0.1 Complex Numbers and the Complex Plane

0.1.1 Review of Complex Numbers

- A complex number takes the form $z = x + iy$ where $x, y \in \mathbb{R}$ and that $i^2 = -1$. We denote this set of numbers as $\mathbb{C}$.
- The **real** and **imaginary** part of z is defined as follows:

$$x = \Re(z) \text{ and } y = \Im(z).$$

- Real numbers like x have $\Im(x) = 0$, while numbers like y have $\Re(y) = 0$. In other words, y is said to be **purely imaginary**.
- Each complex number can be represented as a point in $\mathbb{R}^2$ with the $x-$ axis representing the **real axis** and the $y-$axis representing the **imaginary axis**.
- The operations that are used with complex numbers works similarly to how we would treat real numbers; that is, they follow commutativity, associativity, and distributivity.
- The addition of two complex numbers works like adding two corresponding vectors in $\mathbb{R}^2$ where you add up each component.
- Multiplication of complex numbers produces a rotation (around the unit circle) and a dilation of said vector. This is seen using the polar form of a complex number.

0.1.2 Notion of Length and Inequalities

- The notion of length in $\mathbb{R}^2$ is also similar to the complex plane. For $z \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we can see that the **absolute value** of a complex number $z = x + iy$ is

$$|z| = (x^2 + y^2)^{1/2}.$$

  This is the distance from the origin to the point (x, y).
- Like in $\mathbb{R}$, the triangle inequality for $\mathbb{C}$ also holds:

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

- Some other inequalities in $\mathbb{C}$ include:

$$|\Re(z)| \leq |z| \text{ and } |\Im(z)| \leq |z|.$$

  Furthermore, the reverse triangle inequality holds:

$$||z| - |w|| \leq |z - w|.$$

- The **complex conjugate** of $z = x + iy$ is defined by

$$\overline{z} = x - iy.$$

  Geometrically, this is seen as a reflection of z over the real axis.
- Furthermore, we say that $z \in \mathbb{C}$ is a real number if and only if $z = \overline{z}$. On the other hand, we say that $z \in \mathbb{C}$ is purely imaginary if and only if $z = -\overline{z}$.

0.1.3 Other Properties and Polar Form

- On top of defining our real and imaginary parts of a complex number, it should be straightforward to prove that
$$\Re(z) = \frac{z+\overline{z}}{2} \text{ and } \Im(z) = \frac{z-\overline{z}}{2i}.$$
- We also see that $|z|^2 = z\overline{z}$ implies that
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$
whenever $z \neq 0$.
- The **polar form** of $z \in \mathbb{C}$ is
$$z = re^{-i\theta}$$
for $r > 0$ and $\theta \in \mathbb{R}$. We denote θ as the **argument** of z; that is, $\theta = \arg(z)$. Note that θ is defined uniquely up to a multiple of 2π.
- Recall that
$$e^{i\theta} = \cos\theta + i\sin\theta$$
and that $|e^{i\theta}| = 1$ as well as $r = |z|$.
- Finally, with these properties, we can see that multiplying $z, w \in \mathbb{C}$ results in
$$zw = rse^{i(\theta+\varphi)}$$
where $z = re^{i\theta}$ and $w = se^{i\varphi}$. The rs is corresponds to the dialation of zw and $\theta + \varphi$ corresponds to a rotation.

0.2 Convergence

Definition 0.2.1 (Convergence in $\mathbb{C}$). A sequence $(z_n) \subset \mathbb{C}$ is said to **converge** to $w \in \mathbb{C}$ if
$$\lim_{n\to\infty} |z_n - w| = 0$$
and we write
$$w = \lim_{n\to\infty} z_n.$$

The two formulae above are equivalent to each other.

- In other words, (z_n) converges to w if and only if the corresponding sequence of points in the complex plane converges to the point that corresponds to w.
- We can check that $(z_n) \to w$ if and only if for $w = a + bi$, we have $(a_n) \to a$ and $(b_n) \to b$; that is, the real and imaginary parts of the sequence of complex numbers converges, respectively.

Definition 0.2.2 (Cauchy Sequence). A sequence (z_n) is said to be a **Cauchy sequence** if
$$|z_n - z_m| \to 0$$
as $n, m \to \infty$.

- In other words, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that $|z_n - z_m| < \varepsilon$ whenever $n, m > N$.
- Similarly to how $\mathbb{R}$ is complete and how all $z \in \mathbb{C}$ are made up of a tuple of real numbers in $\mathbb{R}^2$, we see that if (z_n) is Cauchy, then every Cauchy complex sequence converges to a limit that is in $\mathbb{C}$.

Theorem 0.2.1. The set of complex numbers $\mathbb{C}$ is complete.

0.3 Sets in the complex plane

Definition 0.3.1 (Open Discs). If $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **open disc** $D_r(z_0)$ **of radius** r **centered at** z_0 to be the set

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Definition 0.3.2 (Closed Discs). If $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **closed disc** $\overline{D_r}(z_0)$ **of radius** r **centered at** z_0 to be the set

$$\overline{D_r}(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$