Math 230A: Homework 4

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1. Consider $\mathbb{R}$ with its standard metric. Let $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Prove that $E' = \{0\}$.

   Proof. Note that $0 \notin E$. To show that $E' = \{0\}$, it suffices to show that for $\varepsilon > 0$
   $$N_\varepsilon(0) \cap E \neq \emptyset$$
   which is equivalent to
   $$(-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$
   Let $\varepsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Thus, $1/m \in (-\varepsilon, \varepsilon) \cap E$ and so $0 \in E'$. ■

2. Consider $\mathbb{R}$ with its standard metric.

   (a) Prove that $\mathbb{N}' = \emptyset$. Is $\mathbb{N}$ closed?

   Solution. Suppose for sake of contradiction that $\mathbb{N}' \neq \emptyset$. Let $n \in \mathbb{N}'$. Let $\varepsilon = 1$. Then $N_1(n) \cap \mathbb{N} \setminus \{n\} \neq \emptyset$ where $N_1(n) = (n-1, n+1)$. But note that $(n-1, n+1)$ only contains one point, namely, n and nothing else. Hence, n must be an isolated point be an isolated point of $\mathbb{N}$ which is a contradiction. Thus, $\mathbb{N} = \emptyset$. Because $\emptyset \subseteq \mathbb{N}$, $\mathbb{N}$ must be a closed set. ■

   (b) Prove that $\mathbb{Q}' = \mathbb{R}$. Is $\mathbb{Q}$ closed?

   Solution. Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. Consider the open interval $(a - \varepsilon, a + \varepsilon)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find at least one $x \in \mathbb{Q}$ such that $x \in (a - \varepsilon, a + \varepsilon)$; that is, we have
   $$N_\varepsilon(a) \cap (\mathbb{Q} \setminus \{a\}) \neq \emptyset.$$
   So, any $a \in \mathbb{R}$ must be a limit point of $\mathbb{Q}$. Therefore, we conclude that $\mathbb{Q}' = \mathbb{R}$. Consequently, $\mathbb{Q}$ must not be closed since its limit points take are outside of $\mathbb{Q}$. ■

3. Consider $\mathbb{R}^2$ with its standard metric. Let $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$. Prove that $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

   Proof. Our goal is to show that $E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$; that is, we will show that
   $$E' = E \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}.$$
   We proceed by considering two cases:

   (1) If (a, b) is such that $a^2 + b^2 > 4$, then we will prove that $(a, b) \notin E'$.

   (2) We will prove that if (a, b) is such that $a^2 + b^2 \leq 4$, then $(a, b) \in E'$; that is,
   $$\forall\ N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}) \neq \emptyset.$$

   With (1), consider a point $p = (a, b)$ that is outside of E. Let $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$. Then we see that

$$0 < \frac{1}{2}(\sqrt{a^2+b^2} - 2) \iff a^2 + b^2 > 4.$$

Hence, we must have that $N_\delta(p) \cap (E \setminus \{p\}) = \emptyset$.

With (2), consider the distance from the origin to any point (a, b) on the boundary of the circle E. Let $\delta = \frac{1}{2}\min\{\varepsilon, 2 - \varepsilon\}$. Suppose $\varepsilon < 2$. Let p be the points on the ray $\overline{OQ}$ where the distance to q is δ. Note that $p \in N_\varepsilon((a, b))$ since the distance between p and (a, b) is δ which is less than ε. Also, $p \in E$ since p is on the ray $\overline{OQ}$. Thus, we have

$$p \in N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\}).$$

On the other hand, if $\varepsilon \geq 2$, then let $\delta = 1$. E is clearly the intersection

$$N_\varepsilon((a, b)) \cap (E \setminus \{(a, b)\})$$

is nonempty with a similar argument shown when $\varepsilon < 2$.

Thus, we conclude that

$$E' = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}.$$

■

4. Let (X, d) be a metric space. Prove that X and $\emptyset$ are both open sets.

Proof. We want to show the following:

(i) X is open

(ii) $\emptyset$ is open

Let $x \in X$. For (i), we want to show that there exists $\delta > 0$ such that $N_\delta(x) \subseteq X$. Pick another point $q \neq x$ such that $q \in X$. Choose $\delta = d(x, q) > 0$. But note that any open neighborhood is a subset of X. So, $N_\delta(x) \subseteq X$ and so X is open.

Note that $\emptyset^\circ = \emptyset$. Hence, $\emptyset$ is an open set. ■

5. Let (X, d) be a metric space. Prove that X and $\emptyset$ are both closed sets.

Proof. We will proving the following statements:

(i) X is closed.

(ii) $\emptyset$ is closed.

Let x be a limit point of X. Then for all $\varepsilon > 0$, we have $N_\varepsilon(x) \cap (X \setminus \{x\}) \neq \emptyset$. But note that $N_\varepsilon(x)$ is an open set that is contained in X. Thus, x must be contained within X which proves (i).

For (ii), note that the complement of $\emptyset$ is just X itself. Since X is open by problem 4, we see that $X^c = \emptyset$ must be closed. ■

6. Consider $\mathbb{R}$ equipped with the discrete metric. Let $E = \{1, 2, 3\}$. Prove that E is open, that is, $E^\circ = E$.

Proof. We will check that for every element x in E that there exists $\delta > 0$ such that $N_\delta(x) \subseteq E$. Choose $\delta = 1$, then we have $d(1, x) = 0 < 1$. Since this is the only point that satisfies δ, we must have that $N_1(1) = \{1\} \subseteq E$. Similarly, $N_1(2) = \{2\} \subseteq E$ and $N_1(3) = \{3\} \subseteq E$. Thus, E must be open. ■

7. Consider $X = \mathbb{R}$ equipped with the standard metric. Let $a < b$.

(a) Prove that the sets $(a, b), (a, \infty)$, and $(-\infty, a)$ are open.

Proof. We will show that the following intervals in $\mathbb{R}$ are open:

(i) (a, b)

(ii) (a, ∞)

(iii) $(-\infty, a)$

Starting with (i), we want to show that there exists a $\delta > 0$ such that $N_\delta(x) \subseteq (a,b)$ where $N_\delta(x) = (x-\delta, x+\delta)$. Let $x \in (a,b)$. Choose $\delta = \frac{1}{2}\min\{|b-x|, |x-a|\}$. Then we have $N_\delta(x) \subseteq (a,b)$. Thus, we conclude that (a,b) is open.

With (ii), fix $x \in (a,\infty)$. Observe that (a,x) is an open interval that is contained within (a,∞). Now, let $y \in (a,x)$. Then choose $\delta = \frac{1}{2}\min\{|x-y|, |y-a|\} > 0$. Hence, we have

$$N_\delta(y) \subseteq (a,x) \subseteq (a,\infty).$$

Thus, we have that (a,∞) is an open set in $\mathbb{R}$.

Lastly, fix $x \in (-\infty, b)$. Consider the open interval (x,b) and let $y \in (x,b)$. Choose $\delta = \frac{1}{2}\min\{|b-y|, |y-x|\} > 0$. Then we have

$$N_\delta(y) \subseteq (x,b) \subseteq (b,\infty)$$

which shows that $N_\delta(y)$ is an open set. ■

(b) Prove that the sets $[a,b], [a,\infty)$, and $(-\infty, b)$ are closed.

Proof. We will showing that the following intervals are closed in $\mathbb{R}$:

(i) $[a,b]$

(ii) $[a,\infty)$

(iii) $(-\infty, b]$

Starting with (i), we want to show that every limit point x of $[a,b]$ is contained in $[a,b]$. Thus, let x be a limit point of x. By definition, for all $\delta > 0$, we have $N_\delta(x) \cap ([a,b] \setminus \{x\}) \neq \emptyset$. So, $N_\delta(x)$ contains a point $q \neq x$ in this intersection such that $q \in [a,b]$. This means that $a \leq q \leq b$. Using the fact that $q \in N_\delta(x)$, we have

$$\begin{aligned} |x-q| < \delta &\iff q - \delta < x < q + \delta \\ &\implies a - \delta \leq x \leq b + \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we conclude that $a \leq x \leq b$ which means that $x \in [a,b]$. Thus, $[a,b]$ is closed.

With (ii), we will employ the same process. Let x be a limit point of $[a,\infty)$. Then for all $\varphi > 0$, we see that $N_\varphi(x) \cap ([a,\infty) \setminus \{x\}) \neq \emptyset$. To this end, pick $p \neq x$ in this intersection such that $p \in [a,\infty)$. Since p is also contained within $N_\varphi(x)$, we see that

$$\begin{aligned} |x-p| < \varphi &\iff p - \varphi < x < p + \varphi \\ &\iff a - \varphi \leq x < p + \varphi. \end{aligned}$$

This implies that $a \leq x < p$ since $\varphi > 0$ is arbitrary. Thus, we have $x \in [a,\infty)$ and so $[a,\infty)$ is closed.

The last case is similar to (ii). Let $\varepsilon > 0$. If x is a limit point of $(-\infty, b]$, then $N_\varepsilon(x) \cap ((-\infty, b]) \setminus \{x\}) \neq \emptyset$ implies that we can pick $q \neq x$ in this intersection such that $q \in (-\infty, b]$. Since $\varepsilon > 0$ is arbitrary, we have

$$q - \varepsilon < x < q + \varepsilon \leq b + \varepsilon$$

implies

$$q < x \leq b.$$

Hence, $x \in (-\infty, b]$ and so $(-\infty, b]$ is a closed set. ■

8. Let (X,d) be a metric space. Let $p \in X$ and $\delta > 0$. Prove that the closed ball $C_\delta(p) = \{x \in X :$

$d(x, p) \leq \delta\}$ is indeed closed.

Proof. Our goal is to show that $C_\delta(p)$ is a closed set. To this end, we will show that $[C_\delta(p)]^c$ is an open set. If this holds, then we can conclude that $C_\delta(p)$ is a closed set. Let $x \in [C_\delta(p)]^c$. Then we have $d(x, p) > \delta$. Our goal is to find $\varepsilon > 0$ such that $N_\varepsilon(y) \subseteq [C_\delta(p)]^c$. Since $d(x, p) > \delta$ we can set $\varepsilon = d(x, p) - \delta > 0$. Using the radius $\varepsilon > 0$, we can construct the neighborhood of x, $N_\varepsilon(x)$. Let $y \in N_\varepsilon(x)$. In order for y to be contained within $[C_\delta(p)]^c$, we have to show that $d(p, y) > \delta$. Using the triangle inequality, we have

$$\begin{aligned} d(x,p) \leq d(x,y) + d(y,p) \Longrightarrow d(y,p) &\geq d(x,p) - d(x,y) \\ &> d(x,p) - \varepsilon && (y \in N_\varepsilon(x)) \\ &= \delta. \end{aligned}$$

Thus, we see that $N_\varepsilon(x) \subseteq [C_\delta(p)]^c$ for some $\varepsilon > 0$. Hence, $[C_\delta(p)]^c$ is open and so $C_\delta(p)$ is closed. ∎

9. Give an example that shows that the equality $\overline{N_\delta(p)} = C_\delta(p)$ is not always true.

Solution. Consider the interval $[0, 1]$ in $\mathbb{R}$ with the discrete metric. Clearly, we see that $1/2 \in [0, 1]$. If we let $\varepsilon = 1$, then

$$N_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) < 1\} = \{1/2\}$$

since the only case when the inequality is satisfied is when $x = 1/2$. If we consider the closure of this neighborhood, we just get

$$\overline{N_1(1/2)} = \{1/2\}.$$

Now, consider the closed ball

$$C_1(1/2) = \{x \in \mathbb{R} : d(x, 1/2) \leq 1\}.$$

Observe that for any $x \in \mathbb{R}$, either $x = 1/2$ or $x \neq 1/2$ in $[0, 1]$, the inequality of the set above we always be satisfied; that is, the set will just be all elements contained in $[0, 1]$. Thus, we see that $C_1(1/2) = [0, 1]$ and, in this case, that $C_1(1/2) \neq \overline{N_1(1/2)}$. ∎

10. Let (X, d) be a metric space. Prove that an arbitrary intersection of closed sets is closed. Prove that a finite union of closed sets is closed.

Proof. Let (X, d) be a metric space. We will prove that

(1) An arbitrary intersection of closed sets is closed.

(2) A finite union of closed sets is closed.

To this end, we will proceed by showing the propositions above.

(1) Let $\{F_\alpha\}$ be an arbitrary collection of closed sets. We want to show that

$$\bigcap_\alpha F_\alpha \text{ is closed.}$$

It suffices to show that the complement of this set is open. Consider the complement

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c. \tag{1}$$

Observe that each F_α^c is open since each F_α is closed. But note that the arbitrary union of open sets is open and so the union of the right-hand side of (1) is open. Thus, the left-hand side of (1) is open. Hence, the complement

$$\Big(\bigcap_{\alpha} F_\alpha\Big)^c \text{ is open} \Longleftrightarrow \bigcap_{\alpha} F_\alpha \text{ is closed}$$

and we are done.

(2) Let $\{F_i : 1 \leq i \leq n\}$ be a finite collection of closed sets F_i. We want to show that $\bigcup_{i=1}^n F_i$ is a closed set. It suffices to show that the complement of this set, that is $\Big(\bigcup_{i=1}^n F_i\Big)^c$ is open. Observe that

$$\Big(\bigcup_{i=1}^n F_i\Big)^c = \bigcap_{i=1}^n F_i^c.$$

Note that each F_i^c is open if and only if each F_i is closed. Thus, the finite intersection of each open set F_i^c is open; that is,

$$\Big(\bigcup_{i=1}^n F_i\Big)^c = \bigcap_{i=1}^n F_i^c \text{ is open.}$$

Hence, the finite union of closed sets

$$\bigcup_{i=1}^n F_i \text{ is closed.}$$

■

11. Let E be a subset of a metric space (X, d). Show that $\overline{E} = \{x \in X : \ \forall x > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}$.

Proof. Let E be a subset of a metric space (X, d). We want to show that

$$\overline{E} = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

Denote

$$A = \{x \in X : \forall \varepsilon > 0 \ N_\varepsilon(x) \cap E \neq \emptyset\}.$$

It suffices to show the following two inclusions:

(1) $\overline{E} \subseteq A$.

(2) $A \subseteq \overline{E}$.

Starting with (1), suppose $x \in \overline{E}$. Then either $x \in E$ or $x \in E'$ (or both). If $x \in E'$, then x is a limit point of E. Then for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Since $E \setminus \{x\} \subseteq E$, we we see that

$$N_\varepsilon(x) \cap E \neq \emptyset$$

which implies that $x \in A$. Suppose $x \in E$. Then for all $\varepsilon > 0$, we see that

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

If x is a limit point and $x \in E$, then we immediately have that

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

by definition of limit point. Thus, we see that in the three cases outlined that $\overline{E} \subseteq A$.

With (2), suppose $x \in A$. We want to show that $x \in \overline{E}$; that is, x is either a limit point of E or an element of E. Suppose that x is NOT an element of E. By definition of A, we see that for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

Note that $x \notin E$ implies that $E \setminus \{x\} = E$ and so we have

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus, x is a limit point of E and so $x \in \overline{E}$. Otherwise, when x is not a limit point of E, then the nonempty intersection

$$N_\varepsilon(x) \cap E$$

implies that $x \in E$ and so $x \in \overline{E}$. Thus, $A \subseteq \overline{E}$. ■

12. Let (X, d) be a metric space. Show that the closure of a bounded set is bounded.

Proof. Let (X, d) be a metric space and suppose $E \subseteq X$ is bounded; that is, there exists $\varepsilon > 0$ and $q \in X$ such that $E \subseteq N_\varepsilon(q)$. We claim that $\overline{E} \subseteq N_{\varepsilon+1}(q)$ so that $\overline{E}$ is bounded. Note that $\overline{E} = E \cup E'$. Let $y \in \overline{E}$. Then either $y \in E$ or $y \in E'$. If $y \in E$ and E i bounded, then

$$d(y, q) \leq \varepsilon < \varepsilon + 1.$$

Thus, $y \in N_{\varepsilon+1}(q)$. If $y \in E'$, then for all $\delta > 0$,

$$N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset.$$

Let $\delta = 1/2$. Then pick a point x inside the intersection

$$N_{1/2}(y) \cap (E \setminus \{y\}).$$

Then $x \in N_{1/2}(y)$ and $x \in E$. If $x \in N_{1/2}(y)$, then $d(x, y) < 1/2$ and $d(x, q) \leq \varepsilon$, respectively. Using the triangle inequality, we see that

$$d(y, q) \leq d(y, x) + d(x, q) < \frac{1}{2} + \varepsilon < 1 + \varepsilon.$$

Thus, $y \in N_{\varepsilon+1}(q)$. Therefore, we see that $\overline{E}$ is bounded in both cases. ■

13. Prove that a nonmepty subset A of a metric space (X, d) is open if and only if it is a union of neighborhoods (open balls).

Proof. ($\Longrightarrow$) Suppose A is an open set in X. We will show that A can be written as a union of open balls; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x)$$

where $N_{\varepsilon_x}(x)$ are open neighborhoods with radius ε_x (ε depends on x) for each $x \in A$. To this end, we will show the following inclusions:

(1) $A \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$

(2) $\bigcup_{x \in A} N_{\varepsilon_x}(x) \subseteq A$

Starting with the inclusion in (1), let $x \in A$. Since A is open, we can find an $\varepsilon_x > 0$ such that $N_{\varepsilon_x}(x) \subseteq A$. Note that $N_{\varepsilon_x}(x) \subseteq \bigcup_{x \in A} N_{\varepsilon_x}(x)$. So, x must be contained in $\bigcup_{x \in A} N_{\varepsilon_x}(x)$ which satisfies (1).

Let $p \in \bigcup_{x \in A} N_{\varepsilon_x}(x)$. Then for some point $x \in A$, p must be contained in some neighborhood $N_{\varepsilon_x}(x)$ with radius $\varepsilon_x > 0$. Since $x \in A$ and A is an open set, we must have $N_{\varepsilon_x}(x) \subseteq A$. Thus, p must be contained in A as well. Thus, the inclusion in (2) is satisfied.

($\Longleftarrow$) Suppose A is the union of open balls in A; that is,

$$A = \bigcup_{x \in A} N_{\varepsilon_x}(x).$$

Since each $N_{\varepsilon_x}(x)$ is open, the union of the right-hand side of the equation above is open. Thus, A must be open as well and we are done.

■

14. On page 9, Rudin implicitly defines a subset $E \subseteq \mathbb{R}$ to be "dense" if it satisfies:

    (i) For all $x, y \in \mathbb{R}$ with $x < y$, there exists $p \in E$ such that $x < p < y$.

    On page 32, he defines a subset E of a metric space X to be "dense" if it satisfies:

    (ii) Every point of X is either a limit point of E or a point of E (that is, $\overline{E} = X$).

    Prove that $E \subseteq \mathbb{R}$ satisfies (i) if and only if it satisfies (ii).

Proof. ($\Longrightarrow$) Suppose $E \subseteq \mathbb{R}$ satisfies the property (i). We want to show that every point in $\mathbb{R}$ must either be a limit point of E or a point in E. To this end, let $x \in \mathbb{R}$ be an arbitrary point. Suppose that x is not an element of E. Thus, we want to show that x is a limit point of E; that is, we will show that

$$\forall \delta > 0 \quad N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Let $\delta > 0$. Then denote the open neighborhood of x as

$$N_\delta(x) = \{y \in \mathbb{R} : |x - y| < \delta\} = (x - \delta, x + \delta).$$

But note that $x - \delta < x + \delta$ implies that there exists a $p \neq x \in E$ such that $x - \delta < p < x + \delta$. Thus, we see that $|x - p| < \delta$ and hence, we see that

$$N_\delta(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Hence, x is a limit point of E. Now, if x is NOT a limit point, then x is an isolated point of E and that $x \in E$.

($\Longleftarrow$) Let $x, y \in \mathbb{R}$ such that $x < y$. We will show that there exists $p \in E$ such that $x < p < y$. Suppose that $\mathbb{R} = \overline{E}$. Since $x \in \mathbb{R}$, we must have $x \in \overline{E}$. That is, for all $\varepsilon > 0$, we have

$$N_\varepsilon(x) \cap E \neq \emptyset.$$

This implies that $N_\varepsilon(x)$ contains a point $p \neq x$ such that $p \in E$. This means that $p \in (x - \varepsilon, x + \varepsilon)$ and so

$$x - \varepsilon < p < x + \varepsilon.$$

Since $x < y$, we have that

$$x - \varepsilon < p < y + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we find that $x < p < y$ for some $p \in E$. ■

15. Let A and B be subsets of a metric space (X, d).

    (a) If $A \subseteq B$, then $A' \subseteq B'$.

Proof. Let $x \in A'$. We want to show that $x \in B'$; that is, x is a limit point of B. By definition, $x \in A'$ implies that for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$. So, pick $p \neq x$ in this intersection such that $p \in A$. But note that $A \subseteq B$. Thus, $p \in B$ and we must have for all $\varepsilon > 0$,

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset.$$

Thus, x is also a limit point of B (that is, $x \in B'$) and we conclude that $A' \subseteq B'$. ■

    (b) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. Suppose $A \subseteq B$. By part (i), we see that $A' \subseteq B'$. Thus,

$$\overline{A} = A \cup A' \subseteq B \cup B' = \overline{B}.$$

Hence, $\overline{A} \subseteq \overline{B}$. ■

(c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Proof. Our goal is to show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cap B}$. Then either $x \in A \cap B$ or $x \in (A \cap B)'$. Suppose $x \in A \cap B$. Then both $x \in A$ and $x \in B$ and so $x \in \overline{A}$ and $x \in \overline{B}$ since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Now, suppose $x \in (A \cap B)'$. Then x is a limit point of $A \cap B$; that is, for all $\varepsilon > 0$, $N_\varepsilon(x) \cap ((A \cap B) \setminus \{x\}) \neq \emptyset$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we must have that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \tag{1}$$

and

$$N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset. \tag{2}$$

Thus, (1) and (2) imply that $x \in A'$ and $x \in B'$ meaning that x is both a limit of point of A and B. Hence, $x \in \overline{A}$ and $x \in \overline{B}$. Thus, we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ in both of these cases. ■

(a) Prove that $A' \cup B' = (A \cup B)'$.

Proof. Our goal is to show that $A' \cup B' = (A \cup B)'$; that is, we need to show that

(1) $A' \cup B' \subseteq (A \cup B)'$

(2) $(A \cup B)' \subseteq A' \cup B'$.

Starting with (1), let $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$. If $x \in A'$, then x is a limit point of A. Hence, for all $\varepsilon > 0$ such that

$$N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset.$$

Then pick a point in this intersection, say, $q \neq x$ such that $q \in A$. Since $A \subseteq A \cup B$, we have $q \in A \cup B$. Thus, we have

$$N_\varepsilon(x) \cap \Big((A \cup B) \setminus \{x\}\Big) \neq \emptyset.$$

Hence, x is a limit point of $A \cup B$ and so $x \in (A \cup B)'$. If $x \in B'$, then for all $\varepsilon > 0$ such that

$$N_\varepsilon(x) \cap B \setminus \{x\} \neq \emptyset.$$

Since $B \subseteq A \cup B$, we have

$$N_\varepsilon(x) \cap \Big((A \cup B) \setminus \{x\}\Big) \neq \emptyset.$$

Then x is a limit point of $A \cup B$; that is, $x \in (A \cup B)'$. Hence, we have that $A' \cup B' \subseteq (A \cup B)'$.

Now, with (2), let $x \in (A \cup B)'$. Then x is a limit point of $A \cup B$. Thus, for all $\varepsilon > 0$ we have

$$N_\varepsilon(x) \cap ((A \cup B) \setminus \{x\}) \neq \emptyset.$$

Pick a point in this intersection, say, $p \neq x$. Note that $p \in A \cup B$. So, either $p \in A$ or $p \in B$. If $p \in A$, then x is a limit point of A. Thus, $x \in A'$ and so $x \in A' \cup B'$. If $p \in B$, then x is a limit point of B and so $x \in B'$. Thus, $x \in A' \cup B'$. Therefore, we have $(A \cup B)' \subseteq A' \cup B'$ which proves (2).

With (1) and (2), we have

$$A' \cup B' = (A \cup B)'.$$

■

16. Let (X, d) be a metric space and let $E \subseteq X$.

(a) Prove that E' is closed.

Proof. Our goal is to show that E' is closed; that is, any limit point of E' is contained within E'. Let x be a limit point of E'. Then for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (E' \setminus \{x\}) \neq \emptyset$. To this end, pick $y \neq x$ in this intersection such that $y \in E'$. But this tells us that y is a limit point of E. Thus, for all $\delta > 0$, $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$. Thus, we can pick $p \neq y$ such that $p \in E$. Note that $p \neq y \neq x$ and $p \in E$ imply that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset.$$

Thus, x is a limit point of E and so $x \in E'$. Hence, E' is closed. ■

(b) Prove that E and $\overline{E}$ have the same limit points.

Proof. To prove that E and $\overline{E}$ have the same limit points, we need to show that $E' = (\overline{E})'$. We will prove the following two inclusions:

(1) $E' \subseteq (\overline{E})'$

(2) $(\overline{E})' \subseteq E'$.

Starting with (1), let $x \in E'$. Then x is a limit point of E. Thus, for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset$. Since $E \subseteq \overline{E}$, we must have that

$$N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset.$$

Thus, $x \in \overline{E}'$ which proves (1).

Now with (2), let $x \in \overline{E}'$. Then for all $\varepsilon > 0$, $N_\varepsilon(x) \cap (\overline{E} \setminus \{x\}) \neq \emptyset$. From this nonempty intersection, pick $y \neq x$ such that $y \in \overline{E}$; that is, $y \in E$ or $y \in E'$. If $y \in E$, then x is a limit point of E and thus $x \in E'$. If $y \in E'$, then y is a limit point of E. Hence, for all $\delta > 0$ $N_\delta(y) \cap (E \setminus \{y\}) \neq \emptyset$. From this, we can pick $q \neq y \neq x$ such that $q \in E$. But this tell us that

$$N_\varepsilon(x) \cap (E \setminus \{x\}) \neq \emptyset,$$

proving that x is a limit point of E and so $x \in E'$. ■

(c) Construct an example that shows E and E' do not necessarily have the same limit points?

Proof. Suppose we have the following set $E = \{x < \frac{1}{n} : n \in \mathbb{N}, x \in \mathbb{R}\}$ and its set of limit points $E' = \{\frac{1}{n} : n \in \mathbb{N}\}$. By problem 1, we see that the set of limit points of E' must be $\{0\}$. Clearly, E and E' do not have the same limit points. ■

17. Construct a bounded set of real numbers with exactly three limit points.

Solution. Consider the sets $A = \{1/n : n \in \mathbb{N}\}, L = \left\{\pi + \frac{1}{n} : n \in \mathbb{N}\right\}, I = \left\{3 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Note that each set is bounded. If we take the union of these sets, then we have the following set

$$A \cup L \cup I$$

which has three limit points which are $0, \pi$, and 3, respectively. ■

18. Let (X, d) be a metric space and $E \subseteq X$.

(a) Prove that E° is always open, that is $(E^\circ)^\circ = E^\circ$.

Proof. To show that E° is open, we will show that $(E^\circ)^\circ$. To do this, we will need to show the following two inclusions:

(1) $(E^\circ)^\circ \subseteq E^\circ$

(2) $E^\circ \subseteq (E^\circ)^\circ$.

Starting with (1), let $x \in (E^\circ)^\circ$. By definition of interior points, there must exists a $\delta > 0$ such that $N_\delta(x) \subseteq E^\circ$. Clearly, we have $x \in E^\circ$ which proves (1).

Now, let $x \in E^\circ$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq E$ where $x \in E$. Suppose for sake of contradiction that x is NOT in the interior of E°. Then for all $\delta > 0$, $N_\delta(x) \nsubseteq E^\circ$. That is, $N_\delta(x) \cap (E^\circ)^c \neq \emptyset$. Since $E^\circ \subseteq E$, it follows that $x \in E^c$. But this is a contradiction because we had assume that $x \in E^\circ$ and so $x \in E$. Thus, we must have x lie in the interior of E°. From (1) and (2), we conclude that E° is open. ■

(b) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Proof. Suppose $G \subseteq E$ and G is open. We will show that $G \subseteq E^\circ$. Let $x \in G$. Since G is open, there exists $\delta > 0$ such that $N_\varepsilon(x) \subseteq G \subseteq E$. Thus, x must be in the interior of E; that is, $x \in E^\circ$. Hence, $G \subseteq E^\circ$. ■

(c) Prove that $E^\circ = \overline{E^c}^c$.

Proof. We will show that $E^\circ = (\overline{E^c})^c$. We will show the following two inclusions:

(1) $E^\circ \subseteq (\overline{E^c})^c$,

(2) $(\overline{E^c})^c \subseteq E^\circ$.

Starting with (1), let $x \in E^\circ$. Then there exists a $\delta > 0$ such that $N_\delta(x) \subseteq E$. This implies that x cannot be in E^c. Otherwise, x is not an interior point of E. Subsequently, we have that for all $\delta > 0$, $N_\delta(x) \cap E^c = \emptyset$. Thus, neither $x \in E^c$ nor $x \in (E^c)'$. Hence, $x \in (\overline{E^c})^c$.

With (2), let $x \in (\overline{E^c})^c$. Then $x \notin \overline{E^c}$. Since $\overline{E^c} = E^c \cup (E^c)'$, we have $x \notin E^c$ and $x \notin (E^c)'$. Then there exists $\delta > 0$ such that

$$N_\delta(x) \cap (E^c \setminus \{x\}) = \emptyset \iff N_\delta(x) \cap E^c = \emptyset.$$

Since $x \notin E^c$, we must have $x \in E$. Thus, $N_\delta(x) \subseteq E$ for some $\delta > 0$ and so $x \in E^\circ$ which proves (2). ■

(d) Give an example that shows that E and $\overline{E}$ do not always have the same interiors.

Solution. Let $E = \mathbb{Q}$ and note that $\overline{E} = \mathbb{R}$ under the standard metric of $\mathbb{R}$. Since $\mathbb{R}$ is open, we know that $\mathbb{R}^\circ = \mathbb{R}$. But the interior of $\mathbb{Q}$ is empty, which is clearly not equal to $\mathbb{R}^\circ = \overline{E}^\circ$. ■

(e) Give an example that shows that E and E° do not always have the same closures.

Proof. Consider the set of rational numbers $\mathbb{Q}$. Then observe that $\mathbb{Q}^\circ$ is empty since for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains irrational numbers which are not contained in $\mathbb{Q}$. Hence, $\overline{\mathbb{Q}^\circ} = \emptyset$. But notice that the closure $\overline{\mathbb{Q}} = \mathbb{R}$. Clearly, $\overline{\mathbb{Q}} \neq \overline{\mathbb{Q}^\circ}$. ■

19. Let (X, d) be a metric space and let $A \subseteq X$. Prove that A is dense in X if and only if any nonempty open set in X has a nonempty intersection with A.

Proof. ($\Longrightarrow$) Let B be a nonempty open set in X. Let $b \in B$. We will show that $A \cap B \neq \emptyset$. Since $b \in X$ and A is dense in X, either

(1) b is a limit point of A or

(2) $b \in A$.

If b is a limit point of A, then for all $\delta > 0$,

$$N_\delta(b) \cap (A \setminus \{b\}) \neq \emptyset.$$

Since B is an open set, we know that $N_\delta(b) \subseteq B$. Together with the fact that $A \setminus \{b\} \subseteq A$, we can conclude that

$$B \cap A \neq \emptyset.$$

On the other hand, $b \in A$ immediately implies that $B \cap A \neq \emptyset$.

($\Longleftarrow$) Let $p \in X$ and let $A \subseteq X$. We need to show that A is dense in X; that is, we need to show that p is either a limit point of A or $p \in A$. Suppose that $p \notin A$. We will show that p is a limit point of A. Let $\varepsilon > 0$ and consider the open neighborhood $N_\varepsilon(p)$. By assumption, the open ball $N_\varepsilon(p)$ contains a nonempty intersection with A; that is,

$$N_\varepsilon(p) \cap A \neq \emptyset.$$

Subsequently, we have

$$N_\varepsilon(p) \cap (A \setminus \{p\}) \neq \emptyset.$$

Thus, p is a limit point of A. On the other hand, if p is not a limit point of A, then p is an isolated point of A. Thus, $p \in A$. ■

Proof. ($\Longleftarrow$) Suppose A has a nonempty intersection with every nonempty open set in X. Thus, we have

$$\begin{aligned} (A^c)^\circ = \emptyset &\Longrightarrow_{\text{18-c)}} \left(\overline{(A^c)^c}\right)^c = \emptyset \\ &\Longrightarrow (\overline{A})^c = \emptyset \\ &\Longrightarrow \overline{A} = X. \end{aligned}$$

Thus, A is dense in X.

($\Longrightarrow$) Suppose $\overline{A} = X$. By problem 11, we see that for any $x \in X$ and for all $\varepsilon > 0$ that

$$N_\varepsilon(x) \cap A \neq \emptyset. \tag{1}$$

If E is any nonempty open set, then let $q \in E$. Since q is an interior point of E, there exists $\delta > 0$ such that

$$N_\delta(q) \subseteq E. \tag{2}$$

By (1) and (2), we must have that $E \cap A \neq \emptyset$. ■

20. Let (X, d) be a metric space and $A_i \subseteq X$ for all $i \in \mathbb{N}$.

    (a) Prove that for all $n \in \mathbb{N}$, we have $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$.

    Proof. First, we will show by induction on $n \in \mathbb{N}$ that

    $$\left(\bigcup_{i=1}^n A_i\right)' = \bigcup_{i=1}^n A_i'.$$

    Let $n = 2$ be our base case, then we see that $(A_1 \cup A_2)' = A_1' \cup A_2'$ by problem 15-(d).

Now, we will suppose that the result holds for the nth case. Thus, we have that

$$\begin{aligned}\Big(\bigcup_{i=1}^{n+1} A_i\Big)' &= \Big(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\Big)' \\ &= \Big(\bigcup_{i=1}^{n} A_i\Big)' \cup A_{n+1}' \\ &= \Big(\bigcup_{i=1}^{n} A_i'\Big) \cup A_{n+1}' \\ &= \bigcup_{i=1}^{n+1} A_i'.\end{aligned}$$

Then for all $i \in \mathbb{N}$, we see that

$$\begin{aligned}\overline{\bigcup_{i=1}^{n} A_i} &= \Big(\bigcup_{i=1}^{n} A_i\Big) \cup \Big(\bigcup_{i=1}^{n} A_i\Big)' \\ &= \Big(\bigcup_{i=1}^{n} A_i\Big) \cup \Big(\bigcup_{i=1}^{n} A_i'\Big) \\ &= \bigcup_{i=1}^{n} A_i \cup A_i' \\ &= \bigcup_{i=1}^{n} \overline{A_i}.\end{aligned}$$

■

(b) Prove that $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$.

Proof. Our goal is to show that $\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$. Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$. Then for some $i \in \mathbb{N}$, we see that $x \in \overline{A_i}$. Then either $x \in A_i$ or $x \in A_i'$. If $x \in A_i$ and $A_i \subseteq \bigcup_{i=1}^{\infty} A_i$, we see that $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$. Suppose $x \in A_i'$. Note that $A_i' \subseteq \Big(\bigcup_{i=1}^{\infty} A_i\Big)'$. So, $x \in \Big(\bigcup_{i=1}^{\infty} A_i\Big)'$. Thus, $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$. Hence, we have

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}.$$

■

(c) Show, by an example, that the inclusion in part (b) can be proper.

Solution. Consider the sequence of open intervals in $\mathbb{R}$, $A_i = (0, 1 + \frac{1}{i})$. Observe that

$$\bigcup_{i=1}^{\infty} \overline{\Big(0, 1 + \frac{1}{i}\Big)} = \bigcup_{i=1}^{\infty} \Big[0, 1 + \frac{1}{i}\Big] = [0, 2).$$

But we have

$$\overline{\bigcup_{i=1}^{\infty} \Big(0, 1 + \frac{1}{i}\Big)} = \overline{(0, 2)} = [0, 2].$$

Clearly, $[0, 2)$ is a proper subset of $[0, 2]$. ■

21. Let (X, d) be a metric space and $A_i \subseteq X$, for all $i \in \mathbb{N}$.

(a) Prove that for all $n \in \mathbb{N}$, we have $\Big(\bigcap_{i=1}^{n} A_i\Big)^{\circ} = \bigcap_{i=1}^{n} A_i^{\circ}$.

Proof. Our goal is to show that $\left(\bigcap_{i=1}^{n} A_i\right)^{\circ} = \bigcap_{i=1}^{n} A_i^{\circ}$; that is, we need to show the following two inclusions:

(1) $\left(\bigcap_{i=1}^{n} A_i\right)^{\circ} \subseteq \bigcap_{i=1}^{n} A_i^{\circ}$.

(2) $\bigcap_{i=1}^{n} A_i^{\circ} \subseteq \left(\bigcap_{i=1}^{n} A_i\right)^{\circ}$.

Starting with (1), let $x \in \left(\bigcap_{i=1}^{n} A_i\right)^{\circ}$. Then there exists $\delta > 0$ such that $N_\delta(x) \subseteq \bigcap_{i=1}^{n} A_i$. Thus, $x \in \bigcap_{i=1}^{n} A_i$ implies that $x \in A_i$ for all $1 \leq i \leq n$. But this tell us that $N_\varepsilon(x) \subseteq A_i$ for all $1 \leq i \leq n$, and so $x \in A_i$. Hence,

$$x \in \bigcap_{i=1}^{n} A_i^{\circ}.$$

Thus, we conclude that

$$\left(\bigcap_{i=1}^{n} A_i\right)^{\circ} \subseteq \bigcap_{i=1}^{n} A_i^{\circ}.$$

With (2), let $x \in \bigcap_{i=1}^{n} A_i^{\circ}$. Then for all $1 \leq i \leq n$, we have $x \in A_i^{\circ}$. By definition, there exists $\delta > 0$ such that $N_\varepsilon(x) \subseteq A_i$ for all $1 \leq i \leq n$. Thus, $x \in A_i$ for all $1 \leq i \leq n$ and so

$$x \in \bigcap_{i=1}^{n} A_i.$$

Subsequently, $N_\delta(x) \subseteq \bigcap_{i=1}^{n} A_i$ and so $x \in \left(\bigcap_{i=1}^{n} A_i\right)^{\circ}$. Therefore, we have

$$\bigcap_{i=1}^{n} A_i^{\circ} \subseteq \left(\bigcap_{i=1}^{n} A_i\right)^{\circ}.$$

Hence, (1) and (2) gives us our desired result. ■

(b) Prove that $\left(\bigcap_{i\in\mathbb{N}} A_i\right)^{\circ} \subseteq \bigcap_{i\in\mathbb{N}} A_i^{\circ}$.

Proof. Let $x \in \left(\bigcap_{i\in\mathbb{N}} A_i\right)^{\circ}$. Then there exists $\delta > 0$ such that $N_\delta(x) \subseteq \bigcap_{i\in\mathbb{N}} A_i$. Hence, $x \in \bigcap_{i\in\mathbb{N}} A_i$ if and only if $x \in A_i$ for all $i \in \mathbb{N}$. But this also tells us that $N_\delta(x) \subseteq A_i$. Hence, $x \in A_i^{\circ}$ for all $i \in \mathbb{N}$. Thus,

$$\left(\bigcup_{i\in\mathbb{N}} A_i\right)^{\circ} \subseteq \bigcup_{i\in\mathbb{N}} A_i^{\circ}.$$

■

(c) Give an example where equality does not hold in (b).

Solution. Consider the sequence of closed intervals A_i in $\mathbb{R}$ defined by

$$A_i = \left[a - \frac{1}{i}, b\right].$$

Then the interior of these closed intervals are

$$A_i^{\circ} = \left(a - \frac{1}{i}, b\right).$$

Now, observe that

$$\bigcup_{i\in\mathbb{N}} A_i^\circ = [a,b).$$

But

$$\left(\bigcup_{i\in\mathbb{N}} A_i\right)^\circ = [a,b]^\circ = (a,b).$$

Hence, we have $(a,b) \subset [a,b)$. ■

22. Let (X,d) be a metric space and $A_i \subseteq X$, for all $i \in \mathbb{N}$.

(a) $\bigcup_{i\in\mathbb{N}} A_i^\circ \subseteq \left(\bigcup_{i\in\mathbb{N}} A_i\right)^\circ$.

Proof. Let $x \in \bigcup_{i\in\mathbb{N}} A_i^\circ$. Then for some $i \in \mathbb{N}$, we have $x \in A_i^\circ$. Then there exists a $\delta > 0$ such that $N_\delta(x) \subseteq A_i$. But note that

$$A_i \subseteq \bigcup_{i\in\mathbb{N}} A_i$$

and so

$$N_\delta(x) \subseteq \bigcup_{i\in\mathbb{N}} A_i.$$

This tells us that $x \in \left(\bigcup_{i\in\mathbb{N}} A_i\right)^\circ$. Therefore,

$$\bigcup_{i\in\mathbb{N}} A_i^\circ \subseteq \left(\bigcup_{i\in\mathbb{N}} A_i\right)^\circ.$$

■

(b) Give an example of a finite collection in which equality does not hold in (a).

Solution. Consider the two closed intervals $A = [0,1]$ and $B = [1,2]$. Note that $A^\circ = (0,1)$ and $B^\circ = (1,2)$. Thus,

$$A^\circ \cup B^\circ = (0,1) \cup (1,2)$$

whereas

$$(A \cup B)^\circ = ([0,2])^\circ = (0,2).$$

So, $(0,1) \cup (1,2) \subseteq (0,2)$ since 1 is not included in $(0,1) \cup (1,2)$. ■