Handout-1- Math-241

Plan: - Go over syllabus
- Introduce metric spaces
- Discuss some topological aspects of metric spaces

Learning Objectives:
- To be able to discuss a metric space
- To be able to discuss topological aspects of a metric space

Metric Spaces

On $\mathbb{R}$, we have the usual notion of distance between $x, y \in \mathbb{R}$ defined by $d(x,y) = |x-y|$. We learned that this function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ enjoys certain properties:

(i) $d(x,y) = 0$ if and only if $x = y$
(Nondegeneracy)

(ii) $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{R}$
(Symmetricity)

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in \mathbb{R}$.
(Triangle inequality)

We would like to extend this idea and define a notion of distance in a general situation by using these properties.

<u>Definition:</u> Let X be a non-empty set. A metric d on X is a function $d: X \times X \longrightarrow \mathbb{R}$ such that

M1: $d(x, y) = 0$ if and only if x and y are same.
(d is non-degenerate)

M2: $d(x, y) = d(y, x)$ for all $x, y \in X$.
(d is symmetric)

M3: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
(d satisfies the triangle inequality)

<u>Class Exercise:</u> Assume that d is a metric on X. Show that $d(x, y) \geq 0$ for all $x, y \in X$.

<u>Class Exercise:</u> Let $X = \{a, b\}$. Is it possible to define a metric on X?

<u>Definition (Metric Space):</u> A metric space is a pair (X, d) where X is a non-empty set and d is a metric on X.

Examples: (i) Let X be a non-empty set. Define

$d: X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are different} \\ 0 & \text{otherwise} \end{cases}$$

Then d is a metric on X (called the discrete metric) and (X, d) is a metric space.

(ii) Let $X = \mathbb{R}$, $d(x,y) = \sqrt{|x-y|}$. Then $(\mathbb{R}, d)$ is a metric space

(This will be a homework problem)

(iii) $X = \mathbb{R}^n$, $\cdot\ d_\infty(\vec{x}, \vec{y}) = \max\{|x_i - y_i| \mid 1 \le i \le n\}$

$\cdot\ d_{Euclid}(\vec{x}, \vec{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^2\right]^{1/2}$

$\cdot\ d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$

are all metrics on $\mathbb{R}^n$.

(iv) Let (X, d) be a metric space. Define

$$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Then $\tilde{d}(x,y)$ is also a metric on X.

(This will be a HW problem).

Topological Aspects of a metric space

Let (X, d) be a metric space. Let $r > 0$.

Definitions: (i) $B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}$ open ball

(ii) $\bar{B}(x_0; r) = \{x \in X \mid d(x, x_0) \le r\}$ closed ball

(iii) $S(x_0; r) = \{x \in X \mid d(x, x_0) = r\}$

sphere.

Remark: $S(x_0; r) = \bar{B}(x_0; r) \setminus B(x_0; r)$

difference of sets.

Class Exercise: (i) Let $X = \mathbb{R}^2$ and $d_\infty(\vec{x}, \vec{y})$

$$= \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Draw $B(\vec{0}; 1)$, $\bar{B}(\vec{0}; 1)$, $S(\vec{0}; 1)$

Definitions (Open and closed sets):

Let (X, d) be a metric space. We say $M \subseteq X$ open if M contains an open ball about each of its points, i.e. if $x_0 \in M$ then we can find $r > 0$ such that $B(x_0; r) \subseteq M$. We say $K \subseteq X$ is closed if $K^c = X \setminus K$ is open.

Facts:
- X is open
- An open ball is open.
- Union of any collection of open sets is open.
- Intersection of a finite collection of open sets is open.