Homework 1

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Problem 1. (a) Define $f : [0, \infty] \to \mathbb{R}$ by $f(t) = \frac{t}{1+t}$ show that f is an increasing function.

(b) Let (X, d) be a metric space. Define $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that $\tilde{d}$ is a metric on X.

Proof. (a) We will show that $f : [0, \infty] \to \mathbb{R}$ defined by $f(t) = \frac{t}{1+t}$ is an increasing function; that is, for any $a, b \in [0, \infty]$ with $a \leq b$, we have

$$\frac{a}{1+a} \leq \frac{b}{1+b}.$$

To this end, let $a, b \in [0, \infty]$ such that $a \leq b$. Observe that

$$\begin{aligned} a \leq b &\iff ab + a \leq ab + b \\ &\iff a(b+1) \leq b(a+1) \\ &\iff \frac{a}{1+a} \leq \frac{b}{1+b}. \end{aligned}$$

Hence, we see that $f(t) = \frac{t}{1+t}$ must be an increasing function.

(b) (i) Let $x, y \in X$. Since (X, d) is a metric space, we know that $d(x, y) = 0$ if and only if $x = y$. Thus, we see that

$$\begin{aligned} \tilde{d}(x, y) = 0 &\iff \frac{d(x, y)}{1 + d(x, y)} = 0 \\ &\iff d(x, y) = 0 \\ &\iff x = y. \end{aligned}$$

Hence, property (ii) is satisfied.

(ii) Let $x, y \in X$. Since $d(x, y) = d(y, x)$ for all $x, y \in X$ (because (X, d) is a metric space), we see that

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = \tilde{d}(y, x).$$

Thus, property (iii) is satisfied.

(iii) Now, we will show that $\tilde{d}$ satisfies the triangle inequality. Let $x, y, z \in X$. We need to consider a few cases when proving the triangle inequality:

(I) $d(x, y) \leq d(x, z)$.

(II) $d(x, y) \leq d(z, y)$

(III) $d(x, y) > d(x, z)$ and $d(x, z) > d(z, y)$.

We proceed with each case as follows:

(I) If $d(x,y) \le d(x,z)$, then by using the monotonicity of $\tilde{d}(x,y)$, we can see that

$$\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \le \frac{d(x,z)}{1+d(x,z)} = \tilde{d}(x,z) \le \tilde{d}(x,z) + \tilde{d}(z,y).$$

(II) Similarly, if $d(x,y) \le d(z,y)$, we have by using the monotonicity of $\tilde{d}(x,y)$ that

$$\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \le \frac{d(z,y)}{1+d(z,y)} = \tilde{d}(z,y) \le \tilde{d}(x,z) + \tilde{d}(z,y).$$

(III) If both $d(x,y) > d(x,z)$ and $d(x,z) > d(z,y)$, then by the triangle inequality property of (X,d), we see that

$$\begin{aligned}\tilde{d}(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &\le \frac{d(x,z)+d(z,y)}{1+d(x,y)} \\ &= \frac{d(x,z)}{1+d(x,y)} + \frac{d(z,y)}{1+d(z,y)} \\ &\le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= \tilde{d}(x,z) + \tilde{d}(z,y).\end{aligned}$$

Thus, we can see that $\tilde{d}(x,y)$ satisfies the triangle inequality.

Since properties (I) through (IV) are satisfied, we can conclude that $\tilde{d}$ is a metric on X. ■

Problem 2. Let $X = \mathbb{R}^n$. For $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. Define $d_1(\vec{x},\vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ and $d_\infty(\vec{x},\vec{y}) = \max\{|x_i - y_i| : 1 \le i \le n\}$. Show that d_1 and d_∞ are metrics on $\mathbb{R}^n$.

Proof. We will show that the following functions

(1) $d_1(\vec{x},\vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$

(2) $d_\infty(\vec{x},\vec{y}) = \max\{|x_i - y_i| : 1 \le i \le n\} = \max_{1\le i\le n} |x_i - y_i|$.

First, we will show that (1) is a metric on $\mathbb{R}^n$.

(i) Suppose $\vec{x} = \vec{y}$. Then observe that

$$\begin{aligned}\vec{x} = \vec{y} &\iff x_i = y_i \ \forall 1 \le i \le n \\ &\iff x_i - y_i = 0 \ \forall 1 \le i \le n \\ &\iff |x_i - y_i| = 0 \ \forall 1 \le i \le n \\ &\iff \sum_{i=1}^{n} |x_i - y_i| = 0 \ \forall 1 \le i \le n \\ &\iff d_1(\vec{x},\vec{y}) = 0.\end{aligned}$$

Thus, property (i) is satisfied.

(ii) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(\vec{y}, \vec{x}).$$

Thus, property (ii) is satisfied.

(iii) Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$. By the triangle inequality of the standard metric $|\cdot|$ on $\mathbb{R}$, we have

$$\begin{aligned} d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} [|x_i - z_i| + |z_i - y_i|] \\ &= \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| \\ &= d_1(\vec{x}, \vec{z}) + d_1(\vec{z}, \vec{y}). \end{aligned}$$

Thus, properties (i) through (ii) of a metric space are satisfied. Hence, d_1 is a metric on $\mathbb{R}^n$.

Now, we will show that (2) is a metric on $\mathbb{R}^n$.

(i) Observe that

$$\begin{aligned} d_\infty(\vec{x}, \vec{y}) = 0 &\implies \max_{1 \leq i \leq n} |x_i - y_i| = 0 \\ &\implies 0 \leq |x_i - y_i| \leq 0 \quad \forall 1 \leq i \leq n \\ &\implies |x_i - y_i| = 0 \quad \forall 1 \leq i \leq n \\ &\implies x_i - y_i = 0 \quad \forall 1 \leq i \leq n \\ &\implies x_i = y_i \quad \forall 1 \leq i \leq n \\ &\implies \vec{x} = \vec{y}. \end{aligned}$$

Conversely, we have

$$\begin{aligned} \vec{x} = \vec{y} &\implies x_i = y_i \quad \forall 1 \leq i \leq n \\ &\implies x_i - y_i = 0 \quad \forall 1 \leq i \leq n \\ &\implies |x_i - y_i| = 0 \quad \forall 1 \leq i \leq n \\ &\implies \max_{1 \leq i \leq n} |x_i - y_i| = 0 \\ &\implies d_\infty(\vec{x}, \vec{y}) = 0. \end{aligned}$$

(ii) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then we have

$$d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq n} |x_i - y_i| = \max_{1 \leq i \leq n} |y_i - x_i| = d_\infty(\vec{y}, \vec{x}).$$

(iii) Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$. Our goal is to show that

$$d_\infty(\vec{x}, \vec{y}) \leq d_\infty(\vec{x}, \vec{z}) + d_\infty(\vec{z}, \vec{y}).$$

Note that

$$|x_i - z_i| \leq \max_{1 \leq i \leq n} |x_i - z_i| \quad \text{and} \quad |z_i - y_i| \leq \max_{1 \leq i \leq n} |z_i - y_i| \qquad (\forall 1 \leq i \leq n)$$

Adding the two inequalities above gives us

$$|x_i - z_i| + |z_i - y_i| \leq \max_{1 \leq i \leq n} |x_i - z_i| + \max_{1 \leq i \leq n} |z_i - y_i|. \qquad (1 \leq i \leq n)$$

Using the triangle inequality property of the standard metric $|\cdot|$ on $\mathbb{R}$, we can see that

$$|x_i - y_i| \le |x_i - z_i| + |z_i - y_i|.$$

Hence, we have

$$|x_i - y_i| \le \max_{1\le i\le n} |x_i - z_i| + \max_{1\le i\le n} |z_i - y_i|.$$

Notice that the right-hand side of the above inequality is an upper bound of the set $\{|x_i - y_i| : x_i, y_i \in \mathbb{R}, 1 \le i \le n\}$. Thus, we see that

$$\max_{1\le i\le n} |x_i - y_i| \le \max_{1\le i\le n} |x_i - z_i| + \max_{1\le i\le n} |z_i - y_i|.$$

By definition, we can conclude that

$$d_\infty(\vec{x}, \vec{y}) \le d_\infty(\vec{x}, \vec{z}) + d_\infty(\vec{z}, \vec{y}).$$

Hence, we can conclude that d_∞ forms a metric on $\mathbb{R}^n$. ■

Problem 3. Let $X = \mathbb{R}^n$ and d_1 and d_∞ be as defined as in problem 2.

(a) (i) Show that $d_\infty(\vec{x}, \vec{y}) \le d_1(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(ii) Let $\vec{x_0} \in \mathbb{R}^n$ and $r > 0$. Let

$$B_1 = \{\vec{x} \in \mathbb{R}^n : d_1(\vec{x}, \vec{x_0}) < r\}$$
$$B_2 = \{\vec{x} \in \mathbb{R}^n : d_\infty(\vec{x}, \vec{x_0}) < r\}.$$

Which one of the following holds and why?

$$B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1.$$

(b) Show that $d_1(\vec{x}, \vec{y}) \le n \cdot d_\infty(\vec{x}, \vec{y})$.

(c) Fix $\vec{x_0} \in \mathbb{R}^n$ and $r > 0$. Prove that

$$\{\vec{x} \in \mathbb{R}^n : d_\infty(\vec{x}, \vec{x_0}) < \frac{r}{n}\} \subseteq \{\vec{x} \in \mathbb{R}^n : d_1(\vec{x}, \vec{x_0}) < r\}.$$

(d) Prove that $M \subseteq \mathbb{R}^n$ is open with respect to the metric d_1 if and only if M is open with respect to the metric d_∞.

Proof. (a) (i) Note that for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have

$$|x_i - y_i| \le \sum_{i=1}^{n} |x_i - y_i|. \qquad (\forall 1 \le i \le n)$$

Furthermore, the right-hand side of the above inequality is an upper bound for the set

$$\{|x_i - y_i| : 1 \le i \le n, \quad \vec{x}, \vec{y} \in \mathbb{R}^n\}.$$

Hence, we have

$$\max_{1\le i\le n} |x_i - y_i| \le \sum_{i=1}^{n} |x_i - y_i| \Longrightarrow d_\infty(\vec{x}, \vec{y}) \le d_1(\vec{x}, \vec{y}).$$

(ii) We claim that $B_2 \subseteq B_1$ holds. Let $\vec{y} \in B_2$. By part (i), we can see that $d_\infty(\vec{y}, \vec{x_0}) \le d_1(\vec{y}, \vec{x_0}) < r$. Hence, we have $\vec{y} \in B_1$ and so $B_2 \subseteq B_1$.

(b) Our goal is to show that $d_1(\vec{x}, \vec{y}) \leq n \cdot d_\infty(\vec{x}, \vec{y})$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Observe that

$$\begin{aligned} d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} \max_{1 \leq i \leq n} |x_i - y_i| \\ &= n \cdot \max_{1 \leq i \leq n} |x_i - y_i| \\ &= n \cdot d_\infty(\vec{x}, \vec{y}). \end{aligned}$$

Thus, we have $d_1(\vec{x}, \vec{y}) \leq n \cdot d_\infty(\vec{x}, \vec{y})$.

(c) Fix $\vec{x_0} \in \mathbb{R}^n$ and $r > 0$. Set

$$\begin{aligned} D_1 &= \{\vec{x} \in \mathbb{R}^n : d_\infty(\vec{x}, \vec{x_0}) < \frac{r}{n}\}, \\ D_2 &= \{\vec{x} \in \mathbb{R}^n : d_1(\vec{x}, \vec{x_0}) < r\}. \end{aligned}$$

We will show that $D_1 \subseteq D_2$. Let $\vec{y} \in D_1$. Then

$$d_\infty(\vec{y}, \vec{x_0}) < \frac{r}{n} \Longleftrightarrow n \cdot d_\infty(\vec{y}, \vec{x_0}) < r.$$

Using part (b), we can write

$$d_1(\vec{y}, \vec{x_0}) < r.$$

Hence, $\vec{y} \in D_2$. So, $D_1 \subseteq D_2$.

(d) ($\Longrightarrow$) Suppose that M is an open set with respect to the metric d_1. We will show that M is open with respect to the metric d_∞; that is, we want to show that for all $\vec{x} \in M$, there exists a $\delta > 0$ such that $B_{d_\infty}(\vec{x}; \delta) \subseteq M$. To this end, let $\vec{x} \in M$. By assumption, we can see that there exists $\hat{\delta} > 0$ such that $B_{d_1}(\vec{x}, \hat{\delta}) \subseteq M$.

We claim that $\hat{\delta}$ can be used as the same δ we were looking for. Indeed, we can see by part (ii) of (a) that $B_{d_\infty}(\vec{x}; \delta) \subseteq B_{d_1}(\vec{x}, \delta) \subseteq M$. Hence, we have M must be open with respect to the metric d_∞.

($\Longleftarrow$) Suppose that M is an open set with respect to the metric d_∞. Our goal is to show that M is open with respect to the metric d_1; that is, for all $\vec{y} \in M$, we need to find a $\delta > 0$ such that $B_{d_1}(\vec{x}; \delta) \subseteq M$. Let $\vec{y} \in M$. By assumption, we can find a $\hat{\delta} > 0$ such that $B_{d_\infty}(\vec{y}; \hat{\delta}) \subseteq M$. If $\vec{x} \in B_{d_\infty}(\vec{y}; \hat{\delta})$, then $d_\infty(\vec{x}, \vec{y}) < \hat{\delta}$. Set $\delta = \hat{\delta} - d_\infty(\vec{x}, \vec{y}) > 0$. It suffices to show that $B_{d_1}(\vec{y}; \delta) \subseteq B_{d_\infty}(\vec{y}; \hat{\delta})$. Let $\vec{z} \in B_{d_1}(\vec{y}; \delta)$. Then

$$d_1(\vec{z}, \vec{y}) < \delta \Longrightarrow d_1(\vec{z}, \vec{y}) < \hat{\delta} - d_\infty(\vec{x}, \vec{y}).$$

Since $d_1(\vec{z}, \vec{y}) < d_1(\vec{z}, \vec{y}) + d_\infty(\vec{x}, \vec{y})$, we can conclude that

$$d_1(\vec{z}, \vec{y}) < \hat{\delta} \Longrightarrow \vec{z} \in B_{d_\infty}(\vec{y}; \hat{\delta}) \Longrightarrow B_{d_1}(\vec{y}; \delta) \subseteq B_{d_\infty}(\vec{y}; \hat{\delta}).$$

Hence, M is open with respect to d_1.

■

Problem 4. Let I be an indexing set and $\{X_i\}_{i \in I}$ be a collection of subsets of X. We define

$$\begin{aligned} \bigcup_{i \in I} X_i &= \{x \in X : x \in X_i \ \text{ for some } \ i \in I\} \\ \bigcap_{i \in I} X_i &= \{x \in X : x \in X_i \ \text{ for all } \ i \in I\}. \end{aligned}$$

Let (X, d) be a metric space.

(i) Let $\{M_i\}_{i \in I}$ be a collection of open sets in X. Show that $\bigcup_{i \in I} M_i$ is also open.

(ii) Let M_1 and M_2 be two open sets in X. Prove that $M_1 \cap M_2$ is open.

(iii) Let $x_0 \in X$ and $r > 0$. Show that $B(x_0; r)$ is open.

(iv) Let $\{K_i\}_{i \in I}$ be a collection of closed sets in X. Prove that $\bigcap_{i \in I} K_i$ is closed.

(v) Let K_1 and K_2 be closed in X. Prove that $K_1 \cup K_2$ is closed.

(vi) Let $x_0 \in X$ and $r > 0$. Prove that $\overline{B}(x_0; r)$ is closed.

(vii) Let $M \subseteq X$. Prove that $\overline{M}$ is closed in X.

(viii) Let $x_0 \in X$ and $r > 0$. Prove that

$$\overline{B(x_0; r)} \subseteq \overline{B}(x_0; r).$$

Is it always true that

$$\overline{B(x_0; r)} = \overline{B}(x_0; r)?$$

Justify your answer.

Proof. (i) Let $x \in \bigcup_{i \in I} M_i$. Our goal is to find a $\delta > 0$ such that $B(x, \delta) \subseteq \bigcup_{i \in I} M_i$. Now, notice that

$$x \in \bigcup_{i \in I} M_i \implies \exists k \in I \text{ such that } x \in M_k.$$

Since M_k is an open set, there exists $\hat{\delta} > 0$ such that $B(x, \hat{\delta}) \subseteq M_k$. But this means that

$$B(x, \hat{\delta}) \subseteq M_k$$

since $M_k \subseteq \bigcup_{i \in I} M_i$.

(ii) Let $x \in M_1 \cap M_2$. Our goal is to show that there exists $\delta > 0$ such that $B(x, \delta) \subseteq M_1 \cap M_2$. If $x \in M_1 \cap M_2$, then $x \in M_1$ and $x \in M_2$. Then $x \in M_1$ and M_1 is open implies there exists $\delta_1 > 0$ such that $B(x, \delta_1) \subseteq M_1$. Likewise, $x \in M_2$ an M_2 is open implies that there exists $\delta_2 > 0$ such that $B(x, \delta_2) \subseteq M_2$. Choose $\delta = \frac{1}{2} \min\{\delta_1, \delta_2\}$. Our goal is to show that $B(x, \delta) \subseteq M_1 \cap M_2$. By the way δ was constructed, observe that

$$B(x, \delta) \subseteq B(x, \delta_1) \subseteq M_1 \text{ and } B(x, \delta) \subseteq B(x, \delta_2) \subseteq M_2.$$

If $y \in B(x, \delta)$, then $y \in M_1$ and $y \in M_2$ by the above. Hence, we have

$$B(x, \delta) \subseteq M_1 \cap M_2.$$

(iii) Our goal is to show that $B(x_0, r)$ is open; that is, we want to show that for every $x \in B(x_0, r)$, there exists $\delta > 0$ such that $B(x, \delta) \subseteq B(x_0, r)$. Let $x \in B(x_0, r)$. Then

$$d(x, x_0) < r \implies r - d(x, x_0) > 0.$$

Choose $\delta = r - d(x, x_0)$. Now, our goal is to show that

$$B(x, \delta) \subseteq B(x_0, r).$$

Let $y \in B(x, \delta)$. Then

$$d(y, x) < \delta = r - d(x, x_0) \implies d(y, x) + d(x, x_0) < r.$$

By the triangle inequality, we can see that

$$d(y, x_0) \leq d(y, x) + d(x, x_0).$$

This implies that

$$d(y, x_0) < r \implies y \in B(x_0, r).$$

Hence, $B(x, \delta) \subseteq B(x_0, r)$.

(iv) Let $\{K_i\}_{i\in I}$ be a collection of closed sets in X. Our goal is to show that $\bigcap_{i\in I} K_i$ is closed. It suffices to show that

$$\Big(\bigcap_{i\in I} K_i\Big)^c \text{ is open.}$$

The above can be rewritten in the following way

$$\Big(\bigcap_{i\in I} K_i\Big)^c = \bigcup_{i\in I} K_i^c.$$

Since each K_i is closed, we can see that K_i^c is open. Using part (i), we can conclude that $\Big(\bigcap_{i\in I} K_i\Big)^c$ is open and so

$$\bigcap_{i\in I} K_i$$

must be closed.

(v) Apply part (iv) to $i \in I$ on $K_1, K_2, \emptyset, \emptyset, \ldots$.

(vi) Our goal is to show that $\overline{B}(x_0, r)$ is a closed set. To this end, we will show that $[\overline{B}(x_0, r)]^c$ is an open set. If this holds, then we can conclude that $\overline{B}(x_0, r)$ is a closed set. Let $x \in [\overline{B}(x_0, r)]^c$. Then we have $d(x, x_0) > r$. Our goal is to find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq [\overline{B}(x_0, r)]^c$. Then $d(x, p) > r$. Choose $\varepsilon = d(x, p) - r > 0$. Using this chosen radius ε, let $y \in B(x, \varepsilon)$. In order for y to be contained within $[\overline{B}(x_0, r)]^c$, we have to show that $d(p, y) > r$. Using the triangle inequality, we have

$$\begin{aligned} d(x, x_0) \le d(x, y) + d(y, x_0) \implies d(y, x_0) &\ge d(x, x_0) - d(x, x_0) \\ &> d(x, x_0) - \varepsilon && (y \in B(x, \varepsilon)) \\ &= r. \end{aligned}$$

Indeed, we can now see that $B(x, \varepsilon) \subseteq [\overline{B}(x_0, r)]^c$. This tells us that $[\overline{B}(x_0, r)]^c$ is open and so $\overline{B}(x_0, r)$ is closed.

(vii) Our goal is to show that $(\overline{E})^c$ is open. We need to show that every point of $(\overline{E})^c$ is an interior point of $(\overline{E})^c$. Let $p \in (\overline{E})^c$. We have

$$\begin{aligned} p \in (\overline{E})^c &\implies p \notin \overline{E} \\ &\implies p \notin (E \cup E') \\ &\implies p \notin E \ \wedge \ p \notin E'. \end{aligned}$$

Note that

$$\begin{aligned} p \notin E' &\implies \exists \varepsilon > 0 \ B(p, \varepsilon) \cap (E \setminus \{p\}) = \emptyset \\ &\implies \exists \varepsilon > 0 \ B(p, \varepsilon) \cap E = \emptyset. \end{aligned} \tag{1}$$

In what follows, we will show that $B(p, \varepsilon) \cap E' = \emptyset$. So, we have

$$\begin{aligned} &B(p, \varepsilon) \cap (E \cup E') = \emptyset \\ &\implies B(p, \varepsilon) \cap \overline{E} = \emptyset \\ &\implies B(p, \varepsilon) \subseteq (\overline{E})^c. \end{aligned}$$

Thus, we have that p is an interior point of $(\overline{E})^c$. It remains to show that $B(p, \varepsilon) \cap E' = \emptyset$. Assume for sake of contradiction that $B(p, \varepsilon) \cap E' \neq \emptyset$. Let $q \in B(p, \varepsilon) \cap E'$. Then we have $q \in B(p, \varepsilon)$ and $q \in E'$. Because $B(p, \varepsilon)$ is an open set, there exists $\delta > 0$ such that $B(q, \delta) \subseteq$

$B(p, \varepsilon)$ and that $B(q, \delta) \cap (E \setminus \{q\}) \neq \emptyset$, respectively. But note that since $B(q, \delta) \subseteq B(p, \varepsilon)$ and $E \setminus \{q\} \subseteq E$ implies that

$$B(p, \varepsilon) \cap E \neq \emptyset.$$

which contradicts (1).

(viii) We can see immediately that $B(x_0, r) \subseteq \overline{B}(x_0, r)$. By part (vi), we can see that $\overline{B}(x_0, r)$ is a closed set. As a consequence, we have that

$$\overline{B(x_0, r)} \subseteq \overline{B}(x_0, r).$$

In general, it is not true that

$$\overline{B(x_0, r)} = \overline{B}(x_0, r).$$

Consider the interval $[0, 1]$ in $\mathbb{R}$ with the discrete metric. Clearly, we see that $1/2 \in [0, 1]$. If we let $\varepsilon = 1$, then

$$B(\frac{1}{2}, 1) = \{x \in \mathbb{R} : d(x, 1/2) < 1\} = \{1/2\}$$

since the only case when the inequality is satisfied is when $x = 1/2$. If we consider the closure of this neighborhood, we just get

$$\overline{B(\frac{1}{2}, 1)} = \{1/2\}.$$

Now, consider the closed ball

$$B(\frac{1}{2}, 1) = \{x \in \mathbb{R} : d(x, 1/2) \leq 1\}.$$

Observe that for any $x \in \mathbb{R}$, either $x = 1/2$ or $x \neq 1/2$ in $[0, 1]$, the inequality of the set above we always be satisfied; that is, the set will just be all elements contained in $[0, 1]$. Thus, we see that $B(\frac{1}{2}, 1) = [0, 1]$ and, in this case, that $\overline{B}(\frac{1}{2}, 1) \neq \overline{B(\frac{1}{2}, 1)}$.

■

Problem 5. In this problem, we will establish some key inequalities that will be useful later in the class.

(i) **Young's Inequality:** Let $a, b \in \mathbb{R}$, $a \geq 0$, $b \geq 0$, and $p > 1$. Let $q = \frac{p}{p-1}$. Then

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}.$$

Prove Young's Inequality.

(ii) **Holder's Inequality:** Let $p > 1$. For $\vec{x} \in \mathbb{R}^n$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, define $\|\vec{x}\|_p = \Big[\sum_{i=1}^{n} |x_i|^p\Big]^{1/p}$. Let $q = \frac{p}{p-1}$. For $\vec{x}, \vec{y} \in \mathbb{R}^n$, prove that

$$\sum_{i=1}^{n} |x_i y_i| \leq \|\vec{x}\|_p \|\vec{y}\|_q$$

where $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

(iii) **Minkowski's Inequality** Let $p, \vec{x}$, and $\vec{y}$ be as in (ii). Prove that

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p.$$

(iv) Let $X = \mathbb{R}^n$, $p > 1$. For $\vec{x}, \vec{y} \in \mathbb{R}^n$, define $d_p(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_p$. Prove that d_p is a metric on $\mathbb{R}^n$.

Proof. (i) Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \in \mathbb{R}$ be nonnegative. We have the following cases:

(1) $a = 0$ and $b = 0$

(2) $a = 0$ and $b > 0$

(3) $a > 0$ and $b = 0$

(4) $a > 0$ and $b > 0$.

We proceed with the proof of the result with the following cases.

(1) If $a = 0$ and $b = 0$, then the result is immediate.

(2) If $a = 0$ and $b > 0$, then we immediately have

$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{b^q}{q}.$$

(3) If $b = 0$ and $a > 0$, then we similarly have

$$ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q} = \frac{a^p}{p}.$$

(4) Suppose $a > 0$ and $b > 0$. By the property of logarithms, we see that

$$ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b}.$$

Also, we see that

$$\ln(a^p) = p \ln a \text{ and } \ln(b^q) = q \ln b.$$

Now, observe that

$$e^{\ln a + \ln b} = e^{\frac{p}{p} \ln a + \frac{q}{q} \ln b} = e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)}.$$

Notice that e^t, when differentiated twice, is a strictly positive function. Thus, e^t is convex for all $t \in \mathbb{R}$ our knowledge of calculus. Thus, we can use Jensen's inequality to conclude that

$$e^{\frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q)} \leq \frac{1}{p} e^{\ln(a^p)} + \frac{1}{q} e^{\ln(b^q)} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

(ii) Let $a = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$ and $b = \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$. Note that if $a = 0$ or $b = 0$, then both sides of the above inequality will be zero. Hence, it suffices to show the result when $a \neq 0$ and $b \neq 0$. For each $i \in \{1, \dots, n\}$ let $u_i = \frac{|x_i|}{a}$ and $v_i = \frac{|y_i|}{b}$. Using part (a), we can see that

$$\begin{aligned}
\sum_{i=1}^n \left| \left(\frac{x_i}{a} \right) \left(\frac{y_i}{b} \right) \right| = \sum_{i=1}^n u_i v_i &\leq \sum_{i=1}^n \left(\frac{u_i^p}{p} + \frac{v_i^q}{q} \right) \\
&= \frac{1}{p} \sum_{i=1}^n \frac{|x_i|^p}{a^p} + \frac{1}{q} \sum_{i=1}^n \frac{|y_i|^q}{b^q} \\
&= \frac{1}{pa^p} \sum_{i=1}^n |x_i|^p + \frac{1}{qb^q} \sum_{i=1}^n |y_i|^q \\
&= \left(\frac{1}{pa^p} \right) a^p + \left(\frac{1}{qb^q} \right) b^q \\
&= \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\sum_{i=1}^{n} \left| \left(\frac{x_i}{a}\right)\left(\frac{y_i}{b}\right) \right| \leq 1 \implies \sum_{i=1}^{n} \left|\frac{x_i}{a}\right|\left|\frac{y_i}{b}\right| \leq 1 \\
&\implies \frac{1}{ab} \sum_{i=1}^{n} |x_i||y_i| \\
&\implies \sum_{i=1}^{n} |x_i y_i| \leq ab = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |y_i|^q\Big)^{\frac{1}{q}}.
\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{n} |x_i y_i| \leq \|\vec{x}\|_p \|\vec{y}\|_q.$$

(iii) Suppose $\vec{x}, \vec{y} \in \mathbb{R}^n$. Notice that if $\sum_{i=1}^{n} |x_i + y_i|^p = 0$, then Minkowski's inequality immediately follows. Hence, it suffices to show the result when $\sum_{i=1}^{n} |x_i + y_i|^p \neq 0$. Note that if $p = 1$, then Minkowski's inequality immediately follows via applying the triangle inequality (on the standard metric $|\cdot|$ on $\mathbb{R}$) and distributing the summation. Thus, suppose further that $p > 1$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$ (where $p > 1$ and q is a positive real number). Hence, we have that

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} (|x_i| + |y_i|)|x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\
&\leq \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\Big)^{\frac{1}{q}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\Big)^{\frac{1}{q}} \\
&\hspace{10cm} \text{(Holder's Inequality)} \\
&= \left(\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}}\right) \Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{\frac{1}{q}}.
\end{aligned}$$

Dividing $\Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{\frac{1}{q}}$ by both sides, we see that

$$\frac{\sum_{i=1}^{n} |x_i + y_i|^p}{\Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{\frac{1}{q}}} \leq \left(\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}}\right)$$

which can be re-written to

$$\Big(\sum_{i=1}^{n} |x_i + y_i|^p\Big)^{1-\frac{1}{q}} \leq \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} |y_i|^p\Big)^{\frac{1}{p}}.$$

Now, observe that

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p}.$$

If we set

$$A = \sum_{i=1}^{n} |x_i + y_i|^p,$$

then we see that

$$A^{1-\frac{1}{q}} = A^{1-\left(1-\frac{1}{p}\right)} = A^{\frac{1}{p}}.$$

Thus, we see that

$$\left(\sum_{i=1}^{n} |x_i + y_i|\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

and so we conclude that

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p.$$

(iv) Let $X = \mathbb{R}^n$ and $p > 1$. We will show that

$$d_p(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_p$$

is a metric on $\mathbb{R}^n$.

(I) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Suppose $d_p(\vec{x}, \vec{y}) = 0$. By definition of $d(\vec{x}, \vec{y})$ and by property (2) of norms, we have

$$\begin{aligned} d_p(\vec{x}, \vec{y}) = 0 &\implies \|\vec{x} - \vec{y}\| = 0 \\ &\implies \vec{x} - \vec{y} = 0 \\ &\implies \vec{x} = \vec{y}. \end{aligned}$$

This shows property (ii).

(II) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then by property (3) of norms, we see that

$$d_p(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \| - (\vec{y} - \vec{x})\| = | - 1|\|\vec{y} - \vec{x}\| = d_p(\vec{y}, \vec{x}).$$

Thus, property (iii) is satisfied.

(III) Let $\vec{x}, \vec{y}, \vec{z} \in V$. Then by the triangle inequality property of norms, we see that

$$\begin{aligned} d_p(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| &= \|(\vec{x} - \vec{z}) + (\vec{z} - \vec{y})\| \\ &\leq \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}\| \\ &= d_p(\vec{x}, \vec{z}) + d_p(\vec{z}, \vec{y}). \end{aligned}$$

Hence, we can conclude that d_p is a metric on $\mathbb{R}^n$. ■