Linear Algebra Notes

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Chapter 1

Vector Spaces

1.1 Bases And Dimension

- Recall that S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

Definition 1 (Basis). A **basis** β for a vector space V is linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Example.
- Recall that the empty set $\emptyset$ is linearly independent and that $\text{span}(\emptyset) = \{0\}$. The empty set $\emptyset$ in this case is the basis for the zero vector space.
- Note that in F^n, the vectors $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0) \ldots, e_n = (0, 0, \ldots, 0, 1)$ form a basis for F^n.
- The basis for $M_{m\times n}(F)$ is the set of matrices E^{ij} such that the only nonzero entry is a 1 in the ith and jth column.
- As we have seen in the last section, the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$.
- In $P(F)$, the set $\{1, x, xx^2, \ldots\}$ is a basis. *Bases are not limited to finite sets. They can be infinite.*

Theorem 1. Let V be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if each $v \in V$ can be unique expressed as a linear combination of vectors in β, that is, expressed in the form

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof. Suppose $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V. Then $\text{span}(\beta) = V$. If $v \in V$, then $v \in \text{span}(\beta)$. Hence, we can write v as a linear combination of vectors in β such that choosing scalars $a_1, a_2, \ldots, a_n \in F$ leads to

$$v = \sum_{i=1}^{n} a_iu_i.$$

Suppose there exists another representation of $v \in V$ such that

$$v = \sum_{i=1}^{n} b_i x_i$$

Hence, observe that

$$\begin{aligned} \sum_{i=1}^{n} a_i x_i &= \sum_{i=1}^{n} b_i y_i \\ \Rightarrow \sum_{i=1}^{n} (a_i - b_i) x_i &= 0. \end{aligned}$$

Since β is linearly independent, we know that $a_i - b_i = 0$ which implies $a_i = b_i$ for all $1 \leq i \leq n$. Hence, v can be expressed as a unique linear combination of vectors in β. ■