Math 230A: Homework 1

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1. Mark each statement True or False.

   1-1) If x and y are elements of an ordered field, then either $x \leq y$ or $y < x$. **True.**

   1-2) Every ordered field has the least upper bound property. **False.**

   1-3) If $E \subseteq \mathbb{R}$ is bounded above and $\alpha = \sup E$, then $\alpha \in E$. **False.**

   1-4) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, then $\alpha = \sup E$. **False.**

   1-5) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \in E$, then $\alpha = \sup E$. **True.**

   1-6) If $E \subseteq \mathbb{R}$ and $\alpha \geq x$, for all $x \in E$, and $\alpha \notin E$, then $\alpha \neq \sup(E)$. **False.**

2. Prove the following: Suppose α is an upper bound for $E \subset \mathbb{R}$. Then $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   Proof. Suppose α is an upper bound for $E \subseteq \mathbb{R}$. For the forwards direction, suppose $\alpha = \sup E$ and let $\varepsilon > 0$. Note that $\alpha - \varepsilon < \alpha$ implies that $\alpha - \varepsilon$ is NOT an upper bound of E. By definition, there must exist an element $x_0 \in E$ such that $x_0 > \alpha - \varepsilon$.

   For the backwards direction, let α be an upper bound of E. We need to show $\alpha = \sup E$; that is, we need to show that α is an upper bound of E and that if $\gamma < \alpha$, then γ is NOT an upper bound of E. Notice that the first property is satisfied by our assumption. Thus, all that is left to show is the second property.

   Suppose $\gamma < \alpha$. Then $\alpha - \gamma > 0$. Choose $\varepsilon = \alpha - \gamma$. Our goal is to find some element $x \in E$ such that $x > \gamma$. By assumption, there exists an element $x_0 \in E$ such that

   $$x_0 > \alpha - \varepsilon = \alpha - (\alpha - \gamma) = \gamma.$$

   Thus, $x_0 > \gamma$ for some $x_0 \in E$. Therefore, γ is NOT an upper bound of E and so we conclude that

   $$\alpha = \sup E.$$

   ■

3. Let $E = \{n/n + 1 : n \in \mathbb{N}\}$. Prove that $\sup E = 1$.

   Proof. To show that 1 is the supremum of E, we need to show that 1 is an upper bound for E and that it satisfies the lemma found in Problem 2. First, we will show that 1 is an upper bound of E. Observe that for any $n \in \mathbb{N}$, we have

   $$\frac{n}{n+1} < \frac{n}{n} = 1.$$

   Thus, E is bounded above by 1.

   Let $\varepsilon > 0$. By the Archimedean Property, there exists an $n \in \mathbb{N}$ such that

   $$\frac{1}{n+1} < \varepsilon.$$

Then observe that
$$\frac{1}{n+1} = \frac{(n+1)-n}{n+1} = 1 - \frac{n}{n+1}.$$
Thus,
$$1 - \frac{n}{n+1} < \varepsilon \iff \frac{n}{n+1} > 1 - \varepsilon.$$
But note that $\frac{n}{n+1} \in E$. Thus, we conclude that $\sup E = 1$ by the lemma found in Problem 2.

■

4. State the analogue of the Very Useful Theorem for infimums.

Solution. Suppose β is a lower bound for $E \subset \mathbb{R}$. Then $\beta = \inf E$ if and only if for all $\varepsilon > 0$, there exists $y_0 \in E$ such that $y_0 < \beta + \varepsilon$. ■

5. Suppose $a, b \in \mathbb{R}$ and $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Prove $a \leq b$.

Proof. Suppose $a, b \in \mathbb{R}$ and $a \leq b+\varepsilon$ for all $\varepsilon > 0$. Suppose for sake of contradiction that $a > b$. Note that $a - b > 0$. Pick $\varepsilon = \frac{a-b}{2}$. Then
$$\begin{aligned} b + \varepsilon = b + \frac{a-b}{2} = \frac{2b+a-b}{2} &= \frac{a+b}{2} \\ &< \frac{a+a}{2} \\ &= \frac{2a}{2} \\ &= a. \end{aligned}$$
But this means that $a > b+\varepsilon$ which is a contradiction of our assumption that $a \leq b+\varepsilon$. Thus, it must be the case that $a \leq b$. ■

6. Complete the following proof.

Theorem (Greatest-lower-bound property of $\mathbb{R}$)**.** Every nonempty subset of A of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf A$ exists and is a real number.

Proof. Continuation of the proof presented in homework. Set $\beta = -\alpha$ where $\alpha = \sup(-A)$. Thus, there exists $a \in A$ such that $-\gamma < -a$. Multiplying by a negative on both sides of this inequality, we get $\gamma > a$ for some $a \in A$. If γ was a lower bound of A, then $\gamma \leq a$ for all $a \in A$. However, we have the negation of this, so we must have that γ is NOT a lower bound of A, which satisfies part (ii). Since $\beta \leq \alpha$ for all $a \in A$ and $\beta < \gamma$ implies γ is not a lower bound for A, we conclude that β must be the infimum of A. ■

7. (i) Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many elements of A.

Proof. Let $\varepsilon > 0$. Our goal is to show that $(\sup A - \varepsilon, \sup A)$ has infinitely many points. Note that as a consequence of the result found in Problem 2, there exists $x_0 \in A$ such that
$$\sup A - \varepsilon < x_0 \leq \sup A.$$

By assumption, we have that $\sup A \notin A$. Thus, we have

$$\sup A - \varepsilon < x_0 < \sup A$$

for some $x_0 \in A$. So, $(\sup A - \varepsilon, \sup A) \cap A$ is nonempty.

Now, we will show, that the set $(\sup A - \varepsilon, \sup A) \cap A$ contains infinitely many elements. Thus, suppose for sake of contradiction that the nonempty set $(\sup A - \varepsilon, \sup A) \cap A$ contains FINITELY many elements. Let us denote these elements by a_i for $1 \leq i \leq n$; that is,

$$(\sup A - \varepsilon, \sup A) \cap A = \{a_1, \dots, a_n : 1 \leq i \leq n\}.$$

For all $1 \leq k \leq n$, let the distance between each a_k and $\sup A$ be defined by

$$d_k = |\sup A - a_k|.$$

Since $a_k \in A$ and $\sup A \notin A$, we must have $|\sup A - a_k| > 0$. Let

$$d = \frac{\min\{d_1, \dots, d_n\}}{2}.$$

But this implies that none of the $a_1, \dots, a_n$ lies inside the interval $(\sup A - d, \sup A)$. That is,

$$A \cap (\sup A - d, \sup A) = \emptyset,$$

implies that $\sup A - d \in \mathrm{UP}(A)$ which is a contradiction because $\sup A - d$ is not an upper bound of A. Thus, $(\sup A - \varepsilon, \sup A) \cap A$ must contain infinitely many elements of A.

■

(ii) Let $B \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below. Show that if $\inf B \notin B$, then for all $\varepsilon > 0$ the open interval $(\inf B, \inf B + \varepsilon)$ contains infinitely many elements of B.

Proof. Let $\varepsilon > 0$. Note that as a consequence of Problem 3, there exists an $\beta \in B$ such that

$$\inf B \leq \beta < \inf B + \varepsilon.$$

Since $\inf B \notin B$, we must have

$$\inf B < \beta < \inf B + \varepsilon.$$

Thus, the set $(\inf B, \inf B + \varepsilon) \cap B$ is nonempty.

Now, we will show that $(\inf B, \inf B + \varepsilon) \cap B$ contains infinitely many points. Thus, suppose for sake of contradiction that the nonempty set $(\inf B, \inf B + \varepsilon) \cap B$ contains FINITELY many points. Denote these points by b_i for $1 \leq i \leq n$; that is

$$(\inf B, \inf B + \varepsilon) \cap B = \{b_i : 1 \leq i \leq n\}$$

For all $1 \leq k \leq n$, let the distance between each b_k and $\inf B$ be defined by

$$d_k = |\inf B - b_k|.$$

Since $b_k \in B$ and $\inf B \notin B$, we must have $|\inf B - b_k| > 0$. Let

$$d = \frac{\min\{d_1, d_2, \dots, d_n\}}{2}.$$

Note that observe that none of the b_i for $1 \leq i \leq n$ lies within the open interval $(\inf B, \inf B + d)$. That is, the set

$$(\inf B, \inf B + d) \cap B = \emptyset$$

implies that $\inf B + d \in \mathrm{LO}(B)$ which is a contradiction because $\inf B + d$ is not a lower bound of B. Thus, the open interval $(\inf B, \inf B + \varepsilon)$ must have infinitely many points. ■

8. Suppose the construction of $\mathbb{R}$ and the proof of the least-upper-bound property for $\mathbb{R}$ can be completed without directly using the well-ordering principle for $\mathbb{N}$. (Of course, statements equivalent to the well-ordering principle might have been used, but we are assuming that the nontrivial fact of their equivalence to the well-ordering principle was not utilized in the construction of $\mathbb{R}$ or in proving its Dedekind completeness.) Use the greatest-lower-bound property of $\mathbb{R}$ and the result of the previous exercises to prove the well-ordering principle for $\mathbb{N}$; that is, prove that every nonempty subset of $\mathbb{N}$ has a minimum.

Proof. Let $E \subseteq \mathbb{N}$ be a nonempty set. Note that E is bounded below by 0, and so $\mathrm{LO}(E) \neq \emptyset$. Since E is nonempty and bounded below, we know that $\inf E$ exists by the greatest-lower-bound property of $\mathbb{R}$.

To show that E contains a minimum, we suppose for contradiction that $\inf E \notin E$. Now, pick $\varepsilon = \frac{1}{5}$. By Part 2 of Problem 7, the open interval $(\inf E, \inf E + \frac{1}{5})$ intersects E at infinitely many points (these points being natural numbers). Suppose we choose an $m \in \mathbb{N}$ such that $m \in (\inf E, \inf E + \frac{1}{5})$. Note that m also lies interval $(m-1, m+1)$. Thus, $(\inf E, \inf E + \varepsilon) \subseteq (m-1, m+1)$. However, we see that $(\inf E, \inf E + \frac{1}{5})$ contains infinitely many natural numbers which cannot happen within the open interval $(m-1, m+1)$ since this interval contains at most only one element, namely m, which is a contradiction. Hence, it must be the case that $\inf E \in E$; that is, E contains a minimum. ■

9. Let $A, B \subseteq \mathbb{R}$ be nonempty, bounded sets and let $c \in \mathbb{R}$. Define the following sets:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{ab : a \in A, b \in B\} \\ cA &= \{ca : a \in A\}. \end{aligned}$$

Prove that

9-1) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Our goal is to show that $A+B$ contains an infimum and that $\inf(A+B) = \inf A + \inf B$. Note that $A+B \neq \emptyset$ since $a+b \in A+B$ and A and B are nonempty sets. Since A and B are both bounded below, there exists $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that

$$a \geq x \text{ for all } a \in A \text{ and } b \geq y \text{ for all } b \in B.$$

Adding these two inequalities together, we see that $a + b \geq x + y$ for all $a + b \in A + B$. Thus, $x + y$ is a lower bound for $A + B$. Since $A + B$ is bounded below, it follows from greatest-lower-bound property that $\inf(A+B)$ exists.

Now, let us show $\inf(A+B) = \inf(A) + \inf(B)$. In order to show that $\inf(A+B) = \inf(A) + \inf(B)$, we need to show that

(i) The set $A+B$ is bounded below by $\inf(A) + \inf(B)$.

(ii) For every $\varepsilon > 0$, there exists $\varphi \in A+B$ such that

$$\varphi < \inf(A) + \inf(B) + \varepsilon.$$

Since A and B are nonempty and bounded below, we know that $\inf(A)$ and $\inf(B)$ exists by the greatest-lower-bound property. Thus, we can see that $a \geq \inf A$ for all $a \in A$ and $b \geq \inf B$ for all $b \in B$. Adding these two inequalities together results in

$$a + b \geq \inf(A) + \inf(B) \quad \text{for all } a + b \in A + B,$$

which shows (i). Now, let $\varepsilon > 0$. Using the lemma found in Problem 3, there exists an $\alpha \in A$ and $\beta \in \beta$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{2}$$

and

$$\beta < \inf(B) + \frac{\varepsilon}{2},$$

respectively. Adding these two inequalities, we get

$$\varphi = \alpha + \beta < \inf(A) + \inf(B) + \varepsilon \quad \text{for some } \varphi \in A + B,$$

showing (ii). Thus, we conclude that $\inf(A+B) = \inf(A) + \inf(B)$. ■

9-2) $\sup(A+B) = \sup(A) + \sup(B)$

Proof. Our goal is to show that $A+B$ contains a supremum and that $\sup(A+B) = \sup A + \sup B$. Note that $A + B \neq \emptyset$ since $a + b \in A + B$ and A and B are nonempty sets. Since A and B are bounded above, there exists $u \in \mathbb{R}$ and $v \in \mathbb{R}$ such that

$$a \leq u \text{ for all } a \in A \text{ and } b \leq v \text{ for all } b \in B, \text{ respectively.}$$

Adding these two inequalities together, we see that $a + b \leq u + v$ for all $a + b \in A + B$. Thus, $u + v$ is a upper bound for $A + B$. Since $A + B$ is bounded above, it follows from least-upper-bound property that $\sup(A+B)$ exists.

In order to show that $\sup(A+B) = \sup(A) + \sup(B)$, we need to show that

(i) $A+B$ is bounded above by $\sup(A) + \sup(B)$ and

(ii) For every $\varepsilon > 0$, there exists a $\lambda \in A + B$ such that

$$\lambda > \sup(A) + \sup(B) - \varepsilon.$$

Since A and B are nonempty and bounded above, we must have that $\sup(A)$ and $\sup(B)$ exists by the least-upper-bound property. Hence, we have that $a \leq \sup(A)$ for all $a \in A$ and $b \leq \sup(B)$ for all $b \in B$. Adding these two inequalities together, gives us

$$a + b \leq \sup(A) + \sup(B)$$

for all $a + b \in A + B$, which proves (i). Let $\varepsilon > 0$. Since $\sup(A)$ and $\sup(B)$

exists, we see that there exists $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{2}$$

and

$$\beta > \sup(B) - \frac{\varepsilon}{2}$$

by the lemma found in Problem 2. Adding these two inequalities results in

$$\lambda = \alpha + \beta > \sup(A) + \sup(B) - \varepsilon$$

for some $\lambda = \alpha + \beta \in A + B$, which shows (ii). Thus, we conclude that $\sup(A + B) = \sup(A) + \sup(B)$. ■

9-3) $\sup(-A) = -\inf(A)$

Proof. We will show that $-A$ contains a supremum and that $\sup(-A) = -\inf(A)$. Note that $-A$ is nonempty because $-a \in -A$ and A is nonempty. Since A is bounded below, we can find an $x \in \mathbb{R}$ such that $a \geq x$ for all $a \in A$. Multiplying by a negative, we see that $-a \leq -x$ for all $-a \in -A$. Thus, $-x$ is an upper bound for $-A$. By the least upper bound property, $\sup(-A)$ exists.

To show that $\sup(-A) = -\inf(A)$, we need to show that

(i) $-A$ is bounded above by $-\inf(A)$ and that

(ii) for all $\varepsilon > 0$, there exists $-\alpha \in -A$ such that

$$-\alpha > -\inf(A) - \varepsilon.$$

Since A is bounded below and nonempty, we know that $\inf(A)$ exists (by the greatest-lower-bound property) and that $a \geq \inf(A)$ for all $a \in A$. Multiplying this inequality by a negative, we get that

$$-a \leq -\inf(A) \quad \text{for all} \quad -a \in -A.$$

Thus, $-A$ is bounded above by $-\inf(A)$, showing (i).

Let $\varepsilon > 0$. Since $\inf(A)$ exists, there exists $\alpha \in A$ such that

$$\alpha < \inf(A) + \varepsilon$$

by the lemma found in Problem 3. By multiplying by a negative on this inequality, we must have that

$$-\alpha > -\inf(A) - \varepsilon$$

for some $-\alpha \in -A$, showing (ii). Thus, we conclude that $\sup(-A) = -\inf(A)$. ■

9-4) $\inf(-A) = -\sup(A)$.

Proof. We will show that $-A$ contains an infimum and that $\inf(-A) = -\sup(A)$. Note that $-A$ is nonempty because $-a \in -A$ and A is nonempty. Since A is bounded above, we can find an $y \in \mathbb{R}$ such that $a \leq y$ for all $a \in A$. Multiplying by a negative, we see that $-a \geq -y$ for all $-a \in -A$. Thus, $-y$ is an lower bound for $-A$. By the greatest-lower-bound property, $\inf(-A)$ exists. Using 9-3), we see that

$$-\sup(A) = -\sup(-(-A)) = -(-\inf(-A)) = \inf(-A).$$

Thus, we conclude that $-\sup(A) = \inf(-A)$. ■

9-5) $\sup(A - B) = \sup(A) - \inf(B)$.

Proof. Observe that

$$\begin{aligned} \sup(A - B) = \sup(A + (-B)) &= \sup(A) + \sup(-B) && \text{(9-2))} \\ &= \sup(A) - \inf(B). && \text{(9-3))} \end{aligned}$$

■

9-6) $\inf(A - B) = \inf(A) - \sup(B)$.

Proof. Observe that

$$\begin{aligned} \inf(A - B) = \inf(A + (-B)) &= \inf(A) + \inf(-B) && \text{(9-1))} \\ &= \inf(A) - \sup(B). && \text{(9-4))} \end{aligned}$$

■

9-7) $\sup(cA) = c\sup(A)$ if $c > 0$.

Proof. Let $c > 0$. We will show that cA contains a supremum and that $\sup(cA) = c\sup(A)$. Note that cA is nonempty because $ca \in cA$ and $A \neq \emptyset$. Since A is bounded above, we can find an $x \in \mathbb{R}$ such that $a \leq x$ for all $a \in A$. Thus, $ca \leq cx$ for all $ca \in cA$. Thus, cx is an upper bound for cA. By the least-upper-bound property, $\sup(cA)$ exists. Now, we will show that $\sup(cA) = c\sup(A)$ by showing that

(i) The set cA is bounded above by $c\sup A$.

(ii) For all $\varepsilon > 0$, there exists an $c\alpha \in cA$ such that

$$c\alpha > c\sup(A) - \varepsilon.$$

First, we show that cA is bounded above by $c\sup(A)$. Since A is nonempty and bounded above, we know that $\sup(A)$ exists (least-upper-bound property). Thus, we see that

$$a \leq \sup(A) \quad \text{for all } a \in A.$$

Multiplying by c, we must have

$$ca \leq c\sup(A) \quad \text{for all } ca \in cA.$$

Thus, cA is bounded above by $c\sup(A)$. Now, let $\varepsilon > 0$. Since $\sup(A)$ exists, there exists $\alpha \in A$ such that

$$\alpha > \sup(A) - \frac{\varepsilon}{c}$$

by the result found in Problem 2. Multiplying by $c > 0$, we must have

$$c\alpha > c\sup(A) - \varepsilon \quad \text{for some } c\alpha \in cA,$$

showing (ii). Thus, we conclude that $c\sup(A) = \sup(cA)$. ■

9-8) $\inf(cA) = c\inf(A)$ if $c > 0$.

Proof. Let $c > 0$. We will show that cA contains an infimum and that $\inf(cA) = c\inf(A)$. Note that cA is nonempty because $ca \in cA$ and $A \neq \emptyset$. Since A is bounded below, we can find a $y \in \mathbb{R}$ such that $a \geq y$ for all $a \in A$. Thus, $ca \geq cy$ for all $ca \in cA$. Thus, cy is an lower bound for cA. By the greatest-lower-bound property, $\inf(cA)$ exists.

To show that $\inf(cA) = c\inf(A)$, we need to show that

(i) $c\inf(A)$ is a lower bound of cA and

(ii) for all $\varepsilon > 0$, there exists a $\gamma \in cA$ such that

$$\gamma < c\inf(A) + \varepsilon.$$

Since $\inf(A)$ exists (because $A \neq \emptyset$ and A is bounded below), we have that $a \geq \inf(A)$ for all $a \in A$. Multiplying by $c > 0$, we have $ca \geq c\inf(A)$ for all $ca \in cA$. Thus, cA is bounded below by $c\inf(A)$. This proves (i).

Let $\varepsilon > 0$. Note the fact that $\inf(A)$ exists implies that there exists an $\alpha \in A$ such that

$$\alpha < \inf(A) + \frac{\varepsilon}{c}$$

by the lemma found in Problem 3. Multiplying by $c > 0$, we have

$$c\alpha < c\inf(A) + \varepsilon$$

for some $\gamma = c\alpha \in cA$, showing (ii). Thus, we conclude that

$$\inf(cA) = c\inf(A).$$

■

9-9) $\sup(cA) = c\inf(A)$ if $c < 0$.

Proof. Let $c < 0$. Observe that

$$\begin{aligned} c\inf(A) &= c\inf(-(-A)) \\ &= -c(\sup(-A)) && (9\text{-}4)) \\ &= \sup((-c)(-A)) && (-c > 0 \text{ and } 9\text{-}7)) \\ &= \sup(cA). \end{aligned}$$

Thus, we conclude that $c\inf(A) = \sup(cA)$ for $c < 0$. ■

9-10) $\inf(cA) = c\sup(A)$ if $c < 0$.

Proof. Let $c < 0$. Observe that

$$\begin{aligned} \inf(cA) &= \inf((-c)(-A)) && (-c > 0) \\ &= -c\inf(-A) && (9\text{-}8)) \\ &= -c(-\sup A) && (9\text{-}4)) \\ &= c\sup A. \end{aligned}$$

Thus, we conclude that

$$\inf(cA) = c\sup A$$

for $c < 0$. ■

9-11) Is it true that $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

Proof. This is false. Consider the sets $A = \{1, 2, 4, 8\}$ and $B = \{-5, -2, -3, -1\}$. Then $AB = \{-5, -4, -12, -8\}$. Observe that $\sup(A) \cdot \sup(B) = 8 \cdot -1 = -8$, but $\sup(AB) = -5 \neq -8 = \sup(A) \cdot \sup(B)$. ∎

10. Recall that $\mathbb{Q}$ is a field; in particular, $\mathbb{Q}$ is closed under addition and multiplication.

10-1) Prove that if $p \in \mathbb{Q}$ and t is an irrational number, then $p + t$ is an irrational number.

Proof. Suppose $p \in \mathbb{Q}$ and t is an irrational number. Suppose for sake of contradiction that $p + t$ is an rational number. Then there exists $x, y \in \mathbb{Z}$ with $y \neq 0$ such that

$$p + t = \frac{x}{y}.$$

Subtracting p on both sides, we get

$$t = \frac{x}{y} - p.$$

Since $\mathbb{Q}$ is a field and $\frac{x}{y}, p \in \mathbb{Q}$, we get that $\frac{x}{y} - p \in \mathbb{Q}$. But this tells us that t is a rational number which is a contradiction. Thus, t must be irrational. ∎

10-2) Complete the following proof.

Theorem. Given any two real numbers $x < y$, there exists an irrational number t satisfying $x < t < y$.

Proof. It follows from $x < y$ that $x - \sqrt{2} < y - \sqrt{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $p \in \mathbb{Q}$ such that $x - \sqrt{2} < p < y - \sqrt{2}$. Adding $\sqrt{2}$ on both sides gives us

$$x < p + \sqrt{2} < y.$$

Since $p \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, we get that $t = p + \sqrt{2}$ is irrational from 10-2). Thus, $x < t < y$ for some irrational number t.

∎

11. Prove the following:

Theorem (Nested Interval Property)**.** For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, that is $\cap_n^\infty I_n \neq \emptyset$.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ because $a_1 \in A$. Since $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$ and that each I_{n+1} is contained within each I_n , we see that each $a_n \in A$ must be bounded above by b_n for all $n \in \mathbb{N}$. Since $A \neq \emptyset$ and bounded above, there must exists a number x such that $x = \sup(A)$. Since x is an upper bound of A, we have $a_n \leq x \leq b_n$. Thus, $x \in I_n$ for all $n \in \mathbb{N}$ which means that

$$x \in \bigcap_{n=1}^{\infty} I_n,$$

proving that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

12. Complete the following proof:

Prove that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.

Proof. Suppose for sake of contradiction $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. So there exists $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. This implies that $x > 0$ and $x < 1/n$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < x.$$

But this contradicts our assumption that $x < 1/n$ holds for all $n \in \mathbb{N}$. Thus, the intersection

$$\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$$

must be empty. ■