Contents

Linear Algebra Exercises

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Chapter 1

Vector Spaces

1.1 Linear Dependence and Linear Independence

Exercise 1.5.1

Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vector in S.

> **Proof.** **True** ■

(b) Any set containing the zero vector is linearly dependent.

> **Proof.** **True** ■

(c) The empty set is linearly dependent.

> **Proof.** **False**. It is linearly independent. ■

(d) Subsets of linearly dependent sets are linearly dependent. **True** by Theorem 6.

(e) Subsets of linearly independent sets are linearly independent.

> **Proof.** **True** by corollary to Theorem 6. ■

(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

> **Proof.** **True** this is by definition. ■

Exercise 1.5.4

In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Proof. Choose a finite amount of scalars $a_1, a_2, \ldots, a_n \in F$ to create the following linear combination:

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = (0, 0, \ldots, 0). \tag{1}$$

To show that the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, we need to show that the scalars $a_1, a_2, \ldots, a_n \in F$ have the trivial representation; that is, $a_1 = a_2 = \cdots = a_n = 0$. Since the jth coordinate of e_j is 1 but 0 in all the other entries, we have that

$$\begin{aligned}
&a_1(1, 0, \ldots, 0) + a_2(0, 1, \ldots, 0) + \cdots + a_n(0, 0, \ldots, 1) \\
&= (a_1, 0, \ldots, 0) + (0, a_2, \ldots, 0) + \cdots + (0, 0, \ldots, a_n) \\
&= (a_1, a_2, \ldots, a_n).
\end{aligned}$$

Hence, we have

$$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

Equating each entry of the left side of the equation above to 0, we find that $a_i = 0$ for all $1 \leq j \leq n$. Hence, the set $\{e_1, e_2, \dots, e_n\}$ is linearly independent. ■

Exercise 1.5.4

Show that the set $\{1, x, x^2, \dots, x^n\}$ is linearly independent in $P_n(F)$.

Proof. Just like the prior exercise, we need to show that we can find scalars $a_0, a_1, \dots, a_n \in F$ such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

where $a_i = 0$ for all $0 \leq i \leq n$. Note that the 0 polynomial is just

$$0 + 0x + 0x^2 + \cdots + 0x^n = 0.$$

Hence, equating coefficients we immediately get that $a_i = 0$ for all $0 \leq i \leq n$. Thus, the set $\{1, x, x^2, \dots, x^n\}$ is linearly independent. ■

Exercise 1.5.6

In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Proof. First, we create a linear combination of a finite amount vectors in $E = \{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ with scalars δ_k for $1 \leq k \leq N$ with $N = mn$ as the number of total entries in each matrix in $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Note that after doing our scalar multiplication and summing up each term, we find that each $\delta_k E_{ij} = \delta_k$ in our linear combination can equated with a corresponding i and j entry in the zero matrix such that $\delta_k = 0$ for all $1 \leq k \leq N$. Hence, E is a linearly independent set. ■

Exercise 1.5.7

Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

Proof. Define W as the linearly independent spanning set of the set of diagonal matrices in $M_{2\times 2}$ where

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

To see why W is a linearly independent set, choose scalars $\delta_1, \delta_2 \in F$ such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing scalar multiplication and vector addition gives us the following equation

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the zero matrix is a diagonal matrix, we know that equation entries where $i = j$yields $\delta_1 = \delta_2 = 0$. Hence, W is a linearly independent set that generates the set of diagonal matrices of $M_{2\times 2}(F)$. ■

Exercise 1.5.8

Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = \mathbb{R}$, then S is linearly independent.

Proof. ∎

(b) Prove that if F has characteristic two, then S is linearly dependent.

Proof. ∎

Exercise 1.5.9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other. I have written two proofs for this:

Proof. Let u and v be distinct vectors in a vector space V.

$(\Rightarrow)$ Since $\{u, v\}$ is a linearly dependent set, we can find scalars $a_1, a_2 \in F$ such that

$$a_1 u + a_2 v = 0 \tag{1}$$

Suppose v is not a multiple of u and choose $a_1 \neq 0$ since $\{u, v\}$ is linearly dependent. We need to show that u is a multiple of v. Solving for u, we get that

$$u = -\frac{a_2}{a_1} v.$$

Hence, u is a multiple of v.

$(\Leftarrow)$ Suppose u or v is a scalar multiple of the other. Assume u is the scalar multiple of v. Then for some $c \neq 0 \in F$, we have $u = cv$. Hence, we have $u - cv = 1u - cv = 0$. This tells us that $\{u, v\}$ is linearly dependent. ∎

Exercise 1.5.12

Prove Theorem 1.6 and its corollary.

Proof. See proof in notes. ∎

Exercise 1.5.13

Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Proof. Let u and v be distinct vectors in V.

For the forwards direction, assume $\{u, v\}$ is a linearly independent set. We need to show that $\{u + v, u - v\}$ is linearly independent. Hence, we need to find $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0. \tag{1}$$

Note that (1) leads to

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu. \end{aligned}$$

Since $\{u, v\}$ is a linearly independent set, we have that

$$au - bv = 0$$

and

$$av + bu = 0$$

for $a = b = 0$. Hence,

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ and so $\{u - v, u + v\}$ is a linearly independent set.

For the backwards direction, suppose $\{u+v, u-v\}$ is linearly independent. We need to show that $\{u, v\}$ is linearly independent. Note that $a, b \in F$ such that

$$a(u+v) + b(u-v) = 0$$

for $a = b = 0$ since $\{u-v, u+v\}$ is linearly independent. Note that

$$\begin{aligned} a(u+v) + b(u-v) &= au + av + bu - bv \\ &= au - bv + av + bu \\ &= 0 + av + bu \\ &= 0. \end{aligned}$$

Thus, $av + bu = 0$ where a, b both zero. Thus, the set $\{u, v\}$ is linearly independent. ■