

# Math 234A Lecture Notes

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# Chapter 1

## Week 1

### 1.1 Lecture 1

#### 1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either  $\mathbb{R}$  or  $\mathbb{C}$ . For example, the polynomial  $x^2 + 1$  does not have a solution in  $\mathbb{R}$ , but it does have a solution in  $\mathbb{C}$ .
- Solving real integrals that may be difficult to deal with using standard techniques developed in  $\mathbb{R}$ ; that is, something like

$$\int_0^\infty \frac{\sin x}{x} dx.$$

- Solving problems in physics, particularly, in the Quantum Field Theory.

#### 1.1.2 What is the goal?

Let's recall some facts about the real number system  $\mathbb{R}$ .

- (i)  $(\mathbb{R}, +, \cdot)$  is a field.
- (ii) We have an order relation on  $\mathbb{R}$ .
- (iii) For all  $x \in \mathbb{R}$ ,  $x^2 + 1 > 0$ . Hence, the polynomial equation  $x^2 + 1 = 0$  does not have any solutions in  $\mathbb{R}$ .

Our goal is to find the "smallest" field  $\mathbb{C}$  such that

- (i)  $\mathbb{R}$  is "contained" in  $\mathbb{C}$ .
- (ii) For any polynomial  $f \in \mathbb{C}$ , there exists a solution for  $f$  in  $\mathbb{C}$ .

Let's assume for a moment that we CAN solve the equation  $x^2 + 1 = 0$ . Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of  $\mathbb{R}$  as a vector space, and using the operations defined on that vector space to define the operations of  $\mathbb{C}$ . Recall from Linear Algebra that  $\zeta$  is just the span of the basis vectors 1 and  $i$ . In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on  $\mathbb{C}$ .

**Definition** (Operations on  $\mathbb{C}$ ). Let  $z, w \in \mathbb{C}$  and set  $z = \alpha_1 + i\beta_1$  and  $w = \alpha_2 + i\beta_2$  for any  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ . We define the two operations, addition  $+$  and multiplication  $\cdot$ , in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that  $\mathbb{C}$  forms a field.

**Proposition.** The defined operations of  $\mathbb{C}$  form a field.

**Proof.** To do. ■

**Lemma** (Existence of a Square Root). Let  $\alpha + i\beta \in \mathbb{C}$ . Then there exists  $\gamma + i\delta \in \mathbb{C}$  such that  $(\gamma + i\delta)^2 = \alpha + i\beta$ .

**Proof.** To do. ■

## 1.2 Lecture 2

### 1.2.1 Topics

- Discuss the complex plane  $\mathbb{C}$  as a working model for complex numbers.
- Discuss the Euclidean Topology on  $\mathbb{C}$ .
- Discuss polar representation of a complex number.

## 1.3 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

**Lemma.** Let  $F$  be a field containing  $\mathbb{R}$  and the equation  $x^2 + 1 = 0$  contains a solution  $i \in F$ .

- (i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then  $\mathbb{C}$  is a **subfield** of  $F$ .

- (ii) Let  $F$  be another field containing  $\mathbb{R}$  and containing a solution  $i'$  of  $x^2 + 1 = 0$  and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then  $\mathbb{C}$  and  $\mathbb{C}'$  are isomorphic as fields.

The second part of this lemma is simply saying that  $\mathbb{C}$  is a unique subfield of  $F$ .

### 1.3.1 A model for $\mathbb{C}$

Let  $\mathbb{C} = \mathbb{R}^2$ . Then define addition  $+$  and multiplication  $\cdot$  as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

**Lemma.** The complex numbers  $\mathbb{C}$  have the following properties

- (i)  $(\mathbb{C}, +, \cdot)$  is a field.
- (ii) Let  $\mathcal{J} : \mathbb{R} \rightarrow \mathbb{C}$  by  $x \rightarrow (x, 0)$ . Then  $\mathcal{J}$  is a subfield and forms an isomorphism between  $\mathbb{R}$  and  $\mathcal{J}(\mathbb{R})$ .

**Proof.** Rough outline of proof:

- (i) Prove that  $\mathcal{J}(\mathbb{R})$  is a subfield.
- (ii) Prove that  $\mathcal{J}(\mathbb{R})$  is a field homomorphism.
- (iii) Prove that  $\mathcal{J}(\mathbb{R})$  is an isomorphism; that is, show that  $\mathcal{J}$  is a bijective map.

■

**Definition** (Constructing a solution for  $x^2 + 1 = 0$ ). Define  $i = (0, 1)$  as our imaginary number in  $\mathbb{C}$  and let  $i^2 = (-1, 0)$ .

**Proposition.** Given  $\alpha, \beta \in \mathbb{R}$ , show that  $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$ .

**Proof.** Let  $\alpha, \beta \in \mathbb{R}$ . Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

**Definition** (Real and Imaginary part of Complex Number). Let  $z = \alpha + i\beta$  for  $\alpha, \beta \in \mathbb{R}$ . Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of**  $z$ , respectively. If  $\Im(z) = 0$ ,  $z$  is a real number, and if  $\Re(z) = 0$ , then we call  $z$  **purely imaginary**.

**Definition** (Complex Conjugate). Let  $z = \alpha + i\beta$  be a complex number. Its complex conjugate is defined as  $\bar{z} = \alpha - i\beta$ .

Geometrically, this is viewed as a reflection of the ordered pair  $z = (\alpha, \beta)$  over the  $x$ -axis.

**Proposition.** For any  $z, w \in \mathbb{C}$ , we have the following properties:

- (i)  $\bar{\bar{z}} = z$ .
- (ii)  $\overline{z \pm w} = \bar{z} \pm \bar{w}$ .
- (iii)  $\overline{zw} = \bar{z} \cdot \bar{w}$ .
- (iv)  $\Re(z) = \frac{1}{2}(z + \bar{z})$ .
- (v)  $\Im(z) = \frac{1}{2i}(z - \bar{z})$ .
- (vi)  $z\bar{z} = \alpha^2 + \beta^2 \geq 0$ .

**Proof.** (i)

■

**Definition** (Modulus of a Complex Number). Let  $z \in \mathbb{C}$ . We define the **modulus**  $|z| = \sqrt{z\bar{z}}$ .

**Proposition** (More Properties of Complex Numbers). Given  $z, w \in \mathbb{C}$ , we have the following properties:

- (i)  $|z| = 0$  if and only if  $z = 0$ .
- (ii)  $|zw| = |z||w|$ .
- (iii)  $|\Re(z)| \leq |z|$ .
- (iv)  $|\Im(z)| \leq |z|$ .
- (v)  $|z + w| \leq |z| + |w|$ .
- (vi) For any  $z \in \mathbb{C}$  and  $z \neq 0$ , then  $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ .

### 1.3.2 Viewing $\mathbb{C}$ as a Metric Space

In  $\mathbb{C}$ , the metric we will be using is  $d(z, w) = |z - w|$ .

**Definition** (Metric Space). For  $z, w \in \mathbb{C}$ , we call  $\mathbb{C}$  a metric space if it satisfies the following properties:

- (i)  $d(z, w) = d(w, z)$ .
- (ii)  $d(z, w) = 0$  if and only if  $z = w$  and  $d(z, w) > 0$  if and only if  $z \neq w$ .
- (iii) For any  $z, w, u \in \mathbb{C}$ , we have
$$d(z, w) \leq d(z, u) + d(u, w).$$

Now we have  $\mathbb{C}$  is a metric space endowed with the metric  $d(z, w) = |z - w|$ .

# Chapter 2

## Week 2

### 2.1 Lecture 3

#### 2.1.1 Topics

- Polar Representation of Complex Numbers
- Convergence of Sequences in  $\mathbb{C}$

#### 2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^\bullet = \{z \in \mathbb{C} : z \neq 0\}$ .
- $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$ .

**Definition** (Polar Representation of Complex Numbers). Let  $(\alpha, \beta) \in \mathbb{R}^2$ . The polar representation of  $(\alpha, \beta)$  is

$$(\alpha, \beta) = \gamma(\cos \varphi, \sin \varphi)$$

with  $\tan \varphi = \frac{\beta}{\alpha}$ . Note that if  $\psi = 2\pi + \varphi$ , then

$$(\alpha, \beta) = \gamma(\cos \psi, \sin \psi)$$

where  $\gamma$  is uniquely defined and  $\varphi$  is defined up to the addition of a multiple of  $2\pi$ .

- $\gamma$  is uniquely defined.
- $\varphi$  is defined up to the addition of a multiple of  $2\pi$ .

**Remark.** This representation may not be unique!

**Proposition.** The map  $\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}^\bullet$  defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos \varphi + i \sin \varphi)$$

is surjective.

**Remark.** The proposition above is a systematic way of saying that if  $z \in \mathbb{C}^\bullet$ , then

$$z = \gamma(\cos \varphi + i \sin \varphi)$$

with  $\gamma = |z|$  and  $\varphi$  can be determined up to a multiple of  $2\pi$ .



If we insist, we can make the polar representation unique by restricting the domain to  $-\pi < \varphi \leq \pi$  where  $\varphi$  is denoted as the **argument of**  $z$ .

**Definition** (Argument and Principle Argument). Let  $z \in \mathbb{C}$  and  $z = \gamma(\cos \varphi + i \sin \varphi)$  be a polar representation of  $z$ . Then  $\varphi$  is called **an argument of**  $z$ . If  $-\pi < \varphi \leq \pi$ , then  $\varphi$  is called **the principal argument of**  $z$  and it is denoted by  $\text{Arg}(z)$ .

**Remark.** For any other domain, we denote the argument by  $\varphi = \arg((x, y))$ .

**Lemma.** Let  $z = \gamma(\cos \varphi + i \sin \varphi)$  and  $w = \gamma'(\cos(\varphi') + i \sin(\varphi'))$  in  $\mathbb{C} \setminus \{0\}$ . Then

$$zw = \gamma\gamma'[\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')].$$

**Proof.** Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos \varphi + i \sin \varphi)(\cos \varphi' + i \sin \varphi') \\ &= \gamma\gamma'[(\cos \varphi \cos \varphi' - \sin \varphi \sin \varphi') + i(\sin \varphi \cos \varphi' + \cos \varphi \sin \varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')). \end{aligned}$$

■

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

**Corollary.** Let  $z \in \mathbb{C}$  with  $z = \gamma(\cos \varphi + i \sin \varphi)$ . Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i \sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos \varphi - i \sin \varphi). \end{aligned}$$

**Corollary** (De Moivre's Theorem). Let  $z = \gamma(\cos \varphi + i \sin \varphi) \in \mathbb{C} \setminus \{0\}$  and let  $n \in \mathbb{Z}$ . Then

$$z^n = \gamma^n(\cos n\varphi + i \sin n\varphi).$$

**Remark.** If  $n$  is a negative integer, then  $z^n = (z^{-1})^{-n}$ .

The corollary above allows us to compute the  $n$ th roots of a non-zero complex number.

**Example** (An example of De Moivre's Theorem). Suppose we have the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find  $z^{10}$ . First, we need to find the angle that makes this complex number. Since the  $x$  and  $y$  coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i \sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i \frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

- (i) **The set of all positive real numbers**  $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$
- (ii) **The set of all complex numbers excluding zero**  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .

**Proposition.** The map  $\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$  defined by

$$(r, \varphi) \rightarrow r(\cos \varphi + i \sin \varphi)$$

is surjective.

**Remark.** This gives us the tool we need to show that every non-zero  $z \in \mathbb{C}$  has a polar representation.

### 2.1.3 Convergence of Sequences in $\mathbb{C}$

**Definition** (Convergence in  $\mathbb{C}$ ). Let  $\{z_n\}_{n=1}^\infty$  be a sequence in  $\mathbb{C}$ . We say that  $\{z_n\}$  converges to  $z \in \mathbb{C}$  if for all  $\varepsilon > 0$ , we can find  $N_\varepsilon \in \mathbb{N}$  such that

$$|z_n - z| < \varepsilon$$

for all  $n \geq N_\varepsilon$ .

If  $\{z_n\}$  converges to  $z$ , then we write  $z_n \rightarrow z$ .

**Proposition** (Properties of Convergent Sequences). Assume  $(z_n) \rightarrow z$  and  $(w_n) \rightarrow w$ .

- (i) Let  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w$ .
- (ii)  $z_n w_n \rightarrow zw$ .
- (iii)  $z_n^{-1} \rightarrow z^{-1}$ .
- (iv)  $(z_n) \rightarrow z$  if and only if  $\Re(z_n) \rightarrow \Re(z)$  and  $\Im(z_n) \rightarrow \Im(z)$  as a sequences in  $\mathbb{R}$ .

**Proof.** Suppose  $(z_n) \rightarrow z$  and  $(w_n) \rightarrow w$ .

- (i) Let  $\alpha, \beta \in \mathbb{C}$ . Let  $\varepsilon > 0$ . Since  $(z_n) \rightarrow z$ , there exists an  $N_1 \in \mathbb{N}$  such that for any  $n \geq N_1$ , we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise,  $(w_n) \rightarrow w$  implies that we can find an  $N_2 \in \mathbb{N}$  such that for any  $n \geq N_2$ , we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose  $N = \max\{N_1, N_2\}$ . Then for any  $n \geq N$ , we must have

$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w.$$

(ii) Let  $\varepsilon > 0$ . Our goal is to show that there exists an  $N \in \mathbb{N}$  such that

$$|z_n w_n - zw| < \varepsilon.$$

Since  $(z_n) \rightarrow z$ , we can find a  $N_1 \in \mathbb{N}$  such that for any  $n \geq N_1$ , we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where  $M > 0$ . Since  $(w_n) \rightarrow w$ , there exists  $N_2 \in \mathbb{N}$  such that for  $n \geq N_2$ , we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose  $N = \max\{N_1, N_2\}$  such that for any  $n \geq N$ , we have

$$\begin{aligned} |z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\ &\leq |w_n| |z_n - z| + |z| |w_n - w| \\ &\leq M |z_n - z| + |z| |w_n - w| \\ &< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we conclude that

$$z_n w_n \rightarrow zw.$$

(iii) Let  $\varepsilon > 0$ . We will show that  $z_n^{-1} \rightarrow z^{-1}$  by showing that there exists an  $N \in \mathbb{N}$  such that for any  $n \geq N$ , we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since  $(z_n) \rightarrow z$ , there must exist an  $N_1 \in \mathbb{N}$  such that for any  $n \geq N_1$ , we have

$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$

Likewise, we can choose  $N_2 \in \mathbb{N}$  such that for any  $n \geq N_2$ , we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose  $N = \max\{N_1, N_2\}$ , and subsequently, let  $n \geq N$ , then we must have

$$\begin{aligned} |z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z| |z_n|} \\ &< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that  $z_n^{-1} \rightarrow z^{-1}$ .

(iv) ( $\implies$ ) Let  $\varepsilon > 0$ . Since  $(z_n) \rightarrow z$ , we can choose  $N \in \mathbb{N}$  such that for any  $n \geq N$ , we have

$$\begin{aligned} |\Re(z_n) - \Re(z)| &= |\Re(z_n - z)| \leq |z_n - z| < \varepsilon \\ |\Im(z_n) - \Im(z)| &= |\Im(z_n - z)| \leq |z_n - z| < \varepsilon. \end{aligned}$$

Hence, the real and imaginary part of  $(z_n)$  converge.

( $\Leftarrow$ ) Letting  $\varepsilon > 1$  again. Our goal is to find an  $N \in \mathbb{N}$  such that for any  $n \geq N$ , we have

$$|z_n - z| < \varepsilon.$$

Since the real and imaginary part of  $z_n$  converge, we know that there exists  $N_1, N_2 \in \mathbb{N}$  such that, we have

$$|\Re(z_n) - \Re(z)| < \frac{\varepsilon}{2} \tag{1}$$

$$|\Im(z_n) - \Im(z)| < \frac{\varepsilon}{2} \tag{2}$$

whenever  $n \geq N_1$  and  $n \geq N_2$ , respectively. Now, choose  $N = \max\{N_1, N_2\}$  such that for any  $n \geq N$ , we have

$$\begin{aligned} |z_n - z| &= |(\Re(z_n) - \Re(z)) + i(\Im(z_n) - \Im(z))| \\ &\leq |\Re(z_n) - \Re(z)| + |\Im(z_n) - \Im(z)| \quad (|i| = 1) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we see that  $(z_n) \rightarrow z$ . ■

# Chapter 3

## Week 3

### 3.1 Lecture 4

#### 3.1.1 Topics

- (i) Convergence of Series
- (ii) Exponential, Sine, and Cosine functions

#### 3.1.2 Convergence of Series

Infinite series of complex numbers. Let  $\{z_n\}_{n=1}^{\infty}$  be a sequence of complex numbers.

Goal: To give meaning to  $\sum_{n=1}^{\infty} z_n$ .

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^n z_k$$

**Definition** (Convergence of Series). Let  $\{z_n\}_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$ . Define

$$s_k = \sum_{k=1}^n z_k = z_1 + \cdots + z_n$$

where  $(s_n)$  is called the **sequence of partial sums** of  $\{z_n\}$ . If  $s_n \rightarrow s$ , then we say that the series  $\sum_{n=1}^{\infty} z_n$  converges and write  $\sum_{n=1}^{\infty} z_n = s$ .

**Example.** Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i} \left[ \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$

$$\frac{1}{2} is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider

$$\frac{1}{2}i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \implies s_n = \frac{1}{i} \left[ 1 - \frac{1}{2^n} \right].$$

Thus, taking the limit as  $n \rightarrow \infty$  gives us

$$\lim_{n \rightarrow \infty} s_n = \frac{1}{i}.$$

**Example.** Assume that  $|z| < 1$ ,  $z \in \mathbb{C}$ . Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since  $|z| < 1$ , we know that

$$\sum_{n=1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n \rightarrow \infty} \sum_{k=1}^n z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as  $n \rightarrow \infty$ , we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

**Definition** (Absolute Convergence of Infinite Series). Let  $(z_n)$  be a sequence of complex numbers. We say that the series  $\sum_{n=1}^{\infty} z_n$  converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

**Remark.** If  $\sum_{n=1}^{\infty} z_n$  converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

**Example.** Let  $z \in \mathbb{C}$ . The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of  $e^z$  found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define  $e^z$  in  $\mathbb{C}$ .

### 3.1.3 Exponential, Sine, and Cosine

**Definition** (Complex Version of Exponential Function). For  $z \in \mathbb{C}$ , we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all  $z \in \mathbb{C}$ .

**Definition** (Sine and Cosine Series). We define  $\sin z$  as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and  $\cos z$  as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

**Definition.** For  $z \in \mathbb{C}$ , we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether  $\exp(z + w) = \exp(z) \cdot \exp(w)$ . Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \quad \textbf{Show this!}.$$



# Chapter 4

## Week 4

### 4.1 Lecture 4

#### 4.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

#### 4.1.2 Class Exercises

- (i) Show that  $\sum_{n=0}^{\infty} \frac{z^n}{n!}$  converges for all  $z$ .
- (ii) Show that  $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$  converges for all  $z$ .
- (iii) Show that  $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$  converges for all  $z$ .

**Definition** (Exponential, Cosine, and Sine). We define

$$\begin{aligned}\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

Our main goal for this lecture is to show that  $\exp(z+w) = \exp(z)\exp(w)$ .

#### 4.1.3 Cauchy Multiplication Theorem

**Theorem** (Cauchy Multiplication Theorem). Assume that  $\sum_{n=0}^{\infty} z_n$  and  $\sum_{n=0}^{\infty} w_n$  converges absolutely. Then

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^n z_k w_{n-k} \right) = \left( \sum_{n=0}^{\infty} z_n \right) \left( \sum_{n=0}^{\infty} w_n \right) \quad (1)$$

where the series on the left-hand side of (1) converges absolutely.

**Proof.** We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is

bounded. Observe that

$$\begin{aligned}
 \sum_{n=0}^N \left| \sum_{i+j=n} z_i w_j \right| &\leq \sum_{n=0}^N \sum_{i+j=n} |z_i w_j| \\
 &= \sum_{0 \leq i+j \leq N} |z_i w_j| \\
 &\leq \sum_{0 \leq i, j \leq N} |z_i w_j| \\
 &= \sum_{i=0}^N |z_i| \sum_{j=0}^N |w_j|.
 \end{aligned}$$

By our assumption, we can see that the sequence of partial sums of  $\sum_{n=0}^{\infty} z_n$  and  $\sum_{n=0}^{\infty} w_n$  are bounded, and thus the left side of the inequality above is bounded. Thus, we see that

$$\sum_{n=0}^N \left| \sum_{i+j=n} z_i w_j \right|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \left| \sum_{n=0}^{2N} \sum_{i+j=n} z_i w_j - \sum_{i=0}^N z_i \sum_{j=0}^N w_j \right| \rightarrow 0$$

as  $N \rightarrow \infty$ . Let us define the following sets

$$\begin{aligned}
 T_N &= \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i \geq 0, j \geq 0, 0 \leq i + j \leq N\} \\
 t_n &= \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq i \leq N, 0 \leq j \leq N\}.
 \end{aligned}$$

Observe that  $T_N \subseteq t_n$  and  $t_N \subseteq T_{2N} \subseteq t_{2N}$ . Thus,

$$\begin{aligned}
 \alpha_N &= \left| \sum_{(i,j) \in T_{2N} \setminus t_N} z_i w_j \right| \leq \sum_{(i,j) \in T_{2N} \setminus t_N} |z_i w_j| \\
 &\leq \sum_{(i,j) \in t_{2N} \setminus t_N} |z_i w_j| \\
 &= \sum_{i=0}^{2N} \sum_{j=0}^{2N} |z_i w_j| - \sum_{i=0}^N \sum_{j=0}^N |z_i w_j| \\
 &= \sum_{i=0}^{2N} |z_i| \sum_{j=0}^{2N} |w_j| - \sum_{i=0}^N |z_i| \sum_{j=0}^N |w_j| \rightarrow 0 \text{ as } N \rightarrow \infty.
 \end{aligned}$$

■

**Corollary.** For any  $z, w \in \mathbb{C}$ , we have  $\exp(z) \cdot \exp(w) = \exp(z + w)$ .

**Proof.** Let

$$C_n = \sum_{k=0}^n \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by  $n!$  on both sides of the equation above, we see that

$$n! C_n = \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} = (z + w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^N C_n = \sum_{n=0}^N \frac{n! C_n}{n!} = \sum_{n=0}^N \frac{(z+w)^n}{n!}.$$

Let  $n \rightarrow \infty$ . Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w).$$

■

## 4.2 Lecture 5

### 4.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

### 4.2.2 Sine and Cosine Functions

Recall that we defined

$$\begin{aligned} \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C} \\ \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}. \end{aligned}$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

**Lemma.** For any  $z \in \mathbb{C}$ , observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2$ .

**Proof.** Let  $z \in \mathbb{C}$ . We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z + i \sin z. \end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

**Corollary.** For any  $z \in \mathbb{C}$ , we see that

- $\exp(x + iy) = e^x(\cos y + i \sin y)$ .
- $\Re(\exp(z)) = e^x \cos y$ .
- $\Im(\exp(z)) = e^x \sin y$ ,  $|\exp(z)| = e^x$ .
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$ .
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$ .

**Proposition.** Show that  $\exp(z) = \exp(w)$  if and only if  $z - w \in 2\pi i\mathbb{Z}$ . **Hint:** Let  $z = x + iy$  and  $w = u + iv$ . Then

$$\begin{aligned}\exp(z) &= e^x(\cos y + i \sin y) \\ \exp(w) &= e^u(\cos v + i \sin v)\end{aligned}$$

and show that  $u = x$  and  $u - v \in 2\pi\mathbb{Z}$ .

**Remark.** This exercise shows that  $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$  is not injective.

- Note that  $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$  is a group homomorphism. From this exercise, we know that  $\ker(\exp) = 2\pi i\mathbb{Z}$ . This means that  $\exp$  is periodic with period  $2\pi i$ .

Our next goal is find an "inverse" of  $\exp$ . We just learned that it is not possible unless we change the "domain" of  $\exp$ .

Let  $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$ .

**Lemma.**  $\exp : S \rightarrow \mathbb{C}^*$  is a bijective map.

**Proof.** Let  $z \in \mathbb{C}^*$  and  $z = x + iy$ . Let  $z = \gamma(\cos \varphi + i \sin \varphi)$  be the polar representation of  $z$  such that  $-\pi < \varphi \leq \pi$ . Define  $w = \ln \gamma + i\varphi$ . Then  $\exp(w) = z$  and so,  $\exp$  is surjective. Moreover,  $\exp$  is injective (on  $S$ ) as well. ■

**Corollary.** For any  $z \in \mathbb{C}^*$ , we can find a unique  $w \in S$  such that  $\exp(w) = z$ .

**Definition.** Given  $z \in \mathbb{C}^*$ , the unique  $w \in S$  is called the principal value of the logarithm of  $z$  and we write  $w = \text{Log}(z)$ .

**Theorem.** There exists a mapping

$$\text{Log} : \mathbb{C}^* \rightarrow \mathbb{C}$$

such that

- (i)  $\exp(\text{Log } z) = z$
- (ii)  $-\pi < \Im(\text{Log } z) \leq \pi$ .

**Definition (Principal Branch).** The function  $\text{Log } Z$  is called the principal branch of the logarithm.

**Problem 1.** Let  $z \in \mathbb{C}^*$ . Find all possible solutions of  $\exp(w) = z$ .

**Solution.** Taking the log of both sides of  $\exp(w) = z$ , we see that

$$w = \log(z) = \ln |z| + i \left( \operatorname{Arg}(z) + 2\pi k \right), \quad k \in \mathbb{Z}.$$

■

**Lemma.** For  $z \in \mathbb{C}^\bullet$ , we have

$$\operatorname{Log}(z) = \ln |z| + i \operatorname{Arg}(z).$$

### 4.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation  $\exp(w) = z$  are given by

$$w = \operatorname{Log}(z) + 2\pi i k \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \operatorname{Log} z + 2\pi i k$$

is a **multivalued function** and  $\log z = \operatorname{Log} z$  if we want the solution to be in  $S$ .

# Chapter 5

## Week 5

### 5.1 Lecture 6

#### 5.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on  $\mathbb{C}$ :

$$d(z, w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify  $\mathbb{C}$  with  $\mathbb{R}^2$  using the map  $\mathbb{C} \rightarrow \mathbb{R}^2$  with  $\alpha + i\beta \rightarrow (\alpha, \beta)$ . Thus, we can visualize  $\mathbb{C}$  using  $\mathbb{R}^2$ .

Recall that definition of continuity for functions in  $\mathbb{R}$ .

**Definition** (continuity in  $\mathbb{R}$ ). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $f$  is **continuous** for each  $a \in \mathbb{R}$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x - a| < \delta$  implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say  $|x - a| < \delta$ , we can also say that  $x \in (a - \delta, a + \delta)$ . Similarly, we have  $|f(x) - f(a)| < \varepsilon$  is equivalent to  $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ .

#### 5.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x, a) < \delta\}$$

which is a more general statement than the one found in the definition above.

**Definition** (Open Ball). Consider the usual metric  $d(z, w) = |z - w|$  in  $\mathbb{C}$ . Let  $\varepsilon > 0$  and  $a \in \mathbb{C}$ . By an **open ball** centered at  $a$ , and radius  $\varepsilon > 0$ , we mean the set

$$B(a, \varepsilon) = \{z \in \mathbb{C} : d(z, a) < \varepsilon\}.$$

**Definition (Open Set).** Let  $D \subseteq \mathbb{C}$ . We say  $D$  is **open** if for each  $a \in D$ , we can find  $\varepsilon > 0$  such that  $B(a, \varepsilon) \subseteq D$ .

**Example.** (i)  $\mathbb{C}$  is open.

(ii) Let  $\varepsilon > 0$  and  $a \in \mathbb{C}$ . Then  $B(a, \varepsilon)$  is also open; that is, open balls are open.

(iii) Let  $D_1$  and  $D_2$  be open sets. Show that  $D_1 \cap D_2$  is open. Let  $x \in D_1 \cap D_2$ . To show that  $D_1 \cap D_2$  is open, we need to find  $\delta > 0$  such that  $B(x, \delta) \subseteq D_1 \cap D_2$ . Since  $x \in D_1 \cap D_2$ , then  $x \in D_1$  and  $x \in D_2$ . Since  $D_1$  and  $D_2$  are open sets, we can find  $\varepsilon > 0$  and  $\varepsilon' > 0$  such that

$$B(x, \varepsilon) \subseteq D_1 \text{ and } B(x, \varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that  $d(x, p) < \varepsilon$  for all  $p \in D_1$  and  $d(x, q) < \varepsilon'$ . Then pick  $\delta = \min\{d(x, p), d(x, q)\}$ . Since  $x$  is in both  $D_1$  and  $D_2$ , we have that

$$B(x, \delta) \subseteq D_1 \cap D_2.$$

Hence,  $D_1 \cap D_2$  is open.

1. The empty set is open.

2. Let  $\{D_\lambda\}_{\lambda \in \Lambda}$  be a collection of open sets, where  $\Lambda$  is an indexing set. Define the set

$$\bigcup_{\lambda \in \Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that  $\bigcup_{\lambda \in \Lambda} D_\lambda$  is also open.

### 5.1.3 Interior Points

**Definition (Interior Point).** Let  $D \subseteq \mathbb{C}$  and  $a \in D$ . We say that  $a$  is an **interior point** of  $D$  if we can find  $\varepsilon > 0$  such that  $B(a, \varepsilon) \subseteq D$ .

**Proposition.**  $D$  is open if and only if each  $a \in D$  is an interior point of  $D$ .

**Example.** Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton  $\{0\}$ . Show that  $\{0\}$  is not open. Note that the intersection of arbitrary many open sets may not be open.

### 5.1.4 Closed sets

**Definition (Closed Set).** We say  $A \subseteq \mathbb{C}$  **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

**Example.** Define a closed ball with center at  $a \in \mathbb{C}$  and radius  $\varepsilon > 0$  by

$$\overline{B}(a, \varepsilon) = \{z \in \mathbb{C} : |z - a| \leq \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that  $\overline{B}(a, \varepsilon)$  is closed.

**Example.** • For any  $a \in \mathbb{C}$ , we say that the singleton  $\{a\}$  is closed.

- If  $A_1, A_2, \dots, A_n$  are closed sets. Then

$$\bigcup_{i=1}^n A_i \text{ is also closed.}$$

We can just show that the complement of this set is open.

- Let  $\{A_\lambda\}_{\lambda \in \Lambda}$  be a collection of closed sets. Define

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\lambda \ \forall \lambda \in \Lambda\},$$

then  $\bigcap_{\lambda \in \Lambda} A_\lambda$  is closed.

**Definition (Boundary Points).** Let  $D \subseteq \mathbb{C}$ . We say  $a \in \mathbb{C}$  is a **boundary point** of  $D$  if for all  $\varepsilon > 0$ ,  $B(a, \varepsilon) \cap D \neq \emptyset$  and  $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$ ; that is,  $B(a, \varepsilon)$  intersects  $D$  and  $\mathbb{C} \setminus D$  non-trivially.

**Example.** Any point on the circle  $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$  is a boundary point of  $B(a, \varepsilon)$ , where  $\varepsilon > 0$ .

**Example (Boundary Points of Singletons).** • Note that the boundary point of the singleton is just the singleton itself.

- Let  $D = \mathbb{C} \setminus \{a\}$ . Just like the first item, the boundary points of this set is just  $\{a\}$ .

**Definition (Accumulation Point).** Let  $D \subseteq \mathbb{C}$ . We say  $a \in \mathbb{C}$  is an **accumulation point** of  $D$  if for any  $\varepsilon > 0$ ,

$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

**Example.** (i) Let  $D = \mathbb{C}$ . Then  $\{0\}$  is an accumulation point of  $D$ .

(ii) Let  $a$  such that  $|a| = 1$ . Then  $a$  is an accumulation point of  $D = \{z : |z| < 1\}$ .

(iii) Let  $D = \{1, i\}$ . Then  $D$  has no accumulation points.

(iv) If  $D = \{w\}$  for some nonzero  $w \in \mathbb{C}$ . Thus,  $D' = \emptyset$ . That is, every singleton set contains no accumulation points. Furthermore, any finite set does not have any accumulation points.

(v) Let  $D = \mathbb{R}$ . Then the accumulation points of  $D$  is just  $\mathbb{R}$  itself. That is, any point of  $\mathbb{R}$  is an accumulation point.

(vi) Let  $D = B(0, 1)$ . The accumulation points are contained in the following set:

$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Another way to define the set of accumulation points of  $D$  is the following:

$$\{a \in \mathbb{C} : \forall \varepsilon > 0, B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset\}.$$

**Definition (Closure).** Let  $D \subseteq \mathbb{C}$ . We denote the **closure** of  $D$  by  $\mathbf{D}$  and define  $\overline{D}$  as

$$\overline{D} = D \cup \{x \in \mathbb{C} : \forall \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

The closure is the smallest closed set that contains  $D$ !



**Lemma.** (i)  $\overline{D}$  is closed in  $\mathbb{C}$ .

(ii) The following statements are equivalent:

(a)  $a \in \mathbb{C}$  is an accumulation point of  $D$

(b) There is a sequence  $(z_n)$  with  $z_n \in D$ ,  $z_n \neq a$  for all  $n \in \mathbb{Z}_+$  and  $z_n \rightarrow a$ .

**Proof.** Proof is in the homework! ■

**Definition (Bounded).** We say  $D \subseteq \mathbb{C}$  is **bounded** if there exists  $R > 0$  such that  $D \subseteq \overline{B}(0, R)$ ; that is,  $|z| \leq R$  for all  $z \in D$ ; that is,  $D \subseteq \overline{B}(0, R)$ .

# Chapter 6

## Week 6

### 6.1 Lecture 7

#### 6.1.1 Topics

Our goal for this lecture is to discuss complex powers of a non-zero complex number.

#### 6.1.2 Complex Powers

Let  $z \in \mathbb{C}^\bullet$ . We want to define  $z^w$  for  $w \in \mathbb{C}$ . Recall from real number system that, for  $x \in \mathbb{R}_+$ ,  $x^a = e^{a \ln x}$ . We want our complex powers to extend this idea.

**Definition.** We define  $z^w = \exp(w \log z)$ . Note that

$$\begin{aligned}\log z &= \text{Log } z + 2\pi i k \\ &= \ln |z| + i(\text{Arg}(z) + 2\pi k), \quad k \in \mathbb{Z}\end{aligned}$$

is a multi-valued function.

For this reason, the complex power of a non-zero complex number will also be multi-valued. Hence, by choosing a fixed  $k$ , we can find a specific value of  $z^w$ .

**Example.** (i) Compute  $i^i$ . Recall that  $\log i = i\left(\frac{\pi}{2} + 2\pi k\right)$ . Hence, we have

$$i^i = \exp(i \log i) = \{e^{\frac{-\pi}{2} + 2\pi k} : k \in \mathbb{Z}\}.$$

If we are just referring to the principal branch, then

$$i^i = e^{\frac{-\pi}{2}}.$$

(ii) Compute:  $(-1)^{1+i}$ . Note that

$$\log(-1) = i\pi + 2\pi i k = i(2k + 1)\pi.$$

Then we see that for all  $k \in \mathbb{Z}$ , we have

$$\begin{aligned}(-1)^{1+i} &= \exp((2k + 1)\pi(i - 1)) \\ &= e^{-(2k+1)\pi} \exp(e^{(2k+1)\pi i}) \\ &= e^{-(2k+1)\pi} \left[ \cos(2k + 1)\pi + i \sin(2k + 1)\pi \right] \\ &= -e^{-(2k+1)\pi}.\end{aligned}$$

Let us summarize some facts about  $\exp(z)$ .

- For all  $x \in \mathbb{R}$ , we have  $\exp(x) = e^x$ . Thus, from now on, we write  $e^z$  in place of  $\exp(z)$ .
- For  $z \in \mathbb{C}^\bullet$ , we have  $\log z$  is a multi-valued function. That is, we have

$$\log z = \ln |z| + i(\operatorname{Arg} z + 2\pi k), \quad k \in \mathbb{Z}.$$

- For  $z \in \mathbb{C}^\bullet$ , we have  $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$  (this is the principal branch  $\log$ ).
- For  $z \in \mathbb{C}^\bullet$ , we have  $z^w = e^{w \log z}$  is also a multi-valued function.

**Example.** • Find all solutions of  $\sin z = 1$ .

- Find all possible solutions of  $\cos z = i$ .

**Problem 2.** Find all solutions of  $e^z = i$ .

**Solution.** ■

## 6.2 Lecture 8

- Limits of a function
- Continuity
- Derivative

### 6.2.1 Functional Limit

**Definition** (Functional Limit). Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$  be a function. Let  $a$  be an accumulation point of  $D$ . Let  $\ell \in \mathbb{C}$ . We say that  $\ell$  is **the limit of  $f(z)$**  as  $z$  approaches to  $a$  if for all  $\varepsilon > 0$ , there is  $\delta > 0$  such that for all  $z \in D$  with  $0 < |z - a| < \delta$ , we have

$$|f(z) - \ell| < \varepsilon;$$

that is,  $z \in D \cap (B(a, \delta) \setminus \{a\})$  implies  $f(z) \in B(\ell, \varepsilon)$ .

**Example.** Let  $f : B(0, 1) \rightarrow \mathbb{C}$  defined by  $f(z) = \frac{iz}{2}$  then  $\lim_{z \rightarrow i} f(z) = -\frac{1}{2}$ .

**Proof.** Observe that

$$|f(z) - \ell| = \left| \frac{i}{2}z - \left(-\frac{1}{2}\right) \right| = \left| \frac{i}{2} \right| |z - i| = \frac{1}{2} |z - i|.$$

So, given any  $\varepsilon > 0$ , we can take any  $0 < \delta < 2\varepsilon$  such that

$$0 < |z - i| < \delta$$

implies

$$|f(z) - \ell| < \varepsilon. \quad \blacksquare$$

Some facts about limits are:

**Theorem.** Let  $D \subseteq \mathbb{C}$  with  $f : D \rightarrow \mathbb{C}$ ,  $g : D \rightarrow \mathbb{C}$  be two functions let us write  $f = u + iv$ .

- (i) Let  $a = \alpha + i\beta$ . Then  $\lim_{z \rightarrow a} f(x) = \ell$  if and only if

$\lim_{(x,y) \rightarrow (\alpha,\beta)} u(x,y) = \Re(\ell)$  and  $\lim_{(x,y) \rightarrow (\alpha,\beta)} v(x,y) = \Im(\ell)$ ; that is,

$$\lim_{z \rightarrow a} f(z) = \lim_{z \rightarrow a} u(z) + i \lim_{z \rightarrow a} v(z).$$

(ii) Let  $\lim_{z \rightarrow a} f(z) = \ell$  and  $\lim_{z \rightarrow a} g(z) = \ell'$  and  $c, d \in \mathbb{C}$ . Then

$$\lim_{z \rightarrow a} (cf(z) + dg(z)) = c\ell + d\ell'.$$

(iii)  $\lim_{z \rightarrow a} (f(z)g(z)) = \lim_{z \rightarrow a} f(z) \cdot \lim_{z \rightarrow a} g(z)$ .

(iv) We have

$$\lim_{z \rightarrow a} \frac{f(z)}{g(z)} = \frac{\lim_{z \rightarrow a} f(z)}{\lim_{z \rightarrow a} g(z)}$$

whenever  $\lim_{z \rightarrow a} g(z) \neq 0$ .

**Problem 3.** Let  $\lim_{z \rightarrow i} f(z) = 2 + i$  and  $\lim_{z \rightarrow i} g(z) = 1 - i$ . Compute the following:

$$\lim_{z \rightarrow i} \left[ (f(z))^3 + \frac{(1+i)g(z)}{z^2} \right].$$

**Solution.** Observe that

$$\begin{aligned} \lim_{z \rightarrow i} \left[ (f(z))^3 + \frac{(1+i)g(z)}{z^2} \right] &= \lim_{z \rightarrow i} (f(z))^3 + \lim_{z \rightarrow i} \frac{(1+i)g(z)}{z^2} \\ &= (2+i)^3 + \frac{(1+i)(1-i)}{i^2} \\ &= (2+i)^3 - 2. \end{aligned}$$

■

**Definition (Limits at Infinity).** (i) Let  $f : D \rightarrow \mathbb{C}$  be a function and  $a \in \mathbb{C}$  be an accumulation point of  $D$ . Then we say  $\lim_{z \rightarrow a} f(z) = \infty$  if for all  $M > 0$ , there exists  $\delta > 0$  such that

$$z \in D \cap (B(a, \delta) \setminus \{a\})$$

implies  $|f(z)| \geq M$ ; that is,  $f$  is unbounded as  $z$  approaches to  $a$ .

(ii) Let  $f$  be a complex function defined on the complement of a ball in  $\mathbb{C}$ . We say  $\lim_{z \rightarrow \infty} f(z) = \ell$  if for all  $\varepsilon > 0$ , there exists  $R > 0$  such that  $|z| > R$  implies

$$|f(z) - \ell| < \varepsilon.$$

**Example.** (i) Consider  $\lim_{z \rightarrow 0} \frac{1}{z} = \infty$ . Let  $M > 0$ . Let  $\delta = \frac{1}{M}$ . Then  $0 < |z| < \delta$  implies that

$$\left| \frac{1}{z} \right| > \frac{1}{\delta} = M.$$

(ii) Consider  $\lim_{z \rightarrow \infty} \frac{1}{z} = 0$ . Let  $\varepsilon > 0$ . Choose  $R = \frac{1}{\varepsilon} > 0$ . Then  $|z| > R$  implies

$$\left| \frac{1}{z} \right| < \frac{1}{R} = \varepsilon.$$

**Problem 4.** (i) Show that  $\lim_{z \rightarrow \infty} f(z) = \ell$  if and only if

$$\lim_{z \rightarrow \infty} f\left(\frac{1}{z}\right) = \ell.$$

(ii)  $\lim_{z \rightarrow a} f(z) = \infty$  if and only if  $\lim_{z \rightarrow a} \frac{1}{f(z)} = 0$ .

(iii) Give a definition of  $\lim_{z \rightarrow \infty} f(z) = \infty$ . Show that  $\lim_{z \rightarrow \infty} f(z) = \infty$  if and only if  $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = \infty$ .

(iv) Compute  $\lim_{z \rightarrow \infty} \frac{z-1}{z+i}$ .

(v) Compute  $\lim_{z \rightarrow \infty} \frac{gz+i}{z^2+z+1}$ .

(vi) Can you compute  $\lim_{z \rightarrow \infty} e^{-z}$ ?

### 6.2.2 Continuity

**Definition (Continuity).** Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$  be a function. We say  $f$  is **continuous at**  $a \in D$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $z \in B(a, \delta) \cap D$ , we have  $f(z) \in B(f(a), \varepsilon)$ .

**Example.** Compute  $\lim_{z \rightarrow \infty} \frac{z-i}{z+i}$ . Observe that

$$\lim_{w \rightarrow 0} \frac{\frac{1}{w} - i}{\frac{1}{w} + i} = \lim_{w \rightarrow 0} \frac{1 - wi}{1 + wi} = 1.$$

**Lemma.** Let  $f : D \rightarrow \mathbb{C}$  be a function. Then the following statements are equivalent.

- (i) If  $f$  is continuous at  $a \in D$ .
- (ii) For any sequence  $(a_n)$  and  $a_n \in D$  such that  $a_n \rightarrow a$ , we have

$$f(a_n) \rightarrow f(a).$$

We say that  $f : D \rightarrow \mathbb{C}$  is continuous if  $f$  is continuous if  $f$  is continuous for every  $a \in D$ .

Some facts that we can prove as an exercise:

**Lemma.** Let  $f : D \rightarrow \mathbb{C}$  with  $D \subseteq \mathbb{C}$ . Then we have

- (1)  $f$  is continuous if and only if  $\Re(f)$  and  $\Im(f)$  are continuous (Note that  $\Re(f)$  and  $\Im(f)$  are functions of two variables in  $\mathbb{R}$ ; that is,  $\Re(f) : D \rightarrow \mathbb{R}$  and  $\Im(f) : D \rightarrow \mathbb{R}$ ).
- (2) Linear combinations and products of continuous functions are continuous.
- (3) Composition of functions are continuous.
- (4) If  $f$  is continuous on  $D$  and  $f(z) \neq 0$  for all  $z \in D$ , then  $\frac{1}{f}$  is continuous on  $D$ .
- (5) If  $f$  is continuous, then  $|f|$  is also continuous (note that  $|f| = \sqrt{(\Re(f))^2 + (\Im(f))^2}$ ).

**Proof.** Left as an exercise. ■

**Example.** (1)  $\exp : \mathbb{C} \rightarrow \mathbb{C}^\bullet$  is continuous.

(2)  $\sin, \cos, \sinh, \cosh$  are all continuous.

(3)  $\text{Arg} : \mathbb{C}^\bullet \rightarrow (-\pi, \pi]$  is continuous on  $\mathbb{C}_- = \mathbb{C}^\bullet \setminus \{\text{negative real axis}\}$  (**this will be on the**

**next homework).**

(4)  $\text{Log} : \mathbb{C}^\bullet \rightarrow \{x + iy : -\pi < y \leq \pi\}$  is continuous on  $\mathbb{C}_-$  (**this will be on the next homework**)

(5)  $\text{Arg}$  and  $\text{Log}$  are both not continuous at any  $z < 0$  (that is, points on negative axis).

(6) Fix  $a_0, a_1, \dots, a_n \in \mathbb{C}$ . Then

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

is continuous on  $\mathbb{C}$ ; that is, any polynomial with complex coefficients will be continuous.

Note that  $\text{Arg} : \mathbb{C}^\bullet \rightarrow (-\pi, \pi] \subseteq \mathbb{R} \subseteq \mathbb{C}$  is a discontinuous.

# Chapter 7

## Week 7

### 7.1 Lecture 10

#### 7.1.1 Topics

- Discuss few leftover limit topics from last lecture.
- Discuss differentiability of a function  $f : D \rightarrow \mathbb{C}$ .

Recall the following lemma:

**Lemma.** Let  $f : D \rightarrow \mathbb{C}$  be a function where  $D \subseteq \mathbb{C}$ , and  $\ell \in \mathbb{C}$ . Then the following statements are equivalent:

(1)  $\lim_{z \rightarrow a} f(z) = \ell$

(2) Define  $\tilde{f} : D \cup \{a\} \rightarrow \mathbb{C}$  by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in D \\ \ell & \text{if } z = a. \end{cases}$$

Then  $\tilde{f}$  is continuous at  $a$ .

#### 7.1.2 Complex Differentiability

**Definition** (Complex Differentiability). Let  $f : D \rightarrow \mathbb{C}$  be a function, and  $a \in D$  such that  $a$  is an accumulation point of  $D \setminus \{a\}$ . We say that  $f$  is **complex differentiable at  $a$**  if the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \text{ exists.}$$

**Remark.** Note that  $D$  may not always be open!

If  $f$  is complex differentiable at  $a$ , we write

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}.$$

We call  $f$  is complex differentiable on  $D$  if the limit above exists for every  $a \in D$ . Furthermore, we can define a function  $f'$  by mapping  $z \in D$  to  $f'(z) \in \mathbb{C}$ . This  $f'$  is called the **complex derivative** of  $f$ .

In our definition of complex differentiability, we are allowed to choose

$$D = [c, d] \subseteq \mathbb{R}$$

which allows us to write  $f$  in terms of real functions  $u(x)$  and  $v(x)$ ; that is, we have

$$f(x) = u(x) + iv(x), \quad x \in [c, d].$$

**Example.** (i) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be defined by  $f(z) = z$ . Let  $a \in \mathbb{C}$ . We will compute  $f'(a)$ . Clearly, we have

$$f'(a) = 1.$$

(ii) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be defined by  $f(z) = \bar{z}$ . If we fix  $\Im(z) = 0$  and  $z \rightarrow 0$  along the real axis, we have

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = 1.$$

If we fix  $\Re(z) = 0$  and let  $z \rightarrow 0$  along the imaginary axis, we have

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = -1.$$

Thus, we see that the function  $f(z) = \bar{z}$  is not complex differentiable at 0.

**Remark.** The definition of complex differentiability depends on the domain  $D$ . In most textbooks, the domain  $D$  of a function in  $\mathbb{C}$  is frequently stated to be an open set. Different properties can arise when we compare real and complex functions defined on open sets in terms of looking at their differentiability.

**Lemma.** Let  $f : D \rightarrow \mathbb{C}$  and  $a \in D$  such that  $a$  is an accumulation point of  $D \setminus \{a\}$ . Suppose that  $f$  is complex differentiable at  $a$ . Define  $g : D \rightarrow \mathbb{C}$  by

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}$$

Then  $g$  is continuous at  $z = a$ .

**Proof.** Left as an exercise. ■

The main take away from this lemma is that we now have the ability to rewrite our function in a different way; that is, we can write  $f$  in terms of

$$f(z) = f(a) + (z - a)g(z)$$

where  $g$  is a continuous at  $a$ .

**Corollary.** If the function  $f$  is complex differentiable at  $a \in D$ , then  $f$  is continuous at  $a$ .

**Problem 5.** Let  $a \in D$  be a accumulation point of  $D \setminus \{a\}$  and  $\ell \in \mathbb{C}$ . Then the following statements are equivalent:

(1)  $f$  is complex differentiable at  $a$  and  $f'(a) = \ell$ .

(2) Define  $\gamma : D \rightarrow \mathbb{C}$  by

$$f(z) = f(a) + \ell(z - a) + \gamma(z);$$

that is,

$$r(z) = [f(z) - f(a)] + \ell(z - a),$$



then

$$\lim_{z \rightarrow a} \left| \frac{\gamma(z)}{z - a} \right| = 0.$$

In this case,  $\ell = f'(a)$ .

**Theorem.** Assume that  $f$  and  $g$  are complex differentiable at  $a$ .

(i)  $f + g, \lambda f$  where  $\lambda \in \mathbb{C}$  are also complex differentiable at  $a$  and

$$(f + g)' = f'(a) + g'(a)$$

and

$$(\lambda f)'(a) = \lambda f'(a).$$

(ii) The product  $fg$  is complex differentiable and  $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$ .

(iii) Assume that  $f(a) \neq 0$ , then  $\frac{1}{f}$  is also complex differentiable at  $a$  and

$$\left(\frac{1}{f}\right)'(a) = \frac{f'(a)}{(f(a))^2}.$$

**Proof.** Left as an exercise. ■

**Theorem (Chain Rule).** Let  $f : D \rightarrow \mathbb{C}$  and  $g : D' \rightarrow \mathbb{C}$  such that  $f(D) \subseteq D'$ . Let  $a \in D$ . Assume that  $f$  is complex differentiable at  $a \in D$  and  $g$  is complex differentiable at  $f(a)$ . Then  $g \circ f$  is complex differentiable at  $a \in D$  and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

**Proof.** We know  $f(z) = f(a) + (z - a)\varphi(z)$  and  $g(w) = g(f(a)) + (w - f(a))\psi(w)$  such that  $\varphi$  is continuous at  $a$  and  $\psi$  is continuous at  $f(a)$ . Moreover,  $\varphi(a) = f'(a)$  and  $\varphi(f(a)) = g'(f(a))$ . Now,  $\psi \circ \varphi$  is continuous at  $a$  and

$$g(f(z)) = g(f(a)) + (f(z) - f(a))\psi(f(z)).$$

Now, for  $z \neq a$ , then

$$\frac{g(f(z)) - g(f(a))}{z - a} = \underbrace{\frac{f(z) - f(a)}{z - a}}_{\text{limit exists for } z \rightarrow a} \underbrace{\psi(f(z))}_{\text{continuous at } a}.$$

This implies that

$$\lim_{z \rightarrow a} \frac{g(f(z)) - g(f(a))}{z - a} = f'(a)\psi(f(a)) = f'(a)g'(f(a))$$

exists. ■

**Example.** (i) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  and  $n$  is a positive integer such that  $f$  is defined as  $f(z) = z^n$ . Then  $f$  is complex differentiable at any  $z \in \mathbb{C}$ . Show that  $f'(z) = nz^{n-1}$ .

(ii) Let  $f : \mathbb{C}^\bullet \rightarrow \mathbb{C}$  be defined by

$$f(z) = \frac{1}{z}.$$

Then we have

$$f'(z) = -\frac{1}{z^2}.$$

(iii) Let  $f : \mathbb{C}^\bullet \rightarrow \mathbb{C}$  defined by  $f(z) = z^{-n}$  where  $n$  is a positive integer. Then  $f$  is complex

differentiable on  $\mathbb{C}^\bullet$  and

$$f'(z) = -zz^{-n-1}.$$

(iv) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be defined by

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n.$$

Then  $f$  is complex differentiable on  $\mathbb{C}$  and

$$f'(z) = \sum_{k=1}^n ka_kz^{k-1}.$$

(v) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  and  $f(z) = e^z$ .

- Problem 6.**
- Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = z^n$ ,  $n \in \mathbb{Z}^+$ , then  $f'(a) = na^{n-1}$  for all  $a \in \mathbb{C}$ .
  - Let  $f : \mathbb{C}^\bullet \rightarrow \mathbb{C}$ ,  $f(z) = z^{-n}$ ,  $n \in \mathbb{Z}^+$ . Show that  $f'(a) = -na^{-n-1}$  for all  $a \in \mathbb{C}^\bullet$ .

# Chapter 8

## Week 8

### 8.1 Handout-11

In this handout, we will relate complex differentiability with differentiability in multivariable calculus.

**Definition.** Let  $f : D \rightarrow \mathbb{C}$ ,  $D \subseteq \mathbb{C}$  be a function. We say that  $f$  is complex differentiable on  $D$  if  $f$  is differentiable at each  $a \in D$ .

If  $f$  is differentiable on  $D$ , then we can define a function  $f' : D \rightarrow \mathbb{C}$  by  $z \mapsto f'(z)$ . The function  $f'$  is called the complex derivative of  $f$ .

**Remark.** Assume that  $D = [a, b] \subseteq \mathbb{R}$ . By abusing notation, let us write  $f(x)$  with  $x \in [a, b]$ . Let  $f(x) = u(x) + iv(x)$ . Then, one can show that  $f$  is differentiable if and only if  $u$  and  $v$  are differentiable. If  $f$  is differentiable, then  $f'(x) = u'(x) + iv'(x)$ .

**Example (Computing Complex Derivatives).** (i) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = z^n$ ,  $n \in \mathbb{Z}^+$ . Then  $f'(z) = nz^{n-1}$ . The result follows from

$$z^n - w^n = (z - w)(z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}).$$

(ii) Let  $P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n$  for  $z \in \mathbb{C}$  and  $a_0, \dots, a_n \in \mathbb{C}$  are constants and  $n \in \mathbb{Z}^+$ . Then

$$P'(z) = \sum_{k=1}^n ka_k z^{k-1}.$$

(iii) Power series: Let  $c_0, c_1, c_2, \dots$  be a sequence of complex numbers and  $a \in \mathbb{C}$ . A series of the form  $\sum_{n=0}^{\infty} c_n(z-a)^n$  is called a power series centered at  $a$  with coefficients  $\{c_n\}_{n=0}^{\infty}$ . Assume that the series converges on an open ball  $B(a, R)$  for some  $R$ ; that is, for each  $z \in B(a, R)$ , the series  $\sum_{n=0}^{\infty} c_n(z-a)^n$  exists. Then we can define  $f : B(a, R) \rightarrow \mathbb{C}$  by

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n.$$

(iv) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = \bar{z}$ . We claim that  $f$  is not complex differentiable at  $z = 0$ . We have  $f(z) - f(0) = \bar{z}$  and thus,

$$\lim_{z \rightarrow 0} \frac{f(z)}{z} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = \begin{cases} 1 & \text{if } z \rightarrow 0 \text{ along the real axis} \\ -1 & \text{if } z \rightarrow 0 \text{ along the imaginary axis} \end{cases}$$

Consider the function  $\text{Log} : \mathbb{C}^\bullet \rightarrow \mathbb{C}$ . We say that it is NOT continuous along the negative real axis. Let

$$D = \mathbb{C}_- = \mathbb{C}^\bullet \setminus \{z \in \mathbb{C} \mid z < 0\}.$$

Then,  $\text{Log} : D \rightarrow \mathbb{C}$  is complex differentiable and

$$(\text{Log } z)' = \frac{1}{z}$$

on  $D$ . Assuming complex differentiability, we can use  $e^{\text{Log } z} = z$  and chain rule:

$$\begin{aligned} \implies e^{\text{Log } z} \cdot (\text{Log } z)' &= 1 \\ \implies (\text{Log } z)' &= \frac{1}{e^{\text{Log } z}} = \frac{1}{z}. \end{aligned}$$