Contents

Linear Algebra Exercises

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Chapter 1

Vector Spaces

1.1 Vector Spaces

Exercise 1.2.1

Label the following statements as true or false.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

Exercise 1.2.7

Let $S = \{0, 1\}$ and $F = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R}$, show that $f = g$ and where $f(t) = 2t+1$, $g(t) = 1+4t-2t^2$, and $h(t) = 5^t + 1$.

Proof. To show that $f = g$, we have to show that for each $s \in S$ that $f(s) = g(s)$. Since $S = \{0, 1\}$, we can just evaluate both f and g for elements in S. Note that

$$f(0) = 2(0) + 1 = 1$$

and likewise,

$$g(0) = 1 + 4(0) - 2(0)^2.$$

Hence, $f(0) = g(0)$. Now let us evaluate both functions f and g at $s = 1$. Hence, we have

$$f(1) = 2(1) + 1 = 3$$

and

$$g(1) = 1 + 4(1) - 2(1)^2 = 3.$$

Thus, we must have $f(s) = g(s)$ for all $s \in S$.

Now, we need to show that $f + g = h$. Like we did above, we have to show that this is the case for all $s \in S$. Note that

$$(f + g)(s) = f(s) + g(s).$$

Hence, we have

$$f(s) + g(s) = 2 + 6s - 2s^2.$$

Evaluating at $s = 0$, we have
$$f(0) + g(0) = 2.$$
and likewise,
$$h(0) = 5^0 + 1 = 2.$$
Hence, $(f + g)(0) = h(0)$. Now let us evaluate $f + g$ at $s = 1$
$$f(1) + g(1) = 2 + 6(1) - 2(1)^2 = 6$$
and likewise, we have
$$h(1) = 5^1 + 1 = 6.$$
Hence, we have $(f + g)(1) = h(1)$. Thus, we have that $f + g = h$ for all $s \in S$.

■

Exercise 1.2.8

In any vector space V, we have
$$(a + b)(x + y) = ax + ay + bx + by$$
for any $x, y \in V$ and any $a, b \in F$.

Proof. Observe the following set equalities:
$$\begin{aligned}(a + b)(x + y) &= a(x + y) + b(x + y) && \text{(VS 8)}\\ &= ax + ay + bx + by. && \text{(VS 7)}\end{aligned}$$
Hence, we have that
$$(a + b)(x + y) = ax + ay + bx + by$$
for any $x, y \in V$ and any $a, b \in F$. ■

Exercise 1.2.10

Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Let V denote the set of all differentiable real-valued functions defined on the real line. We need to show that V is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined in Example 3.

(VS 1) Let $f, g \in V$. We need to show that $f + g = g + f$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Since $f(x)$ and $g(x)$ are also real numbers, we have
$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$
Hence, we have $f + g = g + f$.

(VS 2) Let $f, g, h \in V$. We need to show that $f + (g + h) = (f + g) + h$. Let $x \in \mathbb{R}$. Since

$f(x), g(x), h(x) \in \mathbb{R}$, we can see that

$$\begin{aligned}(f+(g+h))(x) &= f(x)+(g+h)(x)\\ &= f(x)+g(x)+h(x)\\ &= (f+g)(x)+h(x)\\ &= ((f+g)+h)(x).\end{aligned}$$

Hence, we have $f+(g+h) = (f+g)+h$.

(VS 3) Observe that $f_0(x) = 0$ for all $x \in \mathbb{R}$ is also a real-valued function that is differentiable. We need to show that $f + f_0 = f$. Hence, we have

$$(f+f_0)(x) = f(x)+f_0(x) = f(x)+0 = f(x).$$

Hence, $f + f_0 = f$ for all $x \in \mathbb{R}$.

(VS 4) Take $c = -1$ and perform a scalar operation with a $f \in V$. Observe that $(-1 \cdot f)(x) = (-1)f(x) = -f(x)$. Denote $g = -f$. Since additive inverses exists in $\mathbb{R}$, we have

$$\begin{aligned}(f-g)(x) &= (f-f)(x)\\ &= f(x)-f(x)\\ &= 0\\ &= f_0(x).\end{aligned}$$

Hence, $f - g = f_0$ for all $x \in \mathbb{R}$.

(VS 5) Let $f \in V$ and let $x \in \mathbb{R}$. We need to show that $1 \cdot f = f$. Since $f(x) \in \mathbb{R}$, we can see that multiplicative identities in $\mathbb{R}$ are also preserved in V; that is, we have

$$(1f)(x) = 1 \cdot f(x) = f(x).$$

Hence, we have $1f = f$ for all $x \in \mathbb{R}$.

(VS 6) Let $a, b \in \mathbb{R}$ and $f \in V$. Let $x \in \mathbb{R}$ be arbitrary. Then observe that

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a(bf)(x).$$

Hence, we have $(ab)f = a(bf)$ for all $x \in \mathbb{R}$.

(VS 7) Let $a \in \mathbb{R}$ and let $f, g \in V$. We need to show that $a(f+g) = af + ag$. Let $x \in \mathbb{R}$. Observe that $f(x), g(x) \in \mathbb{R}$ imply

$$\begin{aligned}(a(f+g))(x) &= a(f+g)(x)\\ &= a(f(x)+g(x))\\ &= af(x)+ag(x)\\ &= (af)(x)+(ag)(x).\end{aligned}$$

Hence, we conclude that $a(f+g) = af + ag$ for all $x \in \mathbb{R}$.

(VS 8) Let $a, b \in \mathbb{R}$ and let $f \in V$. Let $x \in \mathbb{R}$. Then we have

$$\begin{aligned}((a+b)f)(x) &= (a+b)f(x) \\ &= af(x) + bf(x) \\ &= (af)(x) + (bf)(x).\end{aligned}$$

Hence, we have $(a+b)f = af + bf$ for all $x \in \mathbb{R}$.

Since all the properties of a vector space have been satisfied, we conclude that V is a vector space. ■

Exercise 1.2.11

Let $V = \{0\}$ consist of a single vector O and define $O + O = O$ and $cO = O$ for each scalar $c \in F$. Prove that V is a vector space over F.

Proof. Let $x, y \in O$. Since $V = \{0\}$, we know that x and y are both the 0 vector. By using the addition defined on V, we can see that

$$x + y = O + O = y + x.$$

Note that $x + y \in V$ implies that $x + y = O + O$ and likewise $y + z = O + O$ for every $x, y, z \in V$. Hence, we have that

$$x + (y + z) = O + (O + O) = (O + O) + O = (x + y) + z.$$

Since V consists of only the zero vector O, we know that (VS 3) and (VS 4) are satisfied. By the same reasoning, (VS 5) is satisfied because

$$1 \cdot O = O.$$

Let $a, b \in F$. Then (VS 6) and (VS 7) are satisfied because

$$(ab)O = 0 = a \cdot 0 = a(bO) \text{ and } a(O + O) = 0 = O + O = aO + aO \text{ respectively.}$$

Let $a, b \in F$ again. Then we have

$$\begin{aligned}(a+b)O &= 0 \\ &= O + O \\ &= aO + bO.\end{aligned}$$

Hence, (VS 8) is satisfied. ■

Exercise 1.2.12

A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for all $t \in \mathbb{R}$. Prove that the set of even functions defined on $\mathbb{R}$ with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof. First, let us show that for every $t \in \mathbb{R}$ that $(f + g)(t)$ is also an even function for every pair of even functions f, g and likewise $(cf)(t)$ is an even function for every $c \in \mathbb{R}$.

Observe that

$$\begin{aligned}(f+g)(-t) &= f(-t)+g(-t)\\ &= f(t)+g(t)\\ &= (f+g)(t).\end{aligned}$$

Hence, the function $f+g$ is also even. Now observe that

$$\begin{aligned}(cf)(-t) &= cf(-t)\\ &= cf(t)\\ &= (cf)(t).\end{aligned}$$

Hence, cf is also even function.

Now we can show that the set of even functions is a vector space.

(VS 1) Let f, g be a pair of real-valued even functions. Let $t \in \mathbb{R}$ be arbitrary. We need to show that $f+g = g+f$. Since $f(t)$ and $g(t)$ are real numbers, observe that

$$(f+g)(t) = f(t)+g(t) = g(t)+f(t) = (g+f)(t).$$

Hence, we have that $f+g = g+f$.

(VS 2) Let f, g, h be even functions and let $t \in \mathbb{R}$ be arbitrary. We need to show that $f+(g+h) = (f+g)+h$. By the same reasoning we used to prove (VS 1), observe that

$$\begin{aligned}f(t)+((g+h)(t)) &= f(t)+(g(t)+h(t))\\ &= (f(t)+g(t))+h(t)\\ &= ((f+g)(t))+h(t)\end{aligned}$$

Hence, we have that $f+(g+h)+(f+g)+h$.

(VS 3) Let f be an even function. Note that the zero function is an even function. Denote the zero function as f_0. For every $t \in \mathbb{R}$, we have $f_0(x) = O$. We need to show that $f + f_0 = f$. Since $f_0(x)$ is just a real number, we have

$$(f+f_0)(t) = f(t)+f_0(t) = f(t)+O = f(t).$$

(VS 4) Let f be an even function and let f_0 be the zero function defined above. We need to show that there exists a even function g such that $f+g = f_0$. Let $t \in \mathbb{R}$ be arbitrary. Note that

$$(f+g)(t) = f(t)+g(t)$$

Since $f(t)$ and $g(t)$ are real numbers and there exists an additive identity in the real numbers, we have that $f(t)+g(t) = f_0(t)$. Hence, (VS 5) is satisfied.

(VS 5) Let $x \in \mathbb{R}$ and f an even function. Then we immediately have $(1 \cdot f)(t) = 1f(t) = f(t)$.

(VS 6) Let $a, b \in \mathbb{R}$. We need to show that $(ab)f = a(bf)$. Hence, let $t \in \mathbb{R}$. Then we have

$$(abf)(t) = a(bf)(t).$$

(VS 7) Let $a \in \mathbb{R}$ and let f, g be even functions. Let $t \in \mathbb{R}$. We need to show that $a(f+g) = af + ag$. Then we have

$$\begin{aligned} a(f+g)(t) &= a[f(t) + g(t)] \\ &= af(t) + ag(t). \end{aligned}$$

Hence, (VS 7) is satisfied.

(VS 8) Now let $a, b \in \mathbb{R}$ and let f be an even function. We need to show that $(a+b)f = af + bf$. Let $t \in \mathbb{R}$ be arbitrary. Observe that

$$\begin{aligned} (a+b)f(t) &= af(t) + bf(t) \\ &= (af)(t) + (bf)(t) \end{aligned}$$

Hence, (VS 8) is satisfied.

■

Exercise 1.2.13

Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, a_2 b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2).$$

Is V is a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We have that V is not a vector space of $\mathbb{R}$. To see why, let $(2,1), (4,2) \in V$ where $x = (2,1)$ and $y = (4,2)$. We will show that (VS 1) does not hold; that is, $x + y \neq y + x$. Hence, observe that

$$(2,1) + (4,2) = (2+1, 2) = (3,2)$$

and

$$(4,2) + (2,1) = (4+2, 2) = (6,2)$$

Hence, we have $x + y \neq y + x$ and so V is **NOT** a vector space. ■

Exercise 1.2.14

Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in C \text{ for } i = 1, 2, \ldots, n)\}$; so V is a vector space over C by Example 1. Is V is a vector space over the field of real numbers with the operations of coordinate-wise addition and multiplication?

Proof. Yes, V where

$$V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$$

is a vector space of $\mathbb{R}$.

(VS 1) Let $x, y \in V$ such that $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$. Since addition is entry-wise in V and each entry in both x and y are elements of $\mathbb{R}$ (where $\mathbb{R}$ is a field), we have $a_i + b_i = b_i + a_i$ for all $i = 1, 2, \ldots, n$. Hence, $x + y = y + x$.

(VS 2) Let $x, y, z \in V$ with x and y as defined as before where z contains entries c_i for all $i = 1, 2, \ldots n$. We can see that the entries of x, y, z are elements of $\mathbb{R}$ so associativity is preserved; that is, $a_i + (b_i + c_i) = (a_i + b_i) + c_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $x + (y + z) = (x + y) + z$.

(VS 3) Since $\mathbb{R}$ contains the zero element 0 and V is the set of n-tuples, there exists an element denoted by O such that this element consisting of entries that only have the zero element 0; that is, $O = (0, 0, \ldots, 0)$. Take $x \in V$. Hence, we have $a_i + 0 = a_i$ for every $i = 1, 2, \ldots, n$. Thus, we must have $x + O = x$.

(VS 4) Let $x \in V$ be arbitrary as defined before. Since every entry in x is an element of $\mathbb{R}$; that is, every $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we know that every entry contains an element c_i such that $a_i + c_i = 0$ for every $i = 1, 2, \ldots, n$. Denote $x' = (c_1, c_2, \ldots, c_n)$. Hence, we have $x + x' = O$.

(VS 5) Let $x \in V$. Every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $1 \cdot a_i = a_i$ which holds for all i. Denote this identity element as I with entries consisting only of 1. Hence, we have $I \cdot x = x$.

(VS 6) Let $e, r \in \mathbb{R}$ and let $x \in V$. For every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $(er)a_i = e(ra_i)$ for all $i = 1, 2, \ldots, n$. By using the operations of scalar multiplication for n-tuples, this tells us that $(er)x = e(rx)$. Hence, (VS 6) is satisfied.

(VS 7) Let $e \in \mathbb{R}$ and let $x, y \in V$. We need to show that $e(x + y) = ex + ey$. Note that $a_i, b_i \in \mathbb{R}$ implies that $e(a_i + b_i) = ea_i + eb_i$. Hence, we have $e(x + y) = ex + ey$.

(VS 8) Let $e, r \in \mathbb{R}$ and $x \in V$ as defined before. We need to show that $(e + r)x = ex + rx$. Since $a_i \in \mathbb{R}$ with $e, r \in \mathbb{R}$, we are guaranteed to have $(e + r)a_i = ea_i + ra_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $(e + r)x = ex + rx$.

■

Exercise 1.2.16

Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over $\mathbb{R}$ by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

Proof. (VS 1) Let $A, B \in V$. Since A, B consist of elements $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $A_{ij} + B_{ij} = B_{ij} + A_{ij}$. Hence, $A + B = B + A$.

(VS 2) Let $A, B, C \in V$ with A, B defined as before and C containing real entries C_{ij}. With the same reasoning used to prove (VS 1), we know that $A_{ij} + (B_{ij} + C_{ij}) = (A_{ij} + B_{ij}) + C_{ij}$. Hence, we have $A + (B + C) = (A + B) + C$.

(VS 3) Let $A \in V$ once again. Since the entries of A imply that there exists an element O such that $A_{ij} + 0 = A_{ij}$, we know that $A + O = A$ where O is the **zero matrix** of V.

(VS 4) Since the real entries of A also consists of an element A'_{ij} such that $A_{ij} + A'_{ij} = 0$, this implies that $A + A' = O$ where A' is the additive inverse matrix of V.

(VS 5) Let $x \in V$ as defined as before. Every entry of A, $A_{ij} \in \mathbb{R}$, has the following property:

$1 \cdot A_{ij} = A_{ij}$ for all for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. The matrix whose entries consists of only one we can define as the **identity matrix** denoted by I where $I_{ij} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, we have $A \cdot I = A$.

(VS 6) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. We need to show that $(rt)A = r(tA)$. Since $A_{ij} \in \mathbb{R}$, entry-wise scalar multiplication implies that $(rt)A_{ij} = r(tA_{ij})$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, we must have $(rt)A = r(tA)$.

(VS 7) Let $r \in \mathbb{Q}$ and $A, B \in V$ as defined before. Since $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $r(A_{ij} + B_{ij}) = rA_{ij} + rB_{ij}$. Hence, we have $r(A + B) = rA + rB$.

(VS 8) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. Since $A_{ij} \in \mathbb{R}$ for all i, j, we must have $(r + t)A_{ij} = rA_{ij} + tA_{ij}$. Hence, we have $(r + t)A = rA + tA$.

Hence, V is a vector space over $\mathbb{Q}$. ■

Exercise 1.2.17

Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinate-wise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (ca_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

Proof. We claim that V is not a vector space over F because V fails to satisfy (VS 5). To see why, let $(1, 2) \in V$. Using (VS 5), we have

$$1 \cdot (1, 2) = (1, 0) \neq (1, 2).$$

Hence, V cannot be a vector space. ■

Exercise 1.2.18

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $F = \mathbb{R}$ with these operations? Justify your answer?

Proof. We claim that V is not a vector space over $\mathbb{R}$ and we will use (VS 1) to show this. Let $x, y \in V$ be defined by $x = (1, 2)$ and $y = (3, 4)$. Observe that

$$x + y = (1, 2) + (3, 4) = (7, 14)$$

and

$$y + x = (3, 4) + (1, 2) = (5, 10).$$

Clearly, we have $x + y = (7, 14) \neq (5, 10) = y + x$ and so (VS 1) does not hold. ■

Exercise 1.2.19

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinate-wise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We claim that V is not a vector space over $\mathbb{R}$. To see why, consider (VS 8). If we let $(0,1) \in V$ with $c = 2+1 = 3$. Observe that

$$(2+1)(0,1) = \left(0, \frac{1}{2+1}\right) = \left(0, \frac{1}{3}\right).$$

Likewise, we have

$$2(0,1) + 1(0,1) = \left(0, \frac{1}{2}\right) + \left(0, 1\right) = \left(0, \frac{3}{2}\right).$$

Notice that $(2+1)(0,1) \neq 2(0,1) + 1(0,1)$. Hence, V cannot be a vector space over $\mathbb{R}$. ■

Exercise 1.2.20

Let V denote the set of all real-valued functions f defined on the real line such that $f(1) = 0$. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Define V as a vector space with the operations of addition and scalar multiplication defined in Example 3. We must show that V is a vector space.

(VS 1) Let $f, g \in V$. This means that $f(1) = 0$ and $g(1) = 0$. We need to show that $f + g = g + f$. Since $f(1), g(1) \in \mathbb{R}$ and commutativity holds in $\mathbb{R}$, we can write

$$\begin{aligned}(f+g)(1) &= f(1) + g(1) \\ &= g(1) + f(1) \\ &= (g+f)(1)\end{aligned}$$

Hence, we have $f + g = g + f$.

(VS 2) Let $f, g, h \in V$ then $f(1) = g(1) = h(1) = 0$. We need to show that $f + (g + h) = (f + g) + h$. Observe that

$$\begin{aligned}\Big(f + (g+h)\Big)(1) &= f(1) + (g+h)(1) \\ &= f(1) + g(1) + h(1) \\ &= (f+g)(1) + h(1) \\ &= \Big((f+g) + h\Big)(1).\end{aligned}$$

Hence, we have $f + (g + h) = (f + g) + h$.

(VS 3) Let $f \in V$. We need to show that $f + f_0 = f$ for some $f_0 \in V$. Since V contains elements of $f \in V$ such that $f(1) = 0$, we can choose f_0 such that $f_0(1) = 0$. We can show that this is indeed the additive inverse of V by writing the following:

$$(f + f_0)(1) = f(1) + f_0(1) = f(1) + 0 = f(1).$$

Hence, we have $f + f_0 = f$.

(VS 4) Let $f \in V$. We need to find an element $g \in V$ such that $f + g = f_0$ By definition of V, $f(1) = 0$. We need to show that $f + g = f_0$ with f_0 defined as before. Choose $g = -f$

as our additive inverse and observe that

$$\begin{aligned}(f+g)(1) &= (f-f)(1)\\ &= f(1)-f(1)\\ &= 0-0\\ &= 0\\ &= f_0(1).\end{aligned}$$

Hence, $g=-f$ an element such that $f+g=f_0$.

(VS 5) Let $f \in V$. By definition of V, we have $f(1)=0$. Since $f(1)\in\mathbb{R}$, we know that $1\cdot f(1)=f(1)$. We need to show that $1f=f$. Observe that

$$(1f)(1) = 1\cdot f(1) = f(1).$$

Hence, (VS 5) holds.

(VS 6) Let $a,b\in\mathbb{R}$ and let $x\in V$. We need to show that $(ab)f=a(bf)$. By using scalar multiplication, we can see that

$$(ab\cdot f)(1) = (ab)f(1) = a(bf(1)) = a(b\cdot f)(1).$$

Hence, (VS 6) must hold.

(VS 7) Let $a\in\mathbb{R}$ and let $f,g\in V$. We need to show that $a(f+g)=af+ag$. Observe that

$$\begin{aligned}a(f+g)(1) &= a(f(1)+g(1))\\ &= af(1)+ag(1)\\ &= (af)(1)+(ag)(1).\end{aligned}$$

Hence, $a(f+g)=af+ag$ and so (VS 7) is satisfied.

(VS 8) Let $a,b\in\mathbb{R}$ and let $f\in V$. Observe that

$$\begin{aligned}(a+b)f(1) &= af(1)+bf(1)\\ &= (af)(1)+(bf)(1)\end{aligned}$$

Hence, $(a+b)f=af+bf$. Thus, V must be a vector space over $\mathbb{R}$. ■

Exercise 1.2.21

Let V and W be vector spaces over a field F. Let

$$Z = \{(v,w) : v\in v \text{ and } w\in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1,w_1)+(v_2,w_2) = (v_1+v_2, w_1+w_2) \text{ and } c(v_1,w_1) = (cv_1, cw_1).$$

Proof. Let V and W be vector spaces over a field F.

(VS 1) Let $x,y\in Z$ where $x=(v_1,w_1)$ and $y=(v_2,w_2)$. Since V and W are vector spaces, commutativity holds. Since the addition defined on both V and W is entry-wise, we can see that

$$\begin{aligned} x + y &= (v_1, w_1) + (v_2, w_2) \\ &= (v_1 + v_2, w_1 + w_2) \\ &= (v_2 + v_1, w_2 + w_1) \\ &= (v_2, w_2) + (v_1, w_1) \\ &= y + x. \end{aligned}$$

Hence, we can see that (VS 1) is satisfied.

(VS 2) Let $x, y, z \in Z$ with x and y as defined before as well as $z = (z_3, w_3)$. Using the entry-wise addition defined for Z, we can see that

$$\begin{aligned} x + (y + z) &= (v_1, w_1) + \Big((v_2, w_2) + (v_3, w_3)\Big) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (x + y) + z. \end{aligned}$$

(VS 3) Since V and W are vector spaces, we know that there exists an additive identity O_V and O_W respectively. Hence, we have $(O_V, O_W) \in Z$ and denote $O_Z = (O_V, O_W)$. Now, let $x \in Z$ as defined before. Observe that

$$\begin{aligned} x + O_Z &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + O_V, w_1 + O_W) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 4) Observe that V and W contain additive inverses for each $v \in V$ and $w \in W$ respectively. Since $x = (v_1, w_1)$ with $v_1 \in V$ and $w_1 \in W$, there exists an additive inverse $v_1' \in V$ and $w_1' \in W$ such that $v_1 + v_1' = O_V$ and $w_1 + w_1' = O_W$. This implies that $(v_1', w_1') \in Z$ which we will denote by x' such that

$$\begin{aligned} x + x' &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + v_1', w_1 + w_1') \\ &= (O_V, O_W) \\ &= O_Z. \end{aligned}$$

(VS 5) Let $x \in Z$. Since V and W are vector spaces (VS 5) implies that $1 \cdot v_1 = v_1$ and $1 \cdot w_1 = w_1$ respectively. Then observe that

$$\begin{aligned} 1 \cdot x &= 1 \cdot (v_1, w_1) \\ &= (1 \cdot v_1, 1 \cdot w_1) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 6) Let $x \in Z$ and $a, b \in F$. Then

$$\begin{aligned}(ab)x &= (ab)(v_1, w_1) \\ &= ((ab)v_1, (ab)w_1) \\ &= (a(bv_1), a(bw_1)) && (V, W \text{ vector space}) \\ &= a(bv_1, bw_1) \\ &= a(bx)\end{aligned}$$

(VS 7) Let $x, y \in Z$ as defined before. Let $a \in F$. Then

$$\begin{aligned}a(x+y) &= a\Big((v_1, w_1) + (v_2, w_2)\Big) \\ &= a\Big((v_1 + v_2, w_1 + w_2)\Big) \\ &= (a(v_1 + v_2), a(w_1 + w_2)) \\ &= (av_1 + av_2, aw_1 + aw_2) \\ &= (av_1, av_2) + (aw_1, aw_2) \\ &= a(v_1, v_2) + a(w_1, w_2) \\ &= ax + ay.\end{aligned}$$

(VS 8) Let $a, b \in F$ and let $x \in V$ as defined before. Since V and W are vector spaces, we know that (VS 8) holds for bot entries $v_1 \in V$ and $w_1 \in W$. Hence, observe that

$$\begin{aligned}(a+b)x &= (a+b)(v_1, w_1) \\ &= \Big((a+b)v_1, (a+b)w_1\Big) \\ &= (av_1 + bv_1, aw_1 + bw_1) \\ &= (av_1, aw_1) + (bv_1, bw_1) \\ &= a(v_1, w_1) + b(v_1, w_1) \\ &= ax + bx.\end{aligned}$$

Hence, Z is a vector space.

■

1.2 Subspaces

Exercise 1.3.3

Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$.

Proof. Let $A, B \in M_{m\times n}(F)$ and let $a, b \in F$ be arbitrary. Using scalar multiplication defined on $M_{m\times n}(F)$, we have

$$\begin{aligned}(aA + bB)^t &= (aA)^t + (bB)^t \\ &= aA^t + bB^t.\end{aligned}$$

Hence, we are done. ■

Exercise 1.3.4

Prove that $(A^t)^t = A$ for each $A \in M_{n\times n}(F)$.

Proof. Let $A \in M_{m\times n}(F)$. By definition of transpose, we have

$$\left((A^t)^t\right)_{ij} = (A^t)_{ji} = A_{ij}$$

for all $1 \leq i, j \leq n$. Hence, $(A^t)^t = A$. ∎

Exercise 1.3.5

Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. Let A be an arbitrary square matrix. Since square matrices are symmetric, we have that $A^t = A$. We need to show that $(A + A^t)^t$. Observe that

$$\begin{aligned}(A + A^t)^t &= A^t + (A^t)^t \\ &= A + A^t.\end{aligned}$$

Hence, we have $A + A^t$ is symmetric. ∎

Exercise 1.3.

Prove that $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$ for any $A, B \in M_{n\times n}(F)$.

Proof. Let $A, B \in M_{n\times n}(F)$ and let $a, b \in F$ be arbitrary. Now, let $i = j$ and observe that

$$\begin{aligned}\text{tr}(aA + bB) &= \sum_{i,j\in\mathbb{N}}^{n} (aA + aB)_{ij} \\ &= \sum_{i,j\in\mathbb{N}}^{n} (aA)_{ij} + (bB)_{ij} \\ &= \sum_{i,j\in\mathbb{N}}^{n} (aA)_{ij} + \sum_{i,j\in\mathbb{N}}^{n} (bB)_{ij} \\ &= \sum_{i,j\in\mathbb{N}}^{n} aA_{ij} + \sum_{i,j\in\mathbb{N}}^{n} bB_{ij} \\ &= a\sum_{i,j\in\mathbb{N}}^{n} A_{ij} + b\sum_{i,j\in\mathbb{N}}^{n} B_{ij} \\ &= a\text{tr}(A) + b\text{tr}(B).\end{aligned}$$

Hence, we conclude

$$\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$$

for any $A, B \in M_{n\times n}(F)$. ∎

Exercise 1.3.7

Prove that diagonal matrices are symmetric matrices.

Proof. Let $A \in M_{n\times n}(F)$ be diagonal. Let $i \neq j$ where $A_{ij} = 0$. We need to show that $A^t = A$. If we apply a transpose on A, we get that $A_{ji} = 0$ since A^t is also diagonal and square. Since $A_{ij} = A_{ji} = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$. Hence, $A^t = A$ ■

Exercise 1.3.8

Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

Proof. We claim that W_1 is a subspace of $\mathbb{R}^3$.

(a) Note that $O_{\mathbb{R}^3} \in W_1$ where $O_{\mathbb{R}^3} = (0, 0, 0)$ because $0 = 3 \cdot 0$ and $0 = -1 \cdot 0$.

(b) Let $x, y \in W_1$ where $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$. We need to show that $x + y \in W_1$. Since $a_1 = 3a_2$ and $a_3 = -a_2$ as well as $b_1 = 3b_2$ and $b_3 = -b_2$, we can write $a_1 + b_1 = 3(a_2 + b_2)$ and $a_3 + b_3 = -(a_2 + b_2)$. Hence, $x + y \in W_1$.

(c) Let $c \in \mathbb{R}$ and $x \in W_1$ with x defined as before. Then observe that $ca_1 = c(3a_2) = 3(ca_2)$ and $ca_3 = c(-a_2) = -(ca_2)$. Hence, $cx \in W_1$.

Since all the properties of a Theorem 3 have been satisfied, we can conclude that W_1 is a subspace of $\mathbb{R}^3$. ■

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - 3a_3 = 1\}$

(e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

(f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Exercise 1.3.9

Let W_1, W_3, W_4 be as in Exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$ and observe that each is a subspace of $\mathbb{R}^3$.

Proof. ■

Exercise 1.3.11

Prove that the set $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1, a_2, \ldots, a_n)\} \in F^n : a_1 + a_2 + \cdots + a_n = 1$ is not.

Proof. We need to show that W_1 is a subspace of F^n. We proceed by satisfying the properties of Theorem 3 to do this.

(a) Note that $O_{F^n} \in W_1$ since $0 + 0 + \cdots + 0 = 0$ n times.

(b) Let $x, y \in W_1$ with $x = (a_1, a_2, \ldots, a_3)$ and $y = (b_1, b_2, \ldots, b_n)$. By definition of W_1, we can see that

$$\begin{aligned}\sum_{i=1}^{n}(a_i+b_i) &= \sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i\\ &= 0+0\\ &= 0.\end{aligned}$$

Hence, $x+y\in W_1$ which tells us that W_1 is closed under addition.

(c) Let $x\in W_1$ and $c\in F$. Then observe that

$$\sum_{i=1}^{n}(ca_i)=c\sum_{i=1}^{n}a_i=c\cdot 0=0.$$

Hence, we have $cx\in F^n$.

We claim that W_2 is not a subspace because W_2 is not closed under addition. Let $(0,1),(1,0)\in F^2$. Observe that $0+1=1$ and $1+0=1$, but $(0+1)+(1+0)=1+1=2$. Hence, $(0,1)+(1,0)\notin W_2$.

■

Exercise 1.3.11

Is the set $W=\{f(x)\in P(F): f(x)=0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $P(F)$ if $n\geq 1$? Justify your answer.

Exercise 1.3.12

Prove that the set of $m\times n$ upper triangular matrices is a subspace of $M_{m\times n}(F)$.

Proof. Let V denote the set of $m\times n$ upper triangular matrices. We will show that V is a subspace of $M_{m\times n}(F)$ using Theorem 3.

(a) The zero matrix O from $M_{m\times n}(F)$ contains entries $O_{ij}=0$ whenever $i>j$. Hence, $O\in V$.

(b) Let $A,B\in V$. By definition of V, A and B are upper triangular where $A_{ij}=0$ and $B_{ij}=0$ whenever $i>j$. Observe that

$$(A+B)_{ij}=A_{ij}+B_{ij}=0+0=0$$

whenever $i>j$. Hence, $A+B\in V$.

(c) Let $c\in F$ and $A\in V$ as defined before. Let $i>j$ and observe that $(cA)_{ij}=cA_{ij}=c\cdot 0=0$. Hence, $cA\in V$.

Since all the properties of theorem 3 have been satisfied, we conclude that V is indeed a subspace of $M_{m\times n}(F)$. ■

Exercise 1.3.13

Let S be nonempty set and F is a field. Prove that for any $s_0\in S$, the set $\{f\in\mathcal{F}(S,F): f(s_0)=0\}$, is a subspace of $\mathcal{F}(S,F)$.

Proof. We will proceed to prove that S is a subspace of $\mathcal{F}(S, F)$ over the field F by satisfying the properties of Theorem 3. Let $V = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$.

(a) Note that the zero function $f_0 \in \mathcal{F}(S, F)$ where $f_0(s_0) = 0$ for any $s_0 \in S$ implies that $f_0 \in V$.

(b) Let $f, g \in V$. By definition of V, $f(s_0) = 0$ and $g(s_0) = 0$ for any $s_0 \in S$. We have $f + g \in V$ since
$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0.$$
Hence, V is closed under addition.

(c) Let $f \in V$ and $c \in F$. We have $cf \in V$ since
$$(cf)(s_0) = cf(s_0) = c \cdot 0 = 0$$
for any $s_0 \in S$. Hence, V is closed under scalar multiplication.

■

Exercise 1.3.14

Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

Proof. We proceed by using Theorem 3 to prove that $\mathcal{C}(S, F)$ is a subspace where S is a nonempty set and F is a field.

(a) Note that the zero vector $f_0 \in \mathcal{F}(S, F)$ is in $\mathcal{C}(S, F)$ because $f_0(x_n) = 0$ where $x_n \in S$ for finitely many n.

(b) Let $f, g \in \mathcal{C}(S, F)$. We need to show that $f + g \in \mathcal{C}(S, F)$. Let $x_n \in S$ for finitely many n. Then using the addition defined on $\mathcal{F}(S, F)$, we can write
$$(f + g)(x_n) = f(x_n) + g(x_n) = 0 + 0 = 0.$$

(c) Let $f \in \mathcal{C}(S, F)$ and $c \in F$. We need to show that $cf \in \mathcal{C}(S, F)$. Let $x_n \in S$ for finitely many n. Using the scalar operation defined on $\mathcal{F}(S, F)$, we can write
$$(cf)(x_n) = cf(x_n) = c \cdot 0 = 0.$$

Since all the properties of Theorem 3 have been satisfied, we conclude that $\mathcal{C}(S, F)$ is indeed a subspace of $\mathcal{F}(S, F)$. ■

Exercise 1.3.15

Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$?

Proof. We claim that the set of all differentiable real-valued functions defined on $\mathbb{R}$ is a subspace of $C(\mathbb{R})$. Denote this set as V.

(a) Note that the zero function f_0 is differentiable for all $x \in \mathbb{R}$ and continuous for all $x \in R$. Hence, $f_0 \in V$.

(b) Let $f, g \in V$. Using the addition operation defined on $C(\mathbb{R})$, we get that the sum $(f+g)(x) = f(x) + g(x)$ differentiable which implies that the sum of functions f, g is also continuous. Hence, $f + g \in V$.

(c) Let $f \in V$ and let $c \in \mathbb{R}$. Then $(cf)(x) = cf(x)$ is differentiable for all $x \in \mathbb{R}$ which means that cf is also continuous. Hence, $cf \in V$.

Hence, V is a subspace of $C(\mathbb{R})$. ■

Exercise 1.3.16

Let $C^n(\mathbb{R})$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $C^n(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof. ■

Exercise 1.3.17

Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. For the forwards direction, let $W \subseteq V$ where V is a vector space and W is a subspace of V. Let $a \in F$ and $x, y \in W$. Since W is a subspace, we know that $O_V \in W$. So, W is nonempty. Since W is closed under addition and multiplication, we get that $x + y \in W$ and $ax \in W$ and we are done.

For the backwards direction, let $W \neq \emptyset$ and $W \subseteq V$. Let $a \in F$ and $x, y \in W$ be arbitrary such that $ax \in W$ and $x + y \in W$. We need to show that W is a subspace of V. We need only show that $O_V \in W$ since W is closed under addition and scalar multiplication. Let $x \in W$. We can pick any $c \in F$ such that $c = 0$. So, we have $c \cdot x = 0 \cdot x = O_w$. Since the zero vector $O_W \in W$ is unique, we must have $O_V = O_W$. Hence, $O_V \in W$ and we conclude that W is a subspace of V. ■

Exercise 1.3.18

Prove that a subset W of a vector space V is a subspace of V if and only if $O \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Proof. ($\Rightarrow$) Let $W \subseteq V$ where W is a subspace of V. Since W is a subspace of V, we know that W is closed under addition and scalar multiplication. Let $a \in F$ and $x, y \in W$. Using the third property of Theorem 3, we can see that $ax \in W$. Since W is closed under addition, we can take $y \in W$ and $ax \in W$ such that $ax + y \in W$. Since W is also a vector space by definition, we know that $O_W \in W$. But $O_W = O_V$ so $O_V \in W$.

($\Leftarrow$) Let $a \in F$ and $x, y \in W$. We want to show that $W \subseteq V$ is a subspace of V. We can do this by using Theorem 3.

(a) By assumption, the zero vector $O_V \in W$.

(b) Let $x, y \in W$. Choose $a = 1$ such that $ax + y = x + y$. Since $ax + y \in W$ and $ax + y = x + y$, we also have $x + y \in W$. Hence, W is closed under addition.

(c) Let $x \in W$ and $O_V \in W$. Let $a \in F$. Then we have $ax + O_V = ax \in W$.

Hence, W is a subspace of V by Theorem 3. ■

Exercise 1.3.19

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. $(\Rightarrow)$ Let $W_1 \cup W_2$ is a subspace of V. We need to show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We proceed by showing the contrapositive. Assume $W_2 \not\subseteq W_1$ and $W_1 \not\subseteq W_2$. We need to show that $W_1 \cup W_2$ is **NOT** a subspace of V. By assumption, $x \in W_1$ is not contained in W_2 as well as $y \in W_2$ is not contained in W_1. This implies that $W_1 \cup W_2 = \emptyset$. Since $W_1 \cup W_2$ is empty where $W_1 \cup W_2$ does not contain O_V, it cannot possibly be a subspace of V.

$(\Leftarrow)$ Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We need to show that $W_1 \cup W_2$ is a subspace of V. We proceed by using Theorem 3 to do this. Without loss of generality, assume $W_1 \subseteq W_2$. The proof will be the same if we use $W_2 \subseteq W_1$.

(a) Since W_1 is a subspace of V, we get that $O_W \in W_1$. Furthermore, $W_1 \subseteq W_2$ implies that $O_V \in W$. Since $O_V \in W_1$ and $O_V \in W_2$, we get that $O_V \in W_1 \cup W_2$ by definition of union.

(b) Let $x, y \in W_1$. Since W_1 is a subspace, we get that $x + y \in W_1$. Since $W_1 \subseteq W_2$, we also get that $x + y \in W_2$. Since both $x + y \in W_1$ and $x + y \in W_2$, we know that $x + y \in W_1 \cup W_2$ by definition of the union.

(c) Let $x \in W_1$ and $c \in F$. Since W_1 is closed under scalar multiplication, we have that $cx \in W_1$. But $W_1 \subseteq W_2$ so W_2 also contains $cx \in W_1$. So we must have $cx \in W_1 \cup W_2$.

Hence, $W_1 \cup W_2$ is a subspace of a vector space V. ■

Exercise 1.3.20

Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$.

Proof. Let W be a subspace of a vector space V. Our goal is to show that the following statement: $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for all $1 \leq i \leq n$. We proceed by induction on $i \geq 1$. Let $i = 1$. Since W is closed under scalar multiplication, we know that $a_1 \in F$ and $w_1 \in W$ implies that $a_1w_1 \in W$. Now let $i = 2$, then $w_1, w_2 \in W$ and $a_1, a_2 \in F$ implies that $a_1w_1 + a_2w_2 \in W$ since W is closed under scalar multiplication and addition. Now, assume that our result holds for all $1 \leq i \leq n$. We want to show that it also holds for $i = n + 1$. By our inductive hypothesis, we know that $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$. Let $a_{n+1} \in F$ and $w_{n+1} \in W$. Using vector addition and scalar multiplication, we know that

$$\Big(a_1w_1 + a_2w_2 + \cdots + a_nw_n\Big) + a_{n+1}w_{n+1} \in W.$$

■

Exercise 1.3.21

Let V denote the vector space of sequences in $\mathbb{R}$, as defined in Example 5 of section 1.2. Show that the set of convergent sequences (a_n) (that is, those for which $\lim\limits_{n\to\infty} a_n$ exists) is a subspace of V.

Proof. We will show that W (the set of convergent sequences in $\mathbb{R}$) is a subspace of V by using Theorem 3.

(a) Note that the zero sequence O_n is zero for all $n = 1, 2, \ldots$. Hence, the limit of O_n converges to 0 and so we have $O_n \in W$.

(b) Let (a_n) and (b_n) be two convergent sequences in $\mathbb{R}$. Then we get that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists. Observe that

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Since the a_n and b_n are both convergent sequences, we also get that the sum $(a_n + b_n)$ also converges. Hence, $(a_n + b_n) \in W$.

(c) Let $t \in F$ and $a_n \in W$ as before. Then we have

$$\lim_{n\to\infty} ta_n = t \lim_{n\to\infty} a_n.$$

Since any constant $t \in F$ multiplied by a convergent sequence is convergent, we also get that the sequence (ta_n) is also convergent. Hence, $ta_n \in W$.

■

Exercise 1.3.22

Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

Proof. Let V be the set of all even functions and let W be the set of all odd functions. We will first prove that V is a subspace of $\mathcal{F}(F_1, F_2)$ and the same with W.

(a) Note that the zero function f_0 from $\mathcal{F}(F_1, F_2)$ is even since $f_0(-t) = 0 = f_0(t)$ for all $t \in F_1$. Hence, $f_0 \in V$.

(b) Let $f, g \in V$. We need to show that $f + g \in V$; that is, we need to show that it is even. Let $t \in F_1$. Then observe that

$$(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(t)$$

Hence, $(f + g)(-t) = (f + g)(t)$ for all $t \in F_1$.

(c) Let $f \in V$ and let $c \in F_2$. Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence, $cf \in V$.

Since all the properties of V are satisfied, we have that V is a subspace of $\mathcal{F}(F_1, F_2)$.

Now we will prove W is a subspace of $\mathcal{F}(F_1, F_2)$.

(a) Note that the zero function f_0 is in W because for any $t \in F_1$ we have $f_0(-t) = 0 = -1 \cdot 0 = -f(t)$. Hence, $f_0 \in W$.

(b) Let $f, g \in W$ and Let $t \in F_1$. Observe that

$$(f+g)(-t) = f(-t) + g(-t) = -f(t) - g(t) - (f+g)(t).$$

Hence, $f + g \in W$.

(c) Let $c \in F_2$ and $f \in W$. Let $t \in F_1$ such that $f(-t) = -f(t)$. Then observe that

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Hence, $cf \in W$.

Hence, W is a subspace of $\mathcal{F}(F_1, F_2)$ ■

Definition 1.2.1 (Sum of Two Sets). If S_1 and S_2 are nonempty subsets of a vector spaces V, then the **sum** S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.

Definition 1.2.2 (Direct Sum). A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Exercise 1.3.23

Let $W_1 + W_2$ is a subspace of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Proof. Let W_1 and W_2 be subspaces of a vector space V. Define $W_1 + W_2$ as the set

$$\{x_1 + y_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}.$$

(a) Since W_1 and W_2 are subspaces of V, we know that the zero vector O_V is contained in both W_1 and W_2. Hence, the sum $O_V = O_V + O_V \in W_1 + W_2$.

(b) Let $u, v \in W_1 + W_2$ with $u = x_1 + y_1$ and $v = x_2 + y_2$. Since W_1 and W_2 are subspaces of V, we know that addition is closed in both subsets W_1 and W_2. Hence, $x_1 + x_2 \in W_1$ and $y_1 + y_2 \in W_2$. Observe that

$$\begin{aligned} u + v &= (x_1 + y_1) + (x_2 + y_2) \\ &= (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2. \end{aligned}$$

Hence, addition is closed in $W_1 + W_2$.

(c) Let $c \in F$ where F is a field and let $u \in W_1 + W_2$ with $u = x_1 + y_1$. Since W_1 and W_2 are subspaces of V, we know that $cx_1 \in W_1$ and $cx_2 \in W_2$. Observe that

$$\begin{aligned} cu &= c(x_1 + y_1) \\ &= cx_1 + cy_1 \in W_1 + W_2. \end{aligned}$$

Hence, $W_1 + W_2$ is a subspace of V. ■

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof. Let X be a subspace of V. Suppose X contains both W_1 and W_2; that is, $W_1 \subseteq X$ and $W_2 \subseteq X$. We must show that $W_1 + W_2 \subseteq X$. Let $u \in W_1 + W_2$ with $u = x_1 + y_1$. By definition, we have $x_1 \in W_1$ and $y_1 \in W_2$. Since $W_1 \subseteq X$ and $W_2 \subseteq X$, we have $x_1 \in X$ and $y_1 \in X$. Since X is a subspace of V and X is closed under addition, we have $x_1 + y_1 \in X$. Hence, $u \in X$ and we conclude $W_1 + W_2 \subseteq X$. ■

Exercise 1.3.24

Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \cdots a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Proof. Let W_1 and W_2 be subspaces of F^n. Since W_1 and W_2 are subspaces, they both contain the zero vector O_{F^n}. Observe that for any element $x \in W_1$, we have the nth element $a_n = 0$. On the other hand, $y \in W_2$ implies that $a_1 = a_2 = \cdots = a_{n-1} = 0$ except for the nth element. Thus, the only element that W_1 and W_2 have in common is the zero vector O_{F^n}. Hence, we have $W_1 \cap W_2 = \{O_{F^n}\}$.

Now, we show $W_1 + W_2 = F^n$. To do this, we need to show the following containments:

(i) $W_1 + W_2 \subseteq F^n$ and

(ii) $F^n \subseteq W_1 + W_2$.

Let $u \in W_1 + W_2$ with $u = x_1 + x_2$ where $x_1 \in W_1$ and $y_1 \in W_2$. Since $W_1 \subseteq F^n$, we know that $x_1 \in F^n$. Likewise, $W_2 \subseteq F^n$ implies that $y_1 \in F^n$. Since F^n is a vector space where addition is closed, we have that $u = x_1 + y_1 \in F^n$. Hence, $W_1 + W_2 \subseteq F^n$.

Now, let $u \in F^n$. Since F^n is a vector space over F, each entry in U (where each entry is an element of a field F) can written and separated using the addition defined in F^n in the following way:

$$\begin{aligned} u &= (a_1, a_2, \ldots, a_n) \\ &= (a_1 + 0, a_2 + 0, \ldots, 0 + a_n) \\ &= (a_1, a_2, \ldots, 0) + (0, 0, \ldots, a_n) \end{aligned}$$

where the first term is an element of W_1 and the second term is an element of W_2. Hence, $u \in W_1 + W_2$.

Since both containments are satisfied, we that $W_1 + W_2 = F^n$. Thus, $W_1 \oplus W_2 = F^n$. ■

Exercise 1.3.25

Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

we have $a_i = 0$ whenever i is even. Likewise, let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

we have $b_i = 0$ whenever i is odd. Prove that $P(F) = W_1 \oplus W_2$.

Proof. Let $f(x) \in W_1$. Then we have

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}a_nx^n$$

where $a_i = 0$ where i is even. Likewise, let $g(x) \in W_2$. Hence,

$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} + b_nx^n$$

where $b_i = 0$ for i odd. This tells us that the only representation that W_1 and W_2 have in common is the zero polynomial $f(x) = 0$ where $a_i = 0$ for all $0 \leq i \leq n$. Hence, $W_1 \cap W_2 = \{O\}$.

Now, we want to show that $W_1 + W_2 = P(F)$; that is, we need to show $P(F) \subseteq W_1 + W_2$ and $W_1 + W_2 \subseteq P(F)$. Starting with the former, let $f \in P(F)$. Observe that for a_i where $1 \leq i \leq n$

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n \\ &= \underbrace{(a_1x + a_3x^3 + \cdots + a_nx^n)}_{\in W_1} + \underbrace{(a_0 + a_2x^2 + a_4x^4 + \cdots + a_nx^n)}_{\in W_2} \end{aligned}$$

The first term of the last equality contains coefficients $a_i = 0$ for even i and the second term contains coefficients $a_i = 0$ for odd i. This implies that $f(x) \in W_1 + W_2$. Hence, $P(F) \subseteq W_1 + W_2$.

Now, let $u(x) \in W_1 + W_2$ with $u(x) = f(x) + g(x)$ with $f(x) \in W_1$ and $g(x) \in W_2$. Since W_1 and W_2 are subsets of $P(F)$, we have that $f(x), g(x) \in P(F)$. Since addition is closed in $P(F)$, we have that $u(x) = f(x) + g(x) \in P(F)$. Hence, $W_1 + W_2 \subseteq P(F)$. Since $W_1 + W_2 = P(F)$ and $W_1 \cap W_2 = \{0\}$, we have $W_1 \oplus W_2 = P(F)$. ■

Exercise 1.3.26

In $M_{m\times n}(F)$ define $W_1 = \{A \in M_{m\times n}(F) : A_{ij} = 0 \text{ whenever } i > j\}$ and $W_2 = \{A \in M_{m\times n}(F) : A_{ij} \text{ whenever } i \leq j\}$. Show that $M_{m\times n}(F) = W_1 \oplus W_2$. ($W_1$ is the set of all triangular matrices as defined in the previous section.)

Proof. **TO DO.** ■

Exercise 1.3.30

Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be *uniquely* written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Proof. Let W_1 and W_2 be subspaces of V. Let $v \in V$. Let $x_1, x_1' \in W_1$ and $x_2, x_2' \in W_2$ such that $v = x_1 + x_2 = x_1' + x_2'$. Since W_1 and W_2 is closed under addition, we know that $x_1 - x_2' \in W_1$ and $x_2 - x_2' \in W_2$. But observe that $x_1 - x_1' = x_2 - x_2' \in W_1 \cap W_2$. Since V is a direct sum of the two subspaces W_1 and W_2, we know that $W_1 \cap W_2 = \{O_V\}$ which implies that $x_1 = x_1'$ and $x_2 = x_2'$. This tells us that every v can be expressed uniquely as $x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$.

Conversely, suppose every vector $v \in V$ can be *uniquely* written as $x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$. We need to show that $W_1 \oplus W_2 = V$. Since $v \in V$ is uniquely expressed in terms of $x_1 + x_2$, the only vector that the two subspaces W_1 and W_2 share is the zero vector.

Hence, $W_1 \cap W_2 = \{O_V\}$. Now, we need to show that $V = W_1 + W_2$; that is, we need to show $V \subseteq W_1 + W_2$ and $W_1 + W_2 \subseteq V$. Suppose $v \in V$. Since $x_1 \in W_1$ and $x_2 \in W_2$, we have that $v \in W_1 + W_2$. Hence, $V \subseteq W_1 + W_2$. Now, let $v \in W_1 + W_2$. Since $v = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$ and $W_1 \subseteq V$ and $W_2 \subseteq V$, we know that x_1 and x_2 are contained within V. Since V is vector space, we know that sum of x_1 and x_2 are contained in V. Hence, $v \in V$ and so $V \subseteq W_1 + W_2$. Thus, $V = W_1 \oplus W_2$. ■

Definition 1.2.3 (Cosets). Let W be a subspace of a vector space V over a field F. For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is called the **coset** of W **containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

Definition 1.2.4 (Addition of Cosets). The **addition** of two cosets $v_1 + W$ and $v_2 + W$ is defined as

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$.

Definition 1.2.5 (Scalar Multiplication of Cosets). The **scalar multiplication** of cosets by scalars of F can be defined as

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$.

Exercise 1.3.31

The following exercises require the application of the definitions above.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

Proof. Suppose $v + W$ is a subspace of V. Then $0_V \in v + W$ which implies that $O_V = v + w$. Hence, $v = -w$ where $-w \in W$ and thus $v \in W$. Conversely, suppose $v \in W$.

(a) Since W is a subspace of V, we know that $0_V \in W$. Since $v \in W$, there exists an element $v' \in W$ such that $v + v' = 0_V$. But this tell us that $O_V \in v + W$.

(b) Let $x, y \in v + W$. By definition, $x = v + w$ and $y = v + w'$ for $w, w' \in W$. Then observe that

$$x + y = (v + w) + (v + w') = v + (v + w + w')$$

where $v + w + w' \in W$ since W is a subspace of V. Hence, $v + W$ is closed under addition.

(c) Let $x \in v + W$. Choose $c \in F$ for which we will show that $cx \in v + W$. Then

$$cx = c(v + w) = cv + cw = v(cv + cw - v)$$

where $cv + cw - v$ since W is a subspace of V.

Hence, $v + W$ is a subspace of V. ■

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Proof. Let $x \in v_1 + W$. Then for $\alpha \in W$, we have $x = v_1 + \alpha$. Since $v_1 + W = v_2 + W$, then $x \in v_2 + W$ implies that for $\beta \in W$, we have $x = v_2 + \beta$. Then observe that

$$v_1 + \alpha = v_2 + \beta \Rightarrow v_1 - v_2 = \beta - \alpha.$$

Since W is a subspace, we have that $\beta - \alpha = v_1 - v_2 \in W$.

Conversely, suppose $v_1 - v_2 \in W$. By part (a), we know that $(v_1 - v_2) + W$ is a subspace of V. Hence, $0_V \in (v_1 - v_2) + W$. This means that there exists $w' \in W$ such that

$$\begin{aligned}
&(v_1 - v_2) + w' = 0_V \\
&\Rightarrow v_1 + w' = v_2 + (w - w) \\
&\Rightarrow v_1 + \underbrace{(w' + w)}_{\in W} = v_2 + \underbrace{w}_{\in W}.
\end{aligned}$$

This tells us that $v_1 + W = v_2 + W$ and we are done. ∎

(c) Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

Proof. Suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$ by part (b). Consequently, we have $(v_1 - v_1') + (v_2 - v_2') \in W$ if and only if $(v_1 + v_2) - (v_1' + v_2') \in W$ since W is subspace. Using part (b) again, we have that

$$(v_1 + v_2) + W = (v_1' + v_2') + W.$$

Using the addition defined on cosets, we get

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W).$$

Since W is a subspace, we have $c(v_1 - v_1') \in W$ if and only if $cv_1 - cv_1' \in W$ for some $a \in F$. By part (b), we get that

$$av_1 + W = av_1' + W$$

which can be re-written to

$$a(v_1 + W) = a(v_1' + W)$$

as our desired result.

∎

(d) Prove that the set $S = \{v + W : v \in V\}$ is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** by $V \setminus W$.

Proof. It suffices to show that S is a subspace of V.

(a) Since W is a subspace of V, we know that $0_V \in W$. Then $0_V + 0_V = 0_V \in S$.

(b) Let $x, y \in S$. Then using the operations defined in part (c), we write

$$x + y = (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W.$$

This tells us that $x + y \in S$.

(c) Let $x \in S$. Our goal is to show that $c \in F$ implies $cx \in S$. Using the scalar multiplication in part (c), we get that

$$cx = c(v_1 + W) = cv_1 + W.$$

Thus, S is a vector space. ■

1.3 Linear Combinations

Exercise 1.4.7

In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

Proof. Denote the set $V = \{e_1, e_2, \ldots, e_n\}$. Our goal is to find scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that for all $e_j \in V$ for $1 \leq j \leq n$,

$$\delta_1 e_1 + \delta_2 e_2 + \cdots + \delta_n e_n = (a_1, a_2, \ldots, a_n). \tag{1}$$

Since $e_j = 1$ for the jth coordinate and the rest of the entries are zeros, we have that

$$\delta_j e_j = \delta_j(0, 0, \underbrace{1}_{j\text{th entry}}, \ldots, 0) = (0, 0, \delta_j, \ldots, 0)$$

for all $1 \leq j \leq n$. Summing up each term on the left side of (1) and equating each coordinate to the right side of (1), we get that $a_j = \delta_j$ for each $1 \leq j \leq n$. Hence, the set V can span the vector space F^n.

■

Exercise 1.4.8

Show that $P_n(F)$ is generated by $\{1, x, \ldots, x^n\}$.

Proof. Let $V = \{1, x, \ldots, x^n\}$. We need to find scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that

$$\delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{1}$$

Immediately, we have that equating coefficients in (1) gives us $a_i = \delta_i$ for all $1 \leq i \leq n$. Hence, V generates $P_n(F)$. ■

Exercise 1.4.9

Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate $M_{2\times 2}(F)$.

Proof. Define V with the given 2×2 matrices above. We need to find scalars $\delta_i \in F$ for all $1 \leq i \leq 4$ such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \delta_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \delta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1}$$

Distributing each scalar δ_i for all $1 \leq i \leq 4$ for each term in (1), summing up each matrix in (1), and equating each term entry-wise, we have that $a_{11} = \delta_1, a_{12} = \delta_2, a_{21} = \delta_3$, and $a_{22} = \delta_4$. Hence, V spans $M_{2\times 2}(F)$. ■

Exercise 1.4.10

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Proof. We need to show that the set $\{M_1, M_2, M_3\}$ spans the set of all symmetric 2×2 matrices. We need to find scalars a_1, a_2, a_3 such that

$$a_1 M_1 + a_2 M_2 + a_3 M_3 = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = A \tag{1}$$

where A is any 2×2 symmetric matrix. Observe that for $i = j$, we have $\delta_{12} = \delta_{21}$. Performing scalar multiplication on matrices M_1, M_2, and M_3, addition of all three terms on (1), and equating entry-wise, we get that $\delta_{12} = \delta_{21} = \delta_3$ and $\delta_{11} = \delta_1$ and $\delta_2 = a_{22}$. Hence, we have

$$\begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^t = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}.$$

Thus, the set $\{M_1, M_2, M_3\}$ spans the set of all symmetric 2×2 matrices. ■

Exercise 1.4.11

Prove that $\text{span}(\{x\}) = \{ax : a \in F\}$ for any vector x in a vector space V. Interpret this result geometrically in $\mathbb{R}^3$.

Proof. We need to show that $\text{span}(\{x\}) = \{ax : a \in F\}$, we need to show two containments; that is, $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$ and $\{ax : a \in F\} \subseteq \text{span}(\{x\})$. Let $v \in \text{span}(\{x\})$. Then observe that we can find $\delta \in F$ such that multiplying by $x \in V$ leads $v = \delta x$. But this means that $v \in \{ax : a \in F\}$ by definition. Hence, $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$. Let $v \in \{ax : a \in F\}$. Then $v = ax$ for some $a \in F$. But this is a linear combination of x that makes v. So $v \in \text{span}(\{x\})$ and hence, $\{ax : a \in F\} \subseteq \text{span}(\{x\})$. This result can be viewed as the scaling of vectors in $\mathbb{R}^3$. ■

Exercise 1.4.12

Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$.

Proof. ($\Rightarrow$) Let W be a subspace of V. To show that $\text{span}(W) = W$, we need to show two containments; that is, $\text{span}(W) \subseteq W$ and $W \subseteq \text{span}(W)$. Clearly, W contains itself. Hence, $\text{span}(W) \subseteq W$ Theorem 5. Let $v \in W$. Since W is a subspace, we can find scalars $a_1, a_2, \ldots . a_n \in F$ and vectors $w_1, w_2, \ldots, w_n \in W$ such that

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$$

using the result in Exercise 1.3.20. But this tells us that $v \in \text{span}(W)$. Hence, $W \subseteq \text{span}(W)$.

($\Leftarrow$) Since the span of any subset of W is a subspace and $W = \text{span}(W)$, we have that W is a subspace as well by Theorem 5. ■

Exercise 1.4.13

Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

Proof. Let S_1 and S_2 be subsets of a vector space V. Let $v \in \text{span}(S_1)$. We can find scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ and $x_1, x_2, \ldots, x_n \in S_1$ such that

$$v = \delta_1x_1 + \delta_2x_2 + \cdots + \delta_nx_n.$$

Since $S_1 \subseteq S_2$, we know that $x_1, x_2, \ldots, x_n \in S_2$ so we must have $v \in \text{span}(S_2)$. Hence, $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Now, let $\text{span}(S_1) = V$. We need to show that $\text{span}(S_2) = V$; that is, we need to show $\text{span}(S_2) \subseteq V$ and $V \subseteq \text{span}(S_2)$. By assumption, $S_2 \subseteq V$ and $\text{span}(S_2)$ is a subspace. Clearly, $\text{span}(S_2) \subseteq V$. Since $\text{span}(S_1) = V$ and $\text{span}(S_1) \subseteq \text{span}(S_2)$, we have $V \subseteq \text{span}(S_2)$. Hence, $\text{span}(S_2) = V$. ■

Exercise 1.4.14

Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sum of two subsets is defined in the exercises of Section 1.3.)

Proof. Let S_1 and S_2 be subsets of a vector space V. We need to show $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$; that is, $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$ and $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. Let $v \in \text{span}(S_1 \cup S_2)$. We can find $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that $x_1, x_2, \ldots, x_n \in S_1 \cup S_2$ implies that

$$v = \delta_1x_1 + \delta_2x_2 + \cdots + \delta_nx_n.$$

Hence, either $x_1, x_2, \ldots, x_n \in S_1$ or $x_1, x_2, \ldots, x_n \in S_2$. If $x_1, x_2, \ldots, x_n \in S_1$, then $v \in \text{span}(S_1)$. Since $\text{span}(S_2)$ is a subspace, we know that $O_V \in \text{span}(S_2)$. Hence, $O_V \in \text{span}(S_1)$ and $v \in \text{span}(S_1)$ imply that $v + O_V = v \in \text{span}(S_1) + \text{span}(S_2)$. The other case follows a similar process. Hence, $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$.

Let $s \in \text{span}(S_1) + \text{span}(S_2)$. Hence, $s = u + v$ where $u \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$. The former implies that we can find scalars $a_1, a_2, \ldots, a_n \in F$ such that $x_1, x_2, \ldots, x_n \in S_1$ where

$$u = \sum_{i=1}^{n} a_ix_i$$

and the latter implies that there exists scalars $b_1, b_2, \ldots, b_n \in F$ such that $y_1, y_2, \ldots, y_n \in S_2$

where

$$v = \sum_{i=1}^{n} b_i y_i.$$

Since both $x_i \in S_1$ and $y_i \in S_2$ for all $1 \le i \le n$, we have $x_i, y_i \in S_1 \cup S_2$ for all $1 \le i \le n$. So we must have $s \in \text{span}(S_1 \cup S_2)$. Hence, $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. ■

Exercise 1.4.15

Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_2 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are not unequal.

Proof. Let S_1 and S_2 be subsets of a vector space V. Let $v \in \text{span}(S_1 \cap S_2)$. Then we can find scalars $a_i \in F$ and vectors $x_i \in S_i \cap S_2$ for all $1 \le i \le n$ such that

$$v = \sum_{i=1}^{n} a_i x_i.$$

If $x_i \in S_1 \cap S_2$ for all $1 \le i \le n$, then $x_i \in S_1$ and $x_i \in S_2$ for all $1 \le i \le n$. This implies that $v \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$. Hence, $v \in \text{span}(S_1) \cap \text{span}(S_2)$. Thus, we conclude that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. ■

Example 1.3.1. Define S_1 as the set

$$\{(1,1,0) \in \mathbb{R}^3\}$$

and S_2 as the set

$$\{(1,1,0),(1,0,1),(0,1,1) \in \mathbb{R}^3\}.$$

Observe that $S_1 \cap S_2 = \{(1,1,0)\}$ and thus the of this set yields $\{a(1,1,0) : a \in F \text{ and } (1,1,0) \in \mathbb{R}^3\}$. The span of S_1 yields the following set $\{a(1,1,0) : a \in F$ and $(1,1,0) \in \mathbb{R}^3\}$ and the span of S_2 yields the following set

$$\{a(1,10) + b(1,0,1) + c(0,1,1) : a,b,c \in F \text{ and } (1,1,0),(1,0,1),(0,1,1) \in \mathbb{R}^3\}.$$

Note that $\text{span}(S_1 \cap S_2) \ne \text{span}(S_1) \cap \text{span}(S_2)$ since $\text{span}(S_1) \cap \text{span}(S_2) \ne \emptyset$.

Now define $S_1 = \{O_V\} = S_2$. Clearly, $S_1 \cap S_2 = \{O_V\}$, $\text{span}(S_1) \cap \text{span}(S_2) = \{O_V\}$, and that $\text{span}(S_1 \cap S_2)$ is also equal to this set. $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

Exercise 1.4.16

Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be *uniquely* written as a linear combination of vectors of S.

Proof. Let $x \in \text{span}(S)$. Suppose there exists two sets of scalars $a_1, a_2, \ldots, a_n \in F$ and $b_1, b_2, \ldots, b_n \in F$ such that whenever $v_1, v_2, \ldots, v_n \in S$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = x \tag{1}$$

and

$$b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = x \tag{2}$$

implies that a_i Setting (1) and (2) together, we have

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n \tag{3}$$

which implies that

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0.$$

Since $v_1, v_2, \ldots, v_n \in S$, we have that $a_i - b_i = 0$ and thus $a_i = b_i$ for all $1 \leq i \leq n$. Hence, every vector in the span of S can be *uniquely* written as a linear combination of vectors of S. ■

1.4 Linear Dependence and Linear Independence

Exercise 1.5.1

Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vector in S.

Proof. **True** ■

(b) Any set containing the zero vector is linearly dependent.

Proof. **True** ■

(c) The empty set is linearly dependent.

Proof. **False**. It is linearly independent. ■

(d) Subsets of linearly dependent sets are linearly dependent.

Proof. This is **False**. We can have a linearly independent subset of a set that is linearly dependent. ■

(e) Subsets of linearly independent sets are linearly independent.

Proof. **True** by corollary to Theorem 6. ■

(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

Proof. **True** this is by definition. ■

Exercise 1.5.4

In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Proof. Choose a finite amount of scalars $a_1, a_2, \ldots, a_n \in F$ to create the following linear combination:

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = (0, 0, \ldots, 0). \tag{1}$$

To show that the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, we need to show that the scalars $a_1, a_2, \ldots, a_n \in F$ have the trivial representation; that is, $a_1 = a_2 = \cdots = a_n = 0$.

Since the jth coordinate of e_j is 1 but 0 in all the other entries, we have that

$$\begin{aligned} &a_1(1,0,\dots,0) + a_2(0,1,\dots,0) + \cdots + a_n(0,0,\dots,1) \\ &= (a_1,0,\dots,0) + (0,a_2,\dots,0) + \cdots + (0,0,\dots,a_n) \\ &= (a_1,a_2,\dots,a_n). \end{aligned}$$

Hence, we have

$$(a_1,a_2,\dots,a_n) = (0,0,\dots,0).$$

Equating each entry of the left side of the equation above to 0, we find that $a_i = 0$ for all $1 \le j \le n$. Hence, the set $\{e_1, e_2, \dots, e_n\}$ is linearly independent. ■

Exercise 1.5.4

Show that the set $\{1, x, x^2, \dots, x^n\}$ is linearly independent in $P_n(F)$.

Proof. Just like the prior exercise, we need to show that we can find scalars $a_0, a_1, \dots, a_n \in F$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where $a_i = 0$ for all $0 \le i \le n$. Note that the 0 polynomial is just

$$0 + 0x + 0x^2 + \cdots + 0x^n = 0.$$

Hence, equating coefficients we immediately get that $a_i = 0$ for all $0 \le i \le n$. Thus, the set $\{1, x, x^2, \dots, x^n\}$ is linearly independent. ■

Exercise 1.5.6

In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is linearly independent.

Proof. First, we create a linear combination of a finite amount vectors in $E = \{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ with scalars δ_k for $1 \le k \le N$ with $N = mn$ as the number of total entries in each matrix in $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$. Note that after doing our scalar multiplication and summing up each term, we find that each $\delta_k E_{ij} = \delta_k$ in our linear combination can equated with a corresponding i and j entry in the zero matrix such that $\delta_k = 0$ for all $1 \le k \le N$. Hence, E is a linearly independent set. ■

Exercise 1.5.7

Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

Proof. Define W as the linearly independent spanning set of the set of diagonal matrices in $M_{2\times 2}$ where

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

To see why W is a linearly independent set, choose scalars $\delta_1, \delta_2 \in F$ such that

$$\delta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing scalar multiplication and vector addition gives us the following equation

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the zero matrix is a diagonal matrix, we know that equation entries where $i = j$yields $\delta_1 = \delta_2 = 0$. Hence, W is a linearly independent set that generates the set of diagonal matrices of $M_{2\times 2}(F)$. ■

Exercise 1.5.8

Let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = \mathbb{R}$, then S is linearly independent.

Proof. ■

(b) Prove that if F has characteristic two, then S is linearly dependent.

Proof. ■

Exercise 1.5.9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other. I have written two proofs for this:

Proof. Let u and v be distinct vectors in a vector space V.

($\Rightarrow$) Since $\{u, v\}$ is a linearly dependent set, we can find scalars $a_1, a_2 \in F$ such that

$$a_1 u + a_2 v = 0 \tag{1}$$

Suppose v is not a multiple of u and choose $a_1 \neq 0$ since $\{u, v\}$ is linearly dependent. We need to show that u is a multiple of v. Solving for u, we get that

$$u = -\frac{a_2}{a_1} v.$$

Hence, u is a multiple of v.

($\Leftarrow$) Suppose u or v is a scalar multiple of the other. Assume u is the scalar multiple of v. Then for some $c \neq 0 \in F$, we have $u = cv$. Hence, we have $u - cv = 1u - cv = 0$. This tells us that $\{u, v\}$ is linearly dependent. ■

Exercise 1.5.12

Prove Theorem 1.6 and its corollary.

Proof. See proof in notes. ■

Exercise 1.5.13

Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Proof. Let u and v be distinct vectors in V.

For the forwards direction, assume $\{u, v\}$ is a linearly independent set. We need to show that $\{u+v, u-v\}$ is linearly independent. Hence, we need to find $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0. \tag{1}$$

Note that (1) leads to

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu. \end{aligned}$$

Since $\{u, v\}$ is a linearly independent set, we have that

$$au - bv = 0$$

and

$$av + bu = 0$$

for $a = b = 0$. Hence,

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ and so $\{u - v, u + v\}$ is a linearly independent set.

For the backwards direction, suppose $\{u + v, u - v\}$ is linearly independent. We need to show that $\{u, v\}$ is linearly independent. Note that $a, b \in F$ such that

$$a(u + v) + b(u - v) = 0$$

for $a = b = 0$ since $\{u - v, u + v\}$ is linearly independent. Note that

$$\begin{aligned} a(u + v) + b(u - v) &= au + av + bu - bv \\ &= au - bv + av + bu \\ &= 0 + av + bu \\ &= 0. \end{aligned}$$

Thus, $av + bu = 0$ where a, b both zero. Thus, the set $\{u, v\}$ is linearly independent. ■

(b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

Proof. For the forwards direction, suppose $\{u, v, w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that

$$a_1 u + a_2 v + a_3 w = 0$$

with $a_1 = a_2 = a_3 = 0$. We need to show that $\{u + v, u + w, v + w\}$ is linearly independent; that is, we need to show that we can find scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0 \tag{1}$$

for $a_1 = a_2 = a_3 = 0$. Observe that (1) can be written in the following way

$$(a_1 u + a_3 v + a_2 w) + (a_1 v + a_2 u + a_3 w) = 0 \tag{2}$$

Since $\{u, v, w\}$ is linearly independent, we know that $a_1 = a_2 = a_3 = 0$. But this also has to mean that $\{u + v, u + w, v + w\}$ is a linearly independent set.

For the backwards direction, suppose $\{u + v, u + w, v + w\}$ is linearly independent. Then choose scalars $a_1, a_2, a_3 \in F$ such that

$$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0. \tag{1}$$

We need to show that $\{u, v, w\}$ is linearly independent. Observe that (1) can be re-written as

$$\begin{aligned}
&(a_1 u + a_3 v + a_2 w) + (a_1 v + a_2 u + a_3 w) = 0 \\
&\Rightarrow 0 + (a_1 v + a_2 u + a_3 w) = 0 \\
&\Rightarrow a_1 v + a_2 u + a_3 w = 0
\end{aligned}$$

where $a_1 = a_2 = a_3 = 0$. Hence, $\{u, v, w\}$ is linearly independent. ■

Exercise 1.5.14

Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

Proof. For the forwards direction, Let S be a linearly dependent. Then we need to show that either $S = \{0\}$ or S contains distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \operatorname{span}(S)$. Suppose there does not exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that $v \in \operatorname{span}(S)$. This tells us that S only contains the singleton $S = \{v\}$. Furthermore, we must require $v = 0$ since S is linearly dependent. Otherwise, $v \neq 0$ would imply that S is linearly independent. Hence, we have $S = \{0\}$. Now suppose $S \neq \{0\}$. Since S is linearly dependent, there exists scalars $a_1, a_2, \ldots, a_{n+1}$ and vectors $v, u_1, u_2, \ldots, u_n$ such that

$$a_1 v + a_2 u_1 + a_3 u_2 + \cdots + a_{n+1} u_n = 0$$

with all $a_1, a_2, \ldots, a_n, a_{n+1}$ not all equal to zero. Solving for v, we get that

$$v = -(a_1^{-1} a_2) u_1 - (a_1^{-1} a_3) u_2 - \cdots - (a_1^{-1} a_{n+1}) u_n.$$

Since v is a linear combination of $u_1, u_2, \ldots, u_n$, we have that $v \in \operatorname{span}(S)$.

Conversely, suppose that either $S = \{0\}$ or there exists distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$. Assume $S = \{0\}$. Then S is linearly dependent because the singleton is the zero vector. Now suppose v is a linear combination of vectors $u_1, u_2, \ldots, u_n$. Then there exists scalars $a_1, a_2, \ldots, a_n \in F$ and distinct vectors $u_1, u_2, \ldots, u_n \in S$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Subtracting v from both sides of this equation yields the following equation

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n - 1v = 0.$$

Since not all scalars in the equation above are zero and $v, u_1, u_2, \ldots, u_n \in S$, we must have that S is a linearly dependent set. ■

Exercise 1.5.15

Prove that a set $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k where $1 \leq k < n$.

Proof. Suppose $u_{k+1} \notin \text{span}(\{u_1, u_2, \ldots, u_k\})$ for all $1 \leq k < n$. Since S is linearly dependent, we know that the zero vector is contained in S. Choose $k = 1$ such that $u_1 = 0$ and we are done. On the other hand, suppose $u_1 \neq 0$. We need to show that $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$. Choose $k = n-1$. Then clearly $n = k+1$. Since S is linear independent, choose scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0. \tag{1}$$

where $a_1, a_2, \ldots, a_k, a_{k+1}$ not all zero. Solving for u_{k+1} by subtracting $a_{k+1}u_{k+1}$ on both sides of (1) and multiplying $-a_{k+1}^{-1}$ on both sides of (1), we end up with the following equation:

$$u_{k+1} = -(a_{k+1}^{-1}a_1)u_1 - (a_{k+1}^{-1}a_2)u_2 - \cdots - (a_{k+1}^{-1}a_k)u_k.$$

This tells us that u_{k+1} can be written as a linear combination of vectors $u_1, u_2, \ldots, u_k$. Hence, $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots u_k\})$.

Conversely, either $u_1 = 0$ or $u_{k+1} \in \text{span}(u_1, u_2, \ldots, u_k)$. Suppose $u_1 = 0$. Then S contains the zero vector so S must be linearly dependent. On the other hand, choose $k = n-1$ where $1 \leq k < n$ such that $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ implies that there exists scalars $a_1, a_2, \ldots, a_k$ such that

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k \\ \Rightarrow u_n &= a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}. \end{aligned} \tag{1}$$

Subtracting u_n on both sides of (1) implies that

$$(a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}) - 1u_n = 0.$$

Since not all scalars in the linear combination above are zero, we know that S must be linearly dependent.

■

Exercise 1.5.16

Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Proof. ($\Rightarrow$) Suppose S is a linearly independent set. Let S' be any finite subset of S. By corollary to Theorem 6, we can see that $S' \subseteq S$ implies that S' is also linearly independent. ($\Leftarrow$) We will proceed by proving the contrapositive. Let $S' \subseteq S$ be a finite subset that is linearly dependent set. We will prove that S is a linearly dependent set. Since $S' \subseteq S$, we have that S must be a linearly dependent set by Theorem 6. ■

Exercise 1.5.17

Let M be a square upper triangular matrix (as defined in Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

Proof. Let M be a square upper triangular matrix. Note that M have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & a_{n-1n} \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

We can define the following columns as vectors $v_1, v_2, \ldots v_n$ where

$$v_1 = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ 0 \end{pmatrix}, \ldots, v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Choose scalars $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that

$$\delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_n v_n = 0$$

where the zero vector 0 is denoted as

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using scalar multiplication and addition, we arrive at the following system of equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} + \cdots + \delta_{n-1} a_{1n-1} + \delta_n a_{1n} &= 0 \\ \delta_2 a_{22} + \delta_3 a_{23} + \cdots + \delta_n a_{2n} &= 0 \\ &\vdots \\ \delta_{n-1} a_{n-1n-1} + \delta_n a_{n-1n} &= 0 \\ \delta_n a_{nn} &= 0. \end{aligned}$$

Since all the diagonal entries of M are non-zero, we can see from the equation above that $\delta_n = 0$ which subsequently tells us that $\delta_{n-1} = 0$. We claim that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ for all $n \geq 1$. We can prove this via induction. Let our base case be $n = 1$. Then we have $\delta_1 a_{11} = 0$ with $a_{11} \neq 0$ implies $\delta_1 = 0$. Now let $n = 2$. Then observe that we have an upper triangular 2×2 matrix such that the linear combination of the columns lead to the following system of linear equations:

$$\begin{aligned} \delta_1 a_{11} + \delta_2 a_{12} &= 0 \\ 0 + \delta_2 a_{22} &= 0. \end{aligned}$$

Observe that $\delta_2 = 0$ which also implies that $\delta_1 = 0$. Now suppose our claim that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ holds for all $n \geq 1$. We want to show that our claim still holds for the $n+1$ case. Observe that $\delta_{n+1} = 0$ derived from an $n+1 \times n+1$ matrix. Using the same process that proved the base case, we find that $\delta_n = \delta_{n+1} = 0$. By our inductive hypothesis, we know that $\delta_1 = \cdots = \delta_{n-1} = \delta_n = 0$ for all $n \geq 1$. This tells us that $\delta_1 = \cdots = \delta_{n+1}$ for all $n \geq 1$. Hence, the columns of M are linearly independent. ■

Exercise 1.5.18

Let S be a set of nonzero polynomials in $P(F)$ such that no two have the same degree. Prove that S is linearly independent.

Proof. Let $0 \leq n \leq k$ such that $S = \{p_0(x), p_1(x), p_2(x), \ldots, p_k(x)\}$ where every $p_n(x)$ non-zero such that no two polynomials in this set have the same degree. Define $p_n(x) = x^n + x^{n+1} + \cdots + x^k$. Choose scalars $a_1, a_2, \ldots, a_k \in F$ such that

$$a_1 p_1(x) + a_2 p_2(x) + \cdots + a_k p_k(x) = 0 \tag{1}$$

We need to show that $\delta_1 = \delta_2 = \cdots = \delta_k = 0$. Observe that (1) implies that

$$a_0 + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_k)x^k = 0. \tag{2}$$

Setting each x^n to both sides of the equation above leads to the following system of linear equations:

$$\begin{aligned} a_0 &= 0 \\ a_0 + a_1 &= 0 \\ a_0 + a_1 + a_2 &= 0 \\ &\vdots \\ a_0 + a_1 + a_2 + \cdots + a_k &= 0. \end{aligned}$$

It can be proved via induction that $a_1 = a_2 = \cdots = a_k = 0$ for all $1 \leq n \leq k$. Hence, S is a linearly independent set. ■

Exercise 1.5.19

Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, A_3^t, \ldots, A_k^t\}$ is also linearly independent.

Proof. Suppose $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent set. Choose a finite set of scalars $\delta_1, \delta_2, \ldots, \delta_k \in F$ such that

$$\delta_1 A_1 + \delta_2 A_2 + \cdots + \delta_k A_k = 0$$

with $\delta_1 = \delta_2 = \cdots = \delta_k = 0$. Apply the transpose to both sides, we know that $0^t = 0$ and $(\delta_i A_i)^t = \delta_i (A_i)^t$ for all $1 \leq i \leq k$. Hence, we have

$$\begin{aligned} &(\delta_1 A_1 + \delta_2 A_2 + \cdots + \delta_k A_k)^t = 0^t \\ &\Rightarrow (\delta_1 A_1)^t + (\delta_2 A_2)^t + \cdots + (\delta_k A_k)^t = 0 \\ &\Rightarrow \delta_1 A_1^t + \delta_2 A_2^t + \cdots + \delta_k A_k^t = 0. \end{aligned}$$

Hence, the set $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent. ■

Exercise 1.5.20

Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where f and g are defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$ with

$r \neq s$. Suppose for sake of contradiction that the set $\{f, g\}$ is a linearly dependent set of vectors. By Exercise 1.5.9, we know that either f is a multiple of g or g is a multiple of f. Assume f is a multiple of g. Hence, there exists a $c \in R$ such that $f(t) = cg(t)$. In other words,

$$f(t) = cg(t) \Rightarrow e^{rt} = ce^{st}.$$

To solve for c, let $t = 0$. Then we have

$$f(0) = cg(0) \Rightarrow 1 = c.$$

Hence, we have

$$e^{rt} = e^{st}.$$

This equality is valid only when $r = s$ because otherwise f would not be a scalar multiple of g. But note that $r \neq s$ by assumption. Hence, we have a contradiction and thus $\{f, g\}$ must be linearly independent. ■

Exercise 1.5.21

Let S_1 and S_2 be disjoint linearly independent subsets of V. Prove that $S_1 \cup S_2$ is linearly dependent if and only if $\text{span}(S_1) \cap \text{span}(S_2) \neq \{0\}$.

Proof. ($\Rightarrow$) We will proceed via contrapositive. Since $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$, let $v \in \text{span}(S_1)$ such that we can find a $w \in \text{span}(S_2)$ such that both $v = w$ where both $v = 0$ and $w = 0$. Since $v \in \text{span}(S_1)$, we can find a finite set of vectors $x_1, x_2, \ldots, x_n \in S_1$ and scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$v = \sum_{i=1}^{n} a_i x_i = 0.$$

Likewise, $w \in \text{span}(S_2)$ implies that we can find $y_1, y_2, \ldots, y_n \in S_2$ and scalars $b_1, b_2, \ldots, b_n \in F$ such that

$$w = \sum_{j=1}^{n} b_j y_j = 0.$$

Observe that

$$v = w \Rightarrow \sum_{i=1}^{n} a_i x_i = \sum_{j=1}^{n} b_j y_j.$$

Then we have

$$\sum_{i=1}^{n} a_i x_i - \sum_{j=1}^{n} b_j y_j = 0.$$

Since S_1 and S_2 are disjoint linearly independent sets, we know that $x_i \notin S_2$ and $y_i \notin S_1$ and that $a_i = 0$ and $b_j = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$ respectively. Hence, $S_1 \cup S_2$ is a linearly independent set.

($\Leftarrow$) We will proceed via contrapositive for this direction as well. Suppose $S_1 \cup S_2$ is linearly independent. Then choose a finite number of distinct vectors $x_1, x_2, \ldots, x_n \in S_1 \cup S_2$ and scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$\underbrace{\sum_{i=1}^{n} a_i x_i}_{\in \text{span}(S_1)} = 0$$

with $a_i = 0$ for all $1 \leq i \leq m$. Since $x_i \in S_1 \cup S_2$, then either $x_i \in S_1$ or $x_i \in S_2$. Without loss of generality, suppose $x_i \in S_1$. Then we know that $x_i \notin S_2$ since $S_1 \cap S_2 = \emptyset$. Since S_2 is linearly independent, choose a finite number of vectors $y_1, y_2, \ldots, y_m \in S_2$ and scalars $b_1, b_2, \ldots, b_n \in F$ such that

$$\underbrace{\sum_{j=1}^{m} b_j y_j}_{\in \operatorname{span}(S_2)} = 0$$

with $b_j = 0$ for all $1 \leq j \leq m$. Observe that

$$\sum_{i=1}^{n} a_i x_i = \sum_{j=1}^{m} b_j y_j = 0.$$

Hence, we have $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{0\}$. ∎

1.5 Bases and Dimension

Exercise 1.6.11

Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for V.

Proof. ($\Rightarrow$) We want to show that $\{u+v, au\}$ and $\{au, bv\}$ is a basis for V; that is, we want to show that $\{u + v, au\}$ and $\{au, bv\}$ is both linearly independent and generates V. We will start by showing that $\{u + v, au\}$ is linearly independent. Choose scalars δ_1, δ_2 such that

$$\delta_1(u + v) + \delta_2(au) = 0 \tag{1}$$

with $\delta_1 = \delta_2 = 0$. Let us algebraically manipulate (1) into the following form:

$$\delta_1 u + \delta_1 v + (\delta_2 a)u = 0.$$

Since $\{u, v\}$ is linearly independent and $a \neq 0$, we get that

$$\delta_1 v + (\delta_2 a)u = 0$$

implies $\delta_1 = 0$ and $\delta_2 a = 0$ such that $\delta_2 = 0$. But this implies that $\{u+v, au\}$ is also linearly independent. To show that $\{u+v, au\}$ spans V, it suffices to show that $V \subseteq \{u+v, au\}$ since the other containment $\{u + v, au\} \subseteq V$ follows immediately. Let $v \in V$. By Theorem 1.7, we know that adjoining an arbitrary vector $w \in V$ but not in $\{u + v, au\}$ creates a linearly dependent set. So, we must have $w \in \operatorname{span}(\{u + v, au\})$ and thus $V \subseteq \operatorname{span}(\{a + v, au\})$.

Now, we want to show that $\{au, bv\}$ is a basis. Choose scalars δ_1, δ_2 such that

$$\delta_1(au) + \delta_2(bv) = 0 \tag{2}$$

such that $\delta_1 = \delta_2 = 0$. We can manipulate (2) by rewriting it in the following form:

$$(\delta_1 a)u + (\delta_2 b)v = 0. \tag{3}$$

Since $\{u, v\}$ is a linearly independent set, we know that $\delta_1 a = \delta_2 b = 0$. Since $a, b \neq 0$, this implies that $\delta_1 = \delta_2 = 0$. Hence, the representation in (2) is trivial and thus the set $\{au, bv\}$ is linearly independent. Since adjoining any $w \in V$ not in $\{au, bv\}$ creates a linearly dependent set, we get that $w \in \text{span}(\{au, bv\})$ by Theorem 1.7. Hence, $\{au, bv\}$ generates V.

■

Exercise 1.6.12

Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V.

Proof. First, we prove that $\{u+v+w, v+w, w\}$ is linearly independent. Choose $\delta_1, \delta_2, \delta_3 \in F$ such that

$$\delta_1(u + v + w) + \delta_2(v + w) + \delta_3 w = 0. \tag{1}$$

We can rewrite (1) in the following way:

$$(\delta_1 u + \delta_2 v + \delta_3 w) + \delta_1(v + w) + \delta_2 w = 0.. \tag{2}$$

Since $\{u, v, w\}$ is also a basis, we know that $\{u, v, w\}$ is also linearly independent. Hence, $\delta_1 = \delta_2 = \delta_3 = 0$. Thus, (1) contains the trivial representation and so $\{u + v + w, v + w, w\}$ is linearly independent.

Now, to prove that $S = \{u + v + w, v + w, w\}$ generates V, it suffices to show that $V \subseteq \text{span}(S)$. Adjoining a vector $x \in V$ but not in S produces a linearly independent set. Hence, Theorem 1.7 implies that $s \in \text{span}(S)$. Hence, S generates V and that S is a basis for V.

■

Exercise 1.6.19

Complete the proof of Theorem 1.8.

Proof. See proof in notes. ■

Exercise 1.6.20

Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)

Proof. Let $\dim(V) = n$. Suppose S is a subset of V such that S generates V. Then S could be either $S = \{0\}$ or $S = \emptyset$. In either case, we find that $\text{span}(\emptyset) = \{0\} = V$ or $\text{span}(\{0\}) = \{0\} = V$. Now, suppose S contains a non-zero vector u_1. Thus, the set $\{u_1\}$ is linearly independent. Suppose we continue adding vectors inductively $u_2, u_3, \ldots, u_k$ into this set such that this process stops at exactly k vectors. We claim that our constructed set

$$L = \{u_1, u_2, \ldots, u_k\}$$

is linearly independent for $k \geq 1$. Suppose we assume that L holds for the kth case. We want to show that it also holds for the $k + 1$ case. Observe that

$$L = \{u_1, u_2, \ldots, u_{k+1}\} = \{u_1, u_2, \ldots, u_k\} \cup \{u_{k+1}\}.$$

If $u_{k+1} = 0$, then L would be linearly dependent. Otherwise $u_{k+1} \neq 0$ and so $\{u_{k+1}\}$ is linearly independent. By inductive hypothesis, we also know that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent. Since $\text{span}(\{u_1, u_2, \ldots, u_k\}) \cap \text{span}(\{u_{k+1}\}) = \{0\}$ and that the two sets are disjoint, we know that $\{u_1, u_2, \ldots, u_k\} \cup \{u_{k+1}\}$ is linearly independent. This ends our induction proof.

Note that we cannot have $S \neq L$ since S could be an infinite set. Since L is a subset of V where $\dim(V) = n$, L can be extended into a basis for V by Corollary 2 of the Replacement Theorem that contains exactly n vectors. ■

(b) Prove that S contains at least n vectors.

Proof. Denote the basis constructed from part (a) as β. Since β is a basis for V, β must contain exactly n vectors. Since $\beta \subseteq S$ and S is a generating set for V, then S must contain at least n vectors. ■

Exercise 1.6.21

Prove that a vectors space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

Proof. Let V be a vector space. For the forwards direction, suppose V is an infinite-dimensional vector space. By definition, V contains a basis β that is infinite-dimensional. By definition, β is also linearly independent. Thus, V contains an infinite linearly independent set.

For the backwards direction, we proceed using the converse. Suppose V is a finite-dimensional vector space. Let $\dim(V) = n$. By definition, V contains a basis β that contains exactly n vectors. Since β is also linearly independent, β is a finite linearly independent subset. ■

Exercise 1.6.22

Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine the necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

Proof. We must have $W_1 \subseteq W_2$ in order for $\dim(W_1 \cap W_2) = \dim(W_1)$. Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Since W_1 and W_2 are subspaces, we must also have $W_1 \cap W_2$ as a subspace. Hence, $W_1 \cap W_2$ is finite-dimensional by Theorem 1.11. This implies that $W_1 \cap W_2$ contains a basis β containing exactly $\dim(W_1 \cap W_2)$ vectors. Since β is a linearly independent subset of W_1, we know that β must contain at most $\dim(W_1)$ vectors. Hence, we have $\dim(W_1 \cap W_2) \leq \dim(W_1)$. Since $W_1 \cup W_2$, then $W_1 \subseteq W_1 \cap W_2$. Since W_1 is finite-dimensional, let α be a basis containing exactly $\dim(W_1)$ vectors. Since $\alpha \subseteq W_1 \cap W_2$ and α is a linearly independent set, α must contain at most $\dim(W_1 \cap W_2)$ amount of vectors. Hence, $\dim(W_1) \leq \dim(W_1 \cap W_2)$. Thus, we have $\dim(W_1) = \dim(W_1 \cap W_2)$.

Conversely, we have $\dim(W_1 \cap W_2) = \dim(W_1)$. By Theorem 1.11, we have $W_1 \cap W_2 = W_1$. Since $W_1 \cap W_2 \subseteq W_2$, we know that $W_1 \subseteq W_2$. ■

Exercise 1.6.23

Let $v_1, v_2, \ldots, v_k, v$ be vectors in a vector space V, and define $W_1 = \text{span}(\{v_1, v_2, \ldots, v_k\})$, and $W_2 = \text{span}(\{v_1, v_2, \ldots, v_k, v\})$.

(a) Find necessary and sufficient conditions on v such that $\dim(W_1) = \dim(W_2)$.

Proof. The condition we need is $v \in W_1$. Since W_1 and W_2 are subspaces, we also have $W_1 \cap W_2$ is a subspace. Hence, theorem 1.11 tells us that $W_1 \cap W_2$ is also finite-dimensional. Suppose $v \in W_1$. Since $v \in W_2$ as well, we have that $W_1 \subseteq W_2$. Now let $v \in W_2$. Then choose scalars $a_1, a_2, \dots, a_k$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_k = v.$$

But this tells us that $v \in W_1$. So, $W_2 \subseteq W_1$ and thus $W_1 = W_2$. By theorem 1.11, $\dim(W_1) = \dim(W_2)$.

Conversely, $\dim(W_1) = \dim(W_2)$. Since $v \in W_2$, this also means that $v \in W_1$ since $W_1 = W_2$ by theorem 1.11. ■

(b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case that $\dim(W_1) \neq \dim(W_2)$.

Proof. If $\dim(W_1) \neq \dim(W_2)$, then $v \notin W_1$. This is just the contrapositive of the statement above. ■

Exercise 1.6.24

Let $f(x)$ be a polynomial of degree n in $P_n(\mathbb{R})$. Prove that for any $g(x) \in P_n(\mathbb{R})$ there exists scalars $c_0, c_1, \dots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f^{(1)}(x) + c_2 f^{(2)}(x) + \cdots + c_n f^{(n)}(x),$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

Proof. Since f is differentiable n times, we can construct the set

$$W = \{f(x), f^{(1)}(x), f^{(2)}(x), \dots, f^{(n)}(x)\}$$

containing $n + 1$ polynomials such that no two polynomials contain the same degree (with each derivative of $f(x)$, the degree decreases by one). Since W is a subset of $P_n(\mathbb{R})$ with no two polynomials having the same degree, we see that following the process seen in example 4 in section 1.5 shows that W is a linearly independent set containing $n+1$ vectors. Hence, W is a basis for $P_n(\mathbb{R})$ such that any $g(x) \in P_n(\mathbb{R})$ by Theorem 1.11. Consequently, $g(x)$ can be expressed in terms of the vectors in W such that

$$g(x) = c_0 f(x) + c_1 f^{(1)}(x) + c_2 f^{(2)}(x) + \cdots + c^n f^{(n)}(x)$$

for unique scalars $c_0, c_1, \dots, c_n$ by Theorem 1.8. ■

Exercise 1.6.29

(a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Proof. Let W_1 and W_2 be subspaces of V. Since W_1 and W_2 are finite-dimensional, we also know that $W_1 + W_2$ is finite-dimensional. Now, we will show that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Consider $W_1 \cap W_2$ and note that $W_1 \cap W_2$ being finite-dimensional implies that it contains a basis $\beta_0 = \{u_1, u_2, \ldots, u_k\}$. We can extend β_0 into a basis for W_1 by adding vectors $v_1, v_2, \ldots, v_m$ into β_0. Denote this new set as β_1. Likewise, we add vectors $w_1, w_2, \ldots, w_p$ into β_0 to make a basis β_2 for W_2. We claim that $\beta = \beta_0 \cup \beta_1 \cup \beta_2$ is a basis for $W_1 + W_2$. First, we will show that β is linearly independent. To do this, we need to show that

$$\sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j v_j + \sum_{\ell=1}^{p} \gamma_\ell w_\ell = 0. \tag{1}$$

Subtracting the third term on both sides of (1) produces the following equation:

$$\sum_{j=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j v_j = -\sum_{\ell=1}^{p} \gamma_\ell w_\ell.$$

Observe that the left-hand side is an element of W_1 while the other side is an element of W_2. Hence, we know that the term on the right-hand side of (1) is also an element of $W_1 \cap W_2$. This implies that

$$-\sum_{\ell=1}^{p} \gamma_\ell w_\ell = \sum_{i=1}^{k} \delta_i u_i$$

which can be re-written as

$$\sum_{i=1}^{k} \delta_i u_i + \sum_{\ell=1}^{p} \gamma_\ell w_\ell = 0.$$

Since β_2 is a basis for W_2, we know that $\delta_i = 0$ and $\gamma_\ell = 0$ implying that $a_i = 0$ and $\gamma_\ell = 0$. We can re-write (1) in the following form:

$$\sum_{j=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j v_j = 0. \tag{2}$$

Since β_1 is a linearly independent set, we get that $a_i = 0$ and $b_j = 0$. Hence, (1) contains the trivial-representation which implies that $\beta = \beta_0 \cup \beta_1 \cup \beta_2$ is a linearly independent set.

Now, we will show that β spans $W_1 + W_2$. Observe that $\text{span}(\beta) \subseteq W_1 + W_2$. Now, we will show $W_1 + W_2 \subseteq \text{span}(\beta)$. Suppose we take a vector $v \in W_1 + W_2$ that is not in β and adjoin this vector in β. Note that $\beta \cup \{v\}$ produces a linearly dependent set that by which Theorem 1.7 implies that $v \in \text{span}(\beta)$ and we are done. Hence, β spans $W_1 + W_2$ and thus β is a basis.

Note that β contains exactly $m+p+k$ vectors. Hence, denote $\dim(W_1+W_2) = m+p+k$ which can be re-written as

$$\begin{aligned}
\dim(W_1 + W_2) &= m + p + k \\
&= (k + m) + (k + p) - k \\
&= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).
\end{aligned}$$

■

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

Proof. Suppose V is a direct sum of W_1 and W_2. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. We need to show that $\dim(V) = \dim(W_1) + \dim(W_2)$. Since $W_1 \cap W_2 = \{0\}$, we know that it contains the empty set $\emptyset$ as the basis for $W_1 \cap W_2$. Hence, $\dim(W_1 \cap W_2) = 0$. Using the formula derived in part (a), we can write

$$\begin{aligned}\dim(V) &= \dim(W_1 + W_2)\\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)\\ &= \dim(W_1) + \dim(W_2)\end{aligned}$$

and we are done.

Conversely, $\dim(V) = \dim(W_1) + \dim(W_2)$ implies that $V = W_1 + W_2$. Using part (a) again, we see that the sum $\dim(V) = \dim(W_1) + \dim(W_2) - 0$ implies that $\dim(W_1 \cap W_2) = 0$ and hence $W_1 \cap W_2$ must be equal to the zero set $\{0\}$ (which we know by definition that $\mathrm{span}(\emptyset) = \{0\}$. Hence, V is a direct sum of W_1 and W_2. ■

Exercise 1.6.31

Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

Proof. Observe that W_1 and W_2 being subspaces of V implies that $W_1 \cap W_2$ is a subspace of V. Hence, $W_1 \cap W_2$ is finite-dimensional. Denote $\dim(W_1 \cap W_2) = k$ and let β be a basis for $W_1 \cap W_2$. Since $W_1 \cap W_2 \subseteq W_2$, we know that β must contain at most $\dim(W_2) = n$. Hence, $\dim(W_1 \cap W_2) \leq n$. ■

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Proof. Using the formula found in part (a) of Exercise 1.3.29, part (a) of this exercise, and $\dim(W_1) \geq \dim(W_2)$, we find that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_1) - \dim(W_1 \cap W_2) \leq \dim(W_1) + \dim(W_2).$$

■

Exercise 1.6.33

(a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cup \beta_2$ is a basis for V.

Proof. Let W_1 and W_2 be subspaces of V. Assume β_1 and β_2 are bases for W_1 and W_2 respectively. We need to show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

Since β_1 and β_2 contain distinct linearly independent vectors, we must have $\beta_1 \cap \beta_2 = \emptyset$. Since V is a direct sum of the W_1 and W_2, we know that $W_1 \cap W_2 = \{0\}$ by definition. Since β_1 and β_2 generate W_1 and W_2 respectively, we must have $\mathrm{span}(\beta_1) \cap \mathrm{span}(\beta_2) = \{0\}$. Now, we have the set $\beta_1 \cup \beta_2$ as a linearly independent set by exercise 1.5.21. Observe that $\mathrm{span}(\beta_1 \cup \beta_2) \subseteq V$ follows immediately. Now, take any $v \in V$ that is not in $\beta_1 \cup \beta_2$ such that adjoining this vector $v \in V$ produces a linearly dependent set. By Theorem 1.7, we have $v \in \mathrm{span}(\beta_1 \cup \beta_2)$. Thus, we have $V \subseteq \mathrm{span}(\beta_1 \cup \beta_2)$. Hence, $\beta_1 \cup \beta_2$ is a generating set for V and we are done. ■

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

Proof. Let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2 respectively. Suppose $\beta_1 \cup \beta_2$ is a basis for V. This tells us that $\beta_1 \cup \beta_2$ is linearly independent. Thus, $\text{span}(\beta_1) \cap \text{span}(\beta_2) = \{0\}$ and hence $W_1 \cap W_2 = \{0\}$ since $\text{span}(\beta_1) = W_1$ and $\text{span}(\beta_2) = W_2$. This tells us that $\dim(W_1 \cap W_2) = 0$. Using the fact that $\beta_1 \cup \beta_2$ is a basis for V that contains exactly $\dim(W_1) + \dim(W_2)$, we get that

$$\dim(V) = \dim(W_1) + \dim(W_2).$$

By part (b) of Exercise 1.6.29, we get that $V = W_1 \oplus W_2$. ∎

Exercise 1.6.34

(a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Proof. Since W_1 is a subspace of a finite-dimensional vector space V, we know that W_1 is also finite-dimensional and $\dim(W_1) \leq \dim(V)$ by Theorem 1.11. Thus, let β be a basis for W_1 and let α be a basis for V. Since α is a generating set consisting of $\dim(V)$ vectors and β is a linearly independent subset of V, we can find a subset σ of α consisting of $\dim(V) - \dim(W_1)$ vectors such that $\beta \cup \sigma$ generates V by the Replacement Theorem. Suppose σ is a basis for a subspace of V denoted by W_2 for which $\dim(W_2) = \dim(V) - \dim(W_1)$. Note that $\beta \cup \sigma$ contains exactly $\dim(V)$ vectors so it is also a basis for V and that $\beta \cap \sigma = \emptyset$. Hence, $\dim(V) = \dim(W_1) + \dim(W_2)$ for which it implies that $V = W_1 \oplus W_2$. ∎

(b) Let $V = \mathbb{R}^2$ and $W_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

Proof. **TO DO.** ∎

Exercise 1.6.35

Let W be a subspace of a finite-dimensional vector space V, and consider the basis $\beta_0 = \{u_1, u_2, \ldots, u_k\}$ for W. Let $\beta_1 = \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ be an extension of this basis to a basis for V.

(a) Prove that $\beta_2 = \{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$ is a basis for V/W.

Proof. To show that β_2 is a basis, we need to show that β_2 is a linearly independent set and a generating set for V. Observe that

$$\sum_{j=k+1}^{n} \delta_j (u_j + W) = W \tag{1}$$

for scalars δ_j for $k+1 \leq j \leq n$. Note that W, in this case, is the zero vector of V/W. Hence, (1) implies that

$$\left[\sum_{j=k+1}^{n} \delta_j u_j\right] + W = W. \tag{2}$$

This implies that

$$\sum_{j=k+1}^{n} \delta_j u_j \in W. \tag{3}$$

Since W contains β_0 as a basis, we can write

$$\sum_{j=k+1}^{n} \delta_j u_j = \sum_{i=1}^{k} \gamma_i u_i$$

for scalars γ_i for all $1 \leq i \leq k$ which can be re-written to

$$\sum_{j=k+1}^{n} \delta_j u_j - \sum_{j=1}^{k} \gamma_i u_i = 0$$

where all $\delta_j = 0$ and $\gamma_i = 0$ since β_1 is a basis for V. Since all $\delta_j = 0$, we get that β_2 is a linearly independent set.

To show that β_2 is a generating set for V/W, we need to show that $\operatorname{span}(\beta_2) = V/W$. Note that the containment $\operatorname{span}(\beta_2) \subseteq V/W$. To show that other containment, let $v \in V$ not in β_2. Observe that adjoining v to β_2 creates a linearly dependent set. By Theorem 1.7, we have $v \in \operatorname{span}(\beta_2)$. Hence, $V \subseteq \operatorname{span}(\beta_2)$.

■

(b) Derive a formula relating $\dim(V), \dim(W)$, and $\dim(V/W)$.

Proof. In part (a), we see that β_2 contains n amount of vectors. Unioning this set with β_1 creates a basis for V that contains $\dim(W) + \dim(V/W)$ vectors. Hence, we must have

$$\dim(V/W) = \dim(V) - \dim(W).$$

■

Chapter 2

Linear Transformations and Matrices

2.1 Linear Transformations, Null spaces, and Ranges

Exercise 2.1.1

Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), and T is a function from V to W.

(a) If T is linear, then T preserves sums and scalar products.

> **Solution.** True. This is by definition. ■

(b) If $T(x+y) = T(x) + T(y)$, then T is linear.

> **Solution.** False. We also need to have $T(cx) = cT(x)$ where $x \in V$ and $c \in F$ in order for T to be a linear map. ■

(c) If T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.

> **Solution.** True. This is Theorem 2.2. ■

(d) If T is linear, then $T(0_V) = 0_W$.

> **Solution.** True by properties of linear maps. ■

(e) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.

> **Solution.** False. This only happens when V and W have equal equal dimensions. ■

(f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

> **Solution.** False. We need T to be injective in order to make this valid. ■

(g) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$.

> **Solution.** True by Corollary to Theorem 2.6. ■

(h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Solution. False. We need to have a basis for V. ■

For Exercises 2 through 6, prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is injective or surjective.

Exercise 2.1.2

$T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

Solution. Our first goal is to show that T is linear. Let $x, y \in \mathbb{R}^3$ and $c \in \mathbb{R}$ where $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$. Then observe that

$$\begin{aligned} T(cx + y) &= T(ca_1 + b_1, ca_2 + b_2, ca_3 + b_3) \\ &= ([ca_1 + b_1] - [ca_2 - b_2], 2(ca_3 + b_3)) \\ &= ([ca_1 - ca_2] + [b_1 - b_2], 2ca_3 + 2b_3) \\ &= (c(a_1 - a_2), 2ca_3) + (b_1 - b_2, 2b_3) \\ &= c(a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3) \\ &= cT(x) + T(y). \end{aligned}$$

Hence, T is linear.

Now, let's compute the bases for both $N(T)$ and $R(T)$. To compute $N(T)$, we need to have all the solutions needed for

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = 0$$

to hold. Solving the equation above gives us the following solutions

$$a_1 = a_2 \text{ and } a_3 = 0.$$

Hence, for any $a \in \mathbb{R}$ we have

$$N(T) = \{(a, a, 0) : a \in \mathbb{R}\}.$$

Note that $N(T)$ is generated by the linearly independent set $\{(1, 1, 0)\}$. Thus, $\{(1, 1, 0)\}$ is a basis for $N(T)$ and contains 1 vector. So, $\text{nullity}(T) = 1$ by the Dimension Theorem. This immediately tells us that T is not injective since $\text{nullity}(T) \neq 0$ and hence $\text{rank}(T) = 2$. But $\text{rank}(T) = \dim(\mathbb{R}^2)$. Hence, T must be surjective. Since $R(T) = \mathbb{R}^2$, the basis is just $\{(1, 0), (0, 1)\}$. ■

Exercise 2.1.8

$T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

Solution. First, we show that T is linear. ■

Exercise 2.1.7

Prove properties $1, 2, 3$, and 4 on page 65.

Proof. See proof in notes. ■

Exercise 2.1.8

Prove that the transformations in Example 2 and 3 are linear.

Proof. First we prove that $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \cos\theta + a_2 \sin\theta)$$

is linear. Let $x, y \in \mathbb{R}^2$ defined by $x = (a_1, a_2)$ and $y = (b_1, b_2)$. Let $c \in F$ such that $cx = (ca_1, ca_2)$. To make the computation less difficult, we have

$$cx + y = (ca_1 + b_1, ca_2 + b_2).$$

Using the definition T_θ now, we have that

$$\begin{aligned} T_\theta(cx + y) &= ((ca_1 + b_1)\cos\theta - (ca_2 + b_2)\sin\theta, \\ &\quad (ca_1 + b_1)\cos\theta + (ca_2 + b_2)\sin\theta) \\ &= (c(a_1 \cos\theta - a_2 \sin\theta) + (b_1 \cos\theta - b_2 \sin\theta), \\ &\quad c(a_1 \cos\theta + a_2 \sin\theta) + (b_1 \cos\theta + b_2 \sin\theta)) \\ &= c(a_1 \cos\theta - a_2 \sin\theta, a_1 \cos\theta + a_2 \sin\theta) \\ &\quad + (b_1 \cos\theta - b_2 \sin\theta, b_1 \cos\theta + b_2 \sin\theta) \\ &= cT_\theta(a_1, a_2) + T_\theta(b_1, b_2) \\ &= cT_\theta(x) + T_\theta(y). \end{aligned}$$

Hence, we get that T_θ is linear.

Using the same process, we show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (a_1, -a_2)$ is linear. That is, we have

$$\begin{aligned} T(cx + y) &= (ca_1 + b_1, -(ca_2 + b_2)) \\ &= (ca_1 + b_1, -ca_2 - b_2) \\ &= c(a_1, -a_2) + (b_1, -b_2) \\ &= cT(a_1, a_2) + T(b_1, b_2) \\ &= cT(x) + T(y). \end{aligned}$$

Hence, T is also linear. ■

Exercise 2.1.10

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$.

Solution. First, observe that $(2, 3) \in \mathbb{R}^2$ can be expressed in terms of a linear combination of $(1, 0)$ and $(1, 1)$. Hence, we need to find scalars $a, b \in \mathbb{R}$ such that

$$a(1, 0) + b(1, 1) = (2, 3).$$

Solving for the scalars gives us the solutions $a = -1$ and $b = 3$. Since T is linear, we can now compute $T(2,3)$ by doing the following:

$$\begin{aligned}T(2,3) &= T(-(1,0)+3(1,1))\\ &= -T(1,0)+3T(1,1)\\ &= -(1,4)+3(2,5)\\ &= (5,11).\end{aligned}$$

Hence, we get that $T(2,3)=(5,11)$.

Note that the domain and codomain have equal finite dimensions. Hence, Theorem 2.5 tells us that T must be one-to-one (injective). ■

Exercise 2.1.11

Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Solution. Note that $\beta = \{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$. By Theorem 2.6, T being linear implies that there exists a unique linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$.

To compute $T(8,11)$, we need to find scalars $a, b \in \mathbb{R}$ such that

$$a(1,1)+b(2,3) = (8,11).$$

Solving for a and b gives us the following linear combination:

$$2(1,1)+3(2,3) = (8,11).$$

Since T is linear, we find that

$$\begin{aligned}T(8,11) &= T(2(1,1)+3(2,3))\\ &= 2T(1,1)+3T(2,3)\\ &= 2(1,0,2)+3(1,-1,4)\\ &= (5,-3,16)\end{aligned}$$

■

Exercise 2.1.13

Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. If $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i)$ for $i = 1, 2, \ldots, k$ then S is linearly independent.

Proof. Consider

$$\sum_{i=1}^{k} a_i v_i = 0 \tag{1}$$

for some scalars $a_1, a_2, \ldots, a_k \in F$. In order to show that S is linearly independent, we need to show that $a_i = 0$ for all $1 \le i \le k$. Since T is linear, we get that $T(0) = 0$ implies

$$T\Big(\sum_{i=1}^{k} a_i v_i\Big) = 0.$$

Since T is linear and S is chosen so that $T(v_i) = w_i$ for $1 \le i \le k$, we get that

$$\sum_{i=1}^{k} a_i T(v_i) = 0 \Leftrightarrow \sum_{i=1}^{k} a_i w_i = 0.$$

Since $\{w_1, w_2, \dots, w_k\}$ is linearly independent, we must have $a_i = 0$ for all $1 \le i \le k$. But this tell us that (1) must have the trivial representation. Hence, S must also be linearly independent. ■

Exercise 2.1.14

Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Prove that T is injective if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Proof. ($\Rightarrow$) Define $T : S \to \mathcal{W}$ where S and $\mathcal{W}$ are linearly independent subsets of V and W respectively. Since T is injective, we have that T is an onto map by Theorem 2.5.

($\Leftarrow$) Suppose T carries linearly independent subsets of V onto linearly independent subsets of W. Our goal is to show that T is injective. Suppose

$$T\Big(\sum_{i=1}^{n} a_i v_i\Big) = T\Big(\sum_{i=1}^{n} b_i v_i\Big) \tag{1}$$

for some scalars a_i, b_i and vectors $v_i \in S$ for all $1 \le i \le k$. Since T is linear and $T : S \to \mathcal{W}$ is onto, we can re-write (1) into the following form:

$$\sum_{i=1}^{k} a_i T(v_i) = \sum_{i=1}^{k} b_i T(v_i) \tag{2}$$

which manipulating again, we get that

$$\sum_{i=1}^{k} (a_i - b_i) T(v_i) = 0. \tag{3}$$

Since $T(v_i) \in W$ and W is linearly independent, we must have $a_i - b_i = 0$ if and only if $a_i = b_i$. Hence, we have

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$$

and so T must be injective. ■

(b) Suppose that T is injective and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

Proof. ($\Rightarrow$) In order to show that $T(S)$ is linearly independent, we must show that

$$\sum_{i=1}^{n} a_i T(v_i) = 0 \tag{1}$$

contains the trivial-representation. Since T is linear, we can write (1) into the following

form
$$T\Big(\sum_{i=1}^{n} a_i v_i\Big) = 0.$$
This implies that
$$\sum_{i=1}^{n} a_i v_i \in N(T).$$
Since T is injective, we know that $N(T) = \{0\}$ (by Theorem 2.4), and so we must have
$$\sum_{i=1}^{n} a_i v_i = 0. \tag{2}$$
But $v_i \in S$ for $1 \le i \le k$ is linearly independent, and so $a_i = 0$ for all $1 \le i \le k$. This tells us that (1) contains the trivial-representation. Hence, $T(S)$ is linearly independent.

($\Rightarrow$) Suppose $T(S)$ is linearly independent. Then observe that
$$\sum_{i=1}^{n} a_i T(v_i) = 0$$
for some scalars a_i and $T(v_i) \in T(S)$ for $1 \le i \le k$ such that $a_i = 0$. Since T is linear and T is injective, we can write
$$T\Big(\sum_{i=1}^{n} a_i v_i\Big) = 0 \Leftrightarrow \sum_{i=1}^{n} a_i v_i = 0.$$
Since $a_i = 0$ and $v_i \in S$, we also find that S is a linearly independent set. ■

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is injective and surjective. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Proof. Since β is a basis for V, we get that $\operatorname{span}(T(\beta)) = R(T)$ by Theorem 2.2. Since T is surjective, we have $R(T) = W$, so $\operatorname{span}(T(\beta)) = W$. Hence, $T(\beta)$ generates W. Since β is a linearly independent subset of V and T is an injective linear map, we must also have $T(\beta)$ as a linearly independent subset of W by part (b). Hence, $T(\beta)$ is a basis for W. ■

Exercise 2.1.15

Recall the definition of $P(\mathbb{R})$ on page 10. Define
$$T : P(\mathbb{R}) \to P(\mathbb{R}) \text{ by } T(f(x)) = \int_0^x f(t)\ dt.$$
Prove that T is linear and injective, but not surjective.

Proof. First, we show that $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined by
$$T(f(x)) = \int_0^x f(t)\ dt.$$

Let $cf(x) + g(x) \in P(\mathbb{R})$ where $c \in F$. Then observe that

$$\begin{aligned} T(cf(x) + g(x)) &= \int_0^x [cf(t) + g(t)]\ dt \\ &= \int_0^x cf(t)\ dt + \int_0^x g(t)\ dt \\ &= c\int_0^x f(t)\ dt + \int_0^x g(t)\ dt \\ &= cT(f(x)) + T(g(x)). \end{aligned}$$

Hence, $T : P(\mathbb{R}) \to P(\mathbb{R})$ is a linear map. Let $f(x), g(x) \in P(\mathbb{R})$, then

$$\begin{aligned} T(f(x)) &= T(g(x)) \\ \int_0^x f(t)\ dt &= \int_0^x g(t)\ dt \\ \frac{d}{dx}\Big[\int_0^x f(t)\ dt\Big] &= \frac{d}{dx}\Big[\int_0^x g(t)\ dt\Big] \\ f(x) &= g(x). \end{aligned}$$

Hence, T is an injective map. To see why T fails to be surjective, observe that $1 \in P(\mathbb{R})$ but we cannot find a polynomial $f(x)$ such that $T(f(x)) = 1$; that is, the integration of any polynomial can never yield 1. ■

Exercise 2.1.16

Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is surjective, but not injective.

Proof. Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x) = \frac{d}{dx}[f(x)]$. For $cf(x)+g(x) \in P(\mathbb{R})$ where $c \in F$, observe that

$$\begin{aligned} T(cf(x) + g(x)) &= \frac{d}{dx}[cf(x) + g(x)] \\ &= \frac{d}{dx}[cf(x)] + \frac{d}{dx}[g(x)] \\ &= c\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \\ &= cT(f(x)) + T(g(x)). \end{aligned}$$

Hence, T is a linear map. Now, we want to show that T is surjective. Define

$$f(x) = \int_0^x g(t)\ dt.$$

Then

$$T(f(x)) = \frac{d}{dx}[f(x)] = \frac{d}{dx}\Big[\int_0^x g(t)\ dt\Big] = g(x)$$

by the Second Fundamental Theorem of Calculus. Hence, T is surjective.

To show that T is not injective, let $x^n + a, x^n + b \in P(\mathbb{R})$ where $a, b \in \mathbb{R}$ such that $a \neq b$ and $n > 0$. Clearly, we have $x^n + a \neq x^n + b$. But, we have

$$T(x^n + a) = \frac{d}{dx}[x^n + a] = nx^{n-1}$$

and

$$T(x^n + b) = \frac{d}{dx}[x^n + b] = nx^{n-1}.$$

Note that $x^n + a \neq x^n + b$ yet $T(x^n + a) = T(x^n + b)$. Hence, T is not injective. ■

Exercise 2.1.17

Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.

Proof. Using the Dimension Theorem, we find that

$$\operatorname{rank}(T) = \dim(V) - \operatorname{nullity}(T).$$

Since $\dim(V) < \dim(W)$, we find that

$$\operatorname{rank}(T) < \dim(W) - \operatorname{nullity}(T) < \dim(W).$$

Hence, T cannot be surjective in this case. ■

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

Proof. Using the Dimension Theorem again and $\dim(V) > \dim(W)$, we find that

$$\operatorname{nullity}(T) = \dim(V) - \operatorname{rank}(T) > \dim(W) - \operatorname{rank}(T) > 0.$$

This means that $\operatorname{nullity}(T)$ can never be zero, otherwise T is injective. Hence, T can never be injective if $\dim(V) > \dim(W)$. ■

Exercise 2.1.20

Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

Proof. First, we prove that $T(V_1)$ is a subspace of W. Let $x, y \in T(V_1)$ and $c \in F$. Since V_1 is a subspace of V, we know that $0_V \in V_1$, $x + y \in V_1$, and $cx \in V_1$. Observe that $T(0_V) = 0_W$ since T is linear. Hence, $0_W \in T(V_1)$. Let $x, y \in T(V_1)$. Since T is linear and $x + y \in V_1$, we have $T(x + y) = T(x) + T(y)$. Hence, we must have $x + y \in T(V_1)$. Now, $cx \in V_1$ implies $T(cx) = cT(x)$. Hence, $cx \in T(V_1)$. This tells us that $T(V_1)$ is a subspace of W.

Now, we show $S = \{x \in V : T(x) \in W_1\}$ is a subspace of V. Since $0_W \in W_1$ (because W_1 is a subspace of W) and T is linear, we have that $T(0_V) = 0_W$. Hence, $0_V \in S$. Now, let $x, y \in S$. Hence, $T(x), T(y) \in W_1$ implies $T(x) + T(y) \in W_1$ since W_1 is a subspace of W. Since T is linear, we have $T(x) + T(y) = T(x + y)$, and so $x + y \in S$. Now, let $c \in F$ and $x \in S$. Again, W_1 is a subspace so $cT(x) \in W_1$. Thus, T being linear implies that $cT(x) = T(cx)$. Hence, $cx \in S$. Thus, S is a subspace of V. ■

Exercise 2.1.21

Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U : V \to V$ by

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) \quad \text{and} \quad U(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

Proof. Let $(x_n), (y_n) \in V$ with $(x_n) = (a_1, a_2, \dots)$ and $(y_n) = (b_1, b_2, \dots)$. Let $c \in F$. Then we have

$$\begin{aligned} T(cx_n + y_n) &= (ca_2 + b_2, ca_3 + b_3, \dots) \\ &= (ca_2, ca_3, \dots) + (b_2, b_3, \dots) \\ &= c(a_2, a_3, \dots) + (b_2, b_3, \dots) \\ &= cT(x_n) + T(y_n). \end{aligned}$$

Hence, $T : V \to V$ is a linear map.

Now with $U : V \to V$ observe that

$$\begin{aligned} U(cx_n + y_n) &= (0, ca_1 + b_1, ca_2 + b_1, \dots) \\ &= (0, ca_1, ca_2, \dots) + (0, b_1, b_2, \dots) \\ &= c(0, a_1, a_2, \dots) + (0, b_1, b_2, \dots) \\ &= cU(x_n) + U(y_n). \end{aligned}$$

Hence, $U : V \to V$ is a linear map. ■

(b) Prove that T is surjective, but not injective.

Proof. Let $\{b_n\} \in V$. Define this sequence as

$$\{b_n\} = (b_1, b_2, \dots).$$

We can construct a sequence such that $b_i = a_{i+1}$ for $i \in \mathbb{N}$. Hence,

$$(b_1, b_2, \dots) = (a_2, a_3, \dots).$$

By definition of T, we get that

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) = (b_1, b_2, \dots).$$

Since $\{b_n\} \in V$, we get that T is surjective.

To see why T is not injective, suppose we take two sequences distinct $\{x_n\}, \{y_n\} \in V$ defined by

$$\{x_n\} = (a, 0, 0, \dots) \text{ and } \{y_n\} = (b, 0, 0, \dots)$$

respectively. But we have $T(\{x_n\}) = 0_n = T(\{y_n\})$ where 0_n is the sequence with all terms being zero. Hence, T cannot be injective. ■

(c) Prove that U is injective, but not surjective.

Proof. Let $\{a_n\}, \{b_n\} \in V$ defined by

$$\{a_n\} = (a_1, a_2, \dots) \text{ and } \{b_n\} = (b_1, b_2, \dots).$$

Then observe that

$$\begin{aligned} U(a_1, a_2, \dots) &= U(b_1, b_2, \dots) \\ (0, a_1, a_2, \dots) &= (0, b_1, b_2, \dots). \end{aligned}$$

Equating entries, we get that $a_i = b_i$ for all $i \in \mathbb{N}$ and so, $\{a_n\} = \{b_n\}$. Hence, U is injective.

Observe that $(x_1, 0, 0, \dots) \in V$ but there does not exists a sequence $\{x_n\} \in V$ such that $U(x_1, x_2, \dots) = (x_1, 0, 0)$. Hence, U is not surjective. ■

Exercise 2.1.22

Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$. Can you generalize this result for $T : F^n \to F$? State and prove an analogous result for $T : F^n \to F^m$.

Proof. Let $(x, y, z) \in \mathbb{R}^3$ arbitrary. Observe that

$$(x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z) = xe_1 + ye_2 + ze_3.$$

since $\beta = \{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$. Since T is linear, we know, by Theorem 2.2, that $\operatorname{span}(T(\beta)) = R(T)$. So, we have

$$\begin{aligned} T(x, y, z) &= T(xe_1 + ye_2 + ze_3) \\ &= T(e_1)x + T(e_2)y + T(e_3)z. \end{aligned}$$

Since $T : \mathbb{R}^3 \to \mathbb{R}$, we know that $T(e_1), T(e_2), T(e_3) \in \mathbb{R}$ are just scalars, so denote $T(e_1) = a, T(e_2) = b, T(e_3) = c$. Hence, we have

$$T(x, y, z) = ax + by + cz.$$

Now for the nth case, our basis β for F^n now contains n. So, we have

$$T(x_1, x_2, x_3, \dots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

for some scalars $a_1, a_2, \dots, a_n \in F$ using the same argument above. ■

Exercise 2.1.23

Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Describe geometrically the possibilities for the null space of T.

Proof. By definition of T described in Exercise 2.1.22, the null space is the described as the set

$$N(T) = \{ax + by + cz = 0 : a, b, c \in \mathbb{R} \text{ and } (x, y, z) \in \mathbb{R}^3\}.$$

This represents two vectors in $\mathbb{R}^3$ where we fix $(x, y, z) \in \mathbb{R}^3$ such that we find a $(a, b, c) \in \mathbb{R}^3$ such that the two vectors are perpendicular to each other. ■

Definition 2.1.1 (Projections). Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. A function $T : V \to V$ is called the **projection on W_1 along** W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$.

Exercise 2.1.24

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$. Include figures for each of the following parts.

(a) Find a formula for $T(a, b)$, where T represents the projection on the $y-$axis along the $x-$axis.

Solution. Since $(a, b) = (a, 0) + (0, b)$ and T represents the projection on the $y-$axis along the $x-$axis, we must have $T(a, b) = (0, b)$ by definition of projections. ■

(b) Find a formula for $T(a, b)$, where T represents the projection on the $y-$axis along the line $L = \{(s, s) : s \in \mathbb{R}\}$.

Solution. Observe that $(a, b) \in \mathbb{R}^2$ can be written in the following form:

$$(a, b) = (a, 0) + (0, b) = (a, a - a) + (0, b) = (0, b - a) + (a, a).$$

So, $T(a, b) = (0, b - a)$ by definition of projection. ■

Exercise 2.1.25

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$.

(a) If $T(a, b, c) = (a, b, 0)$, show that T is the projection on the $xy-$plane along the $z-$axis.

Solution. Observe that $(a, b, c) \in \mathbb{R}^3$ can be written in the following form:

$$(a, b, c) = \underbrace{(a, b, 0)}_{\in \mathbb{R}^2} + \underbrace{(0, 0, z)}_{\text{z-axis}}. \tag{1}$$

Since $T(a, b, c) = (a, b, 0)$, we know that T represents the projection on the $xy-$plane along the $z-$axis. ■

(b) Find a formula for $T(a, b, c)$, where T represents the projection on the $z-$axis along the $xy-$plane.

Solution. Let $(a, b, c) \in \mathbb{R}^3$. Then the projection on the $z-$axis along the $xy-$plane should be $T(a, b, c) = (0, 0, c)$ since $(a, b, c) = (a, b, 0) + (0, 0, c)$ where $(a, b, 0) \in \mathbb{R}^2$ and $(0, 0, c)$ is a point on the $z-$axis. ■

(c) If $T(a, b, c) = (a - c, b, 0)$, show that T is the projection on the $xy-$plane along the line $L = \{(a, 0, a) : a \in \mathbb{R}\}$.

Solution. Let $(a, b, c) \in \mathbb{R}^3$. Then we can write this vector in the following way:

$$\begin{aligned}(a, b, c) &= (a, b, 0) + (0, 0, c)\\ &= (a - c + c, b, 0) + (0, 0, c)\\ &= (a - c, b, 0) + (c, 0, c)\end{aligned}$$

where $(a - c, b, 0) \in \mathbb{R}^2$ and $(c, 0, c) \in L$. Since $T(a, b, c) = (a - c, b, 0)$, T must be the projection on the $xy-$plane along the line L. ■

Exercise 2.1.26

Using the notation in the definition above, assume that $T : V \to V$ is the projection on W_1 along W_2.

(a) Prove that T is linear and $W_1 = \{x \in V : T(x) = x\}$.

Proof. Let $x, y \in V$. We need to show that T is linear. Since T is the projection on W_1 along W_2, we must have $x = x_1 + x_2$ and $y = y_1 + y_2$ with $T(x) = x_1$ and $T(y) = y_1$ respectively. Observe that

$$\begin{aligned} cx + y &= c(x_1 + x_2) + (y_1 + y_2) \\ &= (cx_1 + y_1) + (cx_2 + y_2). \end{aligned}$$

Let $c \in F$. Since $T : V \to V$ is the projection on W_1 along W_2, we must have

$$T(cx + y) = cx_1 + y_1 = cT(x) + T(y).$$

Hence, T is linear.

Now, let's show that $W_1 = \{x \in V : T(x) = x\}$. Denote $S = \{x \in V : T(x) = x\}$. Let $x \in W_1$. Since T is the projection on W_1 along W_2 and $x = x + 0_V$, we have $T(x) = x$. Hence, $W_1 \subseteq S$. Conversely, let $x \in S$. Then $T(x) = x$. Since T is the projection on W_1 along W_2, this would mean that $x \in W_1$. Hence,

$$W_1 = \{x \in V : T(x) = x\}.$$

■

(b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

Proof. Let's show that $W_1 = R(T)$. Let $x \in W_1$. Since $W_1 \subseteq V$, we know that $x \in V$ where $x = x + 0_V$. Since T is the projection on W_1 along W_2, we have that $T(x) = x$. Hence, $x \in R(T)$. Conversely, $x \in R(T)$. Then for some $z \in V$, we have $T(x) = z$. Since T is the projection on W_1 along W_2, we have that $x = z + 0_V$ implies that $z \in W_1$. Thus, $x \in W_1$ and so,

$$W_1 = R(T).$$

Now, let's show that $W_2 = N(T)$. Let $x \in W_2$. Since T is a projection on W_1 along W_2, we have $x = 0_V + x$ implies $T(x) = 0_V$. This tell us that $x \in N(T)$. On the other hand, assume $x \in N(T)$. Then $T(x) = 0_V$. Since $x \in V$ and $W_1 + W_2 = V$, we have $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$. Since $T(x) = 0_V$, either $x_1 = 0$ or $x_2 = 0$. Assume $x_2 = 0$, then $x = x_1$ and $x \in W_1$ which is a contradiction. If $x_1 = 0$, then $x_2 = x$, so $T(x_2) = 0_V$. Hence, $x \in W_2$. Thus,

$$W_2 = N(T).$$

■

(c) Describe T if $W_1 = V$.

Solution. If $W_1 = V$, then we simply have $T(x) = x$ as described in part (a). Moreover, $N(T) = \{0\}$ since $V = W_1 + W_2$ implies $N(T) = W_2 = \{0\}$. Thus, T must be injective. ■

(d) Describe T if W_1 is the zero subspace.

Solution. If W_1 is the zero subspace, then $\dim(W_1) = 0$. This implies that $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2) = \dim(W_2)$ (note that $\dim(W_1 \cap W_2) = 0$). So, $\dim(V) = \dim(W_2)$ implies that $V = W_2$. Since T is a projection on W_1 along W_2, then $T(x) = 0_V$ for all $x \in V$. ■

Exercise 2.1.27

Suppose that W is a subspace of a finite-dimensional vector space V.

(a) Prove that there exists a subspace W' and a function $T : V \to V$ such that T is a projection on W along W'.

Proof. Since V is finite-dimensional and W is a subspace of V, we know by Exercise 1.6.34 that there exists a subspace W' such that $W_1 \oplus W_2 = V$. Note that any $x \in V$ can defined by $x = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in W'$. Define the function

$$T : V \to V \text{ by } T(x) = x_1.$$

Since $W + W' = V$ and $W \cap W' = \{0\}$, we get a unique representation of any $x \in V$. Thus, we have that T is well-defined and that T is a projection on W along W'. ■

Remark. I am not quite sure how to show that T is a projection here. I am not sure if showing that T is well-defined is enough to do it.

(b) Give an example of a subspace W of a vector space V such that there are two projections on W along two (distinct) subspaces.

Solution. ■

Definition 2.1.2 (Invariance). Let V be a vector space, and let $T : V \to V$ be linear. A subspace W of V is said to be T**-invariant** if $T(x) \in W$ for every $x \in W$, that is, $T(W) \subseteq W$. If W is $T-$invariant, we define the **restriction of** T **on** W to be the function $T_W : W \to W$ defined by $T_W(x) = T(x)$ for all $x \in W$.

Exercises 28-32 assume that W is a subspace of a vector space V and that $T : V \to V$ is linear. *Warning:* Do not assume that W is $T-$invariant or that T is a projection unless explicitly stated.

Exercise 2.1.28

Prove that the subspaces $\{0\}, V, R(T)$, and $N(T)$ are all $T-$invariant.

Proof. (a) Let $x \in \{0\}$. Since T is linear, we must have $x = 0_V$ such that $T(0_V) = 0_V$. Hence, $\{0\}$ is $T-$invariant.

(b) Since V is a subspace of itself, and $T(x) \in V$ for all $x \in V$, we have that V is $T-$invariant.

(c) Let $x \in R(T)$. Then there exists $w \in V$ such that $T(x) = w \in R(T)$. Hence, $R(T)$ is $T-$invariant.

(d) Let $x \in N(T)$. Then by definition of $N(T)$, we have $T(x) = 0_V \in N(T)$. Hence, $N(T)$ is $T-$invariant.

■

Exercise 2.1.29

If W is $T-$invariant, prove that T_W is linear.

Proof. Since W is $T-$invariant, we have a restriction of T on W such that $T_W : W \to W$ is defined by $T_W(x) = T(x)$ for all $x \in W$. Let $cx + y \in W$. Since T is linear, we know that

$$\begin{aligned} T_W(cx+y) &= T(cx+y) \\ &= cT(x) + T(y) \\ &= cT_W(x) + T_W(y). \end{aligned}$$

Hence, T_W is linear. ■

Exercise 2.1.30

Suppose that T is the projection on W along some subspace W'. Prove that W is $T-$invariant and that $T_W = I_W$.

Proof. Let $x \in W$. Since T is the projection on W along some subspace W', we must have $T(x) = x$. Since $x \in W$, this must imply that $T(x) \in W$ as well and so, we have that $T-$invariant. In fact, the linearity of T and W being $T-$invariant implies that T is linear (by exercise 29) and that

$$T_W(x) = T(x) = x = I_W(x)$$

for any arbitrary $x \in W$. Hence, we also have that $T_W = I_W$. ■

Exercise 2.1.31

Suppose that $V = R(T) \oplus W$ and W is $T-$invariant.

(a) Prove that $W \subseteq N(T)$.

Proof. ■

(b) Show that if V is finite-dimensional, then $W = N(T)$.

Proof. ■

(c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

Proof. ■