0.1 Lecture 10

Theorem (E is open relative to Y). Let (X, d) be a metric space and $E \subseteq Y \subseteq X$ and $Y \neq \emptyset$. E is open relative to Y if and only if there exists an open set $G_0 \subseteq X$ such that $E = G \cap Y$.

Proof. ($\Longrightarrow$) Assume that E is open relative to Y. Our goal is to show that there exists an open set $G \subseteq X$ such that $E = G \cap Y$. Since E is open relative to Y, every $a \in E$, we have that a is an interior point of E; that is, there exists $\varepsilon_0 > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$. Hence, for all $a \in E$, there exists $\varepsilon_a > 0$ such that $N_{\varepsilon_a}(a) \cap Y \subseteq E$. Let $G = \bigcup_{a \in E} N_{\varepsilon_a}(a)$. Clearly, G is open in X since

(1) For all $a \in E$, $N_{\varepsilon_a}(a)$ is a neighborhood and so it is open in X.

(2) A union of open sets is open in any metric space.

In what follows, we will prove that $E = G \cap Y$. Note that

$$G \cap Y = \Big(\bigcup_{a \in E} N_{\varepsilon_a}(a)\Big) \cap Y = \bigcup_{a \in E} \Big(N_{\varepsilon_a} \cap Y\Big) \subseteq \bigcup_{a \in E} E = E.$$

Suppose $b \in E$. We have $b \in N_{\varepsilon_b}(b)$ and thus $b \in G$. Furthermore, $b \in E$ implies $b \in Y$ since $E \subseteq Y$. Thus, we see that $E \subseteq G \cap Y$.

($\Longleftarrow$) Assume that there exists $G \subseteq X$ such that $E = G \cap Y$. We want to show that E is open relative to Y. Our goal is to show that for all $a \in E$, there exists $\varepsilon > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$.

So, let $a \in E$ be given. Our goal is to find $\varepsilon > 0$ such that

$$N_\varepsilon(a) \cap Y \subseteq E.$$

By assumption, we have

$$\begin{aligned} a \in E = G \cap Y &\Longrightarrow a \in G \\ &\Longrightarrow_{G \text{ is open}} \exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \subseteq G. \end{aligned}$$

Hence, we see that

$$N_\varepsilon(a) \cap Y \subseteq G \cap Y = E.$$

■

Theorem. (i) If $E \subseteq \mathbb{R}$ is bounded above, then $\sup E \in \overline{E}$.

(ii) If $E \subseteq \mathbb{R}$ is bounded below, then $\inf E \in \overline{E}$.

Proof. Here we will prove (1) and the proof of (2) is completely analogous.

Since E is bounded above, we have that $\sup E$ exists and is a real number by the least upper bound property of $\mathbb{R}$. Let $\alpha = \sup E$. Our goal is to show that $\alpha \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,

$$N_\varepsilon(\alpha) \cap E \neq \emptyset. \qquad \text{(Exercise 11 of HW4)}$$

Let $\varepsilon > 0$ be given. Since $\alpha = \sup E$, we know that there exists $x \in E$ such that $\alpha - \varepsilon < x$. Hence, there exists $x \in E$ such that

$$\alpha - \varepsilon < x \leq \alpha < \alpha + \varepsilon.$$

Hence, $\alpha - \varepsilon < x < \alpha + \varepsilon$; that is, $x \in N_\varepsilon(\alpha)$. Therefore, $N_\varepsilon(\alpha) \cap E \neq \emptyset$. ■

Definition (Open Covers). Let (X, d) be a metric space and $E \subseteq X$. A collection of sets $\{O_\alpha\}_{\alpha \in \Lambda}$ is said to be an **open cover** of E if

(i) for every $\alpha \in \Lambda$, O_α is open in X.

(ii) $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$.

Example. Consider $(\mathbb{R}, |\bullet|)$ and the subset $E = [0, \infty)$. The collection $\{E_n\}_{n\in\mathbb{N}}$ defined by for all $n \in \mathbb{N}$, we have

$$E_n = \left(\frac{-1}{n}, n\right)$$

is an open cover of E. The reason is as follows:

(i) In homework 4, we proved that every open interval is an open set with the standard metric on $\mathbb{R}$.

(ii) $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Indeed, let $a \in E$. If $a = 0$, then a belongs E_n for all $n \in \mathbb{N}$. So, we have

$$a \in \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, if $a \neq 0$, then $\frac{1}{a} > 0$. By the Archimedean Property of $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a}$. So, $a < n$. Clearly,

$$\frac{-1}{n} < 0 < a < n.$$

Hence, $a \in E_n = \left(\frac{-1}{n}, n\right)$. Consequently,

$$a \in \bigcup_{m=1}^{\infty} E_m.$$

Definition (Compactness). Let (X, d) be a metric space and $K \subseteq X$. We say that K is **compact** if every open cover of K has a finite subcover; that is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover of K, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Example. Let (X, d) be a metric space and $E \subseteq X$. If E is finite, then E is compact. The reason is as follows:

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover. Our goal is to show that this open cover has a finite subcover; that is, we can choose $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

If $E = \emptyset$, there is nothing to prove. Otherwise, $E \neq \emptyset$, we can denote the elements of E by $x_1, \ldots, x_n$. So,

$$E = \{x_1, \ldots, x_n\}.$$

We have

$$x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

and so continue this process until you have reached the nth element of E. Hence, we have

$$x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}.$$

Thus, we have that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$$

and so E is compact.

Example. Let $(\mathbb{R}, |\cdot|)$ and

$$E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Prove that E is compact (In general, if $a_n \to a$ in $\mathbb{R}$, then the set $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact).

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover. Notice that $0 \in E$ and $E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha$ implies that

$$O \in \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_0 \in \Lambda \text{ such that } O \in O_{\alpha_0}.$$

So, if $O \in O_{\alpha_0}$ and O_{α_0} is open, we have

$$\exists \varepsilon > 0 \text{ such that } (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}.$$

By the Archimedean Property of $\mathbb{R}$, we see that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. So, for all $n \geq m$, we have $\frac{1}{n} < \varepsilon$. Hence, for all $n \geq m$, we have $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}$. Note that $1 \in E$ so there exists $\alpha_1 \in \Lambda$ such that $1 \in O_{\alpha_1}$. Similarly, $\frac{1}{2} \in E$ implies that there exists $\alpha_2 \in \Lambda$ such that $\frac{1}{2} \in O_{\alpha_2}$. Continue this process until, we have

$$\frac{1}{m-1} \in E \Longrightarrow \exists \alpha_m \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}}.$$

Thus, we conclude that

$$E \subseteq \bigcup_{n=0}^{\infty} O_{\alpha_n}$$

and so E is compact.

Remark. If X itself is compact, we say that (X, d) is a compact metric space; that is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that

$$X = \bigcup_{\alpha\in\Lambda} O_\alpha$$

then there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$X = \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Theorem (Compactness implies Closed)**.** Let (X, d) be a metric space and $K \subseteq X$ is compact. Then K is compact.

Proof. It is enough to show that K^c is open; that is, we need to find $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq K^c$ for every $x \in K^c$. That is, we want to show that

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\Longrightarrow a \notin K \\ &\Longrightarrow \forall x \in K \ d(x,a) > 0 \end{aligned}$$

For all $x \in K$, let $\varepsilon_x = \frac{1}{4}d(x,a)$. Clearly, we have

$$\forall x \in K \ N_{\varepsilon_x} \cap N_{\varepsilon_x}(a) = \emptyset.$$

Note that $\{N_{\varepsilon_x}(x)\}_{x\in K}$ is an open cover for K. Since K is compact, there is a finite subcover. That is, there exists $x_1, \ldots x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{x_i}}(x_i)$$

and of course

$$N_{\varepsilon_{x_1}}(x_1) \cap N_{\varepsilon_{x_1}}(a) = \emptyset.$$

If we continue this process until the nth step, we have that

$$N_{\varepsilon_{x_n}}(x) \cap N_{\varepsilon_{x_n}}(a) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}\}$. Clearly, $N_\varepsilon(a) \subseteq N_{\varepsilon_{x_i}}(a) \forall 1 \leq i \leq n$. Hence, we have

$$N_{\varepsilon_{x_1}} \cap N_\varepsilon(a) = \emptyset$$

and similarly for the nth step, we have

$$N_{\varepsilon_{x_n}}(x) \cap N_\varepsilon(a) = \emptyset.$$

Therefore, we have

$$N_\varepsilon(a) \cap \left[N_{\varepsilon_{x_1}} \cup \cdots \cup N_{\varepsilon_{x_n}}\right] = \emptyset.$$

So,

$$N_\varepsilon(a) \cap K = \emptyset.$$

■

0.2 Lecture 11

0.2.1 Topics

- Metric subspace
- Theorem 2.35
- Theorem 2.33
- Theorem 2.37
- Theorem 2.36

Theorem. Let (X,d) be a metric space and let $K \subseteq X$ be a compact set, and $E \subseteq K$ is closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Our goal is to show that this cover contains a finite subcover. Note that, E is closed by assumption and so E^c is open. Thus, we have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cup E^c.$$

Thus, E^c together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set K. Since K is compact, this open cover contains a finite subcover. So, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$K \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

Considering that $E \subseteq K$, we can write

$$E \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

However, $E \cap E^c \neq \emptyset$, so

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

So, we have $O_{\alpha_1}, \dots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. ■

Corollary. If F is closed and K is compact, then $F \cap K$ is compact.

Proof. Note $F \cap K$ is a closed subset of the compact set K. Since K is closed and F is closed along with $F \cap K \subseteq K$. Thus, K contains a finite subcover of $F \cap K$. Thus, $F \cap K$ is compact. ■

Theorem. Let (X, d) be a metric space and Suppose $K \subseteq Y \subseteq X$. We have K is compact relative to X if and only if K is compact relative to Y.

Proof. ($\Longleftarrow$) Assume that K is compact relative to Y. Our goal is to show that K is compact relative to X. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X that covers K. Our goal is to show that this cover has a finite subcover. Note that the set $K = K \cap Y$ since $K \subseteq Y$. Hence, we have

$$K = K \cap Y \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y). So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in (Y, d^Y) that covers E. Since K is compact relative to Y, there exists a finite subcover; that is, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n} (O_{\alpha_i} \cap Y) \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

($\Longrightarrow$) Assume that K is compact relative to X. Our goal is to show that K is compact relative to Y. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in (Y, d^Y) that covers K. Our goal is to show that this cover contains a finite subcover. It follows from Theorem 2.30 that for all $\alpha \in \Lambda$, there exists an open set $O_\alpha \subseteq X$ such that $G_\alpha = O_\alpha \cap Y$. We have

$$K \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y) = \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$

Thus, $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for K i n the metric space (X, d). Since K is compact, there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Hence, we see that

$$K = K \cap Y \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y = \bigcup_{i=1}^{n} (O_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} G_{\alpha_i}.$$

That is, this is a finite subcover we were looking for and we are done. ■

Note that the property of compactness gets preserved in X, but when we are referring to open and closed sets these properties do not get preserved in X.

Theorem. Let (X, d) be a metric space. If E is an infinite subset of a compact set K, then E has a limit point in K. (In fact, these two statements are equivalent, but we will only prove the forwards direction)

Proof. Let (X, d) be a metric space. Assume that $K \subseteq X$ is compact and E is infinite. We will show that E has a limit point in K. Assume for sake of contradiction that E has no limit point in K; that is, every limit point in K will never be a limit point of E. In other words, for all $a \in E$, $a \notin E'$. Furthermore, for all $b \in K \setminus E$, we have $b \notin E'$. Therefore, for all $a \in E$, there exists $\varepsilon_a > 0$ such that

$$N_{\varepsilon_a} \cap (E \setminus \{a\}) = \emptyset. \tag{1}$$

Likewise, for all $b \in K \setminus E$, there exists $\delta_b > 0$ such that

$$N_{\delta_b} \cap (E \setminus \{b\}) = \emptyset \iff N_{\delta_b} \cap E = \emptyset. \tag{2}$$

Notice that (1) implies that

$$N_{\varepsilon_a} \cap E = \{a\}.$$

Clearly, K is a subset of $\bigcup_{a\in E} N_{\varepsilon_a}(a) \cup \Big(\bigcup_{b \in K\setminus E} N_{\delta_b}(b)\Big)$. Since K is compact, there exists $a_1, \ldots, a_n \in E$ and $b_1, \ldots, b_m \in K \setminus E$ such that

$$K \subseteq \Big(\bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i)\Big) \cup \Big(\bigcup_{i=1}^{m} N_{\delta_{b_i}}(b_i)\Big).$$

Since for all $b \in K \setminus E$ and the fact that $N_{\delta_b}(b) \cap E = \emptyset$, we have

$$E \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i).$$

Hence,

$$\begin{aligned} E &= E \cap \Big[\bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i)\Big] \\ &= \bigcup_{i=1}^{n} (E \cap N_{\varepsilon_{a_i}}(a_i)) \\ &= \bigcup_{i=1}^{n} \{a_i\} \\ &= \{a_i : 1 \leq i \leq n\}. \end{aligned}$$

But this implies that E is finite which is a contradiction of our assumption that E is infinite. ■

Remark. Let (X, d) be a metric space and $K \subseteq X$. The following statements are equivalent:

(i) K is compact.

(ii) Every infinite subset of K has a limit point in K (limit point compactness).

(iii) Every sequence in K has a subsequence that converges to a point in K (sequential compactness).

Example. Consider the set of intervals $A_i = [i, \infty)$, then

$$\bigcap_{i=1}^{\infty} A_i = \emptyset.$$

Corollary. If $K_1 \supseteq K_2 \supseteq \ldots$ is a sequence of nonempty compact sets, and every finite intersection of these sets is nonempty, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. Suppose for sake of contradiction that the intersection

$$\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset.$$

Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \Big(\bigcap_{\alpha \in \Lambda} K_\alpha \Big) = \emptyset.$$

So,

$$K_{\alpha_0} \subseteq \Big(\bigcup_{\alpha \in \Lambda} K_\alpha \Big)^c \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} K_\alpha^c.$$

Note that K_α^c is open because K_α is closed. So, $\{K_\alpha^c\}$ is an open cover of K_{α_0}. Since K_{α_0} is compact, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{n} K_{\alpha_i}^c \subseteq \Big(\bigcap_{i=1}^{n} K_{\alpha_i} \Big)^c.$$

So, we have

$$K_{\alpha_0} \cap \Big(\bigcap_{i=1}^{n} K_{\alpha_i} \Big) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty.

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Theorem. Let $\{K_\alpha\}$ a collection of compact subsets of X. If the intersection of any finite subcollection is nonempty, then

$$\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset.$$

Proof. ■

Theorem (Nested Interval Property). If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. ■