Understanding Analysis Textbook Solutions

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September 7, 2023

Contents

Chapter 1

The Real Numbers

1.1 The Axiom of Completeness

Exercise 1.3.3

(a) Let $A \neq \emptyset$ and bounded below, and define

$$B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\},$$

Show that $\inf A = \sup B$.

Proof. Our goal is to show that both $\inf A \leq \sup B$ and $\inf A \geq \sup B$. Since $B \neq \emptyset$ and bounded above, we have that the $\sup B$ exists. First we want to show that $\inf A \leq \sup B$. By definition of $\sup B$, it is the greatest lower bound of B. Since $A \neq \emptyset$ and bounded below, we have that the $\sup B$ is greater than any lower bound of A. Hence, we have that $\inf A \leq \sup B$. Now we want to show that $\inf A \geq \sup B$. Suppose for sake of contradiction that $\inf A < \sup B$. Since $A \neq \emptyset$ and bounded below, we have that

$$a - \varepsilon \geq \inf A \tag{1}$$

for some $a \in A$. Our goal is to show that there exists that some $a \in A$ is less than $\sup B$. Hence, choose $\varepsilon = \sup B + a$ such that (1) and $\inf A < \sup B$ implies that

$$\begin{aligned} a - \varepsilon &< \sup B \\ a - (\sup B + 3a) &< \sup B \\ a &< \sup B. \end{aligned}$$

But this is a contradiction since every element in A has to be bigger than B i.e $b > a$ for all $b \in B$. Hence, it must be the case that $\inf A \geq \sup B$. Hence, we have that $\inf A = \sup B$. ■

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Proof. There is no need to assert that greatest lower bounds exist as part of the axiom because we can always seperate a set A that is bounded below into a set B that just consists of lower bounds from A. Since the infimum is just the greatest lower bound, it is equivalent to taking the supremum of a set of lower bounds. We can do this because every element in B is bounded above by every element in A which is permitted by the Axiom of Completeness. ■

Exercise 1.3.4

Let $A_1, A_2, A_3, ...$ be a collection of nonempty sets, each of which is bounded above.

1. Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\cup_{k=1}^{n} A_k\right)$.

Proof. For $\sup(A_1 \cup A_2)$, we have

$$\sup(A_1 \cup A_2) = \sup\{A_1, A_2\}$$

and for $\sup\big(\cup_{k=1}^{n} A_k\big)$, we have

$$\sup\big(\cup_{k=1}^{n} A_k\big) = \sup\{A_k\}$$

for $k \in \mathbb{N}$. ■

2. Consider $\sup\big(\cup_{k=1}^{\infty} A_k\big)$. Does the formula in (a) extend to the infinite case?

Proof. No, because then $\cup_{k=1}^{\infty} A_k$ would be an unbounded set. ■

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set

$$cA = \{ca : a \in A\}.$$

(a) If $c \geq 0$, show that $\sup A(cA) = c \sup A$.

Proof. Suppose $c \geq 0$. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Denote $\alpha = \sup A$. By definition, we have that $\alpha \geq a$ for all $a \in A$. Multiplying by $c \geq 0$, we have that

$$c\alpha \geq ca$$
$$c \sup A \geq ca$$

for all $a \in A$. This shows that $c \sup A$ is an upper bound for cA.

Now we want to show that this upper bound is the least upper bound in cA. Hence, take any upper bound in $b \in A$ such that $ca \leq b$. This implies that $a \leq b/c$. Since $\alpha = \sup A$ is the least upper bound for A, we have that $\sup A \leq b/c$ which further implies that $c \sup A \leq b$ showing that it is the least upper bound in cA. Hence, we have that $\sup A(cA) = c \sup A$. ■

(b) Postulate a similar type of statement for $\sup(cA)$ for the cases $c < 0$.

Proof. For cases $c < 0$, we have $\sup(cA) = c \inf A$. ■

Exercise 1.3.6

Suppose that $A, B \neq \emptyset$ and bounded above. Prove that

$$\sup(A + B) = \sup A + \sup B$$

We prove this proposition using two methods. One deals with direct application of the definition and the other deals with using lemma 1.3.8.

Proof. Our goal is to show that

$$\sup(A + B) = \sup A + \sup B$$

We know that since $A, B \neq \emptyset$ and bounded above, we have that $\sup A, \sup B$ exists. we denote the supremums by the following

$$\sup A = \alpha,$$
$$\sup B = \beta.$$

It suffices to show that following

$$\sup(A + B) \leq \sup A + \sup B \tag{1.1}$$

and

$$\sup(A + B) \geq \sup A + \sup B \tag{1.2}$$

We first show (2) first then we will show (1) next. Suppose we have abirtrary $x \in A$ and $y \in B$. Because $A, B \neq \emptyset$ and bounded above, we know that the set $A + B$ is also non-empty and bounded above which means its supremum $\sup(A + B)$ also exists. Hence, we know that

$$x + y \leq \sup(A + B)$$

Subtracting $y \in B$ to the other side of this inequality will yield

$$x \leq \sup(A + B) - y$$

But we know that since $x \in A$ and $\sup A \geq a$ for all $a \in A$, we have that

$$\sup A \leq \sup(A + B) - y.$$

Likewise, we isolate $y \in B$ to the other side and note that $b \leq \sup B$ for all $b \in B$. Then we get the following:

$$\begin{aligned} y &\leq \sup(A + B) - \sup A \\ \sup B &\leq \sup(A + B) - \sup A \end{aligned}$$

But this implies that

$$\sup A + \sup B \leq \sup(A + B)$$

Now we show (1). By lemma 1.3.8, we know that for all $\varepsilon > 0$, we have that

$$\begin{aligned} \sup A - \frac{\varepsilon}{2} &< a \\ \sup B - \frac{\varepsilon}{2} &< b \end{aligned}$$

for some $a \in A$ and $b \in B$. Adding these two together we have that

$$\sup A + \sup B - \varepsilon < a + b$$

But we also know that a and b are bounded above by their respective supremums so

$$\sup A + \sup B - \varepsilon < a + b \leq \sup A + \sup B$$

Setting $\varepsilon = \sup A + \sup B - \sup(A + B)$. Hence, we have that

$$\sup(A + B) \leq \sup A + \sup B.$$

Since we have (1) and (2), we see that

$$\sup(A + B) = \sup A + \sup B$$

■

Exercise 1.3.7

Prove that if a is an upper bound for A, and $a \in A$, then $\sup A = a$.

Proof. We want to show that $a \leq \sup A$ and $a \geq \sup A$. We start with the former. Since $A \neq \emptyset$ and bounded above, we have that the $\sup A$ exists. Label this supremum as $\sup A = \beta$. For every $\varepsilon > 0$, we have that there exists $b \in A$ such that $\sup A - \varepsilon \leq b$. Choose $\varepsilon = 2 \sup A - a - b$ such that

$$\begin{aligned} \sup A - 2 \sup A + a + b &\leq b \\ - \sup A + a &\leq 0 \\ \Rightarrow a &\leq \sup A \end{aligned}$$

Now for the latter case, since $\sup A = \beta$ is the least upper bound of A and $a \in A$, it follows immediately that $a \leq \sup A$ for all $a \in A$. Hence, $\sup A = a$

■

Exercise 1.3.8

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Suppose $\sup A < \sup B$. Since we have $\sup B$, by lemma 1.3.8 we can say that for every $\varepsilon > 0$, there exists $b \in B$ such that

$$\sup B - \varepsilon < b \tag{1}$$

Choose $\varepsilon = \sup B - \sup A$. We can do this because $\sup A < \sup B$. Hence, (1) implies

$$\begin{aligned} \sup B - \varepsilon &< b \\ \sup B - (\sup B - \sup A) &< b \\ \sup A &< b. \end{aligned}$$

By definition, $\sup A$ is the least upper bound for A. Since $\sup A \geq a$ for all $a \in A$, it follows that from (1) that $a < b$ for all $a \in A$. Hence, for some $b \in B$, b is an upper bound for A.

■

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Exercise 1.3.10 (Cut Property)

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

Proof. Suppose A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$. By Axiom of Completeness, A and B are bounded above and below respectively. This implies that their supremum and infimums exists.

Firstly, we want to show that there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$. Since $a < b$ for all $a \in A$ and $b \in B$, every $b \in B$ is an upper bound for A. Denote B as the set of upper bounds for A. Hence, there must exist $c \in B$ such that c is the **least upper bound** for A due to the Axiom of Completeness. Furthermore, note that $\sup A \in B$ and not in A since $A \cap B = \emptyset$ which means $\sup A \in \mathbb{R}$. Hence, $\sup A \leq b$. But $x \in A$ so $x \leq \sup A$.

Now we want to show there exists $c \in \mathbb{R}$ such that $x \geq c$. Since every $a \in A$ is a lower bound for B and that $B \neq \emptyset$, there must exist an element in A such that it is the **greatest lower bound** for B. Denote this element as $c = \inf B$. Hence, $\inf B \geq a$ for all $a \in A$. Furthermore, $\inf B \in A$ and not in B since $A \cap B = \emptyset$ so $\inf B \in \mathbb{R}$ when we union A and B together. Since $x \in B$, we have that $\inf B \leq x$.

Furthermore, B is nonempty and bounded below and A is the set of lower bounds for B, we have that $\inf B = \sup A = c \in \mathbb{R}$. ■

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove that $\sup E$ exists.

Proof. Assume $\mathbb{R}$ possesses the Cut Property and let $E \neq \emptyset$ that is bounded above. Suppose we have that $E \subseteq \mathbb{R}$. Since $\mathbb{R}$ possesses the cut property, we can find $c \in \mathbb{R}$ such that $x \leq c$ if $x \in E$. Since $A \cap B = \emptyset$, $c \in A \cup B = \mathbb{R}$. Hence, either $c \in A$ or $c \in B$. If $c \in A$, then c is not an upper bound for E since every $a \in A$ is less than every $b \in B$. Furthermore, if $c \in A$ and A is the set of lower bounds for B, then it would contradict that c is an upper bound for E. Thus, we must have $c \in B$. Since $c \in B$, B is the set of upper bounds for E, and $E \neq \emptyset$ and bounded above, $c \in B$ is the smallest element in B which makes it the **least upper bound** for E. Hence, $c = \sup E$ exists. ■

1.2 Consequences of Completeness

Exercise 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

Proof. Suppose $a, b \in \mathbb{Q}$. Then $p, q, m, n \in \mathbb{Z}$ such that $n, q \neq 0$. Hence, $a = \frac{p}{q}$ and $b = \frac{m}{n}$. Adding $a+b$ will give us

$$\begin{aligned} a+b &= \frac{p}{q} + \frac{m}{n} \\ &= \frac{pn+mq}{qn}. \end{aligned}$$

Since $pq+mn, qn \in \mathbb{Z}$ with $q, n \neq 0$, we have that $a+b \in \mathbb{Q}$. Now we multiply a and b together. Then we have

$$\begin{aligned} ab &= \frac{p}{q} \cdot \frac{m}{n} \\ &= \frac{pm}{qn}. \end{aligned}$$

Since $pm, qn \in \mathbb{Z}$ and $q, n \neq 0$, we have that $ab \in \mathbb{Q}$. ∎

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Suppose for sake of contradiction that $at = r$ where $r \in \mathbb{Q}$. Solving for t, we have that $t = \frac{r}{a}$. But this tells us that $t \in \mathbb{Q}$ since $r, a \in \mathbb{Q}$ which is a contradics our assumption that $t \in \mathbb{I}$.

∎

(c) Part (a) can be summarised by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Proof. We can say that $s+t$ is an irrational number while st can either be rational or irrational depending if $s = t$ or $s \neq t$. If $s = t$, then st is rational and if $s \neq t$, then st is irrational. ∎

Exercise 1.4.2

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. **Show that** $s = \sup A$.

Proof. Since $A \neq \emptyset$ and bounded above, we have that $\sup A$ exists. Since $s + \frac{1}{n}$ for all $n \in \mathbb{N}$ is an upper bound for A, we have that

$$\sup A \leq s + \frac{1}{n} \tag{1}$$

for all $n \in \mathbb{N}$. On the other hand, $s - \frac{1}{n}$ is a lower bound for A. Hence,

$$\sup A > s - \frac{1}{n} \tag{2}$$

for all $n \in \mathbb{N}$. We have (1) and (2) imply

$$s - \frac{1}{n} < \sup A \leq s + \frac{1}{n}. \tag{3}$$

This means that either $\sup A < s, \sup A > s$, or $\sup A = s,$. If $\sup A < s$, then $s - \sup A > 0$. Using the Archimedean Property, we can find an $n \in \mathbb{N}$ such that

$$s - \sup A > \frac{1}{n}$$

but this means that $\sup A < s - \frac{1}{n}$ which contradicts (3). On the other hand, if $\sup A > s$, then $\sup A - s > 0$. Using the Archimedean property again, we can find an $n \in \mathbb{N}$ such that

$$\sup A - s > \frac{1}{n}$$

but this means that $\sup A > s + \frac{1}{n}$ which is a contradiction since $\sup A < s + \frac{1}{n}$ from (3). Hence, it must be that $\sup A = s$. ■

Exercise 1.4.3

Prove that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion for the theorem to hold.

Proof. Suppose $x \in (0, \frac{1}{n})$, then $x > 0$. By the Archimedean Property, we can find an $N \in \mathbb{N}$ that is sufficiently large such that $x > \frac{1}{N}$. But this means that $x \in (0, 1/n)$ for all $n \in \mathbb{N}$. Hence, $x \notin \cap_{n=1}^{\infty}(0, \frac{1}{n})$ and then

$$\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset.$$

■

Exercise 1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup T = b$.

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Hence, we use lemma 1.3.8 and the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to state that for every $\varepsilon > 0$, there exists $r \in \mathbb{Q}$ such that $b - \varepsilon < r < b$. But this means that $r \in T$ and $b - \varepsilon$ is not an upper bound for T. Hence, $\sup T = b$.

■

Another proof for this:

Proof. Let $a < b$ where $a, b \in \mathbb{R}$. Consider the following set $T = \mathbb{Q} \cap [a, b]$. We want to show that $\sup T = b$. By definition, b is an upper bound for T since $a < b$. All we need to show is that b is the least upper bound. Since $a < b$ where $a, b \in \mathbb{R}$, we can find $x \in \mathbb{Q}$ such that $a < x < b$. Since $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ with $n \neq 0$, we have that $na < m < nb$. But note that nb is another upper bound for T for n sufficiently large and $nb > b$ implying that b is the least upper bound of T. Hence, $\sup T = b$. ■

Exercise 1.4.5

Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. Consider the real numbers $a - \sqrt{p}$ and $b - \sqrt{p}$ where p is any prime number. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that

$$a - \sqrt{p} < r < b - \sqrt{p}$$

for some $r \in \mathbb{Q}$. Adding $\sqrt{p}$ to both sides, we have that

$$a < r + \sqrt{p} < b.$$

But know that $r + \sqrt{p} \in \mathbb{I}$ by (c) of Exercise 1.4.1. Hence, $t = r + \sqrt{p}$. We can follow the same procedure for trancendental numbers and make this conclusion. ■

Exercise 1.4.7

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Proof. Now we want to show the other case that $\alpha^2 < 2$ cannot happen. Now we need to produce an element in T such that it is less than α, thereby showing that α is not the least upper bound of T. Hence, we construct the following element

$$\left(\alpha - \frac{1}{n}\right) \in T.$$

Squaring this quantity will give us the following

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Like we did before, we get to choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} > \frac{\alpha^2 - 2}{2\alpha}$$

to make

$$\left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2 - (\alpha^2 - 2) = 2.$$

But this shows that $\alpha - \frac{1}{n_0} < \alpha$ showing that α and that our constructed element contradicts that fact that α is the least upper bound. ∎

Exercise 1.4.6

Recall that a set B is dense in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set $\{r \in \mathbb{Q} : q \leq 10\}$

Proof. Yes, since $a < \frac{p}{10} < \frac{p}{q} < b$. ∎

(b) The set of all rationals p/q such that q is a power of 2.

Proof. Yes since $a < \frac{p}{2^n} < b$ for $n \in \mathbb{N}$. ∎

(c) The set of all rationals p/q with $10|p| \geq q$

Proof. ∎

1.3 Cardinality

Exercise 1.5.1

Finish the following proof for Theorem 1.5.7.

Proof. Assume B is a countable set. So there exists a map $f : \mathbb{N} \to B$ such that f is surjective and injective. Let $A \subseteq B$ be an infinite subset of B. We want to show that A is countable. That is, A is both

1. injective
2. surjective.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Let $g : \mathbb{N} \to A$ be the map defined by

$$g(1) = f(n_1).$$

To show injectivity of g, we proceed via induction on the index $i \in \mathbb{N}$. Let the base case be $i = 2$. Then suppose $g(1) = g(2)$. By definition of g and injectivity of f, we have that

$$g(1) = g(2)$$
$$f(n_1) = f(n_2)$$
$$n_1 = n_2.$$

But this means that $n_2 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Hence, g is injective. Now for the inductive step, assume this holds for every $1 \leq i \leq k - 1$. We want to show that this holds for $i = k$. Suppose that

$$g(1) = g(k).$$

By defintion of g and injectivity of f, we have that

$$f(n_1) = f(n_k)$$
$$n_1 = n_k.$$

But this also means that $n_k = \min\{n \in \mathbb{N} : f(n) \in \mathbb{N}\}$. Hence, g is injective.

Now we want to show that g is surjective. Note that we have

$$g(i) = A \cap \{f(n_1), f(n_2), f(n_3), ..., f(n_k)\}.$$

Then by definition of g, we have that $g(i) = f(n_i)$. Since f is surjective, there exists some $b \in B$ such that $f(n_i) = b$. But since $n_i = \{n_i \in \mathbb{N} : f(n_i) \in A\}$, we have that $f(n_i) \in A$ so g is surjective as well. Hence, we have that g is both injective and surjective which means that $\mathbb{N} \sim A$. Therefore, A is countable.

■

Exercise 1.5.2

Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable.

The proposition is: $\mathbb{Q}$ is uncountable.

Proof. Assume for contradiction that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3\}$ and as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while the nested interval property implies that this intersection is nonempty. This contradiction implies $\mathbb{Q}$ must therefore be uncountable. ■

Proof. I think the main issue with this proof is when the author assumed that the set of rationals are closed. Since $\mathbb{Q}$ contains irrantional numbers within each subset of the $\mathbb{Q}$ as well as real numbers, $\mathbb{Q}$ cannot be closed. Hence, we cannot apply the nested interval property here. ■

Exercise 1.5.3

Prove theorem 1.5.8

Theorem 1. (a) If $A_1, A_2, ...A_n$ are each countable sets, then the union of

$$A_1 \cup A_2 \cup ... \cup A_m$$

is countable.

(b) If A_n is a countable set for each $n \in \mathbb{N}$, the $\bigcup_{n=1}^{\infty} A_n$ is countable.

1. First, prove statement (i) for two countable sets, A_1 and A_2.

   Proof. Suppose A_1 and A_2 are countable sets. Then $\mathbb{N} \sim A_1$ and $\mathbb{N} \sim A_2$. Furthermore, we have that the maps $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to A_2$ are bijective. Our goal is to show the union $A_1 \cup A_2$ is also countable i.e we need to show that the map $h : \mathbb{N} \to A_1 \cup A_2$ is bijective.

Before we proceed, let us replace A_2 with the following set B_2 defined as

$$B_2 = A_2 \setminus A_1 = \{h(n) \in A_2 : h(n) \notin A_1\}.$$

Now our following map is $h : \mathbb{N} \to A_1 \cup B_2$ (this is equivalent to $A_1 \cup A_2$)and define it as follows

$$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k+1 \in B_2 \end{cases}$$

Suppose we have $n_1, n_2 \in \mathbb{N}$ and $h(n_1) = h(n_2)$. Since f and g are injective, we have

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

This shows that h is injective (the same process can be applied to g when $h \in B_2$). Note that $A_1 \cap B_2 = \emptyset$ because otherwise h would not be well defined. Now we need to show that h is surjective. Since f and g are surjective, there exists either $x \in A_1$ or $x \in B_2$ such that $h(n) = f(n) = x$ or $h(n) = g(n) = x$. Hence, we have that h is surjective. Since h is a bijective map, we now have that $\mathbb{N} \sim A_1 \cup B_2$.

Suppose we use induction on the index $i \in \mathbb{N}$. Since we have already proven the base case for two countable sets, let us assume $A_1, A_2, ..., A_k$ are all countable sets such that for $i \leq k-1$, the union $A_1 \cup A_2 ... \cup A_{k-1}$ is countable. Let's set $A' = A_1 \cup A_2 ... \cup A_{k-1}$. Our goal is to show that the union $A' \cup A_k$ is countable. Let's define the map $h : \mathbb{N} \to A' \cup B'$ such that

$$B' = A_k \setminus A' = \{h(n) \in A_k : x \notin A'\}.$$

and

$$h(n) = \begin{cases} f(k) \text{ if } n = 2k \\ g(k) \text{ if } n = 2k+1 \end{cases}$$

Let $n_1, n_2 \in \mathbb{N}$. Since A' and A_k are countable sets, we have that

$$\begin{aligned} h(n_1) &= h(n_2) \\ f(n_1) &= f(n_2) \\ n_1 &= n_2. \end{aligned}$$

Hence, h is injective. Now we want to show that h is surjective. If either $h(n) \in A'$ or $h(n) \in A_k$, then since $f : \mathbb{N} \to A'$ and $g : \mathbb{N} \to A_k$ are surjective functions, we have that there exists $x \in A_k$ or $x \in A'$ such that $h(n) = x$. Hence, h is surjective as well. Since h is now a bijective function, we conclude that the union $A_1 \cup A_2 ... \cup A_k$ is countable.

■

2. Explain why induction cannot be used to prove part (1) of Theorem 1.5.8 from part (2)

Proof. We cannot use induction on part (2) of theorem 1.5.8 because the index itself $n \in \mathbb{N}$ is infinite and induction only works only finite n. ■

Proof of the second part of theorem

Proof. Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of countable sets. Define the union

$$S = \bigcup_{n\in\mathbb{N}} S_n.$$

for all $n \in \mathbb{N}$. Assume each S_n is disjoint. Otherwise, let S_1 such that for each $n \geq 1$, define

$$S'_{n+1} = S_{n+1} \setminus S_n = \{x \in S_{n+1} : x \notin S_n\}.$$

This is to ensure that our following map is well-defined. let F_n denote the set of all injections from $S_n \to \mathbb{N}$ Let $\varphi : S \to \mathbb{N} \times \mathbb{N}$ be the map that is defined by

$$\varphi(x) = (n, f_n(x))$$

where $n \in \mathbb{N}$ smallest guranteed by the Well-Ordering Principle. Since each f_n is an injection, it follows that φ is also an injection. Since $\mathbb{N}\times\mathbb{N}$ is countable, there exists an injection $\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. Composing the two functions φ and α, we have that $\alpha\circ\varphi : S \to \mathbb{N}$ is an injection. Since $\mathbb{N}\times\mathbb{N} \sim \mathbb{N}$, we know that α is also surjective. Hence, the composition $\alpha \circ \varphi$ is also surjective. Therefore, we have that S is countable.

■

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Proof. Let (a, b) any interval. Then define the function $f : (a, b) \to \mathbb{R}$ as

$$f(x) = x^2$$

Our objective is to show that f is injective and surjective. To show that f is injective, we need to let $x_1, x_2 \in (a, b)$. Then suppose

$$f(x_1) = f(x_2).$$

Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^2 &= x_2^2 \\ x_1 &= x_2. \end{aligned}$$

This shows that f is injective. Now we want to show that f is surjective. Then there exists $\sqrt{y} \in (a, b)$. Let

$$x = \sqrt{y}.$$

Then we have that

$$\begin{aligned} x^2 &= y \\ f(x) &= y. \end{aligned}$$

Hence, f is surjective. Since $f : (a, b) \to \mathbb{R}$ is a bijective function, we have that $(a, b) \sim \mathbb{R}$. ■

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof. Let $(0, \infty) = \{x : x > 0\}$. Our goal is to show that $(a, \infty) \sim \mathbb{R}$. To show this, we need to show the map $f : (a, \infty) \to \mathbb{R}$ is bijective. Define $f$as the following:

$$f(x) = \ln(x).$$

Then suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in (a, \infty)$. Then

$$\begin{aligned} \ln(x_1) &= \ln(x_2) \\ x_1 &= x_2 \end{aligned}$$

Hence, we have that f is an injective function. Now we want to show that f is surjective. Then let $e^y = x \in (0, \infty)$. Then taking the natural log of both sides, we have that $\ln(x) = y$. Hence, we have that f is a surjective function. Since f is a bijective function, we know that $(0, \infty) \sim \mathbb{R}$. ■

(c) Using open intervals makes it more covenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Proof. We want to show that $[0, 1) \sim (0, 1)$. Define the map $f : [0, 1) \to (0, 1)$ as

$$f(x) = \frac{1}{x-1}$$

Our goal is to show that this map is bijective. Hence, we need to show that this map is both injective and surjective.

To show that f is injective. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in [0, 1)$. Then we have that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{1}{x_1 - 1} &= \frac{1}{x_2 - 1} \\ x_1 - 1 &= x_2 - 1 \\ x_1 &= x_2. \end{aligned}$$

Hence, f is injective.

To show that f is surjective, suppose we have $x - 1 = \frac{1}{y}$. Then

$$y = \frac{1}{x-1}.$$

But we have that $f(x) = \frac{1}{x-1}$ so we have

$$f(x) = \frac{1}{x-1} = y.$$

Hence, f is surjective.

Since f is bijective, we have that $[0, 1) \sim (0, 1)$. ■

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

Proof. $A \sim A$ because A is a bijection onto itself (same elements map to the same elements of the same set). ■

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Proof. If $A \sim B$, then the map $f : A \to B$ is a bijection. Meaning we can map unique eleemnts from A to unique elements to B. Since there is unique mapping of elements from $A \to B$ thenn we would expect to see the same thing when we map the same elements from $B \to A$. ■

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Proof. Suppose we have three sets A, B, and C. Suppose $A \sim B$ and $B \sim C$ then we have two maps $f : A \to B$ and $g : B \to C$ that are bijective. Composing the two functions we get $g \circ f : A \to C$. We want to show that this mapping is also bijective. Let $x_1, x_2 \in A$ then suppose $g \circ f(x_1) = g \circ f(x_2)$. By definition of composition, we have

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) \\ f(x_1) &= f(x_2) && (g \text{ is injective}) \\ x_1 &= x_2. && (f \text{ is injective}) \end{aligned}$$

Hence, $g \circ f$ is an injective function. Now we want to show that $g \circ f$ is a surjective mapping. Since f is surjective, there exists a $y \in B$ such that $f(x) = y$. Since g is also surjective, there exists a $z \in C$ such that $g(y) = c$. Hence, we have that $g(f(x)) = c$ which means $g \circ f$ is a surjective mapping. Therefore, $A \sim C$. ■

Exercise 1.5.11

[Shroder-Bernstein Theorem] Assume there exists an injective function $f : X \to Y$ and another injective function $g : Y \to X$. Show that $X \sim Y$. The strategy is to partition X and Y into components

$$\begin{aligned} X &= A \cup A' \\ Y &= B \cup B' \end{aligned}$$

with $A \cup A' = \emptyset$ and $B \cup B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'

1. Explain how achieving this would lead to a proof that $X \sim Y$.

   Proof. Taking disjoint sets prevents the problem of an element from either map mapping to two elements onto it's image. Thus, allowing us to have a well-defined function. Having two injective maps also would lead to the $X \sim Y$ because composing these two functions would allow us take a unique mapping from one element from each other. ■

2. Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is similar collection in Y.

   Proof. We set proceed by induction on $n \in \mathbb{N}$ and let $P(n)$ be the statement that $\{A_n : n \in \mathbb{N}\}$ and $\{f(A_n) : n \in \mathbb{N}\}$ are pairwise disjoint. Define

   $$A_{n+1} = g(f(A_n))$$

   and for each $n \in \mathbb{N}$. Note that $g(f(A_{n+1})) = A_{n+2}$.

   Let our base case be $n = 1$. Then $A_2 = g(f(A_1))$. By definition of A_1, we have that $x \in X$ but not in $g(Y)$. If $x \notin g(Y)$ then $x \notin g(f(A_1))$ as well. Hence, we have that $A_1 \cap A_2 = \emptyset$ Now assume $P(n)$ holds for $n \leq k - 1$. Define

   $$A_{n+1} = A_{n+1} \setminus g(f(A_{n+1})) = \{x \in A_{n+1} : x \notin g(f(A_{n+1}))\}$$

   Since A_n is pairwise disjoint for each $n \leq k - 1$, we also have that

   $$A_{k-1} = A_{k-1} \setminus g(f(A_{k-1})) = \{x \in A_{k-1} : x \notin g(f(A_{k-1}))\}.$$

   But notice that $g(f(A_{k-1})) = A_k$ which tells us that $A_{k-1} \cap A_k$ are also pairwise disjoint. Thus, A_n for each $n \in \mathbb{N}$ is pairwise disjoint.

   ■

Chapter 2

Sequences and Series

2.1 The Algebraic and Order Limit Theorems

Exercise 2.3.1

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

Proof. Suppose $x_n \geq 0$ and $x_n \to 0$. In order to show that $\sqrt{x_n} \to 0$, it suffices to show that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|\sqrt{x_n} - 0| < \varepsilon.$$

Choose $N \in \mathbb{N}$. Suppose $x_n = 0$ for all $n \in \mathbb{N}$, then $(\sqrt{x_n}) = 0$ for all $n \geq N$ which means that $(\sqrt{x_n}) \to 0$. Suppose $x_n > 0$ for all $n \in \mathbb{N}$, then observe that since $(x_n) \to 0$ and (x_n) bounded, we have

$$\begin{aligned}
|\sqrt{x_n} - 0| &= |\sqrt{x_n}| \\
&= \left|\frac{x_n}{\sqrt{x_n}}\right| \\
&= \left|\frac{x_n - 0}{\sqrt{x_n}}\right| \\
&= \frac{|x_n - 0|}{\sqrt{x_n}} \\
&< \sqrt{M}\frac{\varepsilon}{\sqrt{M}} \\
&= \varepsilon
\end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to 0$. ■

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $(x_n) \to x$. We want to show that $(\sqrt{x_n}) \to x$. Suppose $x_n = 0$ and suppose $N \in \mathbb{N}$ such that for every $n \geq N$, then we have the first case above where $x = 0$ and $(\sqrt{x_n}) \to 0$. Now suppose $x_n > 0$ and choose $N \in \mathbb{N}$ such that for every $n \geq N$, then observe that since $(x_n) \to x$ and (x_n) is bounded by an integer $M > 0$, we have that

$$\begin{aligned}|\sqrt{x_n}-\sqrt{x}| &= \left|\frac{x_n - x}{\sqrt{x_n}+\sqrt{x}}\right| \\ &= \frac{|x_n - x|}{|\sqrt{x_n}+\sqrt{x}|} \\ &< (\sqrt{M}+\sqrt{x})\frac{\varepsilon}{(\sqrt{M}+\sqrt{x})} \\ &= \varepsilon.\end{aligned}$$

Hence, it follows that $(\sqrt{x_n}) \to \sqrt{x}$. ■

Exercise 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n-1}{3}\right) \to 1$;

Proof. Suppose $(x_n) \to 2$. Our goal is to show that property above. It suffices to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{2x_n - 1}{3} - 1\right| < \varepsilon.$$

Choose $N \in \mathbb{N}$ and suppose $n \geq N$

$$\begin{aligned}\left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 4}{3}\right| \\ &= \left|\frac{2}{3}(x_n - 2)\right| \\ &= \left|\frac{2}{3}\right||x_n - 2| \\ &< \frac{2}{3}\cdot\frac{3\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, it follows that

$$\left(\frac{2x_n - 1}{3}\right) \to 1.$$

■

(b) $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$.

Proof. We want to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| < \varepsilon.$$

Choose $N \in \mathbb{N}$ and assume $n \geq N$. Since $(x_n) \to 2$, we can write

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &= \frac{|x_n - 2|}{2|x_n|}. \end{aligned} \tag{1}$$

Since $(x_n) \to 2$, we can set $\varepsilon = 1$ so that we can lower bound the denominator of (1) using

$$2 - \varepsilon < |x_n| \Rightarrow 1 < |x_n|.$$

Then we can set $N = \max\{1, \varepsilon/2\}$ so that

$$\frac{|x_n - 2|}{2|x_n|} < \frac{2\varepsilon}{2} = \varepsilon$$

which satisifes our desired property. ■

Exercise 2.3.3

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Theorem 2. Squeeze Theorem If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = \ell$, then $\lim y_n = \ell$.

Proof. Suppose $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and suppose $\lim x_n = \lim z_n = \ell$. We want to show that $\lim y_n = \ell$. By the Order Limit Theorem, we have $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ implies that $\ell \leq y_n \leq \ell$ for all $n \in \mathbb{N}$. But this means that $y_n = \ell$ for all $n \in \mathbb{N}$. Hence, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$

$$|y_n - \ell| = |\ell - \ell| = 0 < \varepsilon.$$

Hence, it follows that $\lim y_n = \ell$. ■

Exercise 2.3.4

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined).

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

Proof. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\
&= \frac{\lim 1 + \lim(2a_n)}{\lim 1 + \lim(3a_n) - \lim(4a_n^2)} \\
&= \frac{1+2\cdot 0}{1+3\cdot 0+4\cdot 0^2} \\
&= 1.
\end{aligned}$$

■

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

Proof. Let $(a_n) \to 0$. Then

$$\begin{aligned}
\lim \left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim \left(\frac{a_n^2+4a_n}{a_n}\right) \\
&= \lim \left(a_n + 4\right) \\
&= \lim a_n + \lim 4 \\
&= 0 + 4 \\
&= 4.
\end{aligned}$$

■

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Proof. Let $(a_n) \to 0$. Then

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right)\\ &= \frac{\lim 2+\lim(3a_n)}{\lim 1+\lim(5a_n)}\\ &= \frac{2+3\cdot 0}{1+5\cdot 0}\\ &= 2.\end{aligned}$$

■

Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence

$$(x_1, y_1, x_2, y_2, ..., x_n, y_n).$$

For the forwards direction, assume (z_n) is a convergent sequence. We want to show that $\lim x_n = \lim y_n$. It suffices to show that given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|x_n - y_n| < \varepsilon.$$

Suppose $(x_n) \to x$ and $(y_n) \to y$, then we can write

$$\begin{aligned}|x_n - y_n| &= |x_n - z_n + z_n - y_n|\\ &< |x_n - z_n| + |z_n - y_n|\\ &= |x_n - z + z - z_n| + |z_n - z + z - y_n|\\ &< |x_n - x| + |x - z_n| + |z_n - y| + |y - y_n|. \end{aligned} \tag{1}$$

By definition, (z_n) is a shuffled sequence and convergent. Hence, $z_n \to x$ and $z_n \to y$. But by the uniqueness of limits, $x = y$ so we have that

$$|x_n - y_n| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

which means $\lim(x_n - y_n) = \lim x_n - \lim y_n = 0$.

Now for the backwards direction, assume $\lim x_n = \lim y_n$. We want to show (z_n) converges i.e for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|z_n - z| < \varepsilon.$$

Exercise 2.3.6

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Proof. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Assume $(1/n) \to 0$ and $\sqrt{x_n} \to \sqrt{x}$.

Then taking the limit of b_n, we have

$$\begin{aligned}
\lim b_n &= \lim(n - \sqrt{n^2+2n}) \\
&= \lim \frac{-2n}{n+\sqrt{n^2+2n}} \\
&= \lim \frac{-2}{1+\sqrt{1+2/n}} \\
&= \frac{\lim(-2)}{\lim(1+\sqrt{1+2/n})} \\
&= \frac{\lim(-2)}{\lim(1)+\lim(\sqrt{1+2/n})} \\
&= \frac{-2}{1+1+0} && ((1/n)\to 0,\ (\sqrt{x_n})\to\sqrt{x}) \\
&= -1.
\end{aligned}$$

Hence, we have $\lim b_n = -1$. Now we can show that b_n does reach this limit.

Let $\varepsilon > 0$. Then choose

$$N = \frac{2}{\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}-1}.$$

Then assumme $n \geq N$. Our goal is to show that

$$|b_n + 1| < \varepsilon.$$

Then

$$\begin{aligned}
n &> \frac{2}{\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}-1} \\
\Rightarrow & \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} - 1 > \frac{2}{n}
\end{aligned}$$

Then we have

$$\begin{aligned}
\sqrt{1+2/n} &< \frac{1+\varepsilon}{1-\varepsilon} \\
(1-\varepsilon)\sqrt{1-2/n} &< 1+\varepsilon \\
(1-\varepsilon)\sqrt{1-2/n} - 1 &< \varepsilon.
\end{aligned}$$

Then we get

$$-1+\sqrt{1+2/n} < \varepsilon(1+\sqrt{1+2/n})$$

and then

$$\begin{aligned}
\frac{-1+\sqrt{1+2/n}}{1+\sqrt{1+2/n}} &< \varepsilon \\
\frac{-2n}{n+\sqrt{n^2+2n}} + \frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}} &< \varepsilon \\
n - \sqrt{n^2+2n} + 1 &< \varepsilon.
\end{aligned}$$

Hence, it follows that $|b_n + 1| < \varepsilon$.

■

Exercise 2.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

Proof. Let $(x_n) \to x$ and let $p(x)$ be a polynomial. Let

$$p(x) = \sum_{i=0}^{m} a_i x^i$$

and

$$p(x_n) = \sum_{i=0}^{m} a_i x_n^i.$$

Our goal is to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|p(x_n) - p(x)| < \varepsilon.$$

Then by part (i) of the Algebraic Limit Theorem, we have

$$\begin{aligned} |p(x_n) - p(x)| &= \left| \sum_{i=0}^{m} a_i x_n^i - \sum_{i=0}^{m} a_i x^i \right| \\ &= \left| \sum_{i=0}^{m} a_i (x_n^i - x^i) \right| \\ &< \sum_{i=0}^{m} |a_i x_n^i - a_i x^i| && \text{(T.I)} \\ &< \sum_{i=0}^{m} \frac{\varepsilon}{m} && (x_n \to x) \\ &= \frac{\varepsilon}{m} \cdot m \\ &= \varepsilon. \end{aligned}$$

Hence, we have $p(x_n) \to p(x)$.

■

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Exercise 2.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

Proof. Let (a_n) be a bounded but not necessarily convergent sequence, and assume $\lim b_n = 0$. We want to show that $\lim(a_n b_n) = 0$. It suffices to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|a_n b_n - 0| < \varepsilon. \tag{1}$$

Since (a_n) bounded, there exists an $M > 0$ such that $|a_n| < M$. Starting with the left side of (1), choose $N \in \mathbb{N}$ such that for every $n \geq N$

$$\begin{aligned} |a_n b_n - 0| &= |a_n||b_n| \\ &< M \cdot \frac{\varepsilon}{M} && (b_n \to 0) \\ &= \varepsilon. \end{aligned}$$

Hence, it follows that $\lim(a_n b_n) \to 0$. We cannot use the Algebraic Limit Theorem here because (a_n) does not necessarily have a defined limit even though it is bounded. ■

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

Proof. It would simply not converge. ■

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Proof. Suppose $a_n \to a$ where $a = 0$ and $b_n \to b$. Our goal is to show that $\lim(a_n b_n) = 0$. Let $\varepsilon > 0$, then choose $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\begin{aligned} |a_n b_n - 0| &< |a_n||b_n| \\ &< \frac{\varepsilon}{M} \cdot M && (a_n \to 0, b_n \to b) \\ &< \varepsilon \end{aligned}$$

Hence, it follows that $\lim(a_n b_n) = 0$. ■

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

Proof. Suppose $a_n = \frac{n}{2n+1}$ and $b_n = \frac{n}{2n+5}$. We have $\lim a_n = \lim b_n$ but $\lim(a_n - b_n) \neq 0$. ■

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

Proof. Let $\varepsilon > 0$. Consider $||b_n| - |b||$. Assume $n \geq N$ then

$$||b_n| - |b|| < |b_n - b| < \varepsilon$$

by reverse triangle inequality and $(b_n) \to b$. ■

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

Proof. Assume $(a_n) \to a$ and $(b_n - a_n) \to 0$. Let $\varepsilon > 0$. By assumption,

$$\begin{aligned} |a_n - a| &< \frac{\varepsilon}{2}, \quad n \geq N_1 \\ |b_n - a_n| &< \frac{\varepsilon}{2}, \quad n \geq N_2. \end{aligned}$$

Hence, choose $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned} |b_n - a| &= |b_n - a_n + a_n - a| \\ &< |b_n - a_n| + |a_n - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $(b_n) \to a$. ■

(d) If $(a_n) \to a$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Proof. Let $\varepsilon > 0$. Choose N so that $a_n \to 0$. Then consider $|b_n - b|$ and observe that

$$|b_n - b| \leq a_n < \varepsilon,$$

Hence, it follows that $(b_n) \to b$. ■

Exercise 2.3.13(Iterated Limits).

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* Limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \text{ and } \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \varepsilon$.

Proof. Let $a_{mn} = m/(m+n)$. We can compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{m}{m+n}\right) \tag{1}$$

and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{m}{m+n}\right). \tag{2}$$

We start with (1). Hence, we have

$$\begin{aligned}\lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{m}{m+n}\right) &= \lim_{n\to\infty} (1)\\ &= 1.\end{aligned}$$

With (2), we have

$$\begin{aligned}\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{m}{m+n}\right) &= \lim_{m\to\infty} (0)\\ &= 0.\end{aligned}$$

■

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exists? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

Proof. Let us try and compute the limit of a_{mn} the same way we did above. Hence, we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{1}{m+n}\right) = \lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{1}{1+n/m}\right) = \lim_{n\to\infty} (1) = 1 \tag{1}$$

and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{1}{m+n}\right) = 1. \tag{2}$$

We claim that for $\lim_{m,n\to\infty} a_{mn}$ to exists, we have to have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

hold. Since (1) and (2) are the equal to each other, we claim that $\lim_{m,n\to\infty} a_{mn}$ exists.

We can try to do the same process for the sequence $a_{mn} = mn/(m^2+n^2)$. Hence, we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{mn}{m^2+n^2}\right) = \lim_{n\to\infty} (0) = 0 \tag{3}$$

and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty} (0) = 0 \tag{4}$$

Since (2) and (3) are equal, we have that $\lim_{m,n\to\infty} a_{mn}$ exists. ■

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but neither iterated limit can be computed.

Proof. Consider

$$a_{m,n} = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}$$

where the iterated limits cannot be computed. ■

1. Assume $\lim\limits_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim\limits_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim\limits_{m\to\infty} b_m = a$.

Proof. Suppose $\lim\limits_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim\limits_{n\to\infty}(a_{mn}) \to b_m$. We want to show that $\lim\limits_{m\to\infty} b_m = a$. Consider $|b_m - a|$. Then fix $m \in \mathbb{N}$ such that for any $m, n \geq N$, we have that

$$\begin{aligned} |b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |b_m - a_{mn}| + |a_{mn} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $(b_m) \to a$. ■

2.2 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$ converges.

Proof. Let (x_n) be the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$. Our goal is to show that (x_n) is convergent. It is sufficient to show that (x_n) is both *monotone* and *bounded*. We first show that (x_n) is *monotone*. We claim that (x_n) is a *decreasing* sequence. Hence, we will show that for all $n \in \mathbb{N}$, we have $x_n > x_{n+1}$. We proceed by inducting on n. Let the base case be $n = 1$. Then we have that

$$x_1 = 3 > x_2 = \frac{1}{4-3} = 1.$$

Hence, we have $x_1 > x_2$. Now we assume that (x_n) is decreasing for all $1 < n \leq k - 1$. We want to show that $x_n > x_k$ for all $n < k$. Since $n \leq k - 1$, we have $x_{k-1} \leq x_n$ by inductive hypothesis. Consider x_k. By definition, we have that $x_k = 1/(4 - x_{k-1})$. Since $x_{k-1} \leq x_n$, then for all $n \in \mathbb{N}$ we have

$$x_k = \frac{1}{4 - x_{k-1}} < \frac{1}{4 - x_n}.$$

Hence, $x_k < x_n$ for all $n \in \mathbb{N}$. This is equivalent to showing $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, (x_n) is a *monotone* sequence. Now we show that (x_n) is *bounded*. Since $3 = x_1 \geq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_{n+1} = 1/(4 - x_n) > 0$, we have that

$$0 < x_n \leq 3.$$

Hence, (x_n) is bounded. Since (x_n) is *monotone* and *bounded*, we have that (x_n) is a convergent sequence by the Monotone Convergence theorem. ■

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Proof. Since (x_n) is *monotone* and *bounded*, then (x_{n+1}) is also *monotone* and *bounded*. By the Monotone Convergence Theorem, we have that (x_{n+1}) is also convergent. Hence, $\lim x_{n+1}$ also exists. ■

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Proof. Since $\lim x_n = \lim x_{n+1}$, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim \frac{1}{4 - x_n} \\ &= \frac{\lim 1}{\lim(4 - x_n)} \\ &= \frac{1}{\lim(4) - \lim x_n} \\ &= \frac{1}{4 - x}. && (\lim x_n = x) \end{aligned}$$

Then we have

$$x = \frac{1}{4 - x}$$

and then

$$x^2 - 4x + 1 = 0$$

which we can solve via the quadratic formula. Hence, we have $x = 2 + \sqrt{3}$. ■

Exercise 2.4.3

Following the model of Exercise 2.4.2, show that the sequence defined by $y_1 = 1$ and $y_{n+1} = 2 - \frac{1}{y_n}$ converges and find the limit.

Proof. Let (y_n) be the sequence defined by $y_1 = 1$ and

$$y_{n+1} = 4 - \frac{1}{y_n}.$$

for all $n \in \mathbb{N}$. We want to show that (y_n) converges. Hence, our goal is to show that (y_n) is *monotone* and *bounded*. We claim that (y_n) is increasing. Hence, we show this by inducting on $n \in \mathbb{N}$. Our goal is to show that $y_n \leq y_{n+1}$ for all $n \in \mathbb{N}$. Let the base case be $n = 1$. Then observe that

$$y_1 = 1 < y_2 = 4 - \frac{1}{1} = 3$$

Hence, we have $y_1 < y_2$.

Now assume that (y_n) is increasing for all $1 \leq n \leq k - 1$. Hence, $y_n \leq y_{k-1}$. Our goal now is to show that $y_n \leq y_k$ for all $n \in \mathbb{N}$. Let's consider y_k. Then by definition of (y_n), we have

$$y_k = 4 - \frac{1}{y_{k-1}}.$$

Since $y_n \le y_{k-1}$, we have

$$y_k = 4 - \frac{1}{y_{k-1}} \ge 4 - \frac{1}{y_n}$$

This shows that $y_k \ge y_n$ for any $n \in \mathbb{N}$. Hence, it follows that y_n is an increasing sequence and, therefore, *monotone*. Now

Now we want to show that (y_n) is *bounded*. Observe that $1 < y_n$ for all $n \in \mathbb{N}$ which means (y_n) contains a lower bound. Furthermore, for each $n \in \mathbb{N}$ we also have that $y_{n+1} = 4 - 1/y_n < 4$ which means that (y_n) also contains an uppper bound. Hence, it follows that

$$1 < y_n < 4$$

for all $n \in \mathbb{N}$. Hence, we have (y_n) is *bounded*. By the Monotone Convergence Theorem, it follows that (y_n) is a convergent sequence.

By last exercise, we know that $\lim y_n = \lim y_{n+1}$. Let's assume $(y_n) \to y$. Our goal is to compute $\lim y_n$. By the Algebraic Limit Theorem, we have

$$\begin{aligned} y = \lim y_n &= \lim\left(4 - \frac{1}{y_n}\right) \\ &= \lim(4) - \lim\left(\frac{1}{y_n}\right) \\ &= 4 - \frac{\lim(1)}{\lim y_n} \\ &= 4 - \frac{1}{y}. \end{aligned}$$

Hence, we have

$$y = 4 - \frac{1}{y}$$

which yields the following quadratic equation set to zero

$$y^2 - 4y + 1 = 0.$$

Solving for y using the quadratic formula yields $y = 2 + \sqrt{3}$ ∎

Exercise 2.4.5 (Calculating Square Roots)

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big).$$

(a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \ge 0$. Conclude that $\lim x_n = \sqrt{2}$.

Proof. Our first goal is to show that $x_n^2 > 2$ for all $n \in \mathbb{N}$. We proceed by inducting on $n \in \mathbb{N}$. Let our base case be $n = 1$. Then

$$\begin{aligned} x_1 = 2 &< x_1^2 \\ &= 4 \\ &< \frac{9}{4} \\ &= \frac{1}{4}\Big(x_1^2 + \frac{4}{x_1^2} + 4\Big) \\ &= x_2^2 \end{aligned}$$

which implies that $2 < x_1^2 < x_2^2$. Now suppose $x_{k-1}^2 > 2$ for all $n \le k - 1$. We want to show that $x_k^2 > 2$ for all $n \in [1, k)$. Consider x_k and then by definition, we have

$$\begin{aligned} x_k^2 &= \frac{1}{4}\Big(x_{k-1}^2 + \frac{4}{x_{k-1}^2} + 4\Big) \\ &> \frac{1}{4}(2+2+4) \\ &= \frac{8}{4} \\ &= 2. \end{aligned}$$

Hence, $x_k^2 > 2$ for all $n \in \mathbb{N}$. Now we want to show that $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Consider $x_n - x_{n+1}$ then observe that since $x_n^2 > 2$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{x_n^2 - 2}{2x_n} \\ &> \frac{2-2}{2\sqrt{2}} \\ &= 0. \end{aligned}$$

Furthermore, when $x^2 = 2$ we get that $x_n - x_{n+1} = 0$. Hence, we have $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, we get that (x_n) is a convergent sequence. Since $\lim x_n = \lim x_{n+1}$, we can show that $\lim x_n = \sqrt{2}$. By the Algebraic Limit Theorem, we have

$$\begin{aligned} x = \lim x_{n+1} &= \lim\Big(\frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big)\Big) \\ &= \frac{1}{2}\lim\Big(x_n + \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(\lim x_n + \lim\frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(x + \frac{2}{x}\Big) \\ &= \frac{1}{2}x + \frac{1}{x} \\ &= \frac{x^2+2}{2x} \end{aligned}$$

which implies that

$$\begin{aligned} x^2 &= 2 \\ \Rightarrow x &= \sqrt{2}. \end{aligned}$$

Hence, we have $\lim x_n = \sqrt{2}$ ∎

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Proof. Let the sequence (x_n) be defined recursively as $x_1 = c$ and

$$x_{n+1} = \frac{1}{c}\Big(x_n + \frac{c}{x_n}\Big).$$

Assume $x_n^2 > c$ for all $n \in \mathbb{N}$ and $x_n - x_{n+1} \geq 0$, then we have $\lim x_n = \sqrt{c}$. ∎

Exercise 2.4.6 (Limit Superior.)

Let (a_n) be a *bounded* sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. Let (a_n) be a *bounded* sequence. Let (y_n) be defined as

$$y_n = \sup\{a_k : k \geq n\}$$

and further denote

$$A_n = \{a_k : k \geq n\}.$$

Our goal is to show that (y_n) converges. Thus, our goal is to show that (y_n) is both *monotone* and *bounded*.

We first show the former. We claim that (y_n) is a *decreasing* sequence. Hence, we need to show that $y_n \geq y_{n+1}$ for all $n \in \mathbb{N}$. We know that for each term y_n, by definition, is the *least upper bound* of the set $\{a_k : k \geq n\}$. By exercise 1.3.4, we know that since $A_{n+1} \subseteq A_n$, we have that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$. Hence, we have that y_n is a *decreasing* sequence.

Now we want to show that (y_n) is *bounded*. Since (a_n) is a *bounded* sequence, there exists an $M > 0$ such that $M \leq a_n \leq y_n$ for all $n \in \mathbb{N}$. Hence, (y_n) is a *bounded* sequence. In this case, it is enough to have (y_n) be bounded below.

Since (y_n) is *monotone* and *bounded*, we have that (y_n) is a convergent sequence.

■

(b) The *limit superior* of (a_n), or $\limsup a_n$ is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Proof. Let a_n be a *bounded* sequence. Then define

$$w_n = \inf\{a_k : k \geq n\}$$

so we can have

$$\lim w_n = \lim \inf\{a_k : k \geq n\}.$$

This limit exists because the terms of (w_n) are *increasing* and that (w_n) is *bounded* since there exists an $L > 0$ such that $w_n \leq a_n \leq L$.

■

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every *bounded* sequence, and give an example of a sequence for which the inequality is strict.

Proof. Suppose that (a_n) is a *bounded* sequence. Since $\liminf a_n$ and $\limsup a_n$ exists, we have that $\liminf a_n \leq a_n \leq \limsup a_n$. Hence,

$$\liminf a_n \leq \limsup a_n.$$

■

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We start by assuming that $\liminf a_n = \limsup a_n$. We want to show that the $\lim a_n$ exists. Define the following:

$$w = \liminf a_n = \lim w_n,$$
$$y = \limsup a_n = \lim y_n.$$

Since w_n and y_n both bound a_n, it follows that $w_n \leq a_n \leq y_n$. Since the $\liminf a_n$ and

$\limsup a_n$ both exists such that $\lim w_n = \lim y_n = \ell$, it follows that $\lim a_n = \ell$ by the *Squeeze Theorem*. Hence, $\lim a_n$ exists.

Now assume the converse. Hence, our goal now is to show that $\liminf a_n = \limsup a_n$. Since (a_n) is a convergent sequence, let $\varepsilon > 0$ such that there exists an $N \in \mathbb{N}$ where for every $n \geq N$, we have

$$|a_n - \ell| < \varepsilon.$$

This is equivalent to saying

$$\ell - \varepsilon \leq a_n \leq \ell + \varepsilon.$$

Since $\varepsilon > 0$ is abitrary, we can conclude that $\liminf a_n = \limsup a_n$.

■

Exercise 2.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Proof. Our goal is to show that $\sum_{n=1}^{\infty} b_n$ diverges. Since $\sum_{n=0}^{\infty} b_{2^n}$ diverges, the sequence of partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2^k}$$

diverges. This implies that t_k is *unbounded*: that is, there exists $M > 0$ such that for some $K \in \mathbb{N}$, we have for all $k \geq K$ we have $t_k > M$. Since $b_n \geq 0$, it suffices to show that the partial sums of $\sum_{n=1}^{\infty} b_n$ are *unbounded*. Let

$$s_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + b_2 + \ldots + b_m.$$

Let us fix m such that we choose k sufficiently large so that $m \geq 2^{k+1} + 1$. Hence, we have that $s_m \geq s_{2^{k-1}+1}$. Observe that

$$\begin{aligned}
s_{2^{k+1}+1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + \ldots + b_{2^{k+1}+1}) \\
&\geq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \ldots + (b_{2k} + \ldots + b_{2k}) \\
&= b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2k} \\
&= t_k.
\end{aligned}$$

This implies that $s_m \geq s_{2^{k+1}+1} \geq t_k > M$ for all $k \geq K$. Hence, we conclude t_k is *unbounded*. Thus, the series

$$\sum_{n=1}^{\infty} b_n.$$

diverges. ■

Exercise 2.4.10 (Infinite Products)

A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \ldots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \ldots \text{ where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$) for positive x will be useful in one direction.)

Exercise 2.4.6 (Limit Superior.)

Let (a_n) be a *bounded* sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. Let (a_n) be a *bounded* sequence. Let (y_n) be defined as

$$y_n = \sup\{a_k : k \geq n\}$$

and further denote

$$A_n = \{a_k : k \geq n\}.$$

Our goal is to show that (y_n) converges. Thus, our goal is to show that (y_n) is both *monotone* and *bounded*.

We first show the former. We claim that (y_n) is a *decreasing* sequence. Hence, we need to show that $y_n \geq y_{n+1}$ for all $n \in \mathbb{N}$. We know that for each term y_n, by definition, is the *least upper bound* of the set $\{a_k : k \geq n\}$. By exercise 1.3.4, we know that since $A_{n+1} \subseteq A_n$, we have that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$. Hence, we have that y_n is a *decreasing* sequence.

Now we want to show that (y_n) is *bounded*. Since (a_n) is a *bounded* sequence, there exists an $M > 0$ such that $M \leq a_n \leq y_n$ for all $n \in \mathbb{N}$. Hence, (y_n) is a *bounded* sequence. In this case, it is enough to have (y_n) be bounded below.

Since (y_n) is *monotone* and *bounded*, we have that (y_n) is a convergent sequence.

■

(b) The *limit superior* of (a_n), or $\limsup a_n$ is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Proof. Let a_n be a *bounded* sequence. Then define

$$w_n = \inf\{a_k : k \geq n\}$$

so we can have

$$\lim w_n = \lim \inf\{a_k : k \geq n\}.$$

This limit exists because the terms of (w_n) are *increasing* and that (w_n) is *bounded* since there exists an $L > 0$ such that $w_n \leq a_n \leq L$.

■

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every *bounded* sequence, and give an example of a sequence for which the inequality is strict.

Proof. Suppose that (a_n) is a *bounded* sequence. Since $\liminf a_n$ and $\limsup a_n$ exists, we have that $\liminf a_n \leq a_n \leq \limsup a_n$. Hence,

$$\liminf a_n \leq \limsup a_n.$$

■

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We start by assuming that $\liminf a_n = \limsup a_n$. We want to show that the $\lim a_n$ exists. Define the following:

$$w = \liminf a_n = \lim w_n,$$
$$y = \limsup a_n = \lim y_n.$$

Since w_n and y_n both bound a_n, it follows that $w_n \le a_n \le y_n$. Since the $\liminf a_n$ and $\limsup a_n$ both exists such that $\lim w_n = \lim y_n = \ell$, it follows that $\lim a_n = \ell$ by the *Squeeze Theorem*. Hence, $\lim a_n$ exists.

Now assume the converse. Hence, our goal now is to show that $\liminf a_n = \limsup a_n$. Since (a_n) is a convergent sequence, let $\varepsilon > 0$ such that there exists an $N \in \mathbb{N}$ where for every $n \ge N$, we have

$$|a_n - \ell| < \varepsilon.$$

This is equivalent to saying

$$\ell - \varepsilon \le a_n \le \ell + \varepsilon.$$

Since $\varepsilon > 0$ is abitrary, we can conclude that $\liminf a_n = \limsup a_n$.

■

Exercise 2.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Proof. Our goal is to show that $\sum_{n=1}^{\infty} b_n$ diverges. Since $\sum_{n=0}^{\infty} b_{2^n}$ diverges, the sequence of partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2^k}$$

diverges. This implies that t_k is *unbounded*: that is, there exists $M > 0$ such that for some $K \in \mathbb{N}$, we have for all $k \ge K$ we have $t_k > M$. Since $b_n \ge 0$, it suffices to show that the partial sums of $\sum_{n=1}^{\infty} b_n$ are *unbounded*. Let

$$s_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + b_2 + ... + b_m.$$

Let us fix m such that we choose k sufficiently large so that $m \ge 2^{k+1} + 1$. Hence, we have that $s_m \ge s_{2^{k-1}+1}$. Observe that

$$\begin{aligned}
s_{2^{k+1}+1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + ... + (b_{2^k} + ... + b_{2^{k+1}+1}) \\
&\ge b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + ... + (b_{2k} + ... + b_{2k}) \\
&= b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2k} \\
&= t_k.
\end{aligned}$$

This implies that $s_m \ge s_{2^{k+1}+1} \ge t_k > M$ for all $k \ge K$. Hence, we conclude t_k is *unbounded*. Thus, the series

$$\sum_{n=1}^{\infty} b_n.$$

diverges. ■

Exercise 2.4.10 (Infinite Products)

A close relative of infinite series is the *infinite product*

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which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \ldots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\ldots \text{ where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

Proof. Let $a_n = 1/n$. Then observe that

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right) = \prod_{k=1}^{n}\frac{k+1}{k}. \tag{1}$$

As we write out the first few terms of the finite product above, we find that the nth term of the denominators of each term is cancelled out. Hence, the explicit formula for (1) above must be

$$\prod_{k=1}^{n}\frac{k+1}{k} = n+1$$

which clearly diverges as $n \to \infty$. If $a_n = 1/n^2$, then we find that

$$\gamma_n = \prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right) = \prod_{k=1}^{n}\frac{k^2+1}{k^2} = \frac{2}{1}\cdot\frac{5}{4}\cdot\frac{10}{9}\cdot\cdots\cdot\frac{n^2+1}{n^2}.$$

I postulate that γ_n converges similarly to how the sequence of partial sums $\sum_{k=1}^{n} 1/k^2$ also converges. ■

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1+x \leq 3^x$) for positive x will be useful in one direction.)

Proof. ($\Rightarrow$) Suppose the sequence of partial products

$$\gamma_n = \prod_{k=1}^{n}(1+a_n)$$

converges. Then we must have (γ_n) is bounded by some $M > 0$. Furthermore, expanding (γ_n), we find that $\gamma_n > \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$. This means $\sum_{k=1}^{n} a_k$ is bounded. Since $a_n \geq 0$ for all n, we also have that $\sum_{k=1}^{n} a_n$ is a monotone sequence. By the Monotone Convergence Theorem, $\sum_{k=1}^{n} a_k$ converges.

($\Leftarrow$) Now suppose the converse. Using the inequality $1+x \leq 3^x$ and the fact that $\sum_{k=1}^{n} a_k$ is a bounded by some $M > 0$, observe that

$$\gamma_n = \prod_{k=1}^{n}(1+a_n) \leq \prod_{k=1}^{n} 3^{a_n} = 3^{\sum_{k=1}^{n} a_k} \leq 3^M.$$

Hence, (γ_n) is a bounded sequence. Now observe that $a_n \geq 0$ implies (γ_n) is increasing. By the Monotone Convergence Theorem, γ_n converges. ■

2.3 Subsequences and the Bolzano-Weierstrass

Exercise 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Proof. The subsubsequence of the bounded subsequence must converge by the *Bolzano-Weierstrass* theorem. ■

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Proof. Let (a_n) be a sequence defined by

$$a_n = \begin{cases} \frac{1}{2n} \text{ if } n = 2k \\ \frac{1}{2n} + 1 \text{ if } n = 2k+1. \end{cases}$$

Note that $1, 0 \notin (a_n)$ but if we take the subsequences $(a_{2k}) = 1/4k$ and $(a_{2k+1}) = 1/(4k+2)+1$, and take their limit, then we end up with the former converging to 0 and the latter converging to 1.

■

(c) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

Proof. Let's define the infinite set

$$A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

and define a subsequence such that we can make a subsequence for each $n \in \mathbb{N}$ where (a_n) hits every value of A_n. ■

(d) A sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\},$$

and no subsequences converging to points outside of this set.

Proof. This is not possible. There exists such a subsequence that does go to 0 but it is not within the infinite set. ■

Exercise 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

Proof. If every proper subsequence of (x_n) converges to x, then $(x_2, x_3, x_4, \ldots)$ also converges to x. Hence, $(x_n) \to x$ by the *uniqueness of limits.* ■

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

Proof. This is just the contrapositve of the statement:

"If (x_n) converges then every subsequence of (x_n) converges as well."

■

(c) If (x_n) contains a divergent subsequence, then there exists two subsequences of (x_n) that converge to different limits.

Proof. This is false. we can find an (x_n) that is not bounded such that we cannot find a subsequence that converges to a limit. ■

(d) If (x_n) is *monotone* and contains a convergent subsequence, then (x_n) converges.

Proof. Assume (x_n) is *monotone* and contains a convergent *subsequence* (x_{n_k}). It suffices to show that (x_n) is *bounded*. Since (x_{n_k}) is convergent, it is also *bounded*. Hence, there exists $M > 0$ such that for all $n_k \in \mathbb{N}$, we have $|x_{n_k}| \leq M$. Since (x_n) *monotone* then either $n \leq n_k$ or $n \geq n_k$ for all $n \in \mathbb{N}$. Hence, we can write either $-M \leq x_n$ or $x_{n_k} \leq M$. But this means that (x_n) is also *bounded*. Since (x_n) both *bounded* and *monotone*, (x_n) is convergent by the *Monotone Convergence Theorem*. Also, (x_n) and (x_{n_k}) converge to the same limit by the *Uniqueness of Limits*. ■

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative propeprty holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + ...$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + ... + a_{n_1}) + (a_1 + a_2 + ... + a_{n_2}) + (a_1 + a_2 + ... + a_{n_3}) + ...$$

leads to a series that also converges to L.

Proof. Our goal is to show that the associative property for a converging infinite series holds. Let us define the terms of the subsequence

$$\begin{aligned} b_1 &= a_1 + a_2 + a_3 + ... + a_{n_1} \\ b_2 &= a_{n_1+1} + a_{n_1+2} + a_{n_1+3} + \cdots + a_{n_2} \\ &\vdots \\ b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}. \end{aligned}$$

Our goal is to show that the subsequence (b_m) converges to L as well. Suppose $\lim s_n = L$. Let the partial sums (t_m) be regrouped in terms of the subsequence above

$$\begin{aligned} t_m &= b_1 + b_2 + \cdots + b_m \\ &= (a_1 + a_2 + a_3 + ... + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \ldots \\ &+ (a_{n_{m-1}+1} + \cdots + a_{n_m}). \end{aligned}$$

Since $\lim s_n = L$, its sequence of partial sums also converge to L. But this means every subsequence of (t_k) also converges to L. Hence, (b_m) converges to L as well. ■

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to associative. Why doesn't our proof in (a) apply to this example?

Proof. We cannot have infinite series be associative if the sequence of partial sums diverges. This means we cannot regroup the terms of our partial sums into a subsequence that converges, since there are divergent subsequences. ■

Exercise 2.5.5

Assume (a_n) is *bounded* sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose for sake of contradiction that $(a_n) \not\to a$. Then there exists $\varepsilon_0 > 0$ such that $|a_n - a| \geq \varepsilon_0$ for all $N \in \mathbb{N}$. Since (a_n) is *bounded*, we can find a subsequence (a_{n_k}) that converges to some $\ell \in \mathbb{R}$. Since $(a_n) \not\to a$, then $(a_{n_k}) \to \ell$ where $\ell \neq a$. Yet we assumed every convergent subsequence of (a_n) converges to the same limit a but $a \neq \ell$ which is a contradiction. Hence, it must be the case that $\lim a_n = a$. ■

Exercise 2.5.6

Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed)

Proof. Let $b \geq 0$. Our goal is to show that $\lim b^{1/n}$ exists. We observe that

$$b > b^{1/2} > b^{1/3} > b^{1/4} > ... > b^{1/n} \geq 0$$

and conclude by induction that $b^{1/n}$ is a *decreasing* sequence. Since $0 \leq b^{1/n} < b$, we can also conclude that $(b^{1/n})$ is a *bounded* sequence. Hence, $(b^{1/n})$ is a convergent sequence. But note that $(b^{1/n}) \to 0$ for all $b^{1/n} \geq 0$ by exercise 2.3.1. Hence, $\lim b^{1/n} = 0$. ■

Exercise 2.5.7

Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof. Suppose $\lim b^n = 0$. Let $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $-1 < b^n < 1$. Then we have $-1 < b < 1$. Hence, $|b| < 1$.

Now let us show the converse. Assume $|b| < 1$; that is, $-1 < b < 1$. Since $0 \leq b < 1$ holds by Example 2.5.3, we can write $\lim b^n = 0$. Suppose $-1 < b < 0$. We observe that

$$b < b^2 < b^3 < b^4 < ... < 0$$

impying that b^n is an *increasing* sequence for all $n \in \mathbb{N}$ for $b^n \in (-1, 0)$. Furthermore, (b^n) is *bounded* since $1 < b^n < 0$. Hence, (b^n) is a convergent sequence by the *Monotone Convergence Theorem*. Hence, $(b^{2n}) \to \ell$ satisfying $b < \ell \leq 0$. Suppose $\lim(b^n) = \ell$. Let (b^{2n}) be a subsequence, then (b^{2n}) also converges to the same limit b. Hence, we have

$$\begin{aligned}\ell = \lim b^n &= \lim b^{2n} \\ &= \lim(b^n \cdot b^n) \\ &= \lim b^n \cdot \lim b^n \\ &= \ell^2\end{aligned}$$

Then by the same process in Example 2.5.3, we have $\lim(b^n) = 0$.

■

Exercise 2.5.6

Let (a_n) be a *bounded* sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Proof. Let (a_n) be a *bounded* sequence. We observe that $S \neq \emptyset$ and S *bounded above* since there exists $M > 0$ such that $|a_n| \leq M$ where $x < a_n \leq M$. By the *Axiom of Completeness*, $s = \sup S$ exists. Then by lemma 1.3.8, let $\varepsilon = 1/n_k$ such that for some $a_{n_k} \in S$, we have

$$s - \frac{1}{n_k} \leq a_{n_k} \leq s$$

Note that we can write $\lim(s - 1/n_k) = s$ by the *Algebraic Limit Theorem*. By the *Squeeze Theorem*, it follows that $a_{n_k} \to s = \sup S$. ■

2.4 The Cauchy Criterion

Exercise 2.6.1

Prove that every convergent sequence is *Cauchy*.

Proof. Assume (x_n) converges to x. To show that (x_n) is *Cauchy*, we must have for every $\varepsilon > 0$, there must exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we must have

$$|x_n - x_m| < \varepsilon.$$

Let $\varepsilon > 0$. Since $(x_n) \to x$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we must have

$$|x_n - x| < \frac{\varepsilon}{2}.$$

Since (x_n) converges to x, every subsequence (x_{n_k}) of (x_n) converges to x. This means for $n_k \geq N$, we also have

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Now consider $|x_n - x_{n_k}|$ and assume $n, n_k \geq N$. Then by the *triangle inequality*, we can write

$$\begin{aligned}
|x_n - x_{n_k}| &= |x_n - x + x - x_{n_k}| \\
&< |x_n - x| + |x - x_{n_k}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, (x_n) is a *Cauchy Sequence*. ∎

Exercise 2.6.2

Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not *monotone*.

Proof. This is possible. Suppose (x_n) is defined such that

$$x_n = \frac{(-1)^n}{n}.$$

We have (x_n) is Cauchy and thus convergent but it is not monotonic. ∎

Takeaway: Just because a sequence is convergent does not generally imply that it is monotonic.

(b) A cauchy sequence with an unbounded subsequence.

Proof. This is not possible since cauchy sequences must be convergent and convergent sequences are bounded which means every subsequence is bounded as well. ∎

(c) A divergent monotone sequence with a Cauchy subsequence.

Proof. This is not possible. A divergent monotone sequence must contain divergent subsequences. Thus, these subsequences cannot be Cauchy by the Cauchy Criterion. ∎

(d) An unbounded sequence containing a subsequence that is Cauchy.

Proof. This is possible. Let's define the following sequence

$$(1, 2, 1, 4, 1, 6, 1, 8, \ldots)$$

where

$$x_n = \begin{cases} 1 \text{ if } n \text{ odd} \\ \text{even if } n \text{ even} \end{cases}.$$

is an *unbounded* sequence. As we can see, if we take the subsequence (x_{2k+1}), then we find the subsequence

$$(1, 1, 1, 1, 1, ...)$$

converges to 1. ■

Exercise 2.6.3

If (x_n) and (y_n) are *Cauchy* sequences, then one easy way to prove that $(x_n + y_n)$ is *Cauchy* is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the *Algebraic Limit Theorem* then implies $(x_n + y_n)$ is convergent and hence *Cauchy*.

(a) Give a direct argument that (x_n+y_n) is a Cauchy sequence that does not use the Cauchy Criterion or the Algbraic Limit Theorem.

Proof. Suppose (x_n) and (y_n) are *Cauchy Sequences*. Our goal is to show that $(x_n + y_n)$ is also a *Cauchy* sequence. Since (x_n) is *Cauchy*, let $\varepsilon > 0$ such that there exists $N_1 \in \mathbb{N}$ for every $m, n \geq N$, we have

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Likewise, there exists $N_2 \in \mathbb{N}$ such that for every $m, n \geq N$, we have

$$|y_n - y_m| < \frac{\varepsilon}{2}.$$

Our goal is to show that

$$|(x_n + y_n) - (x_m + y_m)| < \varepsilon$$

Now choose $N = \max\{N_1, N_2\}$ such that $n, m \geq N$ and using the triangle inequality, we write

$$\begin{aligned}
|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\
&< |x_n - x_m| + |y_n - y_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we have $(x_n + y_n)$ is also *Cauchy*. ■

(b) Do the same for the product $(x_n y_n)$.

Proof. Suppose (x_n) and (y_n) are both *Cauchy*. This implies (x_n) and (y_n) are *bounded* as well as their subsequences (x_{n_k}) and (y_{b_k}). Hence, there exists $N_1, N_2 \in \mathbb{N}$ such that for every $n_k \geq N_1, N_2$, Since (x_{n_k}) converges and (y_n) are Cauchy sequences, it follows that they are also *bounded*. Hence, there exists $M_1, M_2 > 0$ such that $|y_n| < M_1$ and $|x_{n_k}| < M_2$ for all n, n_k. Our goal is to show

$$|x_n y_n - x_{n_k} y_{n_k}| < \varepsilon.$$

Choose $N = \max\{N_1, N_2\}$ such that for every $n, n_k \geq N$

$$\begin{aligned}
|x_n y_n - x_{n_k} y_{n_k}| &= |x_n y_n - x_{n_k} y_n + x_{n_k} y_n - x_{n_k} y_{n_k}| \\
&= |y_n(x_n - x_{n_k}) + x_{n_k}(y_n - y_{n_k})| \\
&< |y_n||x_n - x_{n_k}| + |x_{n_k}||y_n - y_{n_k}| \\
&< M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\
&= \varepsilon.
\end{aligned}$$

Hence, $(x_n y_n)$ is a *Cauchy sequence*. ■

Exercise 2.6.4

Let (a_n) and (b_n) be *Cauchy sequences*. Decide whether each of the following sequences is a *Cauchy sequence*, justifying each conclusion.

(a) $c_n = |a_n - b_n|$.

Proof. We claim that (c_n) is a *Cauchy* sequence. Let $\varepsilon > 0$. We want to show that given some $N \in \mathbb{N}$, if $n, m \geq N$, then we have

$$|c_n - c_m| < \varepsilon.$$

Then, by the *Reverse Triagle Inequality*

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |(a_n - b_n) - (a_m - b_m)| \\
&= |(a_n - a_m) + (b_n - b_m)| \\
&< |a_n - a_m| + |b_n - b_m| \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, (c_n) is *Cauchy sequence.* ■

(b) $c_n = (-1)^n a_n$.

Proof. This is false. Consider the *Cauchy sequence*

$$a_n = (1, 1, 1, , 1, ...)$$

If we take $c_n = (-1)^n a_n$, then (c_n) is not *Cauchy sequence* since

$$(1, -1, 1, -1, 1, ...)$$

is not. ■

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Proof. This is false. Consider (a_n) defined by the alternating sequence

$$a_n = \frac{(-1)^n}{n}.$$

This sequence is *Cauchy* but (c_n) is not because we have for all $n \in \mathbb{N}$

$$c_n = \left[\left[\frac{(-1)^n}{n}\right]\right] = \begin{cases} 0 \text{ if } n = 2k \\ -1 \text{ if } n = 2k+1. \end{cases}$$

which diverges. ■

Exercise 2.6.5

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \varepsilon$.

Decide which of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

Proof. False. Take the sequence $a_n = n$ and note that $|a_{n+1} - a_n| < \varepsilon$ given any $\varepsilon > 0$, but a_n is *unbounded.* ■

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Proof. Suppose (x_n) and (y_n) are *pseudo-Cauchy.* We want to show that $(x_n + y_n)$ is also *pseudo-Cauchy.* Let $\varepsilon > 0$. Choose $N = \max\{N_1, N_2\}$ such that for every $n \geq N$, we have

$$\begin{aligned}|(x_{n+1} + y_{n+1}) - (x_n + y_n)| &= |(x_{n+1} - x_n) + (y_{n+1} - y_n)| \\ &< |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

by the *Triangle Inequality.* Hence, $(x_n + y_n)$ are *pseudo-Cauchy.* ■

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness ans the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without manking any appeal to the Archimedean Property. This shows that BW, AOC, and MCT are all equivalent.

Proof. Suppose (x_n) is a *bounded* and *monotone* sequence. Our goal is to show that $(x_n) \to x$. By assumption, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \to x$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for every $n_k \geq N$, we have

$$|x_{n_k} - x| < \varepsilon.$$

Since (x_n) is *monotone*, then either $n_k \geq n$ or $n \geq n_k$. If $n \geq n_k \geq N$ for all $n \in \mathbb{N}$, then $|x_n - x| < \varepsilon$. If $n_k \geq n$, then for any choice of $n \geq N$, we observe that

$$|x_n - x| \leq |x_{n_k} - x| < \varepsilon.$$

Hence, we conclude that (x_n) is a convergent sequence. ■

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

Proof. Assume the *Cauchy Criterion* holds. We want to show that there exists (x_{n_k}) such that $(x_{n_k}) \to x$. Since (x_n) *bounded above* and *non-empty*, $x = \sup(x_n)$ exists. Furthermore, $(x_n) \to x$ since (x_n) is *Cauchy.* Since n_k is an *increasing* set of natural numbers and (x_n) is *bounded above*, we have that

$$x_n - \frac{1}{n_k} \leq x_{n_k} \leq x.$$

By the *Squeeze Theorem*, we have $(x_{n_k}) \to x$. ■

2.5 Properties of Infinite Series

Exercise 2.7.1

Proving the *Alternating Series Test* amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \ldots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n). Different characterizations of completeness lead to different proofs.

(a) Prove the *Alternating Series Test* by showing that (s_n) is a *Cauchy Sequence.*

Proof. Let (a_n) be a *decreasing sequence* and suppose $(a_n) \to 0$. We want to show that the *Alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ meets the *Cauchy Criterion.*

We first need to show that for every $n > m$, we have the property

$$0 \leq |a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n| \leq |a_{m+1}|$$

Hence, we proceed by induction on k. Note that

$$\sum_{k=m+1}^{n} (-1)^{k+1} a_k = a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_n.$$

Let our base case be $P(1)$. Then $a_{m+1} \geq 0$. For $P(2)$, we have $a_{m+1} \geq a_{m+2}$ for all m since (a_n) is a *decreasing sequence*. Suppose this holds for all $m \leq k-1$. We want to show that this holds for $P(k)$. Since (a_n) is *decreasing*, we have that $a_{k-1} \geq a_k$. Hence, $a_{k-1} - a_k \geq 0$. Since $P(k-1)$ holds where

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_{k-1} \leq a_{m+1}.$$

But this means that every term leading up to a_k is bounded by a_{m+1}. Hence,

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - ... \pm a_k \leq a_{m+1}.$$

Let $\varepsilon > 0$. All is left to show is that

$$\left| \sum_{k=1}^{n} (-1)^{k+1} a_k \right| < \varepsilon.$$

Hence, for some $N \in \mathbb{N}$, let $n > m \geq N$ and $(a_n) \to 0$ such that

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (-1)^{k+1} a_k \right| &\leq |a_{m+1}| \\ &< \varepsilon. \end{aligned}$$

Hence, the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ meets the *Cauchy Criterion*. ■

(b) Supply another proof for this result using the Nested Interval Property.

Proof. Suppose (a_n) is *decreasing* sequence and $(a_n) \to 0$. Our goal is to show the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. Since (a_n) is *decreasing*, we can use the *Nested Interval Property* to construct closed intervals $I_n = [s_n, s_{n+1}]$ such that the length of these intervals is $|s_n - s_{n+1}| \leq a_n$. The *Nested Interval Property* gurantees the following property that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

where $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Hence, $S \in \mathbb{R}$ can be our candidate limit since $S \in I_n$ for all n. Let $\varepsilon > 0$. Since $(a_n) \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$

$$|s_n - S| \leq a_n < \varepsilon.$$

■

Hence, $(s_n) \to S$.

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the *Monotone Convergence Theorem* leads to a third proof for the *Alternating Series Test*.

Proof. Define the subsequence of partial sums (s_{2n}) as

$$\sum_{k=1}^{n} (-1)^{2k} a_{2k}.$$

Since (a_n) is a *decreasing sequence*, we have that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned} s_1 &= a_2 \geq 0 \\ s_2 &= a_2 + a_4 \geq s_1 \\ s_3 &= a_2 + a_4 + a_6 \geq s_2 \\ &\vdots \\ s_n &= a_2 + a_4 + a_6 + \ldots + a_{2n}. \end{aligned}$$

We can see that s_{2n} is an *increasing sequence*. Also, $|s_{2n}| < M$ since (a_n) is a *bounded sequence*. Hence, we can conclude that the subsequence of partial sums (s_{2n}) is converges to some $S \in \mathbb{R}$.

We can show that (s_{2n+1}) converges to S as well. Since $s_{2n+1} = s_{2n} + a_{2n+1}$, we can use the *Algebraic Limit Theorem* to say that

$$\begin{aligned} \lim(s_{2n+1}) &= \lim(s_{2n} + a_{2n+1}) \\ &= \lim(s_{2n}) + \lim(a_{2n+1}) \\ &= S + 0 \\ &= S. \end{aligned}$$

Since $(s_{2n}) \to S$ and $(s_{2n+1}) \to S$, we have $(s_n) \to S$ as well. ■

Exercise 2.7.4

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the *Cauchy Criterion* for Series.

Proof. Suppose (a_k) and (b_k) are sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Assume $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to show that $\sum_{k=1}^{\infty} a_k$ converges. Define the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ as

$$t_n = \sum_{k=1}^{n} a_k.$$

Let $\varepsilon > 0$. Since $a_k \leq b_k$ and $\sum_{n=1}^{\infty} b_n$ converges, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, we have

$$\begin{aligned} |t_n - t_m| &= \left| \sum_{k=m+1}^{n} a_k \right| \\ &\leq \left| \sum_{k=m+1}^{n} b_k \right| \\ &< \varepsilon. \end{aligned}$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ converges. Note that part (ii) is just the contrapositive of part (i). Hence, it is also true. ■

(b) Give another proof for the *Comparison Test*, this time using the *Monotone Convergence Theorem*.

Proof. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Our goal is to use the *Monotone Convergence Theorem* to show that $\sum_{n=1}^{\infty} a_n$ converges i.e our goal is to show that the sequence of partial sums $t_n = \sum_{k=1}^{n} a_n$ is *bounded* and *monotone*.

Since the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$ are *bounded* and $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, it follows that we have $|t_n| \leq M$ as well.

Now we want to show that (t_n) is a *decreasing sequence*. Since $\sum_{n=1}^{\infty} b_n$ is convergent, we know that $b_n \to 0$. Since $a_n \geq 0$ and $(b_n) \to 0$, the terms (t_n) must also be *decreasing*. Hence, $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$.

Since (t_n) is both *decreasing* and *bounded*, it follows that $\sum_{n=1}^{\infty} a_n$ is a convergent ■

Exercise 2.7.4

Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

Proof. Take $\sum x_n = (-1)^n$ and $\sum y_n = 1/n$. These two series diverge but $\sum x_n y_n = (-1)^n/n$ converges. ■

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

Proof. Take the convergent series $\sum 1/n^2$ and the bounded sequence $y_n = \sin(n)$. We have $\sum x_n y_n = \sum \sin(n)/n^2$ is divergent by the comparison test. ■

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converges but $\sum y_n$ diverges.

Proof. This is impossible. By the Algebraic Series Theorem, we cannot have $\sum(x_n + y_n)$ converge without $\sum y_n$ converging as well. ■

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

Proof. By the comparison test, $\sum(-1)^n x_n$ diverges. ■

Exercise 2.7.5

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Corollary (P-Series Test). The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

Proof. We start with the backwards direction. Suppose $p > 1$. Our goal is to show that $\sum_{n=1}^{\infty} 1/n^p$ converges. Notice that $b_n = 1/n^p$ where $b_n \geq 0$ and b_n *decreasing*. By the *Cauchy Condensation Test*, we can prove that

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^p}\right)^n.$$

converges. Since $p > 1$, we have that

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^{p-1} = \sum_{n=0}^{\infty} \left(\frac{1}{2^p}\right)^n.$$

Since $|r| = |1/2^p| < 1$, we know that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ is a *Geometric Series*. By the *Cauchy Condensation Test*, we can say that $\sum_{n=1}^{\infty} b_n$ converges.

For the forwards direction, since $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges, the only reasonable choice of p is when $p > 1$ or else it is *Harmonic Series* which diverges. ■

Exercise 2.7.6

Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about *subvergent* series:

(a) If (a_n) is *bounded*, then $\sum a_n$ *subverges.*

Proof. This is a valid proposition since the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ are bounded which implies that the sequence of partial sums contains a subsequence partial sums that is convergent. Hence, we can say that $\sum a_n$ is a *subvergent* series. ■

(b) All convergent series are *subvergent.*

Proof. This is valid since the sequence of partial sums for a convergent series converges and hence all of the possible subsequence of partial sums for the series converges to the same limit. ■

(c) If $\sum |a_n|$ *subverges*, then $\sum a_n$ *subverges* as well.

Proof. This is not valid. ■

(d) If $\sum a_n$ *subverges*, then (a_n) has a convergent subsequence.

Proof. This is not valid. ■

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

Proof. Suppose for sake of contradiction that $\sum a_n$ converges. Hence, $(a_n) \to 0$. This means that $\lim(na_n) = 0$ but this contradicts our assumption that $\lim(na_n) = l \neq 0$. Hence, the series $\sum a_n$ must diverge. ■

Another why is to use the limit assumption directly.

Proof. Suppose $a_n > 0$ and $\lim(na_n) = l$. We want to show that $\sum a_n$ diverges. Since $\lim(na_n) = l \neq 0$, let $\varepsilon = 1$ such that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ for all n, we have

$$|na_n - l| < 1 \Leftrightarrow a_n < \frac{1+l}{n}.$$

This implies that

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n}.$$

Note that $\sum \frac{1+l}{n}$ is not a *p-series* since n^p where $p = 1$. Hence, the series $\sum \frac{1+l}{n}$ diverges. Hence, we have that $\sum a_n$ is also a divergent series by the comparison test. ■

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Proof. Suppose $a_n > 0$ and $\lim(n^2 a_n)$ exists. Suppose $\lim(n^2 a_n) = L$ for some $L \in \mathbb{R}$. Let $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$|n^2 a_n - L| < \varepsilon.$$

Hence, we have

$$n^2 a_n - L < 1 \Leftrightarrow a_n < \frac{1+l}{n^2} \tag{1}$$

Our goal is to show via *comparison test* that the series $\sum a_n$ converges. From (1), we have

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1+l}{n^2}.$$

Observe that the series $\sum \frac{1+l}{n^2}$ is a *p-series* test which converges. Hence, the series $\sum a_n$ converges by the *Comparison test.* ■

Exercise 2.7.8

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ *converges absolutely*, then $\sum a_n^2$ also *converges absolutely.*

Proof. Since $\sum a_n$ converges absolutely, then we have the series $\sum |a_n|$ converges. In order for $\sum a_n^2$ to converge absolutely, we need to show that $\sum |a_n^2|$ converges. Furthermore, (a_n) is a *bounded* sequence. Hence, there exists $M > 0$ such that $|a_n| \leq M$. Since there exists

$N \in \mathbb{N}$, for any $n \geq N$, we can write

$$\begin{aligned}\sum |a_n^2| &= \sum |a_n \cdot a_n| \\ &= \sum |a_n| \cdot |a_n| \\ &\leq \sum M \cdot |a_n| \\ &= M \sum |a_n|\end{aligned}$$

We know by the *Algebraic Limit Theorem* for series that $M \sum |a_n|$ converges. Hence, the series $\sum a_n^2$ converges absolutely by the *Comparison Test*. ■

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

Proof. Since (b_n) converges, we have that (b_n) is also *bounded*. Hence, there exists $M > 0$ such that for all n we have $b_n \leq M$. Hence, we have

$$\sum a_n b_n \leq M \sum a_n.$$

By the *Algebraic Limit Theorem* for series, we have that $M \sum a_n$ converges. Since $a_n b_n \leq M a_n$, we have that the series $\sum a_n b_n$ also converges by the *Comparison test*. ■

(c) If $\sum a_n$ *converges conditionally*, then $\sum n^2 a_n$ diverges.

Proof. This is false. Consider the series $\sum \frac{(-1)^n}{n^2}$ which *converges conditionally* but note that $\sum n^2 \frac{(-1)^n}{n^2} = \sum (-1)^n$ diverges. ■

Exercise 2.7.9 (Ratio Test).

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the *Ratio Test* states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

Proof. There exists $N \in \mathbb{N}$ such that $n \geq N$ because $\lim |\frac{a_{n+1}}{a_n}| = r$. This means that $|\frac{a_{n+1}}{a_n}|$ is *bounded*. Hence, we have that $|\frac{a_{n+1}}{a_n}| \leq r'$ which means that $|a_{n+1}| \leq r'|a_n|$. ■

(b) Why does $|a_N| \sum (r')^n$ converge?

Proof. The series $|a_N| \sum (r')^n$ converges because $|r'| < 1$ which means that $|a_N| \sum (r')^n$ is a *geometric series* which converges. ■

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Proof. Consider the series $\sum |a_n|$ and the fact that

$$\sum |a_n| \leq |a_N| \sum (r')^n$$

for all $n \geq N$. Since the right hand series is *geometric* which converges, we can conclude that $\sum |a_n|$ also converges by the comparison test. Hence, the series $\sum a_n$ converges *absolutely* and thus the series $\sum a_n$ converges.

■

Exercise 2.7.10

Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

Proof. No, this does not converge. Look at part (a) of Exercise 2.4.10. ■

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10}$ certainly converges. Why? Does it converge to zero?

Proof. Suppose we define

$$\gamma_n = \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots \frac{2n-1}{2n}.$$

Observe that γ_n is a decreasing sequence as well as bounded by $1/2$. Using MCT, we know that the sequence of partial products above is a convergent sequence. ■

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3).

Proof. Define the sequence of partial products (γ_n) as

$$\gamma_n = \prod_{k=1}^{n} \frac{4k^2}{(2k-1)(2k+1)} = \left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots \frac{4n^2}{(2n-1)(2n+1)}.$$

Observe that γ_n is bounded by $\gamma_1 = 4/3$. Hence, we have $0 \leq \gamma_n \leq 4/3$ We can induct on $n \in \mathbb{N}$ to show that γ_n is decreasing. Hence, we must have γ_n convergent by MCT. ■

Exercise 2.7.12 (Summation-by-parts)

Let (x_n) and (y_n) be sequences, let $s_n = \sum_{k=1}^{n} x_k$ and set $x_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Proof. Using the observation that $x_j = s_j - s_{j-1}$ and the fact that the two series

$$\sum_{j=m}^{n} s_j y_{j+1}, \tag{1}$$

$$\sum_{j=m}^{n} s_{j-1} y_j \tag{2}$$

that cancel every term except nth term $(s_n y_{n+1})$ of (1) and the mth term $(s_{m-1} y_m)$ of (2) , we can write

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m}^{n} (s_j y_j - s_{j-1} y_j) \\
&= \sum_{j=m}^{n} (s_j y_j - s_j y_{j+1} + s_j y_{j+1} - s_{j-1} y_j) \\
&= \sum_{j=m}^{n} (s_j y_{j+1} - s_{j-1} y_j) + \sum_{j=m}^{n} s_j (y_j - y_{j+1}) \\
&= \sum_{j=m}^{n} s_j y_{j+1} - \sum_{j=m}^{n} s_{j-1} y_j + \sum_{j=m}^{n} s_j (y_j - y_{j+1}) \\
&= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}). \qquad (1)
\end{aligned}$$

■

Exercise 2.7.13 (Abel's Test).

Abel's test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = \sum_{k=1}^{n} x_k$.

Proof. Using Exercise 2.7.12, let $n = 1$ where $s_0 = 0$. Then we have that

$$\begin{aligned}
\sum_{k=1}^{n} x_k y_k &= s_n y_{n+1} - s_0 y_1 + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) \\
&= s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}).
\end{aligned}$$

■

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's test.

Proof. Since (y_n) is a bounded and monotonically decreasing sequence where $y_k \geq y_{k+1} \geq 0$, we know that (y_n) must be a convergent sequence by the Monotone Convergence Theorem. Furthermore, for all $k \in \mathbb{N}$, we have $y_k - y_{k+1} \geq 0$. Since s_n converges, we know that there exists some $M > 0$ such that $|s_n| \leq M$. Hence, we also have

$$0 \leq |s_k||y_k - y_{k+1}| \leq M(y_k - y_{k+1}).$$

The left right side of the inequality above forms the following telescoping series

$$\sum_{k=1}^{n} M(y_k - y_{k+1}) = M(y_1 - y_{n+1}).$$

Since (y_n) converges, the limit of the above equation must also be convergent. Since the sequence of partial sums of

$$\sum_{n=1}^{\infty} s_k(y_k - y_{k+1})$$

converges absolutely, we can now use it in our argument to prove that the series

$$\sum_{n=1}^{\infty} x_n y_n \tag{1}$$

satisfies the Cauchy Criterion. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for any $n > m \geq N$, we have that

$$\begin{aligned}
\Big|\sum_{k=m+1}^{n} x_k y_k\Big| &= \Big|s_n y_{n+1} - s_m y_{m+1} + \sum_{k=m+1}^{n} s_k(y_k - y_{k+1})\Big| \\
&\leq |s_n y_{n+1} - s_m y_{m+1}| + \Big|\sum_{k=m+1}^{n} s_k(y_k - y_{k+1})\Big| \\
&\leq |y_{n+1}||s_n - s_m| + |s_m||y_{n+1} - y_{m+1}| \\
&+ \Big|\sum_{k=m+1}^{n} s_k(y_k - y_{k+1})\Big| \\
&\leq y_1|s_n - s_m| + M|y_{n+1} - y_{m+1}| \\
&+ \Big|\sum_{k=m+1}^{n} s_k(y_k - y_{k+1})\Big| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since (1) satisfies the Cauchy Criterion, we must conclude that (1) converges.

■

2.6 Double Summations and Products of Infinite Series

Exercise 2.8.1

Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{mn}$. How does this value compare to the two iterated values for the sum already computed?

The double summation from section 2.1 is $a_{ij} = \frac{1}{2^{j-i}}$ where $\{a_{ij} : i, j \in \mathbb{N}\}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

Proof. To find $\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{mn}$, we first need to define the sequence of partial sums. We can fix j (the rows of the matrix) and define the sequence of partial sums for the series $\sum_{i,j=1}^{\infty} a_{ij}$ as

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}}\right) = -2 + \frac{1}{2^{n-1}}$$

which taking the limit leads to

$$\lim_{n\to\infty} \left(-2 + \frac{1}{2^{n-1}}\right) = -2.$$

■

Exercise 2.8.2

Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some $b_i \in \mathbb{R}$, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges.

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \tag{1}$$

converges. This means that the (1) meets the *Cauchy Criterion*. Let $\varepsilon > 0$. This implies that there exists $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| < \varepsilon.$$

Consider $\left|\sum_{(i,j)\in A(m,n)} a_{ij}\right|$ where

$$A(m,n) = \{(i,j) : 1 \leq i \leq j \leq n\}.$$

Using the *Triangle Inequality*, we find that j

$$\begin{aligned}\left|s_{mm} - s_{nn}\right| &= \left|\sum_{(i,j)\in A(m,n)} a_{ij}\right| \\ &\leq \sum_{(i,j)\in A(m,n)} |a_{ij}| \\ &< \varepsilon.\end{aligned} \tag{2}$$

Since (2) meets the *Cauchy Criterion* for series, we know that $\sum_{m,n}^{\infty} a_{ij}$ must be *Cauchy* and thus must converge as well. ∎

Another proof using the Comparison Test goes something like this

Proof. Suppose the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges. This means that for each $i \in \mathbb{N}$ the infinite series

$$\sum_{j=1}^{\infty} a_{ij} = r_i$$

for some $r_i \in \mathbb{R}$. Hence, we have the infinite series

$$\sum_{i=1}^{\infty} r_i. \tag{1}$$

Our goal is to show that (1) converges. Suppose we look at the terms

$$|r_i| = \Big|\sum_{j=1}^{\infty} a_{ij}\Big|.$$

Note by the *Triangle Inequality* that

$$\sum_{i=1}^{\infty} |r_i| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|.$$

by assumption the infinite series to the right converges. Hence, the series to the left must also converge by the *Comparison Test*. Since $\sum |r_i|$ converges, then the series $\sum r_i$ converges by the *Absolute Convergence Test* . ■

Theorem 3. Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover, we have that

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1) , define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|.$$

■

Exercise 2.8.3

(a) Prove that (t_{nn}) converges.

Proof. From our definition of t_{nn} above we have

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|.$$

We want to show for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|t_{nn} - L| < \varepsilon$. By assumption, we know that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| \tag{1}$$

converges absolutely which implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges. Note that $t_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \to s_n = \sum_{i=1}^{\infty} r_i$ for some $r_i \in \mathbb{R}$. Furthermore, we have $s_n \to L$ since (1) converges. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for any

$n \geq N$, we have that

$$\begin{aligned}|t_{nn} - L| &= |t_{nn} - s_n + s_n - L| \\ &\leq |t_{nn} - s_n| + |s_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, the sequence of partial sums (t_{nn}) converges. ■

Another way we can prove this is to use the Monotone Convergence Theorem.

Proof. Our goal is to show that (t_{nn}) is converges to L. That is, our goal is to show that (t_{nn}) is bounded and monotone. We know that (t_{nn}) is monotone since all t_{nn} are non-negative terms and that $\sum_{n,m}^{\infty} |a_{ij}| = L$ where $L \geq 0$. To show that (t_{nn}) is bounded note that

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m}\sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Hence, (t_{nn}) is a bounded sequence. By the Monotone Convergence Theorem, (t_{nn}) converges. ■

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Proof. Suppose (t_{nn}) is a Cauchy Sequence. Then for some $N \in \mathbb{N}$ we have that for any $n \geq m > N$

$$|t_{nn} - t_{mm}| < \varepsilon.$$

We can rewrite this in the following way to say that

$$|\sum_{n,m} t_{ij}| < \varepsilon.$$

Our goal is to show that

$$|s_{nn} - s_{mm}| < \varepsilon.$$

Hence, for any $n \geq m > N$, we have that

$$\begin{aligned}|s_{nn} - s_{mm}| &\leq |t_{nn} - t_{mm}| \\ &= \left|\sum_{n,m} t_{ij}\right| \\ &< \varepsilon.\end{aligned}$$

Hence, (s_{nn}) converges. ■

Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges. In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij},$$

Because $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Exercise 2.8.4

(a) Let $\varepsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\varepsilon}{2} < t_{mn} \leq B$.

Proof. Since (t_{mn}) bounded, we can say that $t_{mn} \leq B$. Since the set

$$\{t_{mn} : m, n \in \mathbb{N}\}$$

is bounded above and non-empty, we also have that $B = \sup\{t_{mn} : m, n \in \mathbb{N}\}$ exists. Hence, for any $\varepsilon > 0$, we have that $B - \frac{\varepsilon}{2}$ is not an upper bound. Hence, there exists some $t_{n_0 m_0}$ such that $B - \frac{\varepsilon}{2} < t_{m_0 n_0} \leq t_{mn}$. Furthermore, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq m > N_1$ since (t_{mn}) converges. Hence, we must have that $B - \frac{\varepsilon}{2} < t_{mn} \leq B$ ■

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \varepsilon$$

for all $m, n \geq N$.

Proof. Consider $|s_{mn} - S| < \varepsilon$. Since $(s_{nn}) \to S$, let $\varepsilon > 0$ such that for some $N_2 \in \mathbb{N}$ we have $n \geq m > N_2$, we have

$$|s_{nn} - S| < \frac{\varepsilon}{2}.$$

Since (s_{nn}) meets the Cauchy Criterion, we have that there exists $N_2 \in \mathbb{N}$ such that for any $n \geq m > N$, we have

$$|s_{nn} - s_{mn}| < \frac{\varepsilon}{2}.$$

Hence, observe that for any $n \geq m > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} - s_{nn} + s_{nn} - S| \\
&\leq |s_{mn} - s_{nn}| + |s_{nn} - S| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we have that $(s_{mn}) \to S$. ■

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 2.8.5

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + ... + r_m) - S| \leq \varepsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

Proof. By exercise 2.8.4, we know that $s_{mn} \to S$. Note that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} r_i \text{ for each } i.$$

Hence, we have

$$\lim_{m,n \to \infty} s_{mn} = \lim_{m \to \infty} \sum_{i=1}^{m} r_i = S$$

which is equivalent to saying that for all $m > N$ for some $N \in \mathbb{N}$ we have that

$$\left|\left(\sum_{i=1}^{m} r_i\right) - S\right| \leq \varepsilon.$$

■

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} a_{ij}$ converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Proof. Using the same process above for summing up the columns of $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ leads to

$$\left|\left(\sum_{j=1}^{n} c_j\right) - S\right| \leq \varepsilon.$$

Hence, we must have that

$$\lim_{n\to\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

■

Chapter 3

Basic Topology of The Real Numbers

3.1 Open and Closed sets

Exercise 3.2.2

Let
$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\}$$
and
$$B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$
Answer the following questions for each set:

(a) What are the limit points?

> **Proof.** The limit points of A is $L = \{-1, 1\}$ and the limit points of B is $L = \{0, 1\}$. ■

(b) Is the set open? Closed?

> **Proof.** The sets A and B are not closed since their limit points are not contained and open since we can create $V_\varepsilon(x) \subseteq A$ while B is not open since $V_\varepsilon(x) \not\subseteq B$ for every $x \in \mathbb{Q}$ however small ε is. ■

(c) Does the set contain any isolated points?

> **Proof.** From part (b), since we cannot find any points near each $x \in \mathbb{Q}$ in B, we have that all the points of B are isolated points. ■

(d) Find the closure of the set.

> **Proof.** The closure of sets A and B are $\overline{A} = A \cup \{-1, 1\}$ and $\overline{B} = B \cup \{0, 1\}$. ■

Exercise 1.2.13

Show De Morgan's Laws where $\{A_i : 1 \leq i \leq n\}$ is a collection of sets such that
$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c \tag{1}$$
$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c \tag{2}$$
for any finite $n \in \mathbb{N}$.

> **Proof.** Our goal is to show that both inclusions hold for (1) and (2). Our first step is to induct on $n \in \mathbb{N}$ to show that
> $$\left(\bigcup_{i=1}^{n} A_i\right)^c \subseteq \bigcap_{i=1}^{n} A_i^c. \tag{1}$$

Let $n = 1$ be the base case. It follows immediately that $A_1^c \subseteq A_1^c$. Let $n = 2$, then it follows that $(A_1 \cup A_2)^c \subseteq A_1^c \cap A_2^c$ by exercise 1.2.5. For the other inclusion, we also have $A_1^c \cap A_2^c \subseteq (A_1 \cup A_2)^c$. Now suppose (1) holds for $1 \leq n \leq k-1$. We want to show that (1) holds for k. Let

$$A' = \bigcup_{n=1}^{k-1} A_n$$

then consider the following

$$\left(\bigcup_{n=1}^{k} A_n\right)^c = \left(A_k \cup \left[\bigcup_{n=1}^{k-1} A_n\right]\right)^c = (A_k \cup A')^c$$

Let $x \in (A_k \cup A')^c$, then we know that $x \notin (A_k \cup A')$. This means that $x \notin A_k$ and $x \notin A'$. Hence, we have $x \in A_k^c$ and $x \in (A')^c$; that is,

$$\begin{aligned}(A_k \cup A')^c &\subseteq A_k^c \cap (A')^c \\ &= A_k^c \cap \left(\bigcup_{n=1}^{k-1} A_n\right)^c. \\ &\subseteq A_k^c \cap \left(\bigcap_{n=1}^{k-1} A_n^c\right) \\ &= A_k \cap (A_{k-1} \cap \dots \cap A_1) \\ &= \bigcap_{n=1}^{k} A_n^c.\end{aligned}$$

Hence, we have

$$\left(\bigcup_{i=1}^{n} A_i\right)^c \subseteq \bigcap_{i=1}^{n} A_i^c.$$

For the other inclusion, suppose the containment

$$\bigcap_{n=1}^{k-1} A_k^c \subseteq \left(\bigcup_{n=1}^{k-1} A_k\right)^c \tag{2}$$

holds for $1 \leq n \leq k-1$. We want to show that (2) holds for k. Consider the finite intersection

$$\bigcap_{n=1}^{k} A_n^c = A_k^c \cap \left(\bigcap_{n=1}^{k-1} A_n^c\right).$$

If we know that $x \notin \bigcap_{n=1}^{k-1} A_n$ and $x \notin A_k$ then $x \notin \left(A_k \cup \left(\bigcap_{n=1}^{k-1} A_n\right)\right)$. Hence, using our inductive hypothesis, we have

$$\begin{aligned}\bigcap_{n=1}^{k} A_n^c &= A_k^c \cap \left(\bigcap_{n=1}^{k-1} A_n^c\right) \\ &\subseteq A_k^c \cup \left(\bigcap_{n=1}^{k-1} A_n\right)^c \\ &\subseteq A_k^c \cup \left(\bigcup_{n=1}^{k-1} A_n^c\right) \\ &= \left(\bigcup_{n=1}^{k} A_n\right)^c\end{aligned}$$

Since both containments hold, we must have

$$\left(\bigcup_{n=1}^{k} A_n\right)^c = \bigcap_{n=1}^{k} A_n^c.$$

The proof to the other equation is similar. ■

Exercise 3.2.4

Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

Proof. Let $A \neq \emptyset$ and bounded above. Since $s = \sup A$ exists we can let $\varepsilon > 0$ such that for some $\alpha \in A$, we have $s - \varepsilon < \alpha$. Our goal is to show that $s \in \overline{A}$. Let (a_n) be a sequence in A such that $a_n \neq s$ for all $n \in \mathbb{N}$. Let $\varepsilon = 1/n$ such that

$$s - \frac{1}{n} < \alpha \leq a_n \leq s.$$

By the Squeeze Theorem, we have $\lim a_n = s = \sup A$. This means $s = \sup A$ is a limit point where $L = \{s\}$ such that $\overline{A} = A \cup L$. Hence, $s \in \overline{A}$.

■

(b) Can an open set contain its supremum?

Proof. An open set A cannot contain its supremum, which is a limit point in part (a), since otherwise A would be a closed set. ■

Exercise 3.2.5

Prove Theorem 3.2.8: Show that a set $F \subseteq \mathbb{R}$ is closed if and only if if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof. ($\Rightarrow$) Let $F \subseteq \mathbb{R}$ be a closed set. Let x be a limit point and let (x_n) be a Cauchy sequence be arbitrary. Since F is a closed set, the limit point $x \in F$; that is, $\lim x_n = x \in F$ where $x_n \neq x$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Let $F \subseteq \mathbb{R}$. We want to show that F is closed. Let (x_n) be a Cauchy sequence contained in F such that $\lim x_n = x \in F$. Note that $x_n \neq x$ for all $n \in \mathbb{N}$. Since all the limit points of F are contained in F, then F must be a closed set. ■

Exercise 3.2.7

Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

Proof. Let L be the set of limit points of A, and suppose that x is a limit point of L. Our goal is to show that x is a limit point of A. Let $V_\varepsilon(x)$ be arbitrary. Let $\varepsilon > 0$, then we know that $V_\varepsilon(x)$ intersects L at a point $\ell \in L$ where $\ell \neq x$. Choose $\varepsilon' > 0$ small enough so that $V_{\varepsilon'}(\ell) \subseteq V_\varepsilon(x)$ and $x \notin V_{\varepsilon'}(\ell)$. Since $\ell \in L$, we know that ℓ is a limit point of Am and therefore x is a limit point of A and thus an element of L. ■

(b) Argue that if x is a limit point $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Proof. Suppose x is a limit point of $\overline{A} = A \cup L$. By definition, we can construct $V_\varepsilon(x)$ such that $V_\varepsilon(x)$ intersects $a \in \overline{A}$ where $a \neq x$. This means $x \in A$ or $x \in L$. If $x \in A$, then $V_\varepsilon(x)$ intersects every point $a \in A$ where $x \neq a$. Hence, x is a limit point of A. If $x \in L$, then we can use the same argument from above to construct an $\varepsilon' > 0$ small enough so that $V_{\varepsilon'}(\ell) \subseteq V_\varepsilon(x)$ where $x \notin V_{\varepsilon'}(\ell)$. Since $\ell \in L$ is a limit point, this means that that $V_{\varepsilon'}(\ell)$ intersects A. But since $x \notin V_{\varepsilon'}(\ell)$ this means that $V_\varepsilon(x)$ intersects A at every point of A that is not x. Hence,

x is a limit point of A. ■

Exercise 3.2.9

A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

Proof. Suppose we show the first equation. Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$ then for all E_λ where $\lambda \in \Lambda$, we must have $x \notin E_\lambda$. But this is equivalent to saying that $x \in E_\lambda^c$ for all $\lambda \in \Lambda$ where $E_\lambda^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Hence, we have

$$x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

Now let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then for all $\lambda \in \Lambda$, we have $x \in E_\lambda^c$. This is true if and only if $x \notin E_\lambda$ for all $\lambda \in \Lambda$. Since $E_\lambda \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda$, we also have that $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. Hence, we have

$$x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c.$$

Hence,

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

The other equation can be shown through a similar process above. ■

(b) Now, provide the details for the proof of Theorem 3.2.14.

Proof. To prove part (i), suppose we have a finite collection of open sets where

$$\{E_i : 1 \le i \le N\}.$$

Since E_i closed, their complements E_i^c is open. Since the finite intersection of open sets is open, we have that

$$\left(\bigcup_{i=1}^N E_i\right)^c = \bigcap_{i=1}^N E_i^c$$

is open. But this means that

$$\bigcup_{i=1}^N E_i$$

is closed.

To prove part (ii), suppose we have an arbitrary collection of closed sets

$$\{E_\lambda : \lambda \in \Lambda\}.$$

Since E_λ is closed, we have that their complement E_λ^c is open. But this means that the union

$$\bigcup_{\lambda\in\Lambda} E_\lambda^c = \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c \tag{1}$$

is also open. But since the complement of the intersection of (1) is open, we have

$$\bigcap_{\lambda \in \Lambda} E_\lambda$$

is closed. ■

Exercise 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. We want to show the following containments

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B},$$
$$\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$$

Suppose $x \in \overline{A \cup B}$. Then x is a limit point of $A \cup B$. Hence, either $x \in A$ or $x \in B$. But x is a limit point so there exists $V_\varepsilon(x)$ that either intersects $a \neq x \in A$ or $b \neq x \in B$. But this means that x is a limit point of A or B. Hence, $x \in \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A} \cup \overline{B}$. Then either $x \in \overline{A}$ or $x \in \overline{B}$. But this means that x is a limit point of A or B which imply that $V_\varepsilon(x)$ intersects elements of both A or B that is not x. Hence, x must be a limit point of either A or B. Hence, $x \in \overline{A \cup B}$. Since both containments are true, we have that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ■

(b) Does this result about closures extend to infinite unions of sets?

Proof. No this result does not extend to infinite unions. Consider the counter-example where we have a closed set $H_n = [1/n, 1] \subseteq \mathbb{R}$ where

$$\bigcup_{i=1}^{\infty} \overline{H_n} = (0, 1] \tag{1}$$

$$\overline{\bigcup_{i=1}^{\infty} H_n} = [0, 1] \tag{2}$$

It is clear that (1) and (2) are not the same sets. ■

Exercise 3.2.14

A dual notion to the closure of a set is the interior of a set. The *interior* of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \exists V_\varepsilon(x) \subseteq E\}.$$

Results about closures and interiors posses a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$. First we show that first statement.

Proof. ($\Rightarrow$) Suppose E is closed. We want to show that $\overline{E} = E$; that is, we want to show that $\overline{E} \subseteq E$ and $\overline{E} \supseteq E$. Note that the first containment follows immediately since $\overline{E}$ is the smallest set containing E. Now we want to show that $\overline{E} \supseteq E$. Let $x \in E$ be a limit point. Since x is a limit point and E is a closed set, we know that x must be contained in E. This means that set of limit points L of E must where $x \in L$ implies that $x \in \overline{E}$. Hence, $\overline{E} = E$.

($\Leftarrow$) It follows that E is closed since $\overline{E}$ contains its limit points and that $\overline{E} = E$. ■

Now we show the second statement

Proof. ($\Rightarrow$) Suppose E is an open set. We must show the following two containments: $E^\circ \subseteq E$ and $E^\circ \supseteq E$. We show the first containment. Let $x \in E^\circ$ be arbitrary. Then there exists $V_\varepsilon(x)$ such that $V_\varepsilon(x) \subseteq E$. Hence, $x \in E$ so we have $E^\circ \subseteq E$. Now we show that second containment. Since E is an open set, let $x \in E$ be arbitrary such that there exists $V_\varepsilon(x) \subseteq E$. But this is by definition the interior of E so we must have $x \in E^\circ$.

($\Leftarrow$) Suppose $E = E^\circ$. We want to show that E is an open set. Let $x \in E$ be arbitrary.

Since $E = E^\circ$, there exists $V_\varepsilon(x)$ such that $V_\varepsilon(x) \subseteq E$. But this means E is an open set by definition. ■

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Show that $\overline{E}^c = (E^c)^\circ$

Proof. We want to show that first equation; that is, we want to show the following two containments $\overline{E}^c \subseteq (E^c)^\circ$ and $\overline{E}^c \supseteq (E^c)^\circ$. First we show the former containment. Let $x \in \overline{E}^c$ be arbitrary. If $x \notin \overline{E}$, then x is not a limit point of E and $x \notin E$. But this means that $x \in (E^c)^\circ$ and hence, $\overline{E}^c \subseteq (E^c)^\circ$. Now we show the second containment. Let $x \in (E^c)^\circ$ be arbitrary. There exists $V_\varepsilon(x) \subseteq E^c$. We can be sure that x is not a limit point of $\overline{E}$ since $\overline{E}$ contains all its limit points. Hence, we must have $x \in \overline{E}^c$. Hence, we have $\overline{E}^c = (E^c)^\circ$. ■

Now we show $(E^\circ)^c = \overline{E^c}$

Proof. We want to show the following two containments; namely, $(E^\circ)^c \subseteq \overline{E^c}$ and $\overline{E^c} \subseteq (E^\circ)^c$.

We start with the first containment. Let $x \in (E^\circ)^c$ be arbitrary. This means $x \notin E^\circ$ and hence for all $\varepsilon-$neighborhoods of x, we have $V_\varepsilon(x) \not\subseteq E$. Our goal is to show that $x \in \overline{E^c}$. If x is not a limit point of E^c, then we just have $x \in E^c$ and hence $x \in \overline{E^c}$. Otherwise, we can prove x is a limit point of E^c. Suppose L is the set of limit points of E^c. Let $\varepsilon' > 0$ be as small as possible and $\ell \in L$ such that $V_{\varepsilon'}(\ell) \subseteq V_\varepsilon(x)$ where $x \notin V_{\varepsilon'}(\ell)$. Since ℓ is a limit point of E^c, $V_{\varepsilon'}(\ell)$ intersects E^c. But this also means $V_\varepsilon(x)$ intersects points of E^c that is not x. Hence, x is a limit point of E^c and thus $x \in \overline{E^c}$.

Now let $x \in \overline{E^c}$ be arbitrary. Then either $x \in E^c$ or $x \in L$ where L denotes the set of limit points of E^c. If $x \in E^c$, then surely $x \notin E^\circ$. Hence, $x \in (E^\circ)^c$. If $x \in L$ and $\overline{E}$ is a closed set, then x cannot be in E°. Hence, x must be in $(E^\circ)^c$. Hence $\overline{E^c} \subseteq (E^\circ)^c$

■

3.2 Compact Sets

Exercises 3.3.1

Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exists and are contained in K.

Proof. Suppose K is compact. By the equivalence theorem, K is bounded. Since K is nonempty and bounded above and below, $\sup K$ exists and $\inf K$ exists by the Axiom of Completeness. We begin by constructing a sequence (x_n) that is contained in K such that we want to show that $(x_n) \to \sup K$. Let $\varepsilon > 0$. Since $\sup K$ is the least upper bound, we can have $\sup K - \varepsilon$ be not an upper bound. But since $(x_n) \subseteq K$, we must have $x_n \leq \sup K$. Together we have

$$\sup K - \varepsilon \leq x_n \leq \sup K.$$

Since $\varepsilon > 0$ is arbitrary, the sequence $(x_n) \to \sup K$. Since K is closed $\sup K$ must be contained in K. By a similar reasoning, we can generate (x_n) so that $(x_n) \to \inf K$ which is also contained in K. Hence, $\sup K$ and $\inf K$ contained in K. ■

Exercise 3.3.2

Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbb{N}$.

Proof. Not compact. We can see that for every $n \in \mathbb{N}$, the sequence $a_n = n$ diverges to infinity and so does its subsequences; that is, if we define the odd and even subsequences (a_n) where both sequences also converge to infinity. ■

(b) $\mathbb{Q} \cap [0, 1]$.

Proof. This set is compact because it is bounded and closed. ■

(c) The Cantor set.

Proof. The cantor set is closed. ■

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

Proof. This set is not compact since the limit point of this set ($L = \{0\}$) is not contained in the set. ■

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \dots\}$.

Proof. This set is compact since its limit point $L = \{1\}$ is contained in the set and the fact that it is bounded by 1 and 0. ■

Exercise 3.3.3

Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Proof. Assume that K is closed and bounded. Our goal is to show that K is compact. Let (a_n) be a sequence contained in K. Since K is bounded, we know that (a_n) is also bounded. Hence, there exists a subsequence (a_{n_k}) such that $(a_{n_k}) \to a$. But since K is closed and a is a limit point of K, we have that $\lim a_{n_k} = a$ is also contained in K. ■

Exercise 3.3.4

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both or neither.

(a) $K \cap F$.

Proof. Compact. ■

(b) $\overline{F^c \cup K^c}$.

Proof. Closed. ■

(c) $K \setminus F = \{x \in K : x \notin F\}$

Proof. Definitely compact. ■

(d) $\overline{K \cap F^c}$.

Proof. Definitely compact. ■

Exercise 3.3.8

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}.$$

This turns out to a reasonable definition between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

Proof. Suppose K and L are disjoint, nonempty compact sets. Our goal is to show that $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$. Since K and L are nonempty compact sets, we also have that K and L are bounded sets. Hence, $\inf A$ exists where

$$A = \{|x - y| : x \in K \text{ and } y \in L\}$$

by the Axiom of Completeness. Then by lemma 1.3.8, we have that for some $x_0 \in K$ and

$y_0 \in L$, consider the distance $d = |x_0 - y_0| \in A$ such that we have

$$d = |x_0 - y_0| > \inf A + \varepsilon$$

for all $\varepsilon > 0$. Since K and L are disjoint and $\varepsilon > 0$ is arbitrary, we have $d = |x_0 - y_0| > 0$. Hence, we have that $d > 0$. ■

Technically, the proof above is not valid yet since we haven't gone over functions and how compactness imply continuity.

Proof. Suppose K and L are non-empty compact sets that are disjoint. Suppose for sake of contradiction that $d = 0$. Since K and L are both compact, let (x_n) and (y_n) contain subsequences (x_{n_k}) and (y_{n_k}) that converge to x_0 and y_0 respectively. We want to show that $\lim |x_{n_k} - y_{n_k}| = |x_0 - y_0|$. Let $\varepsilon > 0$. Then for every $n_k > N$ for some $N \in \mathbb{N}$, we have that

$$\begin{aligned} ||x_{n_k} - y_{n_k}| - |x_0 - y_0|| &\leq |(x_{n_k} - y_{n_k}) - (x_0 - y_0)| \\ &= |(x_{n_k} - x_0) + (y_0 - y_{n_k})| \\ &\leq |x_{n_k} - x_0| + |y_{n_k} - y_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $d = |x_0 - y_0| = 0$, we have that $\lim |x_{n_k} - y_{n_k}| = d = 0$. But this means that $K \cap L \neq \emptyset$ since K and L are closed sets which is a contradiction. Hence, we must have $d > 0$. ■

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Proof. Basically the argument above but taking away the fact that K and L are compact sets. ■

Exercise 3.3.9

Follow these steps to prove the final implication in Theorem 3.3.8. Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

Proof. Let I_1 be a half of I_0 whose intersection with K does not have a finite subcover, so that $I_1 \cap K$ cannot be finitely covered and $I_1 \subseteq I_0$. Then bisect I_1 into two closed intervals A_2 and B_2, and again let $A_2 = I_2$ such that $A_2 \cap K$ does not have a finite subcover. Otherwise, $B_2 = I_2$ such that $B_2 \cap K$ does not contain a finite subcover. Hence, $I_2 \subseteq I_1 \subseteq I_0$. We can repeat this process of bisecting each interval I_n and determining which closed set does not have a finite subcovers until we have our desired sequence I_n such that $\lim I_n = 0$. ■

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

Proof. Since K is compact, $K \cap I_n$ is also compact for each $n \in \mathbb{N}$. Hence, we know that $\bigcap_{n=1}^{\infty} I_n \cap K$ is non-empty, and there exists $x \in K \cap I_n$ for all n. ■

(c) Because $x \in K$, there must exists an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Proof. Let $x \in K$ and let O_{λ_0} be an open set that contains x. Since O_{λ_0} is an open set, we can construct an $\varepsilon-$neighborhood such that $V_{\varepsilon_0}(x) \subseteq O_{\lambda_0}$. Now choose n_0 such that for every $n \geq n_0$, $|I_{n_0}| < \varepsilon_0$. Hence, I_{n_0} is contained within a single open set O_{λ_0} which means it has a finite subcover. This contradiction tells us that K must have had a finite subcover. ■

Exercise 3.3.10

Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine-Borel Theorem. Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for

$[a,b]$ and define S to be the set of all $x \in [a,b]$ such that $[a,x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

> **Proof.** Let S be defined as the set of all $x \in [a,b]$ such that $[a,x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$. Since $[a,b]$ is a closed interval, we can define a sequence of points (x_n) that converges to $x \in [a,b]$. Since $\{O_\lambda : \lambda \in \Lambda\}$ is an open cover for $[a,b]$, we know that there must exists a finite cover for $[a,x]$. Hence, S is nonempty and S is bounded since $a \le x \le b$. By the Axiom of Completeness, $s = \sup S$ exists. ■

(b) Now show $s = b$, which implies $[a,b]$ has a finite subcover.

> **Proof.** Since $[a,b]$ is a closed and bounded interval, it follows that $[a,b]$ is a compact set. By exercise 3.3.1, $[a,b]$ must contain its supremum. Hence, $\sup[a,b] \in S$ and hence, $[a,b]$ must have a finite subcover. ■

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

> **Proof.** Let K be a closed and bounded set. From (a) and (b), K must have finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$. ■

Other solutions (not mine).

(b) Now show $s = b$, which implies $[a,b]$ has a finite subcover.

> **Proof.** Suppose for contradiction that $s < b$, letting $s \in O_{\lambda_0}$ implies $[a,s]$ is finitely coverable since we can take the finite cover of an $x \in O_{\lambda_0}$ with $x < s$. This causes a contradiction however since there exists points $y > s$ with $y \in O_{\lambda_0}$ meaning $[a,y]$ is also finitely coverable. Therefore, the only option is $s = b$, since any $s < b$ fails. ■

(c) We must also consider the case where y does not exists; that is, there exists a "gap". Let $y = [s,b] \cap K$ and suppose $y \neq s$. Since $y \in [s,b] \cap K$ we know

$$[a,y] \cap K = ([a,s] \cap K) \cup ([s,y] \cap K) = [a,s] \cap K \cup y.$$

Therefore if $\{O_{\lambda_1}, \dots, O_{\lambda_n}\}$ covered $[a,s]$ then letting $y \in O_{\lambda_{n+1}}$ would give the finite cover $\{O_{\lambda_1}, \dots, O_{\lambda_{n+1}}\}$ contradicting the assumption that $s < b$, therefore $s = b$ is the only option, and so K can be finitely covered.

3.3 Perfect Sets and Connected Sets

Exercise 3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

> **Proof.** $P \cap K$ always compact but not always perfect. This is because $P \cap K$ is always a closed and bounded set. ■

Exercise 3.4.5

Let A and B be nonempty subsets of $\mathbb{R}$. Show that if there exists disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

> **Proof.** Let A and B be nonempty subsets of $\mathbb{R}$. Suppose there exists sets U and V such that $U \cap V = \emptyset$. Let $x \in U$. Since U is an open set, let $x \in U^\circ$. Hence, there exists $V_\varepsilon(x)$ such that $V_\varepsilon(x) \subseteq U$. Let x be a limit point of A such that there exists $(x_n) \to x$ where $x_n \neq x$ for all $n \in \mathbb{N}$. Doing the same process for the set V suppose there exists (y_n) is a sequence contained in B such that $(y_n) \to y$ where $y \in V$ is an interior point such that there exist $V_\varepsilon(y) \subseteq V$.
>
> Since $U \cap V = \emptyset$ and x and y are interior points of U and V respectively, it follows that $V_\varepsilon(x) \cap V_\varepsilon(y) = \emptyset$. But this means neither limit point of A nor B is contained in the other. Hence, A and B must be separated sets. ■

Exercise 3.4.6

Prove Theorem 3.4.6.

Proof. ($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected and suppose A and B are disjoint. Since E is connected, we have $\overline{A} \cap B \neq \emptyset$ and $\overline{B} \cap A \neq \emptyset$. Without loss of generality, let $x \in \overline{A} \cap B$. Since A and B are disjoint, we must not have $x \in A$. Hence, $x \in B$ else A and B would not be disjoint. Hence, x is a limit point of A. Hence, there exists $(x_n) \subseteq A$ such that $(x_n) \to x$ where $x \neq x_n$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Our goal now is to show the converse; that is, show that A and B are not separated i.e $\overline{A} \cap B \neq \emptyset$ and $\overline{B} \cap A \neq \emptyset$. Suppose there exists $(x_n) \subseteq A$. By assumption $(x_n) \to x$ such that $x \in B$. Since $\overline{A}$ is a closed set, we have that $x \in \overline{A}$. But this means that $\overline{A} \cap B \neq \emptyset$. The argument is similar when $(x_n) \subseteq B$. Hence, $\overline{B} \cap A \neq \emptyset$. But this means that $E = A \cup B$ is connected. ■

Exercise 3.4.7

A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exists separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

Proof. Since the rational numbers are dense in $\mathbb{I}$, we can construct the following interval $x < c < y$ where $c \in \mathbb{I}$. We can set A and B such that

$$A = (-\infty, c) \cap \mathbb{Q} \quad \text{and} \quad B = \mathbb{Q} \cap (c, +\infty)$$

Let $x \in A$ and $y \in B$, then neither set is empty and neither set contains a limit point of the other. Since $A \cap B = \mathbb{Q}$, we must have $\mathbb{Q}$ as a totally disconnected set unless $c \in \mathbb{Q}$ which is not. ■

(b) Is the set of irrational numbers totally disconnected?

Proof. The set of irrational numbers is totally disconnected because we can always find $x \in \mathbb{Q}$ such that for every $z, y \in \mathbb{I}$, we have $z < x < y$. Thus, we can follow the same argument above to produce two sets that are separated. ■

Chapter 4

Functional Limits and Continuity

4.1 Functional Limits

Exercise 4.2.1

(a) Supply the details for how Corollary 4.1.1.1 follows from the Sequential Criterion for Functional Limits in Theorem 4.1.1 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

Proof. Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$. Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. By the Sequential Criterion for functional limits, let $(x_n), (y_n) \subseteq A$ such that $(x_n) \to c$ and $(y_n) \to c$ where $x_n, y_n \neq c$ implying that $f(x_n) \to L$ and $g(y_n) \to M$. By the Algebraic Limit Theorem, we can state that

$$\lim[f(x_n) + g(y_n)] \lim f(x_n) + \lim g(y_n) = L + M.$$

Hence, we have that

$$\lim_{x \to c}[f(x) + g(x)] = L + M$$

by the Sequential Criterion for Functional Limits. ■

(b) Now, write another proof of Corollary 4.1.1.1 part (ii) directly from Definition for functional limits without using the sequential criterion in Theorem 4.1.1.

Proof. We can also prove the addition rule for Functional Limits directly from the definition. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then for some $\delta > 0$, suppose c is a limit point of A such that $0 < |x - c| < \delta$. We want to show that for any arbitrary $\varepsilon > 0$ that

$$|(f(x) + g(x)) - (L + M)| < \varepsilon.$$

Hence, choose $\delta = \min\{\delta_1, \delta_2\}$

$$\begin{aligned}
|(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we have

$$\lim_{x \to c}[f(x) + g(x)] = L + M.$$

■

(c) Repeat (a) and (b) for Corollary 4.1.1.1 part (iii).

Proof. Let $(x_n), (y_n) \subseteq A$ such that $(x_n) \to c$ and $(y_n) \to c$ for some limit point c of A where we have $\lim f(x_n) = L$ and $\lim g(y_n) = M$. By the Algebraic Limit Theorem, we have that

$$\lim[f(x_n)g(y_n)] = \lim f(x_n) \cdot \lim g(y_n) = L \cdot M.$$

By the Sequential Criterion, this also means that

$$\lim_{x \to c}[f(x)g(x)] = L \cdot M.$$

Now we can prove the same fact but this time we use only the Definition of functional limit. Let $f : A \to \mathbb{R}$. Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$ where c is a limit point of A. Let $\varepsilon > 0$. Our goal is to show that

$$|f(x)g(x) - LM| < \varepsilon.$$

Since $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - g(x)L + g(x)L - LM| \\
&= |g(x)(f(x) - L) + L(g(x) - M)| \\
&\leq |g(x)||f(x) - L| + L|g(x) - M| \\
&< M + 1 \cdot \frac{\varepsilon}{2(M+1)} + L \cdot \frac{\varepsilon}{2L} \\
&= \varepsilon.
\end{aligned}$$

Hence, we have that

$$\lim_{x \to c} f(x)g(x) = LM.$$

■

Exercise 4.2.2

For each stated limit, find the largest possible $\delta-$neighborhood that is a proper response to the given ε challenge.

(a) $\lim_{x \to 3}(5x - 6) = 9.$

Proof. Since $\varepsilon = 1$, we know that

$$|(5x - 6) - 9| < 1.$$

To solve for δ, we do the following

$$\begin{aligned}
|(5x - 6) - 9| &= |5(x - 3)| < 1 \\
&= 5|x - 3| < 1
\end{aligned}$$

which implies that

$$|x - 3| < \frac{1}{5} = \delta.$$

Hence, the largest possible $\delta-$neighborhood that is a proper response to the given ε challenge.

■

(b) $\lim_{x \to 4} \sqrt{x} = 2$, where $\varepsilon = 1$.

Proof. Since $\varepsilon = 1$, we know that

$$|\sqrt{x} - 2| < 1.$$

To get our δ, we just do the following manipulations

$$\begin{aligned} \sqrt{x}-2 &< 1 \\ \sqrt{x} &< 3 \\ x &< 9 \\ |x-4| &< 5=\delta. \end{aligned}$$

Hence, our ε response is $\delta = 5$. ■

(c) $\lim_{x\to\pi} [[x]] = 3$, where $\varepsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)

Proof. Let $\varepsilon = 1$. We want to generate a $\delta > 0$ that satisfies the ε challenge. Since $[[\pi]] = 3$, our max $\delta-$neighborhood can only have $\delta = \pi - 3$ ■

(d) We have (c) but this time with $\varepsilon = .01$.

Proof. It would still remain as $\delta = \pi - 3$. ■

Exercise 4.2.5

Use Definition 4.1.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

Proof. Suppose there exists a $\delta > 0$ such that $0 < |x-2| < \delta$. Let $\varepsilon > 0$. Then observe that by using definition 4.1.1 that

$$\begin{aligned} |f(x)-10| &= |(3x+4)-10| \\ &= 3|x-2| \\ &< 3\delta \\ &= 3\cdot\frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

■

(b) $\lim_{x\to 0} x^3 = 0$.

Proof. Suppose there exists $\delta > 0$ such that $0 < |x| < \delta$. By applying the definition of functional limits, choose $\delta = (\varepsilon)^{1/3}$

$$\begin{aligned} |f(x)-0| &= |x^3| \\ &< \delta^3 \\ &= \varepsilon. \end{aligned}$$

Hence, we have $|f(x)-0| < \varepsilon$ which implies that

$$\lim_{x\to 0} x^3 = 0.$$

■

(c) $\lim_{x\to 2}(x^2+x-1) = 5$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\begin{aligned}
|f(x) - 5| &= |(x^2 + x - 1) - 5| \\
&= |x^2 + x - 6| \\
&= |x + 3||x - 2| \\
&< |x + 3|\delta \\
&< 3 \cdot \frac{\varepsilon}{3}. && = \varepsilon.
\end{aligned}$$

■

(d) $\lim_{x \to 3} 1/x = 1/3$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\begin{aligned}
|f(x) - \frac{1}{3}| &= |\frac{1}{x} - \frac{1}{3}| \\
&= \left|\frac{x-3}{3x}\right| \\
&= \frac{|x-3|}{3|x|} \\
&< 12\varepsilon \cdot \frac{1}{12} \\
&= \varepsilon.
\end{aligned}$$

■

Exercise 4.2.6

Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

Proof. This is true since δ that has been constructed is the largest possible neighborhood that one can produce as a response to the ε challenge. ■

(b) If $\lim_{x \to a} f(x) = L$ and a happens to be in the domain of f, then $f(a) = L$.

Proof. This is false. I have trouble thinking of an example. Will get back to this one soon. ■

(c) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.

Proof. This is true based on the Algebraic Functional Limit Theorem. ■

(d) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Proof. This is not true in general since we can easily produce $g(x)$ that is unbounded. For this to work, we would need $g(x)$ to be bounded. ■

Exercise 4.2.7

Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \le M$ for all $x \in A$. Show that if $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} g(x)f(x) = 0$ as well.

Proof. Let $g : A \to \mathbb{R}$. Suppose $\lim_{x \to c} g(x) = 0$ and suppose $f(x)$ is bounded; that is, there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in A$. Using the Functional Limit Definition, suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Hence, we have

$$\begin{aligned} |f(x)g(x) - 0| &= |f(x)g(x)| \\ &= |f(x)||g(x)| \\ &\leq M|g(x)| \\ &< M \cdot \frac{\varepsilon}{M} \\ &= \varepsilon. \end{aligned}$$

It follows from this that $\lim_{x\to c} f(x)g(x) = 0$. ■

Exercise 4.2.9 (Infinite Limits).

The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge response style of Definition 4.1.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\varepsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition 1. We say $\lim_{x\to c} f(x) = \infty$ to mean that for all $M > 0$, we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

Proof. Let $M > 0$ and $f(x) = 1/x^2$. Choose $\delta = 1/\sqrt{M}$. Since $0 < |x| < \delta$, we have

$$x^2 < \delta^2 \Leftrightarrow \frac{1}{x^2} > \frac{1}{\delta^2}.$$

But this means that

$$f(x) = \frac{1}{x^2} > \frac{1}{\delta^2} = \frac{1}{1/M} = M.$$

Hence, we have

$$\lim_{x\to 0} f(x) = \infty.$$

■

(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show that $\lim_{x\to\infty} 1/x = 0$.

Definition 2. Let $f : A \to \mathbb{R}$. We say $\lim_{x\to\infty} f(x) = L$ to mean for any $\varepsilon > 0$, there exists $x_0 \in A$ where $A \subseteq \mathbb{R}$ such that for any $x \geq x_0$, we have

$$|f(x) - L| < \varepsilon.$$

Proof. We start with some sketch work for what x_0 might be. Observe that

$$\begin{aligned} |f(x) - L| &= |\frac{1}{x} - 0| \\ &= \frac{1}{x} \\ &< \varepsilon. \end{aligned}$$

Solving for x, we get that

$$x > \frac{1}{\varepsilon}.$$

Hence, $x_0 = 1/\varepsilon$.

Let $\varepsilon > 0$ and let $f(x) = 1/x$. Choose $x_0 = 1/\varepsilon$. Suppose $x > x_0 = 1/\varepsilon$. Then we have that

$$x > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{x} < \varepsilon.$$

Hence, we have that

$$|f(x) - 0| < \varepsilon.$$

This means $\lim_{x\to\infty} f(x) = 0$. ■

Exercise 4.2.10

Introductory calculus courses typically refer to the *right-hand limit* of a function as the limit obtained by "letting x approach a from the right-hand side"

(a) Give a proper definition in the style of Definition 4.1.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = M.$$

Definition 3. We say that

$$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = M$$

if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ and $0 < |c - x| < \delta$ we have

$$|f(x) - L| < \varepsilon,$$
$$|f(x) - M| < \varepsilon$$

respectively.

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal to L.

Proof. Suppose $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have

$$|f(x) - L| < \varepsilon. \tag{1}$$

This denotes our right-hand limit. Note that this also works if we flip the order of our δ assumption. Hence, we have $0 < |a - x| < \delta$ such that (1) holds. ■

But this means our right-hand and left-hand limits are equal to each other.

Now we show the converse. Suppose the right-hand and left-hand limits are equal to each other; that is,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

We want to show that $\lim_{x\to a} f(x) = L$. Since both $|x - a| < \delta$ and $|a - x| < \delta$ hold for both limits, we immediately have that

$$|f(x) - L| < \varepsilon.$$

Hence, we have $\lim_{x\to a} f(x) = L$.

Exercise 4.2.11(Squeeze Theorem).

Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Suppose $f(x) \leq g(x) \leq h(x)$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show

$$\lim_{x\to c} g(x) = L$$

as well.

Proof. Let $(x_n), (y_n), (z_n) \subseteq A$. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c of A. Using the Sequential Criterion for Functional Limits, let $f(x_n) \to L$ and $h(z_n) \to L$. There exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$f(x_n) \leq g(y_n) \leq h(z_n) \Leftrightarrow L \leq g(y_n) \leq L.$$

Hence, we have $g(y_n) \to L$ by the Squeeze Theorem for Sequences. By the Sequential Criterion, we must have $\lim_{x \to c} g(x) = L$ as well. ■

Another proof using the definition of Functional Limits directly.

Proof. Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Since $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$, we know that

$$\lim_{x \to c} [h(x) - f(x)] = 0.$$

Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Since $f(x) \leq g(x) \leq h(x)$, we have

$$\begin{aligned}
|g(x) - L| &\leq |h(x) - L| \\
&= |h(x) - f(x) + f(x) - L| \\
&\leq |h(x) - f(x)| + |f(x) - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

But, this means that $\lim_{x \to a} g(x) = L$. ■

Another proof

Proof. Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Suppose there exists $\delta > 0$ such that $0 < |x - c| < \delta$. Choose $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Since $f(x) \leq g(x) \leq h(x)$, we have

$$\begin{aligned}
|g(x) - L| &= |g(x) - h(x) + h(x) - L| \\
&\leq |g(x) - f(x) + f(x) - h(x)| + |h(x) - L| \\
&\leq |g(x) - f(x)| + |f(x) - h(x)| + |h(x) - L| \\
&\leq |h(x) - f(x)| + |f(x) - L| + |L - h(x)| + |h(x) - L| \\
&\leq |h(x) - L| + |L - f(x)| + 2|h(x) - L| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{2\varepsilon}{4} \\
&= \varepsilon.
\end{aligned}$$

Hence, it follows that

$$\lim_{x \to a} g(x) = L.$$

■

4.2 Continuous Functions

Exercise 4.3.1

Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

Proof. Let $\varepsilon > 0$. Suppose $c = 0$. Let $c \in A$ and $g(x) = \sqrt[3]{x}$. Since $|x| < \delta$, we can say that $|\sqrt[3]{x}| < \sqrt[3]{\delta}$. Then choose $\delta = \varepsilon^3$ such that

$$\begin{aligned}|g(x) - g(c)| &= |\sqrt[3]{x} - 0|\\ &= |\sqrt[3]{x}|\\ &< \sqrt[3]{\varepsilon^3}\\ &= \varepsilon.\end{aligned}$$

Hence, we have that

$$\lim_{x\to 0} \sqrt[3]{x} = 0.$$

■

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ will be helpful.)

Proof. Now let $c \neq 0$. Let $\varepsilon > 0$ and suppose $|x - c| < \delta$. By the given identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, we can write

$$\begin{aligned}|g(x) - g(c)| &= |\sqrt[3]{x} - \sqrt[3]{c}|\\ &= \left|x^{3/9} - c^{3/9}\right|\\ &= |x^{1/9} - c^{1/9}||x^{2/9} + x^{2/9}c^{2/9} + c^{2/9}|.\end{aligned}$$

Since $|x - c| < \delta$, we have

$$\begin{aligned}x^{1/9} - c^{1/9} &< (\delta + c)^{1/9} - c^{1/9} &(1)\\ x^{2/9} + x^{1/9}c^{1/9} + c^{2/9} &< (\delta + c)^{2/9} + x^{1/9}c^{1/9} + c^{1/9} &(2)\\ x^{1/9}c^{1/9} &< (\delta + c)^{1/9}c^{1/9}. &(3)\end{aligned}$$

By using the identity again, we write

$$\begin{aligned}|g(x) - g(c)| &= |x^{1/9} - c^{1/9}||x^{2/9} + x^{2/9}c^{2/9} + c^{2/9}|\\ &< [(\delta + c)^{1/9} - c^{1/9}][(\delta + c)^{2/9} + (\delta + c)^{1/9}c^{1/9} + c^{1/9}]\\ &= (\delta + c)^{3/9} - c^{3/9}\\ &= (\delta + c)^{1/3} - c^{1/3}. &(4)\end{aligned}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$ such that (4) implies

$$(\delta + c)^{1/3} - c^{1/3} = (\delta^3 - c + c)^{1/3} - c^{1/3} = \varepsilon + c^{1/3} - c^{1/3} = \varepsilon.$$

Hence, we have $|g(x) - g(c)| < \varepsilon$ implying that

$$\lim_{x\to c} g(x) = g(c).$$

■

Exercise 4.3.3

(a) Supply a proof for Theorem 4.2.3 using the $\varepsilon - \delta$ characterization of continuity.

Proof. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Assume that the range

$$f(A) = \{f(x) : x \in A\}$$

is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.

Suppose f is continuous at $c \in A$ and g is continuous at $f(c) \in B$. Let $\varepsilon > 0$. Since f is continuous at $c \in A$, we can construct $\delta > 0$ such that $|x - c| < \delta$ where

$$|f(x) - f(c)| < \varepsilon.$$

Let $\varepsilon = \delta$. Then assume

$$|f(x) - f(c)| < \delta$$

since g is continuous at $f(c) \in B$. Then we immediately have that

$$|g \circ f(x) - g \circ f(c)| = |g(f(x)) - g(f(c))| < \varepsilon.$$

Hence, we have

$$\lim_{x \to c} g \circ f(x) = g \circ f(c).$$

■

(b) Give another proof of this theorem using the sequential characterization of continuity.

Proof. Let $f(x_n) \subseteq f(A)$ and $(x_n) \subseteq A$ where the image

$$f(A) = \{f(x) : x \in A\}.$$

Let $\varepsilon > 0$. Since $(x_n) \to c$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|f(x_n) - f(c)| < \varepsilon.$$

But $g(f(x_n0) \to g(f(c))$ since g is continuous at $f(c) \in B$ so we end up having

$$|g \circ f(x_n) - g \circ f(c)| = |g(f(x_n)) - g(f(c))| < \varepsilon.$$

Hence, we have

$$\lim g \circ f(x_n) = g \circ f(c).$$

Note that this fact follows immediately if $f(x_n) = f(c)$ and $x_n = c$. ■

Exercise 4.3.5

Show using definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Since $c \in A$ and c is an isolated point of $A \subseteq \mathbb{R}$, we must have $x = c$ and $f(x) = f(c)$ where $x \in A$ is an arbitrary point. This follows from the fact that there exists an $\varepsilon-$neighborhood such that $V_\varepsilon(x) \cap A = \emptyset$. Let $\varepsilon > 0$. Then choose $\delta = \varepsilon$ such that whenever $|x - c| < \delta$, we have that

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$$

■

Exercise 4.3.6

Using the $\varepsilon - \delta$ characterization of continuity (and thus using no previous results about sequences), show that the linear function $f(x) = ax + b$ is continuous at every point $x \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ and let $c \in \mathbb{R}$ be arbitrary. Choose $\delta = \varepsilon/a$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned}
|f(x) - f(c)| &= |(ax + b) - (ac + b)| \\
&= |a(x - c)| \\
&= a|x - c| \\
&< a \cdot \frac{\varepsilon}{a} \\
&= \varepsilon.
\end{aligned}$$

Hence, every linear function $f(x) = ax + b$ is continuous at any point $c \in \mathbb{R}$. ■

Exercise 4.3.7

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show K is a closed set.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Let $c \in K$. Since h is continuous on $\mathbb{R}$ we can use the Sequential Criterion. Let $x_n \in K$. There exists $(x_n) \subseteq K$ such that $(x_n) \to c$ implying that $h(x_n) \to h(c)$. Since every $x \in K$ has the property that $h(x) = 0$ and $c \in K$, we have that $h(c) = 0$. Hence, $h(x_n) \to h(c)$ is contained in K. Thus, K is closed. ■

Exercise 4.3.10

Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|].$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots f_n(x)\}$$

is a continuous function.

Proof. Suppose $f_1, f_2, \ldots, f_n$ are continuous functions. We proceed by inducting on n. Let our base case be $n = 2$. Then we have our continuous functions f_1, f_2. By using the $\varepsilon - \delta$ definition for continuity, we have that

$$\begin{aligned}
|g(x) - g(c)| &= \left|\frac{1}{2}\Big[(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|\Big]\right. \\
&\quad \left. - \frac{1}{2}\Big[(f_1(c) + f_2(c)) + |f_1(c) - f_2(c)|\Big]\right| \\
&= \left|\frac{1}{2}\Big[(f_1(x) - f_1(c)) + (f_2(x) - f_2(c))\Big]\right. \\
&\quad \left. + \frac{1}{2}\Big[|f_1(x) - f_2(x)| + |f_1(c) - f_2(c)|\Big]\right| \\
&\le \frac{1}{2}\Big[|f_1(x) - f_1(c)| + |f_2(x) - f_2(c)| \\
&\quad + |(f_1(x) - f_2(x)) - (f_1(c) - f_2(c))|\Big]
\end{aligned}$$

■

Exercise 4.3.11 (Contraction Mapping Theorem).

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \le c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and let $x, c \in \mathbb{R}$ where c is a limit point of A. We want to show that f is continuous on $\mathbb{R}$; that is, we want to show that

$$\lim_{x \to c} f(x) = f(c).$$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/c$ such that whenever $|x - c| < \delta$ we have

$$|f(x) - f(c)| \le k|x - c| < k \cdot \frac{\varepsilon}{k} = \varepsilon.$$

Hence, we have

$$\lim_{x \to c} f(x) = f(c).$$

■

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), ...).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence, we may let $y = \lim y_n$.

Proof. Observe that for any fixed $n \in \mathbb{N}$,

$$|y_{m+1} - y_{m+1}| = |f(y_m) - f(y_{m+1})| \leq c|y_m - y_{m+1}|.$$

We can show this via induction that

$$\begin{aligned} |y_{m+1} - y_{m+2}| &\leq c|y_m - y_{m+1}| \\ &\leq c^2|y_{m-1} - y_m| \\ &\vdots \\ &\leq c^m|y - 1 - y_2|. \end{aligned}$$

The fact that $0 < c < 1$ means that $\sum_{n=1}^{\infty} c^n$ converges (because it is a geometric series) enables us to conclude that (y_n) is Cauchy sequence. To see this, observe that

$$\begin{aligned} |y_m - y_n| &= |y_m - y_{m+1} + y_{m+1} + \cdots + y_{n-1} - y_n| \\ &\leq c^{m-1}|y_1 - y_2| + c^m|y_1 - y_2| + \dots c^{n-2}|y_1 - y_2| \\ &= c^{m-1}|y_1 - y_2|(1 + c + \cdots + c^{n-m-r}) \\ &< c^{m-1}|y_1 - y_2|\Big(\frac{1}{1-c}\Big). \end{aligned}$$

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ large enough so that $c^{N-1} < \varepsilon 1 - c/|y_1 - y_2|$. Then the previous calculation shows that $n > m \geq N$ implies $|y_m - y_n| < \varepsilon$; that is, we have

$$\begin{aligned} |y_{m+1} - y_{n+1}| &\leq |y_m - y_n| \\ &< c^{N-1}|y_1 - y_2|\Big(\frac{1}{1-c}\Big) \\ &< \frac{\varepsilon(1-c)}{|y_1 - y_2|} \cdot \frac{|y_1 - y_2|}{1-c} \\ &= \varepsilon. \end{aligned}$$

Hence, (y_n) is Cauchy Sequence. ■

(c) Prove that y is a fixed point of f; that is, $f(y) = y$ and that it is unique in this regard.

Proof. Since f is continuous on $\mathbb{R}$ and $(y_n) \subseteq \mathbb{R}$ is a Cauchy sequence and that $\lim y_n = y$ for all $n \in \mathbb{N}$, we have that

$$f(y_n) = y_{n+1} \to f(y) = y.$$

Hence, y is a fixed point of f. ■

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$ then the sequence $(x, f(x), f(f(x)), ...)$ converges to y defined in (b).

Proof. Let us fix $x \in \mathbb{R}$ where $(x_n) \subseteq \mathbb{R}$ is a sequence of points. Since the f is defined like $f(x_n) = x_{n+1}$. Since $x \in \mathbb{R}$ is fixed, we know that $f(x_n) \to f(x)$ and that $(x_{n+1}) \to x$. Hence, $f(x) = x$ as defined in (b). ■

Exercise 4.3.12

Let $F \subseteq \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Proof. Let $F \subseteq \mathbb{R}$ be a nonempty closed set. Let $x \in F$. Since F is a closed, let (x_n) be a Cauchy sequence such that $(x_n) \to x$ that is contained in F. But since $g : F \to \mathbb{R}$, we must have

$$|g(x_n) - g(x)| < \varepsilon$$

by the Sequential Criterion of Continuity. This means g is continuous on all of $\mathbb{R}$.

Let $x \notin F$. Suppose for sake of contradiction that $g(x) = 0$. Let $(x_n) \subseteq F$ where $x_n \in F$. Since F is closed, (x_n) is a Cauchy sequence such that $(x_n) \to x$. But since g is continuous x has to be contained in F which is a contradiction. Hence, it must be that $g(x) \neq 0$. ∎

Exercise 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Proof. By using the linearity property of f, we have $f(0 + 0) = f(0) + f(0)$ which implies $f(0) = 0$. For the inverse property, suppose $x \in \mathbb{R}$ such that $x + (-x) = 0$. Then using the linearity property of f, we have $f(x) + (-x)) = f(x) + f(-x)$. Since $f(0) = 0$, we have that $f(x) + f(-x)$ implies $f(-x) = -f(x)$. ∎

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

Proof. Let $k = f(1)$. We proceed to show $f(n) = kn$ by inducting on $n \in \mathbb{N}$. Let our base case be $n = 1$. If $n = 1$, then $f(1) = k$. Now suppose $f(n)$ holds for $1 \leq n \leq \ell - 1$. Then

$$\begin{aligned} f(\ell) &= k\ell \\ &= k(\ell + 1 - 1) \\ &= k(\ell - 1) + k \\ &= f(\ell - 1) + f(1). \end{aligned}$$

Hence, $f(n) = kn$ for all $n \in \mathbb{N}$. To prove $f(z) = kz$ for all $z \in \ Z$ we can just prove it for all negative integers and zero. This is easy to see when $z = 0$. It's also easy to see that f holds for $z \in \mathbb{Z}^-$. Since $f(-x) = -f(x)$, we have that $f(-\ell) = -f(\ell)$. Since $f(n)$ holds for all $n \in \mathbb{Z}^+$. Hence, $f(z) = nz$ for all $z \in \mathbb{Z}$. Before proving $f(r) = kr$ for any rational number r, let us consider $1/n$ where $n \in \mathbb{N}$. Note that

$$\begin{aligned} k &= f(1) \\ &= f\Big(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\Big) \\ &= nf\Big(\frac{1}{n}\Big). \end{aligned}$$

Applying this for any given $r \in \mathbb{Q}$, we can see that

$$\begin{aligned} f(m/n) &= f\Big(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\Big) \\ &= mf\Big(\frac{1}{n}\Big) \\ &= k\Big(\frac{m}{n}\Big). \end{aligned}$$

We can prove that this holds for any rational number $r < 0$ by using a similar strategy to the used to prove the negative integers case above. ■

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Proof. Assume f is continuous at $x = 0$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/k$ such that whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |kx - kc| \\ &= k|x - c| \\ &< k \cdot \frac{\varepsilon}{k} \\ &= \varepsilon. \end{aligned}$$

Hence, f is continuous for $c \neq 0$. But f is also continuous at $x = 0$. Hence, f is continuous at every point in $\mathbb{R}$ and thus $f(x) = kx$ for all $x \in \mathbb{R}$. ■

4.3 Continuous Functions on Compact Sets

Exercise 4.4.1

(a) Show that $f(x) = x^3$ is continuous on all $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ and let $c \in \mathbb{R}$. Choose $\delta = \min\{1, \varepsilon/|(c+1)^2 + (c+1)c + c^2|\}$. Let $f(x) = x^3$. Then whenever $|x - c| < \delta$, we have that

$$\begin{aligned} |f(x) - f(c)| &= |x^3 - c^3| \\ &= |x - c||x^2 + xc + c^2| \\ &< \frac{\varepsilon}{|(c+1)^2 + (c+1)c + c^2|} \cdot |(c+1)^2 + (c+1)c + c^2| \\ &= \varepsilon. \end{aligned}$$

Hence, we have that $\lim_{x \to c} f(x) = f(c)$ for any $c \in \mathbb{R}$. ■

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbb{R}$.

Proof. Let $(x_n) = n$ and $(y_n) = n + 1/n$, then

$$|x_n - y_n| = \left|\frac{1}{n}\right| \to 0.$$

Then we have that

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left|n^3 - \left(n + \frac{1}{n}\right)^3\right| \\ &= \left| - \left(3n + \frac{3}{n} + \frac{1}{n^3}\right)\right| \\ &= 3n + \frac{3}{n} + \frac{1}{n^3} \\ &\geq 3. \end{aligned}$$

Hence, there exists $\varepsilon_0 = 3$ such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$. which implies $f(x) = x^3$ is not uniform continuous. ■

(c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

Proof. Let (a, b) be a bounded subset of $\mathbb{R}$. Then let $\varepsilon > 0$. Let $x, y \in [a, b]$. Choose $\delta =$.

Hence, whenever $|x - y| < \delta$ we have that

$$\begin{aligned}
|f(x) - f(y)| &= |x^3 - y^3| \\
&= |x - y||x^2 + xy + y^2| \\
&< \delta \cdot 3b^2 \\
&= \frac{\varepsilon}{3b^2} \cdot 3b^2 \\
&= \varepsilon.
\end{aligned}$$

Hence, we have that f is uniformly continuous on the bounded set $(a, b) \subseteq \mathbb{R}$. ■

Exercise 4.4.2

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

Proof. No it is not. Let $(x_n), (y_n)$ be two sequences such that $x_n = 1/2n$ and $y_n = 1/(2n+1)$. Observe that $|x_n - y_n| \to 0$ since both sequences tend to zero. Now we have that

$$|f(x_n) - f(y_n)| = |2n - 2n - 1| = 1 = \varepsilon_0.$$

Hence, $f(x)$ is not uniformly continuous on $(0, 1)$. ■

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on (0,1)?

Proof. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. Assume $f(x)$ is defined on $(0, 1)$. Then whenever $|x-y| < \delta$, we have that

$$\begin{aligned}
|g(x) - g(y)| &= |\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| \\
&= \frac{|x^2 - y^2|}{|\sqrt{x^2 + 1} + \sqrt{y^2 + 1}|} \\
&= \frac{|x - y||x + y|}{|\sqrt{x^2 + 1} + \sqrt{y^2 + 1}|} \\
&< \delta \cdot \frac{|x + y|}{|\sqrt{x^2 + 1} + \sqrt{y^2 + 1}|} \\
&< \delta \cdot \frac{2}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we have that $g(x)$ is uniform continuous on $(0, 1) \subseteq \mathbb{R}$. ■

Exercise 4.4.3

Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \delta/2$ such that whenever $|x - y| < \delta$ for any $x, y \in [1, \infty)$ we have that

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \frac{|y^2 - x^2|}{|x^2 y^2|} \\
&= \frac{|x - y||x + y|}{|x^2||y^2|} \\
&\leq \frac{|x - y|}{|x^2||y^2|}(|x| + |y|) && \text{(Triangle Inequality)} \\
&< \delta \cdot \frac{1}{|x^2 y^2|}(|x| + |y|) && (|x - y| < \delta) \\
&= \delta \cdot \left(\frac{1}{xy^2} + \frac{1}{x^2 y}\right) \\
&< \frac{\varepsilon}{2} \cdot 2 && (x \geq 1 \text{ and } \delta = \varepsilon/2) \\
&= \varepsilon.
\end{aligned}$$

Hence, $f(x) = 1/x^2$ is a continuous function on $[1, \infty)$. Now we want to show that $f(x)$ is not uniformly continuous on $(0, 1]$. Hence, let $(x_n), (y_n)$ be two sequences that are contained in $(0, 1]$. Defined these two sequences as follows:

$$\begin{aligned}
x_n &= \frac{1}{\sqrt{2n}}, \\
y_n &= \frac{1}{\sqrt{2n + 1}}.
\end{aligned}$$

Since (x_n) and (y_n) both tend towards zero, we have that $|x_n - y_n| \to 0$. Now consider $|f(x_n) - f(y_n)|$. We want to construct an ε_0 such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Hence, observe that

$$\begin{aligned}
|f(x_n) - f(y_n)| &= \left|\left(\frac{1}{1/\sqrt{2n}}\right)^2 - \left(\frac{1}{1/\sqrt{2n + 1}}\right)^2\right| \\
&= |2n - 2n - 1| \\
&= 1 \\
&= \varepsilon_0.
\end{aligned}$$

Hence, we have that $f(x)$ cannot be uniformly continuous on the interval $(0, 1]$. ■

Exercise 4.4.4

Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

Proof. Let f be a continuous function $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$. Since $[a, b]$ are compact sets, and f is continuous on $[a, b]$, we have $f(a) \leq f(x) \leq f(b)$. Since $f > 0$ for all $a \leq x \leq b$, we have that

$$\frac{1}{f(b)} \leq \frac{1}{f(x)} \leq \frac{1}{f(a)}$$

which means $1/f$ is a bounded function on $[a, b]$. ■

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

Proof. Let f be a uniformly continuous function on a bounded set A. Hence, let $\varepsilon = 1$. There exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have that

$$|f(x) - f(y)| < \varepsilon.$$

Since A is bounded, there exists a subsequence $(x_n) \to x$ where x is a limit point of A. Suppose for sake of contradiction that $f(A)$ is unbounded. Let $\delta = 1$. Since f is unbounded, we have that whenever $|x_n - x_m| < 1$ where $n \neq m$, we have

$$|f(x_n) - f(x_m)| > 1.$$

But this means that our function is not uniformly continuous on A which is a contradiction. Thus, f must have a bounded range.

■

(c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

Proof. This is false. Suppose we have the function f defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{I}. \end{cases}$$

We have that for any compact set K, we have that every element of x gets mapped to $f(K)$ but $f(x)$ is not a continuous function. ■

Exercise 4.4.5

Assume that g is defined on an open interval (a, c) a nd it is known to be uniformly continuous on $(a, b]$ and $[b, c)$. Let $x \in (a, b]$ and $y \in [b, c)$. Note that $(a, c) = (a, b] \cup [b, c)$. Since f is uniformly continuous on $(a, b]$, we have that for any $x, b \in (a, b]$, there exists $\delta_1 > 0$ such that whenever $|x - b| < \delta_2$, we have that

$$|f(x) - f(b)| < \frac{\varepsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that whenever $|b - y| < \delta_2$ such that whenever

$$|f(b) - f(y)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$ such that whenever $|x - y| < \delta$, we have that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(b) + f(b) - f(y)| \\ &= |f(x) - f(b)| + |f(b) - f(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $f(x)$ is uniformly continuous on (a, c).

Exercise 4.4.7

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. Let $\varepsilon > 0$ and let $x, y \in [0, \infty)$. If $x, y = 0$, then it immediately follows that f is uniformly continuous since choosing $\delta = \varepsilon$ implies that whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |\sqrt{0} - \sqrt{0}| = 0 < \varepsilon.$$

Suppose $x, y \neq 0$. Then choose $\delta = \varepsilon \cdot 2\sqrt{c}$ for any $c \in [0, \infty)$ such that whenever $|x - y| < \delta$ we have that

$$\begin{aligned}|f(x)-f(y)| &= |\sqrt{x}-\sqrt{y}| \\ &= \Big|\frac{x-y}{\sqrt{x}+\sqrt{y}}\Big| \\ &= \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \\ &\leq \frac{|x-y|}{2\sqrt{c}} \\ &< \frac{\delta}{2\sqrt{c}} \\ &= \varepsilon.\end{aligned}$$

Hence, we have that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$. ■

Exercise 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

Proof. Let $f(x) = 1/x(x + 1)$. The range of $f(x)$ is $(0, 1)$. ■

(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

Proof. Let the following function $f(x)$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{4}) \\ 2x - \frac{1}{2} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ 1 & \text{if } x \in (3/4, 1) \end{cases}$$

■

(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Proof. ■

Exercise 4.4.9

Definition 4. Lipschitz Functions A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\Big|\frac{f(x)-f(y)}{x-y}\Big| \leq M$$

for all $x \neq y \in A$.

Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

Proof. Suppose $f : A \to \mathbb{R}$ is Lipschitz and let $x \neq y \in A$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon/M$. Assume $|x - y| < \delta$. Using the fact that f is Lipschitz, we have

$$\Big|\frac{f(x)-f(y)}{x-y}\Big| \leq M.$$

But this is equivalent to

$$|f(x) - f(y)| \leq M|x - y| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Hence, we have that $|f(x)-f(y)| < \varepsilon$ for all $x \neq y \in A$ implying that $f : A \to \mathbb{R}$ is a uniformly continuous function. ■

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Proof. No the converse statement is not true. Take the function $f(x) = \sqrt{x}$. This function contains a slope that gets arbitrarly steep as the values of x tend toward infinity. Another example is the function $g(x) = 1/x^2$. We can see that if $g(x)$ is defined on $(0, 1)$ then the slope of the $g(x)$ gets arbitrary big as $x \to 0$. ■

Exercise 4.4.10

Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x),\ f(x)g(x),\ \frac{f(x)}{g(x)},\ g \circ f(x).$$

(a) $f(x) + g(x)$.

Proof. The addition of two uniformly continuous functions $f(x), g(x)$ defined on the common domain A is continuous. Let $\varepsilon > 0$. Then define $\delta = \min\{\delta_1, \delta_2\}$ such that whenever

$$|x - y| < \delta_1,$$
$$|x - y| < \delta_2$$

we have that

$$\begin{aligned}|f(x) + g(x) - (f(y) - g(y))| &= |(f(x) - f(y)) + (g(x) - g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, the sum of two uniformly continuous functions is uniformly continuous. ■

(b) $f(x)g(x)$.

Proof. Not necessarily true unless both of the functions are both bounded.

■

(c) $f(x)/g(x)$ where $g(x) > 0$.

Proof. Like the product of two uniformly continuous functions, the quotient is also not necessarily true unless both $g(x)$ and $f(x)$ are bounded below and above respectively. ■

(d) $f(g(x))$.

Proof. Given $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ uniformly continuous, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in A so that composition is defined on A. Let $\varepsilon > 0$. Let $x \neq y \in A$. Since $f(x)$ is uniformly continuous on A, choose $\varepsilon = \delta$ such that whenever $|f(x) - f(y)| < \delta$ and $g : A \to \mathbb{R}$ being uniformly continuous on A, we have that

$$\begin{aligned}|g \circ f(x) - g \circ f(y)| &= |g(f(x)) - g(f(y))| \\ &< \varepsilon.\end{aligned}$$

■

Exercise 4.4.11 (Topological Characterization of Continuity).

Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Proof. $(\Leftarrow)$ Let us construct an $\varepsilon-$neighborhood around $g(c)$; that is, $V_\varepsilon(g(c))$. Suppose $V_\delta(c)$ with $x \in V_\delta(c)$. Since $g^{-1}(O)$ is an open set, we have that $V_\delta(c) \subseteq g^{-1}(O)$. But this means that $x \in g^{-1}(O)$ implying that $g(x) \in V_\varepsilon(g(c))$ since $g(x) \in O$.

$(\Rightarrow)$ Let $V_\varepsilon(g(c))$ where c is a limit point of O. Since g is a continuous function, there exists $V_\delta(c)$ such that whenever $x \in V_\delta(c)$, we have that $g(x) \in V_\varepsilon(g(c))$. Since O is an open set, we have that $V_\varepsilon(g(c)) \subseteq O$. But this means that $g(x) \in O$ as well implying that x is also an element of $g^{-1}(O)$. Hence, we have that $V_{\delta(c)} \subseteq g^{-1}(O)$ and thus $g^{-1}(O)$ is an open set. ■

4.4 The Intermediate Value Theorem

Exercise 4.5.3

Definition 5. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A.

Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property, then f is continuous on $[a, b]$.

Proof. Let f be an increasing function on $[a, b]$. Since f satisfies the intermediate value property, we know that $f(a) \leq f(c) \leq f(b)$ for some $c \in (x, y)$. Since we know that $f(a) \leq f(c)$, let us suppose two cases; that is, let us suppose either $f(c) - \varepsilon/2 < f(a)$ or $f(a) \leq f(c) - \varepsilon/2$. Then the former implies that we can set $x_1 = a$ and the latter implies that we can set $f(c) - \varepsilon/2 = f(x_1)$. Taking the latter case, we can set $f(x_1) = f(c) - \varepsilon/2$ and let $x \in (x_1, c]$ because f satisfies the intermediate value property. Hence, we have that

$$f(c) - \varepsilon/2 \leq f(x) \leq f(c). \tag{1}$$

Likewise, we know that $f(c) \leq f(b)$. Let us suppose two cases again; either $f(b) < f(c) + \varepsilon/2$ or $f(b) \geq f(c) + \varepsilon/2$. The former we can set $b = x_2$ and the latter we can set $f(x_2) = f(c) + \varepsilon/2$ because f satisfies the intermediate value theorem. If we let $x \in [c, x_2)$, then we have that

$$f(c) \leq f(x) \leq f(c) + \varepsilon/2 = f(x_2). \tag{2}$$

Taking (1) and (2) together then choose $\delta = \min\{x_1 - c, x_2 - c\}$, we have that

$$\varepsilon/2 \leq f(x) - f(c) \leq \varepsilon/2$$

which is equivalent to

$$|f(x) - f(c)| \leq \varepsilon/2.$$

■

Exercise 4.5.5

(a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

Proof. Considering the special case where f is a continuous function satisfying the property that $f(a) < 0 < f(b)$. Our goal is to show that $f(c)$ for some $c \in (a, b)$. Define the set

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

Note that $f(b)$ is an upper bound for K and we know $a \in K$ since $f(a) < 0$. Hence, we have that $\sup K$ exists by the Axiom of Completeness. Our goal is to show that $\sup K = f(c) = 0$. Consider the cases where $f(c) > 0, f(c) < 0$, and $f(c) = 0$. We will show that the first two cases contradict our notion that $\sup K$ is the least upper bound. Assume $f(c) < 0$ for some $c \in (a, b)$. But this means that $f(c)$ would not be an upper bound of K since there exists some α such that $\sup K < \alpha$where α is not an upper bound of K. Assume $f(c) > 0$ for some $c \in (a, b)$. This implies that $f(c) > 0$ is an upper bound of K but is not the least upper bound of K which is a contradiction. Hence, it must be that $f(c) = 0$. ■

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Proof. Consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$ and consider the midpoint $z = (a + b)/2$. If $f(z) \geq 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and positive at the right. We can extend this inductively to produce a sequence of closed intervals $I_n = [a_n, b_n]$ with the property described above to make a nested sequence

$$I_0 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots .$$

Since the intersection of these nested intervals I_n for all n is nonempty, we can find a $c \in \bigcup_{n=0}^{\infty} I_n$ such that $f(c) = 0$ since $f(x)$ is a continuous function. ■

Exercise 4.5.6

Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

Proof. Define $g(x) = f(x) - f(x + 1/2)$. Note that g is continuous over $[0, 1/2]$. Hence, we have

$$\begin{aligned} g(0) &= f(0) - f(1/2) && (1) \\ g(1/2) &= f(1/2) - f(1) \\ &= -g(0) && (2) \end{aligned}$$

By the Intermediate Value Theorem, we can see that there must exists $c \in [0, 1/2]$ such that $g(c) = 0$. Hence, we must have $g(c) = f(c) - f(c+1/2) = 0$ which implies $f(c) = f(c+1/2)$. ■

(b) Show that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

Proof. Define $g(x_n) = f(x_n) - f(x_n + 1/n)$. Note that $g(x_n)$ is a continuous function over $[0, 1]$. Since g is continuous, let $(x_n) \to 0$ and observe that $g(0) = f(0) - f(0) = 0$ and let $(x_n) \to 1$ such that $g(1) = f(1) - f(1) = 0$. Hence, we can see that $f(0) = f(1) = 0$. Hence, we can use the Intermediate Value Theorem to state that there exists $c \in [0, 1]$ such that $g(c) = 0$. Hence, we have

$$g(c) = 0 \Leftrightarrow f(x_n) = f(x_n + 1/n).$$

■

Proof. For fixed $n \in \mathbb{N}$, define $g(x_n) = f(x_n) - f(x_n + 1/n)$ such that $|x_n - y_n| = 1/n$ where $y_n = x_n + 1/n$. Note that g is continuous over $[0, (n - 1)/n]$. We want to show that there exists a root in the interval $[0, (n - 1)/n]$. Consider the following

$$\begin{aligned} \sum_{k=0}^{n-1} g\left(\frac{k}{n}\right) &= \sum_{k=0}^{n-1} f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \\ &= f(1) - f(0) \\ &= 0 \end{aligned}$$

and note that if there exists $0 \leq k \leq n-1$ such that $g(k/n) = 0$ then we are done. Otherwise, if $g(k/n) \neq 0$ for $0 \leq k \leq n - 1$, then there must exist $0 \leq k_1 \leq n - 1$ and $0 \leq k_2 \leq n - 1$ such that $g(k_1/n)$ and $g(k_2/n)$ have opposite sign. Hence, we can use the intermediate value theorem to posit the existence of $c \in [0, (n - 1)/n]$ such that $g(c) = 0$. Hence, we have

$$f(x_n) = f\left(x_n + \frac{1}{n}\right).$$

■

Exercise 4.5.7

Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0,1]$.

Proof. Let g be defined by $g(x) = x - f(x)$. We can see that g is continuous over $[0,1]$ since $f(x)$ is continuous over $[0,1]$. Now observe that

$$g(0) = -f(x), \tag{1}$$
$$g(1) = 1 - f(x). \tag{2}$$

Since $g(1) = 1 + g(0)$, we have that $g(1) \geq g(0)$. Since g is continuous over $[0,1]$, we can use the Intermediate Value Theorem to find a $c \in [0,1]$ such that $g(c) = 0$. Hence, we must have

$$g(c) = c - f(c) = 0 \Leftrightarrow f(c) = c.$$

■

Exercise 4.5.8 (Inverse Functions).

If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous on an interval $[a,b]$ and injective, then f^{-1} is also continuous.

Proof. Assume f is continuous on an interval $[a,b]$ and injective. Let $\varepsilon > 0$. Assume f is monotone so that we can use the intermediate value theorem to posit the existence of an y_0 such that it is between $f(x)$ and $f(y)$. Choose f to be an increasing function such that there exists some $x_1 \in [a,b]$ such that $f(x_1) < f(c)$. Hence, there exists y_1 in the image of f such that $x_1 = f^{-1}(y_1) = c - \varepsilon$ where $x_1 = c - \varepsilon < c$. Let $x \in (x_1, c]$ such that

$$x_1 = c - \varepsilon < x \leq c. \tag{4.1}$$

Likewise, there exists some $x_2 \in [a,b]$ such that $f(x_2) > f(c)$. Define $x_2 = c + \varepsilon$ such that $x_2 > c$. Let $x \in [c, x_2)$, we have that

$$c \leq x < x_2 = c + \varepsilon. \tag{4.2}$$

Since f is injective, we have $f^{-1}(x') = x$ for all $x' \in \operatorname{im}(f)$ and $f^{-1}(c') = c$ for some $c' \in \operatorname{im}(f)$. Hence, we have that

$$|f^{-1}(x') - f^{-1}(c')| < \varepsilon$$

whenever $|f(x) - f(c)| < \delta$. ■

4.5 Sets of Discontinuity

Exercise 4.6.3

State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Proof. We say $\lim_{x \to c^-} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |c - x| < \delta$. Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(y_n) = M$ for all sequences (y_n) satisfying $y_n < c$ and $\lim(y_n) = c$. ■

Theorem 4. Given $f : A \to \mathbb{R}$ and a limit point c of A, $\lim_{x \to c} f(x) = L$ if and only if

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

Exercise 4.6.4

Supply a proof for this proposition.

Proof. Let $f : A \to \mathbb{R}$ and a limit point c of A. Assume $\lim_{x\to c} f(x) = L$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. To show that the right-hand limit equals L. Let $x > c$. Then immediately, we have that $0 < |x - c| < \delta$ implies $\lim_{x\to c^+} f(x) = L$ since $|f(x) - L| < \varepsilon$. For the left-hand limit, suppose $x < c$. Then

$$0 < |x - c| < \delta \Leftrightarrow 0 < |c - x| < \delta$$

implies $|f(x) - L| < \varepsilon$ holds. Hence, $\lim_{x\to c^-} f(x) = L$.

Conversely, the fact that $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} = L$ implies for some $\delta > 0$ that for any $x > c$ or $x < c$ that $0 < |x - c| < \delta$ holds. Hence, we have that $|f(x) - L| < \varepsilon$; that is, $\lim_{x\to c} f(x) = L$. ■

Generally speaking, discontinuities can be divided into three categories:

(i) If $\lim_{x\to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at c is called *removable*.

(ii) If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then f has a *jump* discontinuity.

(iii) If $\lim_{x\to c} f(x)$ does not exist for some other reason, then the discontinuity at c is called an *essential* discontinuity.

We now have the proper characteristics to describe our set of discontinuities D_f for an arbitrary monotone function f.

Exercise 4.6.5

Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Proof. Without loss of generality, let f be an increasing function. Then for all $x < y$, $f(y) \geq f(x)$. If $\lim_{x\to c} f(x) = L$, but $L \neq f(c)$, then either $f(c) < L$ or $f(c) > L$. In either case, there exists an $x \in A$ such that for all $y \in A$, we have $f(x) > f(y)$ or $f(x) < f(y)$. But this contradicts our assumption that f is increasing. Hence, the discontinuity cannot be a *removable* discontinuity. If there exist an *essential discontinuity* then f cannot be monotone since either the left-hand limit or the right-hand limit does not exists. Hence, the only type of discontinuity an increasing function can have is a jump discontinuity. ■

Chapter 5

The Derivative

5.1 Derivatives and Intermediate Value Property

Exercise 5.2.1

Supply proofs for parts (i) and (ii) of Theorem 5.2.4.

(i) $(f+g)'(c) = f'(c) + g'(c)$.

Proof. Assume f and g are functions that are both differentiable at some point $c \in A$. Since $(f+g)(x) = f(x) + g(x)$ and the Algebraic Function Limit Theorem, we have that

$$\begin{aligned}(f+g)'(x) &= \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x-c} \\ &= \lim_{x\to c} \frac{(f(x) - f(c)) + (g(x) - g(c))}{x-c} \\ &= \lim_{x\to c} \Big(\frac{f(x) - f(c)}{x-c} + \frac{g(x) - g(c)}{x-c} \Big) \\ &= \lim_{x\to c} \frac{f(x) - f(c)}{x-c} + \lim_{x\to c} \frac{g(x) - g(c)}{x-c} \\ &= f'(c) + g'(c).\end{aligned}$$

■

(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

Proof. Since f is differentiable at $c \in A$, we have that

$$\begin{aligned}(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x) - (kf)(c)}{x-c} \\ &= \lim_{x\to c} \frac{kf(x) - kf(c)}{x-c} \\ &= \lim_{x\to c} \frac{k(f(x) - f(c))}{x-c} \\ &= k \cdot \lim_{x\to c} \frac{f(x) - f(c)}{x-c} \\ &= kf'(c).\end{aligned}$$

■

Exercise 5.2.2

Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

Proof. Take $g(x) = |x|$ and $f(x) = \frac{1}{|x|}$ where $(fg)(x) = 1$ is just a constant function that is differentiable everywhere (including zero). ■

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

Proof. Take $f(x) = \frac{1}{x}$ and $g(x) = x^2$. Their product $(fg)(x) = x$ is differentiable at $x = 0$. ■

(c) A function f not differentiable at zero and a function g differentiable at zero when $f + g$ is differentiable at zero.

Proof. This is impossible since $f = (f + g) - f$ would be differentiable at zero. ■

(d) A function f is differentiable at zero but not differentiable at any other point.

Proof. Let $f(x)$ be defined by the following:

$$f(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

■

Exercise 5.2.3

(a) Use the definition of the derivative to produce the proper formula for the derivative of $h(x) = 1/x$.

Proof. Let $h : A \to \mathbb{R}$ be a function defined on an interval A. Let $c \neq 0$ in A be arbitrary. Using the definition of the derivative, we have that

$$\begin{aligned} h'(c) &= \lim_{x\to c} \frac{h(x) - h(c)}{x - c} \\ &= \lim_{x\to c} \frac{1/x - 1/c}{x - c} \\ &= \lim_{x\to c} \frac{-(x - c)}{xc} \cdot \frac{1}{x - c} \\ &= \lim_{x\to c} -\frac{1}{xc} \\ &= \frac{-1}{c^2}. \end{aligned}$$

Hence, we have that $h'(c) = -1/c^2$ for any $c \neq 0$ in A. ■

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

Proof. Define $h = (g)^{-1}$. By the result in part (a) and the Chain Rule, we have that $h'(c) = \frac{-1}{[g(c)]^2} \cdot g'(c)$. Then by the Product Rule, we have that

$$\begin{aligned} (f \cdot h)'(c) &= f'(c)h(c) + h'(c)f(c) \\ &= \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{[g(c)]^2} \\ &= \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}. \end{aligned}$$

■

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (ii).

Proof. Let f and g be differentiable functions where $g(x) \neq 0$ for all $x \in A$. Note that since f and g are differentiable, they are also continuous on A. Hence, $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. Then observe that

$$\begin{aligned}
(f/g)'(c) &= \lim_{x \to c} \frac{(f/g)(x) - (f/g)(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)/g(x) - f(c)/g(c)}{x - c} \\
&= \lim_{x \to c} \left[\frac{1}{g(x)g(c)} \cdot \frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c}\right] \\
&= \lim_{x \to c} \left(\frac{1}{g(x)g(c)}\right) \lim_{x \to c} \left(\frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c}\right) \\
&= \lim_{x \to c} \left(\frac{1}{g(x)g(c)}\right) \lim_{x \to c} \left(g(x)\frac{f(x) - f(c)}{x - c} - f(x)\frac{g(x) - g(c)}{x - c}\right) \\
&= \lim_{x \to c} \left(\frac{1}{g(x)g(c)}\right) \left(\lim_{x \to c} g(x)\frac{f(x) - f(c)}{x - c} - \lim_{x \to c} f(x)\frac{g(x) - g(c)}{x - c}\right) \\
&= \frac{1}{[g(c)]^2} \cdot (g(c)f'(c) - f(c)g'(c)).
\end{aligned}$$

■

Exercise 5.2.4

Follow these steps to provide a slightly modified proof of the Chain Rule.

(a) Show that a function $h : A \to \mathbb{R}$ is differentiable at $a \in A$ if and only if there exists a function $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = \ell(x)(x - a)$$

for all $x \in A$.

Proof. ($\Rightarrow$) Suppose $h : A \to \mathbb{R}$ is differentiable at $a \in A$. We want to show that there exists $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies the following:

$$h(x) - h(a) = \ell(x)(x - a).$$

Since h is differentiable at $a \in A$, we know that the following

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

exists. But we can define a function $\ell(x)$ inside the limit as

$$\ell(x) = \frac{h(x) - h(a)}{x - a} \Leftrightarrow h(x) - h(a) = \ell(x)(x - a).$$

But we know that h is continuous since h is differentiable so we have that $\ell(x)$ is also continuous.

($\Leftarrow$) Now we want to show the converse. Suppose there exists $\ell : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = \ell(x)(x - a)$$

for all $a \in A$. Since ℓ is continuous, we now for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - a| < \delta$, we have that

$$|\ell(x) - \ell(a)| < \varepsilon. \tag{1}$$

But since

$$\ell(x) = \frac{h(x)-h(a)}{x-a}$$

we know that the limit

$$\lim_{x\to a} \frac{h(x)-h(a)}{x-a}$$

exists. Hence, h is differentiable at a. ■

(b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5.

Proof. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. Let f be a differentiable function at $c \in A$ and let $g : B \to \mathbb{R}$ be a differentiable function at $f(c) \in B$.

Since g is differentiable at $f(c) \in B$, we can construct the function ℓ such that

$$\ell(f(x)) = \frac{g(f(x)) - g(f(c))}{f(x)-f(c)}. \tag{1}$$

By part (a), we know that $\ell(f(x))$ is a continuous function and that it satisfies the following property

$$g(f(x)) - g(f(c)) = \ell(f(x))(f(x) - f(c)) \tag{2}$$

for all $f(c) \in B$. Dividing (2) by $(x - c)$, we get that

$$(g \circ f)'(c) = \frac{g(f(x)) - g(f(c))}{x-c} = \ell(f(x))\frac{f(x)-f(c)}{x-c}. \tag{3}$$

Assuming our result in part (a) and the fact that f and g are both differentiable at $c \in A$ and $f(c) \in B$ respectively, taking the limit of (3) leads to the conclusion that $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$. ■

Exercise 5.2.5

Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

Proof. The function f is continuous for all $a > 0$. ■

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

Proof. The function f is differentiable at zero for all $a > 1$ and yes the derivative function is continuous in the same conditions. ■

(c) For which values of a is f twice-differentiable?

Proof. The function f is twice-differentiable whenever $a > 2$. ■

Exercise 5.2.6

Let g be defined on an interval A, and let $c \in A$.

(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h\to 0} \frac{g(c+h) - g(c)}{h}.$$

Proof. Since g is differentiable at $c \in A$ and A is an interval, we can always define the difference quotient between two points, in this case $c + h$ and c, and still have the limit as $h \to 0$ and still have $g'(c)$. ■

(b) Assume A is open. If g is differentiable at $c \in A$, show that

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

Proof. Let g be differentiable at $c \in A$. We want to show that

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}. \tag{1}$$

Let $\varepsilon > 0$. Since A is open, there exists $\delta > 0$ such that $V_h(c) \subseteq A$. Let $h \in V_h(c)$. Through a few algebraic maniupulations, we get that

$$\begin{aligned}
\left|\frac{g(c+h) - g(c-h)}{2h} - g'(c)\right| &= \frac{1}{2}\left|\frac{g(c+h) - g(c)}{h} + \frac{g(c) - g(c-h)}{h} - 2g'(c)\right| \\
&= \frac{1}{2}\left|\left(\frac{g(c+h) - g(c)}{h} - g'(c)\right) - \left(\frac{g(c) - g(c-h)}{h} - g'(c)\right)\right| \\
&\leq \frac{1}{2}\left(\left|\frac{g(c+h) - g(c)}{h} - g'(c)\right| + \left|\frac{g(c) - g(c-h)}{h} - g'(c)\right|\right) \\
&< \frac{1}{2}(2\varepsilon) = \varepsilon.
\end{aligned}$$

This means that we can write $g'(c)$ as (1) above. ■

Exercise 5.2.7

Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.

Exercise 5.2.8

Review the definition of uniform continuity. Given a differentiable function $f : A \to \mathbb{R}$, let's say that f is *uniformly differentiable* on A if, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \varepsilon$$

whenever $0 < |x - y| < \delta$.

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? How about $g(x) = x^3$?

Proof. $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$ but $g(x) = x^3$ is not. ■

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

Proof. Let $f : A \to \mathbb{R}$ be uniformly differentiable on an interval A. Let $\varepsilon > 0$ and let $c \in A$. We want to show that f' is continuous on A; that is, whenever $|x - c| < \delta$ for some $\delta > 0$, we have that

$$|f'(x) - f'(c)| < \varepsilon.$$

By using the fact that f is uniformly differentiable on an interval A, we have

$$\begin{aligned}|f'(x)-f'(c)| &= \left|f'(x)-\frac{f(x)-f(c)}{x-c}+\frac{f(x)-f(c)}{x-c}-f'(c)\right| \\ &\le \left|f'(x)-\frac{f(x)-f(c)}{x-c}\right|+\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, we have that f is continuous on A. ■

(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Proof. Not necessarily. It is differentiable on $[0, 1]$, but not uniformly differentiable since the δ response gets progressively smaller as we compute $g_2'(x)$ as $x \to 0$. ■

Exercise 5.2.11

Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$ and a point $y \in (a, b)$ where $g(y) < g(b)$.

Proof. To prove $g(a) > g(x)$ for some $x \in (a, b)$, let us construct the sequence $(x_n) \subseteq (a, b)$ such that $x_n \to x$. We can do this because g being differentiable on $[a, b]$ immediately implies that g is continuous on $[a, b]$. Since $g'(a) < 0$, we know that

$$\lim_{n\to\infty}\frac{g(x_n)-g(a)}{x_n-a} < 0. \tag{1}$$

Since g is continuous on $[a, b]$, we have that (1) implies

$$\frac{g(x)-g(a)}{x-a} < 0 \Leftrightarrow g(x) < g(a).$$

Likewise, we can construct the sequence $(y_n) \subseteq (a, b)$ such that $(y_n) \to y$. Since g is continuous we know that

$$\lim_{n\to\infty}\frac{g(y_n)-g(b)}{y_n-b} > 0$$

implies that

$$\frac{g(y)-g(b)}{y-b} > 0 \Leftrightarrow g(y) > g(b).$$

■

(b) Now complete the proof of Darboux's Theorem started earlier.

Proof. Without loss of generality, suppose $g(x) < g(y)$. Since g is a continuous function on (a, b), there exists L such that $g(x) < L < g(y)$ where $g(c) = L$ for some $c \in (a, b)$. Taking the derivative of both sides leads to $g'(x) < 0 < g'(y)$. But by definition, we have that $g'(x) = f'(x) - \alpha$ and thus we have $f'(c) = \alpha$. ■

Exercise 5.2.12 (Inverse functions).

If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8, we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

where $y = f(x)$.

Proof. Since $f : [a, b] \to \mathbb{R}$ is a one-to-one function, there exists an inverse function f^{-1} which is defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. Since f is differentiable on $[a, b]$ with $f(x) \neq 0$ for all $x \in [a, b]$, we know that

$$\begin{aligned}(f^{-1})'(y) &= \lim_{y \to f(c)} \frac{f^{-1}(y) - f^{-1}(f(c))}{y - f(c)} \\ &= \lim_{y \to f(c)} \frac{x - c}{y - f(c)} \\ &= \lim_{x \to c} \frac{1}{\frac{f(x) - f(c)}{x - c}} \qquad (1) \\ &= \frac{1}{f'(c)}.\end{aligned}$$

The reason why we can exchange the limits on (1) is due to the fact that f^{-1} is continuous on the range of f.

■

5.2 The Mean Value Theorem

Exercise 5.3.1

Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Proof. Let f be differentiable on a closed interval $[a, b]$ and f' continuous on $[a, b]$. Let us define our closed interval as A. Let $x \neq y \in [a, b]$. By the Mean Value Theorem, we can find a $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Since f' is continuous on a compact set $[a, b]$, the range of f' is also bounded. Hence, there exist $M > 0$ such that $|f'(c)| \leq M$. Hence, we have that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M.$$

■

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Proof. Since f' is continuous on a compact set $[a, b]$, then it follows that f' attains a maximum and minimum on $[a, b]$. Hence, there is some $x_0 \in [a, b]$ such that $|f'(c)| \leq |f'(x_0)|$. By assumption, $|f'(x_0)| < 1$. Hence, we have that

$$|f'(c)| = \left|\frac{f(x) - f(y)}{x - y}\right| \leq |f'(x_0)| = s.$$

This means that

$$|f(x) - f(y)| \leq s|x - y|$$

and hence, f is contractive. ■

Exercise 5.3.2

Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is injective on A. Provide an example that the converse statement need not be true.

Proof. Let f be differentiable on an interval A. Let $x \neq y \in A$. By the Mean Value Theorem, we can find a $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}. \tag{1}$$

Since $f'(c) \neq 1$ on A and $x \neq y$, (1) implies that $f(x) \neq f(y)$. Hence, f is an injective function. ■

Exercise 5.3.3

Let h be differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

Proof. Since h is a differentiable function that is defined on $[0, 3]$, we can find a $d \in (1, 3)$ such that

$$h'(d) = \frac{h(3) - h(1)}{3 - 1} = \frac{2 - 2}{3 - 1} = 0.$$

Since $h'(d) = 0$, we know that $h(d)$ must be a constant. Hence, $h(d) = d$ for some $d \in [0, 3]$. ■

(b) Argue that at some point c we have $h'(c) = 1/3$.

Proof. Since h is a differentiable function defined on $[0, 3]$, we can use the Mean Value Theorem to state that there exists a $c \in (0, 3)$ such that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{1}{3}.$$

■

(c) Argue that $h'(x) = 1/4$ at some point in the domain.

Proof. Since there exists $c \in (0, 3)$ such that $h'(c) = 1/3$ and $h'(d) = 0$ for some $d \in (a, b)$, there exists $\ell \in (0, 3)$ such that $h'(\ell) = 1/4$ by Darboux's theorem. ■

Exercise 5.3.4

Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(a) If $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(0) = 0$ and $f'(0) = 0$.

Proof. Suppose $f(x_n) = 0$ for all $n \in \mathbb{N}$. Since f is differentiable on the interval A, we know that f is also continuous on A. Since $(x_n) \to 0$ for $x_n \neq 0$, we know that $f(x_n) = 0$ implies $f(0) = 0$. Since f is differentiable, $f(0) = 0$, and $x_n \neq 0$, we have that

$$\begin{aligned} f'(0) &= \lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n} \\ &= 0. \end{aligned}$$

■

(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Proof. Suppose f is twice-differentiable at zero. This means $f'(x_n) = 0$ as well. Since $x_n \neq 0$ and $f'(0) = 0$, we have

$$f''(0) = \lim_{n\to\infty} \frac{f'(x_n) - f'(0)}{x_n} = 0.$$

■

Exercise 5.3.5

(a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem.

Proof. Suppose f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). By the Mean Value Theorem, we can find a $c \in (a, b)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a}, \\ g'(c) &= \frac{g(b) - g(a)}{b - a}. \end{aligned}$$

Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Note that h is also continuous on $[a, b]$ and differentiable on the open interval (a, b) since f and g are both continuous on $[a, b]$ and differentiable on (a, b). Using the Mean Value Theorem again, we can find a $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}. \tag{1}$$

Using algebraic manipulations, we end up having $h(b) - h(a) = 0$ implying that $h'(c) = 0$. Expanding the right side of (1), we have that

$$\begin{aligned} \frac{h(b) - h(a)}{b - a} &= (f(b) - f(a)) \cdot \frac{g(b) - g(a)}{b - a} - (g(b) - g(a)) \cdot \frac{f(b) - f(a)}{b - a} \\ &= (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c). \end{aligned}$$

Since $h'(c) = 0$, we arrive at

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If $g' \neq 0$ for all $x \in (a, b)$, then our conclusion can be stated as follows:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

■

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve).

Proof. Graphically, we can find a tangent through $(f(a), g(a))$ and $(f(b), g(b))$ such that it, meaning $g's$ slope is parallel to the tangent of f. Another way to explain this is to set $f = x$, $g = y$, and $t = a$. Then the Mean Value Theorem states that

$$\frac{x'(t)}{y'(t)} = \frac{dx}{dy} = \frac{x(b) - x(a)}{y(b) - y(a)}$$

which is just the familiar Mean Value Theorem for parametric curves. ■

Exercise 5.3.6

(a) Let $g : [0, a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.

Proof. Suppose g is a differentiable function defined on $[0, a]$. Since g is continuous on $[0, a]$, differentiable on (a, b) and $g(0) = 0$, we use the Mean Value Theorem to state that there exists

$c \in (a, b)$ such that

$$g'(c) = \frac{g(x) - g(0)}{x - 0} = \frac{g(x)}{x}.$$

Since $|g'(x)| \leq M$, we know that

$$|g'(c)| = \left|\frac{g(x)}{x}\right| \leq M \Leftrightarrow |g(x)| \leq Mx.$$

■

(b) Let $h : [0, a] \to \mathbb{R}$ be twice-differentiable, $h'(0) = h(0) = 0$ and $|h"(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

Proof. Let h be twice-differentiable with $h'(0) = h(0) = 0$ as well as $|h"(x)| \leq M$ for all $x \in [0, a]$. Since h is twice-differentiable, we can find a $c \in (a, b)$ such that

$$h"(c) = \frac{h'(x) - f(0)}{x - 0} = \frac{h'(x)}{x}.$$

Since we are also given $|h'(x)| \leq M$, we know that $|h(x)| \leq Mx$. Since $|h"(x)| \leq M$ for all $x \in [0, a]$, we have that

$$\begin{aligned} |h"(c)| \leq M \Leftrightarrow \left|\frac{h'(x)}{x}\right| &\leq \frac{M}{2} \\ \Leftrightarrow |h'(x)| &\leq \frac{M}{2}x \\ \Leftrightarrow |h(x)| &\leq \frac{M}{2}x^2. \end{aligned}$$

■

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Proof. Let h be differentiable three times. Suppose $h"(0) = h'(0) = h(0) = 0$. It is a similar process as above. ■

Exercise 5.3.7

A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof. Suppose for sake of contradiction that f contains more than one fixed point. Let $a, b \in A$ be fixed points of f. Then we have $f(a) = a$ and $f(b) = b$. Suppose $f'(x) \neq 1$ for any $x \in A$. Since f is differentiable on A, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.$$

But this tell us that $f'(c) = 1$ which contradicts our assumption that $f'(x) \neq 1$ on A. Hence, a and b must be the same fixed point. ■

Exercise 5.3.8

Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x \to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Proof. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. We want to show that $f'(0)$ exists and is equal to L. Define $h(x) = f(x) - f(0)$ and $g(x) = x$. If we take the following limit

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{h(x)}{g(x)}$$

we notice that $f'(0) = 0/0$ which prompts us to use L'Hopital's rule for limits. Hence, if we take differentiate $h(x)$ and $g(x)$ and then take the limit of their quotient $h'(x)/g'(x)$, we get that $h'(x) = f'(x)$ and $g'(x) = 1$. But note that $h'(x)/g'(x) = f'(x)$. Since $\lim_{x\to 0} f'(x) = L$, we know that $\lim_{x\to 0} \frac{h'(x)}{g'(x)}$ must also equal to L. But this means that by L'hopital's rule that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = L.$$

■

Exercise 5.3.9

Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the $0/0$ case of L'Hopital's Rule under this stronger Hypothesis.

Proof. Since f' and g' are continuous at a, we know that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)} = L.$$

But we know f and g are also differentiable at a which means f and g are also continuous at a. Since $x \neq a$ for all $x \in A$ where A is an interval, we know that

$$\begin{aligned} L &= \frac{f'(a)}{g'(a)} \\ &= \frac{\lim_{x\to a} f(x) - f(a)}{\lim_{x\to a} g(x) - g(a)} \\ &= \lim_{x\to a} \frac{f(x)}{g(x)} && (f(a) = g(a) = 0) \end{aligned}$$

Hence, we have that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$. ■

Exercise 5.3.11

(a) Use the Generalized Mean Value Theorem to furnish a proof of the $0/0$ case of L'Hopital's Rule (Theorem 5.3.6).

Proof. Assume f and g continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. Suppose $f(a) = g(a) = 0$, $g'(x) \neq 0$ for all $x \neq a$, and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $0 < |x - a| < \delta$. Let $x \in (a, a + \delta)$. Since f is differentiable on A, there exists $c \in (x, a)$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $\lim_{x\to a} f'(x)/g'(x) = L$, we have that

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| < \varepsilon.$$

Since $g(a) = f(a) = 0$, we have that

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon.$$

Hence, we have that $\lim_{x\to a} f(x)/g(x) = L$. ■

Exercise 5.3.12

If f is twice differentiable on an open interval containing a and f" is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f"(a).$$

Proof. Suppose f is twice differentiable on an open interval containing a and f'' is continuous at a. Let $\varepsilon > 0$. Our goal is to show that

$$\left|\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - f"(a)\right| < \varepsilon$$

whenever $|h| < \delta$ for some $\delta > 0$. We first observe, through algebraic manipulation, that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{1}{h} \cdot \left(\frac{f(a+h) - f(a)}{h} - \frac{f(a) - f(a-h)}{h}\right). \tag{1}$$

Since f is twice differentiable on an interval containing a and f" continuous at a, there exist $t_1 \in (a, a+h)$ and $t_2 \in (a-h, a)$ such that

$$f'(t_1) = \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad f'(t_2) = \frac{f(a) - f(a-h)}{h}.$$

Likewise, there exists $x \in (a-h, a+h)$ such that,

$$f"(x) = \frac{f'(t_1) - f'(t_2)}{h}.$$

Since f" is continuous at a, we can choose $\delta = \min\{\delta_1, \delta_2\}$ such that whenever $|h| < \delta$, we have that

$$\begin{aligned}\left|\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - f"(a)\right| &= \left|\frac{f'(t_1) - f'(t_2)}{h} - f"(a)\right| \\ &= |f"(x) - f"(a)| \\ &< \varepsilon.\end{aligned}$$

Hence, we can write that

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f"(a).$$

■

Exercise 5.4.1

Sketch a graph of $(1/2)h(2x)$ on $[-2, 3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

as n gets larger.

Proof. The graph of $h_1(x)$ is just the Sawtooth function $h(x)$ but with a maximum height of $1/2$ and the length of the period is just 1. Notice that as n gets bigger and bigger our height for $h_n(x)$ is just $1/2^n$ while the period is $1/2^{n-1}$. The slopes of all the segments that make up $h_n(x)$ is just ± 1 for all $n \in \mathbb{N}$. ■

Let's define

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

which we claim as having the property that it is continuous for all $x \in \mathbb{R}$ but non-differentiable for all $x \in \mathbb{R}$.

Exercise 5.4.2

Fix $x \in \mathbb{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges and thus $g(x)$ is properly defined.

Proof. Since the maximum height of $h_n(x)$ is $1/2^n$, it follows that for all $n \in \mathbb{N}$, the sequence of partial sums is bounded; that is

$$0 \leq \frac{1}{2^n} h_n(2^n x) \leq \frac{1}{2^n}.$$

Since $\frac{1}{2^n}$ produces a geometric series which converges, we know that $\frac{1}{2^n} h_n(2^n x)$ must also converge by the Comparison Test. This means the series that defines $g(x)$ converges and is, therefore, well defined. ■

We can ask a couple of questions about functions that are defined by infinite series.

(a) Certainly, finite sum of a continuous functions is continuous but is the infinite sum of continuous functions necessarily continuous? We will see that this is not always the case in the next chapter.

Exercise 5.4.4

As the graph in Figure 5.7 suggests, the structure of $g(x)$ is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.

(a) how do we know g attains a maximum value M on $[0, 2]$? What is the value?

Proof. Since g is continuous on a compact set $[0, 2]$, we know that g must attain its maximum and minimum on the interval. Since g converges, we can use associativity to redefine g in the following way:

$$f_n(x) = h_{2n}(x) + h_{2n+1}(x)$$

for all $n \in \mathbb{N}$. A scaling argument is used to show that $g(x) \leq \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3}$. Hence, the max of $g(x) = \frac{4}{3}$. ■

Exercise 5.4.5

Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$

and use this to prove that $g'(0)$ does not exist.

Proof. For $g'(0)$ to exist, the sequential criterion for limits requires that

$$g'(0) = \lim_{m \to \infty} \frac{g(x_m) - g(0)}{x_m - 0}$$

exist for any sequence $(x_m) \to 0$. Let us fix $m \in \mathbb{N}$ and consider $x_m = 1/2^m$. Then plugging this in to g, we get that

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}).$$

Now we have to consider a few cases. Suppose $n > m$, then $h(2^{n-m}) = 0$ because the sawtooth function is zero for even n. If $n \leq m$, then we have $h(x) = x$ and we get that

$$\frac{1}{2^n} h(^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$$

by the definition of $h(x)$. This means we can represent $g(x_m)$ as a finite sum

$$g(x_n) = \sum_{n=0}^{m} \frac{1}{2^m}.$$

Using the difference quotient, we can see that

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{\sum_{n=0}^{m} 1/2^m}{1/2^m} = \sum_{n=0}^{m} 1 = m + 1.$$

Notice that the $g(x_m) = m + 1$ is a sequence that diverges, we know that g is not differentiable at zero and therefore does not exist. ■

Although intuition might lead us astray by telling us that $g'(0) = \infty$, we need to remember that for $x_m = -(1/2^m)$ we can use the same argument above to produce a difference quotient that approaches $-\infty$. This is that "cusp" that we see at $x = 0$ in the graph of g.

Using the same argument above, we can prove g' does not exists for $x = 1$ and $x = 1/2$. In fact, $g'(x)$ does not exist for any $x \in \mathbb{Q}$ such that $x = p/2^k$ where $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. These values of $x \in \mathbb{Q}$ are called *dyadic* points. For every x that is *dyadic*, h_n contains a corner at x as long as $n \geq k$.

What if x is not *dyadic*? For fixed values of $m \in \mathbb{N} \cup \{0\}$, x can be between two dyadic points; that is,

$$\frac{p_m}{2^m} < x < \frac{p_m + 1}{2^m}.$$

Setting $x_m = p_m/2^m$ and $y_m = (p_m)+1)/2^m$, we can repeat this for each m that produces two sequences (x_m) and (y_m) that satisfy the following:

(a) $\lim x_m = \lim y_m = x$,

(b) $x_m < x < y_m$.

We can prove the following lemma:

Lemma 1. Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, then

$$f'(a) = \lim_{n \to \infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

Proof. ■

We can use this lemma to show that $g'(x)$ does not exist.

Chapter 6

Sequences and Series of Functions

6.1 Uniform Convergence of a Sequence of Functions

Exercise 6.2.1

Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

Proof. Let us take the limit of (f_n). A well know trick we can use is by dividing by $(1/n)/(1/n)$; that is, we have that

$$\begin{aligned}\lim_{n\to\infty} f_n &= \lim_{n\to\infty} \frac{nx}{1+nx^2} \\ &= \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} \\ &= \frac{1}{x}.\end{aligned}$$

■

(b) Is the convergence uniform on $(0, \infty)$?

Proof. We claim that (f_n) is not uniform on $(0, \infty)$. Let $\varepsilon > 0$. Then

$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \frac{1}{x(1+nx^2)}.$$

The right side of this equation implies

$$\frac{1}{x(1+nx^2)} < \varepsilon.$$

After a few algebraic manipulations, we can choose $N \in \mathbb{N}$ such that

$$N > \left(\frac{1}{\varepsilon} - x\right)\frac{1}{x^3}$$

which is clearly dependent on x. ■

(c) Is the convergence uniform on $(0, 1)$?

Proof. We cannot have a lower bound that gets rid of the dependency of our choice of N along $(0, 1)$. Hence, (f_n) does not converge uniformly along $(0, 1)$. ■

(d) Is the convergence uniform on $(1, \infty)$?

Proof. We claim that $(f_n) \to f$ is uniform on $(1, \infty)$. Since (f_n) is defined on $(0, 1)$, we can lower bound

$$\frac{1}{x(1+nx^2)} < \frac{1}{n+1}.$$

along the interval $(1, \infty)$ which produces the following

$$\frac{1}{\varepsilon} < n+1.$$

This implies that for some $N \in \mathbb{N}$, we have

$$N > \frac{1}{\varepsilon} - 1.$$

We can see that our choice of N does not depend on x. Hence, we have that $(f_n) \to f$ uniformly along $(1, \infty)$. ■

Exercise 6.2.2

(a) Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, ..., \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n. Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at zero?

Proof. Each f_n is continuous at $x = 0$, but f is not continuous at $x = 0$ since $1/n \to 0$ as $n \to \infty$. Finally, we have that f_n does not converge uniformly on $\mathbb{R}$. ■

(b) Repeat this exercise using the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Each g_n as well as its limit g are continuous at $x = 0$. Furthermore, g converges uniformly on $\mathbb{R}$. ■

(c) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Each h_n continuous at $x = 0$ but its limit h is not continuous at $x = 0$. We have h_n does not converge uniformly on $\mathbb{R}$. ■

Exercise 6.2.3

For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

(a) Find the pointwise limit on $[0, \infty]$.

Proof. Let us find the pointwise limit of $g_n(x)$. If we take the limit of (g_n), we have

$$\lim_{n\to\infty} \frac{x}{1+x^n} = 0.$$

This limit cannot be uniform on $[0,\infty)$ since our choice of N depends on x.

Now let us find the limit of (h_n). If $x \geq 1/n$, we have $\lim\limits_{n\to\infty} h_n(x) = 1$. If $0 \leq x < 1/n$, then we have

$$\lim_{n\to\infty} h_n(x) = 0.$$

Just like $g_n(x)$, $h_n(x)$ cannot converge on $[0,\infty)$ since our choice of N depends on x. ■

Exercise 6.2.5

Using the Cauchy Criterion for convergent sequences of real numbers, supply a proof for Theorem 6.2.5. (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Proof. ($\Rightarrow$) Let (f_n) be defined on $A \subseteq \mathbb{R}$. Define $f_n(x) = y_n$, $f_m(x) = y_m$, and $f(x) = L$. We want to show that (f_n) is Cauchy. Let $\varepsilon > 0$. Since $(f_n) \to f$ uniformly, there exists $N \in \mathbb{N}$ such that for any $n, n \geq N$, we have

$$|y_n - L| < \varepsilon,$$
$$|y_m - L| < \varepsilon.$$

But since (f_n) is just a sequence of real numbers that converges, we have that for any $m, n \geq N$,

$$|y_n - y_m| < \varepsilon.$$

($\Leftarrow$) Suppose for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ and $x \in A$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

This means that (f_n) is a Cauchy sequence. Since (f_n) is a sequence of real numbers on A, we know that (f_n) is also convergent. Since $(f_n) \to f$ whenever $n, m \geq N$ and $x \in A$, we have that $(f_n) \to f$ uniformly. ■

Exercise 6.2.6

Assume $f_n \to f$ on a set A. Theorem 6.2.6 is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

Proof. We want to show that this is true when $f_n \to f$ uniformly. Let $\varepsilon > 0$. Suppose f_n is uniformly continuous. Then for any $n \geq N$ for some $N \in \mathbb{N}$, we have that

$$|f_n(x) - f_n(y)| < \varepsilon.$$

we know that Choose $N = \max\{N_1, N_2, N_3\}$ such that for any $n \geq N$ and $x, y \in A$, we have that

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, $(f_n) \to f$ uniformly. ■

(b) If each f_n is bounded, then f is bounded.

Proof. Suppose f_n is bounded, then there exists $M > 0$ such that $|f_n(x)| \leq M$ for any $n \in \mathbb{N}$. Assume $f_n \to f$ uniformly. Then let $\varepsilon = 1$ and let $x \in A$ be arbitrary, there exists $N \in \mathbb{N}$

such that for any $n \geq N$, we have

$$|f(x) - f_n(x)| < 1 \Leftrightarrow |f(x)| < 1 + |f_n(x)| \leq M + 1 = M'.$$

Hence, f is bounded. ■

Exercise 6.2.7

Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Proof. Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions $f_n(x) = f(x + \frac{1}{n})$. Let $(x_n), (y_n) \subseteq K$. Define $y_n = x + \frac{1}{n}$ and $x_n = x$. Since f is uniformly continuous, we know that $|x_n - y_n| \to 0$ implies that $|f(x_n) - f(y_n)| \to 0$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\begin{aligned} |f_n(x) - f(x)| &= |f(x + \frac{1}{n}) - f(x)| \\ &= |f(y_n) - f(x_n)| \\ &< \varepsilon. \end{aligned}$$

Hence, $(f_n) \to f$ uniformly. ■

Exercise 6.2.8

Let (g_n) be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $(1/g_n)$ converges uniformly on K to $1/g$.

Proof. Let $\varepsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, we have

$$\left| \frac{1}{g_n(x)} - \frac{1}{g(x)} \right| < \varepsilon.$$

Since $(g_n) \to g$ uniformly on K, we know that whenever $n \geq N$ and $x \in K$, for some $N \in \mathbb{N}$, we have that

$$|g_n(x) - g(x)| < \varepsilon.$$

Furthermore, (g_n) continuous on compact set K for each $n \in \mathbb{N}$. This means that for each $n \in \mathbb{N}$, we have that $g_n(x)$ has a bounded range. Hence, there exists $M > 0$ such that $|g_n(x)| \geq M$. Similarly, $|g(x)| \geq L$ for some $L > 0$. Then choose $N = ML \cdot \varepsilon$ such that for any $n \geq N$, we have that

$$\begin{aligned} \left| \frac{1}{g_n(x)} - \frac{1}{g(x)} \right| &= \left| \frac{g(x) - g_n(x)}{g_n(x) \cdot g(x)} \right| \\ &= \frac{|g_n(x) - g(x)|}{|g_n(x)||g(x)|} \\ &< \frac{ML \cdot \varepsilon}{ML} = \varepsilon. \end{aligned}$$

Hence, g_n converges to g uniformly on K. ■

There is another way to prove this using the Cauchy Criterion.

Proof. Let $\varepsilon > 0$. It suffices to show that $1/g_n \to 1/g$ uniformly by showing $(1/g_n)$ satisfies the Cauchy Criterion. Since g_n is a bounded sequence of functions, there exists $M > 0$ such that

$|g_n| \geq M$. Choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have

$$\begin{aligned}\left|\frac{1}{g_n(x)} - \frac{1}{g_m(x)}\right| &= \left|\frac{g_m(x) - g_n(x)}{g_n(x)g_m(x)}\right| \\ &= \frac{|g_m(x) - g_n(x)|}{|g_n(x)||g_m(x)|} \\ &< \frac{M^2\varepsilon}{M^2} \\ &= \varepsilon.\end{aligned}$$

Since $(1/g_n)$ is a Cauchy sequence of functions, we know that $(1/g_n)$ must converge uniformly. ■

Exercise 6.2.9

Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is uniformly convergent sequence of functions.

Proof. Let $\varepsilon > 0$ and let $x \in \mathbb{R}$. Since (f_n) and (g_n) are uniformly convergent sequences of functions, there exists $N_1, N_2 \in \mathbb{N}$ such that for any $n \geq N$ where $N = \max\{N_1, N_2\}$, we have that

$$\begin{aligned}|f_n(x) + g_n(x) - (f(x) + g(x))| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\ &\leq |f_n(x) - f(x)| + |g_n(x) + g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

■

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

Proof. Let either (g_n) or (f_n) be unbounded. Take, for example, (f_n) defined as

$$f_n = x^2 + \frac{1}{n}$$

on $\mathbb{R}$. We can see that $f_n \to f$ where f is also unbounded. Hence, $(f_n g_n)$ cannot be uniformly convergent. ■

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

Proof. Suppose there exists $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and let $x \in \mathbb{R}$. It suffices to show that $(f_n g_n)$ is Cauchy. Since $(f_n) \to f$ and $(g_n) \to g$ uniformly, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have that

$$\begin{aligned}|f_n(x)g_n(x) - f_m(x)g_m(x)| &= |f_n(x)g_n(x) - f_m(x)g_n(x) + f_m(x)g_n(x) - f_m(x)g_m(x)| \\ &\leq |g_n(x)||f_n(x) - f(x)| + |f_m(x)||g_n(x) - g(x)| \\ &< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

Hence, $(f_n g_n)$ converges uniformly by the Cauchy Criterion. ■

Exercise 6.2.10

This exercise and the next explore the partial converse of the Continuous Limit Theorem. Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Proof. Suppose f is continuous on $[a, b]$, f_n increasing, and $(f_n) \to f$ pointwise on $[a, b]$. Since f continuous on $[a, b]$, we will have a mapping from the closed interval $[a, b]$ to another closed interval

$[c,d]$. Our intention is to split $[c,d]$ into small intervals so that we may use the fact that f_n is increasing to create a bound that is at most a size of ε_1. By the Order Limit Theorem, we know that f_n increasing also implies that f is also increasing.

Let $\varepsilon > 0$. Define that for each x_i, we have $f(x_i) = y_i$. Let $y_1 = c$ and $y_n = d$, where our small intervals our defined as everything in between; that is, for every $k \in \mathbb{N}$, we have $[y_{k_2}, y_{k+1}]$ implies $|y_{k+1} - y_k| < \varepsilon_1$. Since $(f_n) \to f$, we can find M_k such that $m_k > M_k$ implies $|y_k - f_{m_k}(x_k)| < \varepsilon_1$. Let $M = \max\{M_1, M_2, \ldots, M_n\}$. Let $m > M$ be arbitrary. Since f_m increasing, we can bound $f_m(x_{i+1} - f_m(x_i))$ by

$$\begin{aligned}|f_m(x_{i+1}) - f_m(x_i)| &= |f_m(x_{i+1}) - y_{i+1} + y_{i+1} - y_i + y_i - f_m(x_i)| \\ &\leq |f_m(x_{i+1}) - y_{i+1}| + |y_{i+1} - y_i| + |y_i - f_m(x_i)| \\ &< \varepsilon_1.\end{aligned}$$

Consider $|f(x) - f_m(x)|$ and let $x \in [x_i, x_{i+1}]$ with i arbitrary. Since f is increasing and f is continuous, we know that $y_i \leq f(x) \leq y_{i+1}$ and hence $f(x) - y_i < \varepsilon_1/3$. Similarly, we have that $f_m(x) - f_m(x_i) \leq f_m(x) - f_m(x_{i+1}) < \varepsilon_1/3$. Hence, observe that

$$\begin{aligned}|f(x) - f_m(x)| &= |f(x) - y_i + y_i - f_m(x_i) + f_m(x) - f_m(x_i)| \\ &\leq |f(x) - y_i| + |y_i - f_m(x_i)| + |f_m(x) - f_m(x_i)| \\ &< \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} \\ &= \varepsilon_1.\end{aligned}$$

Hence, $f_n \to f$ uniformly. ∎

Exercise 6.2.11

Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$, the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

Proof. Since f and f_n are both continuous, we know that $g_n = f - f_n$ must also continuous by the Algebraic continuity theorem. Since $f_n \to f$ pointwise and $g_n = f - f_n$, we must also have that $(g_n) \to 0$. Since f_n increasing for all $n \in \mathbb{N}$, we have that g_n is an decreasing sequence of functions. Hence, we can restate our hypothesis in the following way

(i) g_n is continuous,

(ii) $(g_n) \to 0$ for all $n \geq N$ for some $N \in \mathbb{N}$,

(iii) g_n is decreasing for all $n \in \mathbb{N}$.

∎

(b) Let $\varepsilon > 0$ be arbitrary, and define $K_n\{x \in K : g_n(x) \geq \varepsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$, and use this observation to finish the argument.

Proof. First, we must show that each K_n is also compact. Our intention is to create a nested sequence of compact sets so that we may extract a contradiction that implies that the infinite intersection of K_n must be empty and not nonempty. To prove K_n is compact, we must prove that K_n is closed and bounded.

To show that K_n is closed, let us assume that $(x_m) \subseteq K_n$ is a convergent sequence in K_n. We need to show that the limit of (x_m) is contained in K_n. Since g_n is a continuous function, we know that $\lim_{m\to\infty} g_n(x_m) = g_n(x)$. This implies that $g_n(x) \geq \varepsilon$ for all $n \in \mathbb{N}$. By definition of K_n, we must have $x \in K_n$. Hence, K_n for all $n \in \mathbb{N}$ is closed. To show that K_n is bounded is straightforward. Since K is a compact set and $K_n \subseteq K$ for all $n \in \mathbb{N}$, K_n must also be bounded. Hence, for each $n \in \mathbb{N}$, K_n is compact.

Since g_n is decreasing, we know that for all $x \in K$ and $n \in \mathbb{N}$ that $g_n(x) \geq g_{n+1}(x)$. This immediately implies that $K_n \supseteq K_{n+1}$. By the Nested Compact Interval theorem, we must have that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset. \tag{1}$$

But if we let x be arbitrary where this x is contained in this intersection, then we have $g_n(x) \geq \varepsilon$, which contradicts our assumption that $(g_n) \to 0$. Hence, the intersection in (1) must be empty for $(g_n) \to 0$; that is, there exists $N \in \mathbb{N}$ such that

$$|g_n(x)| < \varepsilon$$

for all $n \geq N$ and $x \in K$. Hence, $(g_n) \to g$ uniformly. ■

6.2 Uniform Convergence and Differentiation

Exercise 6.3.1

Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

Proof. First, we show that (g_n) converges uniformly on $[0, 1]$. We observe that since (g_n) is defined on $[0, 1]$, we know that we must have

$$\frac{x^n}{n} \leq \frac{1}{n}. \tag{1}$$

Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$ such that for any $n \geq N$ and $x \in [0, 1]$, we have that

$$n > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n} < \varepsilon.$$

We know by (1) that the above will become

$$\left|\frac{x^n}{n} - 0\right| = \frac{x^n}{n} \leq \frac{1}{n} < \varepsilon.$$

Since x and $n \in \mathbb{N}$ were arbitrary, we know that (g_n) must converge uniformly and that

$$\lim g_n(x) = 0.$$

Now we want to show that g is a differentiable function. Let $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have that

$$\left|\frac{g(x) - g(c)}{x - c} - 0\right| = 0 < \varepsilon.$$

Hence, g is differentiable and that $g'(x) = 0$ for all $x \in [0, 1]$. ■

(b) Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

Proof. We want to show that (g'_n) converges on $[0, 1]$. Computing the derivative of (g'_n) leads to the following

$$g'_n(x) = x^{n-1}.$$

Let $x \in [0, 1]$. If $x = 1$, then $g'_n \to 1$. If $0 \leq x < 1$, then $g'_n \to 0$. Putting everything together, our limit function $g'(x)$ can be written as

$$g'(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since our convergence of $g_n'(x)$ depends on our choice of x in $[0, 1]$, we must have that $g_n(x)$ does not converge uniformly. This means that setting $h = \lim g_n$ will produce $h \neq g'$. ■

Exercise 6.3.2

Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbb{R}$.

Proof. First, define $f_n(x) = x^2 + \frac{1}{n}$ and observe that $\lim f_n(x) = x^2$. Since $h_n(x) = \sqrt{f_n(x)}$, we can compute the pointwise limit of $h_n(x)$ as

$$\begin{aligned} \lim h_n(x) &= \lim \sqrt{f_n(x)} \\ &= \sqrt{x^2} \\ &= |x|. \end{aligned}$$

Now we want to show that this convergence is uniform. Our goal is to show that (h_n) is a Cauchy sequence. Let $x \in \mathbb{R}$ arbitrary and $\varepsilon > 0$. Since $h_n(x) \to h(x)$ pointwise where $h(x) = |x|$, choose $N \in \mathbb{N}$ such that for any $m, n \geq N$, we have that

$$\begin{aligned} |h_n(x) - h_m(x)| &= |h_n(x) - h(x) + h(x) - h_m(x)| \\ &\leq |h_n(x) - h(x)| + |h(x) - h_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since (h_n) is a Cauchy sequence of functions, we must have $(h_n) \to h$ uniformly by the Cauchy Criterion. ■

(b) Note that each h_n is differentiable. Show $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is *not* uniform on any neighborhood of zero.

Proof. Our goal is to show that $g(x) = \lim h_n'(x)$ exists. Assume each h_n is differentiable. First, we compute h_n'. Let $x \in \mathbb{R}$ be arbitrary. By the Chain Rule, we have

$$h_n'(x) = \frac{x}{\sqrt{x^2 + 1/n}}. \tag{1}$$

Since $\lim h_n(x) = |x|$ pointwise, taking the limit of (1) produces

$$\begin{aligned} \lim h_n'(x) &= \lim \frac{x}{\sqrt{x^2 + 1/n}} \\ &= \frac{x}{|x|}. \end{aligned}$$

We are certain that the convergence of $h_n'(x)$ is not uniform because $g(x) = x/|x|$ is defined as

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

which means that the convergence of $h_n'(x)$ is depend on our choice of $x \in \mathbb{R}$. ■

Exercise 6.3.3

Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

(a) Find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

Proof. To find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum, we use the Interior Extremum Theorem to find $x \in \mathbb{R}$ such that $f_n'(x) = 0$. Computing the derivative of $f_n(x)$, we use a combination of the chain rule and product rule to get

$$f_n'(x) = \frac{1-nx^2}{(1+nx)^2}.$$

Setting $f_n'(x) = 0$, we find that

$$1 - nx^2 = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{n}}$$

for any $n \in \mathbb{N}$. Denote these points on $\mathbb{R}$ where $\alpha = 1/\sqrt{n}$ and $\beta = -1/\sqrt{n}$. We find that $f_n(x)$ attains its maximum when $\alpha = 1/\sqrt{n}$ and minimum when $\beta = \pm 1/\sqrt{n}$. Furthermore, we have

$$f_n(\alpha) = \frac{1}{2\sqrt{n}}$$

and

$$f_n(\beta) = \frac{1}{2\sqrt{n}}.$$

To show that (f_n) converges uniformly on $\mathbb{R}$, it is enough to show that (f_n) satisfies the Cauchy Criterion. First, we observe that the sequence $\alpha_n = \frac{1}{\sqrt{2n}}$ is a Cauchy sequence. Hence, choose $N \in \mathbb{N}$ such that for any $n, m \geq N$, we have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &= \left|\frac{x}{1+nx^2} - \frac{x}{1+mx^2}\right| \\
&\leq \frac{1}{2}\left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right| \\
&< \frac{1}{2} \cdot 2\varepsilon = \varepsilon.
\end{aligned}$$

Since (f_n) satisfies the Cauchy Criterion, we must have a uniform convergence of (f_n) to f. Letting $n \to \infty$, we get that the limit function f is just $f = 0$. ■

(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Proof. Let $f = \lim f_n$. By last part, we get that

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}.$$

Taking the limit of $f_n'(x)$ produces an ∞/∞ case that can be remedied by using L'Hopital's rule. Let $\alpha_n(x) = 1 - nx^2$ and $\beta_(x) = (1+nx^2)^2$. Then taking the derivative of both of these functions produces

$$\alpha_n'(x) = -2nx \text{ and } \beta_n'(x) = 4nx(1+nx^2).$$

This implies that

$$\lim_{n\to\infty} \frac{\alpha_n'(x)}{\beta_n'(x)} = \lim_{n\to\infty} \frac{-1}{2(1+nx^2)} = 0 = f'(x)$$

which holds for all $x \in \mathbb{R}$. ■

Exercise 6.3.4

Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbb{R}$.

Proof. Our first goal is to show that $h_n \to 0$. Let $\varepsilon > 0$. We observe that $|\sin(nx)| \leq 1$. Choose $N = 1/\varepsilon^2$ such that for any $n \geq N$, we have that

$$\begin{aligned} \left|\frac{\sin(nx)}{\sqrt{n}} - 0\right| &= \left|\frac{\sin(nx)}{\sqrt{n}}\right| \\ &= \frac{|\sin(nx)|}{\sqrt{n}} \\ &\leq \frac{1}{\sqrt{n}} \\ &< \varepsilon. \end{aligned}$$

Since our choice of $N \in \mathbb{N}$ does not depend on $x \in \mathbb{R}$, we know that the convergence of $h_n(x)$ must be uniform.

Now let us show that the sequence of derivatives (h_n') diverges for every $x \in \mathbb{R}$. First, we compute $h_n'(x)$ which results in

$$h_n'(x) = \sqrt{n}\cos(nx).$$

Let $x \in \mathbb{R}$ be arbitrary. Since $x_n = \sqrt{n}$ an unbounded sequence in $\mathbb{R}$, we know that $h_n'(x)$ is also unbounded despite $|\cos(nx)| \leq 1$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. This means that $h_n'(x)$ is unbounded which implies that it diverges for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. ∎

Exercise 6.3.5

Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

Proof. First we compute $g(x)$. Set $g(x) = \lim g_n(x)$. Then taking the limit as $n \to \infty$ produces

$$\begin{aligned} \lim g_n(x) &= \lim \frac{nx + x^2}{2n} \\ &= \lim \left(\frac{x}{2} + \frac{x^2}{2n}\right) \\ &= \lim \left(\frac{x}{2}\right) + \lim \left(\frac{x^2}{2n}\right) \\ &= \frac{x}{2}. \end{aligned}$$

This means $g(x) = x/2$ and hence $g'(x) = 1/2$. ∎

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

Proof. First we compute $g_n'(x)$. Using our derivative rules, we get

$$\begin{aligned} g_n'(x) &= \frac{1}{2n}\Big(n + 2x\Big) \\ &= \frac{1}{2} + \frac{x}{n} \end{aligned}$$

which holds for all $n \in \mathbb{N}$.

To show that (g'_n) converges uniformly on every interval $[-M, M]$, we need to show that (g'_n) satisfies the Cauchy Criterion. Let $\varepsilon > 0$. Choose $N = 1/\varepsilon$ such that for any $n \geq N$ and for any $x \in [-M, M]$, we have

$$\begin{aligned}
|g'_n(x) - g'(x)| &= \left|\frac{1}{2} + \frac{x}{n} - \frac{1}{2}\right| \\
&= \left|\frac{x}{n}\right| \\
&\leq \frac{M}{n} \\
&< M \cdot \frac{\varepsilon}{M} = \varepsilon.
\end{aligned}$$

Since our choice of N only depends on ε, we know that $(g'_n) \to g'$ uniformly. ■

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Proof. First, we compute $f(x)$ by taking the limit as $n \to \infty$ of $f_n(x)$. Observe that

$$\lim_{n\to\infty} \frac{nx^2 + 1}{2n + x} = \frac{x^2}{2}.$$

Then we compute $f'(x)$ so we have

$$f'(x) = x.$$

Another way to compute the derivative of $f(x)$ is by differentiating $f_n(x)$ for each $n \in \mathbb{N}$ and then showing that the differentiated sequence of functions is uniformly convergent. By differentiating $f_n(x)$ using the product rule and chain rule, we get

$$\begin{aligned}
f'_n(x) &= \frac{2nx}{2n + x} - \frac{nx^2 + 1}{(2n + x)^2} \\
&= \frac{2nx(2n + x) - (nx^2 + 1)}{(2n + x)^2} \\
&= \frac{4n^2x + nx^2 + 1}{4n^2 + 4nx + x^2}.
\end{aligned}$$

Now we want to show that $f'_n(x)$ actually converges to its limit uniformly so that we shall show that

$$\lim_{n\to\infty} f'_n(x) = f'(x).$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have that

$$\begin{aligned}
|f'_n(x) - f'(x)| &= \left|\frac{4n^2x + nx^2 + 1}{(2n + x)^2} - x\right| \\
&= \left|\frac{1 - 3nx^2 - x^3}{(2n + x)^2}\right| \\
&= \frac{3nx^2 + x^3 - 1}{4n^2 + 4nx + x^2} \\
&\leq \left|\frac{1 - 3nM^2 - M^3}{4n^2 + 4nM + M^2}\right| \\
&< \varepsilon.
\end{aligned}$$

The first inequality holds because $f'_n(x)$ is defined on the closed interval $[-M, M]$ and the second inequality holds because

$$\frac{1 - 3nM^2 - M^3}{4n^2 + 4nM + M^2} \to 0.$$

Since this convergence of $f_n'(x) \to f'(x)$ does not depend on $x \in [-M, M]$, we know that the $f_n' \to f$ uniformly. Hence, we know that

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

by Theorem 6.3.3. ■

Exercise 6.3.6

Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbb{R}$.

(a) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbb{R}$.

Proof. Let $f_n(x)$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } \text{ even} \end{cases}.$$

If we differentiate $f_n(x)$ then we just get a sequence of derivatives f_n' that converges uniformly to 0. But $f_n(x)$ does not converge anywhere. ■

(b) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Proof. This is not possible. The differentiability of (f_n) for all $x \in [a, b]$ is required for f_n' to converge uniformly. ■

Exercise 6.3.7

Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that for any $x \in [a, b]$ and $m, n \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| \le |(f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)| + |f_n(x_0) - f_m(x_0)|.$$

Proof. Proof is right under Theorem 6.3.2. ■

6.3 Series of Functions

Exercise 6.4.1

Supply the details for the proof of the Weierstrass M-test (Corollary 6.4.5).

Proof. Let $M_n > 0$ be a real number satisfying $|f_n(x)| \le M_n$ for all $x \in A$.Let $\varepsilon > 0$. Our goal is to use the Cauchy Criterion for Uniform Convergence of Series to prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly. Suppose $\sum_{n=1}^{\infty} M_n$ converges, then for some $N \in \mathbb{N}$, we have for any $n > m \ge N$ implies

$$\left| \sum_{k=m+1}^{n} M_k \right| < \varepsilon.$$

Since $M_n > 0$, this result can be restated as follows

$$\sum_{k=m+1}^{n} M_k < \varepsilon.$$

By letting $n > m \ge N$ and $x \in A$ arbitrary as before, we can use the triangle inequality to state that

$$
\begin{aligned}
\Big|\sum_{k=m+1}^{n} f_k(x)\Big| &\leq \sum_{k=m+1}^{n} |f_k(x)| \\
&\leq \sum_{k=m+1}^{n} M_k \\
&< \varepsilon.
\end{aligned}
$$

By the Cauchy Criterion for Uniform Convergence, we know that $\sum_{n=1}^{\infty} f_m(x)$ must converge uniformly on A, ■

Exercise 6.4.2

Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

Proof. Since $\sum_{n=1}^{\infty} g_n$ converges uniformly, we know that

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_k(x)$$

converges uniformly. If we consider the case when $n = m - 1$, then we have that

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_k(x) = \lim_{n\to\infty} \Big(g_n(x) + \sum_{k=1}^{n-1} g_n(x) \Big). \tag{1}$$

Furthermore, (1) implies that

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_k(x) = \lim_{n\to\infty} g_n(x) + \lim_{n\to\infty} \sum_{k=1}^{n-1} g_n(x). \tag{2}$$

Since the sequence of partial sums for $\sum_{n=1}^{\infty} g_n(x)$, we know that subtracting the second term on the right side of (2) results in

$$\lim_{n\to\infty} g_n(x) = 0.$$

Hence, $g_n \to 0$. ■

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof. Let $x \in A$ be arbitrary. Since $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ satisfies the Cauchy Criterion for uniform convergence, we know that for any $n > m \geq N$ and $x \in A$, we have that

$$
\begin{aligned}
\Big|\sum_{k=m+1}^{n} f_n(x)\Big| &\leq \Big|\sum_{k=m+1}^{n} g_n(x)\Big| \\
&< \varepsilon.
\end{aligned}
$$

Hence, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly. ■

(c) If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A, then there exists constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Proof. Suppose our series is defined by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tag{1}$$

where $f_n(x) = (-1)^{n+1}/n$. We can see by the Alternating Series test that (1) converges uniformly. But note that $|f_n(x)| \leq 1/n = M_n$ which produces the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges.

■

Exercise 6.4.3

(a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.

Proof. Our goal is to use the Term-by-term Continuity Theorem to show that g is continuous on $\mathbb{R}$; that is, we want to show that g_n is continuous and $\sum_{n=0}^{\infty} g_n$ converges uniformly. Since $\cos(x)$ is a continuous function, we know that for each $n \in \mathbb{N}$, g_n must be a sequence of continuous functions. All that is left is to show that $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly. We shall do this by using the Weierstrass M-test. Since $|\cos(2^n x)| \leq 1$ for all $x \in \mathbb{R}$, we know that

$$\sum_{n=0}^{\infty} \left| \frac{\cos(2^n x)}{2^n} \right| \leq \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n. \tag{1}$$

Since the right side of (1) is a geometric series that converges, we know that the series $\sum_{n=0}^{\infty} g_n(x)$ must converge uniformly to $g(x)$. Hence, $g(x)$ is continuous on all of $\mathbb{R}$. ■

(b) The function g was cited in Section 5.4 as an example of a nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

(b) We can define the series above via the sawtooth function from section 5.4 where

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} 2^n h(2^n x). \tag{1}$$

If we look at the terms $h_n(x)$, we can see that

$$|h_n(x)| \leq \frac{1}{2^n} \tag{2}$$

which forms a geometric series on the right side of (2). Hence, we must have (1) converge uniformly via the Weierstrass M-test.

Exercise 6.4.4

Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Proof. Let $h_n(x) = \frac{x^{2n}}{(1+x^{2n})}$ be the terms of the series

$$\sum_{n=0}^{\infty} h_n(x).$$

When $|x| \geq 1$, we observe that the terms of (h_n) does not converge to 0 as $n \to \infty$. If $|x| < 1$, then we have that

$$|h_n(x)| \leq x^{2n}.$$

Since $|x| < 1$, x^{2n} forms a geometric series that converges. Hence, we must have that the series of $h_n(x)$ must be uniformly convergent. Furthermore, for any $0 \leq a < 1$, we will find that the infinite series of $h_n(x)$ converges uniformly where

$$\sum_{n=0}^{\infty} \frac{a^{2n}}{1 + a^{2n}}.$$

■

Exercise 6.4.5

(a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \dots$$

is continuous on $[-1, 1]$.

Proof. Using the Weierstrass M-test, we have that

$$\sum_{n=1}^{\infty} \left| \frac{x^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{1}$$

Since the series on the right side of (1) is a P-series, we know that it must converge which means that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

must converge. Note the sequence of functions

$$h_n(x) = \frac{x^n}{n^2}$$

is continuous for each $n \in \mathbb{N}$ because for each $n \in \mathbb{N}$ $h_n(x)$ is just a polynomial which is continuous. Hence, we must have that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is continuous on $[-1, 1]$. ■

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{n} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-test to prove that f is continuous at x_0.

Proof. Fix x_0, then consider any $|x_0| < 1$. Using the Weierstrass M-test, we will find that the terms

$$\left| \frac{x_0^n}{n} \right| \leq |x_0^2|^n$$

forms a geometric series for which the right side will converge implying that the series of the left side will converge. ■

Exercise 6.4.7

Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

Proof. We will use the term-by-term differentiability theorem to show that

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

is differentiable. We need to show that (f_k) is differentiable for every $k \in \mathbb{N}$, $\sum_{k=1}^{\infty} f_n'(x)$ converges uniformly to some $g(x)$ in $\mathbb{R}$, and that for some $x_0 \in [a,b] \subseteq \mathbb{R}$ that $\sum_{k=1}^{\infty} f_k(x_0)$ converges to $f(x_0)$.

Note that for any $k \in \mathbb{N}$, we know that $f_k(x)$ is differentiable since $\sin(kx)$ is a differentiable function for all $x \in \mathbb{R}$. Now we compute $f_n'(x)$ which results in

$$f_n'(x) = \frac{\cos(kx)}{k^2}.$$

Since $|\cos(kx)| \leq 1$, we can state that

$$\sum_{k=1}^{\infty} \left| \frac{\cos(kx)}{k^2} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{1}$$

Since the p-series on the right of (1) converges, we know that

$$\sum_{k=1}^{\infty} f_k'(x)$$

must converge uniformly to some $g(x)$ on $\mathbb{R}$. By the same reasoning, we can show that $\sum_{k=1}^{\infty} f_n(x)$ converges for all $x \in \mathbb{R}$; that is, since $|\sin(kx)| \leq 1$, we know that

$$\sum_{k=1}^{\infty} \left| \frac{\sin(kx)}{k^3} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$$

where the p-series on the right converges which implies that $\sum_{k=1}^{\infty} f_n(x)$ converges. By the Term-by-term differentiability theorem, we know that

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

must converge uniformly to a differentiable function $f(x)$. Furthermore, we know that $f'(x)$ is continuous since f_n' for all $n \in \mathbb{N}$ is continuous and

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges uniformly on $\mathbb{R}$. ■

(b) Can we determine if f is twice-differentiable? We claim that f is not twice-differentiable. If we compute $f"_n(x)$ and end up with the series

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

, then the series above fails the Weierstrass M-test since the constant series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges (this is a harmonic series).

Exercise 6.4.8

Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Proof. The function f is defined on every $x \in \mathbb{R}$. We first claim that f is continuous. We do this by showing that

$$\sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}$$

converges uniformly and that each f_n is continuous. Since $|\sin(k/x)| \leq |kx|$, we know that

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k/x)}{k} \right| \leq \sum_{k=1}^{\infty} \frac{|x/k|}{k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} 1/k^2$ converges, we know that

$$\sum_{k=1}^{\infty} f_n(x)$$

must converge uniformly. Since $\sin(x)$ is continuous (trig functions are continuous), we know that $f(x)$ must be a continuous function.

We clam that f is also differentiable. To do this we show that

$$\sum_{k=1}^{\infty} f_n(x_0)$$

converges to some function $f(x_0)$ and then show that

$$\sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}$$

converges uniformly where $f_n'(x) = \cos(x/k)/k^2$. Using the same process to show that $\sum_{k=1}^{\infty} f_n$ converges but only for some $x_0 \in (a, b)$, we know that

$$\sum_{k=1}^{\infty} f_n(x_0) = f(x_0).$$

Now to show that $\sum_{k=1}^{\infty} f_n'(x)$ converges uniformly to some $\ell(x)$ in $\mathbb{R}$, we can see that $|\cos(x/k)| \leq 1$ such that

$$\sum_{k=1}^{\infty} \left| \frac{\cos(k/x)}{k^2} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since the series on the right of the above converges (p-series), we have that $\sum_{k=1}^{\infty} f_n'(x)$ must converge uniformly using the Weierstrass M-test. Since (f_n) differentiable and $\sum_{k=1}^{\infty} f_n'(x)$ converges uniformly to some $\alpha(x)$ on $\mathbb{R}$, we know that $f(x)$ must be differentiable.

We can use the same process above to show that f is twice-differentiable. ■

Exercise 6.4.9

Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.

Proof. Note that (h_n) is a sequence of rational functions which are continuous. Our goal is to show that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly where

$$h_n(x) = \frac{1}{x^2 + n^2}$$

differentiable for all $n \in \mathbb{N}$. Let $x \in \mathbb{R}$ be arbitrary. Since $h_n(x)$ reaches a maximum at $x = 0$, we know that $h_n(x) \leq 1/n^2$. Thus,

$$\sum_{n=1}^{\infty} \left| \frac{1}{x^2 + n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and the fact that $\sum_{n=1}^{\infty} 1/n^2$ is a p-series which converges, we know that $\sum_{n=1}^{\infty} h_n(x)$ must converge uniformly to $h(x)$. This means that h must be continuous on $\mathbb{R}$ by the Term-by-term Continuity Theorem. ∎

(b) Is h differentiable? If so, is the derivative function h' continuous?

Proof. We claim that h is a differentiable function. First we compute $h_n'(x)$. Using our differentiation rules, we arrive at

$$h_n'(x) = \frac{-2x}{(x^2 + n^2)^2}.$$

Our goal is to show that

$$\sum_{n=1}^{\infty} h_n'(x)$$

converges uniformly to some $\ell(x)$ in $\mathbb{R}$. Since $h_n(x)$ is also secondly differentiable, we can use the Interior Extremum Theorem to find the points on $\mathbb{R}$ such that $h"(x) = 0$ such that $h_n'(x)$ is at its maximum. Differentiating $h_n'(x)$ again, we arrive arrive at

$$h_n"(x) = \frac{6x^2 - 2n^2}{(x^2 + n^2)^3}.$$

Setting $h_n''(x_0) = 0$ produces the following values where $h'(x)$ reaches its extrema:

$$x_0 = \pm \frac{n}{\sqrt{3}}.$$

Plugging this point into $h_n'(x)$ where

$$h_n'(x_0) = \frac{9}{8\sqrt{3}n^3}.$$

Hence, we can bound $h_n'(x)$ by this value. So we have that

$$\sum_{n=1}^{\infty} |h_n"(x)| \leq \sum_{n=1}^{\infty} \frac{9}{8\sqrt{3}n^3}.$$

Since the series on the right side of the above inequality converges, we know that by the Weierstrass M-test that

$$\sum_{n=1}^{\infty} h_n'(x) = \ell(x).$$

Furthermore, since $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly for all $x \in \mathbb{R}$, we know that $h(x)$ must be twice-differentiable.

∎

6.4 Power Series

Exercise 6.5.1

Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

Proof. ■

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Proof. ■

Exercise 6.5.3

Use the Weierstrass M-test to prove Theorem 6.5.2.

Proof. Suppose a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0. Then the series $\sum_{n=0}^{\infty} |a_n x_0^n|$ converges. Let $x \in [-c, c]$ where $c = |x_0|$. We proceed via the Weierstrass M-test to show that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly. We observe that

$$|a_n x^n| \leq a_n c^n = a_n |x_0|^n = a_n |x_0^n|.$$

This tells us that

$$\sum_{n=0}^{\infty} |a_n x^n| \leq \sum_{n=0}^{\infty} a_n |x_0^n|.$$

Since the right side of the above inequality converges, we know that $\sum_{n=0}^{\infty} a_n x^n$ must converge uniformly on any $x \in [-c, c]$. ■

Exercise 6.5.3 (Term-by-term Antidifferentiation).

Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$, find a power series representation for g.

Proof. First we show that $F(x)$ converges uniformly on $(-R, R)$. Since $\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$, we know that the sequence of partial sums must be bounded by some $M > 0$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for any $n > m \geq N$, we have that

$$\begin{aligned}
\left| \sum_{k=m+1}^{n} \frac{a_k}{k+1} x^{k+1} \right| &= \left| \sum_{k=m+1}^{n} (a_k x^k) \left(\frac{x}{k+1} \right) \right| \\
&\leq M \left(\frac{x}{m+2} \right) \\
&< M \left(\frac{R}{m+2} \right) \\
&< \varepsilon
\end{aligned}$$

by Abel's Lemma. Since the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges on $(-R, R)$, we know that $F(x)$ must be continuous on $(-R, R)$ and differentiable by Theorem 6.5.6. Hence, we can differentiate according to the formula in Theorem 6.5.6 to write that

$$F'(x) = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

■

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Proof. Suppose g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$; that is,

$$g'(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Since $f(x) = F'(x)$ and the fact that Antiderivatives are not unique, we can write

$$g(x) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

where c is some constant. ■

Exercise 6.5.5

(a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

Proof. Let x satisfy $0 < s < 1$. Let us proceed by inducting on n where $P(n)$ is statement that for every $n \geq 1$, we have that ns^{n-1} is bounded. Let our base case be $n = 1$. Then using the fact that $0 < s < 1$, we have that $0 < ns^{n-1} < 1$. Now suppose $P(n)$ holds for every $n \leq k-1$. We want to show that $P(n)$ holds for the kth case. Observe that

$$(k-1)s^{k-2} = \frac{ks^k - s^k}{s^2}$$

which means that for any $0 < s < 1$, we have

$$ks^k \leq s^2(k-1) + s^k < (k-1) + 1 = k.$$

Hence, we have that ns^{n-1} is bounded for all $n \geq 1$.

■

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Proof. Let $x \in (-R, R)$ be arbitrary. Pick $t \in (-R, R)$ such that $|x| < t < R$. By using this bound on t, we can write the following

$$|na_n x^{n-1}| = |nx^{n-1} \cdot a_n| \leq |nt^{n-1} \cdot a_n|.$$

From part (a), we know that $0 < s < 1$ implies that ns^{n-1} is bounded. Hence, there exists $M > 0$ such that

$$nt^{n-1} = ns^{n-1}\left(\frac{t}{s}\right)^{n-1} \leq M\left(\frac{t}{s}\right)^{n-1}.$$

Using the Weierstrass M-test, we can see that the bound above forms the following series

$$\sum_{n=1}^{\infty} Ma_n \left(\frac{t}{s}\right)^{n-1}. \tag{1}$$

Since $\sum_{n=0}^{\infty} a_n x^n$ converges on (-R, R), we know that (1) must converge and thus

$$\sum_{n=0}^{\infty} |n a_n x^{n-1}|$$

must also converge. ■

Exercise 6.5.6

Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for all } |x| < 1.$$

Use the results about power series in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.

Proof. ■

Exercise 6.5.7

Let $\sum a_n x^n$ be a power series with $a_n \neq 0$ and assume

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all $x \in (-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful).

Proof. Since

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

where $L \neq 0$, we know that the sequence (a_{n+1}/a_n) must be bounded by some $M > 0$; that is, $|\frac{a_{n+1}}{a_n}| \leq M$. This implies that

$$|a_{n+1}| \leq M|a_n|.$$

■

Now choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have that

$$|a_N x^N| \leq M|a_n||x|^n. \tag{1}$$

Since $L \neq 0$ and $|x| < 1/L$, we must have that

$$M|a_n||x|^n < M a_n \left(\frac{1}{L}\right)^n. \tag{2}$$

Since the right side of (2) forms the geometric series

$$\sum M a_n \left(\frac{1}{L}\right)^n,$$

we know that

$$\sum a_n x^n$$

must converge absolutely by the Weierstrass M-test and hence it must converge uniformly for every $x \in (-1/L, 1/L)$.

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.

Proof. Let $L = 0$ and let $x \in \mathbb{R}$ be arbitrary. Since $L < 1$, we can $r' \in (L, 1)$ such that

$$|a_n| \leq |a_N|(r')^n.$$

Using the Weierstrass M-test, we know that $r'x < 1$ must imply that

$$|a_n x^n| \leq |a_N|(r'x)^n \leq |a_N|.$$

Since $L < 1$, we know that the series

$$\sum_{N=1}^{\infty} |a_N|$$

converges and hence the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

must also converge for any $x \in \mathbb{R}$.

■

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\}.$$

(General properties of the *limit superior* are discussed in Exercise 2.4.7.)

Proof. Let $L' \neq 0$. Since $s_n \to L'$, we know that s_n must be bounded. Hence, there exists some $M > 0$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Let $x \in (-1/L', 1/L)$. Using the Weierstrass M-test, we can bound $\frac{a_{n+1}}{a_n} x^n$ by its supremum (i.e the sequence s_n) such that

$$\begin{aligned} \left|\frac{a_n}{a_N} \cdot x^n\right| &\leq |s_n||x|^n \\ &\leq M\Big(\frac{1}{L'}\Big)^n. \end{aligned}$$

Since the last inequality forms the following geometric series

$$\sum M\Big(\frac{1}{L'}\Big)^n.$$

Hence, the power series

$$\sum a_n x^n$$

must converge on $(-1/L', 1/L')$. To show that the power series converges for all $x \in \mathbb{R}$, we can employ the same process in part (b) since $\lim |a_{n+1}/a_n| = \lim s_n$. ■

Exercise 6.5.8

(a) Show that power series representation are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n \in \mathbb{N}$.

Proof. Since

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \tag{1}$$

for all $x \in (-R, R)$, we know that the series above must also be differentiable and follow the formula below

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n b_n x^{n-1} \tag{2}$$

by Theorem 6.5.6. If $n = 0$, we see that (1) implies

$$a_0 = b_0.$$

Now suppose $n = 1$, then (2) implies that

$$a_1 = b_1.$$

An inductive argument can be used to show that for every $n \geq 0$ with n corresponding to the nth derivative of f, we can show that $a_n = b_n$ for all $n \geq 0$. ∎

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Proof. Assume $f'(x) = f(x)$. Since

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in (-R, R)$, we know that f must also be differentiable and satisfy the following formula

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Furthermore, we know that f is *infinitely differentiable*. Since $f'(x) = f(x)$ we can change indices of the power series representation of $f'(x)$ so that we may use the proposition from part (a) to have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

implies $a_n = b_n$ for all $n \geq 0$ where

$$b_n = (n+1) a_{n+1}.$$

Note that $f(0) = 1$ which implies that $a_0 = a_1$ since $a_n = (n+1)a_{n+1}$ for all $n \geq 0$. Since each a_n is defined recursively, we know that for $n = 2$ that

$$a_2 = \frac{a_1}{2} = \frac{1}{1 \cdot 2}.$$

For $n = 3$, we apply the same reasoning to get

$$a_3 = \frac{a_2}{3} = \frac{1}{1 \cdot 2 \cdot 3}.$$

Likewise, $n = 4$ gives us

$$a_4 = \frac{a_3}{4} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}.$$

Continuing the pattern for $n \geq 0$, we arrive at the form of a_n which is just

$$a_n = \frac{f(0)}{n!}.$$

∎

Exercise 6.5.10

Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $(x_n) \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbb{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

Proof. Since the series

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

converges on $(-R, R)$; that is, uniformly for any compact set $K \subseteq (-R, R)$ and the fact that $g_n(x) = b_n x^n$ is continuous for all $n \geq 0$, we know by the Continuity theorem that $g(x)$ must be a continuous function. Since $(x_n) \to 0$ with $x_n \neq 0$ and the fact that $g(x_n) = 0$ for all $n \in \mathbb{N}$, we know that $g(x_n) \to 0$. But this is equivalent to saying that

$$\lim_{x \to 0} g(x) = 0 = g(0).$$

But we need to show that $g(x) = 0$ for all $x \in (-R, R)$. Since $g(x)$ converges on $(-R, R)$, we know that $g(x)$ must be differentiable and must follow

$$g'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1}.$$

Since $g(x_n) = 0$ for all $n \in \mathbb{N}$, it must also follow that the differentiability of g implies that $g'(x_n) = g(x_n) = 0$. Let $i \geq 0$ be the ith derivative of g, (we know this holds since g is infinitely differentiable). Hence, part (a) implies that for every $i \geq 0$, we must have

$$g^i(x_n) = g^{i+1}(x_n) = 0$$

From part (a) of exercise 6.5.8, we know that the for all $i \geq 0$ and the fact that $g(x_n) = 0$ for all $n \in \mathbb{N}$, $b_i = b_{i+1} = 0$

$$b_i = b_{i+1}$$

for all $i \geq 0$. Hence, for every b_n must be zero showing that for any $x \in (-R, R)$ that $g(x) = 0$. ■

Exercise 6.5.11

A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable* to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1^-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

Proof. Let $\varepsilon > 0$. To show that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges to L, we will use the Cauchy Criterion and Abel's lemma. Note that since $x \in [0, 1)$, x^k forms a monotonically decreasing sequence that contains terms that are greater than or equal to zero. Suppose there exists an $N \in \mathbb{N}$ such that for any $n > n \geq N$, then by Abel's Lemma

$$\begin{aligned} \left| \sum_{k=m+1}^{n} a_k x^k \right| &\leq \varepsilon x^k \\ &< \varepsilon. \end{aligned}$$

Hence, we must have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converge to L uniformly for all $x \in [0, 1)$. Since each term of $\sum_{n=0}^{\infty} a_n x^n$ is continuous and $f(x)$ converges uniformly for all $[0, 1]$, we know that $f(x)$ is continuous on this interval. This means that the functional limit $\lim_{x \to 1^-} f(x) = L$ exists and furthermore $f(1) = L$. ■

(b) Show that $\sum_{n=0}^{\infty} (-1)^n$ is Abel-summable and find the sum.

Proof. ■

6.5 Taylor Series

Exercise 6.6.1

The derivation in Example 6.6.1 shows the Taylor series for $\arctan(x)$ is valid for all $x \in (-1, 1)$. Notice, however, that the series also converges when $x = 1$. Assuming that $\arctan(x)$ is continuous, explain why the value of the series at $x = 1$ must necessarily be $\arctan(1)$. What interesting identity do we get in this case?

Proof. We know the series equals the value of $\arctan(1)$ at $x = 1$ because we know that the term-by-term antidifferentiation of the series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \tag{1}$$

that converges for all $x \in (-1, 1)$ produces the series

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{2}$$

that also converges for $x \in (-1, 1)$. Since we have convergence of (2) for all $x \in (-1, 1)$ and the fact that each term in the series representation of (2) is continuous shows that at $x = 1$, we must have equality of the series representation to $\arctan(1)$. This leads us to the interesting identity that

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

■

Exercise 6.6.2

Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

(a) $x\cos(x^2)$

Proof. For this problem, we will use the $\frac{d}{dx}$ notation to compute our derivatives. We know that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Since $\sin(x)$ is an infinitely differentiable function, we can take the derivative of its Taylor expansion. Hence, we have

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \frac{d}{dx}\Big[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\Big] \\
&= \sum_{n=0}^{\infty} \frac{d}{dx}\Big[\frac{(-1)^n}{(2n+1)!} x^{2n+1}\Big] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\end{aligned}$$

This means that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Substituting $x = x^2$ and multiplying by x gives us

$$x\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+1}$$

which holds for any $x \in (-R, R)$ where $R > 0$. ∎

(b) $x/(1+4x^2)^2$

Proof. Take the function $f(x) = \frac{1}{1-x}$ which has the following Taylor Expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which is defined for all $|x| < 1$. Taking the derivative of $f(x)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$$

which has a Taylor series expansion of

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Letting $x = -4x^2$ and multiplying by x, we arrive at the following Taylor series expansion

$$\frac{x}{(1+4x^2)^2} = \sum_{n=1}^{\infty} (-4)^{n-1} nx^{2n-1}$$

which holds for all $x \in (-R, R)$. ∎

(c) $\log(1+x^2)$

Proof. Our goal is to use the Term-by-term Anti-differentiation to write a Taylor series expansion for $\log(1+x^2)$. Set $F(x) = \log(1-x)$ and $f(x) = \frac{1}{1-x}$. Since the series expansion for $f(x)$; that is,

$$f(x) = \sum_{n=0}^{\infty} x^n$$

is defined for any $|x| < 1$, we have that

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

which satisfies $F'(x) = f(x)$. Letting $x = -x^2$ we get that

$$\log(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{2n+2}.$$

∎

Exercise 6.6.3

Derive the formula for the Taylor Coefficients given in Theorem 6.6.2.

Proof. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$ Since f is infinitely differentiable by Theorem 6.5.7, we can take derivatives of f where $f^n(0) = n! a_n$ implies that

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

∎

Exercise 6.6.4

Explain how Lagrange's Remainder Theorem can be modified to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log(2).$$

Proof. Let $f(x) = \log(1+x)$. Since f is $N+1$ differentiable on $(0, R)$, there exists a point c satisfying $|c| < 1$ where the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

by Lagrange's Remainder Theorem. Our goal is to show that the error function $E_N(1)$ converges uniformly to 0 on $(0, R)$. Hence, we have

$$|E_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} \right|$$

where $f^{(N+1)}(c) = N!/(1+c)^{N+1}$ implies

$$|E_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} \right| = \frac{1}{(N+1)(1+c)^{N+1}} < \frac{1}{N+1} \to 0$$

uniformly on $(0, R)$. Hence, we have that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log(2).$$

■

Exercise 6.6.5

(a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

Proof. To generate the Taylor coefficients for $f(x) = e^x$ we can just use the formula given to us via Theorem 6.6.2 and the fact that $f^{(n)}(0) = f^{(n+1)}(0)$ for all $n \geq 0$ where $f^{(n)}(0) = e^0 = 1$, to write

$$a_n = \frac{f(0)}{n!} = \frac{1}{n!}.$$

Hence, we can define the following power series

$$S_N(x) = \sum_{N=0}^{\infty} \frac{x^N}{N!}.$$

To show that $S_n(x) \to f(x)$ where $f(x) = e^x$, we will use Lagrange's Remainder theorem. Given an $x \in (-R, R)$ non-zero, suppose there exists a point c satisfying $|c| < |x|$ such that $E_N(x) = e^x - S_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)x^{N+1}}{(N+1)!} = \frac{e^c x^{N+1}}{(N+1)!}.$$

Since $x \in [-R, R]$, we can produce the following bound

$$|E_N(x)| = \left| \frac{e^c x^{N+1}}{(N+1)!} \right| \leq \frac{e^c R^{N+1}}{(N+1)!}.$$

Since the term on the right side converges to zero uniformly on $(-R, R)$, we know that $E_N(x) \to 0$ which means that $S_N(x) \to e^x$ uniformly on $(-R, R)$. ■

(b) Verify the formula $f'(x) = e^x$.

Proof. If we take the Taylor Expansion of e^x which is defined for all $x \in [-R, R]$, we differentiate

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

via Theorem 6.5.7 to get the following series representation

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

We can reorder our indices to get

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

■

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Proof. To generate the series for e^{-x}, let $x = -x$. Then

$$f(-x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

To attain the Taylor expansion of $e^x \cdot e^{-x}$ we can multiply the two series together. Since we are just collecting powers of x, we can try and form a formula for the summation of the coefficients of the Taylor expansions of e^x and e^{-x}. By using the formula from section 2.7 where

$$\sum_{i=0}^{n} d_k$$

with

$$d_n = \sum_{i=0}^{n} a_i b_{i-n}.$$

Then we have

$$\begin{aligned} e^x \cdot e^{-x} &= \Big(\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big)\Big(\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!}\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} \frac{(-1)^{n-i}}{i!(n-i)!}\Big) x^n. \end{aligned}$$

■

Exercise 6.6.6

Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

introduced at the end of this section.

(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find g"(0).

Proof. Since $x \neq 0$ we can use the Chain Rule to get

$$g'(x) = \frac{2}{x^3} e^{-1/x^2}.$$

Next, we compute $g"(0)$ using the definition of the derivative. Then we have

$$\begin{aligned} g"(0) &= \lim_{x \to 0} \frac{g'(x) - g'(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{2/x^4}{e^{1/x^2}} \end{aligned}$$

Since we have a limit that produces an ∞/∞ situation, we can use L'hopital to get

$$\lim_{x \to 0} \frac{4}{x^2 e^{1/x^2}}.$$

But this in itself causes another ∞/∞ situation as $x \to 0$. Hence, we can use L'Hopital's rule again to get

$$\lim_{x \to 0} \frac{4}{e^{1/x^2}}$$

which goes to zero as $x \to 0$. Hence, we have that $g"(0) = 0$. ■

(b) Compute $g"(x)$ and $g^{(3)}(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)}(x)$ at points different from zero.

Proof. Since we have $g^{(1)}(x) = 2e^{-1/x^2}/x^3$, we can compute $g^{(2)}(x)$ for $x \neq 0$ using our differentiation rules. Hence, we have

$$\begin{aligned} g^{(2)}(x) &= \frac{2e^{-1/x^2}}{x^6}(2 - 3x^2), \\ g^{(3)}(x) &= \frac{2e^{-1/x^2}}{x^9}(4 - 18x^2 + 12x^4). \end{aligned}$$

For our nth derivative, we have that for $x \neq 0$

$$g^{(n)}(x) = \begin{cases} \frac{2e^{-1/x^2}}{x^3} & \text{if } n = 1 \\ \frac{2e^{-1/x^2}}{x^{3n}} \sum_{i=0}^{n} (-1)^i a_i x^{2i} & \text{if } n > 1 \end{cases}$$

where $a_n \in \mathbb{R}$. ■

(c) Construct a general statement argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Proof. We can use induction to prove that $g^{(n)}(0) = g^{(n+1)}(0) = 0$ for all $n \geq 1$ to show that $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. We can compute the nth derivative of g by using the description outlined in part (b). Then use the definition of the derivative to produce a ∞/∞ so that we may use L'Hopital to show that $g^{(n)}(0) = g^{(n+1)}(0) = 0$. Hence, $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. ■

Exercise 6.6.8

Here is a weaker form of Lagrange's Remainder Theorem whose proof is arguably more illuminating than the one for the stronger result.

(a) First establish a lemma: If g and h are differentiable on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$, then $g(t) \leq h(t)$ for all $t \in [0, x]$.

Proof. Let $t \in [0, x]$. Since g and h are differentiable on $[0, x]$, we can use the Mean Value Theorem so that choosing $c \in (0, t)$ implies

$$\begin{aligned} g'(c) &= \frac{g(t) - g(0)}{t - 0}, \\ h'(c) &= \frac{h(t) - h(0)}{t - 0}. \end{aligned}$$

Since $c \in [0, x]$, we have $g'(c) \leq h'(c)$. But we have $g(0) = h(0)$ so

$$g'(c) \leq h'(c) \Leftrightarrow \frac{g(t) - g(0)}{t} \leq \frac{h(t) - h(0)}{t}.$$

Multiplying through by t and adding $g(0)$ from both sides (keeping in mind that $g(0) = h(0)$), we arrive at $g(t) \leq h(t)$. ■

(b) Let f, S_N, and E_N be as Theorem 6.6.3, and take $x \in (0, R)$. If $|f^{(N+1)}(t)| \leq M$ for all $t \in [0, x]$, show

$$|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}.$$

Proof. Let f, S_N, and E_N be as Theorem 6.6.3. Let $x \in (0, R)$ and let $x_{N+1} \in [0, x]$. By part (a), we know that

$$\frac{E_N(x)}{x^{N+1}} \leq \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}$$

where $x_{N+1} \in (0, x^N) \subseteq \cdots \subseteq (0, x)$ and

$$E_N^{(N+1)}(x_{N+1}) = f^{(N+1)}(x_{N+1}) - S_N^{(N+1)}(x_{N+1}).$$

Since $S_N^{(N+1)}(x) = 0$ past the nth derivative, we know that

$$E_N^{(N+1)}(x_{N+1}) = f^{(N+1)}(x_{N+1}).$$

Now set $x^{N+1} = t$. Since $|f^{(N+1)}(t)| \leq M$ by assumption, we must have

$$|E_N(x)| \leq \frac{|f^{(N+1)}(x_{N+1})|x^{N+1}}{(N+1)!} \leq \frac{Mx^{N+1}}{(N+1)!}.$$

■

Exercise 6.6.9 (Cauchy's Remainder Theorem).

Let f be differentiable $N + 1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x - a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

Proof. Letting $a = x$, we have

$$\begin{aligned} S_N(x, x) &= c_0 \\ &= \frac{f^{(0)}(x)}{0!} \\ &= f^{(0)}(x) \end{aligned}$$

which implies that

$$E_N(x, x) = f(x) - f(x) = 0.$$

■

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E_N'(x, a) = \frac{-f^{(N+1)}(a)}{N!}(x - a)^N.$$

Proof. Since $E_N(x,a) = f(x) - S_N(x,a)$, differentiating with respect to a leads to $E_N'(x,a) = -S_N'(x,a)$. Then

$$\begin{aligned}
-S_N'(x,a) &= -\Big[\sum_{n=1}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n - \frac{f^{(n)}(a)}{(n-1)!}x^{n-1}\Big] \\
&= \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} - \sum_{n=1}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= \sum_{n=0}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n - \sum_{n=1}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= -\frac{f^{(N+1)}(a)}{n!}(x-a)^n.
\end{aligned}$$

Hence, we have that

$$E_N'(x,a) = -\frac{f^{(N+1)}(a)}{n!}(x-a)^n.$$

■

(c) Show

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some $c \in (0,x)$. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Proof. Using the Mean Value Theorem, there exists a $c \in (0,x)$ such that

$$E_N'(x,c) = \frac{E_N(x) - E_N(x,0)}{x-0}.$$

But note that $E_N'(x,c) = 0$ since $E_N(x) - E_N(x,0) = 0$ by part (a). Furthermore, we know that part (b) must also imply that

$$E_N'(x,c) = -\frac{f^{(N+1)}(c)}{N!}(x-c)^N.$$

So we must have

$$\frac{f^{(N+1)}(c)}{N!}(x-c)^N = \frac{E_N(x) - E_N(x,0)}{x} = 0$$

which some algebraic manipulation leads us to our desired conclusion

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x.$$

■

Exercise 6.7.1

Assuming WAT, show that if f is continuous on $[a,b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a,b]$.

Proof. Using the Weierstrass Approximation Theorem, we can let $\varepsilon = \frac{1}{n}$. By choosing an $N = 1/\varepsilon$ such that $n \geq N$, we can have a sequence of polynomials (p_n) such that

$$|p_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

■

Exercise 6.7.2

Prove Theorem 6.7.3.

Proof. We can partition the closed interval $[a, b]$ into

$$a = x_0 < x_1 < \cdots < x_n = b$$

where each subinterval is defined as $[x_{i-1}, x_i]$ where $i \in \mathbb{N}$. Since f is continuous over $[a, b]$ which is a compact set, we know that f must be uniformly continuous on $[a, b]$. Hence, f takes on a maximum and a minimum value on $[a, b]$. We can do this on each subinterval of $[a, b]$ where

$$f(x_{i-1}) \leq f(x) \leq f(x_i) \Leftrightarrow f(x) - f(x_{i-1}) \leq f(x_i) - f(x_{i-1}).$$

We can define $\phi(x)$ at the endpoints of $[a, b]$ to be linear as a way of interpolating between the endpoints of each subinterval. Then for any $x \in (a, b)$, let q be the largest segment endpoint that is less than x, and r be the following segment endpoint. Using the uniform continuity of f over $[a, b]$, we can choose $\delta > 0$ such that whenever $|x - q| < \delta$, we have

$$|f(x) - \phi(q)| \leq |\phi(q) - \phi(r)| < \varepsilon.$$

■

Exercise 6.7.3

(a) Find the second degree polynomial $p(x) = q_0 + q_1 + q_2x^2$ that interpolates the three points $(-1, 1)$, $(0, 0)$, and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.

Proof. Using the points given to us, we can set up a system of linear equations where

$$\begin{aligned} 1 &= q_0 - q_1 + q_2 \\ 0 &= q_0 \\ 1 &= q_0 + q_1 + q_2. \end{aligned}$$

Solving this set of equations gives us the coefficients

$$\begin{aligned} q_0 &= 0 \\ q_1 &= 0 \\ q_2 &= 1 \end{aligned}$$

which gives us the following interpolating quadratic polynomial

$$p(x) = x^2.$$

■

Exercise 6.7.4

Show that $f(x) = \sqrt{1-x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot \cdots \cdot 2n}$$

for $n \geq 1$.

Proof. Since $f(x) = \sqrt{1-x}$ is infinitely differentiable we can use Taylor's Formula

$$a_n = \frac{f^{(n)}(0)}{n!}$$

to produce the coefficients of the Taylor series of $f(x)$. Note that $a_0 = 1$ because $f(0) = \sqrt{1-0} = 1$. Taking the first derivative of f, we find that

$$f^{(1)}(x) = \frac{-(1-x)^{-1/2}}{2}$$

which produces the Taylor coefficient

$$a_1 = \frac{-1}{2}.$$

We can take the second derivative ($n = 2$)

$$f^{(2)}(x) = \frac{-1}{2}(1-x)^{-3/2}$$

which produces the Taylor Coefficient at $n = 2$

$$a_2 = \frac{f^{(2)}(0)}{2!} = \frac{-1}{4}.$$

For $n \geq 1$, we find that

$$f^{(n)}(x) = \frac{1}{2^n}(1-x)^{-(2n-1)/2}\prod_{i=1}^{n} 2i-3$$

where plugging in $x = 0$ yields

$$f^{(n)}(0) = \frac{1}{2^n}\prod_{i=1}^{n} 2i-3.$$

Then using Taylor's formula, we have for $n \geq 1$

$$\begin{aligned}
a_n &= \frac{f^{(n)}(0)}{n!} \\
&= \frac{1}{2^n n!}\prod_{i=1}^{n}(2i-3) \\
&= \frac{\prod_{i=1}^{n}(2i-3)}{\left(\prod_{i=1}^{n} 2\right)\left(\prod_{i=1}^{n} i\right)} \\
&= \prod_{i=1}^{n}\frac{2i-3}{2i}.
\end{aligned}$$

■

Our goal now is to show that the error function of $f(x) = \sqrt{1-x}$ for all $x \in [-1, 1]$ where

$$E_N(x) = f(x) - \sum_{n=0}^{N} a_n x^n$$

goes to 0 uniformly as $N \to \infty$. Normally, we can use Lagrange's Remainder Theorem to show that this is the case. But this is an unfruitful approach since fixing $x \in (0, 1]$ produces a situation where the max of $f(x)$ is largest at $x = c$ where $(x/1-x)^{N+1/2}$ grows exponentially to infinity whenever $x > 1/2$; that is

$$\begin{aligned}
E_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} \\
&= \frac{1}{(N+1)!}\left(\frac{-1\cdot 3\cdot 5\cdots(2N-1)}{2^{N+1}(1-c)^{(N+1)/2}}\right) \\
&= \left(\frac{-1\cdot 3\cdot 5\cdots(2N-1)}{2\cdot 4\cdot 6\cdots(2N+2)}\right)\left(\frac{x}{1-c}\right)^{(N+1)/2} x^{1/2}.
\end{aligned}$$

Exercise 6.7.5

(a) Follow the advice in Exercise 6.6.9 to prove the Cauchy form of the remainder:

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.

Proof. See exercise 6.6.9 for the solution. ∎

(b) Use this result to prove

$$\sqrt{1-x} = \sum_{n=0}^{N} a_n x^n$$

is valid for all $x \in (-1, 1)$.

Proof. Let $x \in (-1, 1)$. Using Cauchy's Remainder Theorem, there exists some $c \in (0, x)$ such that

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x.$$

Our goal is to make our error function $E_N(x) \to 0$. Note that

$$f^{(N+1)}(c) = \frac{(1-c)^{-(2N+1)/2}}{2^{N+1}} \prod_{n=1}^{N+1} 2n - 3.$$

Then using Cauchy's Remainder Theorem, we have

$$\begin{aligned}
|E_N(x)| &= \left| \frac{\prod_{n=1}^{N+1} 2n-3}{2^{N+1} N!} (1-c)^{-(2N+1)/2} (x-c)^N x \right| \\
&= \left| \frac{\prod_{n=1}^{N+1} 2n-3}{2 \cdot \prod_{n=1}^{N} 2n} (1-c)^{-(2N+1)/2} (x-c)^N x \right| \\
&< \left| \frac{\prod_{n=1}^{N+1} 2n-3}{2 \cdot \prod_{n=1}^{N} 2n} (1-c)^{-(2N+1)/2} (1-c)^N \right| \\
&= \frac{|\prod_{n=1}^{N+1} 2n-3|}{2 \cdot \prod_{n=1}^{N} 2n} (1-c)^{-1/2} \\
&< \frac{(1-c)^{-1/2}}{\prod_{n=1}^{N} 2n} \to 0.
\end{aligned}$$

Hence, we conclude that $E_N(x) \to 0$ which implies that

$$\sqrt{1-x} = \sum_{n=0}^{N} a_n x^n$$

is valid for all $x \in (-1, 1)$. ∎

Exercise 6.7.6

(a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$. Show $c_n < \frac{2}{\sqrt{2n+1}}$.

Proof. We proceed by induction to show

$$c_n < \frac{2}{\sqrt{2n+1}} \tag{1}$$

for all $n \geq 1$. Let our base case be $n = 1$, then

$$c_1 = \frac{1}{2} < \frac{2}{\sqrt{3}}.$$

Now let us assume that (1) holds for $n \geq 1$. Let us show that (1) holds for the $n+1$ case. Using the definition of c_n, observe that

$$\begin{aligned}
c_{n+1} &= \frac{1 \cdot 3 \cdot 5 \cdots 2n+1}{2 \cdot 4 \cdot 6 \cdots 2n+2} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)} \\
&= c_n \cdot \frac{2n+1}{2n+2} \\
&< \frac{2}{\sqrt{2n+1}} \cdot \frac{2n+1}{2n+2} \\
&= \frac{2n+1}{n+1\sqrt{2n+1}} \\
&< \frac{2}{\sqrt{2n^2+3n+1}} \\
&< \frac{2}{\sqrt{2n+3}}.
\end{aligned}$$

Hence, we conclude that c_n satisfies the inequality

$$c_n < \frac{2}{\sqrt{2n+1}}$$

for all $n \geq 1$. ■

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Exercise 6.7.4.

Proof. Our goal is to show that $\sum_{n=0}^{\infty} a_n$ converges absolutely where

$$a_n = \prod_{i=1}^{n} \frac{2i-3}{2i}$$

is the sequence of Taylor coefficients. Then observe that

$$\begin{aligned}
a_n &= \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{(2n-1)}{(2n-1)} \\
&= \frac{-1}{(2n-1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \\
&= -\frac{c_n}{2n-1}.
\end{aligned}$$

Since $c_n < \frac{2}{\sqrt{2n-1}}$ for $n \geq 1$, we can write

$$|a_n| = \frac{c_n}{2n-1} < \frac{2}{(2n-1)\sqrt{2n+1}}$$

which creates a series

$$\sum_{n=1}^{\infty} \frac{2}{(2n-1)\sqrt{2n+1}}$$

that converges via the root test. Hence, we have $\sum_{n=0}^{\infty} a_n$ converges absolutely. ■

(c) Carefully explain how this verifies that equation (1) holds for all $x \in [-1, 1]$.

Proof. Since $\sum_{n=0}^{\infty} a_n$ converges absolutely, we can use the Weirstrass M-test to show that

$$\sum_{n=0}^{\infty} a_n x^n$$

converges uniformly on $[-1, 1]$. Observe that for any $x \in [-1, 1]$, we have

$$|a_n x^n| \leq a_n.$$

Since the right hand side of the above inequality produces a series that converges absolutely (from part(b)), we know that the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

must converge uniformly on $[-1, 1]$. ■

Our goal is to find polynomials that approximate the absolute value function on an interval containing the non-differentiable point at the origin.

Exercise 6.7.7

(a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\varepsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \varepsilon$$

for all $x \in [-1, 1]$.

Proof. Let $\varepsilon > 0$. Note that $|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)}$ ha a series representation

$$\sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n \tag{1}$$

where

$$a_n = \prod_{i=1}^{n} \frac{2i - 3}{2i}$$

which holds for all $x \in [-1, 1]$. Since the right hand side of (1) is just a polynomial, we can set

$$q(x) = \sum_{n=0}^{\infty} a_n (1 - x^2)^n$$

such that

$$||x| - q(x)| = |\sqrt{1 - (1 - x^2)} - q(x)| < \varepsilon.$$

■

(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

Proof. Let $c = \max\{a, b\}$ and let $x \in [-1, 1]$. Then let $\varepsilon/c > 0$ such that there exists a polynomial $q(x)$ such that

$$\left| \left| \frac{x}{c} \right| - q\left(\frac{x}{c}\right) \right| < \frac{\varepsilon}{c}. \tag{1}$$

Then let $x \in [a,b]$ be arbitrary. Then multiplying by c on both sides of (1), we get that

$$\left|\left|\frac{x}{c}\right| - q\left(\frac{x}{c}\right)\right| < \frac{\varepsilon}{c} \Leftrightarrow \left||x| - c \cdot q\left(\frac{x}{c}\right)\right| < \varepsilon$$

where we have found the polynomial $c \cdot q(x/c)$. Hence, we have

$$||x| - q(x)| < \varepsilon$$

for all $x \in [a,b]$. ∎

Exercise 6.7.8

(a) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1,1]$.

Proof. We know that $h_a(x)$ can be uniformly approximated with a polynomial $q(x)$ since

$$|x-a| = \sqrt{(x-a)^2} = \sqrt{1-(1-(x-a)^2)} \tag{1}$$

is continuous on $[-1,1]$ and the fact that (1) contains a Taylor series representation on $[-1,1]$. Using the same techniques as exercise 6.7.7, we can show that given an $\varepsilon > 0$that we can find a polynomial $p(x)$ such that

$$|h_a(x) - q(x)| < \varepsilon.$$

∎

(b) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show that there exists constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1,1]$.

Proof. Define $\phi(x)$ at the endpoints of the interval $[-1,1]$. Then using polynomial approximation of $h_{a_n}(x)$, we can define the series

$$\sum_{i=1}^{n} b_{i-1} h_{a_{i-1}}(x)$$

where the coefficients b_0 and b_{n-1} for $n \geq 1$ can be found by

$$b_{i-1} = \frac{\phi(a_i) - \phi(a_{i-1})}{a_i - a_{i-1}} - b_{n-1}.$$
$$b_0 = \frac{\phi(a_1) - \phi(a_0)}{a_1 - a_0}.$$

Putting everything together, we define

$$\phi(x) = \phi(-1) + \sum_{i=1}^{n} b_{i-1} h_{a_{i-1}}(x).$$

∎

(c) Complete the proof of WAT for the interval $[-1,1]$, and then generalize to an arbitrary interval $[a,b]$.

Proof. Let $\varepsilon > 0$ and let f be a continuous function over the compact set $[-1,1]$. Since f is continous over $[-1,1]$, we know that f must also be a uniformly continuous function over $[-1,1]$. Our goal is to find some polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon.$$

Using the definition of the polygonal function $\phi(x)$ from above, we know that f can be approximated by $\phi(x)$. Hence, we have

$$|f(x) - \phi(x)| < \frac{\varepsilon}{2}.$$

Furthermore, $\phi(x)$ can also be uniformly approximated by $p(x)$ to get

$$|\phi(x) - p(x)| < \frac{\varepsilon}{2}.$$

Using the triangle inequality, we can write that

$$\begin{aligned}
|f(x) - p(x)| &= |f(x) - \phi(x) + \phi(x) - p(x)| \\
&\leq |f(x) - \phi(x)| + |\phi(x) - p(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

which holds for all $x \in [-1, 1]$. To extend this result for all $x \in [a, b]$, we can follow a similar approach to part (c) of Exercise 6.6.7. ■

Chapter 7

The Riemann Integral

7.1 The Definition of the Riemann Integral

Exercise 7.2.1

Let f be a bounded function on $[a, b]$, and let P be an arbitrary parititon of $[a, b]$. First, explain why $U(f) \geq L(f, P)$. Now, prove Lemma 7.2.6.

Proof. By definition, we know that

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

and $U(f, P)$, for any given partition $P \in \mathcal{P}$, is an upper bound of the set $\mathcal{L} = \{L(f, P) : P \in \mathcal{P}\}$. Since $U(f)$ is any upper bound of $\mathcal{L}$ and $L(f)$ is the *least upper bound* then, we know that $U(f) \geq L(f, P)$. Then given an arbitrary $\varepsilon > 0$, we can find some partition $P_\varepsilon \in \mathcal{P}$ such that

$$L(f) - \varepsilon < L(f, P_\varepsilon) \leq U(f)$$

which implies that

$$L(f) \leq U(f) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we know that $U(f) \geq L(f)$. ■

Exercise 7.2.3 (Sequential Criterion for Integrability)

(a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^\infty$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0,$$

and in this case $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

Proof. First we prove the forwards direction. We can find a *common refinement* where $Q = P_1 \cup P_2$ such that $P_1 \subseteq Q$ and $P_2 \subseteq Q$. Given that f is bounded, let $\varepsilon > 0$ such that the following statements can be made:

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2} \Leftrightarrow U(f, P_1) - U(f) < \frac{\varepsilon}{2}$$
$$L(f, P_2) > L(f) - \frac{\varepsilon}{2} \Leftrightarrow L(f) - L(f, P_2) < \frac{\varepsilon}{2}.$$

Now assume $L(f) = U(f)$. Then for some $N \in \mathbb{N}$, we assume that for any $n \geq N$, we have

$$\begin{aligned}
|U(f, P_n) - L(f, P_n)| &\leq |U(f, P_1) - L(f, P_2)| \\
&= |U(f, P_1) - U(f) + U(f) - L(f) + L(f) - L(f, P)| \\
&\leq |U(f, P_1) - U(f)| + |U(f) - L(f)| + |L(f) - L(f, P)| \\
&= |U(f, P_1) - U(f)| + |L(f) - L(f, P)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Now we shall prove the backwards direction. Choose $\varepsilon = 1/n$. Assume there exists a partition P_n such that for some $N \in \mathbb{N}$, we have that for any $n \geq N$

$$U(f) - L(f) \leq U(f, P_n) - L(f, P_n) < \frac{1}{n} \to 0.$$

But by Theorem 1.2.6, we have that $U(f) = L(f)$. Hence, f must be integrable. ■

(b) For each n, let P_n be the partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ will be useful.

Proof. Let $f(x) = x$. Then for each $n \in \mathbb{N}$, let $x_k = k/n$. Then using the formula $1 + 2 + 3 + \cdots + n = n(n + 1)/2$, we attain the formula for the *Upper Sum*

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \frac{k}{n}\Big(\frac{k}{n} - \frac{k-1}{n}\Big) \\
&= \sum_{k=1}^{n} \frac{k}{n^2} \\
&= \frac{n(n+1)}{2n^2} \\
&= \frac{n+1}{2n}.
\end{aligned}$$

Similarly, we can let $x_{k-1} = (k - 1)/n$ to attain the formula for the *Lower Sum*

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \frac{k-1}{n}\Big(\frac{k-1}{n} - \frac{k-2}{n}\Big) \\
&= \sum_{k=1}^{n} \frac{k-1}{n^2} \\
&= \frac{n^2 + n - 2}{2n^2}.
\end{aligned}$$

Hence, we have the following formulas:

$$\begin{aligned}
U(f, P_n) &= \frac{n^2 + n}{2n^2} \\
L(f, P_n) &= \frac{n^2 + n - 2}{2n^2}
\end{aligned}$$

■

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

Proof. To show that $f(x)$ integrable on $[0,1]$, all we need to show is

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

Indeed, we have

$$\begin{aligned}\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)] &= \lim_{n\to\infty}\frac{1}{n^2}\\ &= 0.\end{aligned}$$

Hence, $f(x)=x$ is integrable on $[0,1]$. For large n, we know that $U(f)\leq 1/2$ and $L(f)\geq 1/2$. Since $U(f)=L(f)$, we know that

$$\int_a^b f=\frac{1}{2}.$$

■

Exercise 7.2.4

Let g be bounded on $[a,b]$ and assume there exists a partition P with $L(f,P)=U(g,P)$. Is g necessarily continuous? Is it integrable? If so, what is the value of $\int_a^b g$.

Proof. Yes, g is continuous since it is just the constant function and constant functions are continuous. Let $\varepsilon>0$. It is integrable becuse there exists a partition P_ε such that $L(g,P_\varepsilon)=U(g,P_\varepsilon)$ implies that

$$U(g,P)-L(g,P)<\varepsilon.$$

Furthemore, we have $\int_a^b g=g(x_k)\Delta x_k$. ■

Exercise 7.2.5

Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n)\to f$ uniformly on $[a,b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Proof. Let $\varepsilon>0$. Since f_n is integrable on $[a,b]$, there exists a partition P_ε such that

$$U(f_n,P_\varepsilon)-L(f_n,P_\varepsilon)<\varepsilon.$$

Since $(f_n)\to f$ uniformly, we can bound the difference of the upper sums between f_n and f by choosing $N_1\in\mathbb{N}$ such that for any $n\geq N_1$, we have

$$U(f,P_\varepsilon)-U(f_n,P_\varepsilon)=(M_k-f_n(\ell_k))\sum_{k=1}^{n}\Delta x_k<\frac{\varepsilon}{3(b-a)}\sum_{k=1}^{n}\Delta x_k$$

for some $\ell_k\in[x_{k-1},x_k]$ and similarly choosing $N_2\in\mathbb{N}$ such that for any $n\geq N_2$

$$L(f_n,P_\varepsilon)-L(f,P_\varepsilon)=(f_n(s_k)-m_k)\sum_{k=1}^{n}\Delta x_k<\frac{\varepsilon}{3(b-a)}\sum_{k=1}^{n}\Delta x_k.$$

Furthermore, we must also choose $N_3\in\mathbb{N}$ such that for any $n\geq N_3$

$$|U(f_n,P_\varepsilon)-L(f_n,P_\varepsilon)|<\frac{\varepsilon}{3(b-a)}.$$

Then we can find an $N = \max\{N_1, N_2, N_3\}$ such that for any $n \geq N$

$$\begin{aligned} U(f, P_\varepsilon) - L(f, P_\varepsilon) &= U(f, P_\varepsilon) - U(f_n, P_\varepsilon) + U(f_n, P_\varepsilon) - L(f_n, P_\varepsilon) \\ &\quad + L(f_n, P_\varepsilon) - L(f, P_\varepsilon) \\ &< \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \varepsilon. \end{aligned}$$

■

Exercise 7.2.6

A *tagged partition* $(P, \{c_k\})$ is one where in addition to a partition P we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. The corresponding *Riemann Sum*,

$$R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

is discussed in Section 7.1, where the following definition is alluded to.

Definition 6. Riemann's Original Definition of the Integral A bounded function f is *integrable* on $[a, b]$ with $\int_a^b f = A$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \varepsilon.$$

Show that if f satisfies Riemann's definition above, then f is integrable in the sense of Definition 7.2.7.

Proof. Let $\varepsilon > 0$ and let $(P_n, \{c_k\})$ be a tagged partition satisfying $\Delta x_k < \delta_1$ for all k. It suffices to show that there exists some $\delta > 0$ such that our arbitrary tagged partition defined above must imply that

$$|U(f, P_n) - L(f, P_n)| < \varepsilon.$$

Let P_1 and P_2 be any two partitions of $[a, b]$. We can attain a common refinement by taking $P_1 \cup P_2 = P_n$ where $P_1 \subseteq P_n$ and $P_2 \subseteq P_n$. Suppose we take tag partitions P_1 and P_2 so that we may pick sampling points ℓ_k and d_k in $[x_{k-1}, x_k]$. Since f is integrable on $[a, b]$, there must exist $\delta_1, \delta_2 > 0$ such that for tagged partitions $(P_n, \{\ell_k\})$ and $(P_1, \{d_k\})$ satisfying $\Delta x_k < \delta_1$ and $\Delta x_k < \delta_2$ respectively, we must have

$$|U(f, P_n) - R(f, P_1)| < \frac{\varepsilon}{4}.$$

Similarly, pick a sampling point z_k in $[x_{k-1}, x_k]$ such that for some $\delta_3 > 0$, we can choose $(P_2, \{z_k\})$ satisfying $\Delta x_k < \delta_3$

$$|R(f, P_2) - L(f, P_n)| < \frac{\varepsilon}{4}.$$

Furthermore, the integrability of f also implies that

$$\begin{aligned} |R(f, P_1) - R(f, P_2)| &\leq |R(f, P_1) - A| + |A - R(f, P_2)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \end{aligned}$$

Hence, we can find $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$ such that for any partition $(P_n, \min\{z_k, \ell_k, d_k\})$ satisfying

$\Delta x_k < \delta$ for all k, we have

$$\begin{aligned}
|U(f,P_n) - L(f,P_n)| &\leq |U(f,P_n) - R(f,P_1)| + |R(f,P_1) - R(f,P_2)| \\
&+ |R(f,P_2) - L(f,P_n)| \\
&\leq |U(f,P_n) - R(f,P_1)| + |R(f,P_1) - A| \\
&+ |A - R(f,P_2)| + |R(f,P_2) - L(f,P_n)| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \varepsilon.
\end{aligned}$$

By the Integrability Criterion, f is integrable on $[a,b]$. ■

Exercise 7.2.7

Let $f : [a,b] \to \mathbb{R}$ be increasing on the set $[a,b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$. Show that f is integrable on $[a,b]$.

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is increasing on $[a,b]$. Since $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, we can take $\Delta x_k < \frac{\varepsilon}{b-a}$. Then let $\varepsilon > 0$ such that for every P_ε, we have

$$\begin{aligned}
U(f,P_\varepsilon) - L(f,P_\varepsilon) &= \sum_{k=1}^{n}(M_k - m_k)\Delta x_k \\
&= \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))\Delta x_k \\
&< \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) \\
&< \frac{\varepsilon}{b-a} \cdot b - a \\
&= \varepsilon.
\end{aligned}$$

■

7.2 Integrating Functions with Discontinuities

Exercise 7.3.1

Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0,1]$.

(a) Show that $L(f,P) = 1$ for every partition P of $[0,1]$.

Proof. By definition of $h(x)$, we know that for any $0 \leq x < 1$, we have $h(x) = 1$. If $x = 1$, then a quick calculation will produce $U(f,P) = 2$. Since $L(f,P) \leq U(f,P)$, we get that $L(f,P) = 1$ for every $0 \leq x < 1$. ■

(b) Construct a partition P for which $U(f,P) < 1 + 1/10$.

Proof. Let $P = \{0, 1 - \frac{1}{11}, 1 + \frac{1}{11}, 1\}$ and let our subintervals be $[0, 1 - \frac{1}{11}], [1 - \frac{1}{11}, 1 + \frac{1}{11}]$, and $[1+\frac{1}{10}, 1]$. Using the definition of $U(f,P)$, we write

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{3} M_k \Delta x_k \\ &= \left(1-\frac{1}{11}\right) + 2\left(\frac{2}{11}\right) - 2\left(\frac{1}{11}\right) \\ &= 1 + \frac{1}{11} \\ &< 1 + \frac{1}{10}. \end{aligned}$$

Hence, we have

$$U(f,P) < 1 + \frac{1}{10}.$$

■

(c) Given $\varepsilon > 0$, construct partition P_ε for which $U(f,P_\varepsilon) < 1+\varepsilon$.

Proof. Let $\varepsilon > 0$ and let $P = \{0, 1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}, 1\}$. Then using the definition of $U(f,P)$, we have

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{3} M_k \Delta x_k \\ &= \left(1-\frac{\varepsilon}{2}\right) + 2\varepsilon - 2\cdot\frac{\varepsilon}{2} \\ &= 1 + \frac{\varepsilon}{2} \\ &< 1+\varepsilon. \end{aligned}$$

Hence, we have

$$U(f,P) < 1+\varepsilon.$$

■

Exercise 7.3.3

Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} . \end{cases}$$

Show that f is integrable on $[0,1]$ and compute $\int_0^1 f$.

Proof. Let $\varepsilon > 0$. To show that f is integrable, it suffices to show $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$ for some partition P_ε of $[0,1]$. Note that $L(f,P) = 0$ for all $x \in [0,1]$. Let us construct P_ε by taking

$$P_\varepsilon = \left\{0, \frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon, 1\right\}.$$

Then taking the definition of $U(f,P_\varepsilon)$, we have that

$$\begin{aligned} U(f,P_\varepsilon) &= \sum_{k=1}^{3} M_k \Delta x_k \\ &= 0\left(\frac{1}{n} - \varepsilon\right) + 2\varepsilon + 0\left(1 - \frac{1}{n} - \varepsilon\right) \\ &= 2\varepsilon. \end{aligned}$$

Since $L(f, P_\varepsilon) = 0$, we have that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon - 0 < \varepsilon.$$

Hence, f is integrable on $[0, 1]$. Note that $U(f) = 0$ and hence, we must have

$$\int_0^1 f = 0.$$

■

Exercise 7.3.4

Let f and g be functions defined on (possibly different) closed intervals, and assume the range of f is contained in the domain of g so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that it is not the case that if f and g are integrable, then $g \circ f$ is integrable.

Proof. Let

$$g(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

and

$$f(x) = \begin{cases} 1/q & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}.$$

The function $g(x)$ is continuous everywhere except at 0 and f is continuous at every $x \in \mathbb{I}$. But the composition of these functions,

$$(g \circ f)(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}$$

which is nowhere continuous and hence not integrable on any interval. ■

Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.

(b) If f is increasing and g is integrable, then $g \circ f$ is integrable.

Proof. Let $f(x) = x^2$ and define

$$g(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Note that f is an increasing function while g is an integrable function. But note that

$$(g \circ f)(x) = \begin{cases} \frac{1}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is non-integrable. ■

(c) If f is integrable and g is increasing, then $g \circ f$ is integrable.

Proof. Let $\varepsilon > 0$ and P_ε be an arbitrary partition of an arbitrary closed interval $[a, b]$. Let us first define $U(g \circ f, P_\varepsilon)$ and $L(g \circ f, P_\varepsilon)$. So we have

$$U(g \circ f, P_\varepsilon) = \sum_{k=1}^{n} g(M_k)\Delta y_k$$
$$L(g \circ f, P_\varepsilon) = \sum_{k=1}^{n} g(m_k)\Delta y_k$$

Let $\varepsilon > 0$. Our goal is to show, using our partition P_ε,

$$U(g \circ f, P_\varepsilon) - L(g \circ f, P_\varepsilon) < \varepsilon.$$

Since g is an increasing function on $[a, b]$ and the range of f is contained in the domain of g, we have that g is a bounded function. But this means that we can create sequence (σ_k) where $\sigma_k = g(t_k)$. Since (σ_k) is increasing and bounded, we know it must be convergent and hence it must be Cauchy. So choose $N \in \mathbb{N}$ such that for any $k, k-1 \geq N$, we

$$\begin{aligned} |M'_k - m'_k| &= |g(t_k) - g(t_{k-1})| \\ &= |\sigma_k - \sigma_{k-1}| \\ &< \varepsilon \end{aligned}$$

where t_k and t_{k-1} are contained in the domain of g. Observe that by our properties of the upper and lower sum of f, we have

$$y_k - y_{k-1} \leq M_k - m_k \quad \text{for all } k.$$

Since f is integrable, we can choose $\varepsilon = b - a$ so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < b - a.$$

Putting everything together, we have

$$\begin{aligned} U(g \circ f, P_\varepsilon) - L(g \circ f, P_\varepsilon) &= \sum_{k=1}^{n}(M'_k - m'_k)\Delta y_k \\ &= \sum_{k=1}^{n}(M'_k - m'_k)\Delta y_k \cdot \frac{\Delta x_k}{\Delta x_k} \\ &= \frac{1}{b-a}\sum_{k=1}^{n}(M'_k - m'_k)\Delta y_k \Delta x_k \\ &\leq \frac{1}{b-a}\sum_{k=1}^{n}(M'_k - m'_k)(M_k - m_k)\Delta x_k \\ &< \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(M_k - m_k)\Delta x_k \\ &< \frac{\varepsilon}{b-a} \cdot b - a \\ &= \varepsilon. \end{aligned}$$

Hence, the composition $g \circ f$ is an integrable function on $[a, b]$ by Theorem 7.2.8. ■

Here is a correction of the above proof

Proof. Suppose f is an integrable function on $[a, b]$ and g is an increasing function on $[a, b]$. Let $\varepsilon > 0$. Then we shall show that for an arbitrary partition P_ε, we have

$$U(g \circ f, P_\varepsilon) - L(g \circ f, P_\varepsilon) < \varepsilon.$$

Let P_ε be an arbitrary partition. Since f is an integrable function, we know that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} [M_k - m_k]\Delta x_k < \varepsilon.$$

Note that the range of f is contained within the domain of g. Define $y_k = f(x_k)$. Since g is an increasing function, we can write

$$\begin{aligned} U(g \circ f, P_\varepsilon) - L(g \circ f, P_\varepsilon) &= \sum_{k=1}^{n} [M_k' - m_k']\Delta y_k \\ &= \sum_{k=1}^{n} [g(y_k) - g(y_{k-1})]\Delta y_k \cdot \frac{\Delta x_k}{\Delta x_k} \\ &= \frac{g(f(b)) - g(f(a))}{b-a} \sum_{k=1}^{n} \Delta y_k \Delta x_k \end{aligned}$$

By the properties of upper and lower sum of f, observe that for every k

$$\Delta y_k \leq M_k - m_k.$$

Then we just have

$$\frac{g(f(b)) - g(f(a))}{b-a} \sum_{k=1}^{n} \Delta y_k \Delta x_k \leq \frac{g(f(b)) - g(f(a))}{b-a} \sum_{k=1}^{n} [M_k - m_k]\Delta x_k$$

Using the assumption that f is integrable on $[a, b]$, we have that the right side of the inequality above leads to

$$U(g \circ f, P_\varepsilon) - L(g \circ f, P_\varepsilon) < \varepsilon.$$

Using Theorem 7.2.8, we must have $g \circ f$ integrable on $[a, b]$.

■

7.3 Properties of the Integral

Exercise 7.4.1

Let f be a bounded function on a set A, and set

$$M = \sup\{f(x) : x \in A\}, \ \ m = \inf\{f(x) : x \in A\},$$

$$M' = \sup\{|f(x)| : x \in A\} \ \text{ and } \ m' = \inf\{|f(x)| : x \in A\}.$$

(a) Show that $M - m \geq M' - m'$.

Proof. Observe that $f(x) \leq M' \leq M$ and likewise $m \leq m'$. Then we have

$$-m \geq -m' \Leftrightarrow M' - m \geq M' - m'. \tag{1}$$

Since $M' \leq M$, (1) implies

$$M - m \geq M' - m'.$$

■

(b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

Proof. Suppose f is integrable on $[a, b]$. Let $\varepsilon > 0$. Then there exists a partition P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} [M_k - m_k]\Delta x_k < \varepsilon.$$

By using part (a), we know that

$$\sum_{k=1}^{n} [M'_k - m'_k]\Delta x_k \leq \sum_{k=1}^{n} [M_k - m_k]\Delta x_k.$$

Then using the same partition P_ε that we found, we have

$$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Hence, we conclude that $|f|$ is an integrable function on $[a, b]$. ■

(b) Provide the details for the argument that in this case we have $|\int_a^b f| \leq \int_a^b |f|$.

Proof. Since $|f|$ integrable, we know that $U(|f|) = L(|f|) = \int_a^b |f|$. Likewise f being integrable implies $U(f) = L(f) = \int_a^b f$. Let $\varepsilon > 0$ and let P be a partition of $[a, b]$. Using the properties of the upper and lower integral, we must have

$$\begin{aligned}
\left| \int_a^b f \right| = |U(f)| &\leq |U(f, P)| \\
&< \Big| L(f, P) + \varepsilon \Big| \\
&\leq |L(f, P)| + \varepsilon \\
&\leq L(|f|, P) + \varepsilon \\
&\leq \int_a^b |f| + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we must have $|\int_a^b f| \leq \int_a^b |f|$. ■

Exercies 7.4.2

(a) Let $g(x) = x^3$, and classify each of the following as positive, negative, or zero.

$$\text{(i)} \quad \int_0^{-1} g + \int_0^1 g \qquad \text{(ii)} \quad \int_1^0 g + \int_0^1 g \qquad \text{(iii)} \quad \int_1^{-2} g + \int_0^1 g.$$

Proof. (i) zero, (ii) zero, (iii) positive ■

(b) Show that if $b \leq a \leq c$ and f is integrable on the interval $[b, c]$, then it is still the case that $\int_a^b f = \int_a^c f + \int_b^c f$.

Proof. Since f is integrable on the interval $[b, c]$, we have

$$\int_b^c f = \int_b^a f + \int_a^c f. \tag{1}$$

Rearranging (1), we have

$$-\int_b^a f = \int_a^c f - \int_b^c f.$$

By using Definition 7.4.1, we know that

$$-\int_b^a f = \int_a^b f \text{ and } -\int_b^c f = \int_c^b f.$$

Hence, we conclude that

$$\int_a^b f = \int_a^c f + \int_b^c f$$

■

Exercise 7.4.3

Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counter-example.

(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

Proof. Define

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ -1 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Notice that $|f|$ is integrable, but not f. ■

(b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.

Proof. We can use Thomae's function in the last section

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \quad \text{is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We see that $g(x) \geq 0$ and $g(x) > 0$ for an infinite number of points $x \in [a, b]$, but $\int_a^b g = 0$. ■

(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Proof. Since g is continuous on $[a, b]$ and $g(x) \geq 0$, we know that g must be integrable on $[a, b]$. Furthermore, g reaches its maximum and minimum on $[a, b]$ since $[a, b]$ is a compact interval. Hence, there exists at least one point y_0 such that $g(y_0)$ is the minimum of g on $[a, b]$. Since g is integrable on $[a, b]$, we have that

$$\begin{aligned} \int_a^b g = L(g) \geq L(g, P) \\ &= \sum_{k=1}^n m_k \Delta x_k \\ &\geq \sum_{k=1}^n g(y_0) \Delta x_k \\ &> 0. \end{aligned}$$

Hence, we conclude that $\int_a^b g > 0$. ■

Exercise 7.4.4

Show that if $f(x) > 0$ for all $x \in [a, b]$ and f is integrable, then $\int_a^b f > 0$.

Proof. Let $x \in [a, b]$. Since f is integrable on $[a, b]$, we have $U(f) = L(f) = \int_a^b f$. Let P be an arbitrary partition of $[a,b]$. Since $f(x) > 0$, we have

$$\int_a^b f = L(f) \geq L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k > 0$$

Hence, $\int_a^b f > 0$. ■

Exercise 7.4.5

Let f and g be integrable functions on $[a,b]$.

(a) Show that if P is any partition of $[a,b]$, then

$$U(f+g,P) \leq U(f,P) + U(g,P). \tag{1}$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

Proof. In exercise 1.3.6, we proved for any two sets $A, B \neq \emptyset$, we have

$$\sup(A+B) \leq \sup A + \sup B.$$

In this context, we have

$$A = \{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } B = \{g(x) : x \in [x_{k-1}, x_k]\}$$

with

$$A + B = \{f(x) + g(x) : x \in [x_{k-1}, x_k]\}.$$

Let P be any partition of $[a,b]$. Then observe that

$$\begin{aligned} U(f+g,P) &= \sum_{k=1}^{n} \sup_{x\in[x_{k-1},x_k]} (f+g)(x)\Delta x_k \\ &\leq \sum_{k=1}^{n} \sup_{x\in[x_{k-1},x_k]} f(x)\Delta x_k + \sum_{k=1}^{n} \sup_{x\in[x_{k-1},x_k]} g(x)\Delta x_k \\ &= U(f,P) + U(g,P). \end{aligned}$$

We have strict inequality whenever $f(x) > g(x)$ and the corresponding inequality to (1) is

$$L(f+g,P) \geq L(f,P) + L(g,P).$$

■

(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.

Proof. See part (i) of Theorem 7.4.2 in the section notes. ■

Exercise 7.4.6

Although not part of Theorem 7.4.2, it is true that the porduct of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(a) If f satisfies $|f(x)| \leq M$ on $[a,b]$, show

$$|(f(x))^2 + (f(y))^2| \leq 2M|f(x) - f(y)|.$$

Proof. Let $x, y \in [a,b]$. Since f satisfies $|f(x)| \leq M$ and $|f(y)| \leq M$, we have

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \leq 2M.$$

Multiplying the inequality above by $|f(x) - f(y)|$ yields

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

∎

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.

Proof. Let $\varepsilon > 0$. Since f is integrable on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \frac{\varepsilon}{2M}.$$

Then using the same partition P_ε and using part (a), we have that

$$\begin{aligned} U(f^2, P_\varepsilon) - L(f^2, P_\varepsilon) &= \sum_{k=1}^{n} [(M_k')^2 - (m_k')^2]\Delta x_k \\ &\leq 2M \sum_{k=1}^{n} [M_k' - m_k']\Delta x_k \\ &< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Hence, f^2 is also integrable on $[a, b]$. ∎

(c) Now show that if f and g are integrable, then fg is integrable. (Consider $(f + g)^2$).

Proof. Observe that

$$(f + g)^2 = f^2 + 2fg + g^2$$

and solving for fg yields

$$fg = \frac{1}{2}[(f + g)^2 - (f^2 + g^2)].$$

Since f and g are integrable, we know by part (i) of Theorem 7.4.2 that $f + g$ is integrable. Let $h = f + g$. Then by part (b), we have that h^2 is integrable as well as f^2 and g^2. By using part (i) and (ii) of Theorem 7.4.2, we find that fg is integrable. ∎

Exercise 7.4.8

For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n, \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that H is integrable and compute $\int_0^1 H$.

Proof. Notice that h_n is a sequence of continuous functions on the set $[0, 1] \setminus \{1/2^n\}$ and that $\sum_{n=1}^{\infty} h_n = 0$ uniformly. By the Integrability Limit Theorem, we conclude that H must be integrale and that $\int_0^1 H(x) = 0$. ∎

Exercise 7.4.9

Let g_n and g be uniformly bounded on $[0, 1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Assume $g_n \to g$ pointwise on $[0, 1]$ and uniformly on any set of the form $[0, \alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Proof. Let $\varepsilon > 0$. Our goal is to show

$$\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$$

which can be done by finding an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\Big|\int_0^1 g_n - \int_0^1 g\Big| < \varepsilon.$$

Suppose $g_n \to g$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$. Looking at g on $[0,\alpha]$, the uniform convergence $g_n \to g$ imply that there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$ and $x \in [0,\alpha]$, we have that

$$|g_n - g| < \frac{\varepsilon}{2\alpha}.$$

Utilizing the integrability of both g_n and g on $[0,1]$, we can also state that $g_n - g$ is integrable by Exercise 7.4.1 and hence, $|g_n - g|$. We can use the triangle inequality and Theorem 7.4.1, to write the following:

$$\begin{aligned}
\Big|\int_0^1 g_n - \int_0^1 g\Big| &= \Big|\int_0^1 g_n - g\Big| \\
&\leq \int_0^1 |g_n - g| \\
&= \int_0^\alpha |g_n - g| + \int_\alpha^1 |g_n - g|.
\end{aligned}$$

We can easily make the first term small by finding $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$\int_0^\alpha |g_n - g| < \frac{\varepsilon}{2\alpha} \cdot \alpha = \frac{\varepsilon}{2}.$$

To make the second term small, we can utilize the integrability of $|g_n - g|$ on $[0,1]$ to state that for any partition P of $[\alpha,1]$, we have

$$\begin{aligned}
\int_\alpha^1 |g_n - g| = U(|g_n - g|) &\leq U(|g_n - g|, P) \\
&= \sum_{k=1}^{n} \sup_{x \in [x_{k-1},x_k]} |g_n - g| \cdot \Delta x_k.
\end{aligned}$$

Since g_n and g are uniformly bounded by a single $M > 0$ satisying $|g(x)| \leq M$ and $|g_n(x)| \leq M$, we know that

$$\sup_{x \in [x_{k-1},x_k]} |g_n - g| \leq 2M.$$

Furthermore, utilizing pointwise convergence of $g_n \to g$ on $[\alpha,1]$, we can say there exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$

$$\sup_{x \in [\alpha,x_1]} |g_1 - g| < \frac{\varepsilon}{4(x_1 - \alpha)}.$$

Then we have

$$\begin{aligned}
\int_\alpha^1 |g_n - g| &\leq \sup_{x \in [\alpha,x_1]} |g_1 - g|(x_1 - \alpha) + \sum_{k=1}^{n} \sup_{x \in [x_{k-1},x_k]} |g_n - g|\Delta x_k \\
&< \frac{\varepsilon}{4(x_1 - \alpha)} \cdot (x_1 - \alpha) + 2M \cdot \frac{\varepsilon}{8M} \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Letting $N = \max\{N_1, N_2\}$, assuming that $n \geq N$, we have that

$$\begin{aligned}
\left|\int_0^1 g_n - \int_0^1 g\right| &= \left|\int_0^1 g_n - g\right| \\
&\leq \int_0^1 |g_n - g| \\
&= \int_0^\alpha |g_n - g| + \int_\alpha^1 |g_n - g| \\
&< \frac{\varepsilon}{2\alpha} \cdot \alpha + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, we conclude that

$$\lim_{n\to\infty} \int_0^1 g = \int_0^1 g.$$

■

Exercise 7.4.10

Assume g is integrable on $[0, 1]$ and continuous at 0. Show

$$\lim_{n\to\infty} \int_0^1 g(x^n)\ dx = g(0).$$

Proof. Let $\varepsilon > 0$. Since g is continuous at 0, we know that g is also integrable at 0 which means that $\int_0^1 g(0) = g(0)$. Furthermore, the continuity of g at 0 implies that there exists $\delta > 0$ and $N \in \mathbb{N}$ such that whenever $|x^n| < \delta$ and $n \geq N$, we have

$$|g(x^n) - g(0)| < \varepsilon.$$

Then observe that

$$\begin{aligned}
\left|\int_0^1 g(x^n) - g(0)\right| &= \left|\int_0^1 g(x^n) - \int_0^1 g(0)\right| \\
&= \left|\int_0^1 g(x^n) - g(0)\right| \\
&\leq \int_0^1 |g(x^n) - g(0)| \\
&< \varepsilon \int_0^1 = \varepsilon.
\end{aligned}$$

■

7.4 The Fundamental Theorem of Calculus

Exercise 7.5.4

Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Proof. Suppose for sake of contradiction that $f(x) > 0$ (the proof is similar for $f(x) < 0$). Since f is continuous, we know that f must also be integrable. Hence, we know that $\int_a^b f > 0$ by Exercise 7.4.4. But this contradicts our assumption that $\int_a^b f = 0$. Hence, $f(x) = 0$ for all $x \in [a, b]$. ■

Exercise 7.5.5

The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 under the additional assumption that the sequence of derivatives is continuous.

Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, b]$. Assuming each f_n' is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a, b]$. Show that $g(x) = f'(x)$.

Proof. Our goal is to show that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

Let $\varepsilon > 0$. Since $f_n' \to g$ uniformly and each f_n' continuous, we must have

$$\lim_{n \to \infty} \int_c^x f_n' = \int_c^x g.$$

But notice that $(f_n) \to f$, so we must have

$$\lim_{n \to \infty} \int_c^x f_n' = \lim_{n \to \infty} [f_n(x) - f_n(c)] = f(x) - f(c).$$

Hence, we have

$$\int_c^x g = f(x) - f(c).$$

Since $(f_n') \to g$ uniformly, there must exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ and $0 < |x - c| < \delta$ for some $\delta > 0$, we have

$$\begin{aligned}
\left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| &= \left|\frac{1}{x - c} \int_c^x [g - f_n']\right| \\
&\leq \frac{1}{x - c} \int_c^x |g - f_n'| \\
&< \frac{\varepsilon}{x - c} \cdot (x - c) \\
&= \varepsilon.
\end{aligned}$$

Then by the same choice of $N \in \mathbb{N}$ as in the original proof of Theorem 6.3.1, we have that for any $n \geq N$

$$\begin{aligned}
\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| &\leq \left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \\
&\quad + \left|\frac{f_n(x) - f_n(c)}{x - c} - f_n'(c)\right| + |f_n'(c) - g'(c)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

■

Exercise 7.5.6 (Integration-by-parts).

(a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)\, dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\, dt. \tag{1}$$

Proof. Let us rewrite (1) by adding the second term on the right side of (1) to both sides so that we may show that

$$\int_a^b [h(t)k'(t) + h'(t)k'(t)]\ dt = h(b)k(b) - h(a)k(a). \tag{2}$$

Since $h(x)$ and $k(x)$ have continuous derivatives, it follows that $h'(x)$ and $k'(x)$ are both integrable. Furthermore, the differentiability of both h and k imply that they are continuous and thus h and k are integrable. Then by exercise 7.4.6, we now that hk' and $h'k$ are integrable and thus their sum $hk' + h'k$ is also integrable. But observe that

$$(hk)' = hk' + h'k$$

by the product rule which means that $(hk)'$ is integrable. By part (i) of theorem 7.5.1, we have

$$\int_a^b (hk)'(t)\ dt = (hk)(b) - (hk)(a) = h(b)k(b) - h(a)k(a). \tag{3}$$

Hence, we can rearrange (3) to

$$\int_a^b h(t)k'(t)\ dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\ dt.$$

■

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

Proof. ■

Exercise 7.5.7

Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that f is continuous. (To get started, set $G(x) = \int_a^x f$.)

Proof. Define $G(x) = \int_a^x f$ and assume f is integrable and G continuous. Since f is continuous, we must have differentiability of G. Using the Mean Value Theorem, we can find a $c \in (a, b)$ such that

$$G'(c) = \frac{G(b) - G(a)}{b - a} = \frac{1}{b - a}\left[\int_a^b f - \int_a^a f\right] = \frac{1}{b - a}\int_a^b f.$$

By assumption of part(i) and part(ii) of the Fundamental Theorem, we also know that $f(x) = F'(x) = G'(x)$ for all $x \in [a, b]$. Thus, we can apply the Mean Value Theorem again to use the same $c \in (a, b)$ to get

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

Since $F'(c) = G'(c)$, we must have

$$\int_a^b f = F(b) - F(a).$$

■

Exercise 7.5.8 (Natural Logarithm and Euler's Constant).

Let

$$L(x) = \int_1^x \frac{1}{t}\ dt,$$

where we consider only $x > 0$.

(a) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.

Proof. By definition, we have

$$L(1) = \int_1^1 \frac{1}{t}\, dt = 0.$$

Since $x > 0$, we know that $g(x) = \frac{1}{x}$ is continuous (uniformly continuous for closed intervals). Hence, L is differentiable by part (ii) of the Fundamental Theorem of Calculus. ■

(b) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)

Proof. Using the Chain Rule, we can differentiate to $g(t) = L(ty)$ to get

$$g'(t) = L'(ty) = \frac{1}{ty} \cdot y = \frac{1}{t}.$$

This tells us that

$$L'(ty) = \frac{1}{t}$$

which integrating both sides on the interval $[1, x]$, we have

$$\int_1^x L'(ty)\, dt = \int_1^x \frac{1}{t}\, dt \Leftrightarrow L(xy) - L(y) = L(x) - L(1).$$

Then, solving for $L(xy)$ by subtracting $L(y)$ on both sides and the fact that $L(1) = 0$ leads us to our result

$$L(xy) = L(x) + L(y).$$

■

(c) Show $L(x/y) = L(x) - L(y)$.

Proof. Using the Chain Rule, observe that

$$L'(1/x) = x \cdot \Big(-\frac{1}{x^2} \Big) = -\frac{1}{x}.$$

Then we have

$$L(1/x) = \int_1^x L'(1/t)\, dt = \int_1^x -\frac{1}{t}\, dt = -\int_1^x \frac{1}{t}\, dt = -L(x).$$

Using part (a) then gives us our result

$$\begin{aligned} L(x/y) &= L(x) + L(1/y) \\ &= L(x) - L(y). \end{aligned}$$

■

(d) Let

$$\gamma_n = \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

Proof. Todo. ■

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting indentity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Proof. Todo. ■

Exercise 7.5.9

Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f is continously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'|$.

Proof. Suppose f' is continuous differentiable. Then consider a partition P of $[a, b]$ where $[x_{k-1}, x_k]$ is a pariticular subinterval. Since f' is continuous, we can use the Fundamental Theorem of Calculus to write

$$\int_{x_{k-1}}^{x_k} f' = f(x_k) - f(x_{k-1}).$$

Then observe that

$$\begin{aligned}
\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \\
&\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| \\
&= \int_a^b |f'|.
\end{aligned}$$

If we take the supremum over all the partitions of P of $[a, b]$, then we will see that $Vf \leq \int_a^b |f'|$. ■

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

Proof. Our objective is to show that $Vf \geq \int_a^b |f'|$. Using the Mean Value Theorem, we can find a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

since f' is continuous. Let P be a partition of $[a, b]$. Then observe that

$$f'(c)(b - a) = f(b) - f(a) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}).$$

Then observe that

$$\begin{aligned}
|f'(c)(b - a)| &= \left| f'(c) \sum_{k=1}^{n} \Delta x_k \right| \\
&= |f(b) - f(a)| \\
&= \left| \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) \right| \\
&\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|
\end{aligned}$$

Since the $|f'(c)||b-a|$ is independent of our partition P, we know that

$$L(|f'|) \leq \left|\sum_{k=1}^{n} f(x_k) - f(x_{k-1})\right| \leq U(|f'|).$$

Since $|f'|$ is continuous and hence integrable, we must have

$$\begin{aligned}\int_a^b |f'| &\leq \left|\sum_{k=1}^{n} f(x_k) - f(x_{k-1})\right| \\ &\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &\leq \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\} \\ &= Vf.\end{aligned}$$

Hence, $\int_a^b |f'| \leq Vf$. Hence, we have

$$\int_a^b |f'| = Vf.$$

■

Exercise 7.5.10 (Change-of-variable Formula)

Let $g : [a,b] \to \mathbb{R}$ be differentiable and assume g' is continuous. Let $f : [c,d] \to \mathbb{R}$ be continuous, and assume that the range of g is contained in $[c,d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure f is the derivative of some function? How about $(f \circ g)g'$?

Proof. Since $f : [c,d] \to \mathbb{R}$ is continuous, part (ii) of FTC implies

$$\alpha(x) = \int_c^x f(t)\ dt$$

is differentiable and that $\alpha'(x) = f(x)$. If we compose $g; [a,b] \to \mathbb{R}$ and α, the differentiability of α and g yield $f(g(x))g'(x)$ which means

$$\alpha(g(x)) = \int_a^x f(g(x))g'(x).$$

■

(b) Prove the Change-of-variable formula

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(t)\ dt.$$

Proof. Since $f : [c,d] \to \mathbb{R}$ is continuous, we know by part (ii) of FTC that

$$\alpha(x) = \int_c^x f(t)\ dt$$

is differentiable and that $\alpha'(x) = f(x)$ for all $x \in [c,d]$. Since the range of g is contained within the domain of f, we know that on the interval $[g(a), g(b)]$, we must have

$$\int_{g(a)}^{g(b)} f(t)\ dt = \alpha(g(b)) - \alpha(g(a)) \tag{1}$$

by exercise 7.5.7. Suppose if compose α and g together and take the derivative using the Chain Rule. Then we get $f(g(x))g'(x)$ which is a continuous function. Then on the interval $[a, b]$, we see that

$$\int_a^b f(g(x))g'(x)\ dx = \alpha(g(b)) - \alpha(g(a)) \tag{2}$$

by exercise 7.5.7. By equality of (1) and (2), we conclude that

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(t)\ dt.$$

■

Chapter 8

Additional Topics

8.1 The Generalized Riemann Integral

Exercise 8.1.1

(a) Explain why both the Riemann sum $R(f,P)$ and $\int_a^b f$ fall between $L(f,P)$ and $U(f,P)$.

Proof. Let $(P,\{c_k\})$ be a tagged partition of $[a,b]$. Then by definition of by the definition of M_k and m_k (the supremums and infimums of each subinterval $[x_{k-1},x_k]$), we know that

$$m_k \le f(c_k) \le M_k$$

which imply that

$$L(f,P) \le R(f,P) \le U(f,P).$$

If f is integrable then $L(f) = U(f) = \int_a^b f$. This means

$$L(f,P) \le \int_a^b f \le U(f,P).$$

■

(b) Explain why $U(f,P') - L(f,P') < \varepsilon/3$.

Proof. Let $\varepsilon > 0$. By the properties of the supremum and infimum, we have

$$U(f,P') < U(f) + \frac{\varepsilon}{6}$$

and

$$L(f,P') > L(f) - \frac{\varepsilon}{6}.$$

Subtracting these two inequalities and assuming f is integrable $(U(f) = L(f))$, we end up with

$$U(f,P') - L(f,P') < (U(f) - L(f)) + \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

■

If we can show $U(f,P) < U(f,P') + \varepsilon/3$ (and similarly $L(f,P') - \varepsilon/3 > L(f,P)$), then it will follow that

$$\left|R(f,P) - \int_a^b f\right| < \varepsilon$$

and the proof will be done. To do this, we can try to estimate the distance between $U(f,P)$ and $U(f,P')$.

Exercise 8.1.2

Explain why $U(f,P) - U(f,P') \ge 0$.

Proof. If $P = P' \cup P_\varepsilon$, then it follows from lemma 7.2.4 that $U(f,P) \geq U(f,P')$ which implies that $U(f,P) - U(f,P') \geq 0$. ■

Observe that for any partition, the upper sum takes on the form

$$U(f,P) = \sum_{k=1}^{n} M_k \Delta x_k$$

which contains a good number of the M_k terms cancel out.

Exercise 8.1.3

(a) In terms of n, what is the largest number of terms of the form $M_k(x_k - x_{k-1})$ that could appear in one of $U(f,P)$ or $U(f,P')$ but not the other?

Proof. Since P_ε consists of $n-1$ points in $[a,b]$ and there are three points, that being the two endpoints and our sampling point c_k, we must have at most $3(n-1)$ points. ■

(b) Finish the proof in this direction by arguing that

$$U(f,P) - U(f,P') < \frac{\varepsilon}{3}.$$

Proof. Observe that for all $k \in \mathbb{N}$ that $M_k \leq 3(n-1)M$ for some $M > 0$ from part (a). Since P is $\delta-$fine, we must have $\Delta x_k < \varepsilon/9nM$. Hence, we must have

$$\begin{aligned}
U(f,P) - U(f,P') &= \sum_{k=1}^{n} M_k \Delta x_k \\
&\leq (3n-3)M \sum_{k=1}^{n} \Delta x_k \\
&< (3n-3)M \cdot \frac{\varepsilon}{9nM} \sum_{k=1}^{n} \\
&= (3n-3) \cdot \frac{\varepsilon}{9} \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

The same argument can be applied to the lower sums $L(f,P)$ and $L(f,P')$. Then observe that

$$L(f,P') - \frac{\varepsilon}{3} < L(f,P) \leq R(f,P) \leq U(f,P) < U(f,P') + \frac{\varepsilon}{3}$$

now holds which completes our proof that

$$\left| R(f,P) - \int_a^b f \right| < \varepsilon.$$

■

($\Leftarrow$) For the backwards direction, we can assume that $\varepsilon - \delta$criterion in Theorem 8.1.2 holds and show that f is integrable. To show this, we must have the upper sums are close to the lower sums. We now know that it is always the case that

$$L(f,P) \leq R(f,P) \leq U(f,P)$$

independent of the tags chosen to compute $R(f,P)$.

Exercise 8.1.4

(a) Show that if f is continuous, then it is possible to pick tags $\{c_k\}_{k=1}^{n}$ so that

$$R(f,P) = U(f,P).$$

Similarly, there are tags for which $R(f,P) = L(f,P)$ as well.

Proof. Let $\{c_k\}_{k=1}^n$ be an arbitrary tag on a partition P. Since f is continuous on the compact set $[a,b]$, we know that f must also be uniformly continuous. Denote the supremums of each subinterval $[x_{k-1}, x_k]$ by $M_k = f(z_k)$ for all k. Let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that whenever $|c_k - z_k| < \delta$, we have

$$f(c_k) - M_k < \frac{\varepsilon}{b-a}.$$

Then observe that for any partition P of $[a,b]$, we have

$$\begin{aligned} R(f,P) - U(f,P) &= \sum_{k=1}^{n} [f(c_k) - M_k] \Delta x_k \\ &< \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x_k \\ &= \frac{\varepsilon}{b-a} \cdot b - a = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we must have $R(f,P) = U(f,P)$. A similar argument can be used to show $R(f,P) = L(f,P)$. ■

(b) If f is not continuous, it may not be possible to find tags for which $R(f,P) = U(f,P)$. Show, however, that given an arbitrary $\varepsilon > 0$, it is possible to pick tags for P so that

$$U(f,P) - R(f,P) < \varepsilon.$$

The analogous statement holds for lower sums.

Proof. Let $\varepsilon > 0$. Let $\{c_k\}_{k=1}^n$ be an arbitrary tag for P. Since $|f|$ is bounded by some $M > 0$, we know that the distance between the supremums of each subinterval M_k and each tag $f(c_k)$ can be bounded by M; that is, we have

$$M_k - f(c_k) \leq 2Mn.$$

Since the partition P is $\delta-$fine, we know that we can choose $\delta = \frac{\varepsilon}{2Mn}$ such that every subinterval $[x_{k-1}, x_k]$ satisfies

$$\Delta x_k < \frac{\varepsilon}{2Mn}.$$

Then observe that

$$\begin{aligned} U(f,P) - R(f,P) &= \sum_{k=1}^{n} [M_k - f(c_k)] \Delta x_k \\ &\leq 2M \sum_{k=1}^{n} \Delta x_k \\ &< 2M \cdot \frac{\varepsilon}{2Mn} \sum_{k=1}^{n} \\ &= \varepsilon. \end{aligned}$$

The same argument can be applied to show

$$R(f,P) - L(f,P) < \varepsilon.$$

■

Exercise 8.1.5

Use the results of the previous exercise to finish the proof of Theorem 8.1.2.

Proof. Let $\varepsilon > 0$. Then let $(P, \{c_k\})$ be a tagged partition. Let $P = P_1 \cup P_2$ be a common refinement. By assumption, we can have

$$\begin{aligned} R(f,P_1) - R(f,P_2) &= \left[R(f,P_1) - A\right] + \left[A - R(f,P_2)\right] \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \end{aligned}$$

By using the results of part (a) and part(b), we have

$$\begin{aligned} U(f,P) - L(f,P) &= \left[U(f,P) - R(f,P_1)\right] + \left[R(f,P_1) - R(f,P_2\right] \\ &+ \left[R(f,P_2 - L(f,P)\right] \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Hence, f is integrable and $A = \int_a^b f$. ■

Exercise 8.1.6

Consider the interval $[0,1]$.

(a) If $\delta(x) = 1/9$, find a $\delta(x)-$fine tagged partition of $[0,1]$. Does the choice of tags matter in this case?

Proof. Since $\delta(x)$ is just a constant, the choice of tags does not matter in this case. ■

(b) Let

$$\delta(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ x/3 & \text{if } 0 < x \leq 1. \end{cases}$$

Construct a $\delta(x)-$fine tagged partition of $[0,1]$.

Proof. Let $P = \{([0,1/7],1/2),(\{1/2,2/3\},0)\},(\{2/3,1\},1)$ is a $\delta(x)-$fine partition, then observe that

$$x_1 - x_0 < \delta(c_1) \Rightarrow \frac{1}{7} < \frac{1}{6}.$$

and

$$x_2 - x_1 < \delta(c_2) \Rightarrow \frac{11}{21} < \frac{1}{4}$$

and then finally,

$$x_3 - x_2 < \delta(c_3) \Rightarrow \frac{1}{3} < \frac{1}{2}.$$

■

Theorem 5. Given a gauge $\delta(x)$ on an interval $[a,b]$, there exists a tagged partition $(P, \{c_k\}_{k=1}^n)$ that is $\delta(x)-$fine.

Proof. Let $I_0 = [a,b]$. It may be possible to find a tag that the trivial partition $P = \{a,b\}$ works. Specifically, if $b - a < \delta(x)$ for some $x \in [a,b]$, then we can set c_1 equal to such an x and notice that $(P, \{c_1\})$ is $\delta(x)-$fine. If no such x exists, then bisect $[a,b]$ into two equal halves. ■

Exercise 8.1.7

Finish the proof of Theorem 8.1.5.

Proof. Let each interval $I_k = [x_{k-1}, x_k]$ and define the gauge $\delta(c_k) = \varepsilon/2^{k-1}$ to be the length of each I_k. Then we for any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that for any $k \geq N$ the length

$$|x_k - x_{k-1}| < \varepsilon$$

since $\varepsilon/2^{k-1}$ converges to 0 as $k \to \infty$. Since this applies for every subinterval, the partition $(P, \{c_k\}_{k=1}^n)$ is $\delta(x)-$fine. ■

Exercise 8.1.8

Finish the argument.

Proof. Let $\varepsilon > 0$. Since f has generalized Riemann integral A_1 and A_2, there exists a gauge $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ such that for each tagged partition that is $\delta(x)-$fine, we must have

$$\begin{aligned} |A_1 - A_2| &= \Big|A_1 - R(f,P) + R(f,P) - A_2\Big| \\ &\leq |A_1 - R(f,P)| + |R(f,P) - A_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the distance $|A_1 - A_2| < \varepsilon$ implies $A_1 = A_2$. ■

Exercise 8.1.9

Explain why every function that is Riemann-integrable with $\int_a^b f = A$ must also have generalized Riemann integral A.

Proof. If f is Riemann-integrable, we know that f must also be bounded by some $M > 0$ which is the same for all the subintervals $[x_{k-1}, x_k]$. This means every partition $(P, \{c_k\}_{k=1}^n)$ is $\delta(x)-$fine. Hence, f also contains a generalized Riemann integral A. ■

The converse statement of the above is not true since Dirichel's function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a non-Riemann-integrable function whose discontinuities are at every point of $\mathbb{R}$.

Theorem 6. Dirichlet's function $g(x)$ is generalized Riemann-integrable on $[0,1]$ with $\int_0^1 g = 0$.

Proof. Let $\varepsilon > 0$. The goal is to construct a gauge $\delta(x)$ on $[0,1]$ such that whenever $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)-$fine tagged partition, it follows that

$$0 \leq \sum_{k=1}^{n} g(c_k)\Delta x_k < \varepsilon.$$

In this context, the gauge $\delta(x)$ represents the restriction on the size of $\Delta x_k = x_k - x_{k-1}$ where $\Delta x_k < \delta(c_k)$. Thus, the Riemann sums for the $g(x)$ consists of products of the form $g(c_k)\Delta x_k$. If we take irrational tags, then $g(c_k) = 0$ by definition of g. Hence, the only case we need to worry about is when we have rational tags.

Let $\{r_1, r_2, r_3, \dots\}$ be a countable set of rational numbers that are contained in $[0,1]$. Then for each r_k, define $\delta(r_k) = \varepsilon/2^{k+1}$. If $x \notin \mathbb{Q}$, then set $\delta(x) = 1$. ■

Exercise 8.1.10

Show that if $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)-$fine tagged partition, then $R(g, P) < \varepsilon$.

Proof. Please Check later. If $c_k \notin \mathbb{Q}$, it follows that $g(c_k) = 0$ for all k. Then it immediately follows that

$$R(g, P) < \varepsilon.$$

Otherwise, Let $\{r_1, r_2, r_3, \dots\}$ be a countable set of rational numbers that are contained in $[0, 1]$. Since $r_k \in \mathbb{Q}$ for all k, we must have $g(r_k) = 1$. Using the definition of $\delta(r_k)$ and the fact that $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)-$ fine tagged partition, we must have

$$0 \leq R(g, P) = \sum_{k=1}^{n} g(r_k)\Delta x_k = \sum_{k=1}^{n} \Delta x_k < \sum_{k=1}^{n} \frac{\varepsilon}{2^{k-1}} < \varepsilon.$$

Hence, $R(g, P) < \varepsilon$. ■

(i) The failure of the Dirichlet's function to be Riemann-integrable is caused by the freedom to choose between $R(g, P) = 1$ and $R(g, P) = 0$ based on either rational or irrational tagged partitions.

(ii) Nonconstant gauges that depend on the value of x on some interval causes us to discriminate based on which tagged partitions qualify as $\delta(x)-$fine which makes it easier to achieve

$$|R(f, P) - A| < \varepsilon$$

for smaller and more deliberately selected set of tagged partitions.

Exercise 8.1.11

Show that

$$F(b) - F(a) = \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})].$$

Proof. Since $F : [a, b] \to \mathbb{R}$ is differentiable at each point in $[a, b]$, we must also have F continuous at each point in $[a, b]$. This means that the sum

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]$$

is telescoping and thus we must have

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] = F(b) - F(a).$$

■

If $\{c_k\}_{k=1}^n$ is a set of tags for P, then we can estimate the difference between the Riemann sum $R(f, P)$ and $F(b) - F(a)$ by

$$\begin{aligned} |F(b) - F(a) - R(f, P)| &= \left| \sum_{k=1}^{n} [F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})] \right| \\ &\leq \sum_{k=1}^{n} |F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})|. \end{aligned}$$

Let $\varepsilon > 0$. Our goal is to construct a gauge $\delta(c)$ such that

$$|F(b) - F(a) - R(f, P)| < \varepsilon$$

for all $(P, \{c_k\})$ that are $\delta(c)-$fine (Using the variable c in the gauge function is more convenient than x in this case.)

Exercise 8.1.12

For each $c \in [a, b]$, explain why there exists a $\delta(c) > 0$ (a $\delta > 0$ depending on c) such that

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \varepsilon$$

for all $0 < |x - c| < \delta(c)$.

Proof. Since F is differentiable, then by we are guaranteed to find a $\delta(c) > 0$ such that whenever $0 < |x - c| < \delta(c)$ it follows that

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \varepsilon.$$

■

Exercise 8.1.13

(a) For a particular $c_k \in [x_{k-1}, x_k]$ of P, show that

$$|F(x_k) - F(c_k) - f(c_k)(x_k - x_{k-1})| < \varepsilon(x_k - c_k)$$

and

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

Proof. Note that the differentiability of F implies that the right hand limit and left hand limit are the same. Hence, we can state that

$$\lim_{x_k \to c_k} \frac{F(x_k) - F(c_k)}{x_k - c_k} = \lim_{x_{k-1} \to c_k} \frac{F(c_k) - F(x_{k-1})}{c_k - x_{k-1}}. \tag{1}$$

Then by definition of the derivative, the right hand side of (1) implies

$$\left|\frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c)\right| < \varepsilon$$

which leads to

$$|F(x_k) - F(c_k) - f(c_k)(x_k - x_{k-1})| < \varepsilon(x_k - c_k)$$

and likewise the left hand side of (1) implies

$$\left|\frac{F(c_k) - F(x_{k-1})}{c_k - x_{k-1}} - f(c)\right| < \varepsilon$$

which also implies

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

■

(b) Now, argue that

$$|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| < \varepsilon(c_k - x_{k-1}).$$

Proof. Let $\varepsilon > 0$. Using algebraic manipulations we can write, collecting terms, and using the results from part (a), we have

$$\begin{aligned} |F(x_k) - F(x_{k-1}) - F(c_k)(x_k - x_{k-1})| &\leq |F(x_k) - F(c_k) - f(c_k)(x_k - c_k)| \\ &\quad + |F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})| \\ &< \varepsilon(x_k - c_k) + \varepsilon(c_k - x_{k-1}) \\ &= \varepsilon(x_k - x_{k-1}). \end{aligned}$$

Then

$$\begin{aligned}|F(b) - F(a) - R(f,P)| &< \varepsilon \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= \varepsilon(b-a)\end{aligned}$$

■

Exercise 8.1.14

(a) Why are we sure that f and $(F \circ g)'$ have generalized Riemann integrals?

Proof. Since F is differentiable and satisfies $F'(x) = f(x)$ for all $x \in g[a,b]$, we know that f must have a Generalized Riemann Integral. Likewise, the differentiability as well as the continuity of F and g guarantee Riemann integrability of their composition which also implies that the $(F \circ g)$ to have a Generalized Riemann integral. ■

(b) Use Theorem 8.1.9 to finish the proof.

Proof. Assume g is differentiable and F differentiable with $F'(x) = f(x)$ for all $x \in g[a,b]$. By part (a), we must have the following

$$\begin{aligned}\int_a^b (f \circ g) \cdot g' &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f.\end{aligned}$$

Hence, we conclude that

$$\int_a^b (f \circ g) \circ g' = \int_{g(a)}^{g(b)} f.$$

■

8.2 Metric Spaces

Exercise 8.2.1

Decide which of the following are metrics on $X = \mathbb{R}^2$. For each, we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the plane.

(a) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(b) $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(c) $d(x,y) = |x_1 x_2 + y_1 y_2|$.

Proof. (a) We claim that $d(x,y)$ is a metric on $X = \mathbb{R}^2$. Let $x', y' \in \mathbb{R}^2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For part (i), suppose $x' \neq y'$. Then observe that by property of the square root, we know that $d(x,y) > 0$. Otherwise, $d(x,y) = 0$.

For part (ii), observe that

$$\begin{aligned}d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \\ &= d(y,x).\end{aligned}$$

For part(iii), let $x, y, z \in \mathbb{R}^2$. Then observe that

$$\begin{aligned} d(x,y) &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \\ &= \sqrt{(x_1-z_1)^2+(x_2-z_2)^2+(z_1-y_1)^2+(z_2-y_2)^2} \\ &\leq \sqrt{(x_1-z_1)^2+(x_2-z_2)^2}+\sqrt{(z_1-y_1)^2+(z_2-y_2)^2} \\ &= d(x,z)+d(z,y). \end{aligned}$$

Hence, we conclude that $d(x,y)$ is a metric on $\mathbb{R}^2$.

(b) We have $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$ is a metric on $\mathbb{R}^2$. For property (i), observe that $d(x,y) > 0$ if either $x \geq y$ or $x < y$. This holds because $|\cdot| > 0$. If $x = y$, then it follows immediately that $d(x,y) = 0$. To show the triangle inequality, we will use the formula

$$\max\{a,b\} = a+b+||a|-|b||.$$

Then observe that for any $x, y, z \in \mathbb{R}^2$, we have

$$\begin{aligned} d(x,y) &= \max\{|x_1-y_1|, |x_2-y_2|\} \\ &= \frac{1}{2}\Big[|x_1-y_1|+|x_2-y_2|+||x_1-x_1|+|x_2-y_2||\Big] \\ &\leq \frac{1}{2}\Big[|x_1-z_1|+|z_1-y_1|+|x_2-z_2|+|z_2-y_2| \\ &+\Big||x_1-z_1|+|z_1-y_1|-|x_2-z_2|+|z_2-y_2|\Big|\Big] \\ &= \frac{1}{2}\Big[|x_1-z_1|+|x_2-z_2|+\Big||x_1-z_1|-|x_2-z_2|\Big|\Big] \\ &+\frac{1}{2}\Big[|z_1-y_1|+|z_2-y_2|+\Big||z_1-y_1|-|z_2-y_2|\Big|\Big] \\ &= \max\{|x_1-z_1|, |x_2-z_2|\}+\max\{|z_1-y_1|, |z_2-y_2|\} \\ &= d(x,z)+d(z,y). \end{aligned}$$

(c) $d(x,y) = |x_1x_2+y_1y_2|$ cannot be a metric since $d(x,y) \neq 0$ for all $x, y \in \mathbb{R}^2$.

■

- The metric in part (a) is the Euclidean distance between two points in a plane.
- $d(x,y) = |x-y|$ is a metric over $\mathbb{R}$ (the main metric we have been working with throughout the book).

Exercise 8.2.2

Let $C[0,1]$ be the collection of continuous functions on the closed interval $[0,1]$. Decide which of the following are metrics on $C[0,1]$.

(a) $d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$.

(b) $d(f,g) = |f(1)-g(1)|$.

(c) $d(f,g) = \int_0^1 |f-g|$.

Proof. (a) Observe that for any two functions $f, g \in C[0,1]$ that are distinct, we know that

$$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\} \geq |f(x)-f(x)| > 0.$$

If $f = g$, then it immediately follows that $d(f,g) = 0$. Hence, property (i) is satisfied.

Observe that part (ii) is satisfied by taking

$$\begin{aligned} d(f,g) &= \sup\{|f(x)-g(x)| : x \in [0,1]\} \\ &= \sup\{|g(x)-f(x)| : x \in [0,1]\} \\ &= d(g,f). \end{aligned}$$

For part (iii), let $f, g, h \in C[0,1]$, then we must have

$$\begin{aligned} d(f,g) &= \sup|f(x)-g(x)| \\ &= \sup|f(x)-h(x)+h(x)-g(x)| \\ &\leq \sup|f(x)-h(x)| + \sup|h(x)-g(x)| \\ &= d(f,h)+d(h,g). \end{aligned}$$

(b) The first property fails (take $f(1) = 1$ and $g(x) = x$).

(c) We claim that $d(f,g) = \int_0^1 |f-g|$ is a metric on $\mathbb{R}^2$. Note that for any two distinct functions $f, g \in C[0,1]$, we must have $|f-g| > 0$. By exercise 7.4.4, we must have $\int_0^1 |f-g| > 0$. Otherwise, $f = g$ implies $\int_0^1 |f-g| = 0$. If $\int_0^1 |f-g| = 0$, then we must have $|f-g| = 0$ and hence, $f = g$. To show the triangle inequality, let $f, g, h \in C[0,1]$ be integrable (since they are part of a set of continuous functions that are bounded). Hence, observe that

$$\begin{aligned} d(f,g) &= \int_0^1 |f-g| \\ &\leq \int_0^1 |f-h|+|h-g| \\ &= \int_0^1 |f-h| + \int_0^1 |h-g| \\ &= d(f,h)+d(h,g). \end{aligned}$$

■

Define the *discrete metric* on any set X where for any $x, y \in X$, let

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Exercise 8.2.3

Verify that the discrete metric is actually a metric.

Proof. Observe that if $x \neq y$, then by definition we must have $\rho(x,y) > 0$. Otherwise, $\rho(x,y) = 0$ by definition. It is clear that $\rho(x,y) = \rho(y,x)$. To show the triangle inequality, let $x, y, z \in X$, then we must have

$$\begin{aligned} \rho(x,y) &= 1+0 \\ &\leq 1+1 \\ &= \rho(x,z)+\rho(z,y). \end{aligned}$$

Hence, $\rho(x,y)$ is a metric on any arbitrary set X. ■

Exercise 8.2.4

Show that a convergent sequence is Cauchy.

Proof. Since $(x_n) \subseteq$ is a Cauchy sequence, we can pick an $N \in \mathbb{N}$ such that for any $n, m \geq N$, we must have

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \text{and} \quad d(x, x_m) < \frac{\varepsilon}{2}.$$

Using the same choice of $N \in \mathbb{N}$ so that $n, m \geq N$, we must have that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, (x_n) is Cauchy. ■

- Notice that this is only the forwards direction of the Cauchy Criterion we studied under $\mathbb{R}$.
- For metric spaces other than $\mathbb{R}$, the converse of the Cauchy Criterion does not necessarily hold.
- We need to develop an ordering of our space similar to how the Axiom of Completeness is used in $\mathbb{R}$ (This is called *completeness*).
- The convergence of Cauchy sequences is taken to be the definition of completeness.

Exercise 8.2.5

(a) Consider $\mathbb{R}^2$ with the discrete metric $\rho(x, y)$ examined in Exercise 8.2.3. What do Cauchy sequences look like in this pace? Is $\mathbb{R}^2$ complete with respect to this metric?

Proof. Cauchy sequences under the discrete metric under $\mathbb{R}^2$ would have $x_n = (x_{n_1}, x_{n_2})$ and $x_m = (x_{m_1}, x_{m_2})$ such that

$$\rho(x_n, x_m) = \begin{cases} 0 & \text{if } x_n = x_m \\ 1 & \text{if } x_{n_i} \neq x_{m_i} \text{ where } 1 \leq i \leq 2. \end{cases}$$

Yes, $\rho(x, y)$ is complete under $\mathbb{R}^2$. ■

(b) Show that $C[0, 1]$ is complete with respect to the metric in Exercise 8.2.2(a).

Proof. The metric from Exercise 8.2.2 (a) is

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Let $f_n, f_m \in C[0, 1]$. The Cauchy sequence under sup norm metric will be

$$d(f_n, f_m) = \sup_{x \in [0,1]} |f_n(x) - f_m(x)|$$

We want to show that the Cauchy sequence of functions (f_n) converges under $C[0, 1]$. Since (f_n) is a Cauchy sequence under $\mathbb{R}$, we know that is satisfies the Cauchy Criterion. Hence, (f_n) must converge uniformly. By choosing $N \in \mathbb{N}$, we can let $m, n \geq N$ and $x \in [0, 1]$ such that

$$\begin{aligned} d(f_n, f) &= \sup_{x \in [0,1]} |f_n(x) - f(x)| \\ &\leq \sup_{x \in [0,1]} |f_n(x) - f_m(x)| + \sup_{x \in [0,1]} |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, the sup norm metric is complete under $C[0, 1]$. ■

(c) Define $C^1[0, 1]$ to be the collection of differentiable functions on $[0, 1]$ whose derivatives are also continuous. Is $C^1[0, 1]$ complete with respect to the metric defined in Exercise 8.2.2(a)?

Proof. No, $C^1[0,1]$ is not complete under metric defined in Exercise 8.2.2 (a). Define

$$h_n' = \frac{x}{\sqrt{x^2+1/n}}.$$

Note that the convergence of $h'n \to h$ where $h(x) = x/|x|$ is not uniform. Hence, we cannot have completeness on $C^1[0,1]$ when we have pointwise convergence instead of uniform convergence. ■

The sup metric is usually written as

$$\|f-g\|_\infty = d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$$

and setting $g = 0$ gives us the "sup norm"

$$\|f\|_\infty = d(f,0) = \sup\{|f(x)| : x \in [0,1]\}.$$

From now on, we will assume that the space $C[0,1]$ is paired with the metric above unless otherwise specified.

Exercise 8.2.6

Which of these functions from $C[0,1]$ to $\mathbb{R}$ (with the usual metric) are continuous?

(a) $g(f) = \int_0^1 fk$ where k is some fixed function in $C[0,1]$.

Proof. We claim that $g(f) = \int_0^1 fk$ where k is some fixed function in $C[0,1]$. Let $\varepsilon > 0$. Under the usual metric under $\mathbb{R}$, suppose there exists a $\delta > 0$ such that $|x-c| < \delta$. Since $f \in C[0,1]$, f is also continuous. Hence, we can use the same δ such that

$$|f(x)-f(c)| < \varepsilon.$$

Since k is a fixed function in $C[0,1]$, we must have

$$\begin{aligned} |g(f(x)) - g(f(c))| &= \left|k\int_0^1 (f(x)-f(c))\,dx\right| \\ &\le M\int_0^1 |f(x)-f(c)|\,dx && (k \text{ is bounded}) \\ &< M\int_0^1 \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Hence, $g(f)$ is continuous in $C[0,1]$. ■

(b) $g(f) = f(1/2)$.

Proof. Let $\varepsilon > 0$. Since $f \in C[0,1]$, we know that f must be continuous. Hence, we can choose a $\delta > 0$ such that for any $|x-c| < \delta$, we have

$$|g(f(x)) - g(f(c))| = |f(1/2) - f(1/2)| = 0 < \varepsilon.$$

Hence, g is continuous on $C[0,1]$. ■

(c) $g(f) = f(1/2)$, but this time with respect to the metric on $C[0,1]$ from Exercise 8.2.2 (c).

Proof. Not continuous. Let $f = 0$ and let $\delta > 0$. Now define

$$h_\delta(x) = \begin{cases} 1/2 & x \in V_\delta(1/2) \\ 0 & \text{otherwise}. \end{cases}$$

Observe that for any $\delta > 0$, we have that $d(h_\delta, f) = \delta$. Using the metric from part (c), we will end up with $d(h_\delta, h) = (h - f)(1/2) = 1/2$. Thus, we can't satisfy $\varepsilon < 1/2$. ■

Exercise 8.2.7

Describe the $\varepsilon-$neighborhoods in $\mathbb{R}^2$ for each of the different metrics described in Exercise 8.2.1. How about the discrete metric?

Proof. ■

Now we are able to define *open sets, limit points,* and *closed sets* like we did before on $\mathbb{R}$ but this time with more general spaces and different metrics. Reframing our definitions of these concepts in $\mathbb{R}$ in terms of a general space X, we

- call a set $O \subseteq X$ *open* if for every $x \in O$ we can find a neighborhood $V_\varepsilon(x) \subseteq O$.
- A point x is a *limit point* of a set A if every $V_\varepsilon(x)$ intersects A in some point other than x. A set C is *closed* if it contains its limit points.

Exercise 8.2.8

Let (X, d) be a metric space.

(a) Verify that a typical $\varepsilon-$neighborhood $V_\varepsilon(x)$ is an open set. Is the set

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$$

a closed set?

Proof. Observe that $V_\varepsilon(x) \subseteq V_\varepsilon(x)$. Hence, $V_\varepsilon(x)$ is an open set. Yes, we have

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$$

is a closed set. To see why, let x be a limit point of $C_\varepsilon(x)$. By Theorem 3.2.2, there exists a sequence $x_n \subseteq C_\varepsilon(x)$ such that $\lim x_n = x$ with $x_n \neq x$ for all $n \in \mathbb{N}$. This means that we can choose an $N \in \mathbb{N}$ such that for all $n \geq N$, we must have $d(x_n, x) \leq \varepsilon$. Hence, $x \in C_\varepsilon(x)$ and so $C_\varepsilon(x)$ must be a closed set. ■

(b) Show that a set $E \subseteq X$ is open if and only if its complement is closed.

Proof. ($\Rightarrow$) Let x be a limit point of E^c. Then for all $\varepsilon-$neighborhoods, the intersection

$$V_\varepsilon(x) \cap E^c$$

is nonempty. Since E is open, we know that $V_\varepsilon(x) \subseteq E$. But we have $E \cap E^c$ is empty so we must have $x \in E^c$ because otherwise, there exists a $V_\varepsilon(x)$ such that $V_\varepsilon(x) \subseteq E$. Hence, E^c must be closed.

($\Leftarrow$) Let E^c be a closed set. Let $x \in E$. Since x is not a limit point of E^c, x must not be a limit point of E^c. This means there exists an intersection $V_\varepsilon(x) \cap E^c$ that is empty. Hence, $V_\varepsilon(x) \subseteq E$ implying that E is open. ■

Exercise 8.2.9

(a) Show that the set $Y = \{f \in C[0, 1] : \|f\|_\infty \leq 1\}$ is closed in $C[0, 1]$.

Proof. Suppose f is a limit point of Y. Let (f_n) be a Cauchy sequence that converges uniformly to f. Choose $\varepsilon = 1$. Then observe that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we must have

$$\|f_n - f\| \leq 1.$$

Then observe that

$$\begin{aligned}|f(x)| &\leq |f_n(x) - f(x) + f(x)| \\ &\leq |f_n(x) - f(x)| + |f(x)| \\ &\leq \|f_n - f\|_\infty \\ &\leq 1.\end{aligned}$$

Since $|f(x)| \leq \|f(x)\|$, we must also have $\|f(x)\| \leq 1$. Hence, f is contained in Y and thus we must have Y closed. ■

(b) Is the set $T = \{f \in C[0,1] : f(0) = 0\}$ open, closed, or neither in $C[0,1]$?

Proof. T is closed. Let $\varepsilon > 0$. Let $f \in C[0,1]$. Then there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\|f\| < \varepsilon.$$

Hence, $f(0) = 0$ which implies $f \in T$. ■

In $\mathbb{R}$, we came across a proposition that a set is compact if and only if it is closed and bounded. For more general metric spaces, however, this proposition only holds true in the forwards direction.

Exercise 8.2.10

(a) Show that if K is compact subset of the metric space (X, d), then K is closed and bounded.

Proof. Since K is a compact subset of the metric space (X, d), every sequence $(x_n) \subseteq K$ contains a subsequence (x_{n_k}) that converges to a limit x that is contained in K. Let (x_n) be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that for any $n > n_k \geq N$, we must have

$$\begin{aligned}d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since $x \in K$ and $(x_n) \to x$, K must be closed. Since every sequence (x_n) converges, we know that every (x_n) is bounded by some $M > 0$. Hence, K must also be bounded. ■

(b) Show that $Y \subseteq C[0,1]$ from Exercise 8.2.9 (a) is closed and bounded but not compact.

Proof. The results from part (a) of Exercise 8.2.9 imply that Y is bounded and closed. To see why Y is not compact, suppose we have a sequence of continuous functions (f_n) defined by $f_n = x^n$. Since $Y \subseteq C[0,1]$, we know that the $(f_n) \to f$ uniformly. But the pointwise limit of $\lim f_n(x)$ is *not continuous* and every subsequence of (f_n) will necessarily converge pointwise to $f \notin C[0,1]$. Hence, Y cannot be compact in $C[0,1]$. ■

- The concept of *equicontinuity* of functions is key to the solution of part (c) above.
- Look back to the Arzeli-Ascoli Theorem in chapter 6 before solving the exercise above.
- The result found in part (b) can only be made possible if, in addition to our assumptions, Y contained a collection of functions that are equicontinuous.

Exercise 8.2.11

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

Proof. See the solution in Exercise 3.2.14. ■

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ c) = \overline{E^c}$.

Proof. See the solution in Exercise 3.2.14. ■

Exercise 8.2.12

(a) show

$$\overline{V_\varepsilon(x)} \subseteq \{y \in X : d(x,y) \leq \varepsilon\},$$

in an arbitrary metric space (X, d).

Proof. Let x be a limit point of $\overline{V_\varepsilon(x)}$. By definition of $\overline{V_\varepsilon(x)}$, we know that $x \in \overline{V_\varepsilon(x)}$. Then, there exist exists a sequence (x_n) such that $x_n \to x$ with $x_n \neq x$ for all $n \in \mathbb{N}$. Hence, for some $N \in \mathbb{N}$ we know that for any $n \geq N$, we have $d(x_n, x) \leq \varepsilon$. But this is the definition of $C_\varepsilon(x)$. Hence, $x \in C_\varepsilon(x)$. ■

(b) To keep things from sounding too familiar, find an example of a specific metric space where

$$\overline{V_\varepsilon(x)} \neq \{y \in X : d(x,y) \leq \varepsilon\}.$$

Proof. Take $(\mathbb{R}, |\cdot|)$. Then observe that $\overline{V_\varepsilon(x)} \neq \{n \in \mathbb{N} : |1/n| \leq \varepsilon\}$ where $V_\varepsilon(x) = \{0\}$. ■

Exercise 8.2.13

If E is a subset of a metric space (X, d), show that E is nowhere-dense in X if and only if $\overline{E}^c$ is dense in X.

Proof. $(\Rightarrow)$ Suppose E is nowhere-dense in X. Then $\overline{E}^\circ$ is empty. Let $\varepsilon > 0$. Then for any $x \in \overline{E}^\circ$ is contained in $(\overline{E}^\circ)^c$. By exercise 8.2.11, we know that $(\overline{E}^\circ)^c = \overline{(\overline{E}^\circ)^c}$. But notice that we must have $\overline{(\overline{E}^\circ)^c} = X$. Hence, $\overline{E}^c$ must be dense in X.

$(\Leftarrow)$ Suppose $\overline{E}^c$ is dense in X. Let $x \in \overline{E}^\circ$. Let $\varepsilon > 0$. Since x is neither an element of nor a limit point of $\overline{E}^c$, we know that for every $V_\varepsilon(x)$, we have $V_\varepsilon(x) \cap \overline{E}$ is empty. This tells us that $\overline{E}^\circ$ is empty. Hence, E is nowhere dense in X. ■

8.3 Euler's Sum

Exercise 8.3.1

Supply the details to show (3) above.

Proof. Plugging in $x = \pi/2$ into (2), we get that

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{2n}\right)\left(1 + \frac{1}{2n}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2}.$$

Taking the reciprocal of the infinite product above, we end up with

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{\pi}{2}.$$

■

Now we will prove why (3) holds. Set

$$b_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\, dx, \text{ for } n = 0, 1, 2, \ \ldots$$

If we look at the $n = 0$ and $n = 1$ case, we can easily obtain the following equations

$$b_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \text{ and } b_1 = \int_0^{\frac{\pi}{2}} \sin(x)\, dx = 1.$$

Exercise 8.3.2

Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$, and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\ dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\ dt.$$

Proof. Refer to the solution in part (a) of Exercise 7.5.6. ■

Exercise 8.3.3

(a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation

$$b_n = \frac{n-1}{n} b_{n-2} \text{ for all } n \geq 2.$$

Proof. Let $h(x) = \sin^n(x)$ and $k'(x) = \sin(x)$. Let $n \geq 2$. Then by the integration-by-parts formula and using the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, we must have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx &= \int_0^{\frac{\pi}{2}} \sin^n(x) \cdot \sin(x)\ dx \\ &= \Big[-\sin^{n-1}(x) \cdot \cos(x) \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot \cos^2(x)\ dx \\ &= \Big[-\sin^{n-1}(x) \cdot \cos(x) \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot [1 - \sin^2(x)]\ dx\end{aligned}$$

The first term on the last equality cancels out and the second term can be expanded into

$$\begin{aligned}\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x) \cdot [1 - \sin^2(x)]\ dx = &\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx \\ &+ \int_0^{\frac{\pi}{2}} (n-1)\sin^n(x)\ dx.\end{aligned}$$

Hence, we end up with

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx = \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\ dx + \int_0^{\frac{\pi}{2}} (n-1)\sin^n(x)\ dx. \tag{1}$$

Finally, subtracting the second term on the right side of (1), simplifying, and dividing by n on both sides gives us our desired result

$$\begin{aligned}b_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\ dx \\ &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\ dx \\ &= \frac{n-1}{n} b_{n-2}.\end{aligned}$$

■

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).

Proof. The first three even terms are

$$\begin{aligned} b_2 &= \frac{1}{2}b_0 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}, \\ b_4 &= \frac{3}{4}b_2 = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16}, \\ b_6 &= \frac{5}{6}b_4 = \frac{5}{6}\cdot\frac{3\pi}{16} = \frac{5\pi}{32}. \end{aligned}$$

The first odd terms are

$$\begin{aligned} b_3 &= \frac{2}{3}b_1 = \frac{2}{3}\cdot 1 = \frac{2}{3} \\ b_5 &= \frac{4}{5}b_3 = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15} \\ b_7 &= \frac{6}{7}b_5 = \frac{6}{7}\cdot\frac{8}{15} = \frac{16}{35}. \end{aligned}$$

■

(c) Write a general expression for b_{2n} and b_{2n+1}.

Proof. Using the formula we derived in part (a), plugging in the desired cases gives us

$$b_{2n} = \frac{2n-1}{2n}b_{2(n-1)} \quad \text{and} \quad b_{2n+1} = \frac{2n}{2n+1}b_{2n-1}.$$

■

For the $(n+1)$th term, we have the following bound $0 \leq \sin^{n+1}(x) \leq \sin^n(x)$ on $[0, \pi/2]$. But this tells us that (b_n) is a decreasing sequence of functions. Since (b_n) is bounded and decreasing, we know that it must converge. It turns out that $(b_n) \to 0$ but this isn't the limit that we want to concern ourselves at the moment.

Exercise 8.3.4

Show

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Walli's product formula in (3).

Proof. For $k \geq 1$, observe that

$$\begin{aligned} \frac{b_{2n}}{b_{2n+1}} &= \frac{(2n-1)(2n+1)}{(2n)(2n)}\cdot\frac{b_{2n-2}}{b_{2n-1}} \\ &= \frac{(2n-1)(2n+1)}{(2n)(2n)}\cdot\frac{(2n-3)(2n-1)}{(2n-2)(2n-2)}\cdot\frac{b_{2n-4}}{b_{2n-3}}. \end{aligned}$$

Notice when expanding the terms on the numerator and the denominator of b_{2n}/b_{2n+1}, we will always have the same coefficient. Hence, the limit of b_{2n}/b_{2n+1} gives us our result that

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1.$$

■

Some techniques to dealing with the notation in (3) is to use the following equations

$$2\cdot 4\cdot 6\cdots(2n) = 2^n n!$$

and

$$1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1) = \frac{(2n+1)!}{2\cdot 4\cdot 6\cdots(2n)} = \frac{(2n+1)!}{2^n n!}.$$

Exercise 8.3.5

Derive the following alternative form of Walli's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Proof. ■

Exercise 8.3.6

Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

for $n \geq 1$.

Proof. Let $f(x) = 1/\sqrt{1-x}$. Using Taylor's coefficient formula, we have the first three derivatives of f

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{2}\cdot(1-x)^{-3/2}, \\ f^{(2)}(x) &= \frac{1}{2}\cdot\frac{3}{2}\cdot(1-x)^{-5/2}, \\ f^{(3)}(x) &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot(1-x)^{-7/2}. \end{aligned}$$

For $n \geq 1$, we can use induction to show

$$f^{(n)}(x) = \Big[\prod_{k=1}^{n} \frac{2k-1}{2k}\Big](1-x)^{-(2n+1)/2}.$$

Plugging in $x = 0$ and using the techniques given to us above, we now have the desired formula

$$c_n = \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n)!}{2^{2n}(n!)^2}$$

where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n.$$

■

Observe that the coefficients above should look familiar to the formulas produced from Walli's product.

Exercise 8.3.7

Show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Proof. The first statement is shown in Exercise 2.7.10. Observe that

$$c_n \leq \frac{1}{2^{2n}} \leq \frac{1}{n}.$$

Since $\sum 1/n$ diverges, we must also have $\sum c_n$ diverge by the Comparison test. ■

Now our goal is to establish at which particular points in the domain of f where

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

is valid. This can be done by using Lagrange's Remainder Theorem.

To properly show that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

holds for all $x \in (-1, 1)$, we need to show that the error function

$$E_N(x) = \frac{1}{\sqrt{1-x}} - \sum_{n=0}^{N} c_n x^n$$

approaches zero as $N \to \infty$. This can be done using Lagrange's Remainder Theorem (Theorem 6.6.3).

Exercise 8.3.8

Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem, show that equation (4) is valid for all $|x| < 1/2$. What goes wrong when we try try to use this method to prove (4) for $x \in (1/2, 1)$?

Proof. Since f is $N+1$ times differentiable on $(-1/2, 1/2)$, there exists a c such that $|c| < |x|$ where the error function $E_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)x^n}{(N+1)!}$$

by Lagrange's Remainder Theorem. Observe that

$$f^{(N+1)}(c) = \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big](1-c)^{-(2N+3)/2} < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\Big(\frac{2}{3}\Big)^{(2N+3)/2}.$$

Since $|x| < 1/2$ and $|c| < |x|$, we can now write

$$E_N(x) < \Big[\prod_{k=1}^{N+1} \frac{2k-1}{2k}\Big]\frac{2^{3/2}}{3^{(2N+3)/2}(N+1)!} \xrightarrow{N\to\infty} 0.$$

Hence, (4) holds for all $x \in (-1, 1)$. If we try to prove $E_N \to 0$ on $(-1/2, 1)$, then we produce a sequence that diverges. ■

Exercise 8.3.9

(a) Show

$$f(x) = f(0) + \int_0^x f'(t)\ dt.$$

Proof. Since f is continuous differentiable for all $t \in (0, x)$, we can use part (i) of FTC to write

$$\int_0^x f'(t)\ dt = f(x) - f(0).$$

Solving for $f(x)$ gives us our desired result

$$f(x) = f(0) + \int_0^x f'(t)\ dt.$$

■

(b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f"(t)(x-t)\ dt.$$

Proof. Taking advantage of f being continuously differentiable $N+1$ times for all $t \in (0, x)$

and using the integration-by-parts formula found in Exercise 8.3.2, we have

$$\begin{aligned}\int_0^x f"(t)(x-t)\ dt &= \Big[f'(t)(x-t)\Big]_0^x + \int_0^x f'(t)\ dt \\ &= -xf'(0) + [f(x) - f(0)].\end{aligned}$$

Solving for $f(x)$ once again, we get our desired result

$$f(x) = f(0) + f'(0)x + \int_0^x f"(t)(x-t)\ dt.$$

■

(c) Continue in this fashion to complete the proof of the theorem.

Proof. Continuing the process in parts (a) and (b) and using the fact that f is $N+1$ times differentiable for all $x \in (-R, R)$, we have that

$$\begin{aligned}f(x) &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt + \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x_k \\ &= \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt + S_N(x).\end{aligned}$$

Subtracting $S_N(x)$ from both sides above and using the fact that $E_N(x) = f(x) - S_N(x)$ gives us our desired result

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\ dt.$$

■

Exercise 8.3.10

(a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1, 1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

Proof. ■

(b) For a general x satisfying $|x| < 1$, show

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\ dt.$$

Proof. Let $|x| < 1$. Let $N = 2$. Observe that

$$f^{(3)}(t) = \frac{15}{8}(1-t)^{-7/2}.$$

Using the Integral Remainder Theorem, we have that

$$\begin{aligned}E_2(x) &= \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\ dt \\ &= \frac{15}{16}\int_0^x (1-t)^{-7/2}(x-t)^2\ dt \\ &= \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\ dt.\end{aligned}$$

■

(c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \le |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking that this overestimate is in fact larger than $|E_2(x)|$ at three computed values from part (a).

Proof. The inequality above is valid since

$$\begin{aligned}\left|\frac{x-t}{1-t}\right| &= \sqrt{\left(\frac{x-t}{1-t}\right)^2}\\ &\le \sqrt{(x-t)^2}\\ &= |x-t|\\ &\le |x|\end{aligned}$$

which holds for t. ■

(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1,1)$.

Proof. Let f be differentiable $N+1$ times. Using the inequality found in part (c) and $|x| < 1$,we can write

$$\begin{aligned}|E_N(x)| &= \frac{1}{N!}\left[\prod_{k=1}^{N+1}\frac{2k-1}{2k}\right]\left|\int_0^x \left(\frac{x-t}{1-t}\right)^N \cdot \frac{1}{(1-t)^{3/2}}\,dt\right|\\ &\le \frac{c_{N+1}}{N!}\int_0^x \left|\frac{x-t}{1-t}\right|^N \cdot \left|\frac{1}{(1-t)^{3/2}}\right| dt\\ &\le \frac{c_{N+1}}{N!}\int_0^x \frac{|x|^N}{(1-t)^{3/2}}\,dt\\ &< \frac{c_{N+1}}{N!}\int_0^x \frac{1}{(1-t)^{3/2}}\,dt\\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}\int_0^x dt\\ &= \frac{c_{N+1}x}{2\sqrt{2}\cdot N!}\\ &< \frac{c_{N+1}}{2\sqrt{2}\cdot N!}.\end{aligned}$$

Since $c_{N+1} = \prod_{k=1}^{N+1}\frac{2k-2}{2k} \to 0$ from Exercise 8.2.7, we can write

$$|E_N(x)| < \frac{c_{N+1}}{2\sqrt{2}\cdot N!} \to 0$$

as $N \to \infty$. Hence, $|E_N(x)| \to 0$ which tells us that $E_N \to f(x)$ uniformly. ■

Now that we have established that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

holds for all $x \in (-1,1)$, we are now in the position to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty}\frac{c_n}{n+1}x^{2n+1}$$

for all $|x| < 1$ using term-by-term anti-differentiation of (4).

Exercise 8.3.11

Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty}\frac{c_n}{2n+1}x^{2n+1} \text{ for all } |x| < 1. \tag{5}$$

Proof. From our result in part (d) of Exercise 8.3.10, we know that substituting $x = x^2$ into

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n$$

give us

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} c_n x^{2n}$$

which holds for all $x \in (-1, 1)$. By assumption, we know that the derivative of $\arcsin(x)$ is $1/\sqrt{1-x}$. Using Term-by-term Antidifferentiation, we get that

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1.$$

■

Exercise 8.3.12

Our work thus far shows that the Taylor series in (5) is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \leq 1$. Carefully, explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1, 1]$.

Proof. Since (5) is valid for all $|x| < 1$, it suffices to show that (5) holds for $x = 1$ and likewise $x = -1$ so that we may show that (5) holds for $|x| \leq 1$ using Theorem 6.5.2. Plugging in $x = 1$ gives us

$$\sum_{n=0}^{\infty} \frac{c_n}{2n+1}.$$

Let's define

$$\gamma_n = \frac{c_n}{2n+1}.$$

Using the Cauchy Condensation Test, we can prove that $\sum 2^n \gamma_{2^n}$ so that $\sum \gamma_n$ converges. Observe that

$$\sum_{n=0}^{\infty} \frac{c_{2^n}}{2^{2n+1}+1} \leq \frac{1}{2} \sum_{n=0}^{\infty} c_{2^n}.$$

Now our goal is to use the Comparison Test to show that the right side of the inequality above converges which will immediately imply that the left side converges. Hence, observe that we have the following bound

$$\begin{aligned} \frac{1}{2} c_{2^n} &= \frac{1}{2} \cdot \frac{(2^{n+1})!}{(2^n!)^2 \cdot 2^{2^{2n+1}}} \\ &\leq \frac{2^{n+1}-1}{2 \cdot (2^n)!} \\ &\leq \frac{2^{n+1}}{2 \cdot n!}. \end{aligned}$$

Observe that the last inequality forms a series that converges via the ratio test. Hence, we must have $\frac{1}{2} \sum c_{2^n}$ converge via the Comparison test. Hence, we must have $\sum c_n/(2n+1)$ converges via the Cauchy Condensation Test. Hence, by Theorem 6.5.2 we must have (5) converge at $x = |1|$ for all $x \in [-1, 1]$. ■

Exercise 8.3.13

(a) Show

$$\int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Walli's product.

Proof. Observe that the series

$$\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta)$$

converges uniformly to θ for all $-\pi/2 \leq \theta \leq \pi/2$. Hence, we are able to move integration from outside the summation to inside the summation. Using this fact, we write

$$\begin{aligned}\int_0^{\pi/2} \theta \; d\theta &= \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta) \; d\theta \\ &= \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \left[\int_0^{\pi/2} \sin^{2n+1}(\theta) \; d\theta \right] \\ &= \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1}. && \text{(Walli's Formula)}\end{aligned}$$

■

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Proof. Looking at the left side of part (a), we can integrate to get

$$\int_0^{\pi/2} \theta \; d\theta = \left[\frac{1}{2}\theta^2\right]_0^{\pi/2} = \frac{\pi^2}{8}.$$

Focusing our attention to the right side of (a), we see that b_{2n+1} can be expanded to

$$b_{2n+1} = \frac{2n}{(2n+1)} \cdot \frac{(2n-2)}{(2n-1)} \cdot \frac{(2n-4)}{(2n-3)} \cdot \frac{(2n-6)}{(2n-5)} \cdot \ldots .$$

Likewise, c_n can be expanded into

$$\begin{aligned} c_n &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &= \frac{(2n)\cdot(2n-1)\cdot(2n-2)\cdot(2n-3)\cdot(2n-4)\cdot\ldots}{2^{2n}(n!)^2}\end{aligned}$$

Notice that when we multiply b_{2n+1} and c_n together as seen in part (a), we see that the $(2n-k)$ terms for k odd cancel, leaving the $(2n-\ell)$ terms for ℓ even on the top. Hence, we have

$$\begin{aligned}
c_n b_{2n+1} &= \frac{(2n)^2 \cdot (2n-2)^2 \cdot (2n-4)^2 \cdot (2n-4)^2 \cdot \ldots}{4^n (n!)^2 (2n+1)} \\
&= \frac{(2n)^2 \cdot (2(n-1))^2 \cdot (2(n-2))^2 \cdot \ldots}{4^n (n!)^2 (2n+1)} \\
&= \frac{4^n \cdot n^2 \cdot (n-1)^2 \cdot (n-2)^2 \cdot \ldots}{4^n (n!)^2 (2n+1)} \\
&= \frac{4^n (n!)^2}{4^n (n!)^2} \cdot \frac{1}{2n+1}. \\
&= \frac{1}{2n+1}.
\end{aligned}$$

Hence, part (a) leads to

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Since the infinite sum in part (a) converges uniformly for all $-\pi/2 \leq \theta \leq \pi/2$, we are free to rearrange the sum however we like. Observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\
&= \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{4n^2} \\
&= \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{1}{4n^2}.
\end{aligned}$$

Subtracting the second term on the right hand side to both sides above and collecting terms and dividing, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

■

Exercise 8.4.1

For each $n \in \mathbb{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

Proof. ■

(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$, and revisit part (a).

Proof. The statement above is clearly true for $n = 1$. Now assume $n\# = \frac{1}{2}n(n+1)$ holds for $1 \leq n \leq k-1$. We want to show that $n\#$ holds for the kth case. By using the definition of

$n\#$, we can write

$$\begin{aligned} k\# &= k + (k-1)\# \\ &= k + \frac{1}{2}k(k-1) \\ &= \frac{1}{2}(k^2 + k) \\ &= \frac{1}{2}k(k+1). \end{aligned}$$

Since $n\# = \frac{1}{2}n(n+1)$ holds for the kth case, we know that it holds for any $n \in \mathbb{N}$. ■

We can replace the discrete variable $n \in \mathbb{N}$ for values of $x \in \mathbb{R}$ and the resulting formula

$$x\# = \frac{1}{2}x(x+1)$$

will still make sense.

Exercise 8.4.2

Verify that the series converges absolutely for all $x \in \mathbb{R}$, that $E(x)$ is differentiable on $\mathbb{R}$, and $E'(x) = E(x)$.

Proof. First we prove that the series above converges absolutely for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Observe that

$$\left|\sum_{n=0}^{\infty} \frac{x^n}{n!}\right| \leq \sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|.$$

By using the ratio test for power series found in section 6.5, we have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| \\ &= \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0. \end{aligned}$$

Since the limit above is 0, we know that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

converges absolutely for all $x \in \mathbb{R}$. Given any compact set in $\mathbb{R}$, we know that the convergence of (1) to $E(x)$ is uniform. Hence, it must be continuous on any $A \subseteq \mathbb{R}$ and differentiable n times. Differentiating

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and reordering indices we find that $E'(x) = E(x)$. ■

Exercise 8.4.3

(a) Use the results of Exercise 2.8.7 and the binomial formula to show that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$. By definition of $E(x+y)$, using the binomial formula, we can write

$$\begin{aligned} E(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k \cdot x^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k}{k!} \cdot \frac{x^{n-k}}{(n-k)!} \\ &= \left[\sum_{m=0}^{\infty} \frac{x^m}{m!}\right]\left[\sum_{k=0}^{\infty} \frac{y^k}{k!}\right]. && (n-k=m) \end{aligned}$$

Since

$$\begin{aligned} E(x) &= \sum_{m=0}^{\infty} \frac{x^m}{m!}, \\ E(y) &= \sum_{k=0}^{\infty} \frac{y^k}{k!} \end{aligned}$$

both converge absolutely (by Exercise 2.8.7), we can write

$$E(x+y) = E(x)E(y).$$

■

(b) Show that $E(0) = 1, E(-x) = 1/E(x)$, and $E(x) > 0$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. The first fact immediately follows when $x = 0$. Now let us show the second fact. Using the first fact and part(a), we can write

$$1 = E(0) = E(x-x) = E(x)E(-x) \Leftrightarrow E(x)E(-x) = 1.$$

Dividing through by $E(x)$ on both sides leads us to our result

$$E(-x) = \frac{1}{E(x)}.$$

For the last fact, observe that $E(x) > 0$ follows immediately when we consider any $x \geq 0$. Suppose we let x be negative, then using the fact that

$$E(-x) = \frac{1}{E(x)}$$

where $E(x) > 0$ for any $x > 0$ implies that $E(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} > 0$. ■

The takeaway here is that the power series $E(x)$ contains all the "normal" properties that is associated with the exponential function e^x.

Exercise 8.4.4

Define $e = E(1)$. Show $E(n) = e^n$ and $E(m/n) = (\sqrt[n]{e})^m$ for all $m, n \in \mathbb{Z}$.

Proof. Let $P(n)$ be the statement that $E(n) = e^n$ for all $n \in \mathbb{Z}$. Let our base case be $n = 1$. Then by definition, we must have $E(1) = e$. Now assume $E(n) = e^n$ holds for all $n \in \mathbb{Z}^+$. We want to show that $E(n+1) = e^{n+1}$ holds. Observe that by part (a) of Exercise 4.4.3, we have

$$\begin{aligned} E(n+1) &= E(n) \cdot E(1) \\ &= e^n \cdot e \\ &= e^{n+1}. \end{aligned}$$

Hence, $E(n) = e^n$ for all $n \in \mathbb{Z}^+$. To show that the statement also holds for all $n \in \mathbb{Z}^-$, we can just multiply n by a negative to get

$$\begin{aligned} E(-n) &= \frac{1}{E(n)} \\ &= \frac{1}{e^n} \\ &= e^{-n}. \end{aligned}$$

Lastly, we show $E(m/n) = (\sqrt[n]{e})^m$ for all $n, m \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$. Observe that

$$E(1) = E\Big(\frac{n}{n}\Big) = (\sqrt[n]{e})^n \Rightarrow E(1/n) = \sqrt[n]{e}.$$

Furthermore, we can rewrite m/n in the following way where

$$\frac{m}{n} = \sum_{i=0}^{m} \frac{1}{n}.$$

Then we see that

$$\begin{aligned} E\Big(\frac{m}{n}\Big) &= E\Big(\sum_{i=0}^{m} \frac{1}{n}\Big) \\ &= E\Big(\frac{1}{n}\Big) \cdot E\Big(\frac{1}{n}\Big) \cdot E\Big(\frac{1}{n}\Big) \cdot \ \cdots \ m \text{ times} \\ &= \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \sqrt[n]{e} \cdot \ \cdots \ m \text{ times} \\ &= (\sqrt[n]{e})^m. \end{aligned}$$

■

To complete our list of properties of e^x, all we need is its behavior as $x \to \pm\infty$.

Definition 7. Given $f : [a, \infty) \to \mathbb{R}$, we say that $\lim_{x\to\infty} f(x) = L$ if, for all $\varepsilon > 0$, there exists $M > a$ such that whenever $x \geq M$ it follows that $|f(x) - L| < \varepsilon$.

Exercise 8.4.5

Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \ldots$. To get started notice that when $x \geq 0$, all the terms in (1) are positive.

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Choose $M = 1/\varepsilon > a$. Then observe that for any $x \geq M$, we have

$$\left|\frac{x^n}{e^x} - 0\right| = \frac{x^n}{e^x} < \frac{x^n}{x^{n+1}} = \frac{1}{x} < \varepsilon.$$

Hence, $\lim_{x\to\infty} x^n e^{-x} = 0$. ■

Exercise 8.4.6

(a) Explain why we know e^x has an inverse function; that is, let's call it $\log(x)$ defined for any real $x > 0$ and satisfying

(i) $\log(e^y) = y$ for all $y \in \mathbb{R}$ and

(ii) $e^{\log(x)} = x$, for all $x > 0$.

Proof. If we are considering $f(x) = e^x$ defined on $(0, \infty)$, then we get that $f(x)$ is a bijective function for all $x \in (0, \infty)$. To see why, suppose we let $x, y \in (0, \infty)$. Since $\log(x)$ is defined for all $x \in (0, \infty)$, we can say that

$$\begin{aligned} E(x) &= E(y) \\ e^x &= e^y \\ \log(e^x) &= \log(e^y) \\ x &= y. \end{aligned}$$

Hence, $E(x) = e^x$ is an injective function. Now let's show surjectivity. Then letting $x = \log(y)$, observe that

$$E(x) = e^x = e^{\log(y)} = y.$$

Hence, $E(x)$ is a surjective function. Since $E(x)$ is both injective and surjective, we know that $E(x)$ must be bijective and thus must have an inverse function. ■

(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)

Proof. Let $y = f(x) = e^x$. Using the result from Exercise 5.2.12, the fact that $f'(x) = e^x$, and $e^{\log(x)}$, we get that

$$\begin{aligned} (\log x)' &= \frac{1}{f'(x)} \\ &= \frac{1}{e^{\log(x)}} \\ &= \frac{1}{x}. \end{aligned}$$

■

(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log(x) + \log(y) \text{ for all } x, y > 0.$$

Proof. Let $x, y \in (0, \infty)$ with $x = e^y$ and $y = e^x$. Our logarithm properties, we then have $\log(x) = y$ and $\log(y) = x$. Then by using the properties of e^x and $\log(x)$, observe that

$$\begin{aligned} \log(xy) &= \log(e^y \cdot e^x) \\ &= \log(e^{y+x}) \\ &= y + x \\ &= \log(x) + \log(y). \end{aligned}$$

Hence, we have

$$\log(xy) = \log(x) + \log(y).$$

■

(d) For $t > 0$ and $n \in \mathbb{N}$, t^n has the usual interpretation as $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \text{ for all } n \in \mathbb{N}.$$

Proof. Let $t > 0$ and $n \in \mathbb{N}$. Observe that $t = e^{\log(t)}$ and then

$$t^n = \left(e^{\log(t)}\right)^n = e^{n \log(t)}.$$

■

Definition 8. Given $t > 0$, define the exponential function t^x to be

$$t^x = e^{x \log t} \text{ for all } x \in \mathbb{R}.$$

Exercise 8.4.7

(a) Show $t^{m/n} = (\sqrt[n]{t})^m$ for all $m, n \in \mathbb{N}$.

Proof. Let $m, n \in \mathbb{N}$. Then

$$t^{m/n} = (t^{1/n})^m = (\sqrt[n]{t})^m.$$

∎

(b) Show $\log(t^x) = x \log t$, for all $t > 0$ and $x \in \mathbb{R}$.

Proof. Let $t > 0$ and $x \in \mathbb{R}$. Then observe that

$$\begin{aligned} t^x = e^{x \log t} &\Rightarrow \log(t^x) = \log(e^{x \log t}) \\ &\Rightarrow \log(t^x) = x \log t. \end{aligned}$$

∎

(c) Show t^x is differentiable on $\mathbb{R}$ and find the derivative.

Proof. Let $x, t \in \mathbb{R}$. To show that $f(x) = t^x$ is differentiable, we can use the definition of differentiability. Using the fact that $t^x = e^{x \log t}$, we have

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{t^x - t^c}{x - c} \\ &= \lim_{x \to c} \frac{e^{x \log t} - e^{c \log t}}{x - c}. \end{aligned}$$

Observe that $g(x) = e^{x \log t}$ is differentiable. Hence, the limit in the last equality exists and therefore $f'(c)$ exists. Using the Chain Rule, we get

$$f'(x) = (t^x)' = (e^{x \log t})' = \log(t) e^{x \log t} = \log(t) t^x.$$

∎

The strategy we have been partaking in so far is a similar to how we would define what $n!$ would mean if it was replaced by $x \in \mathbb{R}$ instead of $n \in \mathbb{N}$.

Exercise 8.4.8

Inspired by the fact that $0! = 1$ and $1! = 1$, let $h(x)$ satisfy

(i) $h(x) = 1$ for all $0 \leq x \leq 1$, and

(ii) $h(x) = xh(x-1)$ for all $x \in \mathbb{R}$.

(a) Find a formula for $h(x)$ on $[1, 2]$, $[2, 3]$, and $[n, n+1]$ for arbitrary $n \in \mathbb{N}$.

Proof. On $[1, 2]$, observe that

$$h(2) = 2 \cdot h(1) = 2 \cdot 1 \cdot h(0) = 2$$

and likewise

$$h(1) = 1.$$

This tells us that $h(x)$ on $[1, 2]$ must be defined as $h(x) = x$. Whereas on $[2, 3]$, we have

$$h(3) = 3 \cdot h(2) = 3 \cdot 2 \cdot h(1) = 3! = 3$$

and

$$h(2) = 2 \cdot h(1) = 2 \cdot 1 \cdot h(0) = 2! = 2.$$

which tells us that $h(x)$ on $[2,3]$ must be defined as $h(x) = x(x-1)$. On $[n, n+1]$, observe that

$$h(n) = n \cdot h(n-1) = n \cdot (n-1) \cdot h(n-2) = n!$$

and

$$h(n+1) = (n+1) \cdot n \cdot (n-1) \cdot (n-2) = (n+1)!.$$

This tells us that $h(x)$ on $[n, n+1]$ will be defined as

$$h(x) = \prod_{i=1}^{n-1} x - i$$

which can be proven using induction. ■

(b) Now do the same for $[-1, 0], [-2, -1]$, and $[-n, -n+1]$.

Proof. ■

(c) Sketch h over the domain $[-4, 4]$.

Proof. To do. ■

Our function above $h(x)$ satisfies $h(n) = n!$ and it is at least continuous for $x \geq 0$. However, we still run into the problem where our piecewise function contains non-differentiable corners. We conclude that from the exercise above that $x!$ will have the same asymptotic behavior as h at negative integers x. Hence, it won't be defined on $x \in \mathbb{Z}^-$.

Exercise 8.4.9

(a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\varepsilon > 0$, there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \varepsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)

Proof. For the forwards direction, suppose that the improper integral $\int_a^\infty f$ converges. Let $\varepsilon > 0$. By assumption, we can find an $M > a$ such that whenever $d > c \geq M$, it follows that

$$\left| \int_a^d f - L \right| < \frac{\varepsilon}{2} \quad \text{whenever } d \geq M > a,$$
$$\left| \int_a^c f - L \right| < \frac{\varepsilon}{2} \quad \text{whenever } c \geq M > a.$$

Observe that

$$\int_c^d f = \int_c^a f + \int_a^d f = \int_a^d f - \int_a^c f.$$

Then we have

$$\begin{aligned} \left| \int_c^d f \right| &= \left| \int_a^d f - \int_a^c f \right| \\ &\leq \left| \int_a^d f - L \right| + \left| L - \int_a^c f \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Now assume the converse. Let $\varepsilon > 0$. We want to show that

$$\lim_{n\to\infty} \int_a^{a+n} f = L.$$

By assumption, there exists a natural number $N > a$ such that whenever $a + n > n \geq N$, we have

$$\begin{aligned} \left|\int_a^{a+n} f - L\right| &= \left|\left(\int_a^n f + \int_n^{a+n} f\right) - L\right| \\ &= \left|\left(\int_a^n f - L\right) + \int_n^{a+n} f\right| \\ &\leq \left|\int_a^n f - L\right| + \left|\int_n^{a+n} f\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

■

(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.

Proof. Let $\varepsilon > 0$. Our goal is to show that there exists an $M > a$ such that whenever $d > c \geq M$, we have

$$\left|\int_c^d f\right| < \varepsilon.$$

Since $0 \leq f \leq g$ and $\int_0^\infty g$ converges, there exists an $M > a$ such that whenever $d > d \geq M$, we have that

$$\left|\int_c^d f\right| \leq \left|\int_c^d g\right| < \varepsilon.$$

Hence, we must have that $\int_0^\infty f$ converges as well. ■

(c) Part (a) is a Cauchy criterion, and part (b) is a comparison test. State and prove an absolute convergence test for improper integrals.

Definition 9. Absolute Convergence Test For Improper Integrals If $\int_0^\infty |f|$ converges, then $\int_0^\infty f$ converges as well.

Proof. Suppose $\int_0^\infty |f|$ converges. Then observe that

$$\left|\int_0^\infty f\right| \leq \int_0^\infty |f|.$$

Since the right-hand side converges, we know that the left-hand side of the inequality also converges via the Comparison test. Hence, $\int_0^\infty f$ converges. ■

Exercise 8.4.10

(a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t}\, dt = 1.$$

Proof. Using part (i) of FTC implies

$$\begin{aligned}\int_0^\infty e^{-t}\ dt &= \lim_{b\to\infty}\int_0^b e^{-t}\ dt\\ &= \lim_{b\to\infty}\Big[-e^{-b}+e^0\Big]\\ &= 0+1\\ &= 1.\end{aligned}$$

■

(b) Show

$$\int_0^\infty e^{-t}\ dt,\ \text{ for all } \alpha > 0. \tag{3}$$

Proof. Let $\alpha > 0$. Using part (i) of FTC, we have

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\ dt &= \lim_{b\to\infty}\int_0^b e^{-\alpha t}\ dt\\ &= \lim_{b\to\infty}\left[\frac{-e^{-bt}}{b}+\frac{e^0}{\alpha}\right]\\ &= 0+\frac{1}{\alpha}\\ &= \frac{1}{\alpha}.\end{aligned}$$

■

Let us now consider the left side of (3). Differentiating the left hand side, we certainly get the following

$$\left[\frac{1}{\alpha}\right]' = \frac{-1}{\alpha^2}.$$

On the right hand side of (3), however, it is not so obvious whether or not we can "distribute" differentiation inside the integral of (3). Let us pretend that we can so we have

$$[e^{-\alpha t}]' = e^{-\alpha t}\cdot(-t).$$

Now let us actually find out if our conjecture that

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\ dt.$$

Exercise 8.4.11

(a) Evaluate $\int_0^b te^{-\alpha t}\ dt$ using the integration-by-parts formula from Exercise 7.5.6. The result will be an expression in α and b.

Proof. Using the integration-by-parts formula, we get that

$$\begin{aligned}\int_0^b te^{-\alpha t}\ dt &= \left[\frac{-t}{\alpha}e^{-\alpha t}\right]_0^b + \frac{1}{\alpha}\int_0^b e^{-\alpha t}\ dt\\ &= \left[\frac{-be^{-\alpha b}}{\alpha}\right] + \frac{1}{\alpha}\left[\frac{-1}{\alpha}e^{-\alpha t}\right]_0^b\\ &= \left[\frac{-be^{-\alpha b}}{\alpha}\right] + \frac{1}{\alpha}\left[\frac{-1}{\alpha}e^{-\alpha b}+\frac{1}{\alpha}\right]\\ &= \frac{-be^{-\alpha b}}{\alpha} - \frac{e^{-\alpha b}}{\alpha^2} + \frac{1}{\alpha^2}\end{aligned}$$

■

(b) Now compute $\int_0^\infty te^{-\alpha t}\ dt$ and verify equation (4).

Proof. Letting $b \to \infty$ in the result in part (a), gives us

$$\int_0^\infty te^{-\alpha t}\ dt = \lim_{b\to\infty} \int_0^b te^{-\alpha t}\ dt = \frac{1}{\alpha^2}.$$

■

Since the above (4) ended up working out, we have to now create a rigorous foundation for why this works.

Exercise 8.4.12

Assume the function $f(x,t)$ is continuous on the rectangle $D = \{(x,t) : a \leq x \leq b, c \leq t \leq d\}$. Explain why the function

$$F(x) = \int_c^d f(x,t)\ dt$$

is properly defined for all $x \in [a,b]$.

Proof. All we need to do is show that F is integrable on D. To do this, we need to show that F is continuous. Let $\varepsilon > 0$ and let $x \in [a,b]$. Since $f : D \to \mathbb{R}$ is continuous at (x_0, t_0), there exists a $\delta > 0$ such that whenever $\|(x,t) - (x_0,t_0)\| < \delta$, it follows that

$$|f(x,t) - f(x_0,t_0)| < \frac{\varepsilon}{d-c}.$$

Then observe that

$$\begin{aligned}
|F(x) - F(u)| &= \left| \int_c^d f(x,t)\ dt - \int_c^d f(u,v)\ dt \right| \\
&= \left| \int_c^d f(x,t) - f(u,v)\ dt \right| \\
&\leq \int_c^d |f(x,t) - f(u,v)|\ dt \\
&< \int_c^d \frac{\varepsilon}{d-c}\ dt = \varepsilon.
\end{aligned}$$

Hence, we have F must be continuous on D and therefore it must be integrable on D. Not only is F continuous but it is uniformly continuous since D is a compact set. ■

There is a direct analogue of continuous functions on compact sets in the $\mathbb{R}^2$ setting.

Theorem 7. If $f(x,t)$ is continuous on D, then $F(x) = \int_c^d f(x,t)\ dt$ is uniformly continuous on $[a,b]$.

Exercise 8.4.13

Prove Theorem 8.4.5.

Proof. Apply the same argument as in exercise 8.4.12. ■

Now let us add the assumption that for every fixed value $t \in [c,d]$, we end up with a differentiable function $f(x,t)$ such that the limit

$$f_x(x,t) = \lim_{z\to x} \frac{f(z,t) - f(x,t)}{z - x}$$

exists for all $(x,t) \in D$. Furthermore, suppose $f_x(x,t)$ is continuous.

Theorem 8. If $f(x,t)$ and $f_x(x,t)$ are continuous on D, then the function $F(x) = \int_c^d f(x,t)\ dt$ is differentiable and

$$F'(x) = \int_c^d f_x(x,t)\ dt.$$

Proof. Fix $x \in [a,b]$ and let $\varepsilon > 0$ be arbitrary. Our goal is to find a $\delta > 0$ such that

$$\Big|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)\ dt\Big| < \varepsilon. \tag{5}$$

whenever $0 < |z-x| < \delta$. ■

Exercise 8.4.14

Finish the proof of Theorem 8.4.6.

Proof. Suppose $f(x,t)$ and $f_x(x,t)$ are continuous on D. Let $\varepsilon > 0$. Observe that

$$\begin{aligned}\frac{F(z)-F(x)}{z-x} &= \frac{1}{z-x}\Big[\int_c^d f(z,t)\ dt - \int_c^d f(x,t)\ dt\Big]\\ &= \int_c^d \frac{f(z,t)-f(x,t)}{z-x}\ dt.\end{aligned}$$

Since $f(x,t)$ is differentiable, there exists $\alpha \in (a,b)$ such that

$$f_x(\alpha,t) = \frac{f(z,t)-f(x,t)}{z-x}$$

by MVT. Since f_x is continuous on D and D is a compact set, we know that f_x is uniformly continuous. Hence, there exists $\delta_1 > 0$ such that whenever $\|(x,t)-(z,t)\| < \delta_1$ where

$$|f_x(\alpha,t) - f_x(x,t)| < \frac{\varepsilon}{d-c}.$$

By assumption, there exists a $\delta_2 > 0$ and then assume $0 < |z-x| < \delta_2$. Then choose $\delta = \min\{\delta_1, \delta_2\}$ such that whenever $0 < |z-x| < \delta$, we have that

$$\begin{aligned}\Big|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)\ dt\Big| &= \Big|\int_c^d \frac{f(z,t)-f(x,t)}{z-x}\ dt - \int_c^d f_x(x,t)\ dt\Big|\\ &= \Big|\int_c^d f_x(\alpha,t)\ dt - \int_c^d f_x(x,t)\ dt\Big|\\ &= \Big|\int_c^d f_x(\alpha,t)\ - f_x(x,t)\ dt\Big|\\ &\le \int_c^d |f_x(\alpha,t) - f_x(x,t)|\ dt\\ &< \frac{\varepsilon}{d-c}\int_c^d\ dt = \varepsilon.\end{aligned}$$

We conclude that $F(x)$ is differentiable and that

$$F'(x) = \int_c^d f_x(x,t)\ dt.$$

■

Exercise 8.4.15

(a) Show that the improper integral $\int_0^\infty e^{-xt}\ dt$ converges uniformly to $1/x$ on the set $[1/2, \infty)$.

Proof. Let $x \in [1/2, \infty)$. Let $\varepsilon > 0$. Taking the integral of $\int_{1/2}^{d} e^{-xt}\, dt$ leads us to

$$\int_{1/2}^{d} e^{-xt}\; dt = \frac{1}{x}(e^{-dx} - e^{(-1/2)x}).$$

Since $\lim\limits_{x\to\infty} 1/x = 0$, we can find a $M > 1/2$ such that

$$\left|\frac{1}{x} - 0\right| < \frac{\varepsilon}{2}$$

whenever $x \geq M$. Furthermore, we know that $f(x,t) = e^{-xt}$ is a uniformly continuous function on $[1/2, \infty)$. Hence, there we know that there exists a $\delta > 0$ such that whenever $\|-1/2 - d\| < \delta$, we have

$$|f(-1/2, t) - f(d, t)| < \frac{\varepsilon}{4}.$$

Using the same choice of $M > 1/2$, we have that

$$\begin{aligned}
\left|\frac{1}{x} - \int_{1/2}^{d} e^{-xt}\; dt\right| &= \left|\frac{1}{x} - \frac{1}{x}\Big(e^{-dx} - e^{(-1/2)x}\Big)\right| \\
&= \left|\frac{1}{x} + \frac{1}{x}\Big(e^{(-1/2)x} - e^{-dx}\Big)\right| \\
&\leq \left|\frac{1}{x}\right| + \frac{1}{x}|e^{(-1/2)x} - e^{-dx}| \\
&\leq \left|\frac{1}{x} - 0\right| + 2|e^{(-1/2)x} - e^{-dx}|. \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, we have that

$$\int_{1/2}^{d} e^{-xt}\; dt \Rightarrow \frac{1}{x}$$

uniformly. ■

(b) Is the convergence uniform on $(0, \infty)$?

Proof. No, the convergence is not uniform. To see why, observe that

$$\left|\frac{1}{x} - \int_{0}^{d} e^{-xt}\; dt\right| = \left|\frac{1}{x} - \frac{1}{x}(e^{-dx} - 1)\right|.$$

We can see that the convergence of $1/x \to 0$ depends on our choice of x because clearly we cannot bound $1/x$ by 0. Hence, the convergence of $\int_0^\infty e^{-xt}\, dt$ on $(0, \infty)$ is pointwise. ■

Exercise 8.4.16

Prove the following analogue of the Weierstrass M-Test for improper integrals: If $f(x,t)$ satisfies $|f(x,t)| \leq g(t)$ and $\int_a^\infty g(t)\, dt$ converges, then $\int_a^\infty f(x,t)\, dt$ converges uniformly on A.

Proof. Our goal is to use the Absolute Comparison test for improper integrals and the comparison test for improper integrals found in Exercise 8.4.9 (b) to show that $\int_0^\infty f(x,t)\, dt$ converges uniformly. Observe that

$$\left|\int_0^\infty f(x,t)\; dt\right| \leq \int_0^\infty |f(x,t)|\; dt \leq \int_0^\infty g(t)\; dt.$$

Since $\int_0^\infty g(t)\, dt$ converges (which means that our choice of M is independent of $x \in [0, \infty)$), we know that $\left|\int_0^\infty f(x,t)\ dt\right|$ must converge absolutely. Hence, we must have $\int_0^\infty f(x,t)\ dt$ converge uniformly by the comparison test. ■

An immediate consequence of Definition 8.4.7 is that the uniform convergence of an improper integral implies that the sequence of functions defined by

$$F_n(x) = \int_c^{c+n} f(x,t)\ dt$$

converges uniformly to $F(x)$ on the closed interval $[a,b]$.

Theorem 9. If $f(x,t)$ continuous on $D = \{(x,t) : a \le x \le b, c \le t\}$, then

$$F(x) = \int_c^\infty f(x,t)\ dt$$

is uniformly continuous on $[a,b]$, provided the integral converges uniformly.

Exercise 8.4.17

Prove Theorem 8.4.8.

Proof. Let $\varepsilon > 0$ and let $x, y \in (a,b)$. Since $f(x,t)$ is continuous on D and D is a compact set, we know that $f(x,t)$ must be uniformly continuous on D. Hence, there exists a $\delta > 0$ such that whenever $\|(x,t)-(y,t)\| < \delta$, we must have

$$|f(x,t) - f(y,t)| < \frac{\varepsilon}{d-c}.$$

Then using the same choice of $\delta > 0$, we can say

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_c^d f(x,t)\ dt - \int_c^d f(y,t)\ dt\right| \\
&= \left|\int_c^d f(x,t) - f(y,t)\ dt\right| \\
&\le \int_c^d |f(x,t) - f(y,t)|\ dt \\
&< \frac{\varepsilon}{d-c}\int_c^d dt = \varepsilon.
\end{aligned}$$

Hence, F is a uniformly continuous function. ■

Theorem 10. Assume the function $f(x,t)$ is continuous on $D = \{(x,t) : a \le x \le b, c \le t\}$ and $F(x) = \int_c^\infty f(x,t)\ dt$ exists for each $x \in [a,b]$. If the derivative function $f_x(x,t)$ exists and is continuous, then

$$F'(x) = \int_c^\infty f_x(x,t)\ dt,$$

provided the integral in (7) converges uniformly.

Exercise 8.4.18

Prove Theorem 8.4.9.

Proof. Let $\varepsilon > 0$. We can use MVT (since F is differentiable) to write

$$\begin{aligned} F'(x) &= \frac{F(z) - f(y)}{z - y} \\ &= \frac{1}{z-y}\int_c^d f(z,t) - f(y,t)\ dt \\ &= \int_c^d \frac{f(z,t) - f(y,t)}{z-y}\ dt \end{aligned}$$

for some $x \in (z, y)$. Since the derivative $f_x(x,t)$ also exists, we can use MVT again to find an $\alpha \in (z, y)$ such that

$$f_x(\alpha, t) = \frac{f(z,t) - f(y,t)}{z-y}.$$

We know that f_x is continuous. Hence, there exists a $\delta > 0$ such that whenever $\|(\alpha, t) - (x, t)\| < \delta$, we have

$$|f_x(\alpha,t) - f_x(x,t)| < \frac{\varepsilon}{d-c}.$$

Since the improper integral $\int_c^d f_x(x,t)\ dt$ converges uniformly, we can find an $M > c$,

$$\begin{aligned} \left|F'(x) - \int_c^d f_x(x,t)\ dt\right| &= \left|\int_c^d f_x(\alpha,t)\ dt - \int_c^d f_x(x,t)\ dt\right| \\ &= \left|\int_c^d f_x(\alpha,t) - f_x(x,t)\ dt\right| \\ &\leq \int_c^d |f_x(\alpha,t) - f_x(x,t)|\ dt \\ &< \frac{\varepsilon}{d-c}\int_c^d\ dt = \varepsilon. \end{aligned}$$

Hence, we conclude that

$$F'(x) = \int_c^\infty f_x(x,t)\ dt.$$

■

Exercise 8.4.19

(a) Although we verified it directly, show how to use the theorems in this section to give a second justification for the formula

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\ dt, \text{ for all } \alpha > 0$$

Proof. Note that $f(\alpha, t) = e^{-\alpha t}$ is continuous on D implies that

$$F(\alpha) = \frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\ dt \tag{1}$$

exists and is also continuous on D. If we differentiate with respect to α, we get that $f_\alpha(\alpha, t)$ is also continuous on D. Thus, we can say that (1) is differentiable and that

$$\frac{1}{\alpha^2} = F'(\alpha) = \int_c^d f_\alpha(\alpha,t)\ dt$$

is defined by Theorem 8.4.9. ■

(b) Now derive the formula

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\ dt, \text{ for all } \alpha > 0.$$

If we set $\alpha = 1$ in equation (8) we get

$$n! = \int_0^\infty t^n e^{-t}\, dt.$$

Proof. We proceed using induction. Observe that for the $n = 1$ case, we know that

$$\int_0^\infty te^{-\alpha t}\, dt = \frac{1}{\alpha^2}$$

by Exercise 8.4.11. Now suppose

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\, dt$$

for $n \geq 1$. We want to show that this holds for the $n + 1$ case; that is, we want to show

$$\frac{(n+1)!}{\alpha^{n+2}} = \int_0^\infty t^{n+1} e^{-\alpha t}\, dt.$$

Observe by the integration-by-parts formula that

$$\begin{aligned}
\int_0^d t^{n+1} e^{-\alpha t}\, dt &= \left[\frac{-t^{n+1}}{\alpha} e^{-\alpha t}\right]_0^d + \frac{(n+1)}{\alpha} \int_0^d t^n e^{-\alpha t}\, dt \\
&= \frac{-d^{n+1}}{\alpha} e^{-\alpha d} + \frac{(n+1)}{\alpha} \int_0^d t^n e^{-\alpha t}\, dt.
\end{aligned}$$

We see that as we take the limit as $d \to \infty$, we have

$$\lim_{d\to\infty} \frac{-d^{n+1}}{\alpha} e^{-\alpha d} = 0.$$

Now using our induction hypothesis, we have

$$\begin{aligned}
\lim_{d\to\infty} \int_0^d t^{n+1} e^{-\alpha t}\, dt &= \frac{(n+1)}{\alpha} \lim_{d\to\infty} \int_0^d t^n e^{-\alpha t}\, dt. \\
&= \frac{(n+1)}{\alpha} \int_0^\infty t^n e^{-\alpha t}\, dt \\
&= \frac{(n+1)}{\alpha} \cdot \frac{n!}{\alpha^{n+1}} \\
&= \frac{(n+1)!}{\alpha^{n+2}}.
\end{aligned}$$

Hence, we conclude that

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\, dt.$$

■

Now we are well on our way to extending the $n!$ on the right-hand and left-hand side of the above equation to $x \in \mathbb{R}$.

Definition 10. For $x \geq 0$, define the *factorial function*

$$x! = \int_0^\infty t^x e^{-t}\, dt.$$

Exercise 8.4.20

(a) Show that $x!$ is an infinitely differentiable function on $(0, \infty)$ and produce a formula for the n^{th} derivative. In particular show that $(x!)'' > 0$.

Proof. Note that $f(x,t) = t^x e^{-t}$ is continuous and so is $f_x(x,t) = \log(t)t^x e^{-t}$. Observe that we can differentiate t^x as many times as we want. Hence, we have that

$$x! = \int_0^\infty t^x e^{-t}\ dt \tag{1}$$

is infinitely differentiable by Theorem 8.4.6. Differentiating (1), we have that

$$\begin{aligned}\frac{d}{dx}[x!] &= \frac{d}{dx}\Big[\int_0^\infty t^x e^{-t}\ dt\Big] \\ &= \int_0^\infty \frac{\partial}{\partial x}[t^x e^{-t}]\ dt \\ &= \int_0^\infty \log(t)t^x e^{-t}\ dt\end{aligned}$$

It can be shown, using induction, that for every $n \geq 1$, that

$$\frac{d^n}{dx^n}[x!] = \int_0^\infty \log^n(t)t^x e^{-t}\ dt.$$

■

(b) Use the integration-by-parts formula employed earlier to show that $x!$ satisfies the functional equation

$$(x+1)! = (x+1)x!.$$

Proof. Using the integration-by-parts formula and the definition of $x!$, we can write

$$\begin{aligned}(x+1)! &= \int_0^\infty t^{x+1}e^{-t}\ dt \\ &= \lim_{d\to\infty}\Big[-t^{x+1}e^{-t}\Big]_0^d + (x+1)\int_0^\infty t^x e^{-t}\ dt \\ &= 0 + (x+1)x! \\ &= (x+1)x!.\end{aligned}$$

Hence, we have

$$(x+1)! = (x+1)x!.$$

■

Theorem 11. Bohr-Mollerup Theorem There is a unique positive function f defined on $x \geq 0$ satisfying
$f(0) = 1$.

(ii) $f(x+1) = (x+1)f(x)$, and

(iii) $\log(f(x))$ is convex.

Since $x!$ satisfies properties (i), (ii), and (iii), it follows that $f(x) = x!$.

Proof. Geometrically, if $[a,b]$ and $[a',b']$ are two intervals in the domain of a convex function ϕ, and $a \leq a'$ and $b \leq b'$, then the slopes the function ϕ would satisfy the following inequality

$$\frac{\phi(b)-\phi(a)}{b-a} \leq \frac{\phi(b')-\phi(a')}{b'-a'}.$$

Since f satisfies properties (i) and (ii) we know $f(n) = n!$ for all $n \in \mathbb{N}$. Now let us fix $n \in \mathbb{N}$ and $x \in (0,1]$. ■

Exercise 8.4.21

(a) Use the convexity of $\log(f(x))$ and the three intervals $[n-1,n]$, $[n,n+x]$, and $[n,n+1]$ to show

$$x\log(n) \leq \log(f(n+x)) - \log(n!) \leq x\log(n+1).$$

Proof. On the interval $[n-1,n]$ and the fact that $f(n)=n!$, we have

$$\begin{aligned}\log(f(n)) - \log(f(n-1)) &= \log(n!) - \log(n-1!)\\ &= \log(n(n-1)!) - \log(n-1)!\\ &= \log(n) + \log(n-1)! - \log(n-1)!\\ &= \log(n).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\log(f(n+1)) - \log(f(n)) &= \log(n+1)! - \log(n)!\\ &= \log((n+1)!n) - \log(n)\\ &= \log(n+1) + \log(n) - \log(n)\\ &= \log(n+1).\end{aligned}$$

Then using the convexity of $\log(f(x))$, we have

$$\log(n) \leq \frac{\log(f(n+x)) - \log(n!)}{x} \leq \log(n+1)$$

which implies

$$x\log(n) \leq \log(f(n+x)) - \log(n!) \leq x\log(n+1).$$

■

(b) Show $\log(f(n+x)) = \log(f(x)) + \log((x+1)(x+2)\dots(x+n))$.

Proof. We can use induction to show the statement above. Let our base case be $n=1$. Then observe that

$$f(x+1) = f(x)(x+1)$$

which implies that

$$\begin{aligned}\log(f(x+1)) &= \log(f(x)(x+1))\\ &= \log(f(x)) + \log((x+1)).\end{aligned}$$

Now assume the statement holds for $n \geq 1$. For the $n+1$ case, observe that

$$\begin{aligned}\log(f(x+(n+1))) &= \log(f((x+n)+1))\\ &= \log(f(x+n)((x+n)+1))\\ &= \log(f(x)) + \log((x+1)(x+2)\dots(x+n)(x+(n+1))).\end{aligned}$$

■

(c) Now establish that

$$0 \leq \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \leq x\log\left(1+\frac{1}{n}\right).$$

Proof. Using our result from part (a), we can subtract $x\log(n)$ on both sides to get

$$0 \leq \log(f(n+x)) - \log(n!) - x\log(n) \leq x\log(n+1) - x\log(n). \tag{1}$$

Now observe the middle of the inequality above. Using our logarithm properties that we derived from earlier sections and using the result from part (b), we get that

$$\begin{aligned}
\log(f(n+x)) - \log(n!) - x\log(n) &= \log(f(x)) + \log((x+1)(x+2)\ldots(x+n)) \\
&\quad - \log(n^x n!) \\
&= \log(f(x)) - \Big(\log(n^x n!) \\
&\quad - \log((x+1)(x+2)\ldots(x+n))\Big) \\
&= \log(f(x)) \\
&\quad - \log\left(\frac{n^x n!}{(x+1)(x+2)\ldots(x+n)}\right).
\end{aligned}$$

Now, focusing on the right side of the inequality in (1), we have

$$\begin{aligned}
x\log(n+1) - x\log(n) &= x\Big(\log(n+1) - \log(n)\Big) \\
&= x\log\left(\frac{n+1}{n}\right) \\
&= x\log\left(1+\frac{1}{n}\right).
\end{aligned}$$

Hence, we have reached our desired result that

$$0 \leq \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \leq x\log\left(1+\frac{1}{n}\right).$$

■

(d) Conclude that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\ldots(x+n)}, \text{ for all } x \in (0,1].$$

Proof. Observe that the right hand side of the inequality found in part (c) converges; that is,

$$\lim_{n\to\infty} x\log\left(1+\frac{1}{n}\right) = 0.$$

Now let $\varepsilon > 0$. Since the limit above converges, we know that we can find an $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in (0,1]$, we have that

$$\left|\log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\ldots(x+n)}\right)\right| < \varepsilon.$$

This implies that

$$\log(f(x)) = \lim_{n\to\infty} \log\left(\frac{n^x n!}{(x+1)(x+2)\ldots(x+n)}\right).$$

However, we know that the limit above holds whenever

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\ldots(x+n)}.$$

■

(e) Finally, show that the conclusion in (d) holds for all $x \geq 0$.

Proof. First, we show that the limit in part (d) converges when $x = 0$ into the limit found in part (d), we get

$$\lim_{n\to\infty} \frac{n!}{1\cdot 2\cdots\ n} = \lim_{n\to\infty} \frac{n!}{n!} = 1! = f(0).$$

Now suppose $x > 0$. By parts (a) and (c), we know that

$$\frac{n^x n!}{(n+x)!}$$

is bounded as well as decreasing for all $n \geq 1$. By MCT, we must have

$$\lim_{n\to\infty} \frac{n^x n!}{(n+x)!} = f(x).$$

■

Exercise 8.4.22

(a) Where does $g(x) = \frac{x}{x!(-x)!}$ equal zero? What other familiar function has the same set of roots?

Proof. We have $g(x) = \frac{x}{x!(-x)!}$ equals zero at every $x \in \mathbb{Z}$. A similar function that has the same roots as $g(x)$ is $f(x) = \sin(\pi x)$. ■

(b) The function e^{-x^2} provides the raw material for the all-important Gaussian bell curve from probability, where it is known that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ Use this fact (and some standard integration techniques) to evaluate $(1/2)!$.

Proof. Using the definition of $x!$, we have

$$\begin{aligned}
(1/2)! &= \int_0^\infty t^{1/2} e^{-t}\, dt \\
&= \lim_{d\to\infty} \int_0^d t^{1/2} e^{-t}\, dt \\
&= \lim_{d\to\infty} \left[t^{1/2} e^{-t}\right]_0^d + \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\, dt \\
&= \int_0^\infty \frac{1}{2} \cdot t^{-1/2} e^{-(\sqrt{t})^2}\, dt && (\text{let } u = \sqrt{t}) \\
&= \int_0^\infty e^{-u^2}\, du && (du = 1/2 \cdot t^{-1/2}\, dt)
\end{aligned}$$

Hence, we have that

$$(1/2)! = \frac{\sqrt{\pi}}{2}.$$

■

(c) Now use (a) and (b) to conjecture a striking relationship between the factorial function and a well-known function from trigonometry.

Proof. The factorial function (most notably the gamma function) can be used to find values of the sine function. It can also be applied to finding values of the Riemann Zeta function found in the last section. ■

Exercise 8.4.23

As a parting shot, use the value for $(1/2)!$ and the Gauss product formula in equation (9) to derive the famous product formula for π discovered by John Wallis in the 1650's:

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

Proof. In the last section, we defined

$$\prod_{i=1}^{n} \frac{2i}{2i-1} = \frac{2^{2n}(n!)^2}{(2n)!}$$

as well as

$$\prod_{i=1}^{n} \frac{2i}{2i+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Using our result from part (b) of Exercise 8.4.22 and the Gauss product formula found in equation (9), we can write

$$\begin{aligned} \frac{\sqrt{\pi}}{2} = (1/2)! &= \int_0^\infty t^{1/2}e^{-t}\,dt \\ &= \lim_{n\to\infty} \frac{n^{1/2}n!}{(\frac{1}{2}+1)(\frac{1}{2}+2)\dots(\frac{1}{2}+n)} \\ &= \lim_{n\to\infty} \frac{n^{1/2}2^n n!}{(2n+1)!}. \end{aligned}$$

By exercise 8.3.5, we have that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Using this fact and the others stated at the beginning of this proof, we have that

$$\begin{aligned} \lim_{n\to\infty} \prod_{i=1}^{n} \frac{2i\cdot 2i}{(2i-1)(2i+1)} &= \Big(\lim_{n\to\infty} \prod_{i=1}^{n} \frac{2i}{2i-1}\Big)\cdot\Big(\lim_{n\to\infty} \prod_{i=1}^{n} \frac{2i}{2i+1}\Big) \\ &= \Big(\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{n^{1/2}(2n)!}\Big)\cdot\Big(\lim_{n\to\infty} \frac{2^{2n}(n!)^2 n^{1/2}}{(2n+1)!}\Big) \\ &= \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2} \\ &= \frac{\pi}{2}. \end{aligned}$$

■

8.4 Fourier Series

Exercise 8.5.1

(a) Verify that

$$u(x,t) = b_n \sin(nx)\cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbb{N}$ and $b_n \in \mathbb{R}$. What goes wrong if $n \notin \mathbb{N}$.

Proof. We first compute the partial derivatives of $u(x,t)$. Observe the first partial derivative with respect to x of $u(x,t)$ is

$$\frac{\partial u}{\partial x} = nb_n \cos(nx)\cos(nt).$$

Then differentiating with respect to the same variable leads to

$$\frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx)\cos(nt).$$

Likewise, differentiating with respect to t leads to

$$\frac{\partial u}{\partial t} = -nb_n \sin(nx)\sin(nt).$$
$$\frac{\partial^2 u}{\partial t^2} = -n^2 b_n \sin(nx)\cos(nt).$$

We can see that (1) is satisfied using our given solution $u(x,t)$. Now, using the boundary conditions found in (2) and the fact that the sine function is 0 with $n\pi$ for all $n \in \mathbb{N}$, we can see that
$$u(0,t) = b_n \sin(0)\cos(nt) = 0$$
and
$$u(\pi,t) = b_n \sin(n\pi)\cos(nt) = 0.$$

Hence, (2) is satisfied. By the same reasoning, we can see that (3) is satisfied.

If $n \notin \mathbb{N}$, then $u(x,t)$ would result in nonzero sine values rendering equations (1), (2), (3) invalid. ∎

(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Proof. Since $\sin(x)$ and $\cos(x)$ are both continuous differentiable up to n times, we can use term-by-term differentiation of the finite sum of these trigonometric functions to show that, indeed, equations (1), (2), and (3) are all satisfied. ∎

Exercise 8.5.2

Using trigonometric identities when necessary, verify the following integrals.

(a) For all $n \in \mathbb{N}$,
$$\int_{-\pi}^{\pi} \cos(nx)\ dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\ dx = 0$$

Proof. We will begin by showing the first integral. Hence, using the fact that $\sin(x)$ is odd and the fact that $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$, we can say
$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\ dx &= \left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{n}[\sin(n\pi) - \sin(-n\pi)] \\ &= \frac{2}{n}\sin(n\pi) \\ &= 0.\end{aligned}$$

Now we show the second integral. Using the fact that $\cos(x)$ is an even function, we can write
$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\ dx &= \left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{n}[\cos(-n\pi) - \cos(n\pi)] \\ &= \frac{1}{n}[\cos(n\pi) - \cos(n\pi)] \\ &= 0.\end{aligned}$$

∎

(b) For all $n \in \mathbb{N}$,
$$\int_{-\pi}^{\pi} \cos^2(nx)\ dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\ dx = \pi.$$

Proof. Using the trigonometric identity $\cos^2(x) = \frac{1}{2}[1+\cos(2x)]$, we can write

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2(2xn)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [1+\cos(2xn)]\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(2xn)\ dx\end{aligned}$$

The first integral evaluates to

$$\frac{1}{2}\int_{-\pi}^{\pi}\ dx = \frac{1}{2}[x]_{-\pi}^{\pi} = \pi$$

Then, using the fact that sine is an odd function and the fact that $\sin(2\pi n) = 0$ for all $n \in \mathbb{N}$, the second integral evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(2xn)\ dx &= \frac{1}{2n}[\sin(2xn)]_{-\pi}^{\pi} \\ &= \frac{1}{2n}[\sin(2\pi n) - \sin(-2\pi n)] \\ &= \frac{1}{2n}[\sin(2\pi)] && (\sin(-2\pi n) = -\sin(2\pi n)) \\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos^2(x)\ dx = \pi.$$

Similarly, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [1-\cos(2xn)]\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\ dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(2xn)\ dx \\ &= \frac{1}{2}\cdot 2\pi - 0 \\ &= \pi.\end{aligned}$$

■

(c) For all $m, n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx = 0.$$

Proof. Let $m, n \in \mathbb{N}$. Suppose $m = n$. Then using a change of variable (letting $u = \sin(mx)$), we get

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\sin(mx)\ dx &= \frac{1}{m}\int_{-\pi}^{\pi} u\ du\\ &= \frac{1}{2m}[\sin^2(\pi x)]_{-\pi}^{\pi}\\ &= \frac{1}{2m}\sin^2(\pi m) + \frac{1}{2m}\sin^2(\pi m)\\ &= \frac{1}{m}\sin^2(\pi m)\\ &= 0.\end{aligned}$$

If $m \neq n$, then using the trigonometric identity

$$\cos(mx)\sin(nx) = \frac{1}{2}[\sin((m+n)x) - \sin((m-n)x)]$$

we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)x) - \sin((m-n)x)]\ dx\\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((m-n)x)\ dx\\ &= \frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} + \frac{\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi}\end{aligned}$$

Using the fact that cosine is an even function we can evaluate the first term which leads to

$$\begin{aligned}\frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)}[\cos((m-n)\pi) - \cos(-(m-n)\pi)]\\ &= \frac{1}{(m-n)}[\cos((m-n)\pi) - \cos((m-n)\pi)]\\ &= 0.\end{aligned}$$

Using the same reasoning, the second term can be written as

$$\begin{aligned}\frac{\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos(-(m+n)\pi)]\\ &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos((m+n)\pi)]\\ &= 0.\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\ dx = 0.$$

Our goal now is to show

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx = 0$$

for all $m \neq n$. Let $m \neq n$. Using the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)],$$

the first integral evaluates to

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos((m+n)x) + \cos((m-n)x)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\ dx \\
&= \frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi}.
\end{aligned}$$

Using the fact that the sine function is odd and the fact that $\sin((m+n)\pi) = 0$ for all $m, n \in \mathbb{N}$, we can write

$$\begin{aligned}
\frac{\sin((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)} \cdot 2\sin((m+n)\pi) \\
&= \frac{\sin((m+n)\pi)}{(m+n)} \\
&= 0
\end{aligned}$$

Likewise, the second term evaluates to

$$\begin{aligned}
\frac{\sin((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)} \cdot 2\sin((m-n)\pi) \\
&= \frac{\sin((m-n)\pi)}{(m-n)} \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0.$$

Now we show the second integral. Similarly, we use the trigonometric identity

$$\sin(a)\sin(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)].$$

Then observe that

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)x) + \sin((m-n)x)]\ dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((m-n)x)\ dx \\
&= \frac{-\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi}.
\end{aligned}$$

Starting with the first term, we can utilize the fact that $\cos(x)$ is an even function to get

$$\begin{aligned}
\frac{-\cos((m+n)x)}{2(m+n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m+n)}[\cos((m+n)\pi) - \cos((m+n)\pi)] \\
&= 0.
\end{aligned}$$

Likewise, the second integral evaluates to

$$\begin{aligned}
\frac{\cos((m-n)x)}{2(m-n)}\Big|_{-\pi}^{\pi} &= \frac{1}{2(m-n)}[\cos((m-n)\pi) - \cos((m-n)\pi)] \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} \sin(mx)\sin(mx)\ dx = 0.$$

■

Exercise 8.5.3

Derive the formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx \text{ and } b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx$$

for all $m \geq 1$.

Proof. Let $m \geq 1$. Multiplying by $\cos(mx)$ on both sides of (6) gives us

$$f(x)\cos(mx) = a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx).$$

Now, taking the integral on the closed interval $[-\pi, \pi]$ on (6) gives us

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx = a_0 &\int_{-\pi}^{\pi}\cos(mx)\ dx \\ &+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}[a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx)]\ dx.\end{aligned}$$

Using exercise 8.5.2, we know that all the integrals in the above equation evaluate to 0 for all $n \in \mathbb{N}$, except when $n = m$ for which the right side of the equation evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx &= \int_{-\pi}^{\pi} a_m\cos^2(mx)\ dx \\ &= a_m\cdot\pi\end{aligned}$$

Solving for a_m gives us our result

$$a_m = \frac{1}{m}\int_{-\pi}^{\pi}\cos^2(mx)\ dx.$$

Using the same reasoning to show the first integral, we have that

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx = 0$$

for all $n \in \mathbb{N}$ except when $n = m$ for which the integral above evaluates to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx = a_0 &\int_{-\pi}^{\pi}\sin(mx)\ dx \\ &+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}[a_n\cos(nx)\sin(mx) + b_n\sin(nx)\sin(mx)]\ dx \\ &= b_m\int_{-\pi}^{\pi}\sin^2(mx)\ dx \\ &= b_m\cdot\pi.\end{aligned}$$

Solving for b_m gives us our desired result

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx.$$

■

Exercise 8.5.4

(a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is *not* uniform on any interval containing 0.

Proof. At the endpoints, $f(x)$ takes on a different value making our convergence pointwise instead of being uniform. ■

(b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?

Proof. Our goal is to produce coefficients a_0, a_m, and b_m so that we have the following representation of $f(x)$; that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

Using the formula for a_0 derived earlier and using the fact that $|x|$ is an even function, we produce

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\ dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\ dx \\ &= \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big|_0^{\pi} \\ &= \frac{\pi}{2}. \end{aligned}$$

Using the symmetry from the absolute value function once again, we get that computing a_m leads to

$$\begin{aligned} a_m &= \frac{2}{\pi}\int_0^{\pi} x\cos(mx)\ dx \\ &= \frac{2}{\pi}\Big[\frac{1}{m}x\sin(mx)\Big]_0^{\pi} - \frac{2}{m\pi}\int_0^{\pi}\sin(mx)\ dx \\ &= \frac{2}{m^2\pi}\cos(mx)\Big|_0^{\pi} \\ &= \frac{2}{m^2\pi}[\cos(m\pi) - 1] \\ &= \begin{cases} 0 & \text{if } m \text{ even} \\ -4/m^2\pi & \text{if } m \text{ odd.} \end{cases} \end{aligned}$$

Since $\sin(mx)$ is an odd function and $|x|$ is an even function, we find that their product $|x|\sin(mx)$ is also odd. Hence, the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx$$

evaluates to zero. Hence, $b_m = 0$. Therefore, $f(x) = |x|$ can be represented by the following

Fourier series
$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$
By looking at the coefficients, we can see that the convergence of (a_n) goes to zero and does not depend on our choice of $x \in [-\pi, \pi]$. ■

Exercise 8.5.5

Explain why h is uniformly continuous on $\mathbb{R}$.

Proof. We know that h is uniformly continuous on $\mathbb{R}$ since we can extend the continuity of h to any interval $(-n\pi, n\pi]$ for all $n \in \mathbb{Z}^+$ as well as having the implication that $\lim_{x\to -n\pi^+} h(x) = h(n\pi)$. ■

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \frac{\varepsilon}{2}$. Notice that the period of $\sin(nx)$ is $2\pi/n$. Hence, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Now consider a particular interval $[a, b]$ of length $2\pi/n$ over which $\sin(nx)$ moves through one complete oscillation.

Exercise 8.5.6

Show that $\left|\int_a^b h(x)\sin(nx)\ dx\right| < \varepsilon/n$, and use this fact to complete the proof.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{\frac{1}{2n}, \frac{\varepsilon}{4mN}\} > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon/2$. Using the period of $\sin(nx)$ which is $2\pi/n$, we can choose an N large enough such that $\pi/n < \delta$ whenever $n \geq N$. Then using the length of $[a, b]$ given to us above and the fact that $|\sin(nx)| \leq 1$, we can write
$$\int_a^b |h(x)||\sin(nx)|\ dx \leq \int_a^b |h(x)|\ dx.$$
Since h is integrable, we know that h must also be bounded. Hence, there exists an $M > 0$ such that $|h(y)| \leq M$ for all $y \in [a, b]$. Now observe that
$$\begin{aligned}
\left|\int_a^b h(x)\sin(nx)\ dx\right| &\leq \int_a^b |h(x)||\sin(nx)|\ dx \\
&= \int_a^b |h(x)|\ dx \\
&< \left(\frac{\varepsilon}{2} + |h(y)|\right)\int_a^b\ dx \\
&= \left(\frac{\varepsilon}{2} + |h(y)|\right)(b-a) \\
&= \frac{\varepsilon}{2}(b-a) + |h(y)|(b-a) \\
&\leq \frac{\varepsilon}{2}(b-a) + M(b-a) \\
&< \frac{\varepsilon}{2}\cdot\frac{2\pi}{n} + M\cdot\frac{2\pi}{n} \\
&< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} \\
&= \frac{\varepsilon}{n}.
\end{aligned}$$
Hence, we have that
$$\left|\int_a^b h(x)\sin(nx)\ dx\right| < \frac{\varepsilon}{n}.$$
We can use the same reasoning to show that if $|\cos(nx)| \leq 1$, then
$$\left|\int_a^b h(x)\cos(nx)\ dx\right| < \frac{\varepsilon}{n}.$$

Letting $\varepsilon > 0$, we know that $\varepsilon/n \to 0$ as $n \to \infty$. This tells us that we can find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\left| \int_a^b h(x) \sin(nx)\ dx \right| < \frac{\varepsilon}{n} \to 0$$

and

$$\left| \int_a^b h(x) \cos(nx)\ dx \right| < \frac{\varepsilon}{n} \to 0.$$

■

Exercise 8.5.7

(a) First, argue why the integral involving $q_x(u)$ tends to zero as $N \to \infty$.

Proof. Observe that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\ du &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (h(u+x) - h(x))\cos(Nu)\ du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(u+x)\cos(Nu)\ du \\ &- \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\cos(Nu)\ du.\end{aligned}$$

Since $h(x)$ continuous on $(-\pi, \pi]$, we know that both

$$\int_{-\pi}^{\pi} h(u+x)\cos(Nu)\ du \to 0$$

and

$$\int_{-\pi}^{\pi} h(x)\cos(Nu)\ du \to 0$$

as $N \to \infty$. Hence, we must also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\ du \to 0$$

as $N \to \infty$

■

(b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Proof. Our goal is to show that $\frac{1}{2}p_x(u)$ is a continuous function so that we may use the Riemann-Lebesgue formula to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\ du = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}p_x(u)\sin(Nu)\ du = 0.$$

We notice that $\frac{1}{2}p_x(u)$ is clearly continuous. Let us show that it is continuous at $x = 0$. Taking advantage of the famous trigonometric limit

$$\lim_{u\to 0} \frac{u/2}{\sin(u/2)} = 1.$$

and the fact that f is differentiable at x, we can write the expression $p_x(u)$ and add a limit as $u \to 0$; that is,

$$\begin{aligned}
\frac{1}{2}\lim_{u\to 0} p_x(u) &= \frac{1}{2}\lim_{u\to 0} \frac{(f(u+x)-f(x))\cos(u/2)}{u\sin(u/2)} \\
&= \lim_{u\to 0} \frac{f(u+x)-f(u)}{2\sin(u/2)} \\
&= \lim_{u\to 0} \frac{(f(u+x)-f(x))}{2\sin(u/2)} \cdot \frac{u}{u} \\
&= \lim_{u\to 0} \frac{(f(u+x)-f(x))\cos(u/2)}{u} \\
&\cdot \lim_{u\to 0} \frac{u/2}{\sin(u/2)} \cdot \lim_{u\to 0} \cos(u/2) \\
&= f'(x)\cdot 1 \cdot 1.
\end{aligned}$$

Hence, we have that $\frac{1}{2}p_x(u)$ is a continuous function. Now, using the Riemann-Lebesgue theorem, we can say that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\lim_{u\to 0} p_x(u)\right)\sin(Nu)\ du &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\lim_{u\to 0} p_x(u)\right)\sin(Nu)\ du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(Nu)\ du \\
&= 0.
\end{aligned}$$

Hence, we also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\ du = 0$$

■

Exercise 8.5.8

Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converges to the same limit. Give an example to show that it is possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Proof. Let $\varepsilon > 0$. Suppose $(x_n) \to x$. We can choose $N \in \mathbb{N}$ such that for any $N \in \mathbb{N}$, we have

$$|x_n - x| < \varepsilon.$$

Then using the same choice of $N \in \mathbb{N}$, we can write

$$\begin{aligned}
|y_n - x| &= \left|\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} x\right| \\
&= \frac{1}{n}\left|\sum_{i=1}^{n}(x_i - x)\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}|x_i - x| \\
&< \frac{\varepsilon}{n}\sum_{i=1}^{n} \\
&= \frac{\varepsilon}{n}\cdot n = \varepsilon.
\end{aligned}$$

■

Exercise 8.5.9

Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)}\left[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}\right]^2.$$

Proof. Let $\theta \neq 2\pi n$ so that we may have

$$D_n(\theta) = \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$$

be defined. Before we begin the derivation, we need a few more identities to make the computations simpler; that is, let us use $\sin(a)\cos(a) = 1/2\sin(2a)$ and $1/2(1+\cos(2a)) = \cos^2(a/2)$. Notice that we may write

$$D_N(\theta) = \frac{1}{2} + \sum_{n=1}^{N}\cos(n\theta) = \frac{1}{2}\Big(\cos(N\theta) + \frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big).$$

Using the identities given to us, let's manipulate the left-hand side of the equation to get

$$\begin{aligned}
\frac{1}{N+1}\Big[\frac{1}{2} + \sum_{n=1}^{N} D_n(\theta)\Big] &= \frac{1}{N+1}\Big[\frac{1}{2} + \frac{1}{2\sin(\theta/2)}\sum_{n=1}^{N}\sin((n+1/2)\theta)\Big] \\
&= \frac{1}{N+1}\Big[\frac{1}{2} + \frac{1}{2\sin(\theta/2)}\Big(\cos(\theta/2)\sum_{n=1}^{N}\sin(n\theta) \\
&\quad + \sin(\theta/2)\sum_{n=1}^{N}\cos(n\theta)\Big)\Big] \\
&= \frac{1}{2(N+1)}\Big[1 + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta) + \sum_{n=1}^{N}\cos(n\theta)\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2} + \sum_{n=1}^{N}\cos(n\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta) + \frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[D_N(\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta) + \frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2}\Big(\cos(N\theta) + \frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big) \\
&\quad + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{n=1}^{N}\sin(n\theta) + \frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)}\Big[\frac{1}{2}\Big(\cos(N\theta) + \frac{\sin(N\theta)\cos(\theta/2)}{\sin(\theta/2)}\Big) \\
&\quad + \frac{\cos(\theta/2)}{\sin(\theta/2)}\Big(\frac{\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2})}{\sin(\theta/2)}\Big) + \frac{1}{2}\Big] \\
&= \frac{1}{2(N+1)\sin^2(\theta/2)}[\mathbf{A}]
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2}(\cos(N\theta)\sin^2(\theta/2) + \sin(\theta/2)\sin(N\theta)\cos(\theta/2)) \\
&\quad + \cos(\theta/2)\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2}) + \frac{1}{2}\sin^2(\theta/2). \qquad (1)
\end{aligned}$$

Working backwards and using the identities we mentioned at the beginning of this proof, we can manipulate $\sin^2((N+1)\frac{\theta}{2})$ to get

$$\begin{aligned}\sin^2((N+1)\theta/2) &= [\sin(N\theta/2)\cos(\theta/2)+\cos(N\theta/2)\sin(\theta/2)]^2\\ &= (\sin^2(N\theta/2)\cos^2(\theta/2)\\ &+ 2\sin(N\theta/2)\cos(N\theta/2)\sin(\theta/2)\cos(\theta/2)\\ &+ \cos^2(N\theta/2)\sin^2(\theta/2))\\ &= \sin^2(N\theta/2)\cos^2(\theta/2)+\frac{1}{2}\sin(\theta)\sin(N\theta)\\ &+ \cos^2(N\theta/2)\sin^2(\theta/2).\end{aligned}$$

Hence, all we need to show is

$$\sin^2((N+1)\theta/2) = \sin^2(N\theta/2)\cos^2(\theta/2)+\frac{1}{2}\sin(\theta)\sin(N\theta)+\cos^2(N\theta/2)\sin^2(\theta/2)$$

and we're done. Manipulating (1), we write

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}(\cos(N\theta)\sin^2(\theta/2)+\sin(\theta/2)\sin(N\theta)\cos(\theta/2))\\ &+ \cos(\theta/2)\sin((N+1)\frac{\theta}{2})\sin(\frac{N\theta}{2})+\frac{1}{2}\sin^2(\theta/2)\\ &= \sin^2(\theta)\frac{1}{2}(1+\cos(N\theta))+\frac{1}{2}\sin(\theta/2)\cos(\theta/2)\sin(N\theta)\\ &+ \cos(\theta/2)\sin(N\theta/2)\sin((N+1)\theta/2)\\ &= \sin^2(\theta)\cos^2(N\theta/2)+\frac{1}{4}\sin(\theta)\sin(N\theta)+\frac{1}{4}\sin(\theta)\sin(N\theta)\\ &+ \sin^2(N\theta/2)\cos^2(\theta).\end{aligned}$$

Hence, we have that

$$\mathbf{A} = \sin^2((N+1)\theta/2)$$

and thus we can conclude that

$$\frac{1}{N+1}\Big[\frac{1}{2}+\sum_{n=1}^{N}D_n(\theta)\Big] = \frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\theta/2)}\Big]^2$$

for $\theta \neq 2\pi n$. ■

Exercise 8.5.10

(a) Show that

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ du.$$

Proof. Let $\theta \neq 2\pi n$ and $u = t - x$. Using our results from Exercise 8.5.9, we can write

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} S_n(x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\Big[\frac{1}{N+1}\sum_{n=0}^{N} D_n(u)\Big]\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{N+1}\Big[\frac{1}{2}+\sum_{n=1}^{N} D_n(u)\Big]\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\Big]^2\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ dx.
\end{aligned}$$

Thus, we have

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ dx.$$

■

(b) Graph the function $F_N(u)$ for several values of N. Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \to 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi, \pi]$).

Proof. We can leave the graphing to your favorite Ti-84 calculator. We shall prove that $F_N \to 0$ uniformly. Let $u \in \{u : |u| \geq \delta\}$ where $\delta > 0$. Let $\varepsilon > 0$. Since $1/2(N+1) \to 0$, there exists an $M \in \mathbb{N}$ such that for any $N \geq M$, we have

$$\left|\frac{1}{2(N+1)}\right| < \varepsilon.$$

Furthermore, the sine function is bounded by 1. Hence, we can write the following

$$\begin{aligned}
\left|\frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\Big]^2\right| &\leq \left|\frac{1}{2(N+1)}\right| \cdot \left|\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\right|^2 \\
&\leq \frac{1}{2(N+1)} \\
&< \varepsilon.
\end{aligned}$$

Hence, we must also have

$$\frac{1}{2(N+1)}\Big[\frac{\sin((N+1)\frac{u}{2})}{\sin(u/2)}\Big]^2 \to 0$$

uniformly. ■

(c) Prove that $\int_{-\pi}^{\pi} F_N(u)\ du = \pi$.

Proof. We will use the fact that $\int_{-\pi}^{\pi} D_N(u)\ du = \pi$ to show the result. Using the definition of $F_N(u)$ and reordering indices, we can (specifically we will be using the left-hand side of the equation) write

$$\begin{aligned}
\int_{-\pi}^{\pi} F_N(u)\ du &= \int_{-\pi}^{\pi} \Big[\frac{1}{N+1}\Big(\frac{1}{2} + \sum_{n=1}^{N} D_n(u)\Big)\Big]\ du \\
&= \frac{1}{N+1}\int_{-\pi}^{\pi} \Big(D_0(u) + \sum_{n=1}^{N} D_n(u)\Big)\ du. \\
&= \frac{1}{N+1}\int_{-\pi}^{\pi} \Big[\sum_{n=1}^{N+1} D_{n-1}(u)\Big]\ du \\
&= \frac{1}{N+1}\sum_{n=1}^{N+1} \Big[\int_{-\pi}^{\pi} D_{n-1}(u)\ du\Big] \\
&= \frac{\pi}{N+1}\sum_{n=0}^{N+1} \\
&= \frac{\pi}{N+1} \cdot N+1 \\
&= \pi.
\end{aligned}$$

Hence, we have

$$\int_{-\pi}^{\pi} F_N(u)\ du = \pi.$$

■

(d) To finish the proof of Fejér's Theorem, first choose $\delta > 0$ so that

$$|u| < \delta \text{ implies } |f(x+u) - f(x)| < \varepsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x.

Proof. Let $\varepsilon > 0$. First, we show that $\sigma_n(x) \to f(x)$ on $|u| \leq \delta$. Using the fact that f is uniformly continuous on $(-\pi, \pi]$, we can write

$$\begin{aligned}
|\sigma_n(x) - f(x)| &= \Big|\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\ du - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F_N(u)\ du\Big| \\
&= \frac{1}{\pi}\Big|\int_{-\pi}^{\pi} (f(u+x) - f(x))F_N(u)\ du\Big| \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(u+x) - f(x)|F_N(u)\ du \\
&< \frac{\varepsilon}{\pi}\int_{-\pi}^{\pi} F_N(u)\ du \\
&= \frac{\varepsilon}{\pi} \cdot \pi = \varepsilon.
\end{aligned}$$

Now suppose $|u| \geq \delta$. Since f is uniformly continuous, we can bound f by some $M > 0$. Hence, we must have

$$|f(u+x) - f(x)| \leq 2M.$$

Then we can use the fact that $F_N \to 0$ on sets such as $|u| \geq \delta$ to write

$$\begin{aligned}
|\sigma_n(x) - f(x)| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(u+x) - f(x)||F_N(u)|\, du \\
&\leq \frac{2M}{\pi}\int_{-\pi}^{\pi} |F_N(u)|\, du \\
&< \frac{2M}{\pi} \cdot \frac{\varepsilon}{4M}\int_{-\pi}^{\pi} du \\
&= \frac{\varepsilon}{2\pi} \cdot 2\pi \\
&= \varepsilon.
\end{aligned}$$

■

8.4.1 Weierstrass Approximation Theorem

Fejér's Theorem gives us a nice and concise argument for the Weierstrass Approximation Theorem found in section 6.7. We can restate here as a reminder.

Theorem 12 (Weierstrass Approximation Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there exists a polynomial $p(x)$ satisfying

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$.

We can use the fact that $\sin(x)$ and $\cos(x)$ is bounded on a compact set to show theorem above.

Exercise 8.5.11

(a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a, b]$ is $[0, \pi]$.

Proof. Let $\varepsilon > 0$. Since $f(x)$ is continuous on $[0, \pi]$, we can pick a $\delta > 0$ such that whenever $|u| < \delta$, we have

$$|f(u+x) - f(x)| < \varepsilon.$$

Observe that

$$\int_0^{\pi} F_N(u)\, du = \frac{1}{2}\int_{-\pi}^{\pi} F_N(u)\, du = \frac{\pi}{2}.$$

We can use σ_N defined in Fejér's Theorem to write

$$\begin{aligned}
|\sigma_N(x) - f(x)| &= \left|\frac{2}{\pi}\int_0^{\pi} f(u+x)F_N(u)\, du - \frac{2}{\pi}\int_0^{\pi} f(x)F_N(u)\, du\right| \\
&= \frac{2}{\pi}\left|\int_0^{\pi} (f(u+x) - f(x))F_N(u)\, du\right| \\
&\leq \frac{2}{\pi}\int_0^{\pi} |f(u+x) - f(x)|F_N(u)\, du \\
&< \frac{2\varepsilon}{\pi}\int_0^{\pi} F_N(u)\, du \\
&= \frac{2\varepsilon}{\pi} \cdot \frac{\pi}{2} = \varepsilon.
\end{aligned}$$

Note that the polynomial we found was just $\sigma_N(x) = p(x)$. Hence, we have that

$$|f(x) - p(x)| < \varepsilon.$$

■

(b) Show how the case for an arbitrary interval $[a, b]$ follows from this one.

Proof. Taking advantage of f's uniformly continuity on $(-\pi, \pi]$, we can extend uniform continuity of f to all of $\mathbb{R}$. Using Fejér's Theorem, we can find a polynomial $p(x)$ (in this case, $p(x) = \sigma_N(x)$) such that

$$|f(x) - p(x)| < \varepsilon$$

for any interval $[a, b]$. ■

8.5 Construction of R from Q

Exercise 8.6.1

(a) Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{t \in \mathbb{Q} : t < r\}$ is a cut.

Proof. For the first property (C1), we show that $C_r \neq \emptyset$ and $C_r \neq \mathbb{Q}$. Suppose $r \in \mathbb{Q}$ is fixed. Now define $t = r - 1$. Then clearly, we have $t < r$. But this means $t \in C_r$. Hence, $C_r \neq \emptyset$. Now, we show $C_r \neq \mathbb{Q}$. Define $s = r + 1 \in \mathbb{Q}$. Then clearly, $s > r$. Hence, $s \notin C_r$ and thus $C_r \neq \mathbb{Q}$.

For the second property (C2), we need to show that if $r \in C_r$, then C_r also contains every rational $q < r$. Suppose we let $t, q \in \mathbb{Q}$ be arbitrary with $t > q$ with $t \in C_r$. Then by definition of C_r, we must have $t < r$. But we have $q < t < r$. Hence, we have $q < r$ for any $q \in \mathbb{Q}$.

Lastly, we need to show that C_r does not have a maximum; that is, for any $x \in C_r$, there exists an $s \in C_r$ with $x < s$. Let $x \in C_r$. Then by definition of C_r, we must have $x < r$. Suppose we take the midpoint between x and r. Then we must have $x < \frac{x+r}{2} < r$. Setting $s = \frac{x+r}{2}$ gives us our desired result. ■

Avoid thinking of cuts as only having this form. Which of the following subsets of $\mathbb{Q}$ are cuts?

(b) $S = \{t \in \mathbb{Q} : t \leq 2\}$

Proof. This is not a cut because S contains a maximum. In this case, the maximum is 2. ■

(c) $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$

Proof. The set T is a cut. First, we show (C1). Observe that $0 \in T$. Hence, $T \neq \emptyset$. Then observe that $2 \notin T$. Hence, $T \neq \mathbb{Q}$.

Next, we show (C2). Let $r \in T$. Then by definition of T, we must have either $r^2 < 2$ or $r < 0$. If $r < 0$, then we can easily pick $q < r$ for any $q < 0$. Otherwise, we have $r^2 < 2$ such that if we let $q \in \mathbb{Q}$ be arbitrary with $q^2 < r^2$. Then we easily have $q < r$.

Finally, we show (C3). Let $r \in T$. Then either we have $r^2 < 2$ or $r < 0$. Consider

$$\begin{aligned}\left(r + \frac{1}{n}\right)^2 &= r^2 + \frac{2r}{n} + \frac{1}{n} \\ &= r^2 + \frac{2r+1}{n}.\end{aligned}$$

Let us set $s = r + \frac{1}{n_0}$. If $r > 0$, then we can pick an n_0 large enough such that

$$\frac{1}{n_0} < \frac{4 - r^2}{2r+1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < r^2 + \frac{2r+1}{n_0} \cdot \frac{4 - r^2}{2r+1} < 4.$$

Hence, we have $r < s < 2$. Otherwise, we choose n_0 large enough so that

$$\frac{1}{n_0} < \frac{-r^2}{2r+1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < 0.$$

This implies that $r < s < 0$. ■

(d) $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$.

Proof. Is a cut. Same reasoning can be used to prove that U is a cut, but we can assume $a \geq 0$ this time. ■

Exercise 8.6.2

Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Proof. Suppose for sake of contradiction that $r \geq s$. Since $r \in A$ and $s \in \mathbb{Q}$, we know that $s < r$. But this means that $s \in A$ which leads to a contradiction. ■

Exercise 8.6.3

Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$, respectively.

Proof. ($\mathbb{N}$) This is not a field since an additive inverse does not exist; that is, $-1 \notin \mathbb{N}$.

($\mathbb{Z}$) The set of integers is not a field because every element in $x \in \mathbb{Z}$ does not contain an inverse except for 1.

($\mathbb{Q}$) Field.

■

Exercise 8.6.4

Show that this defines an ordering on $\mathbb{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Proof. Observe that the first property (o1) follows immediately. For the second property (o2), suppose $A \leq B$ and $B \leq A$. Then we have $A \subseteq B$ and $B \subseteq A$. This implies that $A = B$. Finally, for the third property (o3), let $A, B, C \in \mathbb{R}$. If $A \leq B$ and $B \leq C$, then we have $A \subseteq B$ and $B \subseteq C$. Then we have $A \subseteq B \subseteq C$. Hence, we have $A \subseteq C$. Hence, we have an ordering on $\mathbb{R}$. ■

Exercise 8.6.5

(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

Proof. For property (c1), let $a \in A$. Then $a = (a - b) + b$ where $a - b \in A$ and $b \in B$ implies $a \in A + B$. Hence, we have that $A + B \neq \emptyset$. Let $c \in C$ where $C \subseteq \mathbb{Q}$, then we find that $A + C \not\subseteq A + B$. Hence, the sum $a + c \notin A + B$. Hence, $A + B \neq \mathbb{Q}$. To show property (c3), Let $a + b \in A + B$ be arbitrary. Since A is a cut, we can find an $\alpha \in A$ such that $a < \alpha$. Likewise, B being a cut implies that there exists $\beta \in B$ such that $b < \beta$. Adding the two inequalities together, we get that $a + b < \alpha + \beta$. Denote the set $S = \{\alpha + \beta : \alpha \in A \text{ and } \beta \in B\}$. Then we have $A + B \subseteq S$ which proves property (c3). Since properties (c1), (c2), and (c3) have been satisfied, we can now conclude that $A + B$ is a cut. ■

(b) Check that addition in $\mathbb{R}$ is commutative (f1) and associative (f2).

Proof. Since $A + B$ is a cut and its elements $a + b$, where $a \in A$ and $b \in B$ are both cuts, subsets of $\mathbb{Q}$ where $\mathbb{Q}$ is an ordered field, we know that addition must be commutative and associative. This means that

$$A + B = B + A.$$

If we have an extra set C that is also a cut, then we would have $(a + b) + c = a + (b + c)$

implying that

$$(A+B)+C = A+(B+C).$$

Hence, we have (f1) and (f2) are satisfied. ■

(c) Show that property (o4) holds.

Proof. Suppose we have $A \subseteq C$ where A and C are both cuts. Let B be a cut. Then $A \subseteq C$ implies $a \leq c$. If we have $b \in B$ be arbitrary, then we can use the ordering of $\mathbb{Q}$ to write $b + a \leq b + c$. But this means that $B + A \subseteq B + C$ and hence property (o4) is satisfied. ■

(d) Show that the cut

$$O = \{p \in \mathbb{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions : $A + O \subseteq A$ and $A \subseteq A + O$.)

Proof. First, we show $A + O \subseteq A$. Let $a + p \in A + O$ be arbitrary. Since $p < 0$, we must have $a + p \leq a$. But this means that $A + O \subseteq A$. For the second inclusion $A \subseteq A + O$, let $a \in A$. Then observe that $a = (a - p) + p$. Since $p < 0$, we can write $a = (a - p) + p$. But since $a - p \in A$ and $p \in O$, we know that $a \in A + O$. Hence, we have $A \subseteq A + O$. ■

Exercise 8.6.6

(a) Prove that $-A$ defines a cut.

Proof. To show the (c1), let $r \in \mathbb{Q}$. Since $\mathbb{Q}$ is a field, we can rewrite r in the following way

$$r = r + (t - t) = (r + t) - t = (t + r) - t < -t$$

with $t \notin A$. Multiplying through the inequality by a negative leads us to $-r > t$. This tells us that $r \in -A$ and we must have $-A \neq \emptyset$. However, this also implies that $-r \notin -A$ and hence, $-A \neq \mathbb{Q}$. To show (c2), let $r \in -A$. Then by definition of $-A$, there exists a $t \notin A$ such that $-r > t$. Suppose we let $-q \in \mathbb{Q}$ be arbitrary with $-q > -r$. Using the fact that $\mathbb{Q}$ is a field, we can multiply the inequality by a negative to get $r > q$ our desired result. Hence, (c2) is satisfied.

Finally, we show (c3). Suppose $r \in -A$. Then there exists $t \notin A$ such that $-r > t$. Multiplying through the inequality by a negative gives us $r < -t$ which is our desired result. Hence, $-A$ defines a cut. ■

(b) What goes wrong if we set $-A = \{r \in \mathbb{Q} : -r \in A\}$?

Proof. If we have $-A$ defined as the set above, then it would fail (c3). We can see this when we define $A = \{r \in \mathbb{Q} : r < 0\}$ and let $-A = \{r \in \mathbb{Q} : r \leq 0\}$ which tells us that $-A$ contains a maximum. ■

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in the definition of a field.

Proof. Let $a \in A$ and $r \in -A$. Let $a + r \in O$. Since $r \in -A$, we know that there exists $t \notin A$ with $t < -r$. Now, we can write $a + r < a - t$. But $-t < 0$ so this tells us that $a + r < 0$. Hence, $a + r \in O$ and we have $A + (-A) \subseteq O$. Now we need to show $O \subseteq A + (-A)$. Let us fix $o \in O$ and finding $a \in A$ such that $a + b = o$. Let $\varepsilon = |o|/2 = -o/2$. Taking advantage of properties (c1) and (c2), we can choose a rational $t \notin A$ such that $t - \varepsilon \in A$. If no such t existed then we would either have $A = \mathbb{Q}$ or $A = \emptyset$. Now, $t \notin A$ implies $-(t + \varepsilon) \in -A$. Then

$$o = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A),$$

and hence we conclude $O \subseteq A + (-A)$. Hence, (f4) is proven. ■

Remark. Another possible way you could show the reverse inclusion in part (c) is to rewrite $o \in O$ such that $o = (o + r) - r$ and show that $0 + r \in -A$ for which, in addition to $-r \in A$, leads to $O \subseteq A + (-A)$. Since there exists $t \notin A$ such that $-r > t$ and $o < 0$, we have

$$o + r < r \Rightarrow o + r < r < -t.$$

Multiplying the inequality by a negative gives us $-(o+r) < t$, which tells us that $o+r \in -A$. Since we also have $-r \in A$, we conclude that $o \in A + (-A)$ and hence $O \subseteq A + (-A)$.

Exercise 8.6.7

(a) Show that AB is a cut and that property (o5) holds.

Proof. To show (c1), let $a, b \in \mathbb{Q}$ with the property that $a, b \geq 0$. Since $\mathbb{Q}$ is a field, we can take the product of a and b and get $a \cdot b \geq 0$ implying that $AB \neq \emptyset$. Likewise, if we take any $p \in \mathbb{Q}$ with $p < 0$ and multiply it by any strictly non-negative $a \in \mathbb{Q}$, then $p \cdot a < 0$ implying that $AB \neq \emptyset$ once again. We know that $AB \neq \emptyset$ since $q \leq 0$ is not in AB. Hence, (c1) is satisfied.

To show (c2), we need to show that for all $q \in \mathbb{Q}$ that $r \in AB$ implies $q < r$. Suppose $r = a \cdot b$ with $a \in A$ and $b \in B$. Since A and B are cuts, we can use (c2) to state that for all $\alpha \in A$ and for all $\beta \in B$, we have such $\alpha < a$ and $\beta < b$ respectively. Then taking the product of these inequalities, we get our desired result that $q = \alpha \cdot \beta < a \cdot b = r$. Hence, (c2) is satisfied.

Finally, we show (c3). Let $r \in AB$ where $r = a \cdot b$ where $a \in A$ and $b \in B$. Since A and B are cuts, we can find $w \in A$ and $z \in B$ such that $r = a \cdot b < z \cdot w = \omega$. Hence, (c3) is satisfied.

Now we need to show that property (o5) holds for AB. Assume $A \geq O$ and $B \geq O$. We know that $O \subseteq A$ and $O \subseteq B$. This means that if we let $p, w \in O$, then $p, w \in A$ and $p, w \in B$. Note that $p \cdot w < 0$. Since $ab \geq 0$, we know that $pw \leq ab$ which implies that $pw \in AB$ and hence, $O \subseteq AB$. ∎

(b) Propose a good candidate for the multiplicative identity on $\mathbb{R}$ and show that this works for all cuts $A \geq O$.

Proof. A good candidate for the multiplicative identity is I. Let us define it as follows:

$$I = \{p \in \mathbb{Q} : p < 1\}.$$

We want to show that $AI = A$; that is, we need to show the following two inclusions:

$$AI \subseteq A \text{ and } A \subseteq AI.$$

Let $x \in AI$. Then we have $xp < x \cdot 1 = x$ which tells us $xp \in A$ and we have $AI \subseteq A$. Now if $x \in A$, then we have $x \cdot 1 > xp$ and hence $x \in AI$. Thus, we have $A \subseteq AI$. Hence, $AI = A$. ∎

(c) Show the distributive property (f5) holds for non-negative cuts. Before moving on to the proof, define $A(B + C)$ as

$$A(B + C) = \{a(b + c) : a \in A, b + c \in B + C \text{ with } a, b + c \geq 0\} \cup \{p \in \mathbb{Q} : p < 0\}.$$

Then let

$$P = \{a(b + c) : a \in A, b + c \in B + C \text{ with } a, b + c \geq 0\}.$$

Proof. The main goal is to show that $A(B + C) = AB + AC$; that is, we need to show the following two inclusions:

$$A(B + C) \subseteq AB + AC \text{ and } AB + AC \subseteq A(B + C).$$

Since $x \in A(B + C)$, then either $x \in P$ or $x \in O$. Suppose we have $x \in P$. Then we have $x = a(b + c)$. Since $\mathbb{Q}$ is an ordered field, we can use the distributive property to write $a(b + c) = ab + ac \in AB + BC$. Hence, $a(b + c) \in AB + BC$ and $A(B + C) \subseteq AB + BC$. Now suppose $x \in O$, then $x < 0$. Since the products AB and BC are non-negative, we know that we must have $ab \geq 0$ and $bc \geq 0$. This tells us that their sum $ab + ac \geq 0$ and hence, we have $x < 0 \leq ab + ac$. Since $x \leq ab + ac$, we can write $O \subseteq AB + AC$.

Next, we show $AB + BC \subseteq A(B + C)$. Let $x \in AB + BC$. Then $x = r_1 + r_2$ where $r_1 \in AB$ and $r_2 \in BC$. If $r_1 \in AB$, then either $r_1 = ab \geq 0$ or $r_1 < 0$. Likewise, $r_2 \in BC$ implies that either $r_2 = ac \geq 0$ or $r_2 < 0$. Suppose $r_1 = ab \geq 0$ and $r_2 = ac \geq 0$. Then we can simply use the distributive property to write $ab + ac = a(b + c) \in A(B + C)$. Hence, $ab + ac \in A(B + C)$. Now suppose $r_1 \geq 0$ and $r_2 < 0$. Then $x = r_1 + r_2 < r_1 + 0$. But note that $AB + O \subseteq O$ which tells us that $x \in A(B + C)$. The case for when $r_2 \geq 0$ and $r_1 < 0$ is similar. If both $r_1 < 0$ and $r_2 < 0$, then $x = r_1 + r_2 < 0 \leq a(b + c)$ and hence, $x \in A(B + C)$. Thus, we can conclude that $AB + BC \subseteq A(B + C)$. ■

Exercise 8.6.8

Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

(a) First, prove that $S \in \mathbb{R}$ by showing that it is a cut.

Proof. First, we show (c1). Since S is defined as the union of nonempty sets A, we also have that $S \neq \emptyset$. Since all the sets within S are just a union of cuts, we know that these cuts also have the property that $A \neq \mathbb{Q}$ and hence $S \neq \mathbb{Q}$.

Next, we show property (c2). Let $x \in S$. Since S is defined as the union of all cuts $A \in \mathcal{A}$, we know that $x \in S$ implies that there exists an A' such that $x \in A'$. Let $q \in \mathbb{Q}$ But since A' is a cut, we know that we can use (c2) to say that $q < x$. Hence, (c2) is satisfied.

Lastly, we show property (c3). Let $x \in S$. Using the same reasoning to show (c2), we know that $x \in A'$ implies that there exists $\alpha \in Q$ such that $x < \alpha$.

Since all the properties of a cut are satisfied, we can now conclude that $S \in \mathbb{R}$ is also a cut. ■

(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Proof. In order to show that S is the least upper bound of $\mathcal{A}$, we need to show that S is an upper bound and that S is the least upper bound.

First, we show S is an upper bound; that is, $S \geq A$ for all $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be arbitrary. Observe that since $S = \bigcup A$, we know that $A \subseteq \bigcup A = S$. But this tells us that $A \leq S$. Since A is arbitrary, we know that S must be an upper bound.

Lastly, we show that S is the *least upper bound*. Let $B \in \mathcal{A}$ be any upper bound. Since each set A in the union $S = \bigcup A$ is bounded by B (that is, $A \subseteq B$), it follows that $S \subseteq B$. But this tells us that $S \leq B$ and hence, S must be the least upper bound of $\mathcal{A}$. ■

Exercise 8.6.9

Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbb{Q} : t < r\}$$

where $r \in \mathbb{Q}$. (See Exercise 8.6.1)

(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbb{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

Proof. Let $r, s \in \mathbb{Q}$. First, we show $C_r + C_s = C_{r+s}$; that is, we need to show that following two inclusions:

$$C_r + C_s \subseteq C_{r+s} \text{ and } C_{r+s} \subseteq C_r + C_s.$$

Let $z \in C_r + C_s$. Then we have $z = x + y$ where $x \in C_r$ and $y \in C_s$. By definition of C_r and C_s, we know that $x < r$ and $y < s$ respectively. Hence, adding both of these inequalities produces $z = x + y < r + s$ which tells us that $z \in C_{r+s}$. Thus, we have $C_r + C_s \subseteq C_{r+s}$.

Now, we show the reverse inclusion. Let $z \in C_{r+s}$. Then $z < r + s$ by definition of C_{r+s}. Note that subtract s from both sides of the inequality to get $z - s < r$. This tells us that $z - s \in C_r$. Likewise, subtract r from both sides of the inequality to get $z - r < s$ which implies that $z - r \in C_s$. We can add these two distinct elements to get

$$(z-r) + (z-s) = 2z - (r+s) < 2(r+s).$$

Dividing by 2 then gives us

$$z - \frac{r+s}{2} < r + s.$$

Hence, $z \in C_r + C_s$. and we conclude that $C_{r+s} \subseteq C_r + C_s$.

Let $r, s \geq 0$ in $\mathbb{Q}$. We want to show $C_r C_s = C_{rs}$. To do this, we need to show the following two inclusions; that is,

$$C_r C_s \subseteq C_{rs} \text{ and } C_{rs} \subseteq C_r C_s.$$

First, let $x \in C_r C_s$. Then either $x = \alpha\beta$ where $\alpha \cdot \beta \geq 0$ where $\alpha \in C_r$ and $\beta \in C_s$ or $x < 0$. If we assume the former, we can take $\alpha \in C_r$ imply $\alpha < r$ and $\beta \in C_s$ imply $\beta < s$. Multiplying these two inequalities together, we must have $\alpha\beta < rs$ which implies $x \in C_{rs}$ and hence $C_r C_s \subseteq C_{rs}$. Suppose $x < 0$. Since $r, s \geq 0$ implies $rs \geq 0$, we know that $x < 0 < rs$. Hence, we must have $x \in C_{rs}$.

Now, we show the reverse inclusion. Suppose $x \in C_{rs}$, then $x < rs$ for $r, s \geq 0$ in $\mathbb{Q}$. Suppose $r, s > 0$ (if $r, s = 0$ then the result follows immediately), then dividing by r on both sides gives us $x/r < s$ which tells us that $x/r \in C_s$. Similarly, we can divide by s to get $x/s < r$ which implies $x/s \in C_r$. Hence, taking the product of these two elements leads to

$$\frac{x}{s} \cdot \frac{x}{r} < (rs)^2 \Leftrightarrow \frac{x}{s^2} \cdot \frac{x}{r^2} < rs.$$

Hence, we must have $x \in C_r C_s$ and thus we have $C_{rs} \subseteq C_r C_s$. ■

(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbb{Q}$.

Proof. Suppose $r \leq s$ in $\mathbb{Q}$. Let $z \in C_r$. Using the fact that $r \leq s$ and using definition of C_r, we have $z < r \leq s$. This tells us that $z \leq s$ which subsequently tells us that $z \in C_s$. Hence, $C_r \subseteq C_s$ which is equivalent to $C_r \leq C_s$. Now, let $C_r \leq C_s$. Suppose for sake of contradiction that $r > s$. Suppose we take the midpoint of r and s. Then we have $a = \frac{r+s}{2}$ which implies $a < r$. This tells us that $a \in C_r$, but $a \notin C_s$ since $a > s$. Hence, we have a contradiction. Hence, we must have $r \leq s$ in $\mathbb{Q}$. ■