0.1 Lecture 12

0.1.1 Topics

- Definition of $K-$cell.
- Theorem: If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.
- Theorem: Every $k-$cell is compact.
- Theorem: Suppose $E \subseteq \mathbb{R}^k$. Then
- Connected sets

  E is closed and bounded $\iff$ E is compact $\iff$ Every infinite subset of E has a limit point in E.
- Theorem: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Corollary. If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of compact sets, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Theorem (Nested Interval Property). If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Definition (K-cell). The set $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ is called a $k-$cell in $\mathbb{R}^k$.

Example. Let $I = [a_1, a_2] \times [a_2, b_2]$ is a $2-$cell in $\mathbb{R}^2$.

Theorem (Nested Cell Property). If $I_1 \supseteq I_2 \supseteq I_3 \cdots$ is a nested sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(n)}, b_1^{(n)}] \times \dots \times [a_k^{(n)}, b_k^{(n)}]$$

Also, let

$$\forall n \in \mathbb{N} \text{ and } \forall 1 \leq i \leq k, \text{ we have } A_i^{(n)} = [a_i^{(n)}, b_i^{(n)}]$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$A_i^{(n)} \supseteq A_i^{(n+1)} \ \forall 1 \leq i \leq k.$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \dots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \dots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \cdots A_k^{(n)}. \end{aligned}$$

Hence, it follows from the nested interval property that there exists

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}$$
$$\exists x_2 \in \bigcap_{i=1}^{\infty} A_2^{(n)}$$
$$\vdots$$
$$\exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}.$$

Thus, by a fact in set theory; that is,

$$(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D).$$

$$\begin{aligned}(x_1, \ldots, x_k) &\in \Big[\bigcup_{n=1}^{\infty} A_1^{(n)}\Big] \times \Big[\bigcap_{n=1}^{\infty} A_2^{(n)}\Big] \times \cdots \times \Big[\bigcap_{n=1}^{\infty} A_k^{(n)}\Big] \\ &\subseteq \bigcap_{n=1}^{\infty} [A_1^{(n)} \times \cdots \times A_k^{(n)}] \\ &= \bigcap_{n=1}^{\infty} I_n.\end{aligned}$$

Hence, we see that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

Theorem. Every $k-$cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for $2-$cells. The proof for a general $k-$cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a $2-$cell. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Let

$$\delta = d(a, b) = \|a - b\| = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$$

. Note that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in I, then

$$\begin{aligned} x_1, y_1 \in [a_1, b_2] &\Longrightarrow |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] &\Longrightarrow |x_2 - y_2| \leq |b_2 - a_2| \end{aligned}$$

which implies that

$$\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta.$$

So, $d(x, y) \leq \delta$. Let us assume for contradiction that I is NOT compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of I that does NOT have a finite subcover; that is, $I \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2} \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i].$$

These subintervals determine 4 $2-$cells. There is at least one of these 4 $2-$cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let us call this $2-$cell as I_1. Notice that

$$\forall x, y \in I_1 \quad \|x-y\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into 4 $2-$cells and continue this process inductively. In this manner, we will obtain a sequence of $2-$cells

$$I, I_1, I_2, I_3, \ldots$$

such that

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \tag{1}$$

$$\forall x, y \in I_n, \quad \|x-y\| \leq \frac{\delta}{2} \tag{2}$$

$$\forall n \in \mathbb{N} \quad I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in I}$$

■

Theorem (Heine-Borel Theorem). Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point of E.

Proof. We will show that $(a) \implies (b) \implies (c) \implies (a)$.

$((a) \implies (b))$ Assume that E is closed and bounded. Our goal is to show that E is compact. Since E is bounded, there exists a $K-$cell, I, such that $E \subseteq I$. Note that by Theorem 2.40, we see that I is compact. By Theorem 2.3.5, E is compact.

$((b) \implies (c))$ Assume that E is compact. Our goal is to show that E is limit point compact; that is, every infinite subset of E has a limit point in E. See proof from last week.

$((c) \implies (a))$ Assume that every infinite subset of E has a limit point in E. Our goal is to show that E is closed and bounded. Suppose for sake of contradiction that E is NOT bounded and NOT closed.

Suppose that E is NOT bounded. In what follows, we will construct a sequence of points $x_1, x_2, \ldots$ in E. Since E is not bounded, we know that

$$\begin{aligned}
E \not\subseteq N_1(0) &\implies \exists x_1 \in E \text{ such that } d(x_1, 0) = \|x_1\|_2 \geq 1.\\
E \not\subseteq N_2(0) &\implies \exists x_2 \in E \text{ such that } d(x_2, 0) = \|x_2\|_2 \geq 2\\
E \not\subseteq N_3(0) &\implies \exists x_3 \in E \text{ such that } d(x_3, 0) = \|x_3\|_2 \geq 3\\
&\vdots\\
E \not\subseteq N_n(0) &\implies \exists x_n \in E \text{ such that } d(x_n, 0) = \|x_n\|_2 \geq n\\
&\vdots
\end{aligned}$$

That is, we have a sequence of points $S = \{x_n : n \in \mathbb{N}\}$ in E with the property that

$$d(x_n, 0) = \|x_n\|_2 \geq n.$$

Note that S is an infinite set; indeed, if S were finite, then

$$S = \{a_1, \ldots, a_m\}.$$

Now, let

$$r = \max\{d(a_n, 0) : 1 \le n \le m\}$$

and let $n \in \mathbb{N}$ be such that $n > r + 1$ (By the Archimedean Property). Since $\|x_n\|_2 \ge n > r + 1$, we can conclude that none of the $a_1, \dots, a_m$ is x_n and this contradicts how S was constructed.

Now, we will show that E is closed. Assume for contradiction that E is NOT closed; that is, $E' \not\subseteq E$. That is, there exists $y_0 \in \mathbb{R}^k$ such that $y_0 \in E'$ but $y_0 \in E$. We will construct a sequence of points $y_1, y_2, \dots$ in E as follows:

$$\begin{aligned}
y_0 \in E' &\implies N_1(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_1 - y_0\| < 1 \\
y_0 \in E' &\implies N_{\frac{1}{2}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_2 - y_0\| < \frac{1}{2} \\
&\vdots \\
y_0 \in E' &\implies N_{\frac{1}{m}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_m - y_0\| < \frac{1}{m} \\
&\vdots
\end{aligned}$$

Let $T = \{y_1, y_2, y_3, \dots\}$ and note that $T \subseteq E$. Note T is infinite (by construction $y_1, y_2, y_3, \dots$ are distinct elements. We claim that if $z \neq y_0$, then $z \in T'$. To this end, we have for all $n \in \mathbb{N}$

$$d(y_0, z) = \|y_0 - z\|_2 \le \|y_0 - y_n\|_2 + \|y_n - z\|_2$$

which implies further that

$$\|y_n - z\|_2 \ge \|y_0 - z\|_2 - \|y_n - y_0\|_2 > \|y_0 - z\|_2 - \frac{1}{n}.$$

Hence, for all $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{1}{2}\|y_0 - z\|_2$, we have

$$\begin{aligned}
d(y_n, z) &> \|y_0 - z\|_2 - \frac{1}{n} \\
&> \|y_0 - z\|_2 - \frac{1}{2}\|y_0 - z\|_2 \\
&= \frac{1}{2}\|y_0 - z\|_2.
\end{aligned}$$

So, for all but finitely many n, we see that

$$d(y_n, z) > \frac{1}{2}\|y_0 - z\|_2.$$

Hence, if we let $\varepsilon = \frac{1}{4}\|y_0 - z\|_2$, then $N_\varepsilon(z) \cap T$ is a finite set which proves $z \notin T'$. But this implies that the only possible limit point of T is y_0, but $y_0 \notin E$. Hence, T is an infinite subset of E with not limit point in E. This contradicts our hypothesis that every infinite subset of E has a limit point of E. ■

Remark. Note that in any general metric space, we have $(a) \implies (b)$ is not necessarily true.

Theorem (Bolzano-Weierstrass Theorem)**.** Let $E \subseteq \mathbb{R}^k$ and E is an infinite set and bounded. Then $E' \neq \emptyset$.

Proof. Suppose that E is bounded. Then there exists a $k-$cell I such that $E \subseteq I$. By Theorem 2.40, we know that I is a compact set. Furthermore, we know that I is limit point compact by Theorem 2.41. So, every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E \neq \emptyset$. ■

0.2 Lecture 13

0.2.1 Topics

- Separated sets, disconnected sets, connected set.
- Theorem: $E \subseteq \mathbb{R}$ is connected if and only if $x, y \in E$ and $z \in (x, y)$ implies $z \in E$.
- Perfect Sets
- Theorem: $P \subseteq \mathbb{R}^k$ is nonempty perfect implies P is uncountable.
- The Cantor Set

Definition (Connected Sets, Disconnected, connected). Let (X, d) be a metric space.

(i) Two sets $A, B \subseteq X$ are aid to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(iii) A set $E \subseteq X$ is said to **disconnected** if it can be written as a union of tow nonempty separated sets A and B; that is, $E = A \cup B$.

(iv) A set $E \subseteq X$ is said to be connected if it is NOT disconnected.

Example ($\mathbb{R}$ with the standard metric). (*) If we have $A = (1, 2)$ and $B = (2, 5)$ are separated, then

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset$$
$$A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.$$

Hence, $E = A \cup B$ is disconnected.

(*) We have $C = (1, 2]$ and $D = (2, 5)$ are disjoint but not separated; that is, we have

$$C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\}$$
$$C \cup D = (1, 5) \text{ is indeed connected.}$$

Theorem. Let $E \subseteq \mathbb{R}$. E is connected if and only if E contains the following property

$$\text{If } x, y \in E \text{ and } x < z < y, \text{ then } z \in E.$$

Proof. ■

Remark (Proposition 3.3.5, "Differential Calculus on Normed Space", Cartan). Let U be an open set in a normed (real) vector space. The following conditions are equivalent:

(a) U is connected

(b) U is path connected (any two points in U can be connected by a path in U)

Remark (Chapter 4, "Introduction" to Topological Manifolds", John Lee). In any metric space,

$$\text{path connected} \Longrightarrow \text{connected}.$$

However, the converse is not always true!

Definition (Perfect Set). Let (X, d) be a metric space. Let $E \subseteq X$. The following are equivalent definitions:

(i) E is said to be **perfect** if $E' = E$

(ii) E is said to be **perfect** if $E' \subseteq E$ and $E \subseteq E'$.

(iii) E is said to be perfect if E is closed and every point of E is a limit point of E.

(iv) E is said to be perfect if E is closed and E does not have any isolated points.

Example.

- $E = [0,1] \implies E' = [0,1]$. Thus, we have $E = E'$ and so E is perfect.
- $E = [0,1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is NOT perfect.
- $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \implies E' = \{0\} \implies E \neq E'$, so E is not perfect.
- Is E' from the last example perfect? Indeed, $E' = \{0\} \implies (E')' = \emptyset$. Thus, $E' \neq (E')' \implies E'$ is NOT perfect.
- $E = \emptyset$ and $E' = \emptyset$. Thus, $E = E'$ so E is perfect.

Theorem. Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Our proof of the theorem above will use the following two lemmas:

Lemma. Let (X,d) be a metric space. Let $E \subseteq X$ be perfect. If V is any open set in X such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

Proof. Let $q \in V \cap E$. Thus, $q \in V$ and $q \in E$. Then $q \in V$ implies that there exists $\delta > 0$ such that $N_\delta(q) \subseteq V$ and $q \in E$ implies $q \in E'$. As a consequence of these two results, we see that $N_\delta(q) \cap E$ is an infinite set. Thus, $V \cap E$ is an infinite set (Here, we are using the fact that an open set intersected with a perfect set is infinite whenever the intersection is nonempty). ■

Lemma. Let $q \in \mathbb{R}^k$. Let $r > 0$. Then $\overline{N_r(q)} = C_r(q)$.

Proof. Note that since $P' = P$ and $P \neq \emptyset$, we have $P' \neq \emptyset$. Thus, P is infinite. Assume for contradiction that P is countable. Let's denote the distinct elements of P by $x_1, x_2, x_3, \ldots$; that is, we can denote

$$P = \{x_1, x_2, x_3, \dots\}.$$

In what follows, we will construct a sequence of neighborhoods $V_1, B_2, V_3, \ldots$ such that

(i) For all $n \in \mathbb{N}$, we have $\overline{V_{n+1}} \subseteq V_n$.

(ii) For all $n \in \mathbb{N}$, $x_n \notin \overline{V_{n+1}}$.

(iii) For all $n \in \mathbb{N}$, $V_n \cap P \neq \emptyset$.

Let's assume that we have constructed these neighborhoods (Construction of neighborhoods). Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset.$$

Note that

(I) $\overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n}$ so $\overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P$ and so $K_{n+1} \subseteq K_n$ for each n.

(II) Since $\overline{V_n}$ is a closed and bounded set in $\mathbb{R}^k$, we have that $\overline{V_n}$ is compact. Furthermore, P being a perfect set implies that P is a closed set. As a consequence of these two facts, we can conclude that $K_n = \overline{V_n} \cap P$ is compact.

Using facts (I) and (II), we can conclude that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset \tag{*}$$

by Theorem 2.3.6. Recall that for all n, we have $K_n \subseteq P$, and so we have

$$\bigcap_{n=1}^{\infty} K_n \subseteq P.$$

In what follows, we will contradict (*). Let $b \in P$ be arbitrary. Then $b = x_m$ for some $m \in \mathbb{N}$. By property (ii), we see $x_m \notin \overline{V_{m+1}}$ and so $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$. This tells us that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

■

Remark (On the construction of neighborhoods $V_1, V_2, \ldots$)**.** Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$ (because $x_1 \in V_1$ and $c_1 \in P$). Our goal is to construct an open neighborhood V_2 such that

(i) $\overline{V_2} \subseteq V_1$,

(ii) $x_1 \notin \overline{V_2}$,

(iii) $V_2 \cap P \neq \emptyset$.

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$. By the first lemma found above, there exists $y_1 \in V_1 \cap P$ such that $y_1 \neq x_1$. Since V_1 is open and $y_1 \in V_1$, there exists $\delta_1 > 0$ such that $N_{\delta_1}(y_1) \subseteq V_1$.

$$\text{Let } r_2 = \tfrac{1}{2} \min\{d(x_1, y_1), \delta_1\}.$$

Let $V_2 = N_{r_2}(y_1)$. We claim V_2 has all the desired properties above. Indeed, we see that

(i) Observe that

$$\begin{aligned} \overline{V_2} = \overline{N_{r_2}(y_1)} &= \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\} \\ &\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1) \\ &\subseteq V_1. \end{aligned}$$

(ii) Notice that $d(x_1, y_1) > r_2$ implies that

$$x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}.$$

(iii) Since $y_1 \in V_2$ and $y_1 \in P$, we clearly have that $V_2 \cap P \neq \emptyset$.

We can construct $V_3, V_4, \ldots$ in a similar manner.

As a consequence of the theorem above, we have

Corollary. The interval $[0, 1]$ is uncountable.

0.2.2 The Cantor Set

The construction of the Cantor set is typically done in stages.

Step 0:

Let $E_0 = [0, 1]$.

Step 1:

Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, that is, remove the middle third, and define

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

The middle third, in this case, will be calculated by the following

$$\left(\frac{1}{3}, \frac{2}{3}\right) = \left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right).$$

Step 2:

Take each of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and remove the middle third of each of those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Likewise, we can remove the middle third move explicitly by calculating the following

$$\left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right) \text{ and } \left(\frac{3(2)+1}{3^2}, \frac{3(2)+2}{3^2}\right).$$

Continue in this manner until we obtain a sequence of compact sets:

$$E_1, E_2, E_3, E_4, \ldots$$

with the following properties

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \ \cdots$

(2) For each $n \in \mathbb{N}$, E_n is the union of 2^n intervals of length $\frac{1}{3^n}$.

The set $P = \bigcap_{n=1}^{\infty} E_n$ is called the **Cantor Set**.

Remark. Notice that in order to obtain E_n, we remove intervals of the form $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ from E_{n-1}; that is, k is such that $0 \leq k$ and $3k+2 < 3^n$.

Theorem (Properties of the Cantor Set)**.** Let P denote the Cantor set. Then

(1) P is compact

(2) P is nonempty

(3) P contains no segment

(4) P is perfect (and so it is uncountable)

(5) P has measure zero.

Proof. (1) Note that P is an intersection of compact sets. Hence, P is compact (see hw5).

(2) It follows from Theorem 2.3.6 that the intersection of a sequence of nested nonempty compact sets is nonempty. (In fact, the endpoint of each interval that appears at any state belong to P)

(3) Our goal is to show that P does NOT contain any set of the form (α, β) (where $0 \leq \alpha, \beta \leq 1$). Note that, by the construction of P, the intervals of the form:

$$I_{k,n} = \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \; n \in \mathbb{N}, 3k+2 < 3^n$$

have no intersection with P. However, (α, β) contains at least one of $I_{k,n}$'s. Indeed, (α, β)

contains $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$.

■