0.1 Lecture 3

0.1.1 Topics

- Polar Representation of complex numbers.
- Convergence of sequences in $\mathbb{C}$.

0.1.2 Polar Representation of Complex Numbers

If $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we can represent this as

$$(x, y) = (\gamma \cos \varphi, \gamma \sin \varphi)$$

where $\gamma = \sqrt{x^2 + y^2}$ and $\tan(\varphi) = \frac{y}{x}$ is a polar representation of (x, y).

Remark. This representation may not be unique!

If we insist, we can make φ unique by restricting the domain to $-\pi < \varphi \leq \pi$ where φ is denoted as the **argument of** z.

Definition (Principle Argument)**.** If $-\pi < \varphi \leq \pi$, we call this angle the **principle argument** which we denote as

$$\varphi = \mathrm{Arg}((x, y)).$$

Remark. For any other domain, we denote the argument by $\varphi = \arg((x, y))$.

Lemma. Let $z = \gamma(\cos \varphi + i \sin \varphi)$ and $w = \gamma'(\cos(\varphi') + \sin(\varphi')$ in $\mathbb{C} \setminus \{0\}$. Then

$$zw = \varphi\varphi'[\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')].$$

Proof. Using the addition formula, we can write

$$\begin{aligned} zw &= \gamma\gamma'(\cos \varphi + i \sin \varphi)(\cos \varphi' + i \sin \varphi') \\ &= \gamma\gamma'[(\cos \varphi cos\varphi' + \sin \varphi \sin \varphi') + i(\sin \varphi \cos \varphi' + \sin \varphi \cos \varphi')] \\ &= \gamma\gamma'(\cos(\varphi + \varphi') + i \sin(\varphi + \varphi')). \end{aligned}$$

■

Corollary (De Moivre's Theorem)**.** Let $z = \gamma(\cos \varphi + i \sin \varphi) \in \mathbb{C} \setminus \{0\}$ and let $n \in \mathbb{Z}$. Then

$$z^n = \gamma^n(\cos n\varphi + i \sin n\varphi).$$

Remark. If n is a negative integer, then $z^n = (z^{-1})^{-n}$.

The corollary above allows us to compute the nth roots of a non-zero complex number.

Example 0.1.1 (An example of De Moivre's Theorem)**.** Suppose we had the complex number

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Suppose we want to find z^{10}. First, we need to find the angle that makes this complex number. Since the x and y coordinates are both positive this means that the angle must lie in the first quadrant. Thus, we have

$$\varphi = \arg(z) = \frac{\pi}{3}.$$

Using De Moivre's Theorem, we can write

$$\begin{aligned} z^{10} &= \cos\left(10 \cdot \frac{\pi}{3}\right) + i \sin\left(10 \cdot \frac{\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Some notations we would like to establish are the following:

(i) **The set of all positive real numbers** $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$

(ii) **The set of all complex numbers excluding zero** $\mathbb{C}^{\cdot} = \mathbb{C} \setminus \{0\}$.

Proposition. The map $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$(r, \varphi) \longrightarrow \gamma(\cos\varphi + i \sin\varphi)$$

is surjective.

Remark. This gives us the tool we need to show that every non-zero $z \in \mathbb{C}$ has a polar representation.

0.1.3 Convergence of Sequences in $\mathbb{C}$

Definition (Convergence in $\mathbb{C}$)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.

If (z_n) converges to z, then we write $z_n \to z$.

Proposition (Properties of Convergent Sequences)**.** Assume $(z_n) \to z$ and $(w_n) \to w$.

(i) Let $\alpha, \beta \in \mathbb{C}$, then $\alpha z_n + \beta w_n \to \alpha z + \beta w$.

(ii) $z_n w_n \to zw$.

(iii) $z_n^{-1} \to z^{-1}$.

(iv) $(z_n) \to z$ if and only if $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$ as a sequences in $\mathbb{R}$.

Proof. ■