Problem-1: Use Cauchy Integral Formula to compute

(a) $$\oint_{\partial B(1,2)} \frac{z^2+3z-1}{z^2+z-6}\,dz$$

(b) $$\oint_{\partial B(0,2)} \frac{e^{\pi z}}{z^2+1}\,dz$$

(c) $$\oint_{\partial B(1,2)} \frac{1}{z^2-5z+4}\,dz$$

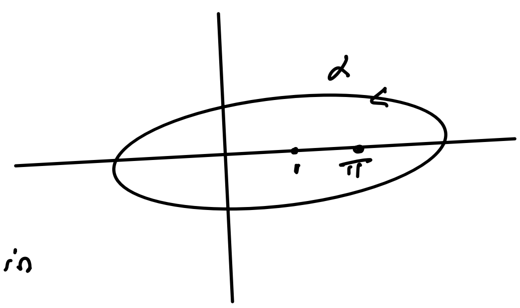
(d) $$\oint_{\partial B(i,1/2)} \frac{\operatorname{Log} z}{z-i}\,dz$$

Problem 2: Use generalized Cauchy Integral formula to evaluate

(a) $$\frac{1}{2\pi i}\oint_{\partial B(0,2)} \frac{z^{10}}{(z-1)^{11}}\,dz$$

(b) $$\frac{1}{2\pi i}\oint_{\alpha} \frac{e^{iz}}{(z-\pi)^5}$$

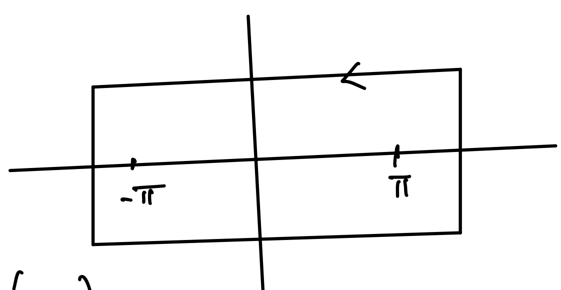
where α is as shown in the figure



(c) $$\oint_{\alpha} \frac{\sin z}{(z^2-\pi^2)^2}\,dz$$

where α is as given in the figure.

(Hint: Think about using partial fraction of $\frac{1}{z^2-\pi^2}$)



Problem 3:

(a) Compute $\displaystyle \frac{1}{2\pi i} \oint_{\partial B(0,1)} \frac{1}{z^n}\, dz.$

(b) Let n be a positive integer.

Show that

$$\frac{1}{2\pi i} \oint_{\partial B(0,1)} \frac{1}{z}\left(z + \frac{1}{z}\right)^n dz = \frac{2^n}{2\pi} \int_0^{2\pi} \cos^n t \, dt$$

(c) Let k be a non-negative integer.

Show that $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \cos^{2k+1} t \, dt = 0$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2k} t \, dt = \frac{(2k)!}{2^{2k} k!^2}$$

Problem-4: Let $f: \mathbb{C} \to \mathbb{C}$ holomorphic. Assume that $|\operatorname{Re} f(z)| \leq M$ for all $z \in \mathbb{C}$. Prove that f must be constant.

Problem-5: Fact: Let $D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$ continuous. If $\int_\Delta f \, dz = 0$ for all triangular path Δ in D such that all points in the interior of Δ lie in D, then f is holomorphic on D.

Morera's theorem

(a) Use Morera's theorem to prove the following: Let $D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$ continuous. Let L be a straight line segment in D. If f is holomorphic on $D \setminus L$, then f is holomorphic on D.

(b) Let $\mathbb{H}_+ = \{ z \in \mathbb{C} \mid Im(z) > 0 \}$

$\mathbb{H}_- = \{ z \in \mathbb{C} \mid Im(z) < 0 \}$

$\mathbb{R}$ = real axis $= \{ z \in \mathbb{C} \mid Im(z) = 0 \}$.

Let $f : \mathbb{H}_+ \cup \mathbb{R} \to \mathbb{C}$ be a continuous function such that $f|_{\mathbb{H}_+}$ is holomorphic. Assume that $f(\mathbb{R}) \subseteq \mathbb{R}$. Define

$$\tilde{f} : \mathbb{C} \to \mathbb{C} \text{ by, } \tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \mathbb{H}_+ \cup \mathbb{R} \\ \overline{f(\bar{z})} & \text{if } z \in \mathbb{H}_- \end{cases}$$

Show that $\tilde{f}$ is holomorphic on $\mathbb{C}$.

Hint: First show $\tilde{f}|_{\mathbb{H}_-}$ is holomorphic and try to use (a).

Problem 6: Compute the radius of convergence of

(i) $\displaystyle\sum_{n=0}^{\infty} n!\, z^n$

(ii) $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ where $a_n = \begin{cases} a^n & n \text{ even} \\ b^n & n \text{ odd} \end{cases}$

$b > a > 0$.

Problem 7: Consider a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in B(0,R)$.

Assume that $f(-z) = f(z)$ for all $z \in B(0,R)$

Show that $a_n = 0$ for all odd n.