

Math 230A: Homework 4

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1. Consider \mathbb{R} with its standard metric. Let $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Prove that $E' = \{0\}$.

Proof. Note that $0 \notin E$. To show that $E' = \{0\}$, it suffices to show that for $\varepsilon > 0$

$$N_\varepsilon(0) \cap E \neq \emptyset$$

which is equivalent to

$$(-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. By the Archimedean Property of \mathbb{R} , there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Thus, $1/m \in (-\varepsilon, \varepsilon) \cap E$ and so $0 \in E'$. ■

2. Consider \mathbb{R} with its standard metric.

- (a) Prove that $\mathbb{N}' = \emptyset$. Is \mathbb{N} closed?

Solution. Suppose for sake of contradiction that $\mathbb{N}' \neq \emptyset$. Let $n \in \mathbb{N}'$. Let $\varepsilon = 1$. Then $N_1(n) \cap \mathbb{N} \setminus \{n\} \neq \emptyset$ where $N_1(n) = (n-1, n+1)$. But note that $(n-1, n+1)$ only contains one point, namely, n and nothing else. Hence, n must be an isolated point be an isolated point of \mathbb{N} which is a contradiction. Thus, $\mathbb{N}' = \emptyset$. Because $\emptyset \subseteq \mathbb{N}$, \mathbb{N} must be a closed set. ■

- (b) Prove that $\mathbb{Q}' = \mathbb{R}$. Is \mathbb{Q} closed?

Solution. Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. Consider the open interval $(a - \varepsilon, a + \varepsilon)$. Since \mathbb{Q} is dense in \mathbb{R} , we can find at least one $x \in \mathbb{Q}$ such that $x \in (a - \varepsilon, a + \varepsilon)$; that is, we have

$$N_\varepsilon(a) \cap (\mathbb{Q} \setminus \{a\}) \neq \emptyset.$$

So, any $a \in \mathbb{R}$ must be a limit point of \mathbb{Q} . Therefore, we conclude that $\mathbb{Q}' = \mathbb{R}$. Consequently, \mathbb{Q} must not be closed since its limit points take are outside of \mathbb{Q} . ■

3. Consider \mathbb{R}^2 with its standard metric. Let $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$. Prove that $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Proof. Our goal is to show that $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$; that is, we will show that

$$E' = E \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}.$$

We proceed by considering two cases:

- (1) If (a, b) is such that $a^2 + b^2 > 4$, then we will prove that $(a, b) \notin E'$.
- (2) We will prove that if (a, b) is such that $a^2 + b^2 \leq 4$, then $(a, b) \in E'$; that is,

$$\forall N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}) \neq \emptyset.$$

With (1), consider a point $p = (a, b)$ that is outside of E . Let $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$. Then we

see that

$$0 < \frac{1}{2}(\sqrt{a^2 + b^2} - 2) \iff a^2 + b^2 > 4.$$

Hence, we must have that $N_\delta(p) \cap (E \setminus \{p\}) = \emptyset$.

With (2), consider the distance from the origin to any point (a, b) on the boundary of the circle E . Let $\delta = \frac{1}{2} \min\{\varepsilon, 2 - \varepsilon\}$. Suppose $\varepsilon < 2$. Let p be the points on the ray \overline{OQ} where the distance to q is δ . Note that $p \in N_\varepsilon((a, b))$ since the distance between p and (a, b) is δ which is less than ε . Also, $p \in E$ since p is on the ray \overline{OQ} . Thus, we have

$$p \in N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}).$$

On the other hand, if $\varepsilon \geq 2$, then let $\delta = 1$. E is clearly the intersection

$$N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\})$$

is nonempty with a similar argument shown when $\varepsilon < 2$.

Thus, we conclude that

$$E' = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}.$$

■

4. Let (X, d) be a metric space. Prove that X and \emptyset are both open sets.

Proof. We want to show the following:

- (i) X is open
- (ii) \emptyset is open

Let $x \in X$. For (i), we want to show that there exists $\delta > 0$ such that $N_\delta(x) \subseteq X$. Pick another point $q \neq x$ such that $q \in X$. Choose $\delta = d(x, q) > 0$. But note that any open neighborhood is a subset of X . So, $N_\delta(x) \subseteq X$ and so X is open.

Note that $\emptyset^\circ = \emptyset$. Hence, \emptyset is an open set.

■

5. Let (X, d) be a metric space. Prove that X and \emptyset are both closed sets.

Proof. We will proving the following statements:

- (i) X is closed.
- (ii) \emptyset is closed.

Let x be a limit point of X . Then for all $\varepsilon > 0$, we have $N_\varepsilon(x) \cap (X \setminus \{x\}) \neq \emptyset$. But note that $N_\varepsilon(x)$ is an open set that is contained in X . Thus, x must be contained within X which proves (i).

For (ii), note that the complement of \emptyset is just X itself. Since X is open by problem 4, we see that $X^c = \emptyset$ must be closed.

■

6. Consider \mathbb{R} equipped with the discrete metric. Let $E = \{1, 2, 3\}$. Prove that E is open, that is, $E^\circ = E$.

Proof. We will check that for every element x in E that there exists $\delta > 0$ such that $N_\delta(x) \subseteq E$. Choose $\delta = 1$, then we have $d(1, x) = 0 < 1$. Since this is the only point that satisfies δ , we must have that $N_1(1) = \{1\} \subseteq E$. Similarly, $N_1(2) = \{2\} \subseteq E$ and $N_1(3) = \{3\} \subseteq E$. Thus, E must be open.

■

7. Consider $X = \mathbb{R}$ equipped with the standard metric. Let $a < b$.

- (a) Prove that the sets (a, b) , (a, ∞) , and $(-\infty, a)$ are open.

Proof. We will show that the following intervals in \mathbb{R} are open:

- (i) (a, b)

(ii) (a, ∞) (iii) $(-\infty, a)$

Starting with (i), we want to show that there exists a $\delta > 0$ such that $N_\delta(x) \subseteq (a, b)$ where $N_\delta(x) = (x - \delta, x + \delta)$. Let $x \in (a, b)$. Choose $\delta = \frac{1}{2} \min\{|b - x|, |x - a|\}$. Then we have $N_\delta(x) \subseteq (a, b)$. Thus, we conclude that (a, b) is open.

With (ii), fix $x \in (a, \infty)$. Observe that (a, x) is an open interval that is contained within (a, ∞) . Now, let $y \in (a, x)$. Then choose $\delta = \frac{1}{2} \min\{|x - y|, |y - a|\} > 0$. Hence, we have

$$N_\delta(y) \subseteq (a, x) \subseteq (a, \infty).$$

Thus, we have that (a, ∞) is an open set in \mathbb{R} .

Lastly, fix $x \in (-\infty, b)$. Consider the open interval (x, b) and let $y \in (x, b)$. Choose $\delta = \frac{1}{2} \min\{|b - y|, |y - x|\} > 0$. Then we have

$$N_\delta(y) \subseteq (x, b) \subseteq (-\infty, b)$$

which shows that $N_\delta(y)$ is an open set. ■

(b) Prove that the sets $[a, b]$, $[a, \infty)$, and $(-\infty, b]$ are closed.

Proof. We will showing that the following intervals are closed in \mathbb{R} :

(i) $[a, b]$ (ii) $[a, \infty)$ (iii) $(-\infty, b]$

Starting with (i), we want to show that every limit point x of $[a, b]$ is contained in $[a, b]$. Thus, let x be a limit point of $[a, b]$. By definition, for all $\delta > 0$, we have $N_\delta(x) \cap ([a, b] \setminus \{x\}) \neq \emptyset$. So, $N_\delta(x)$ contains a point $q \neq x$ in this intersection such that $q \in [a, b]$. This means that $a \leq q \leq b$. Using the fact that $q \in N_\delta(x)$, we have

$$\begin{aligned} |x - q| < \delta &\iff q - \delta < x < q + \delta \\ &\implies a - \delta \leq x \leq b + \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we conclude that $a \leq x \leq b$ which means that $x \in [a, b]$. Thus, $[a, b]$ is closed.

With (ii), we will employ the same process. Let x be a limit point of $[a, \infty)$. Then for all $\varphi > 0$, we see that $N_\varphi(x) \cap ([a, \infty) \setminus \{x\}) \neq \emptyset$. To this end, pick $p \neq x$ in this intersection such that $p \in [a, \infty)$. Since p is also contained within $N_\varphi(x)$, we see that

$$\begin{aligned} |x - p| < \varphi &\iff p - \varphi < x < p + \varphi \\ &\iff a - \varphi \leq x < p + \varphi. \end{aligned}$$

This implies that $a \leq x < p$ since $\varphi > 0$ is arbitrary. Thus, we have $x \in [a, \infty)$ and so $[a, \infty)$ is closed.

The last case is similar to (ii). Let $\varepsilon > 0$. If x is a limit point of $(-\infty, b]$, then $N_\varepsilon(x) \cap ((-\infty, b] \setminus \{x\}) \neq \emptyset$ implies that we can pick $q \neq x$ in this intersection such that $q \in (-\infty, b]$. Since $\varepsilon > 0$ is arbitrary, we have

$$q - \varepsilon < x < q + \varepsilon \leq b + \varepsilon$$

implies

$$q < x \leq b.$$

Hence, $x \in (-\infty, b]$ and so $(-\infty, b]$ is a closed set. ■

8. Let (X, d) be a metric space. Let $p \in X$ and $\delta > 0$. Prove that the closed ball $C_\delta(p) = \{x \in X :$

$d(x, p) \leq \delta\}$ is indeed closed.

Proof. Our goal is to show that $C_\delta(p)$ is a closed set. To this end, we will show that $[C_\delta(p)]^c$ is an open set. If this holds, then we can conclude that $C_\delta(p)$ is a closed set. Let $x \in [C_\delta(p)]^c$. Then we have $d(x, p) > \delta$. Our goal is to find $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq [C_\delta(p)]^c$. Since $d(x, p) > \delta$ we can set $\varepsilon = d(x, p) - \delta > 0$. Using the radius $\varepsilon > 0$, we can construct the neighborhood of x , $N_\varepsilon(x)$. Let $y \in N_\varepsilon(x)$. In order for y to be contained within $[C_\delta(p)]^c$, we have to show that $d(y, p) > \delta$. Using the triangle inequality, we have

$$\begin{aligned} d(x, p) &\leq d(x, y) + d(y, p) \implies d(y, p) \geq d(x, p) - d(x, y) \\ &> d(x, p) - \varepsilon & (y \in N_\varepsilon(x)) \\ &= \delta. \end{aligned}$$

Thus, we see that $N_\varepsilon(x) \subseteq [C_\delta(p)]^c$ for some $\varepsilon > 0$. Hence, $[C_\delta(p)]^c$ is open and so $C_\delta(p)$ is closed. ■

9. Give an example that shows that the equality $\overline{N_\delta(p)} = C_\delta(p)$ is not always true.

Solution. Consider the interval $[0, 1]$ in \mathbb{R} with the discrete metric. Clearly, we see that $1/2 \in [0, 1]$. If we let $\varepsilon = 1$, then

$$N_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) < 1\} = \{1/2\}$$

since the only case when the inequality is satisfied is when $x = 1/2$. If we consider the closure of this neighborhood, we just get

$$\overline{N_1(1/2)} = \{1/2\}.$$

Now, consider the closed ball

$$C_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) \leq 1\}.$$

Observe that for any $x \in \mathbb{R}$, either $x = 1/2$ or $x \neq 1/2$ in $[0, 1]$, the inequality of the set above we always be satisfied; that is, the set will just be all elements contained in $[0, 1]$. Thus, we see that $C_1(1/2) = [0, 1]$ and, in this case, that $C_1(1/2) \neq \overline{N_1(1/2)}$. ■

10. Let (X, d) be a metric space. Prove that an arbitrary intersection of closed sets is closed. Prove that a finite union of closed sets is closed.

Proof. Let (X, d) be a metric space. We will prove that

- (1) An arbitrary intersection of closed sets is closed.
- (2) A finite union of closed sets is closed.

To this end, we will proceed by showing the propositions above.

- (1) Let $\{F_\alpha\}$ be an arbitrary collection of closed sets. We want to show that

$$\bigcap_{\alpha} F_{\alpha} \text{ is closed.}$$

It suffices to show that the complement of this set is open. Consider the complement

$$\left(\bigcap_{\alpha} F_{\alpha}\right)^c = \bigcup_{\alpha} F_{\alpha}^c. \tag{1}$$

Observe that each F_{α}^c is open since each F_{α} is closed. But note that the arbitrary union of open sets is open and so the union of the right-hand side of (1) is open. Thus, the left-hand side of (1) is open. Hence, the complement

$$\left(\bigcap_{\alpha} F_{\alpha}\right)^c \text{ is open} \iff \bigcap_{\alpha} F_{\alpha} \text{ is closed}$$

and we are done.

- (2) Let $\{F_i : 1 \leq i \leq n\}$ be a finite collection of closed sets F_i . We want to show that $\bigcup_{i=1}^n F_i$ is a closed set. It suffices to show that the complement of this set, that is $\left(\bigcup_{i=1}^n F_i\right)^c$ is open. Observe that

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c.$$

Note that each F_i^c is open if and only if each F_i is closed. Thus, the finite intersection of each open set F_i^c is open; that is,

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c \text{ is open.}$$

Hence, the finite union of closed sets

$$\bigcup_{i=1}^n F_i \text{ is closed.}$$

■

11. Let E be a subset of a metric space (X, d) . Show that $\overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}$.

Proof. Let E be a subset of a metric space (X, d) . We want to show that

$$\overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Denote

$$A = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

It suffices to show the following two inclusions:

- (1) $\overline{E} \subseteq A$.
- (2) $A \subseteq \overline{E}$.

Starting with (1), suppose $x \in \overline{E}$. Then either $x \in E$ or $x \in E'$ (or both). If $x \in E'$, then x is a limit point of E . Then for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Since $E \setminus \{x\} \subseteq E$, we see that

$$N_\varepsilon(x) \cap E \neq \emptyset$$

which implies that $x \in A$. Suppose $x \in E$. Then for all $\varepsilon > 0$, we see that

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

If x is a limit point and $x \in E$, then we immediately have that

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

by definition of limit point. Thus, we see that in the three cases outlined that $\overline{E} \subseteq A$.

With (2), suppose $x \in A$. We want to show that $x \in \overline{E}$; that is, x is either a limit point of E or an element of E . Suppose that x is NOT an element of E . By definition of A , we see that for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

Note that $x \notin E$ implies that $E \setminus \{x\} = E$ and so we have

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus, x is a limit point of E and so $x \in \overline{E}$. Otherwise, when x is not a limit point of E , then the nonempty intersection

$$N_\varepsilon(x) \cap E$$

implies that $x \in E$ and so $x \in \overline{E}$. Thus, $A \subseteq \overline{E}$. ■

12. Let (X, d) be a metric space. Show that the closure of a bounded set is bounded.

Proof. Let (X, d) be a metric space and suppose $E \subseteq X$ is bounded; that is, there exists $\varepsilon > 0$ and $q \in X$ such that $E \subseteq N_\varepsilon(q)$. We claim that $\overline{E} \subseteq N_{\varepsilon+1}(q)$ so that \overline{E} is bounded. Note that $\overline{E} = E \cup E'$. Let $y \in \overline{E}$. Then either $y \in E$ or $y \in E'$. If $y \in E$ and E is bounded, then

$$d(y, q) \leq \varepsilon < \varepsilon + 1.$$

Thus, $y \in N_{\varepsilon+1}(q)$. If $y \in E'$, then for all $\delta > 0$,

$$N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset.$$

Let $\delta = 1/2$. Then pick a point x inside the intersection

$$N_{1/2}(y) \cap (E \setminus \{y\}).$$

Then $x \in N_{1/2}(y)$ and $x \in E$. If $x \in N_{1/2}(y)$, then $d(x, y) < 1/2$ and $d(x, q) \leq \varepsilon$, respectively. Using the triangle inequality, we see that

$$d(y, q) \leq d(y, x) + d(x, q) < \frac{1}{2} + \varepsilon < 1 + \varepsilon.$$

Thus, $y \in N_{\varepsilon+1}(q)$. Therefore, we see that \overline{E} is bounded in both cases. ■

13. Prove that a nonempty subset A of a metric space (X, d) is open if and only if it is a union of neighborhoods (open balls).

Proof. (\Rightarrow) Suppose A is an open set in X . We will show that A can be written as a union of open balls; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x)$$

where $N_{\varepsilon_x}(x)$ are open neighborhoods with radius ε_x (ε depends on x) for each $x \in A$. To this end, we will show the following inclusions:

$$(1) A \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$$

$$(2) \bigcup_{x \in A} N_{\varepsilon_x}(x) \subseteq A$$

Starting with the inclusion in (1), let $x \in A$. Since A is open, we can find an $\varepsilon_x > 0$ such that $N_{\varepsilon_x}(x) \subseteq A$. Note that $N_{\varepsilon_x}(x) \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$. So, x must be contained in $\bigcup_{x \in A} N_{\varepsilon_x}(x)$ which satisfies (1).

Let $p \in \bigcup_{x \in A} N_{\varepsilon_x}(x)$. Then for some point $x \in A$, p must be contained in some neighborhood $N_{\varepsilon_x}(x)$ with radius $\varepsilon_x > 0$. Since $x \in A$ and A is an open set, we must have $N_{\varepsilon_x}(x) \subseteq A$. Thus, p must be contained in A as well. Thus, the inclusion in (2) is satisfied.

(\Leftarrow) Suppose A is the union of open balls in A ; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x).$$

Since each $N_{\varepsilon_x}(x)$ is open, the union of the right-hand side of the equation above is open. Thus, A must be open as well and we are done. ■

14. On page 9, Rudin implicitly defines a subset $E \subseteq \mathbb{R}$ to be "dense" if it satisfies:

- (i) For all $x, y \in \mathbb{R}$ with $x < y$, there exists $p \in E$ such that $x < p < y$.

On page 32, he defines a subset E of a metric space X to be "dense" if it satisfies:

- (ii) Every point of X is either a limit point of E or a point of E (that is, $\overline{E} = X$).

Prove that $E \subseteq \mathbb{R}$ satisfies (i) if and only if it satisfies (ii).

Proof. (\Rightarrow) Suppose $E \subseteq \mathbb{R}$ satisfies the property (i). We want to show that every point in \mathbb{R} must either be a limit point of E or a point in E . To this end, let $x \in \mathbb{R}$ be an arbitrary point. Suppose that x is not an element of E . Thus, we want to show that x is a limit point of E ; that is, we will show that

$$\forall \delta > 0 \quad N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Let $\delta > 0$. Then denote the open neighborhood of x as

$$N_\delta(x) = \{y \in \mathbb{R} : |x - y| < \delta\} = (x - \delta, x + \delta).$$

But note that $x - \delta < x + \delta$ implies that there exists a $p \neq x \in E$ such that $x - \delta < p < x + \delta$. Thus, we see that $|x - p| < \delta$ and hence, we see that

$$N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Hence, x is a limit point of E . Now, if x is NOT a limit point, then x is an isolated point of E and that $x \in E$.

(\Leftarrow) Let $x, y \in \mathbb{R}$ such that $x < y$. We will show that there exists $p \in E$ such that $x < p < y$. Suppose that $\mathbb{R} = \overline{E}$. Since $x \in \mathbb{R}$, we must have $x \in \overline{E}$. That is, for all $\varepsilon > 0$, we have

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

This implies that $N_\varepsilon(x)$ contains a point $p \neq x$ such that $p \in E$. This means that $p \in (x - \varepsilon, x + \varepsilon)$ and so

$$x - \varepsilon < p < x + \varepsilon.$$

Since $x < y$, we have that

$$x - \varepsilon < p < y + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we find that $x < p < y$ for some $p \in E$. ■

15. Let A and B be subsets of a metric space (X, d) .

- (a) If $A \subseteq B$, then $A' \subseteq B'$.

Proof. Let $x \in A'$. We want to show that $x \in B'$; that is, x is a limit point of B . By definition, $x \in A'$ implies that for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$. So, pick $p \neq x$ in this intersection such that $p \in A$. But note that $A \subseteq B$. Thus, $p \in B$ and we must have for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset.$$

Thus, x is also a limit point of B (that is, $x \in B'$) and we conclude that $A' \subseteq B'$. ■

- (b) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. Suppose $A \subseteq B$. By part (i), we see that $A' \subseteq B'$. Thus,

$$\overline{A} = A \cup A' \subseteq B \cup B' = \overline{B}.$$

Hence, $\overline{A} \subseteq \overline{B}$. ■

- (c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Proof. Our goal is to show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cap B}$. Then either $x \in A \cap B$

or $x \in (A \cap B)'$. Suppose $x \in A \cap B$. Then both $x \in A$ and $x \in B$ and so $x \in \overline{A}$ and $x \in \overline{B}$ since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Now, suppose $x \in (A \cap B)'$. Then x is a limit point of $A \cap B$; that is, for all $\varepsilon > 0$, $N_\varepsilon(x) \cap ((A \cap B) \setminus \{x\}) \neq \emptyset$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we must have that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \quad (1)$$

and

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset. \quad (2)$$

Thus, (1) and (2) imply that $x \in A'$ and $x \in B'$ meaning that x is both a limit point of A and B . Hence, $x \in \overline{A}$ and $x \in \overline{B}$. Thus, we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ in both of these cases. ■

(a) Prove that $A' \cup B' = (A \cup B)'$.

Proof. Our goal is to show that $A' \cup B' = (A \cup B)'$; that is, we need to show that

$$(1) \quad A' \cup B' \subseteq (A \cup B)'$$

$$(2) \quad (A \cup B)' \subseteq A' \cup B'.$$

Starting with (1), let $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$. If $x \in A'$, then x is a limit point of A . Hence, for all $\varepsilon > 0$ such that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset.$$

Then pick a point in this intersection, say, $q \neq x$ such that $q \in A$. Since $A \subseteq A \cup B$, we have $q \in A \cup B$. Thus, we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Hence, x is a limit point of $A \cup B$ and so $x \in (A \cup B)'$. If $x \in B'$, then for all $\varepsilon > 0$ such that

$$N_\varepsilon(x) \cap B \setminus \{x\} \neq \emptyset.$$

Since $B \subseteq A \cup B$, we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Then x is a limit point of $A \cup B$; that is, $x \in (A \cup B)'$. Hence, we have that $A' \cup B' \subseteq (A \cup B)'$.

Now, with (2), let $x \in (A \cup B)'$. Then x is a limit point of $A \cup B$. Thus, for all $\varepsilon > 0$ we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Pick a point in this intersection, say, $p \neq x$. Note that $p \in A \cup B$. So, either $p \in A$ or $p \in B$. If $p \in A$, then x is a limit point of A . Thus, $x \in A'$ and so $x \in A' \cup B'$. If $p \in B$, then x is a limit point of B and so $x \in B'$. Thus, $x \in A' \cup B'$. Therefore, we have $(A \cup B)' \subseteq A' \cup B'$ which proves (2).

With (1) and (2), we have

$$A' \cup B' = (A \cup B)'. \quad \blacksquare$$

16. Let (X, d) be a metric space and let $E \subseteq X$.

(a) Prove that E' is closed.

Proof. Our goal is to show that E' is closed; that is, any limit point of E' is contained within E' . Let x be a limit point of E' . Then for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (E' \setminus \{x\}) \neq \emptyset$. To this end, pick $y \neq x$ in this intersection such that $y \in E'$. But this tells us that y is a limit point of E . Thus, for all $\delta > 0$, $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$. Thus, we can pick $p \neq y$

such that $p \in E$. Note that $p \neq y \neq x$ and $p \in E$ imply that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus, x is a limit point of E and so $x \in E'$. Hence, E' is closed. ■

(b) Prove that E and \overline{E} have the same limit points.

Proof. To prove that E and \overline{E} have the same limit points, we need to show that $E' = (\overline{E})'$. We will prove the following two inclusions:

$$(1) E' \subseteq (\overline{E})'$$

$$(2) (\overline{E})' \subseteq E'.$$

Starting with (1), let $x \in E'$. Then x is a limit point of E . Thus, for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset$. Since $E \subseteq \overline{E}$, we must have that

$$N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset.$$

Thus, $x \in \overline{E}'$ which proves (1).

Now with (2), let $x \in \overline{E}'$. Then for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset$. From this nonempty intersection, pick $y \neq x$ such that $y \in \overline{E}$; that is, $y \in E$ or $y \in E'$. If $y \in E$, then x is a limit point of E and thus $x \in E'$. If $y \in E'$, then y is a limit point of E . Hence, for all $\delta > 0$, $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$. From this, we can pick $q \neq y \neq x$ such that $q \in E$. But this tells us that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset,$$

proving that x is a limit point of E and so $x \in E'$. ■

(c) Construct an example that shows E and E' do not necessarily have the same limit points?

Proof. Suppose we have the following set $E = \{x < \frac{1}{n} : n \in \mathbb{N}, x \in \mathbb{R}\}$ and its set of limit points $E' = \{\frac{1}{n} : n \in \mathbb{N}\}$. By problem 1, we see that the set of limit points of E' must be $\{0\}$. Clearly, E and E' do not have the same limit points. ■

17. Construct a bounded set of real numbers with exactly three limit points.

Solution. Consider the sets $A = \{1/n : n \in \mathbb{N}\}$, $L = \{\pi + \frac{1}{n} : n \in \mathbb{N}\}$, $I = \{3 + \frac{1}{n} : n \in \mathbb{N}\}$. Note that each set is bounded. If we take the union of these sets, then we have the following set

$$A \cup L \cup I$$

which has three limit points which are 0, π , and 3, respectively. ■

18. Let (X, d) be a metric space and $E \subseteq X$.

(a) Prove that E° is always open, that is $(E^\circ)^\circ = E^\circ$.

Proof. To show that E° is open, we will show that $(E^\circ)^\circ = E^\circ$. To do this, we will need to show the following two inclusions:

$$(1) (E^\circ)^\circ \subseteq E^\circ$$

$$(2) E^\circ \subseteq (E^\circ)^\circ.$$

Starting with (1), let $x \in (E^\circ)^\circ$. By definition of interior points, there must exist a $\delta > 0$ such that $N_\delta(x) \subseteq E^\circ$. Clearly, we have $x \in E^\circ$ which proves (1).

Now, let $x \in E^\circ$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq E$ where $x \in E$. Suppose for sake of contradiction that x is NOT in the interior of E° . Then for all $\delta > 0$, $N_\delta(x) \not\subseteq E^\circ$. That is, $N_\delta(x) \cap (E^\circ)^c \neq \emptyset$. Since $E^\circ \subseteq E$, it follows that $x \in E^c$. But this is a contradiction because we had assumed that $x \in E^\circ$ and so $x \in E$. Thus, we must have x lie in the interior of E° . From (1) and (2), we conclude that E° is open. ■

- (b) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Proof. Suppose $G \subseteq E$ and G is open. We will show that $G \subseteq E^\circ$. Let $x \in G$. Since G is open, there exists $\delta > 0$ such that $N_\delta(x) \subseteq G \subseteq E$. Thus, x must be in the interior of E ; that is, $x \in E^\circ$. Hence, $G \subseteq E^\circ$. ■

- (c) Prove that $E^\circ = \overline{E^c}^c$.

Proof. We will show that $E^\circ = (\overline{E^c})^c$. We will show the following two inclusions:

$$(1) E^\circ \subseteq (\overline{E^c})^c,$$

$$(2) (\overline{E^c})^c \subseteq E^\circ.$$

Starting with (1), let $x \in E^\circ$. Then there exists a $\delta > 0$ such that $N_\delta(x) \subseteq E$. This implies that x cannot be in E^c . Otherwise, x is not an interior point of E . Subsequently, we have that for all $\delta > 0$, $N_\delta(x) \cap E^c = \emptyset$. Thus, neither $x \in E^c$ nor $x \in (E^c)'$. Hence, $x \in (\overline{E^c})^c$.

With (2), let $x \in (\overline{E^c})^c$. Then $x \notin \overline{E^c}$. Since $\overline{E^c} = E^c \cup (E^c)'$, we have $x \notin E^c$ and $x \notin (E^c)'$. Then there exists $\delta > 0$ such that

$$N_\delta(x) \cap (E^c \setminus \{x\}) = \emptyset \iff N_\delta(x) \cap E^c = \emptyset.$$

Since $x \notin E^c$, we must have $x \in E$. Thus, $N_\delta(x) \subseteq E$ for some $\delta > 0$ and so $x \in E^\circ$ which proves (2). ■

- (d) Give an example that shows that E and \overline{E} do not always have the same interiors.

Solution. Let $E = \mathbb{Q}$ and note that $\overline{E} = \mathbb{R}$ under the standard metric of \mathbb{R} . Since \mathbb{R} is open, we know that $\mathbb{R}^\circ = \mathbb{R}$. But the interior of \mathbb{Q} is empty, which is clearly not equal to $\mathbb{R}^\circ = \overline{E}^\circ$. ■

- (e) Give an example that shows that E and E° do not always have the same closures.

Proof. Consider the set of rational numbers \mathbb{Q} . Then observe that \mathbb{Q}° is empty since for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains irrational numbers which are not contained in \mathbb{Q} . Hence, $\overline{\mathbb{Q}^\circ} = \emptyset$. But notice that the closure $\overline{\mathbb{Q}} = \mathbb{R}$. Clearly, $\overline{\mathbb{Q}} \neq \overline{\mathbb{Q}^\circ}$. ■

19. Let (X, d) be a metric space and let $A \subseteq X$. Prove that A is dense in X if and only if any nonempty open set in X has a nonempty intersection with A .

Proof. (\implies) Let B be a nonempty open set in X . Let $b \in B$. We will show that $A \cap B \neq \emptyset$. Since $b \in X$ and A is dense in X , either

(1) b is a limit point of A or

(2) $b \in A$.

If b is a limit point of A , then for all $\delta > 0$,

$$N_\delta(b) \cap (A \setminus \{b\}) \neq \emptyset.$$

Since B is an open set, we know that $N_\delta(b) \subseteq B$. Together with the fact that $A \setminus \{b\} \subseteq A$, we can conclude that

$$B \cap A \neq \emptyset.$$

On the other hand, $b \in A$ immediately implies that $B \cap A \neq \emptyset$.

(\impliedby) Let $p \in X$ and let $A \subseteq X$. We need to show that A is dense in X ; that is, we need to show that p is either a limit point of A or $p \in A$. Suppose that $p \notin A$. We will show that p is a limit point of A . Let $\varepsilon > 0$ and consider the open neighborhood $N_\varepsilon(p)$. By assumption, the open ball $N_\varepsilon(p)$ contains a nonempty intersection with A ; that is,

$$N_\varepsilon(p) \cap A \neq \emptyset.$$

Subsequently, we have

$$N_\varepsilon(p) \cap (A \setminus \{p\}) \neq \emptyset.$$

Thus, p is a limit point of A . On the other hand, if p is not a limit point of A , then p is an isolated point of A . Thus, $p \in A$. ■

Proof. (\Leftarrow) Suppose A has a nonempty intersection with every nonempty open set in X . Thus, we have

$$\begin{aligned} (A^c)^\circ = \emptyset &\implies_{18-c)} \left(\overline{(A^c)^c} \right)^c = \emptyset \\ &\implies \overline{(A^c)^c} = \emptyset \\ &\implies \overline{A} = X. \end{aligned}$$

Thus, A is dense in X .

(\Rightarrow) Suppose $\overline{A} = X$. By problem 11, we see that for any $x \in X$ and for all $\varepsilon > 0$ that

$$N_\varepsilon(x) \cap A \neq \emptyset. \quad (1)$$

If E is any nonempty open set, then let $q \in E$. Since q is an interior point of E , there exists $\delta > 0$ such that

$$N_\delta(q) \subseteq E. \quad (2)$$

By (1) and (2), we must have that $E \cap A \neq \emptyset$. ■

20. Let (X, d) be a metric space and $A_i \subseteq X$ for all $i \in \mathbb{N}$.

(a) Prove that for all $n \in \mathbb{N}$, we have $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$.

Proof. First, we will show by induction on $n \in \mathbb{N}$ that

$$\left(\bigcup_{i=1}^n A_i \right)' = \bigcup_{i=1}^n A_i'.$$

Let $n = 2$ be our base case, then we see that $(A_1 \cup A_2)' = A_1' \cup A_2'$ by problem 15-(d). Now, we will suppose that the result holds for the n th case. Thus, we have that

$$\begin{aligned} \left(\bigcup_{i=1}^{n+1} A_i \right)' &= \left(\bigcup_{i=1}^n A_i \cup A_{n+1} \right)' \\ &= \left(\bigcup_{i=1}^n A_i \right)' \cup A_{n+1}' \\ &= \left(\bigcup_{i=1}^n A_i' \right) \cup A_{n+1}' \\ &= \bigcup_{i=1}^{n+1} A_i'. \end{aligned}$$

Then for all $i \in \mathbb{N}$, we see that

$$\begin{aligned}\overline{\bigcup_{i=1}^n A_i} &= \left(\bigcup_{i=1}^n A_i \right) \cup \left(\bigcup_{i=1}^n A_i \right)' \\ &= \left(\bigcup_{i=1}^n A_i \right) \cup \left(\bigcup_{i=1}^n A'_i \right) \\ &= \bigcup_{i=1}^n A_i \cup A'_i \\ &= \bigcup_{i=1}^n \overline{A_i}.\end{aligned}$$

■

(b) Prove that $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$.

Proof. Our goal is to show that $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$. Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$. Then for some $i \in \mathbb{N}$, we see that $x \in \overline{A_i}$. Then either $x \in A_i$ or $x \in A'_i$. If $x \in A_i$ and $A_i \subseteq \bigcup_{i=1}^{\infty} A_i$, we see that $x \in \bigcup_{i=1}^{\infty} A_i$. Suppose $x \in A'_i$. Note that $A'_i \subseteq \left(\bigcup_{i=1}^{\infty} A_i \right)'$. So, $x \in \left(\bigcup_{i=1}^{\infty} A_i \right)'$. Thus, $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$. Hence, we have

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}.$$

■

(c) Show, by an example, that the inclusion in part (b) can be proper.

Solution. Consider the sequence of open intervals in \mathbb{R} , $A_i = (0, 1 + \frac{1}{i})$. Observe that

$$\bigcup_{i=1}^{\infty} \overline{\left(0, 1 + \frac{1}{i}\right)} = \bigcup_{i=1}^{\infty} \left[0, 1 + \frac{1}{i}\right] = [0, 2].$$

But we have

$$\overline{\bigcup_{i=1}^{\infty} \left(0, 1 + \frac{1}{i}\right)} = \overline{(0, 2)} = [0, 2].$$

Clearly, $[0, 2)$ is a proper subset of $[0, 2]$.

■

21. Let (X, d) be a metric space and $A_i \subseteq X$, for all $i \in \mathbb{N}$.

(a) Prove that for all $n \in \mathbb{N}$, we have $\left(\bigcap_{i=1}^n A_i \right)^{\circ} = \bigcap_{i=1}^n A_i^{\circ}$.

Proof. Our goal is to show that $\left(\bigcap_{i=1}^n A_i \right)^{\circ} = \bigcap_{i=1}^n A_i^{\circ}$; that is, we need to show the following two inclusions:

$$(1) \left(\bigcap_{i=1}^n A_i \right)^{\circ} \subseteq \bigcap_{i=1}^n A_i^{\circ}.$$

$$(2) \bigcap_{i=1}^n A_i^{\circ} \subseteq \left(\bigcap_{i=1}^n A_i \right)^{\circ}.$$

Starting with (1), let $x \in \left(\bigcap_{i=1}^n A_i \right)^{\circ}$. Then there exists $\delta > 0$ such that $N_{\delta}(x) \subseteq \bigcap_{i=1}^n A_i$. Thus, $x \in \bigcap_{i=1}^n A_i$ implies that $x \in A_i$ for all $1 \leq i \leq n$. But this tell us that

$N_\varepsilon(x) \subseteq A_i$ for all $1 \leq i \leq n$, and so $x \in A_i$. Hence,

$$x \in \bigcap_{i=1}^n A_i^\circ.$$

Thus, we conclude that

$$\left(\bigcap_{i=1}^n A_i \right)^\circ \subseteq \bigcap_{i=1}^n A_i^\circ.$$

With (2), let $x \in \bigcap_{i=1}^n A_i^\circ$. Then for all $1 \leq i \leq n$, we have $x \in A_i^\circ$. By definition, there exists $\delta > 0$ such that $N_\delta(x) \subseteq A_i$ for all $1 \leq i \leq n$. Thus, $x \in A_i$ for all $1 \leq i \leq n$ and so

$$x \in \bigcap_{i=1}^n A_i.$$

Subsequently, $N_\delta(x) \subseteq \bigcap_{i=1}^n A_i$ and so $x \in \left(\bigcap_{i=1}^n A_i \right)^\circ$. Therefore, we have

$$\bigcap_{i=1}^n A_i^\circ \subseteq \left(\bigcap_{i=1}^n A_i \right)^\circ.$$

Hence, (1) and (2) gives us our desired result. ■

(b) Prove that $\left(\bigcap_{i \in \mathbb{N}} A_i \right)^\circ \subseteq \bigcap_{i \in \mathbb{N}} A_i^\circ$.

Proof. Let $x \in \left(\bigcap_{i \in \mathbb{N}} A_i \right)^\circ$. Then there exists $\delta > 0$ such that $N_\delta(x) \subseteq \bigcap_{i \in \mathbb{N}} A_i$. Hence, $x \in \bigcap_{i \in \mathbb{N}} A_i$ if and only if $x \in A_i$ for all $i \in \mathbb{N}$. But this also tells us that $N_\delta(x) \subseteq A_i$. Hence, $x \in A_i^\circ$ for all $i \in \mathbb{N}$. Thus,

$$\left(\bigcap_{i \in \mathbb{N}} A_i \right)^\circ \subseteq \bigcap_{i \in \mathbb{N}} A_i^\circ.$$

■

(c) Give an example where equality does not hold in (b).

Solution. Consider the sequence of closed intervals A_i in \mathbb{R} defined by

$$A_i = \left[a - \frac{1}{i}, b \right].$$

Then the interior of these closed intervals are

$$A_i^\circ = \left(a - \frac{1}{i}, b \right).$$

Now, observe that

$$\bigcup_{i \in \mathbb{N}} A_i^\circ = [a, b).$$

But

$$\left(\bigcup_{i \in \mathbb{N}} A_i \right)^\circ = [a, b]^\circ = (a, b).$$

Hence, we have $(a, b) \subset [a, b)$. ■

22. Let (X, d) be a metric space and $A_i \subseteq X$, for all $i \in \mathbb{N}$.

(a) $\bigcup_{i \in \mathbb{N}} A_i^\circ \subseteq \left(\bigcup_{i \in \mathbb{N}} A_i \right)^\circ$.

Proof. Let $x \in \bigcup_{i \in \mathbb{N}} A_i^\circ$. Then for some $i \in \mathbb{N}$, we have $x \in A_i^\circ$. Then there exists a $\delta > 0$ such that $N_\delta(x) \subseteq A_i$. But note that

$$A_i \subseteq \bigcup_{i \in \mathbb{N}} A_i$$

and so

$$N_\delta(x) \subseteq \bigcup_{i \in \mathbb{N}} A_i.$$

This tells us that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right)^\circ$. Therefore,

$$\bigcup_{i \in \mathbb{N}} A_i^\circ \subseteq \left(\bigcup_{i \in \mathbb{N}} A_i\right)^\circ.$$

■

- (b) Give an example of a finite collection in which equality does not hold in (a).

Solution. Consider the two closed intervals $A = [0, 1]$ and $B = [1, 2]$. Note that $A^\circ = (0, 1)$ and $B^\circ = (1, 2)$. Thus,

$$A^\circ \cup B^\circ = (0, 1) \cup (1, 2)$$

whereas

$$(A \cup B)^\circ = ([0, 2])^\circ = (0, 2).$$

So, $(0, 1) \cup (1, 2) \subseteq (0, 2)$ since 1 is not included in $(0, 1) \cup (1, 2)$.

■