0.1 Lectures 13-14

Theorem (Integration by Parts)**.** Let $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ are differentiable and let $u' \in R[a, b]$ and $v' \in R[a, b]$. Then we have

(1) $uv' \in R[a, b]$

(2) $u'v \in R[a, b]$

(3) $\displaystyle\int_a^b uv'\ dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v\ dx.$

Proof. (1) Since $u : [a, b] \to \mathbb{R}$ is differentiable, we have $u \in C[a, b]$. So, we have $u \in R[a, b]$. By assumption, $v' \in R[a, b]$ and so we can conclude that $uv' \in R[a, b]$.

(2) Using the same argument above, we have $uv' \in R[a, b]$.

(3) By the product rule, we have

$$(uv)' = u'v + uv'.$$

In particular, since $(uv)'$ is a sum of integrable functions, it belongs to $R[a, b]$. Now, we integrate both sides

$$\int_a^b (uv)'\ dx = \int_a^b u'v\ dx + \int_a^b uv'\ dx. \tag{I}$$

According to FTC I, we have

$$\int_a^b (uv)'\ dx = [uv]_{x=a}^{x=b} = u(b)v(b) - u(a)v(a). \tag{II}$$

Hence, we have (I) and (II) imply that

$$u(b)v(b) - u(a)v(a) = \int_a^b u'v\ dx + \int_a^b uv'\ dx$$

which further implies that

$$\int_a^b uv'\ dx = u(b)v(b) - u(a)v(a) - \int_a^b u'v\ dx.$$

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0.2 Lectures 15-16

0.2.1 Topics