, then $f_n(x) = 0$. Clearly, we see that $f_n \to f$.

(2) If $x \in [0,1]$ is a rational number, then $c = \frac{p}{q}$ for some nonnegative $p, q \in \mathbb{Z}$. Then for all $n > q$, $n!x \in \mathbb{Z}$. Hence, for all $n > q$ $f_n(c) = 1$. From here, it immediately follows that $f_n \to f$.

Hence, (1) and (2) imply that $f_n \to f$ pointwise. ■

Problem 11 (i)**.** For all $n \geq 1$ define $f_n : [0, \infty) \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } 0 \leq x \leq n \\ 0 & \text{if} x > n \end{cases}$$

Prove that $f_n \to f$ uniformly where $f : [0, \infty) \to \mathbb{R}$ is defined by $f \equiv 0$.

Proof. If $x > n$, $f(x) = 0$. Clearly, $f_n \to 0$ uniformly. Otherwise, assume that $0 \leq x \leq n$. Then $f_n(x) = \frac{1}{n}$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Hence, for any $n > N$, we have

$$|f_n(x) - 0| = |f_n(x)| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Hence, $f_n \to 0$ uniformly. ■

Problem 12 (ii)**.** Show that $\lim\limits_{n\to\infty} \int_0^\infty f_n \, dx \neq \int_0^\infty f \, dx$.

Proof. Note that

$$\begin{aligned} \lim_{n\to\infty} \int_0^\infty f_n(x) \, dx &= \lim_{n\to\infty} \int_0^\infty \frac{1}{n} \, dx \\ &= \lim_{n\to\infty} \left[\lim_{R\to\infty} \int_0^R \frac{1}{n} \, dx \right] \\ &= \lim_{n\to\infty} \left[\lim_{R\to\infty} \frac{R}{n} \right] = \infty. \end{aligned}$$

On the other hand, we have

$$\int_0^\infty f \, dx = \lim_{R\to\infty} \left[\int_0^R (0) \, dx \right] = \lim_{R\to\infty} (0) = 0.$$

Clearly, we have that

$$\lim_{n\to\infty} \int_0^\infty f_n \, dx \neq \int_0^\infty f \, dx.$$

■

Problem 13 ((i))**.** For all $n \geq 1$ define $f_n : [-1,1] \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+n^2x^2}$. Prove that f_n converges uniformly to $f : [-1,1] \to \mathbb{R}$ defined by $f \equiv 0$.

Proof. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in [-1, 1]$ and for all $n > N$, we have

$$|f_n(x) - 0| < \varepsilon.$$

Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2N} < \frac{1}{N} < \varepsilon.$$

Then from our hint, we can see that for any $n > N$, we have

$$\begin{aligned} |f_n(x) - 0| &= \left| \frac{x}{1 + n^2x^2} \right| \\ &= \frac{|x|}{1 + n^2x^2} \\ &\leq \frac{|x|}{2n|x|} \\ &= \frac{1}{2n} \\ &< \frac{1}{2N} \\ &< \varepsilon. \end{aligned}$$

hence, we can see that $f_n \to f$ uniformly on $[-1, 1]$. ■

Problem 14. Prove that f_n' converges pointwise to $g : [-1, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < |x| \leq 1 \end{cases}.$$

Proof. Since each f_n is differentiable, we have that

$$\begin{aligned} f_n'(x) &= \frac{1}{1 + n^2x^2} - \frac{x}{(1 + n^2x^2)^2} \cdot 2n^2x \\ &= \frac{(1 + n^2x^2) - 2n^2x^2}{(1 + n^2x^2)^2}. \end{aligned}$$

Hence, we have

$$f_n'(x) = \frac{1 - n^2x^2}{(1 + n^2x^2)^2}$$

and note that

$$f_n'(0) = \frac{1}{1} = 1.$$

Clearly, if $x = 0$, then $f_n'(0) \to g(0)$. Otherwise, suppose $0 < |x| \le 1$. Then we have

$$\begin{aligned}
|f_n'(x) - 0| &= |f_n'(x)| \\
&= \left| \frac{1 - n^2x^2}{(1 + n^2x^2)^2} \right| \\
&= \frac{|1 - n^2x^2|}{(1 + n^2x^2)^2} \\
&\le \frac{2(1 - n|x|)}{4n^2|x|^2} \\
&= \frac{1}{n^2|x|^2} - \frac{1}{2n|x|} \\
\underbrace{\to}_{\text{ALT}} 0 + 0 = 0.
\end{aligned}$$

Using the Squeeze Theorem, we have that as $n \to \infty$, we have

$$|f_n'(x)| \to 0.$$

Clearly, the convergence above depends on x. Thus, the $f_n' \to 0$ pointwise. ■

Problem 15 (iii). Does f_n' converge uniformly to g.

Solution. No, because the pointwise limit of f_n' in part (ii) is NOT a continuous function. ■

Problem 16. Prove the following theorem.

Theorem. Assume that for each $n \in \mathbb{N}$, $f_n : [a, b] \to \mathbb{R}$ is differentiable, there exists $x_0 \in [a, b]$ such that $(f_n(x_0))_{n \ge 1}$ converges, and (f_n') converges uniformly on $[a, b]$. Then (f_n) converges uniformly on $[a, b]$.

Proof. Our goal is to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$ and $\forall x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Without loss of generality $x_0 < x$. By the Mean Value Theorem, there exists an $\hat{x} \in (x_0, x)$ such that

$$f_n'(\hat{x}) = \frac{f_n(x) - f_n(x_0)}{x - x_0} \Longrightarrow f_n(x) - f_n(x_0) = f_n'(\hat{x})(x - x_0)$$

and similarly, we have

$$f_m'(\hat{x}) = \frac{f_m(x) - f_m(x_0)}{x - x_0} \Longrightarrow f_m(x) - f_m(x_0) = f_m'(\hat{x})(x - x_0).$$

Subtracting these two quantities gives us

$$\begin{aligned}
&f_n(x) - f_n(x_0) - (f_m(x) - f_m(x_0)) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) \\
&\Longrightarrow f_n(x) - f_m(x)q - (f_n(x_0) - f_m(x_0)) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) \\
&\Longrightarrow f_n(x) - f_m(x) = (f_n'(\hat{x}) - f_m'(\hat{x}))(x - x_0) + f_n(x_0) - f_m(x_0).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |(f_n'(\hat{x})) - f_m'(\hat{x})(x - x_0) + f_n(x_0) - f_m(x_0)| \\
&\le |f_n'(\hat{x}) - f_m'(\hat{x})||x - x_0| + |f_n(x_0) - f_m(x_0)|.
\end{aligned}$$

Since each f_n is differentiable on $[a, b]$, we know that each f_n is continuous on $[a, b]$. Therefore, f_n is continuous at $x_0 \in [a, b]$. That is, there exists a $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|f_n(x) - f_n(x_0)| < \varepsilon.$$

Furthermore, we can see by our assumption that if $(f_n(x_0))_{n\geq 1}$ converges, we have that $(f_n(x_0))_{n\geq 1}$ is a Cauchy sequence. That is, there exists an $N_1 \in \mathbb{N}$ such that for any $n > m > N_2$, we have

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Since (f'_n) converges uniformly, there exists an $N_2 \in \mathbb{N}$ such that for any $n > m > N_1$ (given $\hat{x} \in [a, b]$)

$$|f'_n(\hat{x}) - f'_m(\hat{x})| < \frac{\varepsilon}{2\delta}.$$

Let $N = \max\{N_1, N_2\}$. Then for any $n > m > N$, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f'_n(\hat{x}) - f'_m(\hat{x})||x - x_0| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\varepsilon}{2 \cdot \delta} \cdot \delta + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, f_n converges uniformly on $[a, b]$ by the Cauchy Criterion. ■