Math 234A Lecture Notes

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Contents

Chapter 1

Week1

1.1 Lecture 1

1.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either $\mathbb{R}$ or $\mathbb{C}$. For example, the polynomial $x^2 + 1$ does not have a solution in $\mathbb{R}$, but it does have a solution $\mathbb{C}$.
- Solving real integrals that may be difficult to deal with using standard techniques developed in $\mathbb{R}$; that is, something like
$$\int_0^\infty \frac{\sin x}{x}\ dx.$$
- Solving problems in physics, particularly, in the Quantum Field Theory.

1.1.2 What is the goal?

Let's recall some facts about the real number system $\mathbb{R}$.

(i) $(\mathbb{R}, +, \cdot)$ is a field.

(ii) We have an order relation on $\mathbb{R}$.

(iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in $\mathbb{R}$.

Our goal is to find the "smallest" field $\mathbb{C}$ such that

(i) $\mathbb{R}$ is "contained" in $\mathbb{C}$.

(ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in $\mathbb{C}$.

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of $\mathbb{R}$ as a vector space, and using the operations defined on that vector space to define the operations of $\mathbb{C}$. Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i. In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \operatorname{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on $\mathbb{C}$.

Definition (Operations on $\mathbb{C}$). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that $\mathbb{C}$ forms a field.

Proposition. The defined operations of $\mathbb{C}$ form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \zeta$. Then there exists $\gamma + i\delta \in \zeta$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

1.2 Lecture 2

1.2.1 Topics

- Discuss the complex plane $\mathbb{C}$ as a working model for complex numbers.
- Discuss the Euclidean Topology on $\mathbb{C}$.
- Discuss polar representation of a complex number.

1.3 Complex Plane $\mathbb{C}$ as a working model for Complex Numbers

Lemma. Let F be a field containing $\mathbb{R}$ and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let
$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ is a **subfield** of F.

(ii) Let F be another field containing $\mathbb{R}$ and containing a solution i' of $x^2 + 1 = 0$ and
$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$
Then $\mathbb{C}$ and $\mathbb{C}'$ are isomorphic as fields.

The second part of this lemma is simply saying that $\mathbb{C}$ is a unique subfield of F.

1.3.1 A model for $\mathbb{C}$

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication $\cdot$ as
$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$
and
$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers $\mathbb{C}$ have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \to \mathbb{C}$ by $x \to (x, 0)$. Then $\mathcal{J}$ is a subfield and forms an isomorphism between $\mathbb{R}$ and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

(i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.

(ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homorphism.

(iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that $\mathcal{J}$ is a bijective map.

■

Definition (Constructing a solution for $x^2 + 1 = 0$)**.** Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number)**.** Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate)**.** Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\overline{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the $x-$axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

(i) $\overline{\overline{z}} = z$.

(ii) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(iii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iv) $\Re(z) = \frac{1}{2}(z + \overline{z})$.

(v) $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $z\overline{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number)**.** Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

Proposition (More Properties of Complex Numbers)**.** Given $z, w \in \mathbb{C}$, we have the following properties:

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|zw| = |z||w|$.

(iii) $|\Re(z)| \leq |z|$.

(iv) $|\Im(z)| \leq |z|$.

(v) $|z + w| \leq |z| + |w|$.

(vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

1.3.2 Viewing $\mathbb{C}$ as a Metric Space

In $\mathbb{C}$, the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space)**.** For $z, w \in \mathbb{C}$, we call $\mathbb{C}$ a metric space if it satisfies the following properties:

(i) $d(z, w) = d(w, z)$.

(ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.

Now we have $\mathbb{C}$ is a metric space endowed with the metric $d(z, w) = |z - w|$.

Chapter 2

Week2

2.1 Lecture 3

2.1.1 Topics

- Polar Representation of complex numbers.
- Convergence of sequences in $\mathbb{C}$.

2.1.2 Polar Representation of Complex Numbers

First, let us introduce some notation:

- $\mathbb{C}^{\cdot} = \{z \in \mathbb{C} : z \neq 0\}$.
- $\mathbb{R}_{+} = \{\alpha \in \mathbb{R} : \alpha > 0\}$.

Recall that any non-zero $(\alpha, \beta) \in \mathbb{R}^2$ can be represented as

$$(\alpha, \beta) = \gamma(\cos\varphi, \sin\varphi)$$

with $\tan\varphi = \frac{\beta}{\alpha}$.

Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos\psi, \sin\psi).$$

- γ is uniquely defined.
- φ is defined up to the addition of a multiple of 2π.

Remark. This representation may not be unique!

Proposition. The map $\mathbb{R}_{+} \times \mathbb{R} \to \mathbb{C}^{\cdot}$ defined by

$$(\gamma, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. The proposition above is a systematic way of saying that if $z \in \mathbb{C}^{\cdot}$, then

$$z = \gamma(\cos\varphi + i\sin\varphi)$$

with $\gamma = |z|$ and φ can be determined up to a multiple of 2π.

If we insist, we can make the polar representation unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Agument and Principle Argument). Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\mathrm{Arg}(z)$.

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos\varphi + i\sin\varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos\varphi + i\sin\varphi)(\cos\varphi' + i\sin\varphi') \\ &= \gamma\gamma'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' + \sin\varphi\cos\varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i\sin(\varphi + \varphi')). \end{aligned}$$

■

The proposition above allows us to visualize the multiplication of complex numbers as two parts:

- Scaling of the modulus.
- Addition of the two angles.

Corollary. Let $z \in \mathbb{C}^{\cdot}$ with $z = \gamma(\cos\varphi + i\sin\varphi)$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{\gamma}(\cos(-\varphi) + i\sin(-\varphi)) \\ &= \frac{1}{\gamma}(\cos\varphi - i\sin\varphi). \end{aligned}$$

Corollary (De Moivre's Theorem). Let $z = \gamma(\cos\varphi + i\sin\varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i\sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example (An example of De Moivre's Theorem). Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i\sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i\sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

2.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. Suppose $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$. Since $(z_n) \to z$, there exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2\alpha}.$$

Likewise, $(w_n) \to w$ implies that we can find an $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2\beta}.$$

Now, choose $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we must have

$$\begin{aligned}
|\alpha z_n + \beta w_n - (\alpha z + \beta w)| &= |\alpha(z_n - z) + \beta(w_n - w)| \\
&\leq \alpha|z_n - z| + \beta|w_n - w| \\
&< \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we can conclude that

$$\alpha z_n + \beta w_n \to \alpha z + \beta w.$$

(ii) Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that

$$|z_n w_n - zw| < \varepsilon.$$

Since $(z_n) \to z$, we can find a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon}{2M}$$

where $M > 0$. Since $(w_n) \to w$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|w_n - w| < \frac{\varepsilon}{2|z|}.$$

Thus, choose $N = \max\{N_1, N_2\}$ such that for any $n \geq N$, we have

$$\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - w_n z + w_n z - zw| \\
&\leq |w_n||z_n - z| + |z||w_n - w| \\
&\leq M|z_n - z| + |z||w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2|z|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, we conclude that

$$z_n w_n \to zw.$$

(iii) Let $\varepsilon > 0$. We will show that $z_n^{-1} \to z^{-1}$ by showing that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|z_n^{-1} - z^{-1}| < \varepsilon.$$

Since $(z_n) \to z$, there must exists an $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - z| < \frac{\varepsilon |z|^2}{2}.$$

Likewise, we can choose $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|z_n| > \frac{|z|}{2}.$$

If we choose $N = \max\{N_1, N_2\}$, and subsequently, let $n \geq N$, then we must have

$$\begin{aligned}
|z_n^{-1} - z^{-1}| &= \frac{|z_n - z|}{|z||z_n|} \\
&< \frac{\varepsilon |z|^2}{2} \cdot \frac{2}{|z|^2} \\
&= \varepsilon.
\end{aligned}$$

Thus, we conclude that $z_n^{-1} \to z^{-1}$.

■

Chapter 3

Week3

3.1 Lecture 4

3.1.1 Topics

(i) Continue discussion of convergence of sequence/series.

(ii) Discuss exponential, sine, and cosine function.

3.1.2 Convergence of Series

Infinite series of complex numbers. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

Goal: To give meaning to $\sum_{n=1}^{\infty} z_n$.

Define

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

Definition (Convergence of Series)**.** If the sequence $\{s_n\}_{n=1}^{\infty}$ converges, we say that the series

$$\sum_{n=1}^{\infty} z_n$$

converges and we write

$$\sum_{n=1}^{\infty} z_n = s.$$

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot i}.$$

Let

$$\begin{aligned} s_n &= \frac{1}{2i} + \frac{1}{2^2 i} + \cdots + \frac{1}{2^n i} \\ &= \frac{1}{i}\Big[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\Big]. \end{aligned}$$

$$is_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \tag{1}$$
$$\frac{1}{2} is_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2}$$

Consider
$$\frac{1}{2} i \cdot s_n = \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$
Then subtracting (1) and (2), we see that
$$is_n - \frac{1}{2} is_n = \frac{1}{2} is_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow s_n = \frac{1}{i}\Big[1 - \frac{1}{2^n}\Big].$$
Thus, taking the limit as $n \to \infty$ gives us
$$\lim_{n\to\infty} s_n = \frac{1}{i}.$$

Example. Assume that $|z| < 1$, $z \in \mathbb{C}$. Compute
$$\sum_{n=1}^{\infty} z^n.$$
Since $|z| < 1$, we know that
$$\sum_{n-1}^{\infty} z^n$$
is a geometric series and that it converges. Using the geometric series sum formula, we see that
$$\sum_{n=1}^{\infty} z^n = \lim_{n\to\infty} \sum_{k=1}^{n} z^k = \frac{z}{1-z}.$$
Consider this alternative way. Suppose we have
$$s_n = z + z^2 + z^3 + \cdots + z^n.$$
and
$$zs_n = z^2 + z^3 + \cdots + z^n + z^{n+1}.$$
Then subtracting these two equations, we see that
$$(1-z)s_n = z - z^{n+1}.$$
Then we have
$$s_n = \frac{z}{1-z} - \frac{z^{n+1}}{(1-z)}.$$
Then taking the limit as $n \to \infty$, we must have
$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now, we will discuss absolute convergence of complex series.

Definition (Absolute Convergence of Infinite Series)**.** Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers
$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Remark. If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then the series

$$\sum_{n=1}^{\infty} z_n$$

converges. But note that the converse may not be true.

Example. Let $z \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely. Note that this is just the series expansion of e^z found in calculus courses. Consider

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

Note that this infinite series is a real series. If use the ratio test, we can consider

$$a_{n+1} = \frac{|z|^{n+1}}{(n+1)!}$$

and

$$a_n = \frac{|z|^n}{n!}$$

and then show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Next, we shall define e^z in $\mathbb{C}$.

3.1.3 Exponential, Sine, and Cosine

Definition (Complex Version of Exponential Function)**.** For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges absolutely for all $z \in \mathbb{C}$.

Definition (Sine and Cosine Series)**.** We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Definition. For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Now, we want to find out whether $\exp(z+w) = \exp(z) \cdot \exp(w)$. Note that this is true, but we want to prove this rigorously! Recall that the Euler Formula is

$$e^{iz} = \cos z + i \sin z \textbf{ Show this!}.$$

3.2 Lecture 5

3.2.1 Topics

- Continue discussion of some important functions.
- Go over some topological notion briefly.

3.2.2 Sine and Cosine Functions

Recall that we defined

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ for } z \in \mathbb{C}$$
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ for } z \in \mathbb{C}.$$

We also know,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma. For any $z \in \mathbb{C}$, observe that

- $\exp(iz) = \cos z + i \sin z$
- $\cos(z) = (\exp(iz) + \exp(-iz))/2$
- $\sin(z) = (\exp(iz) - \exp(-iz))/2.$

Proof. Let $z \in \mathbb{C}$. We know the series

$$\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

converges absolutely. Thus, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i)^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n} z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
&= \cos z + i \sin z.
\end{aligned}$$

Take the real and imaginary part the formulas stated in the lemma will follow. ■

Corollary. For any $z \in \mathbb{C}$, we see that

- $\exp(x + iy) = e^x(\cos y + i \sin y)$.
- $\Re(\exp(z)) = e^x \cos y$.

- $\Im(\exp(z)) = e^x \sin y$, $|\exp(z)| = e^x$.
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proposition. Show that $\exp(z) = \exp(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$. **Hint:** Let $z = x + iy$ and $w = u + iv$. Then

$$\exp(z) = e^x(\cos y + i \sin y)$$
$$\exp(w) = e^v(\cos v + i \sin v)$$

and show that $u = x$ and $u - v \in 2\pi\mathbb{Z}$.

Remark. This exercise shows that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is not injective.

- Note that $\exp : \mathbb{C} \to \mathbb{C}^{\cdot}$ is a group homomorphism. From this exercise, we know that $\ker(\exp) = 2\pi i\mathbb{Z}$. This means that exp is periodic with period $2\pi i$.

Our next goal is find an "inverse" of exp. We just learned that it is not possible unless we change the "domain" of exp.

Let $S = \{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}$.

Lemma. $\exp : S \to \mathbb{C}^{\cdot}$ is a bijective map.

Proof. Let $z \in \mathbb{C}^{\cdot}$ and $z = x + iy$. Let $z = \gamma(\cos\varphi + i \sin\varphi)$ be the polar representation of z such that $-\pi < \varphi \leq \pi$. Define $w = \ln\gamma + i\varphi$. Then $\exp(w) = z$ and so, exp is surjective. Moreover, exp is injective (on S) as well. ■

Corollary. For any $z \in \mathbb{C}^{\cdot}$, we can find a unique $w \in S$ such that $\exp(w) = z$.

Definition. Given $z \in \mathbb{C}^{\cdot}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.

Theorem. There exists a mapping

$$\mathrm{Log} : \mathbb{C}^{\cdot} \to \mathbb{C}$$

such that

(i) $\exp(\mathrm{Log}\, z) = z$

(ii) $-\pi < \Im(\mathrm{Log}\, z) \leq \pi$.

Definition (Principal Branch)**.** The function $\mathrm{Log}\, Z$ is called the principal branch of the logarithm.

Proposition. Let $z \in \mathbb{C}^{\cdot}$. Find all possible solutions of $\exp(w) = z$.

Lemma. For $z \in \mathbb{C}^{\cdot}$, we have

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

3.2.3 Notion of logarithm

For the proposition above, we can now see that the solutions given by the equation $\exp(w) = z$ are given by

$$w = \mathrm{Log}(z) + 2\pi ik \text{ for } z \in \mathbb{Z}$$

where

$$\log z = \mathrm{Log}\, z + 2\pi ik$$

is a **multivalued function** and $\log z = \mathrm{Log}\, z$ if we want the solution to be in S.

Chapter 4

Week 5

4.1 Lecture 6

4.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on $\mathbb{C}$:

$$d(z, w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ using the map $\mathbb{C} \to \mathbb{R}^2$ with $\alpha + i\beta \to (\alpha, \beta)$. Thus, we can visualize $\mathbb{C}$ using $\mathbb{R}^2$.

Recall that definition of continuity for functions in $\mathbb{R}$.

Definition (continuity in $\mathbb{R}$). Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **continuous** for each $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say $|x - a| < \delta$, we can also say that $x \in (a - \delta, a + \delta)$. Similarly, we have $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

4.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x, a) < \delta\}$$

which is a more general statement than the one found in the definition above.

Definition (Open Ball). Consider the usual metric $d(z, w) = |z - w|$ in $\mathbb{C}$. Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a, and radius $\varepsilon > 0$, we mean the set

$$B(a, \varepsilon) = \{z \in \mathbb{C} : d(z, a) < \varepsilon\}.$$

Definition (Open Set). Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Example. (i) $\mathbb{C}$ is open.

(ii) Let $\varepsilon > 0$ and $a \in \mathbb{C}$. Then $B(a, \varepsilon)$ is also open; that is, open balls are open.

(iii) Let D_1 and D_2 be open sets. Show that $D_1 \cap D_2$ is open. Let $x \in D_1 \cap D_2$ To show that $D_1 \cap D_2$ is open, we need to find $\delta > 0$ such that $B(x, \delta) \subseteq D_1 \cap D_2$. Since $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$. Since D_1 and D_2 are open sets, we can find $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$B(x, \varepsilon) \subseteq D_1 \text{ and } B(x, \varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that $d(x, p) < \varepsilon$ for all $p \in D_1$ and $d(x, q) < \varepsilon'$. Then pick $\delta = \min\{d(x, p), d(x, q)\}$. Since x is in both D_1 and D_2, we have that

$$B(x, \delta) \subseteq D_1 \cap D_2.$$

Hence, $D_1 \cap D_2$ is open.

1. The empty set is open.
2. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a collection of open ets, where Λ is an indexing set. Define the set

$$\bigcup_{\lambda \in \Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that $\bigcup_{\lambda \in \Lambda} D_\lambda$ is also open.

4.1.3 Interior Points

Definition (Interior Point). Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq D$.

Proposition. D is open if and only if each $a \in D$ is an interior point of D.

Example. Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton $\{0\}$. Show that $\{0\}$ is not open. Note that the intersection of arbitrary many open sets may not be open.

4.1.4 Closed sets

Definition (Closed Set). We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

Example. Define a closed ball with center at $a \in \mathbb{C}$ and radius $\varepsilon > 0$ by

$$\overline{B}(a, \varepsilon) = \{z \in \mathbb{C} : |z - a| \leq \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that $\overline{B}(a, \varepsilon)$ is closed.

Example.
- For any $a \in \mathbb{C}$, we say that the singleton $\{a\}$ is closed.
- If $A_1, A_2, \ldots, A_n$ are closed sets. Then
$$\bigcup_{i=1}^{n} A_i \text{ is also closed.}$$
We can just show that the complement of this set is open.
- Let $\{A_\Lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Define
$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\Lambda \ \forall \lambda \in \Lambda\},$$
then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.

Definition (Boundary Points)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a, \varepsilon) \cap D \neq \emptyset$ and $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a, \varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.

Example. Any point on the circle $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$ is a boundary point of $B(a, \varepsilon)$, where $\varepsilon > 0$.

Example (Boundary Points of Singletons)**.**
- Note that the boundary point of the singleton is just the singleton itself.
- Let $D = \mathbb{C} \setminus \{a\}$. Just like the first item, the boundary points of this set is just $\{a\}$.

Definition (Accumulation Point)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,
$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

Example.
(i) Let $D = \mathbb{C}$. Then $\{0\}$ is an accumulation point of D.
(ii) Let a such that $|a| = 1$. Then a is an accumulation point of $D = \{z : |z| < 1\}$.
(iii) Let $D = \{1, i\}$. Then D has no accumulation points.

Definition (Closure)**.** Let $D \subseteq \mathbb{C}$. We denote the **closure of** D by **D** and define $\overline{D}$ as
$$\overline{D} = D \cup \{x \in D : \exists \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

Lemma.
(i) $\overline{D}$ is closed in $\mathbb{C}$.
(ii) The following statements are equivalent:
  (a) $a \in \mathbb{C}$ is an accumulation point of D
  (b) There is a sequence (z_n) with $z_n \in D$, $z_n \neq a$ for all $n \in \mathbb{Z}_+$ and $z_n \to a$.

Proof. ■

Definition (Bounded)**.** We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$.

Chapter 5

Week 6

5.1 Lecture 7

5.1.1 Topics

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5.2 Lecture 8

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