Math 234A: Homework 1

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1. (i) **(Parallelogram identity)** Let $z, w \in \mathbb{C}$. Show that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Proof. Let $z, w \in \mathbb{C}$ with $z = x + iy$ and $w = u + iv$ with $x, y \in \mathbb{R}$ and $u, v \in \mathbb{R}$. Our goal is to show that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Consider $|z-w|^2$ and notice that

$$z - w = (x-u) + i(y-v).$$

By definition of the modulus, we have

$$\begin{aligned}
|z-w|^2 &= (z-w)\overline{z-w} \\
&= ((x-u) + i(y-v))((x-u) - i(y-v)) \\
&= (x-u)^2 + (y-v)^2 \\
&= x^2 - 2xu + u^2 + y^2 - 2yv + u^2 \\
&= (x^2 + y^2) - 2(xu + yv) + (u^2 + v^2) \\
&= |z|^2 - 2(xu + yv) + |w|^2.
\end{aligned}$$

Note that

$$z + w = (x+u) + i(y+v).$$

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= ((x+u) + i(y+v))((x+u) - i(y+v)) \\
&= (x+u)^2 + (y+v)^2 \\
&= x^2 + 2xu + u^2 + y^2 + 2yv + v^2 \\
&= |z|^2 + 2(xu + yv) + |w|^2.
\end{aligned}$$

Adding these two moduli together gives us

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 = 2(|z|^2 + |w|^2)$$

which is our desired result. ■

(ii) **(Binomial Expansion)**: Let $z, w \in \mathbb{C}$ and n be a positive integer. Show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. Let $z, w \in \mathbb{C}$. We proceed via induction on $n \in \mathbb{Z}^+$ to show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Let $n = 1$ be our base case. Then we have

$$\begin{aligned}\sum_{k=0}^{1} \binom{n}{k} z^k w^{n-k} &= \binom{1}{0} z^0 w + \binom{1}{1} z^1 w^0 \\ &= (z+w)^1,\end{aligned}$$

which tells us that the result holds in our base case. Now, suppose the result holds for nth case. We will show the result holds for the $n+1$ case. By our induction hypothesis, we see that

$$\begin{aligned}(z+w)^{n+1} &= (z+w)(z+w)^n \\ &= (z+w) \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1}.\end{aligned}$$

Reordering indices in the first summation by setting $m = k+1$, we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1} &= \sum_{m=1}^{n+1} \binom{n}{m-1} z^m w^{(n+1)-m} \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} z^k w^{(n-k)+1}.\end{aligned}$$

Then separating the first and last term of each summation, respectively, we have

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{n} z^n w + \sum_{m=1}^{n} \binom{n}{m-1} z^m w^{(n-k)+1} + \sum_{k=1}^{n} \binom{n}{k} z^k w^{n-k+1} + \binom{n}{0} w^{n+1} \\ &= \binom{n}{n} z^n w + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] z^k w^{(n+1)-k} + \binom{n}{0} w^{n+1}.\end{aligned}$$

Using the fact that **(Need to show this!)**

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

and collecting the first and last terms of the summation, we see that

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{0} z^0 w^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} z^k w^{(n+1)-k} + \binom{n}{n} z^{n+1} w^0 \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} z^k w^{(n+1)-k}\end{aligned}$$

which completes our induction argument. ■

2. For $z, w \in \mathbb{C}$. Define $\langle z, w \rangle = \Re(z\overline{w})$. (If we think of $\mathbb{C}$ as two dimensional real vector space, then $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{C}$).

   (i) Cauchy Schwarz Inequality: Show that $|\langle z, w \rangle|^2 \leq |z|^2 |w|^2$ for all $z, w \in \mathbb{C}$.

Proof. First, we would like to show that for any $z, w \in \mathbb{C}$

$$\langle z, w \rangle^2 + \langle iz, w \rangle^2 = |z|^2|w|^2.$$

Let $z, w \in \mathbb{C}$. By definition of $\langle z, w \rangle$, we see that

$$\langle z, w \rangle^2 + \langle iz, w \rangle^2 = (xu + yv)^2 + (xv - uy)^2. \tag{1}$$

Observe that

$$(xu + yv)^2 = (xu)^2 + 2xuyv + (yv)^2 \tag{2}$$

and

$$(xv - uy)^2 = (xv)^2 - 2xuyv + (uy)^2. \tag{3}$$

Then adding (2) and (3) results in

$$\begin{aligned}
\langle z, w \rangle^2 + \langle iz, w \rangle^2 &= (xu)^2 + (yv)^2 + (xv)^2 + (uy)^2 \\
&= x^2u^2 + y^2v^2 + x^2v^2 + u^2y^2 \\
&= u^2(x^2 + y^2) + v^2(x^2 + y^2) \\
&= (x^2 + y^2)(u^2 + v^2) \\
&= |z|^2|w|^2.
\end{aligned}$$

Now, we need to show that $|\langle z, w \rangle|^2 \leq |z|^2|w|^2$. By using the result above, we can write

$$|\langle z, w \rangle|^2 \leq |\langle z, w \rangle|^2 + |\langle iz, w \rangle|^2 = |z|^2|w|^2$$

and we are done. ■

(ii) Triangle Inequalities: Show

$$|z + w| \leq |z| + |w|$$

and

$$||z| - |w|| \leq |z - w|$$

for all $z, w \in \mathbb{C}$.

Proof. Let $z, w \in \mathbb{C}$. We will first show that $|z + w| \leq |z| + |w|$. First, we will show the following results:

$$|z + w|^2 = |z|^2 + 2|\langle z, w \rangle| + |w|^2 \tag{1}$$

and

$$|z - w|^2 = |z|^2 - 2|\langle z, w \rangle| + |w|^2. \tag{2}$$

Let $z = x + iy$ and $w = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that

$$z + w = (x + u) + i(y + v)$$

and

$$z - w = (x - u) + i(y - v).$$

Using the definition of the modulus, we see that

$$\begin{aligned}
|z + w|^2 = (z + w)\overline{(z + w)} &= ((x + u) + i(y + v))((x + u) - i(y + v)) \\
&= (x + u)^2 + (y + v)^2 \\
&= x^2 2xu + u^2 + y^2 + 2yv + v^2 \\
&= (x^2 + y^2) + 2(xu + yv) + (y^2 + v^2) \\
&\qquad \text{(since } \Re(z\overline{w}) = xu + yv\text{)} \\
&= |z|^2 + 2\langle z, w \rangle + |w|^2. \qquad \text{(since } \langle z, w \rangle = \Re(z\overline{w})\text{)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|z-w|^2 = (z-w)\overline{(z-w)} &= ((x-u)+i(y-v))((x-u)-i(y-v)) \\
&= (x-u)^2+(y-v)^2 \\
&= x^2-2xu+u^2+y^2-2yv+v^2 \\
&= x^2+y^2-2(xu+yv)+v^2 \\
&= (x^2+y^2)-2\langle z,w\rangle+(v^2+u^2) \\
&= |z|^2-2\langle z,w\rangle+|w|^2.
\end{aligned}$$

Now, let us prove that $|z+w| \leq |z|+|w|$. Consider $|z+w|^2$. By part (a), we see that

$$\begin{aligned}
|z+w|^2 &= |z|^2+2\langle z,w\rangle+|w|^2 \\
&\leq |z|^2+2zw+|w|^2 \\
&\leq |z|^2+2|z||w|+|w|^2 \\
&= (|z|+|w|)^2.
\end{aligned}$$

By taking the square root of both sides, we see that

$$|z+w| \leq |z|+|w|.$$

To show the second inequality, consider $|z-w|^2$. Then using part (a) again, we have

$$\begin{aligned}
|z-w|^2 &= |z|^2-2\langle z,w\rangle+|w|^2 \\
&\geq |z|^2-2|z||w|+|w|^2 \\
&= (|z|-|w|)^2.
\end{aligned}$$

By taking the square root of both sides, we see that

$$|z-w| \geq ||z|-|w||.$$

■

3. **(Lagrange Identity)** Let $z_1,\ldots,z_n,w_1,\ldots,w_n \in \mathbb{C}$. Show that

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2 - \sum_{1\leq i<j\leq n}|z_i\overline{w_j}-z_j\overline{w_i}|^2.$$

Use this to deduce that

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2.$$

Proof. We will show that

$$\sum_{1\leq i<j\leq n}|z_i\overline{w_j}-z_j\overline{w_i}|^2 = \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2 - \left|\sum_{k=1}^{n} z_k w_k\right|^2.$$

Observe that

$$\begin{aligned}
\sum_{1\le i<j\le n} |z_i\overline{w_j} - z_j\overline{w_i}|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} |z_i\overline{w_j} - z_j\overline{w_i}|^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (z_i\overline{w_j} - z_j\overline{w_i})(\overline{z_i}w_j - \overline{z_j}w_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[|z_i|^2|w_j|^2 - 2z_j\overline{z_i}\overline{w_i}w_j + |z_j|^2|w_i|^2\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |z_i|^2|w_j|^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n} z_j\overline{z_i}\overline{w_i}w_j + \sum_{i=1}^{n}\sum_{j=1}^{n} |z_j|^2|w_i|^2 \\
&= \sum_{i=1}^{n}|z_i|^2\sum_{j=1}^{n}|w_i|^2 + \sum_{i=1}^{n}|w_i|^2\sum_{j=1}^{n}|z_j|^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n} z_j\overline{z_i}\overline{w_i}w_j \\
&= \sum_{i=1}^{n}|z_i|\sum_{i=1}^{n}|w_i|^2 - \sum_{i=1}^{n}\overline{z_i}\overline{w_i}\sum_{i=1}^{n} z_iw_i && (\text{for } i=j) \\
&= \sum_{i=1}^{n}|z_i|^2\sum_{i=1}^{n}|w_i|^2 - \left|\sum_{i=1}^{n} z_iw_i\right|^2.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_kw_k\right|^2 &= \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2 - \sum_{1\le i<j\le n}|z_i\overline{w_j} - z_j\overline{w_i}|^2 \\
&\le \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2.
\end{aligned}$$

■

4. Express the following complex number in the form $\alpha + i\beta$:

(i) $(1+i)^{-1}$

Solution. Observe that

$$(1+i)^{-1} = \frac{1}{1+i}$$

and that

$$\frac{1}{1+i}\cdot\frac{(1-i)}{(1-i)} = \frac{(1-i)}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i.$$

■

(ii) $(1+i)/2i$

Solution. Observe that

$$\frac{(1+i)}{2i} = \frac{1}{2i}(1+i) = \frac{1}{2i} + \frac{1}{2} = \frac{1}{2} - \frac{1}{2}i.$$

■

(iii) $(5+5i)^{10}$

Solution. Let $z = 1+i$. Observe that we can write

$$(5+5i)^{10} = 5^{10}(1+i)^{10}.$$

Note that

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Furthermore, we have

$$\tan^{-1}(1/1) = \tan^{-1}(1) = \frac{\pi}{4}.$$

Using De Moivre's formula, we can write

$$\begin{aligned} z^{10} &= (\sqrt{2})^{10}(\cos(10\theta) + i\sin(10\theta) \\ &= (\sqrt{2})^{10}(\cos(5\pi/2) + i\sin(5\pi/2)) \qquad\qquad = (\sqrt{2})^{10} i. \end{aligned}$$

Then we have

$$(5 + 5i)^{10} = 5^{10}(\sqrt{2})^{10} i = 312500000i.$$

■

(iv) $\left(\frac{2+i}{3-2i}\right)^2$

Solution. Our first step is to get $\frac{2+i}{3-2i}$ in terms of $\alpha + i\beta$. Thus, observe that

$$\frac{2+i}{3-2i} = \frac{2+i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{7i+4}{13} = \frac{4}{13} + i\frac{7}{13}.$$

Furthermore, we have

$$\left(\frac{4}{13} + i\frac{7}{13}\right)^2 = \frac{1}{169}(4+7i)^2 = \frac{1}{169}(16 + 46i - 49) = \frac{1}{169}(-33 + 46i).$$

Thus, we have that

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{-33}{169} + \frac{46}{169} i$$

■

(v) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

Solution. Denote $z = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$. Then observe that

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}.$$

Furthermore, we have

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

Using De Moivre's formula, we have that

$$z^3 = 1^3 \cdot \left(\cos\left(3 \cdot \frac{2\pi}{3}\right) + i\sin\left(3 \cdot \frac{2\pi}{3}\right)\right) = \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1.$$

■

5. Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ where $n \in \mathbb{Z}^+$ and

$$w = \gamma^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]$$

where $k \in \mathbb{Z}$. Show that $w^n = z$.

Proof. Note that for any $n \in \mathbb{N}$ that

$$z^n = (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

Thus, we have

$$\begin{aligned} w^n &= \left(\gamma^{1/n}\left[\cos\left(\frac{\varphi+2\pi k}{n}\right)+i\sin\left(\frac{\varphi+2\pi k}{n}\right)\right]\right)^n \\ &= (\gamma^{1/n})^n\left[\cos\left(\frac{\varphi+2\pi k}{n}\right)+i\sin\left(\frac{\varphi+2\pi k}{n}\right)\right]^n \\ &= \gamma\left[\cos\left(n\cdot\frac{\varphi+2\pi k}{n}\right)+i\sin\left(n\cdot\frac{\varphi+2\pi k}{n}\right)\right] \\ &= \gamma(\cos\varphi+i\sin\varphi) \\ &= z \end{aligned}$$

which ends our proof. ■

6. **(Computing fourth roots)**: Find your distinct complex numbers w such that $w^4 = z$ for
   (i) $z = i$.

   Solution. ■

   (ii) $z = -i$.

   Solution. ■

   (iii) $z = 1$.

   Solution. ■

   (iv) $z = -1$.

   Solution. ■

7. Sketch the following sets in $\mathbb{C}$.
   (i) $\zeta = \{z \in \mathbb{C} : \Re((1+i)z - 2) = 0\}$.

   Solution. ■

   (ii) Let $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$ with $b\bar{b} - a > 0$ and

   $$\zeta = \{z \in \mathbb{C} : a|z|^2 + \bar{b}z + b\bar{z} + c = 0\}.$$

   Solution. ■

   (iii) $\zeta = \{z \in \mathbb{C} : |z - i| = 2\}$.

   Solution. ■

8. Let $z, a \in \mathbb{C}$.
   (i) Show that $|1 - z\bar{a}|^2 - |z - a|^2 = (1 - |z|)^2(1 - |a|^2)$.

   Proof. ■

   (ii) Assume that $|a| < 1$. Show that

   $$|z| < 1 \Leftrightarrow \left|\frac{z-a}{1-\bar{a}z}\right| < 1$$

and

$$|z| = 1 \Leftrightarrow \left| \frac{z-a}{1-\bar{a}z} \right| = 1.$$

Proof. ■