0.1 The Derivative of a Real Function

Definition (5.1). Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$, form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, \ t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t).$$

Theorem (5.2). Let f be defined on $[a, b]$. If f differentiable at a point $x \in [a, b]$, then f is continuous at x.

Theorem (5.3). Suppose f and g are defined on $[a, b]$ and are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f'(x)f(x)}{g^2(x)}$ with $g(x) \neq 0$.

Theorem (5.5). Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

0.2 Mean Value Theorems

Definition (5.7). Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. Local minima are defined likewise.

Theorem (5.8). Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

Theorem (5.9). If f and g are continuous real functions on $[a, b]$ which are differentiable on (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the end points.

Theorem (5.10). If f is a real continuous function on $[a, b]$ which is differentiable on (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(x).$$

Theorem (5.11). Suppose f is differentiable on (a, b).

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

0.3 The Continuity of Derivatives

Theorem (5.12). Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

Corollary (5.12). If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

0.4 L'Hopital's Rule

Theorem (5.13). Suppose f and g are real and differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$

If

(a) $f'(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or if

(b) $g(x) \to +\infty$ as $x \to a$,

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$