0.1 Lecture 3

0.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

0.1.2 Review, Existence of Roots

Proposition. There is no rational number whose square is 2.

Theorem. There is a unique positive real number α satisfying $\alpha^2 = 2$.

Proof. (i) **Uniqueness:** Suppose there are two of them α_1 and α_2. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that A is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

■

Remark. A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number α such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

0.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

Definition (Usual Way of Defining Functions)**.** Let A and B be two sets. A **function** from A to B denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

Definition (The Correct Way of Defining Functions)**.** Let A and B be two sets. A function from A to B is a triple (f, A, B) where f is a **relation** form A to B satisfying

(i) $\mathrm{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, A is called the **domain** of f and B is called the **codomain** of f)

Example 0.1.1. Let $A = \emptyset$ and B be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to B is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

Definition (Image, Range, Onto (Surjective)). Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of f as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of f as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say f is **Onto (Surjective)**.

Definition (Preimage). Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of D under f is denoted by

$$f^{-1}(D) = \{x \in A : f(x)\}$$

Definition (One-to-One (Injective)). Consider a function $f : A \to B$. We call f one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of A.

0.1.4 Equivalent Sets

Definition. Let A and B be two sets. We say that A and B have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

Remark.

- An injective and surjective mapping is a bijective mapping.
- A and B have the same cardinal number

$$\begin{aligned}
&= A \text{ and } B \text{ have the same cardinality} \\
&= A \text{ and } B \text{ can be put in the } \textbf{one-to-one correspondence} \\
&= \operatorname{card} A = \operatorname{card} B \\
&= A \text{ and } B \text{ are equivalent} \\
&= A \text{ and } B \text{ are equipotent}
\end{aligned}$$

Example 0.1.2. Consider $\{1, 2, 3\} \sim \{a, b, c\}$. Indeed, the function $f : \{1, 2, 3\} \to \{a, b, c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

Example 0.1.3. $\mathbb{N} \sim \{2, 4, 6, \dots\}$. Indeed, the function $f : \mathbb{N} \to \{2, 4, 6, \dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

Example 0.1.4. $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

Example 0.1.5. $(-\infty, \infty) \sim (0, \infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty, \infty)$ and $(0, \infty)$.

Example 0.1.6. $(0, \infty) \sim (0, 1)$. Indeed, the function $f : (0, \infty) \to (0, 1)$ defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

Example 0.1.7. $[0, 1) \sim (0, 1)$. Indeed, the function $f : [0, 1) \to (0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if} x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

Definition ($\sim$ is an equivalence relation)**.** Let A and B be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}$$
$$\mathbb{N} = \{1, 2, 3, \dots\}$$

0.1.5 Finite, Infinite, Countable, At most countable

Definition (Finite, Infinite, Countable, At most countable)**.** Let A be any set.

(a) We say that A is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number n.

   (*) When $A \sim \mathbb{N}_n$, we say A has n elements and we write $\text{card}(A) = n$.

   (*) Also, we set $\text{card}(\emptyset) = 0$.

(b) The set A is said to be **infinite** if it is not finite.

(c) The set A is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set A is said to be **uncountable** if it is neither countable or finite.

(e) The set A is said to be **at most countable** if it is either finite or countable.

Remark. Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$A \text{ is finite} \Leftrightarrow B \text{ is finite}$$
$$A \text{ is countable} \Leftrightarrow B \text{ is countable}$$
$$A \text{ is uncountable} \Leftrightarrow B \text{ is uncountable}$$

(iii) The union of two finite sets is finite. If A is infinite and B is infinite, then $A \setminus B$ is infinite.

(iv) If A **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.