

Linear Algebra Notes

Lance Remigio

December 8, 2023

Contents

1	Vector Spaces	5
1.1	Vector Spaces	5
1.1.1	Basics	5
1.1.2	Non-examples	8
1.1.3	Basic Extensions from Definition	8
1.2	Subspaces	9
1.3	Linear Combinations and Systems of Linear Equations	12
1.4	Linear Dependence and Linear Independence	16
1.4.1	Motivation	16
1.4.2	Examples of Linearly Dependent Sets	17
1.4.3	Examples of Linear Independent Sets	17
1.5	Bases And Dimension	19

Chapter 1

Vector Spaces

1.1 Vector Spaces

1.1.1 Basics

Definition 1.1.1 (Vector Spaces). A **vector space** (or **linear space**) over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y , in V there is a unique element ax in V , such that the following conditions hold:

- (VS 1) For all $x, y \in V$, $x + y = y + x$ (commutativity of addition).
- (VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (associativity of addition).
- (VS 3) There exists an element in V denoted by O such that $x + O = x$ for each $x \in V$.
- (VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = O$.
- (VS 5) For each element $x \in V$, we have $1x = x$.
- (VS 6) For each $a, b \in F$ and each element $x \in V$, then $(ab)x = a(bx)$.
- (VS 7) For each element $a \in F$ and each pair $x, y \in V$, we have $a(x + y) = ax + ay$.
- (VS 8) For each pair $a, b \in F$ and each $x \in V$, we have $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x , respectively.

- The elements of a field F are called **scalars** and the elements of a vector space V are called **vectors** (these should not be confused!).
- Every vector space will always be defined over a given field, mostly defined over the real numbers \mathbb{R} or the complex numbers \mathbb{C} unless otherwise noted.
- Every vector space should specify the operations of addition and scalar multiplication.

Definition 1.1.2 (n-tuples). An object of the form (a_1, a_2, \dots, a_n) , where the entries a_1, a_2, \dots, a_n are elements of a field F , is called an **n-tuple** with entries from F . The elements a_1, a_2, \dots, a_n are called **entries** or **components** of the n -tuple.

Definition 1.1.3. We say that two n -tuples, (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) , are **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

Example 1. The set of all n -tuples with entries from a field F denoted by F_n is a vector space. To see why, suppose $u, v \in F_n$ where $u = (a_1, a_2, \dots, a_n)$ and $v = (b_1, b_2, \dots, b_n)$. If we take term-by-term

addition of the entries in both u and v , then we end up with

$$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and likewise,

$$cu = (ca_1, ca_2, \dots, ca_n).$$

These same set of operations define \mathbb{R}^3 as a vector space over \mathbb{R} and likewise, \mathbb{C}^2 is a vector space over \mathbb{C} .

- Note that vectors in F^n can be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

rather than **row vectors** (a_1, a_2, \dots, a_n) .

- 1-tuples are just scalars or are just an element from F .

Definition 1.1.4. An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry a_{ij} with $(1 \leq i \leq m, 1 \leq j \leq n)$ is an element of F . We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \dots, a_{in}$ compose the i th **row** of the matrix, and the entries $a_{1j}, a_{2j}, \dots, a_{mj}$ compose the j th **column** of the matrix.

- The rows make a vector space which we denote F^n .
- Likewise, the columns make a vector space we denote F^m .

Definition 1.1.5 (Zero Matrix). The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O .

Definition 1.1.6 (Square Matrix). A matrix is **square** if the number of rows and columns of a matrix are equal.

- Just like our tuple example, the set of all $m \times n$ matrices with entries from a field F form a vector space. Denote this vector space as $M_{m \times n}(F)$ endowed with two operations; that is, **matrix addition** and **scalar multiplication**. Suppose for $A, B \in M_{m \times n}(F)$ and $c \in F$, we have

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

and

$$(cA)_{ij} = cA_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. In other words, the two operations can be performed entry-wise.

- The operations from our tuple case extends very naturally to $M_{m \times n}(F)$. In other words, if we add two matrices A_{ij} and B_{ij} , then we would expect to that $A_{ij} + B_{ij} \in M_{m \times n}(F)$ as well and likewise for the scalar multiplication case.

Definition 1.1.7 (Set of All Functions). Let S be any nonempty set and F be any field, and let $\mathcal{F}(S, F)$ denote the set of all functions from S to F .

Definition 1.1.8. Two functions $f, g \in \mathcal{F}(S, F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$.

The set of functions above forms a vector space with the operations of addition and scalar multiplication defined for every $f, g \in \mathcal{F}(S, F)$ and $c \in F$ with

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$

Definition 1.1.9 (Set of All Polynomials). A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each $a_k \in F$ is called the **coefficient** of x_k .

Definition 1.1.10 (Zero Polynomial). We call $f(x) = 0$ the **zero polynomial** if $a_n = a_{n-1} = \cdots = a_0 = 0$.

Definition 1.1.11 (Degree). The **degree** of a given polynomial f is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Definition 1.1.12 (Equality of Polynomials). We call two polynomials f, g , where

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

are **equal** if $m = n$ and $a_i = b_i$ for all $i = 0, 1, \dots, n$.

Suppose we have $c \in F$ and say we evaluated the polynomial $f \in F$ at c . Then we would have the following

$$f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

where $f(c) \in F$.

Definition 1.1.13 (Basic Operations of Polynomials). Define polynomial **addition** $f + g$ as the following:

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

Let $c \in F$. Let scalar **multiplication** be defined by

$$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0.$$

The operations above form a vector space for $P(F)$ (the set of all polynomials).

Definition 1.1.14 (Sequences). A **sequence** in F is a function $\sigma : \mathbb{Z}^+ \rightarrow F$. A given sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \dots$ is denoted (a_n) .

Let V be the set of all sequences $\sigma(n) \in F$. For every $(a_n), (b_n) \in V$ with $t \in F$, we have the following operations

$$(a_n) + (b_n) = (a_n + b_n) \text{ and } t(a_n) = (ta_n).$$

1.1.2 Non-examples

Example 2. Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$ where R is a field. For every $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Note that S is **NOT** a vector space since (VS 1), (VS 2), and (VS 8) fail.

Example 3. Let S be the same set as in the last example. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$$

and

$$c(a_1, a_2) = (ca_1, 0).$$

Note that S is **NOT** a vector space with these operations since (VS 3), (VS 4), and (VS 5) fail.

1.1.3 Basic Extensions from Definition

Theorem 1.1.1 (Cancellation Law for Vector Addition). If $x, y, z \in V$ such that $x + z = y + z$, then $x = y$.

Proof. There exists a vector $v \in V$ such that $z + v = O$ (VS 4). Thus, we have

$$\begin{aligned} x &= x + O \\ &= x + (z + v) \\ &= (x + z) + v \\ &= (y + z) + v \\ &= y + (z + v) \\ &= y + O \\ &= y. \end{aligned}$$

Hence, we have $x = y$. ■

Corollary. The vector described O described in (VS 3) is unique.

Proof. Let $x \in V$. Suppose there exists two elements $O, O' \in V$ such that

$$x + O = x \tag{1}$$

and

$$x + O' = x. \tag{2}$$

Our objective is to show that $O = O'$. Equating (1) and (2), we find that

$$x + O = x + O' \tag{3}$$

Using the Cancellation Law for Vector Addition, (3) implies that $O = O'$. Hence, O is a unique vector in V . ■

Corollary. The vector y described in (VS 4) is unique.

Proof. Let $x \in V$. Suppose there exists $y, y' \in V$ such that

$$x + y = O \text{ and } x + y' = O.$$

Equating the two equations above, we get that

$$x + y = x + y'.$$

Applying the Cancellation Law for Vector Addition once again, we get that $y = y'$. Hence, y is a unique vector in V . ■

Theorem 1.1.2. In any vector space V , the following statements are true:

- (a) $0x = O$ for any $x \in V$.
- (b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.
- (c) $a0 = 0$ for any $a \in F$.

Proof. (a) Using (VS 8), (VS 3), and (VS 1), it follows that

$$\begin{aligned} 0x + 0x &= (0 + 0)x \\ &= 0x \\ &= 0x + O \\ &= O + 0x. \end{aligned}$$

Hence, $0x = O$ by Theorem 1.1.

- (b) The vector $-(ax)$ is the unique element of V such that $ax + [-(ax)] = O$. If we know that $ax + (-a)x = O$, we can use Corollary 2 to Theorem 1.1 to state that $(-a)x = -(ax)$. But by (VS 8), we have

$$ax + (-ax) = [a + (-a)]x = 0x = O$$

by (a). Hence, we have that $(-a)x = -(ax)$. In particular, we have $(-1)x = -x$. Thus, (VS 6) implies that

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

- (c) The proof of (c) is similar to the proof of (a). ■

1.2 Subspaces

Definition 1.2.1 (Subspaces). A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication.

The most simple examples of subspaces of V is V itself and the set containing just the zero vector. The latter is denoted as the **zero subspace** of V .

We don't have to check all the vector space properties to prove that a subset W of V is a subspace of V . This is because vectors in any subset of V already satisfy all the properties of a vector space. Hence, we need only check that W is closed under

1. **Addition:** $x + y \in W$ whenever $x \in W$ and $y \in W$.
2. **Scalar Multiplication:** $cx \in W$ whenever $c \in F$ and $x \in W$.
3. W contains a zero vector.
4. Each vector in W has an additive inverse in W .

Theorem 1.2.1 (Subspaces). Let V be a vector space and W a subset of V . Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V .

- (a) $O \in W$.

(b) $x + y \in W$ whenever $x \in W$ and $y \in W$.

(c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Proof. Suppose W is a subspace of V . Since W is also a vector space with the operations of addition and scalar multiplication defined in V . Hence, we know that W is closed under addition and scalar multiplication. Note that W contains a zero vector $0' \in W$ such that for any $x \in W$, we have $x + 0' = x$. Since x is also in V (since $W \subseteq V$), we know that $x + 0 = x$. Using the cancellation, we can see that $x + 0' = x + 0$ implies $0' = 0$. Hence, properties (a), (b), and (c) are satisfied.

Conversely, suppose conditions (a), (b), and (c) hold. We need to show that W is a subspace. This tells us that W is closed under addition and scalar multiplication in addition to containing the zero vector. Let $x \in W$. Since W is closed under scalar multiplication, we know that $(-1)x \in W$. By part (b) of theorem 2, we know that $1(-x) = -x \in W$. Hence, W contains an additive inverse and we are done. ■

The theorem above provides a simpler way to determine whether a given subset of a vector space is a subspace. Instead of using the definition to show that a give subset is a subspace, it is more common to use the result above.

Definition 1.2.2 (Transpose). The **transpose** of A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$.

Example 4.

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Definition 1.2.3 (Symmetric Matrices). A **symmetric matrix** is a matrix A such that $A^t = A$.

- The easiest example of a symmetric matrix would be a square matrix where $i = j$.
- The set W of all symmetric matrices in $M_{n \times n}(F)$ is a subspace of $M_{n \times n}(F)$ since the conditions of Theorem 1.3 hold.

We can show that W is indeed a subspace.

1. The zero matrix is equal to its transpose and hence belongs to W .
2. Suppose $A \in W$ and $B \in W$. Hence, $A^t = A$ and $B^t = B$. Hence, we have

$$(A + B)^t = A^t + B^t = A + B$$

which implies that $A + B \in W$.

3. We have $A \in W$ implies $A^t = A$. Now, let $a \in F$. Then we have that $(aA)^t = aA^t = aA$. Hence, $aA \in W$.

Example 5. Let n be non-negative integer, and let $P_n(F)$ consist of all polynomials in $P(F)$ having degree less than or equal to n . We get that $P_n(F)$ is a subspace because:

1. Zero polynomial has degree -1, it is in $P_n(F)$.
2. The sum of two polynomials with degrees less than or equal to n is another a polynomial less than or equal to n .
3. The product of a scalar and a polynomial of degree less than or equal to n is a polynomial less than or equal to n .

This tells us that $P_n(F)$ is a subspace of the space of all polynomials.

Example 6. Let $C(\mathbb{R})$ denote the set of all continuous real-valued functions defined on \mathbb{R} . We know that $C(\mathbb{R})$ is a subset of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in Example 3. We will show that $C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

1. Note that the zero function $f(x) = 0$ for all $x \in \mathbb{R}$ of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a continuous real-valued function.
2. Let $f, g \in C(\mathbb{R})$. Since the sum of f and g is also continuous for all $x \in \mathbb{R}$, we have that $f + g \in C(\mathbb{R})$.
3. Let $c \in \mathbb{R}$ and $f \in C(\mathbb{R})$. We have that for any $x \in \mathbb{R}$, $(cf)(x) = cf(x)$ is a continuous function. Hence, property (c) is satisfied.

Hence, $C(\mathbb{R})$ is a subspace.

Definition 1.2.4 (Upper Triangular). An $m \times n$ matrix A is called **upper triangular** if all its entries below the diagonal entries are zero; that is, if $A_{ij} = 0$ whenever $i > j$.

Example 7. Let B be an upper triangular 3×4 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{pmatrix}$$

Definition 1.2.5 (Diagonal Matrix). An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$; that is, if all its non-diagonal entries are zero.

Example 8. Let A be diagonal 3×3 matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Example 9. The set of diagonal matrices is a subspace of $M_{n \times n}(F)$. To see why, we have

1. Let O be the zero matrix of $M_{n \times n}(F)$. Since O is also a diagonal matrix, we know that O must be in the set of diagonal matrices.
2. Let A, B in the set of diagonal matrices. Let $i \neq j$ such that

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0.$$

Hence, $A + B$ is a diagonal matrix and so addition is closed.

3. Let A be a diagonal matrix as before and let $c \in F$. Let $i \neq j$ again, and observe that

$$(cA)_{ij} = cA_{ij} = c \cdot 0 = 0.$$

Hence, cA is a diagonal matrix and so scalar multiplication is closed.

Since the set of diagonal matrices satisfies all properties of theorem 3, we conclude that it is indeed a subspace of $M_{n \times n}(F)$.

Definition 1.2.6 (Trace). The **trace** of an $n \times n$ matrix M , denoted $\text{tr}(M)$, is the sum of the diagonal entries of M ; that is,

$$\text{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

The set of all $n \times n$ matrices that have a trace equal to zero is a subspace of $M_{n \times n}(F)$ (proved in Exercise 6).

Example 10 (Non-example). Denote V as the set of matrices in $M_{m \times n}(\mathbb{R})$ having non-negative entries. The subset V is not a subspace because it is not closed under scalar multiplication. We can see this by multiplying any matrix in V by a negative number and observe that the entries of said matrix are no longer all non-negative.

We can create subspaces out of other subspaces.

Theorem 1.2.2. Any intersection of subspaces of a vector space V is a subspace of V .

Proof. Let C be a collection of subspaces of V , and let W denote the intersection of the subspaces in C . Since every subspace contains the zero vector and the intersection $W \neq \emptyset$, the zero vector $0 \in W$. Let $a \in F$ and $x, y \in W$. Since each subspace of C is closed under addition and scalar multiplication, it follows that $x + y$ and ax are contained each subspace in C and hence W must be closed under addition and closed under scalar multiplication. Thus, W is a subspace of V by theorem 3. ■

A natural question to ask is whether or not the union of subspaces of a vector space V is a subspace of V too. We can see that indeed the union of subspaces of V satisfy the first first and third property of theorem 3. However, property 2 need not be satisfied all the time. In fact, the union can only be a subspace of V if and only if one the subspaces is a subset of the other.

1.3 Linear Combinations and Systems of Linear Equations

Definition 1.3.1 (Linear Combinations). Let V be a vector space and S a nonempty subset of V . A vector $v \in V$ is called **linear combination** of vectors of S if there exist a finite number of vectors $u_1, u_2, \dots, u_n \in S$ and scalars $a_1, a_2, \dots, a_n \in F$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

In this case, we say that v is a linear combination of $u_1, u_2, \dots, u_n \in V$ and call $a_1, a_2, \dots, a_n \in F$ the **coefficients** of the linear combination.

- An easy example of a vector expressed as a linear combination is the zero vector O where $0v = O$ for each $v \in V$.
- Determining whether a vector is a linear combination of other vectors often involves solving a system of linear equations.

Example 11. Suppose we wanted to express the vector $(2, 6, 8) \in \mathbb{R}^3$ as a linear combination of

$$u_1 = (1, 2, 1), u_2 = (-2, -4, -2), u_3 = (0, 2, 3), u_4 = (2, 0, -3), \text{ and } u_5 = (-3, 8, 16).$$

Our goal is to find scalars a_1, a_2, a_3, a_4 and a_5 such that

$$(2, 6, 8) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5.$$

Doing a bit of algebra, we would need to solve the following system of linear equations

$$\begin{aligned}a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\2a_1 - 4a_2 + 2a_3 + 8a_5 &= 6 \\a_1 - 2a_2 + 3a_3 + 16a_5 &= 8\end{aligned}$$

Solving the system of equations above involves three types of operations:

1. Interchanging the order of any two equations in the system;
2. multiplying any equation in the system by some non-zero constant;
3. adding a constant multiple of any equation to another equation in the system.

We will learn in a later section that the operations listed above do not change the set of solutions to the original system.

Example 12. We claim that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \text{ and } 3x^3 - 5x^2 - 4x - 9$$

in $P_3(\mathbb{R})$, but that

$$3x^3 - 2x^2 + 7x + 8$$

is not. In the first case we wish to find scalars a and b such that

$$\begin{aligned}2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) \\&\quad + b(3x^3 - 5x^2 - 4x - 9) \\&= (a + 3b)x^3 + (-2a - 5b)x^2 \\&\quad + (-5a - 4b)x + (-3a - 9b).\end{aligned}$$

Thus, we have the following system of linear equations:

$$\begin{aligned}a + 3b &= 2 \\-2a - 5b &= -2 \\-5a - 4b &= 12 \\-3a - 9b &= -6.\end{aligned}$$

Adding the appropriate multiples of the first equation to the others in order to eliminate a , we find that

$$\begin{aligned}a + 3b &= 2 \\b &= 2 \\11b &= 22 \\0b &= 0.\end{aligned}$$

Then we get that $a = -4, b = 2, 0 = 0, 0 = 0$. Hence, we have

$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$

Looking at the second case now, we observe that using the preceding technique leads us to the following system of linear equations

$$\begin{aligned}a + 3b &= 3 \\ -2a - 5b &= -2 \\ -5a - 4b &= 7 \\ -3a - 9b &= 8.\end{aligned}$$

Eliminating a as before yields the following:

$$\begin{aligned}a + 3b &= 3 \\ b &= 4 \\ 11b &= 22 \\ 0 &= 17.\end{aligned}$$

The presence of the non-sensical result in the last equality tells us that there are no solutions to the system and that the second polynomial cannot be written as a linear combination of the two polynomials.

We can denote the set of all linear combinations of some set of vectors in the following:

Definition 1.3.2 (Span). Let S be a nonempty subset of a vector space V . The **span** of S , denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S . For convenience, we define $\text{span}(\emptyset) = \{0\}$.

Some immediate examples of spans are:

Example 13. In \mathbb{R}^3 , the span of the set $S = \{(1, 0, 0), (0, 1, 0)\}$ consist of all vectors in \mathbb{R}^3 such that for some scalars $a, b \in \mathbb{R}$, we have

$$a(1, 0, 0) + b(0, 1, 0) = (a, b, 0).$$

This tells us that S contains all the points in the xy -plane. One can show that S is a subspace of \mathbb{R}^3 .

Theorem 1.3.1. The span of any subset S of a vector space V is a subspace of V that contains S . Moreover, any subspace of V that contains S must also contain the span of S .

Proof. Suppose $S = \emptyset$. Then the span of S is just $\text{span}(\emptyset) = \{0\}$ which is a subspace in which S is contained in. Moreover, $\text{span}(\emptyset) = \{0\}$ is always contained in any subspace X of V because every subspace contains the zero vector 0 . Suppose $S \neq \emptyset$, then S contains a vector z . We need to show that $\text{span}(S)$ is a subspace of V .

- (a) Since $S \neq \emptyset$, we know that S contains a vector v such that $0v = 0$. Hence, $0 \in \text{span}(S)$.
- (b) Let $x, y \in \text{span}(S)$. We need to show that $x + y \in \text{span}(S)$. If $x \in \text{span}(S)$, then we can find $a_1, a_2, \dots, a_n \in F$ and $x_1, x_2, \dots, x_n \in S$ such that

$$a_1x_1 + a_2x_2 + \dots + a_nx_n.$$

Likewise, $y \in \text{span}(S)$ implies that we can find scalars b_1, b_2, \dots, b_m and vectors y_1, y_2, \dots, y_m such that

$$b_1y_1 + b_2y_2 + \dots + b_my_m.$$

Then

$$x + y = a_1x_1 + a_2x_2 + \dots + a_nx_n + b_1y_1 + b_2y_2 + \dots + b_my_m$$

is a linear combination and thus $x + y \in \text{span}(S)$.

(c) Note that $c \in F$ implies that

$$\begin{aligned} cx &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (ca_1)x_1 + (ca_2)x_2 + \cdots + (ca_n)x_n \end{aligned}$$

is a linear combination and thus $cx \in \text{span}(S)$.

Hence, $\text{span}(S)$ is a subspace of V . Now we need to show that S is contained within $\text{span}(S)$; that is, $S \subseteq \text{span}(S)$. Let $v \in S$. Then using (VS 5), we can see that $1 \cdot v = v$ is a linear combination; so we have $v \in \text{span}(S)$ and hence, the span of S contains S .

Now let W be any subspace of V that contains S . We need to show that $\text{span}(S) \subseteq W$. Let $v \in \text{span}(S)$. Then we can find scalars $a_1, a_2, \dots, a_n \in F$ and $x_1, x_2, \dots, x_n \in S$ such that

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Since $S \subseteq W$, we know that $x_1, x_2, \dots, x_n \in W$. Using exercise 20 from section 1.3 and using the same set of scalars $a_1, a_2, \dots, a_n \in F$, we have $v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$. Hence, $\text{span}(S) \subseteq W$. ■

Definition 1.3.3. A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we also say that the vectors of S generates (or span) V .

Example 14 (Vectors in \mathbb{R}^3). The vectors $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ generate \mathbb{R}^3 since any given vector $v \in \mathbb{R}^3$ is a linear combination of the three given vectors. Furthermore, there exists scalars $r, s, t \in \mathbb{R}$ such that

$$r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3) = v$$

where

$$r = \frac{1}{2}(a_1 + a_2 - a_3), s = \frac{1}{2}(a_1 - a_2 + a_3) \text{ and } t = \frac{1}{2}(-a_1 + a_2 + a_3).$$

Example 15 (Polynomials). The polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$, and $-x^2 - 4x + 4$ generate $P_2(\mathbb{R})$ because each of the three given polynomials belongs to $P_2(\mathbb{R})$ and each polynomial $ax^2 + bx + c \in P_2(\mathbb{R})$ is a linear combination of these three. It can be shown that we can find $a, b, c \in \mathbb{R}$ such that

$$a(x^2 + 3x - 2) + b(2x^2 + 5x - 3) + c(-x^2 - 4x + 4) = ax^2 + bx + c.$$

Example 16 (Matrices). Note that the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate $M_{2 \times 2}(\mathbb{R})$ because an every $A \in M_{2 \times 2}(\mathbb{R})$ can be expressed as a linear combination of the four given matrices found below where there exists scalars $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where it can be show that

$$\begin{aligned} a &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} - \frac{2}{3}a_{22} \\ b &= \frac{1}{3}a_{11} + \frac{1}{3}a_{12} - \frac{2}{3}a_{21} + \frac{1}{3}a_{22} \\ c &= \frac{1}{3}a_{11} - \frac{2}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22} \\ d &= -\frac{2}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}. \end{aligned}$$

A non-example set of matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

do not generate $M_{2 \times 2}(\mathbb{R})$ since all of the matrices have equal diagonal entries. Thus, every the set of matrices above cannot generate every $A \in M_{2 \times 2}(\mathbb{R})$.

1.4 Linear Dependence and Linear Independence

1.4.1 Motivation

Suppose V is a vector space over a field F and that $W \subseteq V$ is a subspace of V .

- Suppose we have a set S that will generate every vector in V as a linear combination of vectors in S .
- It is desirable to find a subset of S that is as small as possible or rather just enough to generate each vector in V .
- The reason for this is to reduce the amount of computations done to represent a vector in V .
- Finding this subset is a matter of expressing one of the vectors in the spanning set as a linear combination of the other vectors in S .
- This will naturally lead to a system of linear equations for which we have to solve for the coefficients.
- The answer to the question is whether this system of equations leads to a solution or not.
- An easier way to answer this question is to find coefficients (not all zero) for which the zero vector can be written as a linear combination of vectors in S .
- Doing this, guarantees that we can write any one of the vectors in S as a linear combination of the others.

Definition 1.4.1 (Linearly Dependence). A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors in $u_1, u_2, \dots, u_n \in S$ and scalars a_1, a_2, \dots, a_n not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case, we also say that the vectors of S are linearly dependent.

- The **trivial representation** of 0 is a linear combination of scalars $a_1, a_2, \dots, a_n \in F$ and distinct vectors $v_1, v_2, \dots, v_n \in S$ where for all $1 \leq i \leq n$, we have $a_i = 0$.
- This tells us that our definition of **linear dependence** implies that 0 is a non-trivial linear combination.
- Any subset of V that contains the zero vector is subsequently linearly dependent since 0 can be written as non-trivial representation; that is, $1 \cdot 0 = 0$.

1.4.2 Examples of Linearly Dependent Sets

Example 17. Consider a subset in \mathbb{R}^4 defined by

$$S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}.$$

To show that S is a linear dependent set, we need to find scalars $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that a_i for all $1 \leq i \leq 4$ not all zero such that

$$a_1(1, 3, -4, 2) + a_2(2, 2, -4, 0) + a_3(1, -3, 2, -4) + a_4(-1, 0, 1, 0) = 0.$$

We can represent this as a system of linear equations with each equation being set equal to zero and solving for the scalars. Solving this system of equations leads to the coefficients $a_1 = 4, a_2 = -3, a_3 = 2$, and $a_4 = 0$. Thus, we have that S is linearly dependent subset of \mathbb{R}^4 and hence we can write any vector in S as a linear combination of the other vectors contained in S .

Example 18. Define a subset of $M_{2 \times 2}(\mathbb{R})$

$$S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}.$$

This set is **linearly dependent** because we can find coefficients a_1, a_2, a_3 such that $a_1 = 5, a_2 = 3$, and $a_3 = -2$ where

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 1.4.2 (Linear Independence). A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

Whereas linear dependence requires solutions that are not all zero, linear independence **requires** all the solutions to be zero. This provides us an easy way to determine if a finite set is linearly independent.

1.4.3 Examples of Linear Independent Sets

Some facts about linearly independent sets include:

- The empty set is linearly independent since we don't have any vectors to take linear combinations of.
- A set consisting of only one non-zero vector; that is, $\{v\}$ is linearly independent.
- If $\{v\}$ is linearly dependent, then it the singleton has to be the zero vector 0 . This is because $au = 0$

$$u = 1 \cdot u = (a^{-1}a)u = a^{-1}(au) = a^{-1} \cdot 0 = 0.$$

Example 19 (A Set of Vectors in \mathbb{R}^4). It can be shown that the set

$$S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$$

is linearly independent by showing that all the coefficients of the linear combination of 0 represent the trivial representation; that is, we have scalars $a_1 = a_2 = a_3 = a_4 = 0$.

Example 20. For $k = 0, 1, \dots, n$, let $p_k = x^k + x^{k+1} + \dots + x^n$. The set

$$\{p_0(x), p_1(x), \dots, p_n(x)\}$$

is linearly independent in $P_n(F)$. It can be shown that for some scalars $a_0, a_1, \dots, a_n \in F$, the following equation

$$a_0 + (a_0 + a_1 + a_2)x^2 + \dots + (a_0 + a_1 + \dots + a_n)x^n = 0$$

has the trivial-representation; that is, $a_i = 0$ for all $1 \leq i \leq n$.

Theorem 1.4.1. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Proof. Let V be a vector space. Suppose S_1 is linearly dependent. Then there exists a finite number of distinct vectors $v_1, v_2, \dots, v_n \in S_1$ and scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0.$$

Since $S_1 \subseteq S_2$, we must have $v_1, v_2, \dots, v_n \in S_2$ as well. If these vectors are linearly dependent, then these vectors are also linearly dependent in S_2 ; that is, we have scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0.$$

Hence, S_2 is linearly dependent. ■

Corollary. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof. Note that this corollary is just the contrapositive of the theorem before it. Hence, S_1 is linearly independent. ■

- Determining whether there exists a minimal generating set for the span of S is related to solving the problem of finding whether some vector in S that can be written in terms of a linear combination of other vectors in S .
- We can see that in a given subset of \mathbb{R}^3 defined by

$$S = \{u_1, u_2, u_3, u_4\}$$

where $u_1 = (2, -1, 4)$, $u_2 = (1, -1, 3)$, $u_3 = (1, 1, -1)$, and $u_4 = (1, -2, 1)$. Note that this S is linearly dependent. This tells us that any vector in S can be written as a linear combination of the others. Suppose we pick u_3 and write as a linear combination of the vectors u_1, u_2 , and u_4 . We would find that the span of these three vectors (denote this set as S') generates the same spanning set S but with a cardinality one less than S ; that is, $\text{span}(S) = \text{span}(S')$.

- If we find that there does not exist a proper subset that is equivalent to the span of S , then S must be a linearly independent set.

Theorem 1.4.2. Let S be a linearly independent subset of a vector space V , and let $v \in V$ but not in S . Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof. (\Rightarrow) Suppose $S \cup \{v\}$ is a linearly dependent set. Then there exists a finite number of scalars $a_1, a_2, \dots, a_n \in F$ and vectors $u_1, u_2, \dots, u_n \in S \cup \{v\}$ such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

Since S is a linearly independent set, we must have $u_i = v$ for some $1 \leq i \leq n$. Choose $i = 1$ (any choice of i will do) such that

$$a_1v + a_2u_2 + \dots + a_nu_n = 0.$$

Solving for v by subtracting a_1v on both sides, multiplying by a_1^{-1} on both sides of the equation,

and distributing by a_1^{-1} yields the following:

$$v = (-a_1^{-1}a_2)u_2 + (-a_1^{-1}a_3)u_3 + \cdots + (-a_1^{-1}a_n)u_n.$$

Since v is a linear combination of vectors $u_1, u_2, \dots, u_n \in S$, we know that $v \in \text{span}(S)$.

(\Leftarrow) Conversely, suppose $v \in \text{span}(S)$. This implies that there exists a finite amount of scalars $a_1, a_2, \dots, a_n \in F$ and $u_1, u_2, \dots, u_n \in S$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Subtracting v from both sides of the equation above gives us the following equation:

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - v = 0.$$

Note that v is not contained in S , so $v \neq u_i$ for all $1 \leq i \leq n$. Since S is a linearly independent set, we know that $a_i = 0$ for all $1 \leq i \leq n$. This implies that the only coefficient that is non-zero is with $-v = -1v$. Hence, the set of vectors $S' = \{u_1, u_2, \dots, u_n, v\}$ is linearly dependent. Since $S' \subseteq S \cup \{v\}$, we know that $S \cup \{v\}$ is also linearly independent by Theorem 6. ■

1.5 Bases And Dimension

- Recall that S is a generating set for a subspace W and no proper subset of S is a generating set for W , then S must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

Definition 1.5.1 (Basis). A **basis** β for a vector space V is linearly independent subset of V that generates V . If β is a basis for V , we also say that the vectors of β form a basis for V .

Example 21. • Recall that the empty set \emptyset is linearly independent and that $\text{span}(\emptyset) = \{0\}$. The empty set \emptyset in this case is the basis for the zero vector space.

- Note that in F^n , the vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, 0, \dots, 0, 1)$ form a basis for F^n .
- The basis for $M_{m \times n}(F)$ is the set of matrices E^{ij} such that the only nonzero entry is a 1 in the i th and j th column.
- As we have seen in the last section, the set $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$.
- In $P(F)$, the set $\{1, x, x^2, \dots\}$ is a basis. *Bases are not limited to finite sets. They can be infinite.*

Theorem 1.5.1. Let V be a vector space and u_1, u_2, \dots, u_n be distinct vectors in V . Then $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V if and only if each $v \in V$ can be unique expressed as a linear combination of vectors in β , that is, expressed in the form

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for unique scalars a_1, a_2, \dots, a_n .

Proof. (\Rightarrow) Suppose $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V . Then $\text{span}(\beta) = V$. If $v \in V$, then $v \in \text{span}(\beta)$. Hence, we can write v as a linear combination of vectors in β such that choosing

scalars $a_1, a_2, \dots, a_n \in F$ leads to

$$v = \sum_{i=1}^n a_i u_i.$$

Suppose there exists another representation of $v \in V$ such that

$$v = \sum_{i=1}^n b_i x_i$$

Hence, observe that

$$\begin{aligned} \sum_{i=1}^n a_i x_i &= \sum_{i=1}^n b_i y_i \\ \Rightarrow \sum_{i=1}^n (a_i - b_i) x_i &= 0. \end{aligned}$$

Since β is linearly independent, we know that $a_i - b_i = 0$ which implies $a_i = b_i$ for all $1 \leq i \leq n$. Hence, v can be expressed as a unique linear combination of vectors in β .

(\Leftarrow) Conversely, let $v \in V$ be expressed as a unique linear combination of vectors in β such that

$$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \quad (1)$$

for unique scalars a_1, a_2, \dots, a_n . We want to show that β is a basis for V ; that is, we want to show that β is a spanning set for V and β is linearly independent. To show that β is a spanning set for V , we need to show that $\text{span}(\beta) \subseteq V$ and $V \subseteq \text{span}(\beta)$. Note that $\beta \subseteq V$ and $\text{span}(\beta)$ is a subspace for V . Hence, $\text{span}(\beta) \subseteq V$. On the other hand, $V \subseteq \text{span}(\beta)$ follows immediately from (1). To show that β is linearly independent, we need to show that for scalars $\delta_1, \delta_2, \dots, \delta_n$, we have

$$\delta_1 u_1 + \delta_2 u_2 + \dots + \delta_n u_n = 0$$

such that $u_i = 0$ for all $1 \leq i \leq n$. Note that $v + 0 = v$. Hence, we can write

$$\begin{aligned} \sum_{i=1}^n a_i u_i + \sum_{i=1}^n \delta_i u_i &= \sum_{i=1}^n a_i u_i \\ \sum_{i=1}^n (a_i + \delta_i) u_i &= \sum_{i=1}^n a_i u_i \end{aligned}$$

Equating each term in the equation above yields the following equality:

$$a_i + \delta_i = a_i \Rightarrow \delta_i = 0$$

for all $1 \leq i \leq n$. But this tells us that β is linearly independent and we are done. ■

- Any vector $v \in V$ can be written as a linear combination of vectors from the basis containing $u_1, u_2, \dots, u_n \in V$.
- This determines a unique n -tuple of scalars (a_1, a_2, \dots, a_n) and conversely, each n -tuple of scalars determines a unique vector $v \in V$ such that each coefficient from the linear combination of u_1, u_2, \dots, u_n is an entry from said tuple.
- For example, in our vector space F^n , n is the number of vectors that should be in the basis for F^n which is indeed the case.
- In this book, we are only concerned with finite bases.

Theorem 1.5.2 (Finite Spanning Set For a Vector Space). If a vector space V is generated by a finite set S , then some subset of S is a basis for V . Hence, V has a finite basis.

Proof. Suppose $S = \emptyset$ or $S = \{0\}$, then $V = \{0\}$ and \emptyset is a subset of S that is a basis for V . If S neither of these choices, then S must contain at least one nonzero vector u_1 ; that is, $S = \{u_1\}$. Since u_1 is nonzero, it follows that S is a linearly independent set. We can continue this process of adding vectors u_2, \dots, u_k into S such that S is a linearly independent set of k vectors. Since S is a finite set, we must end with the linearly independent set $\beta = \{u_1, u_2, \dots, u_n\}$. There are two cases for which this occurs, either $\beta = S$ or $\beta \subseteq S$:

- (i) Suppose that $\beta = S$ (remember that β is a finite set by construction). Then we have S is a linearly independent set and spanning set for V (since S is a finite set that generates V). Hence, S is a finite basis for V .
- (ii) Suppose $\beta \subseteq S$ is a linearly independent set such that adding $v \in S$ where $v \notin \beta$ makes a linearly dependent set. We claim that β is the desired subset of S that is a basis for V . Then we have two cases; that is, either $v \in \beta$ or $v \notin \beta$. Since $\beta \subseteq S$, we know by Theorem 1.5 that $\text{span}(\beta) \subseteq S$ (This applies for both cases). It suffices to show that $S \subseteq \text{span}(\beta)$. If $v \in \beta$, then surely $v \in \text{span}(\beta)$. Hence, we have $S \subseteq \text{span}(\beta)$. Suppose $v \notin \beta$. Since β is a linearly independent set, then by Theorem 1.7, we have that $\beta \cup \{v\}$ being linearly dependent implies that $v \in \text{span}(\beta)$. Hence, $S \subseteq \text{span}(\beta)$. Thus, that both cases implies that β is a spanning set for V . ■

This theorem tells us that any spanning set of a vector space V can be reduced to a finite basis for V . This is illustrated in the following examples.

Example 22. Define

$$S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$$

which can be shown to generate \mathbb{R}^3 . The idea is to create a proper subset of S such that none of the vectors in β are multiples of each other. In other words, we want a set that is linearly independent; that is, our choices of vectors determine whether a set will be linearly dependent or independent (we want the latter to hold). Say, we pick $(2, -3, 5)$ as our first vector in our subset β . Right away, we can exclude $(8, -12, 20)$ since it is a multiple of $(2, -3, 5)$. Otherwise, including it would make β linearly dependent (see exercise 9 from section 1.5). Next, add the vectors $(1, 0, -2)$ and $(0, 2, -1)$ since they are not multiples of the other. Narrowing down our set to

$$\beta = \{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}.$$

Note that adding the vector $(7, 2, 0)$ makes β linearly dependent, so $(7, 2, 0)$ is excluded from the list. Hence, we have arrived at a subset of S such that $\beta \subseteq S$ is both a linearly independent set and spanning set for \mathbb{R}^3 .

The following theorem and its corollaries are the most important results in the Chapter 1.

Theorem 1.5.3 (Replacement Theorem). Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V .

Proof. Let us proceed the proof via induction on m . Let $m = 0$ be our base case. Then we find that $L = \emptyset$ is linearly independent set with exactly 0 vectors. Letting $H = G$ gives us the desired result (since G contains exactly n vectors). Now suppose that the theorem holds for some integer $m \geq 0$. We will show that the theorem holds for the $m + 1$ case. Let $L = \{v_1, v_2, \dots, v_{m+1}\}$ be a linearly independent subset of V consisting of exactly $m + 1$ vectors. By the corollary to Theorem 1.6, we find that $L' = \{v_1, v_2, \dots, v_m\}$ is a linearly independent set (because $L' \subseteq L$ and L is

linearly independent). Using our induction hypothesis, we can conclude that $m \leq n$ and that there exists a subset $H' = \{u_1, u_2, \dots, u_{n-m}\}$ of G such that $L' \cup H'$ generates V . Thus there exists scalars $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_{n-m}$ such that

$$v_{m+1} = a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + b_2 u_2 + \dots + b_{n-m} u_{n-m}. \quad (1)$$

Note that $n - m > 0$, **unless** v_{m+1} is a linear combination of v_1, v_2, \dots, v_m which by Theorem 1.7 contradicts the assumption that L is a linearly independent set. Hence, $n > m$; that is, $n \geq m + 1$. Furthermore, some b_i , say b_1 is nonzero, for otherwise we obtain the same contradiction. Solving (1) for u_1 , we get

$$\begin{aligned} u_1 = & (-b_1^{-1} a_1) v_1 + (-b_1^{-1} a_2) v_2 + \dots + (-b_1^{-1}) v_m + (b_1^{-1}) v_{m+1} \\ & + (-b_1^{-1} b_2) u_2 + \dots + (-b_1^{-1} b_{n-m}) u_{n-m}. \end{aligned}$$

Let $H = \{u_2, u_3, \dots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$ and because $v_1, v_2, \dots, v_m, u_2, \dots, u_{n-m}$ are clearly in $\text{span}(L \cup H)$, we have that

$$L' \cup H' \subseteq \text{span}(L \cup H).$$

Since $L' \cup H'$ generates V and the fact that $L' \cup H' \subseteq \text{span}(L \cup H)$ (note that $\text{span}(L \cup H)$ is also a subspace), we know Theorem 1.5 that $\text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$. Since $\text{span}(L' \cup H')$ generates V , we know that $V \subseteq \text{span}(L' \cup H') \subseteq \text{span}(L \cup H)$. Observe that $\text{span}(L \cup H) \subseteq V$ is true by default. Hence, $\text{span}(L \cup H)$ generates V and that H contains $(n - m) - 1 = n - (m + 1)$ vectors which concludes our induction proof. ■