Analysis Rudin Notes

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July 6, 2024

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List of Theorems

Chapter 1

Basic Topology

1.1 Finite, Countable, and Uncountable Sets

1.1.1 Function Concepts

Definition 1.1.1 (Functions, Domains, Values, and Range)**.** Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element of x of A there is a associated, in some manner, an element of B, which we denote by $f(x)$.

- The function f is said to be a *function* from A into B;
- The set A is called the *domain* of f;
- The elements $f(x)$ are called the *values* of f;
- The set of values of f is called the *range* of f

Definition 1.1.2 (Images, Onto)**.** Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, then $f(E)$ is called the set of all elements $f(x)$, for $x \in E$.

- We call $f(E)$, the *image* of E under f.
- We call $f(A)$ the *range* of f. Clearly, we have $f(A) \subset B$.
- If $f(A) = B$, we say that f maps *onto* B.

Definition 1.1.3 (Inverse Images, One-to-one)**.**
- If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f.
- If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B.
- Another way to state this is to say that for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ is called a 1-1 function.
- Alternatively, if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ is also called a 1-1 function (this is just contrapositive of the last statement).

Definition 1.1.4 (Correspondence). If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinality*, or, that A and B are *equivalent*. For this, we write $A \sim B$.

This relation contains the following properties:

- **Reflexive**: $A \sim A$.
- **Symmetric**: If $A \sim B$, then $B \sim A$.
- **Transitive:** If $A \sim B$ and $B \sim C$, then $A \sim C$.

Definition 1.1.5 (Finite, Infinite, Countable, Uncountable, At most countable). For any positive integer n, let $\mathbb{N}_n$ be the set whose elelemts are the integers $1, 2, \ldots, n$; let $\mathbb{N}$ be the set consisting of all positive integers. For any set A, we say:

(a) A is *finite* if $A \sim \mathbb{N}_n$ for some n (the empty set is also considered to be finite).

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim \mathbb{N}$.

(d) A is *uncountable* if A is neither finite n or countable.

(e) A is *at most countable* if A is finite or countable.

Proposition 1.1.1. Every infinite subset of a countable set A is countable.

Proof. Suppose $E \subset A$, and E is infinite. Arrange the elements of x of A as a sequence (x_n) of distinct elements. Construct a sequence (n_k) as follows: Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots n_{k-1}$ with $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$ with $(k = 1, 2, 3, \ldots)$, we obtain a 1-1 correspondence between E and $\mathbb{N}$. Thus, by definition, we see that E is an infinite subset of A that is countable. ■

Definition 1.1.6. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

- We can have sets whose elements are also sets.
- To make this easier to understand, we usually denote these kinds of sets as a collection of sets.

Definition 1.1.7 (Union). The *union* of the sets E_α is defined to be the set S such that $x \in S$ if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If our collection of sets A is finite, then we can use the notation

$$S = \bigcup_{m=1}^{n} E_m$$

for $E_1, E_2, \dots, E_n \in A$

On the other hand, when the collection of sets A contains a countable number of elements, then we can use the notation

$$S = \bigcup_{m=1}^{\infty} E_m.$$

The ∞ on the top of the union symbol should not be confused with $+\infty$ and $-\infty$.

Definition 1.1.8 (Intersection). The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if $x \in E_\alpha$ for every $\alpha \in A$, we have

$$P = \bigcap_{\alpha \in A} E_\alpha.$$

Like the union, A can either have a finite collection of sets or a countable collection of sets. Thus, we have

$$P = \bigcap_{m=1}^{n} E_m \text{ and } P = \bigcap_{m=1}^{\infty} E_m,$$

respectively.

Definition 1.1.9 (Nonempty Intersections and Disjoint Sets). If we have $A \cap B \neq \emptyset$, then we say that A and B *intersect*. Otherwise, we say that they are *disjoint*.

Here are some list of algebraic properties of sets:

- **Commutativity**: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- **Associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Distributivity:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \subset A \cup B$.
- $A \cap B \subset A$.
- $A \cup \emptyset$ and $A \cap \emptyset = \emptyset$.
- If $A \subset B$, then
$$A \cup B = B, \quad A \cap B = A$$

Theorem 1.1.1. Let $\{E_n\}$ with $n \in \mathbb{N}$ be a countable collection of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n,$$

Then S is countable.

Proof. Let every set E_n be arranged in a sequence (x_{n_k}) with $k = 1, 2, 3, \ldots,$. We can consider an infinite array such that, in each row, we have all the elements of each E_n. If we take the diagonal entries, starting from left to right, we can rearrange these entries into a sequence

$$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; \ldots \ ,$$

starting from $n = 2$. Notice how the sum of each index in the sequence adds up to the index of the sequence above. Thus, there exists a subset of T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable, using our result about infinite subsets of countable sets. Since each $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus S is countable. ■

Corollary. Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Then

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is at most countable, for T is equivalent to a subset of

$$\bigcup_{n=1}^{\infty} B_n,$$

where $B_n \in A$.

Theorem 1.1.2. Let A be a countable set, and let B_n be the set of all $n-$tuples $(a_1, \ldots, a_n)$, where $a_k \in A (k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_m is countable.

1.2 Metric Spaces

Definition 1.2.1 (Metric Space). A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$ called the *distance* from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

- Suppose a subset Y of X is a metric space, with the same distance function.
- Y must also be a metric space with the same distance function as X (metric).

Definition 1.2.2.

- By the *segment* (a, b) we mean the set of all $x \in \mathbb{R}$ such that $a < x < b$.
- We call an *interval* $[a, b]$ to mean the set of all $x \in \mathbb{R}$ such that $a \leq x \leq b$ for $a, b \in \mathbb{R}$ with $a < b$.

- We call a *half-open interval* either $[a, b)$ or $(a, b]$ to mean $a \leq x < b$ and $a < x \leq b$, respectively.