Math-234-Homework-4

Problem-1: (i) Let $D \subseteq \mathbb{C}$, $f: D \to \mathbb{C}$ be a function, $a \in \mathbb{C}$ an accumulation point of D, and $l \in \mathbb{C}$.

Prove that the following statements are equivalent.

(a) $\lim_{z \to a} f(z)$ exists and $\lim_{z \to a} f(z) = l$.

(b) Define $\tilde{f}: D \cup \{a\} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ l & \text{if } z = a \end{cases}$$

Then $\tilde{f}$ is continuous at D.

(This problem says that $\lim_{z \to a} f(z)$ exists $\Longleftrightarrow$ f can be extended to a function $\tilde{f}$ on $D \cup \{a\}$ that is continuous at a.)

(ii) Let $D \subseteq \mathbb{C}$, $f: D \to \mathbb{C}$ be a function, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$, and $l \in \mathbb{C}$. Use (i) to prove the following statements are equivalent.

(a) f is complex differentiable at a and $f'(a) = l$.

(b) Define $g: D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ l & \text{if } z = a \end{cases}$$

Then g is continuous at a.

When f is complex differentiable then $g(a) = f'(a)$.

Use this to establish f is complex differentiable at a if and only if there exists a function $g: D \to \mathbb{C}$ such that g is continuous at a and $f(z) = f(a) + (z-a)g(z)$ for all $z \in D$.

Problem-2: Use problem-1 (ii) to in the followings.

Let $D \subseteq \mathbb{C}$ and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$.

(i) Suppose $f, g: D \to \mathbb{C}$ are complex differentiable at a. Show that fg is complex differentiable at a and

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

(ii) Suppose that $f: D \to \mathbb{C}$ complex differentiable at a and $f(z) \neq 0$ for all $z \in D$. Show that $\frac{1}{f}$ is also complex differentiable at a and

$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{(f(a))^2}$$

Problem-3: Let $D \subseteq \mathbb{C}$, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$ and $f: D \to \mathbb{C}$ be a function. Let $D' \subseteq \mathbb{C}$ such that $f(D) \subseteq D'$; $f(a)$ is an accumulation point of $D' \setminus \{f(a)\}$. Let $g: D' \to \mathbb{C}$. Assume that f is complex differentiable at a and g is complex differentiable at $f(a)$. Show that $g \circ f$ is complex differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$

Problem-4: (i) Assume that $D \subseteq \mathbb{C}$, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. Let $f, g : D \to \mathbb{C}$ such that both f and g are complex differentiable at a; $f(a) = 0$, $g(a) = 0$; and $g'(a) \neq 0$.

Show that $\lim\limits_{z \to a} \dfrac{f(z)}{g(z)} = \dfrac{f'(a)}{g'(a)}$

(ii) Compute $\lim\limits_{z \to i} \dfrac{z^2+1}{z^4-1}$ and

$$\lim_{z \to i} \frac{z^3 + (1-3i)z^2 + (i-3)z + 2+i}{z-i}$$

Problem-5 (i) Let $P, Q : \mathbb{C} \to \mathbb{C}$ be polynomial functions of degree m and n respectively, where m and n are positive integers. Define $A = \{z \in \mathbb{C} \mid Q(z) = 0\}$. Let $D = \mathbb{C} \setminus A$. Show that $f(z) = \dfrac{P(z)}{Q(z)}$ is complex differentiable on D.

(ii) Determine the largest $D \subseteq \mathbb{C}$ on which the following functions are complex differentiable.

(a) $z \mapsto \dfrac{1}{z^3+1}$ (b) $z \mapsto z^2 + \dfrac{1+i}{z}$ (c) $z \mapsto \dfrac{1}{e^z - 1}$

(d) $z \mapsto \operatorname{Im}(z)$

Problem-6: Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. Let $a \in D$ such that a is accomulation point of $D \setminus \{a\}$. Assume that f is complex differentiable at a.

Define $D^* = \{ z \mid \bar{z} \in D \}$ and $g : D^* \to \mathbb{C}$ by $g(z) = \overline{f(\bar{z})}$. Show that g is complex differentiable at $\bar{a} \in D^*$ and

$$g'(\bar{a}) = \overline{f'(a)}.$$

Problem-7: (i) Let $\mathbb{C}_- = \mathbb{C} - \{ z \in \mathbb{C} / z < 0 \}$

Show that for $z \in \mathbb{C}_-$

$$\operatorname{Arg}(z) = \begin{cases} \cos^{-1}\left(\dfrac{\operatorname{Re}(z)}{|z|}\right) & \text{if } \operatorname{Im}(z) \geq 0 \\[2ex] -\cos^{-1}\left(\dfrac{}{|z|}\right) & \text{if } \operatorname{Im}(z) < 0. \end{cases}$$

(ii) Show that $\operatorname{Arg} : \mathbb{C}_- \to \mathbb{R} \subseteq \mathbb{C}$ is continuous.

(iii) Show that $\operatorname{Log} : \mathbb{C}_- \to \mathbb{C}$ is continuous.

Problem-8 (i) Let $D, D' \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$ and $g: D' \to \mathbb{C}$ continuous. Moreover, assume $f(D) \subseteq D'$ and $g(f(z)) = z$ for all $z \in D$. Let $a \in D$ and $b = f(a) \in D'$. Show that if g is complex differentiable at b and $g'(b) \neq 0$, then f is complex differentiable at a and $f'(a) = \dfrac{1}{g'(b)}$.

(ii) Show that $\mathrm{Log}: \mathbb{C}_- \to \mathbb{C}$ is complex differentiable and

$$(\mathrm{Log}\, z)' = \frac{1}{z}.$$