Problem 1. Prove that $(\mathbb{R}^n, d_\infty)$ is complete.

Proof. Let $(\vec{x_k})$ be a Cauchy sequence in $\mathbb{R}^n$. By a result found in quiz 1, it follows that (x_{ik}) for $1 \leq i \leq n$ is also Cauchy. Since $\mathbb{R}$ is a complete metric space with respect to the standard metric on $\mathbb{R}$, we find that each x_{ik} is also a convergent sequence. By another result in quiz 1, it follows that $(\vec{x_k})$ is a convergent sequence. Hence, $\mathbb{R}^n$ with respect to the d_∞ metric is complete. ■

Problem 2. (i) Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Prove that $\pi_i : \mathbb{R}^n \to \mathbb{R}$ defined by $\pi_i(\vec{x}) = x_i$ are continuous maps with respect to d_∞ on $\mathbb{R}^n$ and the standard metric on $\mathbb{R}$.

(ii) Prove that π_i in (i) are continuous maps with respect to d_{euclid} on $\mathbb{R}^n$ and the standard metric on $\mathbb{R}$.

Proof. (i) Our goal is to show that π_i is a continuous map with respect to d_∞ on $\mathbb{R}^n$; we will do this via the sequential criterion of continuity. Suppose $\vec{x_k} \to \vec{x}$ for some $\vec{x}$ in $\mathbb{R}^n$. By a result found in quiz 1, we can see that $x_{ik} \to x_i$ for $1 \leq i \leq n$. By definition of π_i, we find that as $n \to \infty$, we get

$$\pi_i(\vec{x_k}) = x_{ik} \to x_i = \pi_i(\vec{x}).$$

Hence, we have that π_i is a continuous map with respect to d_∞ and the standard metric on $\mathbb{R}$.

(ii) Our goal is to show that $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to d_{euclid}. Let $(\vec{x_k})$ be a sequence in $\mathbb{R}^n$ such that $\vec{x_k} \to \vec{x}$. Since π_i is continuous on $\mathbb{R}^n$ with respect to the d_∞ metric, we have $d_\infty(\vec{x_k}, \vec{x}) \to 0$. Notice that

$$0 \leq d_{\text{euclid}}(\vec{x_k}, \vec{x}) \leq (n)^{1/2} d_\infty(\vec{x_k}, \vec{x}). \tag{1}$$

Now, $d_\infty(\vec{x_k}, \vec{x}) \to 0$ implies that $d_{\text{euclid}}(\vec{x_k}, \vec{x}) \to 0$ by applying the squeeze theorem to (1). Thus, we can see that

$$\begin{aligned}
|\pi_i(\vec{x_k}) - \pi_i(\vec{x})| &= \Big(|\pi_i(\vec{x_k}) - \pi_i(\vec{x})|^2 \Big)^{1/2} \\
&= \Big(|x_{ik} - x_i|^2 \Big)^{1/2} \\
&\leq \Big(\sum_{i=1}^{n} |x_{ik} - x_i|^2 \Big)^{1/2} \\
&= d_\infty(\vec{x_k}, \vec{x}) \to 0.
\end{aligned}$$

Hence, we conclude that

$$|\pi_i(\vec{x_k}) - \pi_i(\vec{x})| \to 0$$

and so π_i is continuous map with respect to d_∞ and the standard metric on $\mathbb{R}$.

■

Problem 3. (i) Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |e^x - e^y|$. Prove that d is a metric on $\mathbb{R}$.

(ii) Prove or disprove: $(\mathbb{R}, d)$ is complete.

Proof. (i)

■

Problem 4. Let $X = \mathbb{N}$ be the set of positive integers.

(i) Let $d(m,n) = |m-n|$. Prove that (X,d) is complete.

(ii) Let $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$. Prove that (X,d) is not complete.

Problem 5. Let $X = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

(i) Define $d(f,g) = \displaystyle\int_0^1 |f(t) - g(t)|\, dt$. Prove that d is a metric on X. Prove that d is a metric on X.

(ii) Prove that (X,d) is not complete.