

Math 234A: Homework 2

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Problem 1

Definition (Cauchy Sequence). A sequence (z_n) is called a **Cauchy Sequence** if for all $\varepsilon > 0$, there exists a positive integer N such that for all $m, n \geq N$,

$$|z_m - z_n| < \varepsilon.$$

Show that a sequence (z_n) in \mathbb{C} is convergent if and only if it is Cauchy.

Proof. (\implies) Let $\varepsilon > 0$. If (z_n) is Cauchy, then we can find an $N \in \mathbb{N}$ such that for any $n, m \geq N$ such that

$$|z_n - z_m| < \varepsilon.$$

Then observe that for any $n, m \geq N$, we have

$$|\Re(z_n) - \Re(z_m)| = |\Re(z_n - z_m)| \leq |z_n - z_m| < \varepsilon$$

and similarly,

$$|\Im(z_n) - \Im(z_m)| = |\Im(z_n - z_m)| \leq |z_n - z_m| < \varepsilon.$$

Therefore, the real and imaginary parts of (z_n) are Cauchy. Since $\Re(z_n)$ and $\Im(z_n)$ are real-valued sequences, they must converge as well. But this holds if and only if (z_n) converges.

(\impliedby) Let $\varepsilon > 0$. Suppose (z_n) converges. This holds if and only if $\Re(z_n)$ and $\Im(z_n)$ converge. But this holds if and only if $\Re(z_n)$ and $\Im(z_n)$ are both Cauchy since they are real-valued sequences. Thus, we can find an $N \in \mathbb{N}$ such that for any $n, m \geq N$, we see that

$$|\Re(z_n) - \Re(z_m)| < \frac{\varepsilon}{2} \tag{1}$$

and

$$|\Im(z_n) - \Im(z_m)| < \frac{\varepsilon}{2}. \tag{2}$$

Using (1) and (2), we see that

$$\begin{aligned} |z_n - z_m| &= |\Re(z_n) + i\Im(z_n) - (\Re(z_m) + i\Im(z_m))| \\ &= |(\Re(z_n) - \Re(z_m)) + i(\Im(z_n) - \Im(z_m))| \\ &\leq |\Re(z_n) - \Re(z_m)| + |\Im(z_n) - \Im(z_m)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that (z_n) is Cauchy. ■

Problem 2

- (i) Let $z_0 = x_0 + i\zeta_0 \in \mathbb{C}$: Define a sequence (z_n) by $z_{n+1} = \frac{1}{2}(z_n + \frac{1}{z_n})$, $n \geq 1$, and $z_1 = \frac{1}{2}(z_0 + \frac{1}{z_0})$. Show that

$$\lim_{n \rightarrow \infty} z_n = \begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0 \end{cases}$$

Proof. Let $z_0 = x_0 + i\zeta_0 \in \mathbb{C}$. Define a sequence (z_n) by

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right), \quad n \geq 1$$

and

$$z_1 = \frac{1}{2}\left(z_0 + \frac{1}{z_0}\right).$$

Our goal is to show that

$$\lim_{n \rightarrow \infty} z_n = \begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0. \end{cases}$$

Suppose $x_0 > 0$. First, we need to show that $z_n \neq 1$ for all $n \in \mathbb{N}$. We proceed via induction. Let our base case be $n = 1$. Then we want to show that $\Re(z_1) > 0$. Using the definition of z_0 , we see that

$$\begin{aligned} z_1 &= \frac{z_0^2 + 1}{2z_0} = \frac{1}{2(x_0^2 + \zeta_0^2)} \cdot [(x_0 + i\zeta_0)^2(x_0 - i\zeta_0) + (x_0 - i\zeta_0)] \\ &= \frac{x_0^3 + x_0\zeta_0^2 + x_0}{2(x_0^2 + \zeta_0^2)} + i \frac{\zeta_0 x_0^3 - \zeta_0}{2(x_0^2 + \zeta_0^2)}. \end{aligned}$$

Since $z_1 = \Re(z_1) + i\Im(z_1)$ and $x_0 > 0$, we see that

$$\Re(z_1) = \frac{x_0^3 + x_0\zeta_0^2 + x_0}{2(x_0^2 + \zeta_0^2)} > 0.$$

Now, suppose that the result holds for the n th case. We will show that the result holds for the $n + 1$ case. Set $\alpha = \Re(z_n)$ and $\beta = \Im(z_n)$. Then by definition of z_{n+1} , we see that

$$\begin{aligned} z_{n+1} &= \frac{z_n^2 + 1}{2z_n} = \frac{1}{2(\alpha^2 + \beta^2)} \cdot [(\alpha + i\beta)(\alpha - i\beta) + (\alpha - i\beta)] \\ &= \frac{\alpha^3 + \alpha\beta^3 + \alpha}{2(\alpha^2 + \beta^2)} + i \frac{\beta\alpha^3 - \beta}{2(\alpha^2 + \beta^2)}. \end{aligned}$$

By equating the real and imaginary parts of z_{n+1} with the right-hand side of the equation above and using the inductive hypothesis that $\alpha = \Re(z_n) > 0$, we see that

$$\Re(z_{n+1}) = \frac{\alpha^3 + \alpha\beta^3 + \alpha}{2(\alpha^2 + \beta^2)} > 0.$$

Thus, we see that $\Re(z_n) > 0$ for all $n \in \mathbb{N}$. Consequently, this tells us that $\Re(z_n) + 1 > 1$ which further tells us that $|z_{n+1} + 1| > 1$. Now, define a new sequence (w_n) such that

$$w_{n+1} = \frac{z_{n+1} - 1}{z_{n+1} + 1}$$

and note that

$$z_{n+1} = \frac{1 + z_n^2}{2z_n}.$$

Our next step is to show that $w_{n+1} = w_n^2$. Thus, observe that

$$\begin{aligned} w_{n+1} &= \frac{\frac{1+z_n^2}{2z_n} - 1}{\frac{1+z_n^2}{2z_n} + 1} = \frac{\frac{z_n^2 - 2z_n + 1}{2z_n}}{\frac{z_n^2 + 2z_n + 1}{2z_n}} = \frac{z_n^2 - 2z_n + 1}{z_n^2 + 2z_n + 1} \\ &= \frac{(z_n - 1)^2}{(z_n + 1)^2} \\ &= \left(\frac{z_n - 1}{z_n + 1} \right)^2 \\ &= w_n^2. \end{aligned}$$

Now, we want to show, as a consequence of the result above, that (w_n) is a decreasing sequence; that is, we want to show that $0 < |w_{n+1}| < |w_n| < 1$. We will induct on $n \in \mathbb{N}$ to show this. Thus, let $n = 1$ be our base case. Observe that

$$\begin{aligned} 0 < |w_{1+1}| = |w_2| = |w_1^2| &= \left| \left(\frac{z_1 - 1}{z_1 + 1} \right)^2 \right| \\ &< \left| \frac{z_1 - 1}{z_1 + 1} \right| \\ &< 1. \end{aligned}$$

Let us now assume that this result holds for the n th case. Thus, we see that

$$0 < |w_{n+2}| = |w_{n+1}^2| = \left| \left(\frac{z_{n+1} - 1}{z_{n+1} + 1} \right)^2 \right| < |w_{n+1}| < |w_n| < 1.$$

Thus, we see that $0 < |w_{n+1}| < |w_n| < 1$ holds for all $n \in \mathbb{N}$. Clearly, we see that (w_n) is bounded and decreasing for all n . Hence, (w_n) must converge, and in this case, we claim that it converges to 0. Observe that

$$|w_n| = e^{\ln |w_n|}.$$

Now, since $0 < |w_n| < 1$, we must have $\ln |w_n| < 0$. By applying the limit as $n \rightarrow \infty$, we can see that

$$\lim_{n \rightarrow \infty} |w_n| = \lim_{n \rightarrow \infty} e^{\ln |w_n|} = 0.$$

Since $|z_{n+1} + 1| > 1$ for all $n \in \mathbb{N}$, we must have that

$$|z_{n+1} - 1| \rightarrow 0$$

as $n \rightarrow \infty$. Thus, we conclude that $\lim_{n \rightarrow \infty} z_n = 1$. Now, assume that $x_0 < 0$. Consider the sequence $(-z_n)$. Then observe that

$$\lim_{n \rightarrow \infty} -z_n = -\lim_{n \rightarrow \infty} z_n = -1.$$

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(ii) Discuss convergence and divergence of

$$z_n = 1 + i \frac{(-1)^n}{n^2} \text{ for } n = 1, 2, \dots$$

Let $\Phi_n = \text{Arg}(z_n)$. Show that $(\Phi_n) \rightarrow 0$.

Proof. Observe that $\Re(z_n) \rightarrow 1$ and $\Im(z_n) = \frac{(-1)^n}{n^2} \rightarrow 0$ as $n \rightarrow \infty$. Since the real part and imaginary part of z_n converge, we know that (z_n) must also converge. In fact, it converges to the following value

$$z_n \rightarrow 1 + 0i = 1.$$

By definition, we know that

$$\Phi_n = \text{Arg}(z_n) = \tan^{-1} \left(\frac{\Im(z_n)}{\Re(z_n)} \right) \rightarrow \tan^{-1}(0) = 0 \text{ as } n \rightarrow \infty.$$

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(iii) Assume that $0 < \gamma < 1$. Show that

$$\begin{aligned} \sum_{n=1}^{\infty} \gamma^n \cos n\theta &= \frac{\gamma \cos \theta - \gamma^2}{1 - 2\gamma \cos \theta + \gamma^2} \\ \sum_{n=1}^{\infty} \gamma^n \sin n\theta &= \frac{\gamma \sin \theta}{1 - 2\gamma \cos \theta + \gamma^2}. \end{aligned}$$

Proof. Set $z = \gamma e^{i\theta}$. Consider the series

$$\sum_{n=1}^{\infty} z^n$$

which converges absolutely for $|z| < 1$ such that

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}. \quad (1)$$

By rearranging terms and using the polar representation of z , we see that

$$\begin{aligned} \sum_{n=1}^{\infty} z^n &= \sum_{n=1}^{\infty} (\gamma e^{i\theta})^n \\ &= \sum_{n=1}^{\infty} [\gamma(\cos \theta + i \sin \theta)]^n \\ &= \sum_{n=1}^{\infty} \gamma^n (\cos \theta + i \sin \theta)^n \\ &= \sum_{n=1}^{\infty} \gamma^n (\cos n\theta + i \sin n\theta) && \text{(De Moivre's Theorem)} \\ &= \sum_{n=1}^{\infty} \gamma^n \cos n\theta + i \sum_{n=1}^{\infty} \gamma^n \sin n\theta. \end{aligned}$$

Thus, we have

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} \gamma^n \cos n\theta + i \sum_{n=1}^{\infty} \gamma^n \sin n\theta. \quad (2)$$

Working with the sum in (1) and using the polar representation of z , we see that

$$\begin{aligned} \frac{z}{1-z} &= \frac{\gamma e^{i\theta}}{1 - \gamma e^{i\theta}} = \frac{(\gamma \cos \theta + i\gamma \sin \theta)}{(1 - \gamma \cos \theta) - i\gamma \sin \theta} \\ &= \frac{(\gamma \cos \theta + i\gamma \sin \theta)}{(1 - \gamma \cos \theta) - i\gamma \sin \theta} \cdot \frac{(1 - \gamma \cos \theta) + i\gamma \sin \theta}{(1 - \gamma \cos \theta) + i\gamma \sin \theta} \\ &= \frac{\gamma \cos \theta - \gamma^2(\sin^2 \theta + \cos^2 \theta) + i\gamma \sin \theta}{1 - 2\gamma \cos \theta + \gamma^2(\sin^2 \theta + \cos^2 \theta)} \\ &= \frac{\gamma \cos \theta - \gamma^2}{1 - 2\gamma \cos \theta + \gamma^2} + i \frac{\gamma \sin \theta}{1 - 2\gamma \cos \theta + \gamma^2}. \end{aligned}$$

Thus, we see that

$$\frac{z}{1-z} = \frac{\gamma \cos \theta - \gamma^2}{1 - 2\gamma \cos \theta + \gamma^2} + i \frac{\gamma \sin \theta}{1 - 2\gamma \cos \theta + \gamma^2}. \quad (3)$$

Equating the real and imaginary parts of (2) and (3), we see that

$$\begin{aligned} \Re\left(\sum_{n=1}^{\infty} z^n\right) &= \sum_{n=1}^{\infty} \gamma^n \cos n\theta = \frac{\gamma \cos \theta - \gamma^2}{1 - 2\gamma \cos \theta + \gamma^2}, \\ \Im\left(\sum_{n=1}^{\infty} z^n\right) &= \sum_{n=1}^{\infty} \gamma^n \sin n\theta = \frac{\gamma \sin \theta}{1 - 2\gamma \cos \theta + \gamma^2}. \end{aligned}$$

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Problem 3

Let (z_n) be a sequence of non-zero complex numbers. Suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Show that $\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = L$.

Proof. Let $\varepsilon > 0$. Our goal is to show that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$\left| |z_n|^{\frac{1}{n}} - L \right| < \varepsilon.$$

Indeed, we can use the fact that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L,$$

to find an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$L - \varepsilon < \left| \frac{z_{n+1}}{z_n} \right| < L + \varepsilon.$$

Now, observe that

$$\begin{aligned} |z_n|^{\frac{1}{n}} &= \left| \frac{z_n}{z_{n-1}} \cdot \frac{z_{n-1}}{z_{n-2}} \cdot \frac{z_{n-2}}{z_{n-3}} \cdots \frac{z_{N+1}}{z_N} \right|^{\frac{1}{n}} \\ &< \left[(L + \varepsilon)^{n-N} \right]^{\frac{1}{n}} |z_N|^{\frac{1}{n}} \\ &= \left[(L + \varepsilon) \right]^{1 - \frac{N}{n}} |z_N|^{\frac{1}{n}} \end{aligned}$$

and similarly, we see that

$$\begin{aligned} |z_n|^{\frac{1}{n}} &= \left| \frac{z_n}{z_{n-1}} \cdot \frac{z_{n-1}}{z_{n-2}} \cdot \frac{z_{n-2}}{z_{n-3}} \cdots \frac{z_{N+1}}{z_N} \right|^{\frac{1}{n}} \\ &> \left[(L - \varepsilon)^{n-N} \right]^{\frac{1}{n}} |z_N|^{\frac{1}{n}} \\ &= \left[(L - \varepsilon) \right]^{1 - \frac{N}{n}} |z_N|^{\frac{1}{n}}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ for both of these inequalities above, we can see that

$$L - \varepsilon < |z_n|^{\frac{1}{n}} < L + \varepsilon \iff \left| |z_n|^{\frac{1}{n}} - L \right| < \varepsilon$$

which is our desired result.

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Problem 4

Determine all $z \in \mathbb{C}$ such that

(i) $\exp(z) = -2$

Solution. Note that $-2 = -2 + 0i$ which means that

$$\text{Arg}(-2) = \tan^{-1}(-2) = \pi.$$

Thus, we see that

$$\begin{aligned} z = \log(-2) &= \{\ln|-2| + i \text{Arg}(-2) + 2\pi ik : k \in \mathbb{Z}\} \\ &= \{\ln 2 + i(\pi + 2\pi k) : k \in \mathbb{Z}\}. \end{aligned}$$

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(ii) $\exp(z) = -i$

Solution. Note that $-i = 0 - i$ which means that

$$\text{Arg}(-i) = \frac{3\pi}{2}.$$

Thus, we see that

$$\begin{aligned} z = \log(-i) &= \{\ln|-i| + i \text{Arg}(-i) + 2\pi ik : k \in \mathbb{Z}\} \\ &= \left\{ \ln(1) + i \left[\frac{3\pi}{2} + 2\pi k \right] : k \in \mathbb{Z} \right\} \\ &= \left\{ i \frac{3\pi}{2} + 2\pi ik : k \in \mathbb{Z} \right\}. \end{aligned}$$

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(iii) $\sin z = 1 - i$

Solution. Using the fact that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

we see that

$$\begin{aligned} \frac{e^{iz} - e^{-iz}}{2i} &= 1 - i \implies e^{iz} - e^{-iz} = 2i + 2 \\ &\implies (e^{iz})^2 - 2(i+1)e^{iz} = 1 \\ &\implies w^2 - 2(i+1)w = 1 \quad (w = e^{iz}) \\ &\implies w^2 - 2(1+i)w + (1+i)^2 = 1 + (1+i)^2 \\ &\implies (w - (1+i))^2 = 1 + (1+i)^2 \\ &\implies (w - (1+i))^2 = 1 + 2i \\ &\implies w_{1,2} = \pm\sqrt{1+2i} + (1+i). \end{aligned}$$

Now, we will convert the first term on the right-hand side in terms of its respective polar representation. Thus, we have

$$\pm\sqrt{1+2i} = \pm 5^{\frac{1}{4}} e^{i \frac{\tan^{-1}(2)}{2}} = \pm 5^{1/4} \left(\cos\left(\frac{\tan^{-1}(2)}{2}\right) + i \sin\left(\frac{\tan^{-1}(2)}{2}\right) \right).$$

Now, set

$$\alpha = \left(5^{1/4} \cos\left(\frac{\tan^{-1}(2)}{2}\right) + 1\right) + i\left(5^{1/4} \sin\left(\frac{\tan^{-1}(2)}{2}\right) + 1\right).$$

and

$$\zeta = \left(1 - 5^{1/4} \cos\left(\frac{\tan^{-1}(2)}{2}\right)\right) + i\left(1 - 5^{1/4} \sin\left(\frac{\tan^{-1}(2)}{2}\right)\right).$$

$$\begin{aligned} e^{iz} = \alpha &\implies z = \frac{1}{i} \log(\alpha) \\ &\implies z = \{-i[\ln|\alpha| + i \operatorname{Arg}(\alpha) + 2\pi ik] : k \in \mathbb{Z}\} \\ &\implies z = \{(\operatorname{Arg}(\alpha) + 2\pi k) - i \ln|\alpha| : k \in \mathbb{Z}\} \end{aligned}$$

and similarly

$$e^{iz} = \zeta \implies z = \{(\operatorname{Arg}(\zeta) + 2\pi k) - i \ln|\zeta| : k \in \mathbb{Z}\}.$$

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(iv) $\cos z = 3 + 4i$

Solution. Using the fact that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

we have

$$\begin{aligned} e^{iz} + e^{-iz} = 2(3 + 4i) &\implies (e^{iz})^2 + 1 = (6 + 8i)e^{iz} \\ &\implies w^2 + 1 = (6 + 8i)w \\ &\implies w^2 - (6 + 8i)w = -1 \quad (w = e^{iz}) \\ &\implies w^2 - (6 + 8i)w + (3 + 4i)^2 = (3 + 4i)^2 - 1 \\ &\implies (w - (3 + 4i))^2 = -1 + (3 + 4i)^2 \\ &\implies w_{1,2} = \pm \sqrt{8(3i - 1)} + (3 + 4i) \end{aligned}$$

Now, let us convert the first term on the right-hand side into its polar form which is

$$\pm \sqrt{8(3i - 1)} = \pm 8^{1/2} \cdot 10^{1/4} e^{i \frac{\tan^{-1}(-3)}{2}} = \pm 8^{1/2} \cdot 10^{1/4} \left(\cos\left(\frac{\tan^{-1}(-3)}{2}\right) + i \sin\left(\frac{\tan^{-1}(-3)}{2}\right) \right)$$

and the second term as

$$3 + 4i = 5e^{i \tan^{-1}(4/3)} = 5 \left[\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + i \sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right) \right].$$

Let us set

$$\varphi = \tan^{-1}\left(\frac{4}{3}\right) \text{ and } \theta = \frac{\tan^{-1}(-3)}{2}.$$

Furthermore, we shall set

$$\beta = 5 \cos \varphi + 8^{1/2} \cdot 10^{1/4} \cos \theta + i \left(5 \sin \varphi + 8^{1/2} \cdot 10^{1/4} \sin \theta \right)$$

and

$$\omega = 5 \cos \varphi - 8^{1/2} \cdot 10^{1/4} \cos \theta + i \left(5 \sin \varphi - 8^{1/2} \cdot 10^{1/4} \sin \theta \right)$$

Now, we have

$$\begin{aligned} e^{iz} = \beta &\implies z = \frac{1}{i} \log(\beta) \\ &\implies z = \{-i[\ln |\beta| + i \operatorname{Arg}(\beta) + 2\pi ik] : k \in \mathbb{Z}\} \\ &\implies z = \{(\operatorname{Arg}(\beta) + 2\pi k) - i \ln |\beta| : k \in \mathbb{Z}\}. \end{aligned}$$

and

$$\begin{aligned} e^{iz} = \omega &\implies z = \frac{1}{i} \log(\omega) \\ &\implies z = \{-i[\ln |\omega| + i \operatorname{Arg}(\omega) + 2\pi ik] : k \in \mathbb{Z}\} \\ &\implies z = \{(\operatorname{Arg}(\omega) + 2\pi k) - i \ln |\omega| : k \in \mathbb{Z}\}. \end{aligned}$$

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Problem 5

Let (z_n) and (w_n) be two sequences of complex numbers. Assume that $z_n = w_n - w_{n+1}$. Show that $\sum_{n=1}^{\infty} z_n$ converges if and only if (w_n) converges. If $\sum_{n=1}^{\infty} z_n$ converges, show that

$$\sum_{n=0}^{\infty} z_n = w_1 - \lim_{n \rightarrow \infty} w_{n+1}.$$

Use this to compute

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)}.$$

Proof. (\Rightarrow) Let (z_n) and w_n be two sequences of complex numbers. Our goal is to show that (w_n) converges. Assume that $z_n = w_n - w_{n+1}$. Since $\sum_{n=1}^{\infty} z_n$ converges, we know that (z_n) must converge to 0. But we must also have that

$$|z_n| = |w_n - w_{n+1}| \rightarrow 0$$

as $n \rightarrow \infty$. Hence, (w_n) must be a Cauchy sequence and so, it must converge by problem 1.

(\Leftarrow) Suppose that (w_n) converges. Let

$$s_n = \sum_{k=1}^n z_k.$$

Our goal is to show that $(s_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $z_n = w_n - w_{n+1}$, we can see that

$$\begin{aligned} |s_n - s_m| &= \left| \sum_{k=m+1}^n z_k \right| = \left| \sum_{k=m+1}^n w_k - w_{k+1} \right| \\ &= |(w_{m+1} - w_{m+2}) + (w_{m+2} - w_{m+3}) + \cdots + (w_n - w_{n+1})| \\ &= |w_{m+1} - w_{n+1}|. \end{aligned}$$

Since (w_n) converges, we must also have that (w_n) is Cauchy. Thus, we see that

$$|s_n - s_m| = |w_{m+1} - w_{n+1}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, (s_n) must converge and hence the infinite series $\sum_{n=1}^{\infty} z_n$ converges. Using this result, we

can now compute

$$\begin{aligned} s_n &= \sum_{k=1}^n z_k = \sum_{k=1}^n (w_k - w_{k+1}) \\ &= (w_1 - w_2) + (w_2 - w_3) + \cdots + (w_n - w_{n+1}) \\ &= w_1 - w_{n+1} \end{aligned}$$

whereby taking the limit gives us

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (w_1 - w_{n+1}) = w_1 - \lim_{n \rightarrow \infty} w_{n+1}.$$

Now, let us compute the following infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)}.$$

First, we have to convert the fraction

$$\frac{1}{(n+i)((n+1)+i)}$$

into its partial fraction decomposition; that is, we want to find coefficients A and B such that

$$\frac{1}{(n+i)((n+1)+i)} = \frac{A}{n+i} + \frac{B}{n+(1+i)}.$$

Using the heaviside method, we know that

$$A = 1 \text{ and } B = -1.$$

Thus, we see that

$$\sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)} = \sum_{n=1}^{\infty} \left[\frac{1}{n+i} - \frac{1}{n+(1+i)} \right].$$

Using the formula we have just proved, we can see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(n+i)((n+1)+i)} &= \frac{1}{(1+i)} - \lim_{n \rightarrow \infty} \frac{1}{n+(1+i)} \\ &= \frac{1}{1+i} - 0 \\ &= \frac{1}{1+i} \\ &= \frac{1}{2}(1-i). \end{aligned}$$

■

Problem 6

For $z, w \in \mathbb{C}$. Show that

$$\text{Log}(zw) = \text{Log}(z) + \text{Log}(w) + 2\pi i k$$

where

$$k = \begin{cases} 0 & \text{if } -\pi < \text{Arg}(z) + \text{Arg}(w) \leq \pi \\ 1 & \text{if } -2\pi < \text{Arg}(z) + \text{Arg}(w) \leq -\pi \\ -1 & \text{if } \pi < \text{Arg } z + \text{Arg } w \leq 2\pi. \end{cases}$$

Proof. Let $z, w \in \mathbb{C}$. We would like to consider three cases:

- (i) $-\pi < \text{Arg}(z) + \text{Arg}(w) \leq \pi$
- (ii) $-2\pi < \text{Arg}(z) + \text{Arg}(w) \leq -\pi$.
- (iii) $\pi < \text{Arg } z + \text{Arg } w \leq 2\pi$.

Starting with case (i), we see that

$$\text{Log}(z) = \ln |z| + \text{Arg}(z)$$

and that $k = 0$. Thus, we see that

$$\begin{aligned} \text{Log}(zw) &= \ln |zw| + \text{Arg}(zw) \\ &= \ln |z||w| + (\text{Arg}(z) + \text{Arg}(w)) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w)) \\ &= \left(\ln(z) + \text{Arg}(z) \right) + \left(\ln(w) + \text{Arg}(w) \right) \\ &= \text{Log}(z) + \text{Log}(w). \end{aligned}$$

With case (ii), we have $k = 1$ so, we have

$$\begin{aligned} \text{Log}(zw) &= \ln |zw| + \text{Arg}(zw) \\ &= \ln |z||w| + (\text{Arg}(z) + \text{Arg}(w) + 2\pi i) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w) + 2\pi i) \\ &= \left(\ln(z) + \text{Arg}(z) \right) + \left(\ln(w) + \text{Arg}(w) \right) + 2\pi i \\ &= \text{Log}(z) + \text{Log}(w) + 2\pi i k. \end{aligned}$$

Lastly, with case (iii), we have $k = -1$ which implies that

$$\begin{aligned} \text{Log}(zw) &= \ln |zw| + \text{Arg}(zw) \\ &= \ln |z||w| + (\text{Arg}(z) + \text{Arg}(w) - 2\pi i) \\ &= [\ln(z) + \ln(w)] + (\text{Arg}(z) + \text{Arg}(w) - 2\pi i) \\ &= \left(\ln(z) + \text{Arg}(z) \right) + \left(\ln(w) + \text{Arg}(w) \right) - 2\pi i \\ &= \text{Log}(z) + \text{Log}(w) - 2\pi i. \end{aligned}$$

■

Problem 7 (Hyperbolic Functions)

For $z \in \mathbb{C}$, define

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}$$

and

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}.$$

Show that

- (i) $\sinh(z) = -i \sin(iz)$, $\cosh(z) = \cos(iz)$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Using the formulas above, we can see that

$$\begin{aligned} -i \sin(iz) &= -i \left[\frac{\exp(i^2 z) - \exp(-i^2 z)}{2i} \right] \\ &= \frac{\exp(z) - \exp(-z)}{2} \\ &= \sinh(z) \end{aligned}$$

and

$$\begin{aligned} \cos(iz) &= \frac{\exp(i^2 z) + \exp(-i^2 z)}{2} \\ &= \frac{\exp(-z) + \exp(z)}{2} \\ &= \cosh(z). \end{aligned}$$

■

(ii) $\sinh(z + w) = \sin(z) \cosh(w) + \cosh(z) \sinh(w)$.

Proof. We will show the formulas above by using the sum formulas for cosine and sine. By part (i), we see that

$$\begin{aligned} \sinh(z + w) &= -i \sin(i(z + w)) \\ &= -i \sin(iz + iw) \\ &= -i \left[\sin(iz) \cos(iw) + \sin(iw) \cos(iz) \right] \\ &= (-i \sin(iz)) \cos(iw) + (-i \sin(iw)) \cos(iz) \\ &= \sinh(z) \cosh(w) + \sinh(w) \cosh(z) \end{aligned}$$

and

$$\begin{aligned} \cosh(z + w) &= \cos(i(z + w)) \\ &= \cos(iz + iw) \\ &= \cos(iz) \cos(iw) - \sin(iz) \sin(iw) \\ &= \cosh(z) \cosh(w) + i^2 \sin(iz) \sin(iw) \\ &= \cosh(z) \cosh(w) + (-i \sin(iz))(-i \sin(iw)) \\ &= \cosh(z) \cosh(w) + \sinh(z) \sinh(w). \end{aligned}$$

■

(iii) $\cosh^2(z) - \sinh^2(z) = 1$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Using part (i), we see that

$$\begin{aligned} \cosh^2(z) - \sinh^2(z) &= \cos^2(iz) - (-\sin^2(iz)) \\ &= \cos^2(iz) + \sin^2(iz) \\ &= 1. \end{aligned}$$

■

(iv) $\cosh(z + 2\pi i) = \cosh(z)$ and $\sinh(z + 2\pi i) = \sinh(z)$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. Observe that

$$\sinh(2\pi i) = -i \sin(2\pi i^2) = -i \sin(-2\pi) = \sin(2\pi) = 0$$

and

$$\cosh(2\pi i) = \cos(2\pi i^2) = \cos(-2\pi) = \cos(2\pi) = 1.$$

Using part (ii), we can see that

$$\begin{aligned}\cosh(z + 2\pi i) &= \cosh(z) \cosh(2\pi i) + \sinh(z) \sinh(2\pi i) \\ &= \cosh(z)\end{aligned}$$

and

$$\begin{aligned}\sinh(z + 2\pi i) &= \sinh(z) \cosh(2\pi i) + \cosh(z) \sinh(2\pi i) \\ &= \sinh(z).\end{aligned}$$

■

(v) $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

Proof. By using the series representation of cosine and sine, we can see that

$$\begin{aligned}\cosh(z) &= \cos(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (iz)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (-1)^n z^{2n} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\end{aligned}$$

and

$$\begin{aligned}\sinh(z) &= -i \sin(iz) = -i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (iz)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-i)(-1)^n}{(2n+1)!} i^{2n+1} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-i)(-1)^n}{(2n+1)!} (-1)^n i z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

■

Problem 8

(a) Find all possible $z \in \mathbb{C}$ so that

$$\sum_{n=1}^{\infty} n|z|^n$$

converges. Use it to discuss the convergence of the series $\sum_{n=1}^{\infty} nz^n$.

Solution. Using the ratio test, denote $a_n = n|z|^n$. Then observe that

$$\begin{aligned}\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)|z|^{n+1}}{n|z|^n} \right| = \frac{n+1}{n} \cdot |z| \\ &= \left(1 + \frac{1}{n} \right) |z|.\end{aligned}$$

If we take the limit as $n \rightarrow \infty$, we see that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) |z| = |z|.$$

Note that by the ratio test, the series

$$\sum_{n=1}^{\infty} n|z|^n$$

converges if $|z| < 1$. Thus, the series

$$\sum_{n=1}^{\infty} nz^n$$

converges for all $|z| < 1$. ■

1. Show that the series $\sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$ converges absolutely for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$. We proceed via the ratio test to show that

$$\sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

converges absolutely. Thus, observe that

$$r_n = \frac{z^{4n+1}}{(2n)!}.$$

Hence, we have

$$\begin{aligned} \left| \frac{r_{n+1}}{r_n} \right| &= \left| \frac{z^{4n+5}}{(2n+2)!} \cdot \frac{(2n)!}{z^{4n+1}} \right| \\ &= \left| \frac{z^4}{(2n+2)(2n+1)} \right| \\ &= \frac{|z|^4}{(2n+2)(2n+1)}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we see that for any fixed $z \in \mathbb{C}$ that

$$\lim_{n \rightarrow \infty} \left| \frac{r_{n+1}}{r_n} \right| = \lim_{n \rightarrow \infty} \frac{|z|^4}{(2n+2)(2n+1)} = 0 < 1.$$

Thus, we see that the infinite series

$$\sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

converges. Now, observe that

$$\sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} = z \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = z \cosh(z^2)$$

and so we can find the sum of this infinite series granted that we pick a $z \in \mathbb{C}$. ■