Homework-5- Math-234

Problem-1 (When a $\mathbb{R}$- linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ induce a $\mathbb{C}$-linear map $\tilde{A} : \mathbb{C} \to \mathbb{C}$?)

(a) Let $L : \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear map i.e.

$$L(\alpha z_1 + \beta z_2) = \alpha L(z_1) + \beta L(z_2) \text{ for all } z_1, z_2 \in \mathbb{C}$$
and for all $\alpha, \beta \in \mathbb{R}$.

Show that L is $\mathbb{C}$- linear (i.e. $L(wz) = w L(z)$ for all $z \in \mathbb{C}$ and for all $w \in \mathbb{C}$) if and only if

$$L(iz) = i L(z) \text{ for all } z \in \mathbb{C}.$$

(b) Let $A : \mathbb{R}^2 \to \mathbb{R}^2$,

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Define $\tilde{A} : \mathbb{C} \to \mathbb{C}$ by

$$\tilde{A}(x + iy) = (ax + by) + i(cx + dy).$$

Show that (i) $\tilde{A}$ is $\mathbb{R}$-linear.

(ii) $\tilde{A}$ is $\mathbb{C}$-linear if and only if $a = d$ and $b = -c$.
(Hint use (a))

(iii) Assume that $\tilde{A}$ is $\mathbb{C}$-linear. Show that $\tilde{A} z = (a + ic) z$ for all $z \in \mathbb{C}$.

Problem-2: Let $D \subseteq \mathbb{C}$, $f : D \to \mathbb{C}$, $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$ and $\ell \in \mathbb{C}$.

Prove that the following statements are equivalent.

(i) f is complex differentiable at a and $f'(a)=\ell$.

(ii) Define $r: D \to \mathbb{C}$ by

$$r(z) = f(z) - f(a) - \ell(z-a)$$

Then $\lim_{z \to a} \frac{r(z)}{z-a} = 0.$

(Hint: (i) $\Rightarrow$ (ii), HW-4 problem-1 (ii) can be useful).

Problem-8: Let $D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$, $f(z) = u(z) + i v(z)$, $u : D \to \mathbb{R}$, $v: D \to \mathbb{R}$, $a \in D$. Define $\hat{D} \subseteq \mathbb{R}^2$ by $\begin{bmatrix} x \\ y \end{bmatrix} \in \hat{D} \Leftrightarrow x+iy \in D$, $\hat{f}: \hat{D} \to \mathbb{R}^2$ by

$$\hat{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x+iy) \\ v(x+iy) \end{bmatrix}$$

Note that u and v can be thought as $u: \hat{D} \to \mathbb{R}$ and $v: \hat{D} \to \mathbb{R}$ by identifying $x+iy$ with $\begin{bmatrix} x \\ y \end{bmatrix}$.

Assume that f is complex differentiable at a and $f'(a) = \ell$. Show that $\hat{f}$ is differentiable in the sense of multivariable calculus and

$$J\left(\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix}, \hat{f}\right) = \begin{bmatrix} \mathrm{Re}(\ell) & -\mathrm{Im}(\ell) \\ \mathrm{Im}(\ell) & \mathrm{Re}(\ell) \end{bmatrix}.$$

Problem-9: Let $D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$ holomorphic.

(a) Suppose that $Re(f)$ is a constant function on D. Show that $f'(z)=0$ for all $z \in D$.

(b) Suppose that $Im(f)$ is a constant function on D. Show that $f'(z)=0$ for all $z \in D$.

(c) Suppose that $|f|$ is a constant function on D. Show that $f'(z)=0$ for all $z \in D$.

Problem 5 Let $D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$, $f(z) = u(z) + i v(z)$.

Define $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$, $\dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i \dfrac{\partial v}{\partial y}$

and
$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right]$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right]$$

Show that f is holomorphic if and only if
$$\frac{\partial f}{\partial \bar{z}} = 0.$$

If f is holomorphic, then $f'(z) = \dfrac{\partial f}{\partial z}$.

Problem 6: Write the following functions in the form $f(z) = u(z) + i v(z)$ and show that each of them are holomorphic by using Cauchy Riemann equation. Make sure to check all condition that imply holomorphic.

(a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \sin z$

(b) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \cosh z$

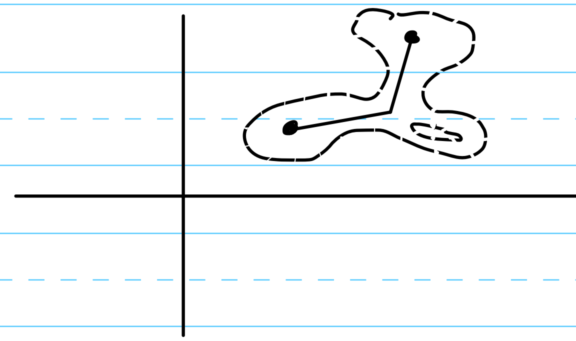
(c) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = e^{-z^2}$

Problem 7: For the following functions $f: \mathbb{C} \to \mathbb{C}$, show that f is not complex differentiable at any $z \in \mathbb{C}$.

(i) $f(z) = \operatorname{Re}(z)$ (ii) $f(x+iy) = 2x + ixy^2$

(iii) $f(z) = e^{\bar{z}}$

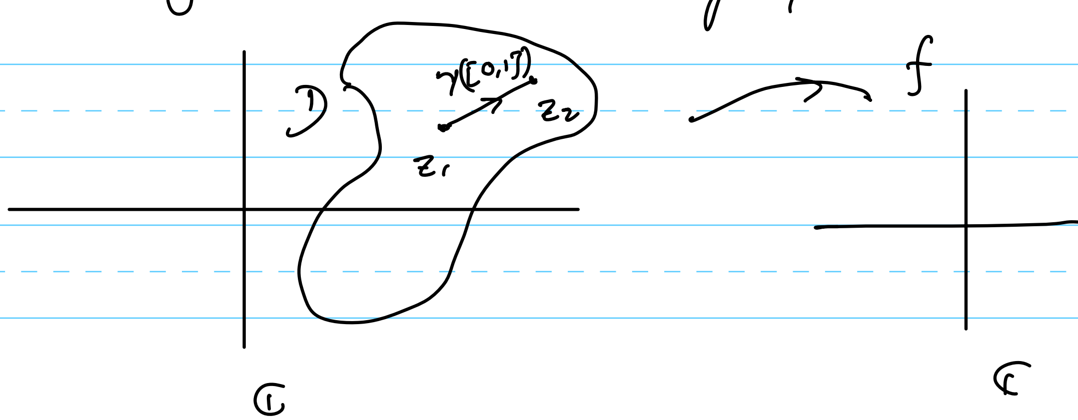
Problem-8: Assume that $D \subseteq \mathbb{C}$ open and D is connected (i.e. any two points in D can be connected by a path consisting of a sequence of line segments, see figure)



Let $f: D \to \mathbb{C}$ holomorphic.

Assume that $f'(z) = 0$ for all $z \in D$. The goal of this problem is to show f is constant.

(a) Let z_1 and z_2 be two points on D such that z_1 and z_2 can be joined by a line segment γ, i.e., $\gamma: [0,1] \to D$, $\gamma(t) = (1-t)z_1 + tz_2$.
Show that f is constant along $\gamma([0,1]) \subseteq D$



Hint: Consider the function $g:[0,1]\to\mathbb{C}$ defined by $g(t)=f(\gamma(t))$. Show that g is constant. Use it to deduce f is constant along $\gamma([0,1])$.

(b) Show that f is constant on D.

Hint: Let z_1 and z_2 in D. Since D is connected, we can find $w_1, \dots, w_k$ in D such that z_1 and w_1 are connected by a line segment; w_i and w_{i+1} are connected by a line segment for $i=1,2,\dots,k-1$; and w_k and z_2 are connected by a line segment. Now, use (a) to show $f(z_1)=f(z_2)$.