Linear Algebra Notes

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Contents

Chapter 1

Vector Spaces

1.1 Linear Dependence and Linear Independence

1.1.1 Motivation

Suppose V is a vector space over a field F and that $W \subseteq V$ is a subspace of V.

- Suppose we have a set S that will generate every vector in V as a linear combination of vectors in S.
- It is desirable to find a subset of S that is as small as possible or rather just enough to generate each vector in V.
- The reason for this is to reduce the amount of computations done to represent a vector in V.
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in S.
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in S.
- Doing this, guarantees that we can write any one of the vectors in S as a linear combination of the others.

Definition 1 (Linearly Dependence)**.** A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors in $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case, we also say that the vectors of S are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $v_1, v_2, \ldots, v_n \in S$ where for all $1 \leq i \leq n$, we have $a_i = 0$.
- This tells us that our definition of **Linear Dependence** implies that 0 is a non-trivial linear combination.
- Any subset of V that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is, $1 \cdot 0 = 0$.

1.1.2 Examples of Linearly Dependent Sets

Example. Consider a subset in $\mathbb{R}^4$ defined by

$$S = \{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4),(-1,0,1,0)\}.$$

To show that S is a linear dependent set, we need to find scalars $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that a_i for all $1 \leq i \leq 4$ not all zero such that

$$a_1(1,3,-3,2) + a_2(2,2,-4,0) + a_3(1,-3,2,-4) + a_4(-1,0,1,0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients $a_1 = 4, a_2 = -3, a_3 = 2$, and $a_4 = 0$. Thus, we have that S is linearly dependent subset of $\mathbb{R}^4$ and hence we can write any vector in S as a linear combination of the other vectors contained in S.

Example. Define a subset of $M_{2\times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} \}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients a_1, a_2, a_3 such that $a_1 = 5, a_2 = 3$, and $a_3 = -2$ where

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 2 (Linear Independence)**.** A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

1.1.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is, $\{v\}$ is linearly independent.
- If $\{v\}$ is linearly dependent, then it the singleton has to be the zero vector 0. This is because $au = 0$
$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

Example (A Set of Vectors in $\mathbb{R}^4$)**.** It can be shown that the set

$$S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of 0 represent the trivial representation; that is, we have scalars $a_1 = a_2 = a_3 = a_4 = 0$.

Example. For $k = 0, 1, \ldots, n$, let $p_k = x^k + x^{k+1} + \cdots + x^n$. The set

$$\{p_0(x), p_1(x), \ldots p_n(x)\}$$

is linearly independent in $P_n(F)$. It can be shown that for some scalars $a_0, a_1, \ldots, a_n \in F$, the following equation

$$a_0 + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n = 0$$

has the trivial-representation; that is, $a_i = 0$ for all $1 \leq i \leq n$.

Theorem 1. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Proof. Let V be a vector space. Suppose S_1 is linearly dependent. Then there exists a finite number of distinct vectors $v_1, v_2, \dots, v_n \in S_1$ and scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Since $S_1 \subseteq S_2$, we must have $v_1, v_2, \dots, v_n \in S_2$ as well. If these vectors are linearly dependent, then these vectors are also linearly dependent in S_2; that is, we have scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Hence, S_2 is linearly dependent. ∎

Corollary. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof. Note that this corollary is just the contrapositive of the theorem before it. Hence, S_1 is linearly independent. ∎

- Determining whether there exists a minimal generating set for the span of S is related to solving the problem of finding whether some vector in S that can be written in terms of a linear combination of other vectors in S.
- We can see that in a given subset of $\mathbb{R}^3$ defined by

  $$S = \{u_1, u_2, u_3, u_4\}$$

  where $u_1 = (2, -1, 4), u_2 = (1, -1, 3), u_3 = (1, 1, -1)$, and $u_4 = (1, -2, 1)$. Note that this S is linearly dependent. This tells us that any vector in S can be written as a linear combination of the others. Suppose we pick u_3 and write as a linear combination of the vectors u_1, u_2, and u_4. We would find that the span of these three vectors (denote this set as S') generates the same spanning set S but with a cardinality one less than S; that is, $\text{span}(S) = \text{span}(S')$.
- If we find that there does not exist a proper subset that is equivalent to the span of S, then S must be a linearly independent set.

Theorem 2. Let S be a linearly independent subset of a vector space V, and let $v \in V$ but not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof. ∎