Linear Algebra Notes

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Contents

Chapter 1

Vector Spaces

1.1 Bases And Dimension

- Recall that S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent.
- Linearly independent sets possess the unique property that every vector that its spanning set generates is unique.
- This is property is what allows generating sets to be the building blocks of vector spaces.

Definition 1 (Basis). A **basis** β for a vector space V is linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Example.
- Recall that the empty set $\emptyset$ is linearly independent and that $\operatorname{span}(\emptyset) = \{0\}$. The empty set $\emptyset$ in this case is the basis for the zero vector space.
- Note that in F^n, the vectors $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0) \ldots, e_n = (0, 0, \ldots, 0, 1)$ form a basis for F^n.
- The basis for $M_{m\times n}(F)$ is the set of matrices E^{ij} such that the only nonzero entry is a 1 in the ith and jth column.
- As we have seen in the last section, the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$.
- In $P(F)$, the set $\{1, x, xx^2, \ldots\}$ is a basis. *Bases are not limited to finite sets. They can be infinite.*

Theorem 1. Let V be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if each $v \in V$ can be unique expressed as a linear combination of vectors in β, that is, expressed in the form

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof. ($\Rightarrow$) Suppose $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V. Then $\operatorname{span}(\beta) = V$. If $v \in V$, then $v \in \operatorname{span}(\beta)$. Hence, we can write v as a linear combination of vectors in β such that choosing scalars $a_1, a_2, \ldots, a_n \in F$ leads to

$$v = \sum_{i=1}^{n} a_iu_i.$$

Suppose there exists another representation of $v \in V$ such that

$$v = \sum_{i=1}^{n} b_i x_i$$

Hence, observe that

$$\begin{aligned}\sum_{i=1}^{n} a_i x_i &= \sum_{i=1}^{n} b_i y_i \\ \Rightarrow \sum_{i=1}^{n} (a_i - b_i) x_i &= 0.\end{aligned}$$

Since β is linearly independent, we know that $a_i - b_i = 0$ which implies $a_i = b_i$ for all $1 \leq i \leq n$. Hence, v can be expressed as a unique linear combination of vectors in β.

($\Leftarrow$) Let $v \in V$. Then v can be uniquely expressed as a linear combination of vectors in $\beta = \{u_1, u_2, \ldots, u_n\}$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \tag{1}$$

for unique a_i for all $1 \leq i \leq n$. Suppose for sake of contradiction that β is **NOT** a basis for V. Then either β is linearly dependent or $\text{span}(\beta) \neq V$. Suppose $\text{span}(\beta) \neq V$, then $v \in V$ cannot be written a linear combination of vectors in β which is a contradiction. Suppose β is linearly dependent. Then either $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some $1 \leq k < n$. If $u_1 = 0$, then (1) can be re-written as

$$v = a_1 0 + a_2 u_2 + \cdots + a_n u_n.$$

This implies that $a_1 \in F$ can be made arbitrary such that v can be written in another representation. But this is a contradiction since we assumed that v contains a unique linear combination. Now, suppose $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$. Then we have

$$u_{k+1} = \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_k u_k.$$

Substituting this equation for u_{k+1} in (1) produces the following equation:

$$\begin{aligned}&v = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k + a_{k+1} u_{k+1} + \cdots a_n u_n \\ &\Rightarrow v = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k + a_{k+1}(\delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_k u_k) \\ &+ \cdots + a_n u_n \\ &\Rightarrow v = (a_1 + a_{k+1}\delta_1) u_1 + (a_2 + a_{k+1}\delta_2) u_2 + \cdots + (a_k + a_{k+1}\delta_k) u_k + \cdots + a_n u_n. \end{aligned} \tag{2}$$

Since v has a unique representation, we must have

$$a_i = a_i + a_{k+1}\delta_i \Rightarrow a_{k+1}\delta_i = 0 \tag{3}$$

for all $1 \leq i \leq k$. Assuming that $a_{k+1} \neq 0$, the only way for $a_{k+1}\delta_i = 0$ is if $\delta_i = 0$ for all $1 \leq i \leq k$. But note that not all δ_i are zero since $\{u_1, u_2, \ldots, u_k\} \subseteq \beta$ is linearly dependent by Exercise 16. Hence, there exists at least one i such that $a_{k+1}\delta_i \neq 0$ which is a contradiction. Hence, β must be a basis for V.

■

- Any vector $v \in V$ can be written as a linear combination of vectors from the basis containing $u_1, u_2, \ldots, u_n \in V$.
- This determines a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ and conversely, each n-tuple of scalars determines a unique vector $v \in V$ such that each coefficient from the linear combination of $u_1, u_2, \ldots, u_n$ is an entry from said tuple.
- For example, in our vector space F^n, n is the number of vectors that should be in the basis for F^n which is indeed the case.

- In this book, we are only concerned with finite bases.

Theorem 2. If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence, V has a finite basis.

Proof. ■