0.1 Complex Numbers and the Complex Plane

0.1.1 Review of Complex Numbers

- A complex number takes the form $z = x + iy$ where $x, y \in \mathbb{R}$ and that $i^2 = -1$. We denote this set of numbers as $\mathbb{C}$.
- The **real** and **imaginary** part of z is defined as follows:

$$x = \Re(z) \text{ and } y = \Im(z).$$

- Real numbers like x have $\Im(x) = 0$, while numbers like y have $\Re(y) = 0$. In other words, y is said to be **purely imaginary**.
- Each complex number can be represented as a point in $\mathbb{R}^2$ with the $x-$ axis representing the **real axis** and the $y-$axis representing the **imaginary axis**.
- The operations that are used with complex numbers works similarly to how we would treat real numbers; that is, they follow commutativity, associativity, and distributivity.
- The addition of two complex numbers works like adding two corresponding vectors in $\mathbb{R}^2$ where you add up each component.
- Multiplication of complex numbers produces a rotation (around the unit circle) and a dilation of said vector. This is seen using the polar form of a complex number.

0.1.2 Notion of Length and Inequalities

- The notion of length in $\mathbb{R}^2$ is also similar to the complex plane. For $z \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we can see that the **absolute value** of a complex number $z = x + iy$ is

$$|z| = (x^2 + y^2)^{1/2}.$$

  This is the distance from the origin to the point (x, y).
- Like in $\mathbb{R}$, the triangle inequality for $\mathbb{C}$ also holds:

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

- Some other inequalities in $\mathbb{C}$ include:

$$|\Re(z)| \leq |z| \text{ and } |\Im(z)| \leq |z|.$$

  Furthermore, the reverse triangle inequality holds:

$$||z| - |w|| \leq |z - w|.$$

- The **complex conjugate** of $z = x + iy$ is defined by

$$\overline{z} = x - iy.$$

  Geometrically, this is seen as a reflection of z over the real axis.
- Furthermore, we say that $z \in \mathbb{C}$ is a real number if and only if $z = \overline{z}$. On the other hand, we say that $z \in \mathbb{C}$ is purely imaginary if and only if $z = -\overline{z}$.

0.1.3 Other Properties and Polar Form

- On top of defining our real and imaginary parts of a complex number, it should be straightforward to prove that
$$\Re(z) = \frac{z+\overline{z}}{2} \text{ and } \Im(z) = \frac{z-\overline{z}}{2i}.$$
- We also see that $|z|^2 = z\overline{z}$ implies that
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$
whenever $z \neq 0$.
- The **polar form** of $z \in \mathbb{C}$ is
$$z = re^{-i\theta}$$
for $r > 0$ and $\theta \in \mathbb{R}$. We denote θ as the **argument** of z; that is, $\theta = \arg(z)$. Note that θ is defined uniquely up to a multiple of 2π.
- Recall that
$$e^{i\theta} = \cos\theta + i\sin\theta$$
and that $|e^{i\theta}| = 1$ as well as $r = |z|$.
- Finally, with these properties, we can see that multiplying $z, w \in \mathbb{C}$ results in
$$zw = rse^{i(\theta+\varphi)}$$
where $z = re^{i\theta}$ and $w = se^{i\varphi}$. The rs is corresponds to the dialation of zw and $\theta + \varphi$ corresponds to a rotation.

0.1.4 Convergence

Definition 0.1.1 (Convergence in $\mathbb{C}$). A sequence $(z_n) \subset \mathbb{C}$ is said to **converge** to $w \in \mathbb{C}$ if
$$\lim_{n\to\infty} |z_n - w| = 0$$
and we write
$$w = \lim_{n\to\infty} z_n.$$

The two formulae above are equivalent to each other.

- In other words, (z_n) converges to w if and only if the corresponding sequence of points in the complex plane converges to the point that corresponds to w.
- We can check that $(z_n) \to w$ if and only if for $w = a + bi$, we have $(a_n) \to a$ and $(b_n) \to b$; that is, the real and imaginary parts of the sequence of complex numbers converges, respectively.

Definition 0.1.2 (Cauchy Sequence). A sequence (z_n) is said to be a **Cauchy sequence** if
$$|z_n - z_m| \to 0$$
as $n, m \to \infty$.

- In other words, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that $|z_n - z_m| < \varepsilon$ whenever $n, m > N$.
- Similarly to how $\mathbb{R}$ is complete and how all $z \in \mathbb{C}$ are made up of a tuple of real numbers in $\mathbb{R}^2$, we see that if (z_n) is Cauchy, then every Cauchy complex sequence converges to a limit that is in $\mathbb{C}$.

Theorem 0.1.1. The set of complex numbers $\mathbb{C}$ is complete.

0.1.5 Sets in the complex plane

- If $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **open disc** $D_r(z_0)$ **of radius** r **centered at** z_0 to be the set
$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$
- If $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **closed disc** $\overline{D_r}(z_0)$ **of radius** r **centered at** z_0 to be the set
$$\overline{D_r}(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$
- The **boundary** of either a closed disc or an open disc is given by the set
$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$
- We denote the **unit disc** as the set
$$\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}.$$
- Given a set $\Omega \subset \mathbb{C}$, a point $z_0 \in \Omega$ is called an **interior point** of Ω if there exists $r > 0$ such that
$$D_r(z_0) \subset \Omega.$$
- A set Ω is **open** if every point in that set is an interior point of Ω.
- A point $z \in \mathbb{C}$ is said to be a **limit point** of the set Ω if there exists a sequence of points $z_n \in \Omega$ such that $z_n \neq z$ and $\lim_{n \to \infty} z_n = z$.
- The **boundary** of a set Ω is equal to its closure minus its interior, and is often denoted by $\partial\Omega$.
- A set Ω is **bounded** if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$; that is, Ω is contained in some large disc.
- If Ω is bounded, we define its **diameter** by
$$\operatorname{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|.$$
- A set Ω is said to be **compact** if it is closed and bounded.

Theorem 0.1.2. The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

Theorem 0.1.3. A set Ω is compact if and only if every open covering of Ω contains a finite subcovering.

The next result will be used to prove Goursat's Theorem (in our study of complex function theory) in a later section.

Proposition 0.1.1. If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that

$$\operatorname{diam}(\Omega_n) \to 0 \text{ as } n \to \infty,$$

there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Proof. Choose a point $z_n \in \Omega_n$ for each n. Since we have sequence of non-empty compact sets in $\mathbb{C}$ such that $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$, we have that (z_n) is Cauchy. Thus, $(z_n) \to w$ and that $w \in \Omega_n$ since each Ω_n for all n is closed. To prove uniqueness, suppose w' is another point that satisfies the same properties as w such that $w \neq w'$. Then this would violate our assumption that $\operatorname{diam}(\Omega_n) \to 0$ because we would have $|w - w'| > 0$. Thus, we must have $w = w'$, proving that w is unique. ∎

- We say that an open set $\Omega \subset \mathbb{C}$ is **connected** if it is not possible to find two disjoint non-empty open sets Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2.$$

- A connected open set in $\mathbb{C}$ is called a **region**.
- A closed set F is connected if one cannot write $F = F_1 \cup F_2$ where F_1 and F_2 are disjoint non-empty closed sets.

0.1.6 Functions on the Complex plane

Continuous Functions

Definition 0.1.3 (Continuity). Let f be a function defined on a set $\Omega \subset \mathbb{C}$. We say that f is **continuous** at the point $z_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, then

$$|f(z) - f(z_0)| < \varepsilon.$$

Definition 0.1.4 (Sequential Definition of Continuity). The function f is said to be continuous on Ω if for every $(z_n) \subset \Omega$ such that $\lim z_n = z_0$, then $\lim f(z_n) = f(z_0)$.

- Recall that sums of continuous and products of continuous functions are continuous.
- A complex function is continuous if and only if the corresponding real and imaginary parts of f is continuous.

Definition 0.1.5 (Maximums/Minimums of Complex Functions). We say that f attains a **maximum** at the point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in \Omega,$$

with the inequality reversed for the definition of a **minimum**.

Theorem 0.1.4 (Continuous Functions on Compact Sets). A continuous function on a compact set Ω is bounded and attains a maximum and minimum on Ω.

0.1.7 Holomorphic Functions

Definition 0.1.6 (Holomorphic at a point). Let $\Omega \subset \mathbb{C}$ be an open set and f is a complex-valued function on Ω. The function f is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. When this limit exists, we denote

$$f'(z_0) = \frac{f(z_0 + h) - f(z_0)}{h}$$

as the derivative of f at z_0.

Note that we're assuming that $h \in \mathbb{C}$ and that $h \neq 0$ with $z_0 + h \in \Omega$, so that the quotient is well-defined.

Note that h is a complex number that may approach 0 from any direction.

Definition 0.1.7 (Holomorphic on a subset of $\mathbb{C}$).

- The function f is said to be **holomorphic on** Ω if f is holomorphic at every point of Ω.
- If C is a closed subset of $\mathbb{C}$, we say that f **holomorphic on** $\mathbb{C}$.
- If C is an open subset, then we say f is **entire**.

- Holomorphic functions have nicer properties than differentiable functions of real variable.
- Differentiability of complex functions imply infinite differentiability.
- Every holomorphic function is analytic; that is, we can power series expansion at every point.
- This is much nicer than real-valued functions, which often have more restrictive properties.

Proposition 0.1.2. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists $a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h)$$

where ψ is a function defined for all h and $\lim_{h \to 0} \psi(h) = 0$.

Proposition 0.1.3. If f and g are holomorphic in Ω, then:

(i) $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$.

(ii) fg is hoolomorphic in Ω and $(fg)' = f'g + fg'$.

(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Complex-valued Functions as Mappings

- Notion of complex differentiability is significantly more different than real differentiability of two variables.
- The existence of the real derivative does not necessarily guarantee that f will be holomorphic.

Proposition 0.1.4. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if we write $F(x, y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

Integration along Curves

Definition 0.1.8 (Parametrization).

- We call a **parametrized curve** as a function $z(t)$ that maps a closed interval $[a, b] \subset \mathbb{R}$ to $\mathbb{C}^2$.
- We say that a parametrized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$.
- At the endpoints $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are **left-handed derivative/right-handed derivative**; that is,

$$z'(a) = \lim_{h \to 0} \frac{z(a + h) - z(a)}{h} \quad \text{and} \quad z'(b) = \lim_{h \to 0} \frac{z(b + h) - z(b)}{h}$$

  with the limit in $z'(a)$ approaching for values $h > 0$ and the limit in $z'(b)$ approaching for values $h < 0$.
- We say that the parametrized curve is **piecewise-smooth** if z is continuous on $[a, b]$, and if there exists points

$$a = a_0 < a_1 < \cdots < a_n = b,$$

  where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$.

Definition 0.1.9.

- The family of parametrizations that are equivalent to $z(t)$ determines a **smooth curve** $\gamma \subset \mathbb{C}$; that is, the image of $[a, b]$ under z with the orientation given by z as t travels from a to b.

- The points $z(a)$ and $z(b)$ are called the **end-points** of the curve and are independent on the parametrization. Since γ carries an orientation, it is natural to say that γ begins at $z(a)$ and ends at $z(b)$.
- We call a smooth or piecewise-smooth curve **closed** if $z(a)$ and $z(b)$ for any parametrization.
- We call a piecewise-smooth curve **simple** if it is **NOT** self-intersecting; that is, $z(t) \neq z(t)$ unless $s = t$.

- We call curves that are closed to begin with simple whenever $z(t) \neq z(s)$ unless $s = t$ or that $s = a$ and $t = b$.
- The endpoints of a curve are independent of parametrization.
- We will call any piecewise-smooth curve as a **curve**.

Definition 0.1.10 (Orientation).
- The **positive orientation** (counterclockwise) is the one that is given by the standard parametrization

$$z(t) = z_0 + re^{it}, \quad \text{where } t \in [0, 2\pi].$$

- On the other hand, the **negative orientation** (clockwise) is give by

$$z(t) = z_0 + re^{-it}, \quad \text{where } t \in [0, 2\pi].$$

Proposition 0.1.5. Integration of continuous functions over curves satisfies the following properties:

(i) It is linear, that is, if $\alpha, \beta \in \mathbb{C}$, then

$$\int_\gamma (\alpha f(z) + \beta g(z))\ dz = \alpha \int_\gamma f(z)\ dz + \beta \int_\gamma g(z)\ dz.$$

(ii) If γ^- is γ with the reverse orientation, then

$$\int_\gamma f(z)\ dz = -\int_{\gamma^-} f(z)\ dz.$$

(iii) One has the inequality

$$\left| \int_\gamma f(z)\ dz \right| \leq \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$$

0.1.8 Exercises

1. Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

This is the complex version of the chain rule.

Proof. ■

2. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. Note that we have $f = u + iv$ and that $x = r\cos\theta$ and $y = r\sin\theta$. Note that

$$\frac{\partial x}{\partial r} = \cos\theta \quad \text{and} \quad \frac{\partial y}{\partial r} = \sin\theta$$

and

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Using the partial chain rule for real valued functions and using the Cauchy-Riemann equations, we can see that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{1}{r}\Big(\frac{\partial v}{\partial y} r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta\Big) \\
&= \frac{1}{r}\Big(\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}\Big) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}$$

which proves our first equation. Similarly, we have

$$\begin{aligned}
\frac{1}{r}\frac{\partial u}{\partial \theta} &= \frac{1}{r}\Big(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\Big) \\
&= \frac{1}{r}\Big(-\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta\Big) \\
&= -\frac{1}{r}\Big(\frac{\partial v}{\partial y} r\sin\theta \frac{\partial v}{\partial x} r\cos\theta\Big) \\
&= -\Big(\frac{\partial v}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial r}\Big) \\
&= -\frac{\partial v}{\partial r}.
\end{aligned}$$

■

3. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$$

where Δ is the **Laplacian**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. Note that

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Big)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big).$$

Thus, we have

$$\begin{aligned} 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= \Big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big) \\ &= \frac{\partial^2}{\partial x^2} - \frac{1}{i^2}\frac{\partial^2}{\partial y^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &= \Delta. \end{aligned}$$

Similarly, we can derive the same result with $4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}$. Thus, we have

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$$

■

4. Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

Proof. To show that the real and imaginary parts of f are **harmonic**, we need to show that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

where u and v are the real and imaginary parts of f, respectively. Since f is holomorphic in the open set Ω, then the Cauchy-Riemann equations must hold; that is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, using Clairaut's Theorem we can see that

$$\begin{aligned} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big) \\ &= \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(-\frac{\partial v}{\partial x}\Big) \\ &= \frac{\partial v^2}{\partial x \partial y} - \frac{\partial v^2}{\partial y \partial x} \\ &= 0. \end{aligned}$$

By a similar argument, we have that

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0.$$

Thus, the real and imaginary parts of f are harmonic. ■

5. Consider the function defined by

$$f(x+iy) = \sqrt{|x||y|}, \quad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Proof. ■

6. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

   (a) $\Re(f)$ is constant;

   (b) $\Im(f)$ is constant;

   (c) $|f|$ is constant;

one can conclude that f is constant.

Proof. Suppose that f is holomorphic in an open set Ω. Set $f = u + iv$ with u, v being functions of real variable.

(a) Suppose $\Re(f)$ is constant. Our goal is to show that

$$\frac{\partial f}{\partial z} = 0$$

in order to show that f is constant. Thus, $\Re(f) = k$ for some constant k. Since f is holomorphic, the Cauchy-Riemann equations hold. Thus, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Since u is a constant, we see that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

Thus, we see that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0,$$

and thus,

$$\frac{\partial f}{\partial y} = 0,$$

by the same process. Therefore, we have that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) = 0 + 0 = 0.$$

which concludes that f is a constant.

(b) Suppose $\Im(f)$ is a constant. Employing the same process found in part (a) gives us our desired result.

(c) Now, suppose that $|f|$ is constant. By definition of $|f|$, we see that

$$k = |f| = \sqrt{u^2 + v^2} \Rightarrow k^2 = u^2 + v^2.$$

Differentiating with respect to x, we see that

$$0 = 2\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0. \tag{1}$$

Similarly, differentiating with respect to y gives us

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

Since the Cauchy-Riemann equations hold, we can see that (1) leads to

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

and similarly,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}.$$

By definition of $\frac{\partial f}{\partial z}$, we see that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\Big) = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{1}{i}\Big(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big)\Big)$$
$$=$$

■