1 Lecture 1

1.1 Topics

- The derivative
- Continuity and Differentiability
- Differentiability Rules

Definition (Differentiability). (*) Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $c \in I$. We say f is **differentiable** at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number).

(*) In this case, the quantity $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ is called the derivative of f at c and is denoted by

$$f'(c), \frac{df}{dx}(c), \left.\frac{df}{dx}\right|_{x=c}$$

(*) If $f : I \to \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. The following are equivalent characterizations of the differentiability:

$$\begin{aligned}
f'(c) = L &\iff \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \\
&\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon \\
&\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |h| < \delta \text{ then } \left| \frac{f(c+h) - f(c)}{h} - L \right| < \varepsilon \\
&\iff \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L
\end{aligned}$$

Theorem (Differentiability Implies Continuous). Let $I \subseteq \mathbb{R}$, $c \in I$, and $f : I \to \mathbb{R}$ is differentiable at c. Then f is continuous at c.

Proof. It suffices to show that $\lim\limits_{x \to c} f(x) = f(c)$. Note that

$$\begin{aligned}
\lim_{x \to c}(f(x) - f(c)) &= \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c}\right](x - c) \\
&= \left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right]\left[\lim_{x \to c}(x - c)\right] \\
&= (f'(c))(0) \\
&= 0.
\end{aligned}$$

So, we have

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c}[f(x) - f(c) + f(c)] \\
&= \lim_{x \to c}(f(x) - f(c)) + \lim_{x \to c} f(c) \\
&= 0 + \lim_{x \to c} f(c) \\
&= 0 + f(c) \\
&= f(c).
\end{aligned}$$

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Corollary. If $f : I \to \mathbb{R}$ is NOT continuous at $c \in I$, then f is NOT differentiable at c.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(i) Prove that f is continuous at 0.

Proof. Our goal is to show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } |x| < \delta \text{ then } |f(x) - f(c)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Note that if $x \notin \mathbb{Q}$,

$$|f(x)| = |0| < \varepsilon.$$

Otherwise, we have $|f(x)| = |x^2| = |x|^2$. IN this case, we claim that $\delta = \sqrt{\varepsilon}$ will work. Indeed, if $|x| < \delta$, then we have

$$|f(x)| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

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(ii) Prove f is discontinuous at all $x \neq 0$.

Proof. Let $c \neq 0$. Our goal is to show that f is discontinuous at c. By the sequential criterion for continuity, it suffices to find a sequence (a_n) such that $a_n \to c$ but $f(a_n) \not\to f(c)$. We will consider two cases; that is, we could either have $c \notin \mathbb{Q}$ or $c \in \mathbb{Q}$.

Suppose $c \notin \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rational numbers (r_n) such that $r_n \to c$. Note that $f(r_n) = r_n^2 \to c^2 \neq 0$, but $f(c) = 0$. Clearly, $f(r_n) \not\to f(c)$ and so f must be discontinuous at c.

Suppose $c \in \mathbb{Q}$. Since the set of irrational numbers is also dense in $\mathbb{R}$, we can find a sequence (s_n) such that $s_n \to c$. Note that $f(s_n) = 0$, but $f(c) = c^2 \neq 0$. Thus, $f(s_n) \not\to f(c)$. Therefore, f must be discontinuous at c. ■

(iii) Prove that f is nondifferentiable at all $x \neq 0$.

Proof. Let $c \neq 0$. Since f is discontinuous at c, we can conclude that f is not differentiable at c. ■

(iv) Prove that $f'(0) = 0$.

Proof. We need to show

$$\lim_{x \to c} \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = 0.$$

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