Understanding Analysis Notes

Lance Remigio

September 7, 2023

Contents

Chapter 1

The Real Numbers

1.1 The Axiom of Completeness

Theorem 1. Every nonempty set of real numbers that is bounded above has a least upper bound.

Definition 1. We call a set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Otherwise, a set is *bounded below* if there exists a $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for every $a \in A$.

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it satisfies the following criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We denote the least upper bound of a set A by calling it the *supremum* of A i.e $\sup(A)$. Similarly, we denote the greatest upper bound of set A by calling it the *infimum* of A i.e $\inf(A)$.

Note that a set can have many upper/lower bounds. But there can only exist one supremum and one infimum. In other words, these bounds are unique. Furthemore, the infimum and supremum need not be in the set.

Consider the following set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

This set is bounded above and below. In addition, we can see that $\sup(A) = 1$ and $\inf(A) = 0$ (this is because each subsequent number in the sequence gets smaller and smaller).

Definition 2. We say that $a_0 \in \mathbb{R}$ is a *maximum* of the set A if $a_0 \in A$ and $a \leq a_0$ for all $a \in A$. Likewise, we say that $a_1 \in \mathbb{R}$ is a *minimum* of A if $a_1 \in A$ and $a \geq a_1$ for every $a \in A$.

If we have an open set $(0, 2)$ then the end points of this set are the infimum and supremum of the set respectively. Note that the maximum and the minimum do not exists because the infimum and the supremum are not in the set. If this set were to be closed, then the supremum and infimum would be in the set which implies that the max and min exists.

Now consider the Example

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

Notice that when we try and search for the supremum for this set, we cannot find one since we can always find a smaller number for an upper bound. One might say that $r = \sqrt{2}$ is the supremum of S but this is false since $r \notin \mathbb{Q}$ and is irrational.

Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and is bounded above. Let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Prove that $\sup(c + A) = c + \sup(A)$

Proof. We use defintion 0.2 to prove this proposition. First, we need to prove that this $\sup(c + A)$ is an upper bound. We have $\sup(A) = s$ for some $s \in A$ if $s \geq a$ for all $a \in A$. We find that adding $c \in \mathbb{R}$ gives us

$$c + s \geq c + a.$$

Hence, we have that $c + s$ is an upper bound for the set $c + A$.

Next, we prove that $\sup(c + A) = c + s$ is the *least upper bound*. We know that $c + s \geq c + a$ for all $a \in A$. Suppose we have another upper bound $b \in A$ such that $c + a \leq b$ for all $a \in A$. Another manipulation gives us $a \leq b - c$ for all $a \in A$. Since $\sup(A) = s$ is the least upper bound for A, it follows that $s \leq b - c$. Hence, we have

$$c + s \leq b \Rightarrow \sup(c + A) = c + \sup(A).$$

■

There is another way to restate part (ii) of defintion 0.2 i.e

Lemma 1. Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $s - \varepsilon < a$.

Proof. For the forward direction, suppose that $s = \sup A$ and consider $s - \varepsilon$. Since s is an upper bound, we have that $s - \varepsilon < s$. This means that $s - \varepsilon$ is not an upper bound. Hence, we can find an element $a \in A$ such that $s - \varepsilon < a$ because otherwise $s - \varepsilon$ would be an upper bound. This concludes the forward direction.

For the backwards direction, assume s is an upper bound. We must satisfy part (ii) of defintion 0.2. Let $\varepsilon > 0$, then $\varepsilon = s - b$. But since any number smaller than s is not an upper bound, we have that $s \leq b$ if b is any other upper bound for S. Hence, $s = \sup A$. ■

Chapter 2

Additional Topics

2.1 A Construction of R from Q

In this section, we will focus on constructing a proof for the existence of the Real Numbers.

Theorem 2 (Existence of Real Numbers)**.** There exists an ordered field in which every nonempty set that is bounded above has a least upper bound. In addition, this field contains $\mathbb{Q}$ as a subfield.

A review of chapter 1 tells us that

(i) We can't do analysis on the set of rational numbers because we don't have the square root defined (nor does it exists) which does not allow us to define the convergence of Cauchy sequences.

A way that we solved this problem is to create the following axiom:

Theorem 3 (Axiom of Completeness)**.** Every nonempty set of real numbers that is bounded above has a least upper bound.

Note that we actually need to demonstrate that we can actually extend the rational numbers to contain nonempty sets that have least upper bounds, thereby making the axiom into a theorem worth proving.

2.1.1 Dedekind Cuts

Let's begin by assuming that all the properties that come with the rational numbers are available to us. Let us also assume that we don't have real numbers at this point.

Definition 3 (Cuts)**.** A subset A of the rational numbers is called a *cut* if it possesses the following three properties:

(c1) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(c2) If $r \in A$, then A also contains every rational $q < r$.

(c3) A does not have a maximum; that is, if $r \in A$, then there exists $s \in A$ with $r < s$.

Exercise 8.6.1

(a) Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{t \in \mathbb{Q} : t < r\}$ is a cut.

Proof. For the first property (C1), we show that $C_r \neq \emptyset$ and $C_r \neq \mathbb{Q}$. Suppose $r \in \mathbb{Q}$ is fixed. Now define $t = r - 1$. Then clearly, we have $t < r$. But this means $t \in C_r$. Hence, $C_r \neq \emptyset$. Now, we show $C_r \neq \mathbb{Q}$. Define $s = r + 1 \in \mathbb{Q}$. Then clearly, $s > r$. Hence, $s \notin C_r$ and thus $C_r \neq \mathbb{Q}$.

For the second property (C2), we need to show that if $r \in C_r$, then C_r also contains every rational $q < r$. Suppose we let $t, q \in \mathbb{Q}$ be arbitrary with $t > q$ with $t \in C_r$. Then by definition of C_r, we must have $t < r$. But we have $q < t < r$. Hence, we have $q < r$ for any $q \in \mathbb{Q}$.

Lastly, we need to show that C_r does not have a maximum; that is, for any $x \in C_r$, there exists an $s \in C_r$ with $x < s$. Let $x \in C_r$. Then by definition of C_r, we must have $x < r$. Suppose we take the midpoint between x and r. Then we must have $x < \frac{x+r}{2} < r$. Setting $s = \frac{x+r}{2}$ gives us our desired result. ■

Avoid thinking of cuts as only having this form. Which of the following subsets of $\mathbb{Q}$ are cuts?

(b) $S = \{t \in \mathbb{Q} : t \leq 2\}$

Proof. This is not a cut because S contains a maximum. In this case, the maximum is 2. ■

(c) $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$

Proof. The set T is a cut. First, we show (C1). Observe that $0 \in T$. Hence, $T \neq \emptyset$. Then observe that $2 \notin T$. Hence, $T \neq \mathbb{Q}$.

Next, we show (C2). Let $r \in T$. Then by definition of T, we must have either $r^2 < 2$ or $r < 0$. If $r < 0$, then we can easily pick $q < r$ for any $q < 0$. Otherwise, we have $r^2 < 2$ such that if we let $q \in \mathbb{Q}$ be arbitrary with $q^2 < r^2$. Then we easily have $q < r$.

Finally, we show (C3). Let $r \in T$. Then either we have $r^2 < 2$ or $r < 0$. Consider

$$\begin{aligned}\left(r + \frac{1}{n}\right)^2 &= r^2 + \frac{2r}{n} + \frac{1}{n} \\ &= r^2 + \frac{2r+1}{n}.\end{aligned}$$

Let us set $s = r + \frac{1}{n_0}$. If $r > 0$, then we can pick an n_0 large enough such that

$$\frac{1}{n_0} < \frac{4 - r^2}{2r+1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < r^2 + \frac{2r+1}{n_0} \cdot \frac{4-r^2}{2r+1} < 4.$$

Hence, we have $r < s < 2$. Otherwise, we choose n_0 large enough so that

$$\frac{1}{n_0} < \frac{-r^2}{2r+1}.$$

Then we have

$$\left(r + \frac{1}{n_0}\right)^2 < r^2 + \frac{2r+1}{n_0} < 0.$$

This implies that $r < s < 0$. ■

(d) $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$.

Proof. ■

Exercise 8.6.2

Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Proof. Suppose for sake of contradiction that $r \geq s$. Since $r \in A$ and $s \in \mathbb{Q}$, we know that $s < r$. But this means that $s \in A$ which leads to a contradiction. ■

Definition 4. Define the *real numbers* $\mathbb{R}$ to be the set of all cuts in $\mathbb{Q}$.

Here we have defined the set $\mathbb{R}$ whose elements are subsets of $\mathbb{Q}$ which is awkward because we think of numbers as elements of a set rather than the sets being elements themselves. Our goal now is to create an algebraic structure on $\mathbb{R}$. This involves answering the following question

"What do we mean by an ordered field?".

2.1.2 Field and Order Properties

Suppose we were given a set F and two elements $x, y \in F$. The *operation* that we impose on F takes x and y as an ordered pair (x, y) and maps it to a third element $z \in F$. Here we are trying to emulate our basic notions of adding and multiplying two numbers.

Definition 5 (Field)**.** A set F is a *field* if there exists two operations — addition $(x + y)$ and multiplication (xy) — that satisfy the following list of conditions:

(f1) (commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

(f2) (associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

(f3) (identities exist) There exist two special elements 0 and 1 with $0 \neq 1$ such that $x + 0 = x$ and $1 \cdot x = x$ for all $x \in F$.

(f4) (inverses exist) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$. If $x \neq 0$, there exists an element x^{-1} such that $xx^{-1} = 1$.

(f5) (distributive property) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Exercise 8.6.3

Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$, respectively.

Proof. ($\mathbb{N}$) This is not a field since an additive inverse does not exist; that is, $-1 \notin \mathbb{N}$.

($\mathbb{Z}$) The set of integers is not a field because every element in $x \in \mathbb{Z}$ does not contain an inverse except for 1.

($\mathbb{Q}$) Field.

■

Just an aside, we can actually use the field properties to show, for example, that for any $x, y, z \in \mathbb{Q}$, we have that $x + y = x + z$ implies $y = z$.

Definition 6. Ordering An *ordering* on a set F is a relation, represented by $\leq$, with the following three properties:

For arbitrary $x, y \in F$, at least one of the statements $x \leq y$ or $y \leq x$ is true.

(o2) If $x \leq y$ and $y \leq x$, then $x = y$.

(o3) If $x \leq y$ and $y \leq z$, then $x \leq z$. Note that writing $y \geq x$ is equivalent to $x \leq y$. The strict inequality $x < y$ is used to mean $x \leq y$ but $x \neq y$.

(o4) If $y \leq z$, then $x + y \leq x + z$.

(o5) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

We have thus far defined $\mathbb{R}$ to be the collection of $\mathbb{Q}$ cuts. Now we want to invent the ordering and basic operations outlined in the last two definitions. The easiest out of these is the ordering. Let A and B be two arbitrary elements of $\mathbb{R}$. Define $A \leq B$ to mean $A \subseteq B$.

Exercise 8.6.4

Show that this defines an ordering on $\mathbb{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Proof. Observe that the first property (o1) follows immediately. For the second property (o2), suppose $A \leq B$ and $B \leq A$. Then we have $A \subseteq B$ and $B \subseteq A$. This implies that $A = B$. Finally, for the third property (o3), let $A, B, C \in \mathbb{R}$. If $A \leq B$ and $B \leq C$, then we have $A \subseteq B$ and $B \subseteq C$. Then we have $A \subseteq B \subseteq C$. Hence, we have $A \subseteq C$. Hence, we have an ordering on $\mathbb{R}$. ■

2.1.3 Algebra in R

Given A and B in $\mathbb{R}$, define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Before trying to verify if $A + B$ defines an operation, we have first ask if $A + B$ defines a cut. Suppose we verify (c2) of our definition of a cut. Suppose A, B are cuts. Let $a + b \in A + B$ be arbitrary and let $s \in \mathbb{Q}$ satisfy $s < a + b$. Then, $s - b < a$, which implies that $s - b \in A$ because A is a cut. But then

$$s = (s - b) + b \in A + B,$$

and (c2) is proved.

Exercise 8.6.5

(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

Proof. For property (c1), let $a \in A$. Then $a = (a - b) + b$ where $a - b \in A$ and $b \in B$ implies $a \in A + B$. Hence, we have that $A + B \neq \emptyset$. Let $c \in C$ where $C \subseteq \mathbb{Q}$, then we find that $A + C \not\subseteq A + B$. Hence, the sum $a + c \notin A + B$. Hence, $A + B \neq \mathbb{Q}$. To show property (c3), Let $a + b \in A + B$ be arbitrary. Since A is a cut, we can find an $\alpha \in A$ such that $a < \alpha$. Likewise, B being a cut implies that there exists $\beta \in B$ such that $b < \beta$. Adding the two inequalities together, we get that $a + b < \alpha + \beta$. Denote the set $S = \{\alpha + \beta : \alpha \in A \text{ and } \beta \in B\}$. Then we have $A + B \subseteq S$ which proves property (c3). Since properties (c1), (c2), and (c3) have been satisfied, we can now conclude that $A + B$ is a cut. ■

(b) Check that addition in $\mathbb{R}$ is commutative (f1) and associative (f2).

Proof. Since $A + B$ is a cut and its elements $a + b$, where $a \in A$ and $b \in B$ are both cuts, subsets of $\mathbb{Q}$ where $\mathbb{Q}$ is an ordered field, we know that addition must be commutative and associative. This means that

$$A + B = B + A.$$

If we have an extra set C that is also a cut, then we would have $(a + b) + c = a + (b + c)$ implying that

$$(A + B) + C = A + (B + C).$$

Hence, we have (f1) and (f2) are satisfied. ■

(c) Show that property (o4) holds.

Proof. Suppose we have $A \subseteq C$ where A and C are both cuts. Let B be a cut. Then $A \subseteq C$ implies $a \leq c$. If we have $b \in B$ be arbitrary, then we can use the ordering of $\mathbb{Q}$ to write $b + a \leq b + c$. But this means that $B + A \subseteq B + C$ and hence property (o4) is satisfied. ■

(d) Show that the cut

$$O = \{p \in \mathbb{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions : $A + O \subseteq A$ and $A \subseteq A + O$.)

Proof. First, we show $A + O \subseteq A$. Let $a + p \in A + O$ be arbitrary. Since $p < 0$, we must have $a + p \leq a$. But this means that $A + O \subseteq A$. For the second inclusion $A \subseteq A + O$, let $a \in A$. Then observe that $a = (a - p) + p$. Since $p < 0$, we can write $a = (a - p) + p$. But since $a - p \in A$ and $p \in O$, we know that $a \in A + O$. Hence, we have $A \subseteq A + O$. ■

Suppose we try and prove additive inverses? Given an $A \in \mathbb{R}$, we need to construct a cut $-A$ such that $A + (-A) = O$. Without the help of the supremum, which we can't even use because it is strictly defined on $\mathbb{R}$ without inherently assuming that it exists, how do we go about defining this set?

Suppose we are given $A \in \mathbb{R}$, then define

$$-A = \{r \in \mathbb{Q} : \text{ there exists } t \notin A \text{ with } t < -r\}.$$

Exercise 8.6.6

(a) Prove that $-A$ defines a cut.

Proof. To show the (c1), let $r \in \mathbb{Q}$. Since $\mathbb{Q}$ is a field, we can rewrite r in the following way

$$r = r + (t - t) = (r + t) - t = (t + r) - t < -t$$

with $t \notin A$. Multiplying through the inequality by a negative leads us to $-r > t$. This tells us that $r \in -A$ and we must have $-A \neq \emptyset$. However, this also implies that $-r \notin -A$ and hence, $-A \neq \mathbb{Q}$. To show (c2), let $r \in -A$. Then by definition of $-A$, there exists a $t \notin A$ such that $-r > t$. Suppose we let $-q \in \mathbb{Q}$ be arbitrary with $-q > -r$. Using the fact that $\mathbb{Q}$ is a field, we can multiply the inequality by a negative to get $r > q$ our desired result. Hence, (c2) is satisfied.

Finally, we show (c3). Suppose $r \in -A$. Then there exists $t \notin A$ such that $-r > t$. Multiplying through the inequality by a negative gives us $r < -t$ which is our desired result. Hence, $-A$ defines a cut. ∎

(b) What goes wrong if we set $-A = \{r \in \mathbb{Q} : -r \in A\}$?

Proof. If we have $-A$ defined as the set above, then it would fail (c3). We can see this when we define $A = \{r \in \mathbb{Q} : r < 0\}$ and let $-A = \{r \in \mathbb{Q} : r \leq 0\}$ which tells us that $-A$ contains a maximum.

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in the definition of a field.

Proof. Let $a \in A$ and $r \in -A$. Let $a + r \in O$. Since $r \in -A$, we know that there exists $t \notin A$ with $t < -r$. Now, we can write $a + r < a - t$. But $-t < 0$ so this tells us that $a + r < 0$. Hence, $a + r \in O$ and we have $A + (-A) \subseteq O$. Now we need to show $O \subseteq A + (-A)$. Let us fix $o \in O$ and finding $a \in A$ such that $a + b = o$. Let $\varepsilon = |o|/2 = -o/2$. Taking advantage of properties (c1) and (c2), we can choose a rational $t \notin A$ such that $t - \varepsilon \in A$. If no such t existed then we would either have $A = \mathbb{Q}$ or $A = \emptyset$. Now, $t \notin A$ implies $-(t + \varepsilon) \in -A$. Then

$$o = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A),$$

and hence we conclude $O \subseteq A + (-A)$. Hence, (f4) is proven. ∎

Remark. Another possible way you could show the reverse inclusion in part (c) is to rewrite $o \in O$ such that $o = (o + r) - r$ and show that $0 + r \in -A$ for which, in addition to $-r \in A$, leads to $O \subseteq A + (-A)$. Since there exists $t \notin A$ such that $-r > t$ and $o < 0$, we have

$$o + r < r \Rightarrow o + r < r < -t.$$

Multiplying the inequality by a negative gives us $-(o + r) < t$, which tells us that $o + r \in -A$. Since we also have $-r \in A$, we conclude that $o \in A + (-A)$ and hence $O \subseteq A + (-A)$.

Suppose we try to create the structure for multiplication using cuts. This can be quite difficult because of the fact that the product of two negative numbers is positive. A way to do this is to define multiplication on non-negative cuts.

Given $A \geq O$ and $B \geq O$ in $\mathbb{R}$, define the product

$$AB = \{ab : a \in A, b \in B \text{ with } a, b \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

Exercise 8.6.7

(a) Show that AB is a cut and that property (o5) holds.

Proof. To show (c1), let $a, b \in \mathbb{Q}$ with the property that $a, b \geq 0$. Since $\mathbb{Q}$ is a field, we can take the product of a and b and get $a \cdot b \geq 0$ implying that $AB \neq \emptyset$. Likewise, if we take any $p \in \mathbb{Q}$ with $p < 0$ and multiply it by any strictly non-negative $a \in \mathbb{Q}$, then $p \cdot a < 0$ implying that $AB \neq \emptyset$ once again. We know that $AB \neq \emptyset$ since $q \leq 0$ is not in AB. Hence, (c1) is satisfied.

To show (c2), we need to show that for all $q \in \mathbb{Q}$ that $r \in AB$ implies $q < r$. Suppose $r = a \cdot b$ with $a \in A$ and $b \in B$. Since A and B are cuts, we can use (c2) to state that for all $\alpha \in A$ and for all $\beta \in B$, we have such $\alpha < a$ and $\beta < b$ respectively. Then taking the product of these inequalities, we get our desired result that $q = \alpha \cdot \beta < a \cdot b = r$. Hence, (c2) is satisfied.

Finally, we show (c3). Let $r \in AB$ where $r = a \cdot b$ where $a \in A$ and $b \in B$. Since A and B are cuts, we can find $w \in A$ and $z \in B$ such that $r = a \cdot b < z \cdot w = \omega$. Hence, (c3) is satisfied.

Now we need to show that property (o5) holds for AB. Assume $A \geq O$ and $B \geq O$. We know that $O \subseteq A$ and $O \subseteq B$. This means that if we let $p, w \in O$, then $p, w \in A$ and $p, w \in B$. Note that $p \cdot w < 0$. Since $ab \geq 0$, we know that $pw \leq ab$ which implies that $pw \in AB$ and hence, $O \subseteq AB$. ■

(b) Propose a good candidate for the multiplicative identity on $\mathbb{R}$ and show that this works for all cuts $A \geq O$.

Proof. We propose the following candidate for the multiplicative identity on $\mathbb{R}$: For $A \geq O$, we have

$$A^{-1} = \{a \in \mathbb{Q} : \exists p \in \mathbb{Q} \Rightarrow a \cdot p^{-1} < 1 \text{ and } p > 0 \text{ with } p^{-1} \notin A\} \cup \{q \in \mathbb{Q} : q < 0\}.$$

■

(c) Show the distributive property (f5) holds for non-negative cuts. Before moving on to the proof, define $A(B+C)$ as

$$A(B+C) = \{a(b+c) : a \in A, b+c \in B+C \text{ with } a, b+c \geq 0\} \cup \{p \in \mathbb{Q} : p < 0\}.$$

Then let

$$P = \{a(b+c) : a \in A, b+c \in B+C \text{ with } a, b+c \geq 0\}.$$

Proof. The main goal is to show that $A(B+C) = AB + AC$; that is, we need to show the following two inclusions:

$$A(B+C) \subseteq AB + AC \text{ and } AB + AC \subseteq A(B+C).$$

Since $x \in A(B+C)$, then either $x \in P$ or $x \in O$. Suppose we have $x \in P$. Then we have $x = a(b+c)$. Since $\mathbb{Q}$ is an ordered field, we can use the distributive property to write $a(b+c) = ab + ac \in AB + BC$. Hence, $a(b+c) \in AB + BC$ and $A(B+C) \subseteq AB + BC$. Now suppose $x \in O$, then $x < 0$. Since the products AB and BC are non-negative, we know that we must have $ab \geq 0$ and $bc \geq 0$. This tells us that their sum $ab + ac \geq 0$ and hence, we have $x < 0 \leq ab + ac$. Since $x \leq ab + ac$, we can write $O \subseteq AB + AC$.

Next, we show $AB + BC \subseteq A(B+C)$. Let $x \in AB + BC$. Then $x = r_1 + r_2$ where $r_1 \in AB$ and $r_2 \in BC$. If $r_1 \in AB$, then either $r_1 = ab \geq 0$ or $r_1 < 0$. Likewise, $r_2 \in BC$ implies that either $r_2 = ac \geq 0$ or $r_2 < 0$. Suppose $r_1 = ab \geq 0$ and $r_2 = ac \geq 0$. Then we can simply use the distributive property to write $ab + ac = a(b+c) \in A(B+C)$. Hence, $ab + ac \in A(B+C)$. Now suppose $r_1 \geq 0$ and $r_2 < 0$. Then $x = r_1 + r_2 < r_1 + 0$. But note that $AB + O \subseteq O$ which tells us that $x \in A(B+C)$. The case for when $r_2 \geq 0$ and $r_1 < 0$ is similar. If both $r_1 < 0$ and $r_2 < 0$, then $x = r_1 + r_2 < 0 \leq a(b+c)$ and hence, $x \in A(B+C)$. Thus, we can conclude that $AB + BC \subseteq A(B+C)$. ■

We can define products of two positive cuts by observing that $-A \geq O$ whenever $O \leq A$. In addition, given any $A \leq O$, we can use property (o4) to write $A + (-A) \leq O + (-A)$ which results in $O \leq -A$. Now let $A, B \in \mathbb{R}$ and define the following possible cases for which we can multiple A and B:

$$AB = \begin{cases} \text{as given} & \text{if } A \geq O \text{ and } B \geq O \\ -[A(-B)] & \text{if } A \geq O \text{ and } B \geq O \\ -[(-A)B] & \text{if } A < O \text{ and } B \geq O \\ (-A)(-B) & \text{if } A < O \text{ and } B < O. \end{cases}$$

We leave these as exercises for you to do. The proofs follow the same pattern as the ones we have done so far with cases usually falling into when terms are either positive or negative.

2.1.4 Least Upper Bounds

We can now use the fact that $\mathbb{R}$ is an ordered field to show that the field is complete. Recall in Chapter 1 that we define "complete" in terms of least upper bounds. Below we restate the definitions seen in Chapter 1.

Definition 7. A set $\mathcal{A} \subseteq \mathbb{R}$ is *bounded above* if there exists a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in \mathcal{A}$. The number B is called an *upper bound* for $\mathcal{A}$.

A real number $S \in \mathbb{R}$ is the *least upper bound* for a set $\mathcal{A} \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) S is an upper bound for $\mathcal{A}$ and

(ii) if B is any upper bound for $\mathcal{A}$, then $S \leq B$.

Exercise 8.6.8

Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

(a) First, prove that $S \in \mathbb{R}$ by showing that it is a cut.

Proof. First, we show (c1). Since S is defined as the union of nonempty sets A, we also have that $S \neq \emptyset$. Since all the sets within S are just a union of cuts, we know that these cuts also have the property that $A \neq \mathbb{Q}$ and hence $S \neq \mathbb{Q}$.

Next, we show property (c2). Let $x \in S$. Since S is defined as the union of all cuts $A \in \mathcal{A}$, we know that $x \in S$ implies that there exists an A' such that $x \in A'$. Let $q \in \mathbb{Q}$ But since A' is a cut, we know that we can use (c2) to say that $q < x$. Hence, (c2) is satisfied.

Lastly, we show property (c3). Let $x \in S$. Using the same reasoning to show (c2), we know that $x \in A'$ implies that there exists $\alpha \in Q$ such that $x < \alpha$.

Since all the properties of a cut are satisfied, we can now conclude that $S \in \mathbb{R}$ is also a cut. ■

(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Proof. In order to show that S is the least upper bound of $\mathcal{A}$, we need to show that S is an upper bound and that S is the least upper bound.

First, we show S is an upper bound; that is, $S \geq A$ for all $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be arbitrary. Observe that since $S = \bigcup A$, we know that $A \subseteq \bigcup A = S$. But this tells us that $A \leq S$. Since A is arbitrary, we know that S must be an upper bound.

Lastly, we show that S is the *least upper bound*. Let $B \in \mathcal{A}$ be any upper bound. Since each set A in the union $S = \bigcup A$ is bounded by B (that is, $A \subseteq B$), it follows that $S \subseteq B$. But this tells us that $S \leq B$ and hence, S must be the least upper bound of $\mathcal{A}$. ■

This finishes the proof that $\mathbb{R}$ is complete. However, we need to solve the problem of the slight mistake in saying that $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. We can solve this issue by saying that $\mathbb{R}$ contains a subfield that looks and acts exactly like $\mathbb{Q}$.

Exercise 8.6.9

Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbb{Q} : t < r\}$$

where $r \in \mathbb{Q}$. (See Exercise 8.6.1)

(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbb{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

Proof. Let $r, s \in \mathbb{Q}$. First, we show $C_r + C_s = C_{r+s}$; that is, we need to show that following two inclusions:

$$C_r + C_s \subseteq C_{r+s} \text{ and } C_{r+s} \subseteq C_r + C_s.$$

Let $z \in C_r + C_s$. Then we have $z = x + y$ where $x \in C_r$ and $y \in C_s$. By definition of C_r and C_s, we know that $x < r$ and $y < s$ respectively. Hence, adding both of these inequalities produces $z = x + y < r + s$ which tells us that $z \in C_{r+s}$. Thus, we have $C_r + C_s \subseteq C_{r+s}$.

Now, we show the reverse inclusion. Let $z \in C_{r+s}$. Then $z < r + s$ by definition of C_{r+s}. Note that subtract s from both sides of the inequality to get $z - s < r$. This tells us that $z - s \in C_r$. Likewise, subtract r from both sides of the inequality to get $z - r < s$ which implies that $z - r \in C_s$. We can add these two distinct elements to get

$$(z - r) + (z - s) = 2z - (r + s) < 2(r + s).$$

Dividing by 2 then gives us

$$z - \frac{r + s}{2} < r + s.$$

Hence, $z \in C_r + C_s$. and we conclude that $C_{r+s} \subseteq C_r + C_s$.

Let $r, s \geq 0$ in $\mathbb{Q}$. We want to show $C_r C_s = C_{rs}$. To do this, we need to show the following two inclusions; that is,

$$C_r C_s \subseteq C_{rs} \text{ and } C_{rs} \subseteq C_r C_s.$$

First, let $x \in C_r C_s$. Then either $x = \alpha\beta$ where $\alpha \cdot \beta \geq 0$ where $\alpha \in C_r$ and $\beta \in C_s$ or $x < 0$. If we assume the former, we can take $\alpha \in C_r$ imply $\alpha < r$ and $\beta \in C_s$ imply $\beta < s$. Multiplying these two inequalities together, we must have $\alpha\beta < rs$ which implies $x \in C_{rs}$ and hence $C_r C_s \subseteq C_{rs}$. Suppose $x < 0$. Since $r, s \geq 0$ implies $rs \geq 0$, we know that $x < 0 < rs$. Hence, we must have $x \in C_{rs}$.

Now, we show the reverse inclusion. Suppose $x \in C_{rs}$, then $x < rs$ for $r, s \geq 0$ in $\mathbb{Q}$. ■

(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbb{Q}$.

Proof. Suppose $r \leq s$ in $\mathbb{Q}$. Let $z \in C_r$. Using the fact that $r \leq s$ and using definition of C_r, we have $z < r \leq s$. This tells us that $z \leq s$ which subsequently tells us that $z \in C_s$. Hence, $C_r \subseteq C_s$ which is equivalent to $C_r \leq C_s$. Now, let $C_r \leq C_s$ and let $z \in C_r$ be arbitrary. By definition of C_r, we have $z < r$. ■