Homework 5

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March 19, 2025

Problem 1. Let $(V, \|\cdot\|)$ be a normed space and Y be a vector subspace of V. Last time, we saw that $V/Y = \{v + Y : v \in V\}$ is also a vector space. Now, assume that Y is closed in $(V, \|\cdot\|)$.

(i) Let v and v' such that $v - v' \in Y$. Show that $\inf_{y\in Y} \|v + Y\| = \inf_{y \in Y} \|v' + y\|$.

Proof. From problem 2(i) of Homework 4, $v - v' \in Y$ implies that $v + Y = v' + Y$. Hence, we have

$$\begin{aligned} v + Y = v' + Y &\implies \|v + y\| = \|v' + y\| \; \forall y \in Y \\ &\implies \inf_{y\in Y} \|v + y\| = \inf_{y\in Y} \|v' + y\|. \end{aligned}$$

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(ii) For $[v] = v + Y \in V/Y$, define

$$\|[v]\|_0 = \inf_{y\in Y} \|v + y\|.$$

Show that $\|\cdot\|_0$ defines a norm on V/W.

Proof. Clearly, we have $\|[v]\|_0 \geq 0$ since $\|\cdot\|$ satisfies property (I).

(I) Suppose $v + Y = 0_{V/W}$ where $[0] = 0_{V/W} = 0_V + Y$. Then by definition of $\|\cdot\|_0$, we have $\|[0]\|_0 = 0$. From part (a), we have

$$\begin{aligned} \|[v]\|_0 = \|[0]\|_0 &\iff \inf_{y\in Y} \|v + y\| = 0 \\ &\iff \|[v]\|_0 = 0. \end{aligned}$$

Hence, the property (I) is satisfied.

(II) Let $\alpha \in F$ where F is a field. Then we have

$$\begin{aligned} \|[\alpha v]\|_0 &= \|\alpha v + Y\|_0 \\ &= \inf_{y\in Y} \|\alpha v + Y\| \\ &= \inf_{y\in Y} \|\alpha(v + Y)\| \\ &= \inf_{y\in Y} |\alpha|\|v + Y\| \\ &= |\alpha| \inf_{y\in Y} \|v + Y\| && (\|\cdot\| \text{ is a norm}) \\ &= |\alpha|\|[v]\|_0. \end{aligned}$$

(III) Let $v_1, v_2 \in V/W$. Then since $\|\cdot\|$ is a norm, we have that

$$\begin{aligned}
\|[v_1+v_2]\|_0 &= \|(v_1+v_2)+Y\|_0 \\
&= \|(v_1+Y)+(v_2+Y)\|_0 \\
&= \inf_{y\in Y} \|(v_1+y_1)+(v_2+y_2)\| \\
&\le \inf_{y\in Y} \Big[\|v_1+y_1\|+\|v_2+y_2\|\Big] \\
&= \inf_{y\in Y} \|v_1+y_1\| + \inf_{y\in Y} \|v_2+y_2\| \\
&= \|[v_1]\|_0 + \|[v_2]\|_0.
\end{aligned}$$

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(iii) For any $v \in V$, show that $\|[v]\|_0 \le \|u\|$.

Proof. By the triangle inequality, we have

$$\|v\| = \|v\| + \|0_Y\| \ge \|v+0_Y\| \ge \inf_{y\in Y} \|v+Y\| = \|[v]\|_0.$$

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(iv) We have a Canonical map $\pi : V \to V \setminus Y$, $\pi(u) = [u]$. Show that π is linear and continuous. Here continuity means that if $\|v_n - v\| \to 0$ in V, then $\|[v_n]-[v]\|_0 \to 0$ in V/W.

Proof. First, we show that π is linear. For any $u_1, u_2 \in V$, we have

$$\begin{aligned}
\pi(u_1+u_1) &= [u_1+u_2] \\
&= (u_1+u_2)+Y \\
&= (u_1+Y)+(u_2+Y) \\
&= [u_1]+[u_2] \\
&= \pi(u_1)+\pi(u_2).
\end{aligned}$$

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