Measure Theory Notes

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January 11, 2025

0.1 Outer Measure

Definition (Length of Open Interval)**.** The **length** $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ for some } a,b\in\mathbb{R} \text{ with } a<b \\ 0 & \text{if } I=\emptyset \\ \infty & \text{if} I=(-\infty,a) \text{ or } I=(a,\infty) \text{ for some } a\in\mathbb{R} \\ \infty & \text{if } I=(-\infty,\infty) \end{cases}.$$

Definition (Outer Measure $|A|$)**.** The **outer measure** $|A|$ of a set $A\subseteq\mathbb{R}$ is defined by

$$|A| = \inf\Big\{\sum_{k=1}^{\infty}\ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A\subseteq\bigcup_{k=1}^{\infty} I_k\Big\}.$$

Proposition (Countable sets have outer measure 0)**.** Every countable subset of $\mathbb{R}$ has outer measure 0.

Proposition (Outer Measure Preserves Order)**.** Suppose A and B are subsets of $\mathbb{R}$ with $A\subseteq B$. Then $|A|\le|B|$.

Definition (Translation; $t+A$)**.** If $t\in\mathbb{R}$ and $A\subseteq\mathbb{R}$, then the **translation** $t+A$ is defined by

$$t+A=\{t+a : a\in A\}.$$

Proposition (Outer Measure is Translation Invariant)**.** Suppose $t\in\mathbb{R}$ and $A\subseteq\mathbb{R}$. Then $|t+A|=|A|$.

Proposition (Countable Subadditivity of Outer Measure)**.** Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\Big|\bigcup_{k=1}^{\infty} A_k\Big| \le \sum_{k=1}^{\infty}|A_k|.$$

Definition (Open Cover)**.** Suppose $A\subseteq\mathbb{R}$.

- A collection $\mathcal{C}$ of open subsets of $\mathbb{R}$ is called an **open cover** of A if A is contained in the union of all the sets in $\mathcal{C}$.
- An open cover $\mathcal{C}$ of A is said to have a **finite subcover** if A is contained in the union of some finite list of sets in $\mathcal{C}$.

Proposition (Outer Measure of a Closed Interval)**.** Suppose $a,b\in\mathbb{R}$, with $a<b$. Then $|[a,b]|=b-a$.