Math 234A: Homework 3

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Problem 1 (Complex Logarithms)**.** Compute the following:

(i) $\operatorname{Log}(i)$ and $\log(i)$

(ii) $\operatorname{Log}(1+i)$ and $\log(1+i)$.

(iii) $\operatorname{Log}(-1)$ and $\log(-1)$.

Note: for $z \in \mathbb{C}^{\bullet}$, $\log z$ is a set not a single number.

Solution. (i) Observe that

$$\begin{aligned}\operatorname{Log}(i) &= \ln|i| + i \operatorname{Arg}(i) \\ &= \ln(1) + i\frac{\pi}{2} \\ &= i\frac{\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\log(i) &= \{\ln|i| + i(\operatorname{Arg}(i) + 2\pi ik) : k \in \mathbb{Z}\} \\ &= \left\{\ln(1) + i\left(\frac{\pi}{2} + 2\pi ik\right) : k \in \mathbb{Z}\right\} \\ &= \left\{i\left(\frac{\pi}{2} + 2\pi k\right) : k \in \mathbb{Z}\right\}.\end{aligned}$$

(ii) Notice that

$$\begin{aligned}\operatorname{Log}(1+i) &= \ln|1+i| + i \operatorname{Arg}(1+i) \\ &= \ln(\sqrt{2}) + i\frac{\pi}{4}\end{aligned}$$

and

$$\log(1+i) = \left\{\ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi k\right) : k \in \mathbb{Z}\right\}.$$

(iii) Observe that

$$\begin{aligned}\operatorname{Log}(-1) &= \ln|-1| + i \operatorname{Arg}(-1) \\ &= \ln|1| + i\pi \\ &= i\pi\end{aligned}$$

and

$$\log(-1) = \{i(\pi + 2\pi k) : k \in \mathbb{Z}\}.$$

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Problem 2 (Complex Powers)**.** Compute the following:

(i) $(1+i)^{3+i}$

(ii) $\left(\frac{1+i}{1-i}\right)^i$

(iii) $(-e)^{i/2}$.

Solution. (i) Observe that

$$(1+i)^{3+i} = \exp((3+i)\log(1+i)).$$

Note that from part (ii) in problem 1, we have

$$\log(1+i) = \Big\{\ln(\sqrt{2}) + i\Big(\frac{\pi}{4} + 2\pi k\Big); k \in \mathbb{Z}\Big\}.$$

Then we see that

$$\begin{aligned}\exp((3+i)\log(1+i)) &= \exp(3+i)\exp(\log(1+i)) \\ &= \Big\{e^3 \cdot e^i \cdot e^{\ln\sqrt{2}} \cdot e^{i\left(\frac{\pi}{4}+2\pi k\right)} : k \in \mathbb{Z}\Big\} \\ &= \Big\{e^3\sqrt{2} \cdot e^{i\left(\frac{5\pi}{4}+2\pi k\right)} : k \in \mathbb{Z}\Big\}.\end{aligned}$$

(ii) Notice that

$$\left(\frac{1+i}{1-i}\right)^i = i^i.$$

So, we must have

$$\begin{aligned}i^i &= \exp(i\log(i)) \\ &= \Big\{\exp\Big(i\Big(\frac{\pi}{2}i + 2\pi k i\Big)\Big) : k \in \mathbb{Z}\Big\} \\ &= \Big\{\exp\Big(-\Big(\frac{\pi}{2} + 2\pi k\Big)\Big) : k \in \mathbb{Z}\Big\}.\end{aligned}$$

(iii) Observe that

$$(-e)^{i/2} = (-1)^{1/2}\exp(i) = i\exp(i).$$

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Problem 3. (a) Let $A \subseteq \mathbb{C}$. Show that the following statements are equivalent.

(i) A is closed.

(ii) For any sequence (a_n) in A such that $a_n \to a \in \mathbb{C}$ implies that $a \in A$.

(iii) A contains all its accumulation points; that is, if $a \in \mathbb{C}$ is an accumulation point of A, then $a \in A$.

(b) Given a set $A \subseteq \mathbb{C}$, we define

$$\mathcal{F}_A = \{F \subseteq \mathbb{C} : F \text{ is closed and } A \subseteq F\}.$$

Define $\overline{A} = \bigcup_{F \in \mathcal{F}_A} F$. Show that $\overline{A} = A \cup A'$ where

$$A' = \{z \in \mathbb{C} : z \text{ is an accumulation point}\}.$$

Proof. (a) To show that all the statements are equivalent, we will show that $(i) \implies (ii) \implies (iii) \implies (i)$.

$(i) \implies (ii)$ Suppose A is closed. Let (a_n) be a sequence in A where $a_n \to a \in \mathbb{C}$ (note that $a_n \neq a$. Our goal is to show that $a \in A$. Suppose for sake of contradiction that $a \notin A$. Then there exists some $\varepsilon > 0$ such that $N_\varepsilon(a) \cap E = \emptyset$; that is, $N_\varepsilon(a) \subseteq A^c$. Hence, we have that $a \in A^c$. But $(a_n) \to a \in \mathbb{C}$ implies that there exists at least one $a_n \neq a$ such that $a_n \in A^c$. However, the sequence (a_n) must be entirely contained in A by assumption which is a contradiction. Thus, $a \in A$.

$(ii) \implies (iii)$ Let (a_n) be a sequence in A where $a_n \neq a \in \mathbb{C}$ where $a \in A$. Our goal is to show that A contains all of its limit points. Let a be a limit point of A. Choose $\varepsilon = 1/n$ and choose $a_n \neq a$ to be a sequence of points in A. Then by assumption, the sequence $(a_n) \to a \in \mathbb{C}$ implies that $a \in A$; that is, we have that

$$N_{1/n}(a) \cap A \neq \emptyset.$$

Because $a \in A$, we can conclude that A must contain all of it's accumulation points.

$(iii) \implies (i)$ Suppose A contains all of its accumulation points. Our goal is to show that A is closed. It suffices to show that A^c is an open set; that is, we need to find an $\delta > 0$ such that $N_\delta(x) \subseteq A^c$ for all $x \in A^c$. To this end, let $x \in A^c$. Then $x \notin A$. This tells us that x cannot be a limit point of A. That is, there exists an $\delta > 0$ such that $N_\delta(x) \cap A = \emptyset$. This implies that $N_\delta(x) \subseteq A^c$ for some $\delta > 0$, and so A^c must be open. Hence, A must be closed.

(b) Our goal is to show that $\overline{A} = A \cup A'$. First, we would like to show two lemmas:

(*) $A \cup A'$ is a closed set.

(**) If F is a closed set and $A \subseteq F$, then $A \cup A' \subseteq F$ as well.

To show that (*) holds, let x be a accumulation point of $A \cup A'$. Our goal is to show that this accumulation point is contained in $A \cup A'$. By definition, we see that for all $\varepsilon > 0$, we have

$$B(x, \varepsilon) \cap ((A \cup A') \setminus \{x\}) \neq \emptyset.$$

To this end, pick a point in this intersection, say, a such that $a \in B(x, \varepsilon)$ and $a \in (A \cup A') \setminus \{x\}$. That is, we have $a \in A$ or $a \in A'$. If $a \in A$, then x is a accumulation point of A, and so $x \in A \cup A'$. If $a \in A'$, then a is a accumulation point of A'. That is, for all $\delta > 0$, we have

$$B(a, \delta) \cap A' \setminus \{a\} \neq \emptyset.$$

Pick a point in this intersection, say, $p \neq a$ such that $p \in A'$. But this implies that x must be a limit point of A, and so $x \in A'$ and thus $A \cup A'$ must be a closed set.

To show that (**) holds, suppose F is a closed set and that $A \subseteq F$. Our goal is to show that $A \cup A' \subseteq F$. Let $x \in A \cup A'$. Then either $x \in A$ or $x \in A'$. If $x \in A$, then $x \in F$ since $A \subseteq F$. On the other hand, if $x \in A'$, then x is a limit point of A. That is, for all $\delta > 0$, we have

$$B(x, \delta) \cap (A \setminus \{x\}) \neq \emptyset.$$

Since $A \subseteq F$, we can see that

$$B(x, \delta) \cap (F \setminus \{x\}) \neq \emptyset$$

which implies that x is a limit point of F. But F is closed, so x must be contained in F. Thus, we have $A \cup A' \subseteq F$ in both cases.

In what follows, we will show that $\overline{A} = A \cup A'$. To do this, we need to show two inclusions:

(1) $\overline{A} \subseteq A \cup A'$

(2) $A \cup A' \subseteq \overline{A}$.

Starting with (1), we see that $A \cup A' \subseteq F$ by (*). But this implies that $A \cup A'$ is the smallest closed set containing F, we must have that

$$A \cup A' \subseteq \bigcap_{F \in \mathcal{F}_A} F = \overline{A}$$

which satisfies (1).

With (2), we want to show that $\overline{A} \subseteq A \cup A'$. Note that $A \cup A'$ is a closed set and $A \subseteq A \cup A'$. Then immediately we see that $\overline{A} \subseteq A \cup A'$, satisfying (2). Thus, we conclude that $\overline{A} = A \cup A'$.

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Problem 4 (Discontinuity of "Arg" Function). (i) Consider the sequence (z_n) with $z_n = -1 + \frac{i}{n}$. Show that $z_n \to -1$.

(ii) Consider the sequence (w_n) with $w_n = -1 - \frac{i}{n}$. Show that $w_n \to -1$ as well.

(iii) Show that $\text{Arg}(z_n) \to \pi$ and $\text{Arg}(w_n) \to -\pi$.

(iv) What did you observe from part (iii)?

Proof. (i) Consider the real and imaginary part of z_n

$$\Re(z_n) = -1 \text{ and } \Im(z_n) = \frac{1}{n}.$$

Clearly, $\Re(z_n) \to -1$ and $\Im(z_n) \to 0$ as $n \to \infty$. Thus, $(z_n) \to -1 + i0 = -1$.

(ii) Similarly, notice that

$$\Re(w_n) \to -1 \text{ and } \Im(w_n) = \tfrac{-1}{n} \to 0$$

as $n \to \infty$. So, $(w_n) \to -1$ as well.

(iii) From parts (i), we see that

$$\text{Arg}(z_n) \to \text{Arg}(-1) = \pi.$$

Consider $\text{Arg}\Big(-1 + \frac{i}{n}\Big)$. Then we see that

$$\text{Arg}(w_n) = \text{Arg}\Big(-1 + \frac{i}{n}\Big) = \text{Arg}(-1) + \text{Arg}\Big(\frac{i}{n}\Big) = -\pi + \tan^{-1}\Big(\frac{i}{n}\Big). \tag{1}$$

If we take the limit as $n \to \infty$ of (1), then we obtain

$$\text{Arg}(w_n) \to -\pi$$

which is different result from part (i).

(iv) I observed that by the sequential criterion of continuity, the argument function $\text{Arg}(z)$ is not a continuous function.

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Problem 5. (i) Let $a \in \mathbb{C}$ and $\varepsilon > 0$. Show that

$$\overline{B(a, \varepsilon)} = \overline{B}(a, \varepsilon).$$

(ii) Let $A = \{x + iy : x, y \in \mathbb{Q}\}$. Show that $\overline{A} = \mathbb{C}$. (Hint: 3(b) can be useful here)

(iii) Let $A, B \subseteq \mathbb{C}$. Show that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Proof. (i) We will show that $\overline{B(a, \varepsilon)} = \overline{B}(a, \varepsilon)$. It suffices to show two inclusions:

(1) $\overline{B(a,\varepsilon)} \subseteq \overline{B}(a,\varepsilon)$

(2) $\overline{B}(a,\varepsilon) \subseteq \overline{B(a,\varepsilon)}$.

With (1), observe that $B(a,\varepsilon) \subseteq \overline{B}(a,\varepsilon)$. Since $\overline{B}(a,\varepsilon)$ is closed, we know by the hint given in part (b) of problem 3 that $\overline{B(a,\varepsilon)} \subseteq \overline{B}(a,\varepsilon)$.

With (2), let $x \in \overline{B}(a,\varepsilon)$. By definition of $\overline{B}(a,\varepsilon)$, we have $d(x,a) \leq \varepsilon$. Then either $d(x,a) < \varepsilon$ or $d(x,a) = \varepsilon$. If $d(x,a) < \varepsilon$, then x is contained in $B(a,\varepsilon)$, and so $x \in \overline{B(a,\varepsilon)}$. Now, suppose $d(x,a) = \varepsilon$. Observe that the closure $\overline{B(a,\varepsilon)}$ contains its boundary points. Thus, $x \in \overline{B(a,\varepsilon)}$. Thus, $\overline{B}(a,\varepsilon) \subseteq \overline{B(a,\varepsilon)}$.

We conclude that (1) and (2) imply $\overline{B(a,\varepsilon)} = \overline{B}(a,\varepsilon)$.

(ii) Let $A = \{x+iy : x,y \in \mathbb{Q}\}$. Our goal is to show that $\overline{A} = \mathbb{C}$. Note that, by problem 3(b), we see that $\overline{A} = A \cup A'$. We need to show the following two inclusions:

(1) $A \cup A' \subseteq \mathbb{C}$

(2) $\mathbb{C} \subseteq A \cup A'$.

Starting with (1), suppose $z \in A \cup A'$. Then either $z \in A$ or $z \in A'$. If $z \in A$, then $z = x+iy$ with $x,y \in \mathbb{Q}$. Since $\mathbb{Q} \subseteq \mathbb{R}$, we see that $x,y \in \mathbb{R}$ and so $z \in \mathbb{C}$. If $z \in A'$, then z is a limit point of A. That is, for all $\varepsilon > 0$

$$B(z,\varepsilon) \cap (A \setminus \{z\}) \neq \emptyset.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that every limit point of $\mathbb{Q}$ is contained in $\mathbb{R}$. Hence, z must be contained in $\mathbb{C}$. So, $A \cup A' \subseteq \mathbb{C}$.

Let $z \in \mathbb{C}$. Then $z = \alpha + i\beta$ with $\alpha,\beta \in \mathbb{R}$. Our goal is to show that $z \in A \cup A'$; that is, either z is a limit point of A or is an element of A. To this end, suppose that z is not an element of A. Note that $\alpha,\beta \in \mathbb{R}$ which are limit points of $\mathbb{Q}$. Hence, z must be a limit point of A. Thus, $z \in A'$ and so, $z \in A \cup A'$.

With (1) and (2), we can conclude that $\overline{A} = \mathbb{C}$.

(iii) Our goal is to show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$; that is, we need to show that

(1) $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

(2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Starting with (1), suppose $x \in \overline{A \cup B}$. Then either $x \in A \cup B$ or $x \in (A \cup B)'$. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then $x \in \overline{A}$ since $A \subseteq \overline{A}$ and so $x \in \overline{A} \cup \overline{B}$. Likewise, if $x \in B$, then $x \in \overline{B}$ since $B \subseteq \overline{B}$. Thus, $x \in \overline{A} \cup \overline{B}$. If $x \in (A \cup B)'$, then x is a limit point of $A \cup B$; that is, for all $\varepsilon > 0$

$$B(x,\varepsilon) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Hence, there exists $q \in B(x,\varepsilon) \cap ((A \cup B) \setminus \{x\})$. Thus, we have $q \in A \cup B$; that is, either $q \in A$ or $q \in B$. If $q \in A$, then x is a limit point of A. Thus, $x \in A'$ and so $x \in \overline{A}$. Hence, $x \in \overline{A} \cup \overline{B}$. If $q \in B$, then x is a limit point of B. Hence, $x \in B'$ and so $x \in \overline{B}$. Thus, $x \in \overline{A} \cup \overline{B}$. Thus, we see that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ which shows (1).

Now, we will show (2). Let $x \in \overline{A} \cup \overline{B}$. Then either $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then $x \in A$ or $x \in A'$. If $x \in A$, then $x \in A \cup B$. Thus, $x \in \overline{A \cup B}$. If $x \in A'$, then x is a limit point of A; that is, for all $\delta > 0$

$$B(x,\delta) \cap (A \setminus \{x\}) \neq \emptyset.$$

Since $A \subseteq A \cup B$, we know that

$$B(x,\delta) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Hence, x is a limit point of $A \cup B$ and so $x \in (A \cup B)'$; that is, $x \in \overline{A \cup B}$. On the other hand, if $x \in \overline{B}$, then the proof is analogous to the case that $x \in A'$. Thus, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Together with (1) and (2), we have that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

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Problem 6. (i) Show that $\mathbb{H} := \{z = x + iy : y > 0\}$ is open.

(ii) Show that $Q_1 := \{z = x + iy : x > 0, y > 0\}$ is open.

(iii) Show that $S = \{x + iy : -\pi < y < \pi\}$

Proof. (i) Our goal is to show that $\mathbb{H}$ is an open set. It suffices to show that $\mathbb{H}^c$ is a closed set. Let (z_n) be a sequence in $\mathbb{H}^c$ such that $(z_n) \to z \in \mathbb{C}$. Our goal is to show that $z \in \mathbb{H}^c$. Note that for $z \in \mathbb{H}^c$, z must have the property that $\Im(z) \leq 0$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|\Im(z_n) - \Im(z)| < \varepsilon \implies |\Im(z)| < |\Im(z_n)| + \varepsilon \leq \varepsilon. \qquad (|\Im(z_n)| \leq 0)$$

Since $\varepsilon > 0$ is arbitrary, we have that $\Im(z) \leq 0$. Hence, $z \in \mathbb{H}^c$.

(ii) Our goal is to show that Q_1 is open by showing that Q_1^c is closed; that is, we need to show that for any sequence $(z_n) \subseteq Q_1^c$, $(z_n) \to z \in \mathbb{C}$ with $z \in Q_1^c$. Note that $z \in Q_1^c$ if $\Im(z) \leq 0$ and $\Re(z) \leq 0$. To this end, let (z_n) be a sequence in Q_1^c that converges to $z \in \mathbb{C}$. Let $\varepsilon > 0$. Our goal is to show that $z \in Q_1^c$. Since $(z_n) \to z$, we know that the real and imaginary part must converge. Thus, choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|\Im(z_n) - \Im(z)| < \varepsilon \tag{1}$$

and

$$|\Re(z_n) - \Re(z)| < \varepsilon. \tag{2}$$

Since $\Re(z_n) \leq 0$ and $\Im(z_n) \leq 0$, (1) and (2) imply that

$$|\Im(z)| < |\Im(z_n)| + \varepsilon \leq \varepsilon$$

and

$$|\Re(z)| < |\Re(z_n)| + \varepsilon \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $|\Re(z)| \leq 0$ and $|\Im(z)| \leq 0$. Thus, $z \in Q_1^c$.

(iii) Our goal is to show that S is open; that is, we need to find $\delta > 0$ such that $B(z, \delta) \subseteq S$ for any $z \in S$. To this end , let $z \in S$. Choose $\delta = \frac{1}{2} \min\{\Im(z) - (-\pi), \pi - \Im(z)\}$. Let $w \in B(z, \delta)$. By the way we chose δ, we have

$$\begin{aligned} |z - w| < \delta &\iff |\Im(z - w)| < \delta \\ &\iff |\Im(w)| < |\Im(z)| + \delta \\ &\iff |\Im(w)| < \pi. \end{aligned}$$

Hence, we see that $w \in S$ and so, S is open.

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Problem 7. (i) Consider the sequence (z_n) defined by $z_n = -1 + \frac{i}{n}$. Compute $\lim_{n\to\infty} \operatorname{Log}(z_n)$.

(ii) Consider the sequence (w_n) defined by $w_n = -1 - \frac{i}{n}$. Compute $\lim_{n\to\infty} \operatorname{Log}(w_n)$.

(iii) What did you observe from (i) and (ii)?

Solution. (i) Note that $|z_n| = \sqrt{1 + \frac{1}{n^2}} \to 1$ as $n \to \infty$. Then observe that

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{Log}(z_n) &= \lim_{n\to\infty} [\ln|z_n| + i\operatorname{Arg}(z_n)] \\
&= \lim_{n\to\infty} \ln|z_n| + i \lim_{n\to\infty} \operatorname{Arg}(z_n) \\
&= \ln(1) + i\pi && \text{(4-(iii))} \\
&= \pi i.
\end{aligned}$$

(ii) Note that $|w_n| = \sqrt{1 + \frac{1}{n^2}} \to 1$ as $n \to \infty$. Then we see that

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{Log}(w_n) &= \lim_{n\to\infty} [\ln|w_n| + i\operatorname{Arg}(w_n)] \\
&= \lim_{n\to\infty} \ln|w_n| + i \lim_{n\to\infty} \operatorname{Arg}(w_n) \\
&= \ln(1) - \pi && \text{(4-(iii))} \\
&= -\pi i
\end{aligned}$$

(iii) I observed that $\operatorname{Log}(z)$ is discontinuous by the Sequential Criterion of Continuity.

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