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# Linear Algebra Exercises

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# Chapter 1

# Vector Spaces

## 1.1 Linear Combinations

**Exercise 1.4.7**

In $F^n$, let $e_j$ denote the vector whose $j$th coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \dots, e_n\}$ generates $F^n$.

**Proof.** Denote the set $V = \{e_1, e_2, \dots, e_n\}$. Our goal is to find scalars $\delta_1, \delta_2, \dots, \delta_n \in F$ such that for all $e_j \in V$ for $1 \le j \le n$,

$$\delta_1 e_1 + \delta_2 e_2 + \cdots + \delta_n e_n = (a_1, a_2, \dots, a_n). \tag{1}$$

Since $e_j = 1$ for the $j$th coordinate and the rest of the entries are zeros, we have that

$$\delta_j e_j = \delta_j(0, 0, \underbrace{1}_{j\text{th entry}}, \dots, 0) = (0, 0, \delta_j, \dots, 0)$$

for all $1 \le j \le n$. Summing up each term on the left side of (1) and equating each coordinate to the right side of (1), we get that $a_j = \delta_j$ for each $1 \le j \le n$. Hence, the set $V$ can span the vector space $F^n$.

■

**Exercise 1.4.8**

Show that $P_n(F)$ is generated by $\{1, x, \dots, x^n\}$.

**Proof.** Let $V = \{1, x, \dots, x^n\}$. We need to find scalars $\delta_1, \delta_2, \dots, \delta_n \in F$ such that

$$\delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{1}$$

Immediately, we have that equating coefficients in (1) gives us $a_i = \delta_i$ for all $1 \le i \le n$. Hence, $V$ generates $P_n(F)$. ■

**Exercise 1.4.9**

Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate $M_{2\times 2}(F)$.

**Proof.** Define $V$ with the given $2 \times 2$ matrices above. We need to find scalars $\delta_i \in F$ for all $1 \le i \le 4$

such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \delta_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \delta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1}$$

Distributing each scalar $\delta_i$ for all $1 \leq i \leq 4$ for each term in (1), summing up each matrix in (1), and equating each term entry-wise, we have that $a_{11} = \delta_1, a_{12} = \delta_2, a_{21} = \delta_3$, and $a_{22} = \delta_4$. Hence, $V$ spans $M_{2\times 2}(F)$. ■

**Exercise 1.4.10**

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric $2 \times 2$ matrices.

**Proof.** We need to show that the set $\{M_1, M_2, M_3\}$ spans the set of all symmetric $2 \times 2$ matrices. We need to find scalars $a_1, a_2, a_3$ such that

$$a_1 M_1 + a_2 M_2 + a_3 M_3 = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = A \tag{1}$$

where $A$ is any $2 \times 2$ symmetric matrix. Observe that for $i = j$, we have $\delta_{12} = \delta_{21}$. Performing scalar multiplication on matrices $M_1, M_2$, and $M_3$, addition of all three terms on (1), and equating entry-wise, we get that $\delta_{12} = \delta_{21} = \delta_3$ and $\delta_{11} = \delta_1$ and $\delta_2 = a_{22}$. Hence, we have

$$\begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^t = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}.$$

Thus, the set $\{M_1, M_2, M_3\}$ spans the set of all symmetric $2 \times 2$ matrices. ■

**Exercise 1.4.11**

Prove that $\text{span}(\{x\}) = \{ax : a \in F\}$ for any vector $x$ in a vector space $V$. Interpret this result geometrically in $\mathbb{R}^3$.

**Proof.** We need to show that $\text{span}(\{x\}) = \{ax : a \in F\}$, we need to show two containments; that is, $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$ and $\{ax : a \in F\} \subseteq \text{span}(\{x\})$. Let $v \in \text{span}(\{x\})$. Then observe that we can find $\delta \in F$ such that multiplying by $x \in V$ leads $v = \delta x$. But this means that $v \in \{ax : a \in F\}$ by definition. Hence, $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$. Let $v \in \{ax : a \in F\}$. Then $v = ax$ for some $a \in F$. But this is a linear combination of $x$ that makes $v$. So $v \in \text{span}(\{x\})$ and hence, $\{ax : a \in F\} \subseteq \text{span}(\{x\})$. This result can be viewed as the scaling of vectors in $\mathbb{R}^3$. ■

**Exercise 1.4.12**

Show that a subset $W$ of a vector space $V$ is a subspace of $V$ if and only if $\text{span}(W) = W$.

**Proof.** ( $\Rightarrow$) Let $W$ be a subspace of $V$. To show that $\text{span}(W) = W$, we need to show two containments; that is, $\text{span}(W) \subseteq W$ and $W \subseteq \text{span}(W)$. Clearly, $W$ contains itself. Hence, $\text{span}(W) \subseteq W$ Theorem 5. Let $v \in W$. Since $W$ is a subspace, we can find scalars $a_1, a_2, \dots . a_n \in F$ and vectors $w_1, w_2, \dots, w_n \in W$ such that

$$v = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \in W$$

using the result in Exercise 1.3.20. But this tells us that $v \in \text{span}(W)$. Hence, $W \subseteq \text{span}(W)$.

( $\Leftarrow$ ) Since the span of any subset of $W$ is a subspace and $W = \text{span}(W)$, we have that $W$ is a subspace as well by Theorem 5. ■

### Exercise 1.4.13

Show that if $S_1$ and $S_2$ are subsets of a vector space $V$ such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

**Proof.** Let $S_1$ and $S_2$ be subsets of a vector space $V$. Let $v \in \text{span}(S_1)$. We can find scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ and $x_1, x_2, \ldots, x_n \in S_1$ such that

$$v = \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n.$$

Since $S_1 \subseteq S_2$, we know that $x_1, x_2, \ldots, x_n \in S_2$ so we must have $v \in \text{span}(S_2)$. Hence, $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Now, let $\text{span}(S_1) = V$. We need to show that $\text{span}(S_2) = V$; that is, we need to show $\text{span}(S_2) \subseteq V$ and $V \subseteq \text{span}(S_2)$. By assumption, $S_2 \subseteq V$ and $\text{span}(S_2)$ is a subspace. Clearly, $\text{span}(S_2) \subseteq V$. Since $\text{span}(S_1) = V$ and $\text{span}(S_1) \subseteq \text{span}(S_2)$, we have $V \subseteq \text{span}(S_2)$. Hence, $\text{span}(S_2) = V$. ∎

### Exercise 1.4.14

Show that if $S_1$ and $S_2$ are arbitrary subsets of a vector space $V$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sum of two subsets is defined in the exercises of Section 1.3.)

**Proof.** Let $S_1$ and $S_2$ be subsets of a vector space $V$. We need to show $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$; that is, $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$ and $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. Let $v \in \text{span}(S_1 \cup S_2)$. We can find $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that $x_1, x_2, \ldots, x_n \in S_1 \cup S_2$ implies that

$$v = \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n.$$

Hence, either $x_1, x_2, \ldots, x_n \in S_1$ or $x_1, x_2, \ldots, x_n \in S_2$. If $x_1, x_2, \ldots, x_n \in S_1$, then $v \in \text{span}(S_1)$. Since $\text{span}(S_2)$ is a subspace, we know that $O_V \in \text{span}(S_2)$. Hence, $O_V \in \text{span}(S_1)$ and $v \in \text{span}(S_1)$ imply that $v + O_V = v \in \text{span}(S_1) + \text{span}(S_2)$. The other case follows a similar process. Hence, $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$.

Let $s \in \text{span}(S_1) + \text{span}(S_2)$. Hence, $s = u + v$ where $u \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$. The former implies that we can find scalars $a_1, a_2, \ldots, a_n \in F$ such that $x_1, x_2, \ldots, x_n \in S_1$ where

$$u = \sum_{i=1}^{n} a_i x_i$$

and the latter implies that there exists scalars $b_1, b_2, \ldots, b_n \in F$ such that $y_1, y_2, \ldots, y_n \in S_2$ where

$$v = \sum_{i=1}^{n} b_i y_i.$$

Since both $x_i \in S_1$ and $y_i \in S_2$ for all $1 \leq i \leq n$, we have $x_i, y_i \in S_1 \cup S_2$ for all $1 \leq i \leq n$. So we must have $s \in \text{span}(S_1 \cup S_2)$. Hence, $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. ∎

### Exercise 1.4.15

Let $S_1$ and $S_2$ be subsets of a vector space $V$. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_2 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are not unequal.

**Proof.** Let $S_1$ and $S_2$ be subsets of a vector space $V$. Let $v \in \text{span}(S_1 \cap S_2)$. Then we can find scalars $a_i \in F$ and vectors $x_i \in S_i \cap S_2$ for all $1 \leq i \leq n$ such that

$$v = \sum_{i=1}^{n} a_i x_i.$$

If $x_i \in S_1 \cap S_2$ for all $1 \leq i \leq n$, then $x_i \in S_1$ and $x_i \in S_2$ for all $1 \leq i \leq n$. This implies that $v \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$. Hence, $v \in \text{span}(S_1) \cap \text{span}(S_2)$. Thus, we conclude that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. ■

**Example.** Define $S_1$ as the set

$$\{(1,1,0) \in \mathbb{R}^3\}$$

and $S_2$ as the set

$$\{(1,1,0),(1,0,1),(0,1,1) \in \mathbb{R}^3\}.$$

Observe that $S_1 \cap S_2 = \{(1,1,0)\}$ and thus the of this set yields $\{a(1,1,0) : a \in F \text{ and } (1,1,0) \in \mathbb{R}^3\}$. The span of $S_1$ yields the following set $\{a(1,1,0) : a \in F \text{ and } (1,1,0) \in \mathbb{R}^3\}$ and the span of $S_2$ yields the following set

$$\{a(1,10) + b(1,0,1) + c(0,1,1) : a,b,c \in F \text{ and } (1,1,0),(1,0,1),(0,1,1) \in \mathbb{R}^3\}.$$

Note that $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$ since $\text{span}(S_1) \cap \text{span}(S_2) \neq \emptyset$.

Now define $S_1 = \{O_V\} = S_2$. Clearly, $S_1 \cap S_2 = \{O_V\}$, $\text{span}(S_1) \cap \text{span}(S_2) = \{O_V\}$, and that $\text{span}(S_1 \cap S_2)$ is also equal to this set. $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

### Exercise 1.4.16

Let $V$ be a vector space and $S$ a subset of $V$ with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of $S$ can be *uniquely* written as a linear combination of vectors of $S$.

**Proof.** Let $x \in \text{span}(S)$. Suppose there exists two sets of scalars $a_1, a_2, \ldots, a_n \in F$ and $b_1, b_2, \ldots, b_n \in F$ such that whenever $v_1, v_2, \ldots, v_n \in S$ such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = x \tag{1}$$

and

$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = x \tag{2}$$

implies that $a_i$ Setting (1) and (2) together, we have

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n \tag{3}$$

which implies that

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0.$$

Since $v_1, v_2, \ldots, v_n \in S$, we have that $a_i - b_i = 0$ and thus $a_i = b_i$ for all $1 \leq i \leq n$. Hence, every vector in the span of $S$ can be *uniquely* written as a linear combination of vectors of $S$. ■