

Math 234A Lecture Notes

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Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either \mathbb{R} or \mathbb{C} . For example, the polynomial $x^2 + 1$ does not have a solution in \mathbb{R} , but it does have a solution in \mathbb{C} .
- Solving real integrals that may be difficult to deal with using standard techniques developed in \mathbb{R} ; that is, something like

$$\int_0^\infty \frac{\sin x}{x} dx.$$

- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system \mathbb{R} .

- (i) $(\mathbb{R}, +, \cdot)$ is a field.
- (ii) We have an order relation on \mathbb{R} .
- (iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in \mathbb{R} .

Our goal is to find the "smallest" field \mathbb{C} such that

- (i) \mathbb{R} is "contained" in \mathbb{C} .
- (ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in \mathbb{C} .

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of \mathbb{R} as a vector space, and using the operations defined on that vector space to define the operations of \mathbb{C} . Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i . In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on \mathbb{C} .

Definition (Operations on \mathbb{C}). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication \cdot , in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that \mathbb{C} forms a field.

Proposition. The defined operations of \mathbb{C} form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \mathbb{C}$. Then there exists $\gamma + i\delta \in \mathbb{C}$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane \mathbb{C} as a working model for complex numbers.
- Discuss the Euclidean Topology on \mathbb{C} .
- Discuss polar representation of a complex number.

1.3 Complex Plane \mathbb{C} as a working model for Complex Numbers

Lemma. Let F be a field containing \mathbb{R} and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

- (i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} is a **subfield** of F .

- (ii) Let F be another field containing \mathbb{R} and containing a solution i' of $x^2 + 1 = 0$ and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} and \mathbb{C}' are isomorphic as fields.

The second part of this lemma is simply saying that \mathbb{C} is a unique subfield of F .

1.3.1 A model for \mathbb{C}

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication \cdot as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers \mathbb{C} have the following properties

- (i) $(\mathbb{C}, +, \cdot)$ is a field.
- (ii) Let $\mathcal{J} : \mathbb{R} \rightarrow \mathbb{C}$ by $x \rightarrow (x, 0)$. Then \mathcal{J} is a subfield and forms an isomorphism between \mathbb{R} and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

- (i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.
- (ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homomorphism.
- (iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that \mathcal{J} is a bijective map.

■

Definition (Constructing a solution for $x^2 + 1 = 0$). Define $i = (0, 1)$ as our imaginary number in \mathbb{C} and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number). Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z , respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate). Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\bar{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the x -axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

- (i) $\bar{\bar{z}} = z$.
- (ii) $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
- (iii) $\overline{zw} = \bar{z} \cdot \bar{w}$.
- (iv) $\Re(z) = \frac{1}{2}(z + \bar{z})$.
- (v) $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
- (vi) $z\bar{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number). Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\bar{z}}$.

Proposition (More Properties of Complex Numbers). Given $z, w \in \mathbb{C}$, we have the following properties:

- (i) $|z| = 0$ if and only if $z = 0$.
- (ii) $|zw| = |z||w|$.
- (iii) $|\Re(z)| \leq |z|$.
- (iv) $|\Im(z)| \leq |z|$.
- (v) $|z + w| \leq |z| + |w|$.
- (vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

1.3.2 Viewing \mathbb{C} as a Metric Space

In \mathbb{C} , the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space). For $z, w \in \mathbb{C}$, we call \mathbb{C} a metric space if it satisfies the following properties:

- (i) $d(z, w) = d(w, z)$.
- (ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have \mathbb{C} is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of Complex Numbers
- Convergence of Sequences in \mathbb{C}

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^\bullet = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos \varphi, \sin \varphi)$$

with $\tan \varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos \psi, \sin \psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π .

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}^\bullet$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos \varphi + i \sin \varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^\bullet$, then

$$z = \gamma(\cos \varphi + i \sin \varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π .

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of z** .

Definition (Argument and Principle Argument). Let $z \in \mathbb{C}^*$ and $z = \gamma(\cos \varphi + i \sin \varphi)$ be a polar representation of z . Then φ is called **an argument of z** . If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of z** and it is denoted by $\text{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos \varphi + i \sin \varphi)$ and $w = \gamma'(\cos(\varphi') + i \sin(\varphi'))$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \gamma\gamma'[\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos \varphi + i \sin \varphi)(\cos \varphi' + i \sin \varphi') \\ &= \gamma\gamma'[(\cos \varphi \cos \varphi' - \sin \varphi \sin \varphi') + i(\sin \varphi \cos \varphi' + \sin \varphi' \cos \varphi)] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')). \end{aligned}$$

■

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^*$ with $z = \gamma(\cos \varphi + i \sin \varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i \sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos \varphi - i \sin \varphi). \end{aligned}$$

Corollary (De Moivre's Theorem). Let $z = \gamma(\cos \varphi + i \sin \varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i \sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the n th roots of a non-zero complex number.

Example (An example of De Moivre's Theorem). Suppose we had the complex number

$$z = \frac{1}{2} + i \frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10} . First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i \sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i \frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

- (i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$
- (ii) **The set of all complex numbers excluding zero** $\mathbb{C}^\cdot = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow r(\cos \varphi + i \sin \varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in \mathbb{C}

Definition (Convergence in \mathbb{C}). Let $\{z_n\}_{n=1}^\infty$ be a sequence in \mathbb{C} . We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z , then we write $z_n \rightarrow z$.

Proposition (Properties of Convergent Sequences). Assume $(z_n) \rightarrow z$ and $(w_n) \rightarrow w$.

- (i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w$.
- (ii) $z_n w_n \rightarrow zw$.
- (iii) $z_n^{-1} \rightarrow z^{-1}$.
- (iv) $(z_n) \rightarrow z$ if and only if $\Re(z_n) \rightarrow \Re(z)$ and $\Im(z_n) \rightarrow \Im(z)$ as a sequences in \mathbb{R} .

Proof. Suppose $(z_n) \rightarrow z$ and $(w_n) \rightarrow w$.

- (i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \rightarrow z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \rightarrow w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned} |\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\ &\leq \alpha|z_n - z| + \beta|w_n - w| \\ &< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \rightarrow \alpha z + \beta w.$$

- (ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \rightarrow z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \rightarrow w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned} |z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\ &\leq |w_n| |z_n - z| + |z| |w_n - w| \\ &\leq M |z_n - z| + |z| |w_n - w| \\ &< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, we conclude that

$$z_n w_n \rightarrow zw.$$

- (iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \rightarrow z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \rightarrow z$, there must exist an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned} |z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z| |z_n|} \\ &< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that $z_n^{-1} \rightarrow z^{-1}$.

- (iv) (\implies) Let $\varepsilon > 0$. Since $(z_n) \rightarrow z$, we can choose $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\begin{aligned} |\Re(z_n) - \Re(z)| &= |\Re(z_n - z)| \leq |z_n - z| < \varepsilon \\ |\Im(z_n) - \Im(z)| &= |\Im(z_n - z)| \leq |z_n - z| < \varepsilon. \end{aligned}$$

Hence, the real and imaginary part of (z_n) converge.

(\impliedby) Letting $\varepsilon > 1$ again. Our goal is to find an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n - z| < \varepsilon.$$

Since the real and imaginary part of z_n converge, we know that there exists $N_1, N_2 \in \mathbb{N}$ such that, we have

$$|\Re(z_n) - \Re(z)| < \frac{\varepsilon}{2} \quad (1)$$

$$|\Im(z_n) - \Im(z)| < \frac{\varepsilon}{2} \quad (2)$$

whenever $n \geq N_1$ and $n \geq N_2$, respectively. Now, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned} |z_n - z| &= |(\Re(z_n) - \Re(z)) + i(\Im(z_n) - \Im(z))| \\ &\leq |\Re(z_n) - \Re(z)| + |\Im(z_n) - \Im(z)| \quad (|i| = 1) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we see that $(z_n) \rightarrow z$.

■

Chapter 3

Week 3

3.1 Lecture 4

3.1.1 Topics

- (i) Convergence of Series
- (ii) Exponential, Sine, and Cosine functions

3.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^n z_k$$

Definition (Convergence of Series). If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i} \left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \quad (1)$$

$$\frac{1}{2}is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \quad (2)$$

Consider

$$\frac{1}{2}i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Then subtracting (1) and (2), we see that

$$is_n - \frac{1}{2}is_n = \frac{1}{2}is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow s_n = \frac{1}{i} \left[1 - \frac{1}{2^n} \right].$$

Thus, taking the limit as $n \rightarrow \infty$ gives us

$$\lim_{n \rightarrow \infty} s_n = \frac{1}{i}.$$

Example. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute

$$\sum_{n=1}^{\infty} z^n.$$

Since $|z| < 1$, we know that

$$\sum_{n=1}^{\infty} z^n$$

is a geometric series and that it converges. Using the geometric series sum formula, we see that

$$\sum_{n=1}^{\infty} z^n = \lim_{n \rightarrow \infty} \sum_{k=1}^n z^k = \frac{z}{1-z}.$$

Consider this alternative way. Suppose we have

$$s_n = z + z^2 + z^3 + \cdots + z^n.$$

and

$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$

Then subtracting these two equations, we see that

$$(1-z)s_n = z - z^{n+1}.$$

Then we have

$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$

Then taking the limit as $n \rightarrow \infty$, we must have

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series). Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in \mathbb{C} .

3.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function). For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series). We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z + w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \quad \textbf{Show this!}.$$

Chapter 4

Week 4

4.1 Lecture 4

4.1.1 Topics

- Continue discussion of convergence of sequences and series.
- Discuss exponential, sine, and cosine function.

4.1.2 Class Exercises

- (i) Show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z .
- (ii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ converges for all z .
- (iii) Show that $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all z .

Definition (Exponential, Cosine, and Sine). We define

$$\begin{aligned}\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

Our main goal for this lecture is to show that $\exp(z+w) = \exp(z)\exp(w)$.

4.1.3 Cauchy Multiplication Theorem

Theorem (Cauchy Multiplication Theorem). Assume that $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converges absolutely. Then

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n z_k w_{n-k} \right) = \left(\sum_{n=0}^{\infty} z_n \right) \left(\sum_{n=0}^{\infty} w_n \right) \quad (1)$$

where the series on the left-hand side of (1) converges absolutely.

Proof. We will show that the sequence of partial sums of (1) satisfy the conditions of the monotone convergence theorem so that it converges. **Why is (1) monotone?** First, we show that (1) is

bounded. Observe that

$$\begin{aligned}
 \sum_{n=0}^N \left| \sum_{i+j=n} z_i w_j \right| &\leq \sum_{n=0}^N \sum_{i+j=n} |z_i w_j| \\
 &= \sum_{0 \leq i+j \leq N} |z_i w_j| \\
 &\leq \sum_{0 \leq i, j \leq N} |z_i w_j| \\
 &= \sum_{i=0}^N |z_i| \sum_{j=0}^N |w_j|.
 \end{aligned}$$

By our assumption, we can see that the sequence of partial sums of $\sum_{n=0}^{\infty}$ and $\sum_{n=0}^{\infty} w_n$ are bounded, and thus the left side of the equation above is bounded. Thus, we see that

$$\sum_{n=0}^N \left| \sum_{i+j=n} z_i w_j \right|$$

converges by the monotone convergence theorem. Next, we will show that

$$\alpha_n = \left| \sum_{n=0}^{2N} \sum_{i+j=n} z_i w_j - \sum_{i=0}^N z_i \sum_{j=0}^N w_j \right| \rightarrow 0$$

as $N \rightarrow \infty$. Let us define the following sets

$$\begin{aligned}
 T_N &= \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i \geq 0, j \geq 0, 0 \leq i + j \leq N\} \\
 t_n &= \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq i \leq N, 0 \leq j \leq N\}.
 \end{aligned}$$

Observe that $T_N \subseteq t_n$ and $t_N \subseteq T_{2N} \subseteq t_{2N}$. Thus,

$$\begin{aligned}
 \alpha_N &= \left| \sum_{(i,j) \in T_{2N} \setminus t_N} z_i w_j \right| \leq \sum_{(i,j) \in T_{2N} \setminus t_N} |z_i w_j| \\
 &\leq \sum_{(i,j) \in t_{2N} \setminus t_N} |z_i w_j| \\
 &= \sum_{i=0}^{2N} \sum_{j=0}^{2N} |z_i w_j| - \sum_{i=0}^N \sum_{j=0}^N |z_i w_j| \\
 &= \sum_{i=0}^{2N} |z_i| \sum_{j=0}^{2N} |w_j| - \sum_{i=0}^N |z_i| \sum_{j=0}^N |w_j| \rightarrow 0 \text{ as } N \rightarrow \infty.
 \end{aligned}$$

■

Corollary. For any $z, w \in \mathbb{C}$, we have $\exp(z) \cdot \exp(w) = \exp(z + w)$.

Proof. Let

$$C_n = \sum_{k=0}^n \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}.$$

Then multiplying by $n!$ on both sides of the equation above, we see that

$$n! C_n = \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} = (z + w)^n$$

by the binomial formula where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, observe that

$$\sum_{n=0}^N C_n = \sum_{n=0}^N \frac{n! C_n}{n!} = \sum_{n=0}^N \frac{(z+w)^n}{n!}.$$

Let $n \rightarrow \infty$. Then we have

$$\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w).$$

■

4.2 Lecture 5

4.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

4.2.2 Sine and Cosine Functions

Recall that we defined

$$\begin{aligned} \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C} \\ \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}. \end{aligned}$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2$.

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z + i \sin z. \end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x + iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.
- $\Im(\exp(z)) = e^x \sin y$, $|\exp(z)| = e^x$.
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and $w = u + iv$. Then

$$\begin{aligned}\exp(z) &= e^x(\cos y + i \sin y) \\ \exp(w) &= e^u(\cos v + i \sin v)\end{aligned}$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is not injective.

- Note that $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that \exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of \exp . We just learned that it is not possible unless we change the "domain" of \exp .

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \rightarrow \mathbb{C}^*$ is a bijective map.

Proof. Let $z \in \mathbb{C}^*$ and $z = x + iy$. Let $z = \gamma(\cos \varphi + i \sin \varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln \gamma + i\varphi$. Then $\exp(w) = z$ and so, \exp is surjective. Moreover, \exp is injective (on S) as well. ■

Corollary. For any $z \in \mathbb{C}^*$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^*$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \text{Log}(z)$.

Theorem. There exists a mapping

$$\text{Log} : \mathbb{C}^* \rightarrow \mathbb{C}$$

such that

- (i) $\exp(\text{Log } z) = z$
- (ii) $-\pi < \Im(\text{Log } z) \leq \pi$.

Definition (Principal Branch). The function $\text{Log } Z$ is called the principal branch of the logarithm.

Problem 1. Let $z \in \mathbb{C}^*$. Find all possible solutions of $\exp(w) = z$.

Solution. Taking the log of both sides of $\exp(w) = z$, we see that

$$w = \log(z) = \ln |z| + i \left(\operatorname{Arg}(z) + 2\pi k \right), \quad k \in \mathbb{Z}.$$

■

Lemma. For $z \in \mathbb{C}^\bullet$, we have

$$\operatorname{Log}(z) = \ln |z| + i \operatorname{Arg}(z).$$

4.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \operatorname{Log}(z) + 2\pi i k \text{ for } z \in \mathbb{C}^\bullet$$

where

$$\log z = \operatorname{Log} z + 2\pi i k$$

is a **multivalued function** and $\log z = \operatorname{Log} z$ if we want the solution to be in S .

Chapter 5

Week 5

5.1 Lecture 6

5.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on \mathbb{C} :

$$d(z, w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify \mathbb{C} with \mathbb{R}^2 using the map $\mathbb{C} \rightarrow \mathbb{R}^2$ with $\alpha + i\beta \rightarrow (\alpha, \beta)$. Thus, we can visualize \mathbb{C} using \mathbb{R}^2 .

Recall that definition of continuity for functions in \mathbb{R} .

Definition (continuity in \mathbb{R}). Let $f : \mathbb{R} \rightarrow \mathbb{R}$. We say that f is **continuous** for each $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say $|x - a| < \delta$, we can also say that $x \in (a - \delta, a + \delta)$. Similarly, we have $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

5.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x, a) < \delta\}$$

which is a more general statement than the one found in the definition above.

Definition (Open Ball). Consider the usual metric $d(z, w) = |z - w|$ in \mathbb{C} . Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a , and radius $\varepsilon > 0$, we mean the set

$$B(a, \varepsilon) = \{z \in \mathbb{C} : d(z, a) < \varepsilon\}.$$

Definition (Open Set). Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Example. (i) \mathbb{C} is open.

(ii) Let $\varepsilon > 0$ and $a \in \mathbb{C}$. Then $B(a, \varepsilon)$ is also open; that is, open balls are open.

(iii) Let D_1 and D_2 be open sets. Show that $D_1 \cap D_2$ is open. Let $x \in D_1 \cap D_2$. To show that $D_1 \cap D_2$ is open, we need to find $\delta > 0$ such that $B(x, \delta) \subseteq D_1 \cap D_2$. Since $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$. Since D_1 and D_2 are open sets, we can find $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$B(x, \varepsilon) \subseteq D_1 \text{ and } B(x, \varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that $d(x, p) < \varepsilon$ for all $p \in D_1$ and $d(x, q) < \varepsilon'$. Then pick $\delta = \min\{d(x, p), d(x, q)\}$. Since x is in both D_1 and D_2 , we have that

$$B(x, \delta) \subseteq D_1 \cap D_2.$$

Hence, $D_1 \cap D_2$ is open.

1. The empty set is open.

2. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a collection of open sets, where Λ is an indexing set. Define the set

$$\bigcup_{\lambda \in \Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that $\bigcup_{\lambda \in \Lambda} D_\lambda$ is also open.

5.1.3 Interior Points

Definition (Interior Point). Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Proposition. D is open if and only if each $a \in D$ is an interior point of D .

Example. Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton $\{0\}$. Show that $\{0\}$ is not open. Note that the intersection of arbitrary many open sets may not be open.

5.1.4 Closed sets

Definition (Closed Set). We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

Example. Define a closed ball with center at $a \in \mathbb{C}$ and radius $\varepsilon > 0$ by

$$\overline{B}(a, \varepsilon) = \{z \in \mathbb{C} : |z - a| \leq \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that $\overline{B}(a, \varepsilon)$ is closed.

Example. • For any $a \in \mathbb{C}$, we say that the singleton $\{a\}$ is closed.

- If A_1, A_2, \dots, A_n are closed sets. Then

$$\bigcup_{i=1}^n A_i \text{ is also closed.}$$

We can just show that the complement of this set is open.

- Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Define

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\lambda \ \forall \lambda \in \Lambda\},$$

then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.

Definition (Boundary Points). Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a, \varepsilon) \cap D \neq \emptyset$ and $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a, \varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.

Example. Any point on the circle $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$ is a boundary point of $B(a, \varepsilon)$, where $\varepsilon > 0$.

Example (Boundary Points of Singletons). • Note that the boundary point of the singleton is just the singleton itself.

- Let $D = \mathbb{C} \setminus \{a\}$. Just like the first item, the boundary points of this set is just $\{a\}$.

Definition (Accumulation Point). Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,

$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

Example. (i) Let $D = \mathbb{C}$. Then $\{0\}$ is an accumulation point of D .

(ii) Let a such that $|a| = 1$. Then a is an accumulation point of $D = \{z : |z| < 1\}$.

(iii) Let $D = \{1, i\}$. Then D has no accumulation points.

(iv) If $D = \{w\}$ for some nonzero $w \in \mathbb{C}$. Thus, $D' = \emptyset$. That is, every singleton set contains no accumulation points. Furthermore, any finite set does not have any accumulation points.

(v) Let $D = \mathbb{R}$. Then the accumulation points of D is just \mathbb{R} itself. That is, any point of \mathbb{R} is an accumulation point.

(vi) Let $D = B(0, 1)$. The accumulation points are contained in the following set:

$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Another way to define the set of accumulation points of D is the following:

$$\{a \in \mathbb{C} : \forall \varepsilon > 0, B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset\}.$$

Definition (Closure). Let $D \subseteq \mathbb{C}$. We denote the **closure** of D by \mathbf{D} and define \overline{D} as

$$\overline{D} = D \cup \{x \in \mathbb{C} : \forall \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

The closure is the smallest closed set that contains D !

Lemma. (i) \overline{D} is closed in \mathbb{C} .

(ii) The following statements are equivalent:

(a) $a \in \mathbb{C}$ is an accumulation point of D

(b) There is a sequence (z_n) with $z_n \in D$, $z_n \neq a$ for all $n \in \mathbb{Z}_+$ and $z_n \rightarrow a$.

Proof. Proof is in the homework! ■

Definition (Bounded). We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$; that is, $D \subseteq \overline{B}(0, R)$.

Chapter 6

Week 6

6.1 Lecture 7

6.1.1 Topics

Our goal for this lecture is to discuss complex powers of a non-zero complex number.

6.1.2 Complex Powers

Let $z \in \mathbb{C}^\bullet$. We want to define z^w for $w \in \mathbb{C}$. Recall from real number system that, for $x \in \mathbb{R}_+$, $x^a = e^{a \ln x}$. We want our complex powers to extend this idea.

Definition. We define $z^w = \exp(w \log z)$. Note that

$$\begin{aligned}\log z &= \text{Log } z + 2\pi i k \\ &= \ln |z| + i(\text{Arg}(z) + 2\pi k), \quad k \in \mathbb{Z}\end{aligned}$$

is a multi-valued function.

For this reason, the complex power of a non-zero complex number will also be multi-valued. Hence, by choosing a fixed k , we can find a specific value of z^w .

Example. (i) Compute i^i . Recall that $\log i = i\left(\frac{\pi}{2} + 2\pi k\right)$. Hence, we have

$$i^i = \exp(i \log i) = \{e^{\frac{-\pi}{2} + 2\pi k} : k \in \mathbb{Z}\}.$$

If we are just referring to the principal branch, then

$$i^i = e^{\frac{-\pi}{2}}.$$

(ii) Compute: $(-1)^{1+i}$. Note that

$$\log(-1) = i\pi + 2\pi i k = i(2k + 1)\pi.$$

Then we see that for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}(-1)^{1+i} &= \exp((2k + 1)\pi(i - 1)) \\ &= e^{-(2k+1)\pi} \exp(e^{(2k+1)\pi i}) \\ &= e^{-(2k+1)\pi} \left[\cos(2k + 1)\pi + i \sin(2k + 1)\pi \right] \\ &= -e^{-(2k+1)\pi}.\end{aligned}$$

Let us summarize some facts about $\exp(z)$.

- For all $x \in \mathbb{R}$, we have $\exp(x) = e^x$. Thus, from now on, we write e^z in place of $\exp(z)$.
- For $z \in \mathbb{C}^\bullet$, we have $\log z$ is a multi-valued function. That is, we have

$$\log z = \ln |z| + i(\operatorname{Arg} z + 2\pi k), \quad k \in \mathbb{Z}.$$

- For $z \in \mathbb{C}^\bullet$, we have $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$ (this is the principal branch \log).
- For $z \in \mathbb{C}^\bullet$, we have $z^w = e^{w \log z}$ is also a multi-valued function.

Example. • Find all solutions of $\sin z = 1$.

- Find all possible solutions of $\cos z = i$.

Problem 2. Find all solutions of $e^z = i$.

Solution. ■

6.2 Lecture 8

- Limits of a function
- Continuity
- Derivative

6.2.1 Functional Limit

Definition (Functional Limit). Let $D \subseteq \mathbb{C}$ and $f : D \rightarrow \mathbb{C}$ be a function. Let a be an accumulation point of D . Let $\ell \in \mathbb{C}$. We say that ℓ is **the limit of $f(z)$** as z approaches to a if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $z \in D$ with $0 < |z - a| < \delta$, we have

$$|f(z) - \ell| < \varepsilon;$$

that is, $z \in D \cap (B(a, \delta) \setminus \{a\})$ implies $f(z) \in B(\ell, \varepsilon)$.

Example. Let $f : B(0, 1) \rightarrow \mathbb{C}$ defined by $f(z) = \frac{iz}{2}$ then $\lim_{z \rightarrow i} f(z) = -\frac{1}{2}$.

Proof. Observe that

$$|f(z) - \ell| = \left| \frac{i}{2}z - \left(-\frac{1}{2}\right) \right| = \left| \frac{i}{2} \right| |z - i| = \frac{1}{2} |z - i|.$$

So, given any $\varepsilon > 0$, we can take any $0 < \delta < 2\varepsilon$ such that

$$0 < |z - i| < \delta$$

implies

$$|f(z) - \ell| < \varepsilon. \quad \text{■}$$

Some facts about limits are:

Theorem. Let $D \subseteq \mathbb{C}$ with $f : D \rightarrow \mathbb{C}$, $g : D \rightarrow \mathbb{C}$ be two functions let us write $f = u + iv$.

- (i) Let $a = \alpha + i\beta$. Then $\lim_{z \rightarrow a} f(x) = \ell$ if and only if

$\lim_{(x,y) \rightarrow (\alpha,\beta)} u(x,y) = \Re(\ell)$ and $\lim_{(x,y) \rightarrow (\alpha,\beta)} v(x,y) = \Im(\ell)$; that is,

$$\lim_{z \rightarrow a} f(z) = \lim_{z \rightarrow a} u(z) + i \lim_{z \rightarrow a} v(z).$$

(ii) Let $\lim_{z \rightarrow a} f(z) = \ell$ and $\lim_{z \rightarrow a} g(z) = \ell'$ and $c, d \in \mathbb{C}$. Then

$$\lim_{z \rightarrow a} (cf(z) + dg(z)) = c\ell + d\ell'.$$

(iii) $\lim_{z \rightarrow a} (f(z)g(z)) = \lim_{z \rightarrow a} f(z) \cdot \lim_{z \rightarrow a} g(z)$.

(iv) We have

$$\lim_{z \rightarrow a} \frac{f(z)}{g(z)} = \frac{\lim_{z \rightarrow a} f(z)}{\lim_{z \rightarrow a} g(z)}$$

whenever $\lim_{z \rightarrow a} g(z) \neq 0$.

Problem 3. Let $\lim_{z \rightarrow i} f(z) = 2 + i$ and $\lim_{z \rightarrow i} g(z) = 1 - i$. Compute the following:

$$\lim_{z \rightarrow i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2} \right].$$

Solution. Observe that

$$\begin{aligned} \lim_{z \rightarrow i} \left[(f(z))^3 + \frac{(1+i)g(z)}{z^2} \right] &= \lim_{z \rightarrow i} (f(z))^3 + \lim_{z \rightarrow i} \frac{(1+i)g(z)}{z^2} \\ &= (2+i)^3 + \frac{(1+i)(1-i)}{i^2} \\ &= (2+i)^3 - 2. \end{aligned}$$

■

Definition (Limits at Infinity). (i) Let $f : D \rightarrow \mathbb{C}$ be a function and $a \in \mathbb{C}$ be an accumulation point of D . Then we say $\lim_{z \rightarrow a} f(z) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that

$$z \in D \cap (B(a, \delta) \setminus \{a\})$$

implies $|f(z)| \geq M$; that is, f is unbounded as z approaches to a .

(ii) Let f be a complex function defined on the complement of a ball in \mathbb{C} . We say $\lim_{z \rightarrow \infty} f(z) = \ell$ if for all $\varepsilon > 0$, there exists $R > 0$ such that $|z| > R$ implies

$$|f(z) - \ell| < \varepsilon.$$

Example. (i) Consider $\lim_{z \rightarrow 0} \frac{1}{z} = \infty$. Let $M > 0$. Let $\delta = \frac{1}{M}$. Then $0 < |z| < \delta$ implies that

$$\left| \frac{1}{z} \right| > \frac{1}{\delta} = M.$$

(ii) Consider $\lim_{z \rightarrow \infty} \frac{1}{z} = 0$. Let $\varepsilon > 0$. Choose $R = \frac{1}{\varepsilon} > 0$. Then $|z| > R$ implies

$$\left| \frac{1}{z} \right| < \frac{1}{R} = \varepsilon.$$

Problem 4. (i) Show that $\lim_{z \rightarrow \infty} f(z) = \ell$ if and only if

$$\lim_{z \rightarrow \infty} f\left(\frac{1}{z}\right) = \ell.$$

(ii) $\lim_{z \rightarrow a} f(z) = \infty$ if and only if $\lim_{z \rightarrow a} \frac{1}{f(z)} = 0$.

(iii) Give a definition of $\lim_{z \rightarrow \infty} f(z) = \infty$. Show that $\lim_{z \rightarrow \infty} f(z) = \infty$ if and only if $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = \infty$.

(iv) Compute $\lim_{z \rightarrow \infty} \frac{z-1}{z+i}$.

(v) Compute $\lim_{z \rightarrow \infty} \frac{gz+i}{z^2+z+1}$.

(vi) Can you compute $\lim_{z \rightarrow \infty} e^{-z}$?

6.2.2 Continuity

Definition (Continuity). Let $D \subseteq \mathbb{C}$ and $f : D \rightarrow \mathbb{C}$ be a function. We say f is **continuous at** $a \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(a, \delta) \cap D$, we have $f(z) \in B(f(a), \varepsilon)$.