Math 234A: Homework 1

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September 8, 2024

Problem 1

(i) **(Parallelogram identity)** Let $z, w \in \mathbb{C}$. Show that
$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Proof. Let $z, w \in \mathbb{C}$ with $z = x + iy$ and $w = u + iv$ with $x, y \in \mathbb{R}$ and $u, v \in \mathbb{R}$. Our goal is to show that
$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$
Consider $|z-w|^2$ and notice that
$$z - w = (x-u) + i(y-v).$$
By definition of the modulus, we have
$$\begin{aligned}
|z-w|^2 &= (z-w)\overline{z-w} \\
&= ((x-u) + i(y-v))((x-u) - i(y-v)) \\
&= (x-u)^2 + (y-v)^2 \\
&= x^2 - 2xu + u^2 + y^2 - 2yv + u^2 \\
&= (x^2 + y^2) - 2(xu + yv) + (u^2 + v^2) \\
&= |z|^2 - 2(xu + yv) + |w|^2.
\end{aligned}$$
Note that
$$z + w = (x+u) + i(y+v).$$
$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= ((x+u) + i(y+v))((x+u) - i(y+v)) \\
&= (x+u)^2 + (y+v)^2 \\
&= x^2 + 2xu + u^2 + y^2 + 2yv + v^2 \\
&= |z|^2 + 2(xu + yv) + |w|^2.
\end{aligned}$$
Adding these two moduli together gives us
$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 = 2(|z|^2 + |w|^2)$$
which is our desired result.

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(ii) **(Binomial Expansion)**: Let $z, w \in \mathbb{C}$ and n be a positive integer. Show that
$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$
where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. Let $z, w \in \mathbb{C}$. We proceed via induction on $n \in \mathbb{Z}^+$ to show that

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Let $n = 1$ be our base case. Then we have

$$\begin{aligned}\sum_{k=0}^{1} \binom{n}{k} z^k w^{n-k} &= \binom{1}{0} z^0 w + \binom{1}{1} z^1 w^0 \\ &= (z+w)^1,\end{aligned}$$

which tells us that the result holds in our base case. Now, suppose the result holds for nth case. We will show the result holds for the $n+1$ case. By our induction hypothesis, we see that

$$\begin{aligned}(z+w)^{n+1} &= (z+w)(z+w)^n \\ &= (z+w)\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1}.\end{aligned}$$

Reordering indices in the first summation by setting $m = k+1$, we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + w \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1} &= \sum_{m=1}^{n+1} \binom{n}{m-1} z^m w^{(n+1)-m} \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} z^k w^{(n-k)+1}.\end{aligned}$$

Then separating the first and last term of each summation, respectively, we have

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{n} z^n w + \sum_{m=1}^{n} \binom{n}{m-1} z^m w^{(n-k)+1} + \sum_{k=1}^{n} \binom{n}{k} z^k w^{n-k+1} + \binom{n}{0} w^{n+1} \\ &= \binom{n}{n} z^n w + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] z^k w^{(n+1)-k} + \binom{n}{0} w^{n+1}.\end{aligned}$$

Using the fact that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

and collecting the first and last terms of the summation, we see that

$$\begin{aligned}(z+w)^{n+1} &= \binom{n}{0} z^0 w^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} z^k w^{(n+1)-k} + \binom{n}{n} z^{n+1} w^0 \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} z^k w^{(n+1)-k}\end{aligned}$$

which completes our induction argument. ■

Problem 2

For $z, w \in \mathbb{C}$. Define $\langle z, w \rangle = \Re(z\overline{w})$. (If we think of $\mathbb{C}$ as two dimensional real vector space, then $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{C}$).

(i) Cauchy Schwarz Inequality: Show that $|\langle z, w\rangle|^2 \leq |z|^2|w|^2$ for all $z, w \in \mathbb{C}$.

Proof. Let $z, w \in \mathbb{C}$. Using the fact that $|\Re(zk)| \leq |z|$ and $|z| = |\overline{z}|$, we see that

$$\begin{aligned} |\langle z, w\rangle|^2 = |\Re(z\overline{w})|^2 &\leq |z\overline{w}|^2 \\ &= |z|^2|\overline{w}|^2 \\ &= |z|^2|w|^2. \end{aligned}$$

Thus, we conclude that $|\langle z, w\rangle|^2 \leq |z|^2|w|^2$. ■

(ii) Triangle Inequalities: Show

$$|z \pm w| \leq |z| + |w|$$

and

$$||z| - |w|| \leq |z \pm w|$$

for all $z, w \in \mathbb{C}$.

Proof. Let $z, w \in \mathbb{C}$. We will first show that $|z + w| \leq |z| + |w|$. First, we will show the following results:

$$|z + w|^2 = |z|^2 + 2|\langle z, w\rangle| + |w|^2 \tag{1}$$

and

$$|z - w|^2 = |z|^2 - 2|\langle z, w\rangle| + |w|^2. \tag{2}$$

Let $z = x + iy$ and $w = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that

$$z + w = (x + u) + i(y + v)$$

and

$$z - w = (x - u) + i(y - v).$$

Using the definition of the modulus, we see that

$$\begin{aligned} |z + w|^2 = (z + w)\overline{(z + w)} &= ((x + u) + i(y + v))((x + u) - i(y + v)) \\ &= (x + u)^2 + (y + v)^2 \\ &= x^2 + 2xu + u^2 + y^2 + 2yv + v^2 \\ &= (x^2 + y^2) + 2(xu + yv) + (y^2 + v^2) && \text{(since } \Re(z\overline{w}) = xu + yv) \\ &= |z|^2 + 2\langle z, w\rangle + |w|^2. && \text{(since } \langle z, w\rangle = \Re(z\overline{w})) \end{aligned}$$

Similarly, we have

$$\begin{aligned} |z - w|^2 = (z - w)\overline{(z - w)} &= ((x - u) + i(y - v))((x - u) - i(y - v)) \\ &= (x - u)^2 + (y - v)^2 \\ &= x^2 - 2xu + u^2 + y^2 - 2yv + v^2 \\ &= x^2 + y^2 - 2(xu + yv) + v^2 \\ &= (x^2 + y^2) - 2\langle z, w\rangle + (v^2 + u^2) \\ &= |z|^2 - 2\langle z, w\rangle + |w|^2. \end{aligned}$$

Now, let us prove that $|z + w| \leq |z| + |w|$. Consider $|z + w|^2$. By part (a), we see that

$$\begin{aligned} |z + w|^2 &= |z|^2 + 2\langle z, w\rangle + |w|^2 \\ &\leq |z|^2 + 2zw + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

By taking the square root of both sides, we see that

$$|z+w| \leq |z| + |w|.$$

Using this fact, we can see that

$$|z-w| = |z+(-w)| \leq |z| + |-w| = |z| + |w|.$$

To show the second inequality, consider $|z-w|^2$. Then using part (a) again, we have

$$\begin{aligned} |z-w|^2 &= |z|^2 - 2\langle z, w\rangle + |w|^2 \\ &\geq |z|^2 - 2|z||w| + |w|^2 \\ &= (|z| - |w|)^2. \end{aligned}$$

By taking the square root of both sides, we see that

$$|z-w| \geq ||z| - |w||.$$

Using this fact, we can say that

$$|z+w| = |z-(-w)| \geq ||z| - |-w|| = ||z| - |w||.$$

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Problem 3

(Lagrange Identity) Let $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Show that

$$\Big|\sum_{k=1}^n z_k w_k\Big|^2 = \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2 - \sum_{1\leq i<j\leq n} |z_i\overline{w_j} - z_j\overline{w_i}|^2.$$

Use this to deduce that

$$\Big|\sum_{k=1}^n z_k w_k\Big|^2 \leq \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2.$$

Proof. We will show that

$$\sum_{1\leq i<j\leq n} |z_i\overline{w_j} - z_j\overline{w_i}|^2 = \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |w_k|^2 - \Big|\sum_{k=1}^n z_k w_k\Big|^2.$$

By rearranging terms, we see that

$$\sum_{i=1}^n \sum_{j=1}^n |z_i\overline{w_j} - z_j\overline{w_i}|^2 = 2\sum_{1\leq i<j\leq n} |z_i\overline{w_j} - z_j\overline{w_i}|^2. \tag{1}$$

Using the definition of modulus and rearranging terms, we can see that

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}|z_i\overline{w_j}-z_j\overline{w_i}|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(z_i\overline{w_j}-z_j\overline{w_i})(\overline{z_i}w_j-\overline{z_j}w_i)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(|z_i|^2|w_j|^2+|z_j|^2|w_i|^2)\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}(z_j\overline{z_i}\overline{w_i}w_j+z_i\overline{z_j}w_i\overline{w_j})\\
&= 2\sum_{i=1}^{n}|z_i|^2\sum_{j=1}^{n}|z_j|^2-2\Big(\sum_{i=1}^{n}z_iw_i\Big)\Big(\sum_{j=1}^{n}\overline{z_jw_j}\Big)\\
&= 2\sum_{i=1}^{n}|z_i|^2\sum_{j=1}^{n}|z_j|^2-2\Big|\sum_{i=1}^{n}z_iw_i\Big|^2.
\end{aligned}$$

Now, dividing 2 on both sides with the right side of (1), we get our desired result

$$\sum_{1\le i<j\le n}|z_i\overline{w_j}-z_j\overline{w_i}|^2=\sum_{i=1}^{n}|z_i|^2\sum_{j=1}^{n}|z_j|^2-\Big|\sum_{i=1}^{n}z_iw_i\Big|^2.$$

Thus, we conclude that

$$\begin{aligned}
\Big|\sum_{k=1}^{n}z_kw_k\Big|^2 &= \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2-\sum_{1\le i<j\le n}|z_i\overline{w_j}-z_j\overline{w_i}|^2\\
&\le \sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|w_k|^2.
\end{aligned}$$

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Problem 4

Express the following complex number in the form $\alpha+i\beta$:

(i) $(1+i)^{-1}$

Solution. Observe that

$$(1+i)^{-1}=\frac{1}{1+i}$$

and that

$$\frac{1}{1+i}\cdot\frac{(1-i)}{(1-i)}=\frac{(1-i)}{1-i^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i.$$

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(ii) $(1+i)/2i$

Solution. Observe that

$$\frac{(1+i)}{2i}=\frac{1}{2i}(1+i)=\frac{1}{2i}+\frac{1}{2}=\frac{1}{2}-\frac{1}{2}i.$$

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(iii) $(5+5i)^{10}$

Solution. Let $z = 1 + i$. Observe that we can write

$$(5 + 5i)^{10} = 5^{10}(1 + i)^{10}.$$

Note that

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Furthermore, we have

$$\tan^{-1}(1/1) = \tan^{-1}(1) = \frac{\pi}{4}.$$

Using De Moivre's formula, we can write

$$\begin{aligned} z^{10} &= (\sqrt{2})^{10}(\cos(10\theta) + i\sin(10\theta) \\ &= (\sqrt{2})^{10}(\cos(5\pi/2) + i\sin(5\pi/2)) \\ &= (\sqrt{2})^{10}i. \end{aligned}$$

Then we have

$$(5 + 5i)^{10} = 5^{10}(\sqrt{2})^{10}i = 312500000i.$$

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(iv) $\left(\frac{2+i}{3-2i}\right)^2$

Solution. Our first step is to get $\frac{2+i}{3-2i}$ in terms of $\alpha + i\beta$. Thus, observe that

$$\frac{2+i}{3-2i} = \frac{2+i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{7i+4}{13} = \frac{4}{13} + i\frac{7}{13}.$$

Furthermore, we have

$$\left(\frac{4}{13} + i\frac{7}{13}\right)^2 = \frac{1}{169}(4 + 7i)^2 = \frac{1}{169}(16 + 46i - 49) = \frac{1}{169}(-33 + 46i).$$

Thus, we have that

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{-33}{169} + \frac{46}{169}i$$

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(v) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

Solution. Denote $z = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$. Then observe that

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}.$$

Furthermore, we have

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

Using De Moivre's formula, we have that

$$z^3 = 1^3 \cdot \left(\cos\left(3 \cdot \frac{2\pi}{3}\right) + i\sin\left(3 \cdot \frac{2\pi}{3}\right)\right) = \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1.$$

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Problem 5

Let $z \in \mathbb{C}^{\cdot}$ and $z = \gamma(\cos\varphi + i\sin\varphi)$ where $n \in \mathbb{Z}^+$ and

$$w = \gamma^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]$$

where $k \in \mathbb{Z}$. Show that $w^n = z$.

Proof. Note that for any $n \in \mathbb{N}$ that

$$z^n = (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi. \tag{1}$$

Using (1), we can see that

$$\begin{aligned} w^n &= \left(\gamma^{1/n}\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]\right)^n \\ &= (\gamma^{1/n})^n\left[\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right]^n \\ &= \gamma\left[\cos\left(n \cdot \frac{\varphi + 2\pi k}{n}\right) + i\sin\left(n \cdot \frac{\varphi + 2\pi k}{n}\right)\right] \\ &= \gamma(\cos\varphi + i\sin\varphi) \\ &= z \end{aligned}$$

which ends our proof. ■

Problem 6

(Computing fourth roots): Find your distinct complex numbers w such that $w^4 = z$ for

(i) $z = i$.

Solution. Note that $z = i = 0 + 1 \cdot i$. For $0 \le k \le 3$, we see that the arguments are

$$\frac{\varphi + 2\pi k}{4} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}.$$

For $k = 0$, we see that

$$\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 + \sqrt{2}/2}{2}} + i\sqrt{\frac{1 + \sqrt{2}/2}{2}}.$$

For $k = 1$, we see that

$$\begin{aligned} \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right) &= -\sqrt{\frac{1 + \cos(5\pi/4)}{2}} + i\sqrt{\frac{1 - \cos(5\pi/4)}{2}} \\ &= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}. \end{aligned}$$

For $k = 2$, we see that

$$\begin{aligned}\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right) &= \cos\left(\frac{1}{2}\cdot\frac{9\pi}{4}\right) + i\sin\left(\frac{1}{2}\cdot\frac{9\pi}{4}\right)\\ &= \sqrt{\frac{1+\cos(9\pi/4)}{2}} + i\sqrt{\frac{1-\cos(9\pi/4)}{2}}\\ &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

Lastly, for $k = 3$, we have

$$\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right) = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}.$$

Thus, the four solutions are

$$\begin{aligned}k = 0;&\ \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}},\\ k = 1;&\ \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}},\\ k = 2;&\ \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}},\\ k = 3;&\ \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

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(ii) $z = -i$.

Solution. Note that $z = -i = 0 - 1 \cdot i$. For $0 \leq k \leq 3$, the arguments are

$$\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}.$$

Let $k = 0$. Then using the result found in problem 5 and the half-angle identity, we have

$$\begin{aligned}\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right) &= -\sqrt{\frac{1+\cos(3\pi/4)}{2}} + i\sqrt{\frac{1-\cos(3\pi/4)}{2}}\\ &= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} + i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}\end{aligned}$$

Let $k = 1$. We get that

$$\cos\left(\frac{\frac{3\pi}{2}+2\pi}{4}\right) + i\sin\left(\frac{\frac{3\pi}{2}+2\pi}{4}\right) = \cos\left(\frac{3\pi}{8}+\frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{8}+\frac{\pi}{2}\right).$$

Using the sum formula, we see that

$$\begin{aligned}\cos\left(\frac{3\pi}{8}+\frac{\pi}{2}\right)+i\sin\left(\frac{3\pi}{8}+\frac{\pi}{2}\right) &= \left[\cos\left(\frac{3\pi}{8}\right)\cos\left(\frac{\pi}{2}\right)-\sin\left(\frac{3\pi}{8}\sin\left(\frac{\pi}{2}\right)\right)\right]\\ &\quad+i\left[\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{3\pi}{8}\sin\left(\frac{\pi}{2}\right)\right)\right]\\ &= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}-i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

For $k=2$, we employ the same process. Thus, we have

$$\begin{aligned}\cos\left(\frac{3\pi}{8}+\pi\right)+i\sin\left(\frac{3\pi}{8}+\pi\right) &= \left[\cos\left(\frac{3\pi}{8}\right)\cos(\pi)-\sin\left(\frac{3\pi}{8}\right)\sin(\pi)\right]\\ &\quad+i\left[\sin\left(\frac{3\pi}{8}\right)\cos(\pi)+\cos\left(\frac{3\pi}{8}\right)\sin(\pi)\right]\\ &= -\cos\left(\frac{3\pi}{8}\right)-i\sin\left(\frac{3\pi}{8}\right)\\ &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}-i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

Lastly, for $k=3$, we have

$$\begin{aligned}\cos\left(\frac{15\pi}{8}\right)+i\sin\left(\frac{15\pi}{8}\right) &= \cos\left(\frac{3\pi}{8}+\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{8}+\frac{3\pi}{2}\right)\\ &= \left[\cos\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{2}\right)-\sin\left(\frac{3\pi}{8}\sin\left(\frac{3\pi}{2}\right)\right)\right]\\ &\quad+i\left[\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{2}\right)+\cos\left(\frac{3\pi}{8}\right)\sin\left(\frac{3\pi}{2}\right)\right]\\ &= \sin\left(\frac{3\pi}{8}\right)-i\cos\left(\frac{3\pi}{8}\right)\\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}+i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

Hence, the solutions are

$$\begin{aligned}k=0;&\ -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}+i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}},\\ k=1;&\ \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}-i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}},\\ k=2;&\ \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}-i\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}},\\ k=3;&\ \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}+i\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

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(iii) $z=1$.

Solution. Note that $z=1=1+i\cdot 0$. For $0\leq k\leq 3$, we have the arguments

$$0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$$

with each argument corresponding to $k \in [0, 3]$. For $k = 0$, observe that

$$\cos(0) + i\sin(0) = 1.$$

For $k = 1$, we have

$$\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i = i.$$

For $k = 2$, we have

$$\cos\pi + i\sin\pi = -1.$$

Lastly, if $k = 3$ we get

$$\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$$

Thus, the solutions are

$$\pm 1, \pm i.$$

■

(iv) $z = -1$.

Solution. Let $0 \le k \le 3$ and $n = 4$. We will compute all the solution along the interval $0 \le k \le 3$. Note that $z = i = 0 + i \cdot 1$ and that $\varphi = \arg z = \pi/2$ and that for $0 \le k \le 3$, we have

$$\frac{\varphi + 2\pi k}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \pi.$$

Using the result in problem 5, observe that for $k = 0$, we have

$$\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

Similarly, for $k = 1$, we have

$$\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

For $k = 2$,

$$\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

For $k = 3$, we have

$$\cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1.$$

Thus, we have

$$z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -1.$$

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Problem 7

Sketch the following sets in $\mathbb{C}$.

(i) $\zeta = \{z \in \mathbb{C} : \Re((1+i)z - 2) = 0\}$.

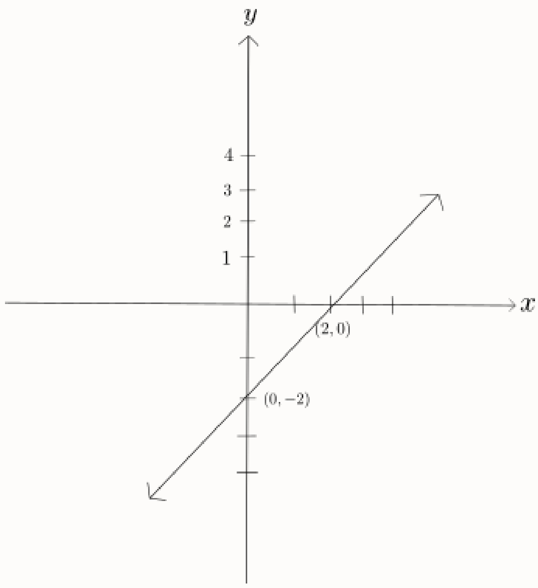
Solution. Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then we see that

$$\begin{aligned}(1+i)z - 2 &= (1+i)(x+iy) - 2 \\ &= (x - y - 2) + i(x + y).\end{aligned}$$

Hence, we have

$$\Re((1+i)z - 2) = x - y - 2 = 0$$

and so, we have linear equation
$$y = x - 2.$$



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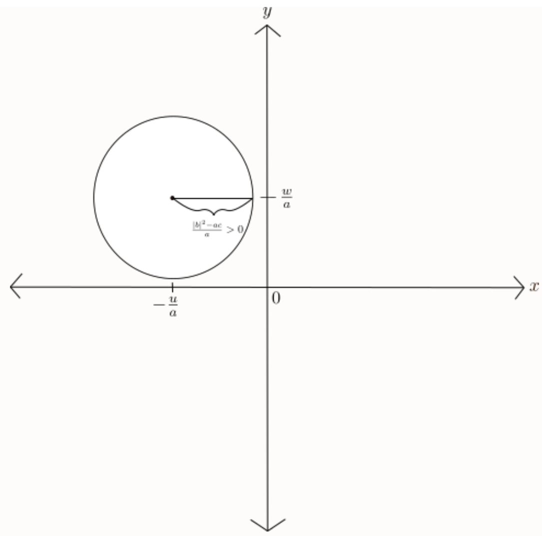
(ii) Let $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$ with $\bar{b}b - ac > 0$ and
$$\zeta = \{z \in \mathbb{C} : a|z|^2 + \bar{b}z + b\bar{z} + c = 0\}.$$

Solution. Let $z = x + iy$ and $b = u + iw$ for $x, y, u, w \in \mathbb{R}$. Then observe that
$$\begin{aligned} 0 = a|z|^2 + \bar{b}z + b\bar{z} + c &= a(x^2 + y^2) + (u - iw)(x + iy) + c \\ &= ax^2 + ay^2 + 2ux + 2wy + c \end{aligned}$$
which imply that
$$\left[x^2 + \frac{2ux}{a} + \left(\frac{u}{a}\right)^2\right] + \left[y^2 + \frac{2wy}{a} + \left(\frac{w}{a}\right)^2\right] = \frac{u^2 + w^2 - ac}{a}$$
which further implies that
$$\left(x^2 + \frac{u}{a}\right)^2 + \left(y^2 - \frac{w}{a}\right)^2 = \frac{|b|^2 - ac}{a}.$$
Set
$$\gamma = \frac{|b|^2 - ac}{a}$$
and so
$$\frac{\left(x^2 + \frac{u}{a}\right)^2}{\gamma} + \frac{\left(y^2 - \frac{w}{a}\right)^2}{\gamma} = 1$$
which is an equation of a circle with radius $\gamma > 0$.



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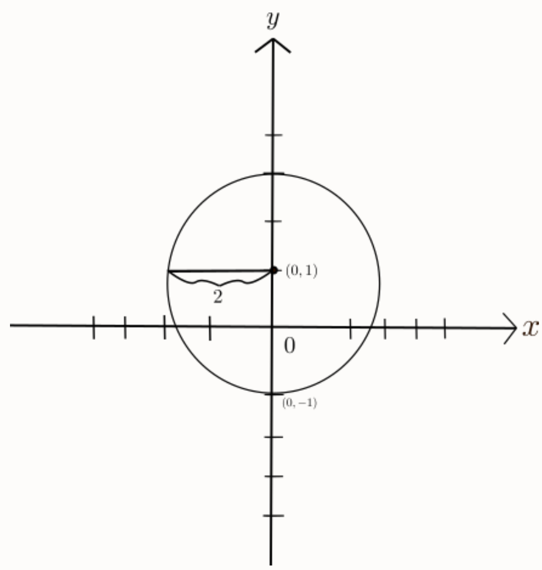
(iii) $\zeta = \{z \in \mathbb{C} : |z - i| = 2\}$.

Proof. Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then we see that

$$\begin{aligned} |z-i|^2 = 4 &\Rightarrow (z-i)\overline{(z-i)} = 4 \\ &\Rightarrow (x + i(y-1))(x - i(y-1)) = 4 \\ &\Rightarrow x^2 + (y-1)^2 = 4 \\ &\Rightarrow \frac{x^2}{4} + \frac{(y-1)^2}{4} = 1 \end{aligned}$$

which is an equation for a circle in $\mathbb{R}^2$. Thus, we have



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Problem 8

Let $z, a \in \mathbb{C}$.

(i) Show that $|1 - z\overline{a}|^2 - |z - a|^2 = (1 - |z|)^2(1 - |a|^2)$.

Proof. Note that $|a|^2 = |\overline{a}|^2$ and that

$$\langle z, a\rangle = \Re(z\overline{a}) = xu + yv = \Re(\overline{z}a) = \langle 1, z\overline{a}\rangle.$$

if $z = z + iy$ and $a = u + iv$ for $x, y, u, v \in \mathbb{R}$. Observe that

$$\begin{aligned} |1 - z\overline{a}|^2 - |z - a|^2 &= [1 - 2\langle 1, z\overline{a}\rangle + |z\overline{a}|^2] - [|z|^2 - 2\langle z, a\rangle + |a|^2] \\ &= |z|^2|a|^2 - |z|^2 - |a|^2 - 1 \\ &= (1 - |z|^2)(1 - |a|^2). \end{aligned}$$

Hence, we see that

$$|1 - z\overline{a}|^2 - |z - a|^2 = (1 - |z|^2)(1 - |a|^2).$$

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(ii) Assume that $|a| < 1$. Show that

$$|z| < 1 \Leftrightarrow \left|\frac{z - a}{1 - \overline{a}z}\right| < 1 \tag{1}$$

and

$$|z| = 1 \Leftrightarrow \left|\frac{z - a}{1 - \overline{a}z}\right| = 1. \tag{2}$$

Proof. Suppose $|z| < 1$ and $|a| < 1$. We will first show that the forwards direction of (1).

$$\left|\frac{z - a}{1 - \overline{a}z}\right| < 1.$$

It suffices to show that

$$|z - a|^2 < |1 - \overline{a}z|^2.$$

By part (i), we see that

$$|z - a|^2 = |1 - z\overline{a}|^2 - (1 - |z|^2)(1 - |a|^2). \tag{3}$$

Since $|z| < 1$ and $|a| < 1$, we see that

$$1 - |z|^2 > 0 \text{ and } 1 - |a|^2 > 0.$$

Thus, we see that

$$-(1 - |z|^2)(1 - |a|^2) < 0.$$

Therefore, (3) implies that

$$|z - a|^2 < |1 - \overline{a}z|^2$$

which further implies that

$$\left|\frac{z - a}{1 - \overline{a}z}\right| < 1.$$

Now, suppose for the backwards direction that

$$\left|\frac{z - a}{1 - \overline{a}z}\right| < 1.$$

Note that this is equivalent to saying

$$|z - a|^2 < |1 - \overline{a}z|^2$$

By using the fact that

$$|z - a|^2 = |z|^2 - 2\langle z, a\rangle + |w|^2$$

and

$$|1 - \overline{a}z|^2 = 1 - 2\langle 1, \overline{a}z\rangle + |\overline{a}z|^2.$$

Then observe that

$$|z-a|^2 < |1-\overline{a}z|^2 \Rightarrow |z|^2 - 2\langle z, a\rangle + |w|^2 < 1 - 2\langle 1, \overline{a}z\rangle + |\overline{a}|^2|z|^2.$$

So, we must have

$$\begin{gathered} |z|^2 - 2\langle z, a\rangle + |a|^2 < 1 - 2\langle 1, \overline{a}z\rangle + |\overline{a}|^2|z|^2 \\ |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \end{gathered}$$

which imply that

$$\begin{aligned} |z|^2 < (1-|a|^2) + |a|^2|z|^2 &\Rightarrow |z|^2(1-|a|^2) < 1-|a|^2 \\ &\Rightarrow |z|^2 < \frac{1-|a|^2}{1-|a|^2} = 1. \end{aligned}$$

Thus, we see that $|z| < 1$. To prove (2), we can replace the inequalities above with an equal sign and the proof will be the same. ■