Contents

Linear Algebra Exercises

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Chapter 1

Vector Spaces

1.1 Linear Dependence and Linear Independence

Exercise 1.5.1

Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vector in S.

> **Proof. True** ∎

(b) Any set containing the zero vector is linearly dependent.
> **Proof. True** ∎

(c) The empty set is linearly dependent.
> **Proof. False**. It is linearly independent. ∎

(d) Subsets of linearly dependent sets are linearly dependent. **True** by Theorem 6.

(e) Subsets of linearly independent sets are linearly independent.
> **Proof. True** by corollary to Theorem 6. ∎

(f) If $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.
> **Proof. True** this is by definition. ∎

Exercise 1.5.4

In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

> **Proof.** Choose a finite amount of scalars $a_1, a_2, \ldots, a_n \in F$ to create the following linear combination:
> $$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = (0, 0, \ldots, 0). \tag{1}$$
> To show that the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, we need to show that the scalars $a_1, a_2, \ldots, a_n \in F$ have the trivial representation; that is, $a_1 = a_2 = \cdots = a_n = 0$. Since the jth coordinate of e_j is 1 but 0 in all the other entries, we have that
>
> $$a_1(1, 0, \ldots, 0) + a_2(0, 1, \ldots, 0) + \cdots + a_n(0, 0, \ldots, 1)$$
> $$= (a_1, 0, \ldots, 0) + (0, a_2, \ldots, 0) + \cdots + (0, 0, \ldots, a_n)$$
> $$= (a_1, a_2, \ldots, a_n).$$
>
> Hence, we have
> $$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

Equating each entry of the left side of the equation above to 0, we find that $a_i = 0$ for all $1 \leq j \leq n$. Hence, the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent. ∎

Exercise 1.5.4

Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

Proof. Just like the prior exercise, we need to show that we can find scalars $a_0, a_1, \ldots, a_n \in F$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where $a_i = 0$ for all $0 \leq i \leq n$. Note that the 0 polynomial is just

$$0 + 0x + 0x^2 + \cdots + 0x^n = 0.$$

Hence, equating coefficients we immediately get that $a_i = 0$ for all $0 \leq i \leq n$. Thus, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent. ∎

Exercise 1.5.6

In $M_{m \times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Proof. First, we create a linear combination of a finite amount vectors in $E = \{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ with scalars δ_k for $1 \leq k \leq N$ with $N = mn$ as the number of total entries in each matrix in $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Note that after doing our scalar multiplication and summing up each term, we find that each $\delta_k E_{ij} = \delta_k$ in our linear combination can equated with a corresponding i and j entry in the zero matrix such that $\delta_k = 0$ for all $1 \leq k \leq N$. Hence, E is a linearly independent set. ∎

Exercise 1.5.7

Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2 \times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

Proof. Define W as the linearly independent spanning set of the set of diagonal matrices in $M_{2 \times 2}$ where

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

To see why W is a linearly independent set, choose scalars $\delta_1, \delta_2 \in F$ such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing scalar multiplication and vector addition gives us the following equation

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the zero matrix is a diagonal matrix, we know that equation entries where $i = j$ yields $\delta_1 = \delta_2 = 0$. Hence, W is a linearly independent set that generates the set of diagonal matrices of $M_{2 \times 2}(F)$. ∎

Exercise 1.5.8

Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = \mathbb{R}$, then S is linearly independent.

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Proof. ■

(b) Prove that if F has characteristic two, then S is linearly dependent.

Proof. ■

Exercise 1.5.9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other. I have written two proofs for this:

Proof. Let u and v be distinct vectors in a vector space V.

($\Rightarrow$) Since $\{u, v\}$ is a linearly dependent set, we can find scalars $a_1, a_2 \in F$ such that

$$a_1 u + a_2 v = 0 \tag{1}$$

Suppose v is not a multiple of u and choose $a_1 \neq 0$ since $\{u, v\}$ is linearly dependent. We need to show that u is a multiple of v. Solving for u, we get that

$$u = -\frac{a_2}{a_1} v.$$

Hence, u is a multiple of v.

($\Leftarrow$) Suppose u or v is a scalar multiple of the other. Assume u is the scalar multiple of v. Then for some $c \neq 0 \in F$, we have $u = cv$. Hence, we have $u - cv = 1u - cv = 0$. This tells us that $\{u, v\}$ is linearly dependent. ■

Exercise 1.5.12

Prove Theorem 1.6 and its corollary.

Proof. See proof in notes. ■

Exercise 1.5.13

Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Proof. Let u and v be distinct vectors in V.

For the forwards direction, assume $\{u, v\}$ is a linearly independent set. We need to show that $\{u + v, u - v\}$ is linearly independent. Hence, we need to find $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0. \tag{1}$$

Note that (1) leads to

$$a(u + v) + b(u - v) = au + av + bu - bv$$
$$= au - bv + av + bu.$$

Since $\{u, v\}$ is a linearly independent set, we have that

$$au - bv = 0$$

and

$$av + bu = 0$$

for $a = b = 0$. Hence,

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ and so $\{u - v, u + v\}$ is a linearly independent set.

For the backwards direction, suppose $\{u + v, u - v\}$ is linearly independent. We need to show that $\{u, v\}$ is linearly independent. Note that $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ since $\{u - v, u + v\}$ is linearly independent. Note that

$$
\begin{aligned}
a(u + v) + b(u - v) &= au + av + bu - bv \\
&= au - bv + av + bu \\
&= 0 + av + bu \\
&= 0.
\end{aligned}
$$

Thus, $av + bu = 0$ where a, b both zero. Thus, the set $\{u, v\}$ is linearly independent. ∎

(b) Let $u, v,$ and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

Proof. For the forwards direction, suppose $\{u, v, w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that
$$a_1 u + a_2 v + a_3 w = 0$$
with $a_1 = a_2 = a_3 = 0$. We need to show that $\{u + v, u + w, v + w\}$ is linearly independent; that is, we need to show that we can find scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0 \tag{1}$$

for $a_1 = a_2 = a_3 = 0$. Observe that (1) can be written in the following way

$$(a_1 u + a_3 v + a_2 w) + (a_1 v + a_2 u + a_3 w) = 0 \tag{2}$$

Since $\{u, v, w\}$ is linearly independent, we know that $a_1 = a_2 = a_3 = 0$. But this also has to mean that $\{u + v, u + w, v + w\}$ is a linearly independent set.

For the backwards direction, suppose $\{u + v, u + w, v + w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0. \tag{1}$$

We need to show that $\{u, v, w\}$ is linearly independent. Observe that (1) can be re-written as

$$
\begin{aligned}
(a_1 u + a_3 v + a_2 w) + (a_1 v + a_2 u + a_3 w) &= 0 \\
\Rightarrow 0 + (a_1 v + a_2 u + a_3 w) &= 0 \\
\Rightarrow a_1 v + a_2 u + a_3 w &= 0
\end{aligned}
$$

where $a_1 = a_2 = a_3 = 0$. Hence, $\{u, v, w\}$ is linearly independent. ∎

Exercise 1.5.14

Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

Proof. For the forwards direction, Let S be a linearly dependent. Then we need to show that either $S = \{0\}$ or S contains distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \text{span}(S)$. Suppose there does not exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \text{span}(S)$. This tells us that S only contains the singleton $S = \{v\}$. Furthermore, we must require $v = 0$ since S is linearly dependent. Otherwise, $v \neq 0$ would imply that S is linearly independent. Hence, we have $S = \{0\}$. Now suppose $S \neq \{0\}$. Since S is linearly dependent, there exists scalars $a_1, a_2, \ldots, a_{n+1}$ and vectors $v, u_1, u_2, \ldots, u_n$ such that
$$a_1 v + a_2 u_1 + a_3 u_2 + \cdots + a_{n+1} u_n = 0$$

with all $a_1, a_2, \ldots, a_n, a_{n+1}$ not all equal to zero. Solving for v, we get that

$$v = -(a_1^{-1}a_2)u_1 - (a_1^{-1}a_3)u_2 - \cdots - (a_1^{-1}a_{n+1})u_n.$$

Since v is a linear combination of $u_1, u_2, \ldots, u_n$, we have that $v \in \text{span}(S)$.

Conversely, suppose that either $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$. Assume $S = \{0\}$. Then S is linearly dependent because the singleton is the zero vector. Now suppose v is a linear combination of vectors $u_1, u_2, \ldots, u_n$. Then there exists scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $u_1, u_2, \ldots, u_n \in S$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Subtracting v from both sides of this equation yields the following equation

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - 1v = 0.$$

Since not all scalars in the equation above are zero and $v, u_1, u_2, \ldots, u_n \in S$, we must have that S is a linearly dependent set. ∎

Exercise 1.5.15

Prove that a set $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k where $1 \leq k < n$.

Proof. Suppose $u_{k+1} \notin \text{span}(\{u_1, u_2, \ldots, u_k\})$ for all $1 \leq k \leq n$. Since S is linearly dependent, we know that the zero vector is contained in S. Choose $k = 1$ such that $u_1 = 0$ and we are done. On the other hand, suppose $u_1 \neq 0$. We need to show that $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$. Choose $k = n - 1$. Then clearly $n = k + 1$. Since S is linear independent, choose scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0. \tag{1}$$

where $a_1, a_2, \ldots, a_k, a_{k+1}$ not all zero. Solving for u_{k+1} by subtracting $a_{k+1}u_{k+1}$ on both sides of (1) and multiplying $-a_{k+1}^{-1}$ on both sides of (1), we end up with the following equation:

$$u_{k+1} = -(a_{k+1}^{-1}a_1)u_1 - (a_{k+1}^{-1}a_2)u_2 - \cdots - (a_{k+1}^{-1}a_k)u_k.$$

This tells us that u_{k+1} can be written as a linear combination of vectors $u_1, u_2, \ldots, u_k$. Hence, $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots u_k\})$.

Conversely, either $u_1 = 0$ or $u_{k+1} \in \text{span}(u_1, u_2, \ldots, u_k)$. Suppose $u_1 = 0$. Then S contains the zero vector so S must be linearly dependent. On the other hand, choose $k = n - 1$ where $1 \leq k < n$ such that $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ implies that there exists scalars $a_1, a_2, \ldots, a_k$ such that

$$u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k$$
$$\Rightarrow u_n = a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}. \tag{1}$$

Subtracting u_n on both sides of (1) implies that

$$(a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}) - 1u_n = 0.$$

Since not all scalars in the linear combination above are zero, we know that S must be linearly dependent.

∎