Math-241-Homework-5

Problem-1: Let $(V, \|\cdot\|)$ be a normed space and Y be a vector subspace of V. Last time, we saw that $V/Y = \{v+Y \mid v \in V\}$ is also a vector space.

Now, assume that Y is closed in $(V, \|\cdot\|)$.

(i) Let v and v' such that $v-v' \in Y$.

Show that $\inf\{\|v+y\| : y \in Y\} = \inf\{\|v'+y\| : y \in Y\}$.

(ii) For $[v] = v+Y \in V/Y$, define

$$\|[v]\|_0 = \inf\{\|v+y\| : y \in Y\}$$

((i) $\Rightarrow$ $\|\cdot\|_0$ is well defined)

Show that $\|\cdot\|_0$ defines a norm on V/W.

(iii) For any $v \in V$. Show that

$$\|[v]\|_0 \leq \|v\|.$$

(iv) We have a Canonical map $\pi : V \to V/Y$, $\pi(v) = [v]$. Show that π is linear and continuous. Here continuity means $\|v_n - v\| \to 0$ in V implies implies that

$$\|[v_n] - [v]\|_0 \to 0 \text{ in } V/W.$$

Problem 2: Consider the normed space $(\ell^\infty, \|\cdot\|_\infty)$.

Define a sequence $e^{(n)}$ in ℓ^∞ by

$$e^{(n)} = (\delta_j^{(n)}), \quad \delta_j^{(n)} = \begin{cases} 1 & \text{if } j=n \\ 0 & \text{otherwise.} \end{cases}$$

(i) Compute $\|e^{(n)} - e^{(n')}\|_\infty$ for $n \neq n'$

(ii) Does $(e^{(n)})$ have a convergent subsequence?

(iii) Prove that $S(\ell^\infty) = \{x = (x_j) \in \ell^\infty / \|x\|_\infty = 1\}$ is closed and bounded but not compact.

Problem-3: Let $(V, \|\cdot\|)$ be a normed space and Y be a subspace of V such that $Y \neq V$. Let $u \in V \setminus Y$.

Define $d(u, Y) = \inf\{\|u-y\| : y \in Y\}$.

(i) Show that if $d(u, Y) = 0$ then $u \in \overline{Y}$.

(ii) Assume that Y is closed. Prove that $d > 0$

Problem-4: Read section 2.6 Kreyszig and write down statements of key theorems, lemmas, and propositions.

Problem 5: (i) Let $T: D(T) \to W$ be a linear operator. Assume that $T^{-1}: R(T) \to D(T)$ exists. Show that if $\{u_1, \cdots, u_n\}$ is linearly dependent on $D(T)$ then $\{Tv_1, \cdots, Tu_n\}$ is linearly dependent on W.

(ii) Let V and W be two vector spaces and $T: V \to W$ be a linear operator. Assume that V and W are finite dimensional and $\dim(V) = \dim(W)$.

Prove that $R(T) = W$ if and only if T^{-1} exists.