Math 230B Lecture Notes

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Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Topics

- The derivative
- Continuity and Differentiability
- Differentiability Rules

Definition (Differentiability). (*) Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $c \in I$. We say f is **differentiable** at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number).

(*) In this case, the quantity $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is called the derivative of f at c and is denoted by

$$f'(c), \frac{df}{dx}(c), \left.\frac{df}{dx}\right|_{x=c}$$

(*) If $f : I \to \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. The following are equivalent characterizations of the differentiability:

$$\begin{aligned} f'(c) = L &\iff \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \\ &\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |h| < \delta \text{ then } \left| \frac{f(c+h) - f(c)}{h} - L \right| < \varepsilon \\ &\iff \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L \end{aligned}$$

Theorem (Differentiability Implies Continuous). Let $I \subseteq \mathbb{R}$, $c \in I$, and $f : I \to \mathbb{R}$ is differentiable at c. Then f is continuous at c.

Proof. It suffices to show that $\lim_{x \to c} f(x) = f(c)$. Note that

$$\begin{aligned}
\lim_{x\to c}(f(x) - f(c)) &= \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right](x-c) \\
&= \left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\left[\lim_{x\to c}(x-c)\right] \\
&= (f'(c))(0) \\
&= 0.
\end{aligned}$$

So, we have

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}[f(x) - f(c) + f(c)] \\
&= \lim_{x\to c}(f(x)-f(c)) + \lim_{x\to c} f(c) \\
&= 0 + \lim_{x\to c} f(c) \\
&= 0 + f(c) \\
&= f(c).
\end{aligned}$$

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Corollary. If $f : I \to \mathbb{R}$ is NOT continuous at $c \in I$, then f is NOT differentiable at c.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(i) Prove that f is continuous at 0.
Proof. Our goal is to show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that if } |x| < \delta \text{ then } |f(x) - f(c)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Note that if $x \notin \mathbb{Q}$,

$$|f(x)| = |0| < \varepsilon.$$

Otherwise, we have $|f(x)| = |x^2| = |x|^2$. IN this case, we claim that $\delta = \sqrt{\varepsilon}$ will work. Indeed, if $|x| < \delta$, then we have

$$|f(x)| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

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(ii) Prove f is discontinuous at all $x \neq 0$.
Proof. Let $c \neq 0$. Our goal is to show that f is discontinuous at c. By the sequential criterion for continuity, it suffices to find a sequence (a_n) such that $a_n \to c$ but $f(a_n) \not\to f(c)$. We will consider two cases; that is, we could either have $c \notin \mathbb{Q}$ or $c \in \mathbb{Q}$.

Suppose $c \notin \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rational numbers (r_n) such that $r_n \to c$. Note that $f(r_n) = r_n^2 \to c^2 \neq 0$, but $f(c) = 0$. Clearly, $f(r_n) \not\to f(c)$ and so f must be discontinuous at c.

Suppose $c \in \mathbb{Q}$. Since the set of irrational numbers is also dense in $\mathbb{R}$, we can find a sequence (s_n) such that $s_n \to c$. Note that $f(s_n) = 0$, but $f(c) = c^2 \neq 0$. Thus, $f(s_n) \not\to f(c)$. Therefore, f must be discontinuous at c. ■

(iii) Prove that f is nondifferentiable at all $x \neq 0$.
Proof. Let $c \neq 0$. Since f is discontinuous at c, we can conclude that f is not differentiable at c. ■

(iv) Prove that $f'(0) = 0$.

Proof. We need to show

$$\lim_{x \to c} \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = 0.$$

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Theorem (Algebraic Differentiability Theorem). Assume that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$ where (I is an interval on $\mathbb{R}$). Then

(i) For all $k \in \mathbb{R}$, kf is differentiable at c, and

$$(kf)'(c) = kf'(c)$$

(ii) $f + g$ is differentiable at c, and

$$(f + gk)'(c) = f'(c) + g'(c)$$

(iii) fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) $\frac{f}{g}$ is differentiable at c provided that $g(c) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

Chapter 2

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Week 4

4.1 Lecture 6

4.1.1 Topics

(1) The definition of Riemann-Stieltjes integral

(2) Refinement of partitions

Definition (Almost Disjoint Intervals)**.** We say that two intervals I and J are **almost disjoint** if either $I \cap J$ is empty or $I \cap J$ has exactly one point.

Definition (Partition)**.** A partition P of an interval $[a, b]$ is a finite set of points in $[a, b]$ that includes both a and b. We always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in an increasing order; so,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Remark. A partition of P of an interval $[a, b]$ is a finite collection of almost disjoint (nonempty) compact intervals whose union is $[a, b]$:

$$P = I_1, I_2, \ldots, I_n$$

where

$$I_1 = [x_0, x_1], \quad I_2 = [x_1, x_2], \quad \cdots \quad I_n = [x_{n-1}, x_n].$$

Again, we denote $x_0 = a$ and $x_n = b$.

Definition (Lower Sum, Upper Sum)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded, $\alpha : [a, b] \to \mathbb{R}$ be increasing, and $P = \{x_0, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. Let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$.

(i) The **Lower Riemann-Stieltjes Sum** of f with respect to the integrator α for the partition P is defined by

$$L(f, \alpha, P) = \sum_{k=1}^{n} m_k(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^{n} m_k \Delta\alpha_k.$$

(ii) The upper **Riemann-Stieltjes sum** of f with respect to the integrator α for the partition P is defined by

$$U(f, \alpha, P) = \sum_{k=1}^{n} M_k(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^{n} M_k \Delta\alpha_k.$$

Definition (Upper R.S Integral, Lower R.s Integral). Let $f : [a, b] \to \mathbb{R}$ be bounded, $\alpha : [a, b] \to \mathbb{R}$ be increasing. Then

(i) The **Upper R.S integral** of f with respect to α (on $[a, b]$) is defined by

$$U(f, \alpha) = \inf_{P \in \Pi} U(f, \alpha, P).$$

Note that the set $\{U(f, \alpha, P) : P \in \Pi\}$ is bounded below by $m(\alpha(b) - \alpha(a))$. So the infimum above is a real number.

(ii) The **Lower R.S Integral** of f with respect to α (on $[a, b]$) is defined by

$$L(f, \alpha) = \sup_{P \in \Pi} L(f, \alpha, P).$$

Note that the set $\{L(f, \alpha, P) : P \in \Pi\}$ the lower sums is bounded above by $M(\alpha(b) - \alpha(a))$. So, the supremum above is a real number.

Definition (Riemann-Stieltjes integrable functions). Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann-Stieltjes integrable** (on $[a, b]$) if

(i) f is bounded

(ii) $L(f, \alpha) = U(f, \alpha)$.

In this case, the R.S integral of f with respect to α, denoted by

$$\int_a^b f \, d\alpha \quad \text{or} \quad \int_a^b f(x) \, d\alpha(x) \quad \text{or} \quad \int_{[a,b]} f \, d\alpha$$

is the common value of $L(f, \alpha)$ and $U(f, \alpha)$. That is,

$$\int_a^b f \, d\alpha = L(f, \alpha) = U(f, \alpha).$$

Definition (Refinement of a Partition).

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9.1 Lectures 13-14

Theorem (Integration by Parts). Let $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ are differentiable and let $u' \in R[a, b]$ and $v' \in R[a, b]$. Then we have

(1) $uv' \in R[a, b]$

(2) $u'v \in R[a, b]$

(3) $\int_a^b uv' \, dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v \, dx.$

Proof. (1) Since $u : [a, b] \to \mathbb{R}$ is differentiable, we have $u \in C[a, b]$. So, we have $u \in R[a, b]$. By assumption, $v' \in R[a, b]$ and so we can conclude that $uv' \in R[a, b]$.

(2) Using the same argument above, we have $uv' \in R[a, b]$.

(3) By the product rule, we have

$$(uv)' = u'v + uv'.$$

In particular, since $(uv)'$ is a sum of integrable functions, it belongs to $R[a, b]$. Now, we integrate both sides

$$\int_a^b (uv)' \, dx = \int_a^b u'v \, dx + \int_a^b uv' \, dx. \tag{I}$$

According to FTC I, we have

$$\int_a^b (uv)' \, dx = [uv]_{x=a}^{x=b} = u(b)v(b) - u(a)v(a). \tag{II}$$

Hence, we have (I) and (II) imply that

$$u(b)v(b) - u(a)v(a) = \int_a^b u'v \, dx + \int_a^b uv' \, dx$$

which further implies that

$$\int_a^b uv' \, dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v \, dx.$$

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9.2 Lectures 15-16

9.2.1 Topics

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Chapter 14

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