Definitions and Theorems

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Operations on $\mathbb{C}$

Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication $\cdot$, in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

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Constructing a Solution for $x^2 + 1 = 0$

Define $i = (0, 1)$ as our imaginary number in $\mathbb{C}$ and let $i^2 = (-1, 0)$.

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Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary of** z, respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

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Complex Conjugate

Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\overline{z} = \alpha - i\beta$.

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Modulus of Complex Number

Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\overline{z}}$.

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Metric Space

For $z, w \in \mathbb{C}$, we call $\mathbb{C}$ a metric space if there exists a function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ that satisfies the following properties:

(i) For any $z, w \in \mathbb{C}$, we have $d(z, w) \geq 0$.

(ii) For any $z, w \in \mathbb{C}$, $d(z, w) = 0$ if and only if $z = w$.

(ii) For any $z, w \in \mathbb{C}$, we have $d(z, w) = d(w, z)$.

(iii) For any $z, w, u \in \mathbb{C}$, we have
$$d(z, w) \leq d(z, u) + d(u, w).$$

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Polar Representation of Complex Numbers

Let $(\alpha, \beta) \in \mathbb{R}^2$. The polar representation of (α, β) is

$$(\alpha, \beta) = \gamma(\cos \varphi, \sin \varphi)$$

with $\tan \varphi = \frac{\beta}{\alpha}$. Note that if $\psi = 2\pi + \varphi$, then

$$(\alpha, \beta) = \gamma(\cos \psi, \sin \psi)$$

where γ is uniquely defined and φ is defined up to the addition of a multiple of 2π.

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Argument and Principle Argument

Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos \varphi + i \sin \varphi)$ be a polar representation of z. Then φ is called **an argument of** z. If $-\pi < \varphi \leq \pi$, then φ is called **the principal argument of** z and it is denoted by $\operatorname{Arg}(z)$. **Click here for the definition**

Convergence of Sequences

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\{z_n\}$ converges to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon$$

for all $n \geq N_\varepsilon$.
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Convergence of Series

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Define

$$s_k = \sum_{k=1}^{n} z_k = z_1 + \cdots + z_n$$

where (s_n) is called the **sequence of partial sums** of $\{z_n\}$. If $s_n \to s$, then we say that the series $\sum_{n=1}^{\infty} z_n$ converges and write $\sum_{n=1}^{\infty} z_n = s$.
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Absolute Convergence of Infinite Series

Let (z_n) be a sequence of complex numbers. We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the series of real numbers

$$\sum_{n=1}^{\infty} |z_n|$$

converges.
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Complex Version of Exponential Function

For $z \in \mathbb{C}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

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Sine and Cosine Series

We define $\sin z$ as

$$\sin z = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$$

and $\cos z$ as

$$\cos z = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!}.$$

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Principal Value of Complex Logarithm

Given $z \in \mathbb{C}^{\bullet}$, the unique $w \in S$ is called the principal value of the logarithm of z and we write $w = \mathrm{Log}(z)$.
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Principal Branch

The function $\mathrm{Log}\, z$ is called the principal branch of the logarithm.
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Open Ball

Consider the usual metric $d(z,w) = |z-w|$ in $\mathbb{C}$. Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a, and radius $\varepsilon > 0$, we mean the set

$$B(a,\varepsilon) = \{z \in \mathbb{C} : d(z,a) < \varepsilon\}.$$

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Open Sets

Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq D$.
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Interior Points

Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq D$.
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Closed Sets

We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.
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Boundary Points

Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a, \varepsilon) \cap D \neq \emptyset$ and $B(a, \varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a, \varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.
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Accumulation Points

Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,

$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

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Closure

Let $D \subseteq \mathbb{C}$. We denote the **closure of** D by **D** and define $\overline{D}$ as

$$\overline{D} = D \cup \{x \in D : \forall \varepsilon > 0 \ B(x, \varepsilon) \cap D \neq \emptyset\}.$$

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Bounded

We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$; that is, $D \subseteq \overline{B}(0, R)$.
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Complex Powers

We define $z^w = \exp(w \log z)$. Note that

$$\begin{aligned} \log z &= \operatorname{Log} z + 2\pi i k \\ &= \ln|z| + i(\operatorname{Arg}(z) + 2\pi k), \ k \in \mathbb{Z} \end{aligned}$$

is a multi-valued function.
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Functional Limit

Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. Let a be an accumulation point of D. Let $\ell \in \mathbb{C}$. We say that ℓ is **the limit of** $f(z)$ as z approaches to a if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $z \in D$ with $0 < |z - a| < \delta$, we have

$$|f(z) - \ell| < \varepsilon;$$

that is, $z \in D \cap (B(a, \delta) \setminus \{a\})$ implies $f(z) \in B(\ell, \varepsilon)$.
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Limits at Infinity

(i) Let $f : D \to \mathbb{C}$ be a function and $a \in \mathbb{C}$ be an accumulation point of D. Then we say $\lim_{z \to a} f(z) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that

$$z \in D \cap (B(a, \delta) \setminus \{a\})$$

implies $|f(z)| \geq M$; that is, f is unbounded as z approaches to a.

(ii) Let f be a complex function defined on the complement of a ball in $\mathbb{C}$. We say $\lim_{z \to \infty} f(z) = \ell$ if for all $\varepsilon > 0$, there exists $R > 0$ such that $|z| > R$ implies

$$|f(z) - \ell| < \varepsilon.$$

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Complex Continuity

Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. We say f **is continuous at** $a \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(a, \delta) \cap D$, we have $f(z) \in B(f(a), \varepsilon)$.
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Complex Differentiability

Let $f : D \to \mathbb{C}$ be a function, and $a \in D$ such that a is an accumulation point of $D \setminus \{a\}$. We say that f is **complex differentiable at** a if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists.}$$

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