

1 Topics

- Topological Continuity
- Continuity and Compactness
- Continuity and Connectedness
- Continuity and Inverse Maps
- Uniform Continuity

So far we have learnt two equivalent descriptions of the concept of continuity for functions $f : (X, d) \rightarrow (Y, \tilde{d})$:

- (1) f is continuous if and only if

$$\forall c \in X \quad \forall \varepsilon > 0 \quad \exists \delta_{\varepsilon, c} > 0 \text{ such that if } d(x, c) < \delta_{\varepsilon, c} \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

- (2) f is continuous if and only if $\forall c \in X, a_n \rightarrow c$ implies $f(a_n) \rightarrow f(c)$.

In undergraduate analysis, most of the sequences we study are sequences in \mathbb{R} . In terms of convergence, we say that

$$a_n \rightarrow c \iff \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } \forall n > N \quad |a_n - c| < \varepsilon.$$

For graduate analysis, convergence can be viewed more generally; that is, we have

$$a_n \rightarrow c \iff \forall N_\varepsilon(c) \quad \exists N \in \mathbb{N} \text{ such that } \forall n > N \quad a_n \in N_\varepsilon(c).$$

2 Topological Continuity

Theorem (Topological Characterization Continuity). Let (X, d) and (Y, \tilde{d}) be metric spaces and let $f : X \rightarrow Y$. The following statements are equivalent:

- (i) f is continuous
- (ii) For every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X .

Proof. $((i) \implies (ii))$ Suppose f is continuous. Our goal is to show that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X . Let B be an open set in Y . Our goal is to show that $f^{-1}(B)$ is open in X . That is, we want to show that every point of $f^{-1}(B)$ is an interior point. Let $p \in f^{-1}(B)$. We need to find an $\delta > 0$ such that $N_\delta^X(p) \subseteq f^{-1}(B)$.

Since f is continuous at p , there exists $\hat{\delta} > 0$ such that

$$\forall x \in N_{\hat{\delta}}^X(p) \quad f(x) \in N_{\hat{\delta}}^Y(f(p)) \subseteq B.$$

Clearly, we have $N_{\hat{\delta}}^X(p) \subseteq f^{-1}(B)$, so we can use this $\hat{\delta}$ as the δ we were looking for.

$((ii) \implies (i))$ Assume that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X . Our goal is to show that f is continuous. We need to show that f is continuous at every point of X . Let $c \in X$. We will prove that f is continuous at c . That is,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that if } x \in N_\delta^X(c) \text{ then } f(x) \in N_\varepsilon^Y(f(c)).$$

Let $\varepsilon > 0$ be given. We need to show that there exists $\delta > 0$ such that

$$N_\delta^X(c) \subseteq f^{-1}(N_\varepsilon^Y(f(c))). \quad (*)$$

Since $N_\varepsilon^Y(f(c))$ is open in Y , it follows from the assumption that $f^{-1}(N_\varepsilon^Y(f(c)))$ is open in X . Since $f^{-1}(N_\varepsilon^Y(f(c)))$ is open in X and $c \in f^{-1}(N_\varepsilon^Y(f(c)))$, we have c is an interior point of $f^{-1}(N_\varepsilon^Y(f(c)))$. Hence, there exists $\delta > 0$ such that $N_\delta^X(c) \subseteq f^{-1}(N_\varepsilon^Y(f(c)))$ as desired. ■

Note that continuous functions does NOT necessarily preserve the following properties:

- open
- closed
- bounded

Some examples of maps that do not preserve the properties above are

- (1) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. Note that $E = (-2, 2)$ is open but $f(E) = [0, 4)$ is not.
- (2) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$. Note that $E = [0, \infty)$ is closed but $f(E) = (0, 1]$ is not.
- (3) $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Note that $E = (0, 1)$ is a bounded set but $f(E) = (1, \infty)$ is not.

3 Continuity and Compactness

Theorem (Continuous Maps Over Compact Sets). Let (X, d) and (Y, \tilde{d}) are metric spaces and let $f : X \rightarrow Y$ be continuous and let E be a compact set in X . Then $f(E)$ is compact in Y .

Proof. Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of $f(E)$. Our goal is to show that this open cover has a finite subcover. We have

$$f(E) \subseteq \bigcup_{\alpha \in I} O_\alpha.$$

So,

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} O_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(O_\alpha).$$

Since $E \subseteq f^{-1}(f(E))$ and $f^{-1}\left(\bigcup_{\alpha \in I} O_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$, we can conclude that

$$E \subseteq \bigcup_{\alpha \in I} f^{-1}(O_\alpha).$$

Now, we have $f : X \rightarrow Y$ is continuous and for each $\alpha \in I$, O_α is open in Y implies that for all $\alpha \in I$, we have $f^{-1}(O_\alpha)$ is open in X . Thus, $\{f^{-1}(O_\alpha)\}_{\alpha \in I}$ is an open cover for E . Since E is compact, we have

$$\exists \alpha_1, \dots, \alpha_n \in I \text{ such that } E \subseteq \bigcup_{i=1}^n f^{-1}(O_{\alpha_i}).$$

Consequently, we have

$$\begin{aligned} f(E) &\subseteq f\left(\bigcup_{i=1}^n f^{-1}(O_{\alpha_i})\right) \\ &= \bigcup_{i=1}^n f(f^{-1}(O_{\alpha_i})) \\ &\subseteq \bigcup_{i=1}^n O_{\alpha_i}. \end{aligned}$$

Thus, we have $\{O_{\alpha_i}\}_{1 \leq i \leq n}$ is a finite subcover for $f(E)$. ■

Theorem (Extreme Value Theorem). Let (X, d) be a compact metric space.

- If $f : (X, d) \rightarrow (Y, \tilde{d})$ is continuous, then $f(X)$ is a closed and bounded set in Y .
- If $f : (X, d) \rightarrow \mathbb{R}$ is continuous, then f attains a maximum values and a minimum value. More precisely, $M = \sup_{x \in X} f(x)$

and $m = \inf_{x \in X} f(x)$ exists, and there exists points $a \in X$ and $b \in X$ such that $f(a) = M$ and

$$f(b) = m.$$

Proof. (i) By the previous theorem, we can see that $f(X)$ must be compact in Y . As we know, every compact set in any metric space is closed and bounded.

(ii) By part (i), $f(X)$ is closed and bounded subset of \mathbb{R} . Since $f(X)$ is a bounded set in \mathbb{R} , $M = \sup f(X) = \sup_{x \in X} f(x)$ and $m = \inf f(X) = \inf_{x \in X} f(x)$ exists. Recall from Theorem 2.28 that $M \in \overline{f(X)}$ and $m \in \overline{f(X)}$. Since $\overline{f(X)} = f(X)$, we can conclude that $M \in f(X)$ and $m \in f(X)$. That is, there exists $a \in X$ such that $f(a) = M$ and there exists $b \in X$ such that $f(b) = m$. ■

4 Continuity and Connectedness

Theorem (Preservation of Connectedness). Let (X, d) and (Y, \tilde{d}) be metric spaces and $f : X \rightarrow Y$ be a continuous map, and let $E \subseteq X$ be a connected set. Then $f(E)$ is connected in Y .

Proof. Assume for contradiction that $f(E)$ is NOT connected. Thus, we can write $f(E)$ as a union of two (nonempty) separated sets A and B :

$$f(E) = A \cup B, \quad \overline{A} \cap B = \emptyset, \quad A \cap \overline{B} = \emptyset.$$

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. In what follows, we will show that G and H form a separation for the set E , which contradicts the assumption that E is connected. We will show that

- (1) G and H are both nonempty
- (2) $\overline{G} \cap H = \emptyset$ (and similarly, $G \cap \overline{H} = \emptyset$)
- (3) $E = G \cup H$

To show (1), we will prove that $f(G) = A$ (similarly, show that $f(H) = B$). We have

(1)

$$\begin{aligned} f(G) &= f(E \cap f^{-1}(A)) \subseteq f(E) \cap f(f^{-1}(A)) \\ &\subseteq f(E) \cap A && (f(E) = A \cup B) \\ &= A. \end{aligned}$$

Then we have $f(G) \subseteq A$. Now, we will show that $A \subseteq f(G)$. Let $y \in A$. Then $y \in f(E)$ and so there exists

$$x \in E \text{ such that } f(x) = y. \quad (*)$$

Then we have

$$f(x) = y \in A \implies x \in f^{-1}(A). \quad (**)$$

Then (*) and (**) imply that

$$\begin{aligned} x \in E \cap f^{-1}(A) &\implies f(x) \in f(E \cap f^{-1}(A)) = f(G) \\ &\implies y \in f(G). \end{aligned}$$

Thus, $A \subseteq f(G)$.

(2) Observe that

$$\begin{aligned}
 G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\
 &= E \cap [f^{-1}(A) \cup f^{-1}(B)] \\
 &= E \cap [f^{-1}(A \cup B)] \\
 &= E \cap [f^{-1}(f(E))] && (f(E) = A \cup B) \\
 &= E && (E \cap f^{-1}(f(E)))
 \end{aligned}$$

(3) To this end, it is enough to show that $f(\overline{G}) \cap f(H) = \emptyset$. Note that $f(H) = B$. So, our goal is to show that $f(\overline{G}) \cap B = \emptyset$. Since $\overline{A} \cap B$ is empty, and so it suffices to show that $f(\overline{G}) \subseteq \overline{A}$. Notice that

$$G = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}).$$

Since f is continuous and \overline{A} is a closed set in Y , we can see that $f^{-1}(\overline{A})$ is closed in X . Thus, we can write

$$G \subseteq f^{-1}(\overline{A}) \implies \overline{G} \subseteq \overline{f^{-1}(\overline{A})} = f^{-1}(\overline{A}).$$

Therefore, we have

$$f(\overline{G}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}.$$

A similar argument shows that $f(\overline{H}) \cap f(G) = \emptyset$. ■

Theorem (Intermediate Value Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous map and let $f(a) \neq f(b)$. Let $L \in \mathbb{R}$ be such that $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = L$.

Proof. Since $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $[a, b]$ is connected. Then, by the preservation of connectedness, we have $f([a, b])$ is connected in \mathbb{R} . This implies that $f([a, b])$ is either a singleton or an interval I in \mathbb{R} . Since $f(a) \neq f(b)$, we know that $f([a, b])$ cannot be a singleton. That is, $f([a, b])$ is an interval I in \mathbb{R} . Note that $f(a), f(b) \in I$ and L is between $f(a)$ and $f(b)$. So, $L \in f([a, b])$. Thus, there exists $c \in [a, b]$ such that $f(c) = L$. But neither $f(a) \neq L$ nor $f(b) \neq L$, so there exists $c \in (a, b)$ such that $f(c) = L$. ■

Note that if $f : X \rightarrow Y$ is continuous and bijective, it is NOT necessarily true that $f^{-1} : Y \rightarrow X$ is continuous.

Example. $f : (-1, 0] \cup [1, 2] \rightarrow [0, 4]$ given by $f(x) = x^2$ is continuous and bijective. However, $f^{-1} : [0, 4] \rightarrow (-1, 0] \cup [1, 2]$ is NOT continuous. Indeed, we see that $[0, 4]$ is connected, but $f^{-1}([0, 4]) = (-1, 0] \cup [1, 2]$ is NOT.

5 Continuity and Inverse Maps

Theorem (Continuous and Bijective maps on Compact Sets). Let (X, d) and (Y, \tilde{d}) are metric spaces, X is compact, and $f : X \rightarrow Y$ is continuous and bijective. Then $f^{-1} : Y \rightarrow X$ is continuous.

Proof. It suffices to show that for every open set $B \subseteq X$, $(f^{-1})^{-1}(B)$ is open in Y . That is, we will show that $f(B)$ is open in Y . Let B be an open set in X . Thus, B^c is closed in X . Since X is a compact set and $B^c \subseteq X$, B^c is compact in X . Since continuity preserves compactness, we have $f(B^c)$ is compact in Y . Since compact sets are closed and bounded, $f(B^c)$ is closed in Y . Now, $[f(B^c)]^c$ is open in Y . Since $[f(B^c)]^c$ is open in Y . Since f is bijective, we have

$$f(B^c) = [f(B)]^c.$$

Thus,

$$f(B) = [(f(B))^c]^c = [f(B^c)]^c$$

is open in Y . ■

Remark (Invariance of Domain). Let $U \subseteq \mathbb{R}^n$ be open and $V \subseteq \mathbb{R}$. Since $f : U \rightarrow V$ is continuous and bijective, then V is open and $f^{-1} : V \rightarrow U$ is continuous.

6 Uniform Continuity

Let A be a proper subset of (X, d) .

Consider the following three questions:

- (1) Given a continuous function $f : A \rightarrow \mathbb{R}$, is it possible to extend f to a continuous function that is defined on all of X .
- (2) Under what conditions can we be certain that a continuous function $f : A \rightarrow \mathbb{R}$ has a continuous extension to all of X .

Theorem (A special case of Tietze Extension Theorem). Let (X, d) be a metric space. Let A be a nonempty closed set in X . If $f : A \rightarrow \mathbb{R}$ is continuous, then f has a continuous extension to all of X .

Theorem. Let (X, d) be a metric space. Let A be a nonempty set in X . If $f : A \rightarrow \mathbb{R}$ is **uniformly continuous** on A , then f can be extended to a continuous function $\tilde{f} : \bar{A} \rightarrow \mathbb{R}$.

Recall that we say that $f : A \subseteq (X, d) \rightarrow (Y, \tilde{d})$ is continuous at $c \in A$:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x, c) < \delta \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

We say that $f : A \subseteq (X, d) \rightarrow (Y, \tilde{d})$ is continuous on A , we have

$$\forall c \in A \forall \varepsilon > 0 \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x, c) < \delta_{\varepsilon, c} \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

In general, the δ in the above statement depends on both ε and c . If the function the desirable property that given an arbitrary $\varepsilon > 0$, one can find a single δ_ε that holds (or stays constant) for all points $c \in A$, then we say that f is uniformly continuous on A .

Definition (Uniform Continuity). Let $f : A \subseteq (X, d) \rightarrow (Y, \tilde{d})$ be a function. We say that f is **uniformly continuous** on A if

$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 \text{ such that } \forall x, y \in A \text{ if } d(x, y) < \delta_\varepsilon, \text{ then } \tilde{d}(f(x), f(y)) < \varepsilon.$$

Remark. We say that f is NOT uniformly continuous on A if

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0 \exists x, y \in A \text{ satisfying } d(x, y) < \delta \text{ but } \tilde{d}(f(x), f(y)) \geq \varepsilon.$$

Clearly, from the above, we can see that uniform continuity is a stronger property than continuity in the sense that if $f : A \rightarrow Y$ is uniformly continuous, then $f : A \rightarrow Y$ is continuous. Note that it is not really productive to talk about uniform continuity at a specific point. Uniform continuity is always discussed in the context to a particular domain. For example, we'll see that $f(x) = x^2$ is NOT uniformly continuous on \mathbb{R} . However, $f(x) = x^2$ is uniformly continuous on $[0, 1]$. This implies determining whether a function is uniform continuous is a matter of analyzing the domain on which the function is defined on.

Example. Prove that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x + 1$ is uniformly continuous on \mathbb{R} .

Our goal is to show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \forall x, y \in \mathbb{R} \text{ if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Clearly, we can take $\delta = \frac{\varepsilon}{2}$ (or any positive number less than $\frac{\varepsilon}{2}$).

The definition of uniform continuity directly implies that if some function $f : A \rightarrow Y$ is uniformly continuous on A , then it is also uniformly continuous on $B \subseteq A$.

Note that tells us that not all continuous functions are uniformly continuous. The following theorem gives a simple criterion for proving the absence of uniform continuity.

Theorem. Let $f : A \subseteq (X, d) \rightarrow (Y, \tilde{d})$. If we can find $\varepsilon_0 > 0$ and two sequences (x_n) and (c_n) in A such that

$$d(x_n, c_n) \rightarrow 0 \text{ and } \forall n \tilde{d}(f(x_n), f(c_n)) \geq \varepsilon_0.$$

Then f is NOT uniformly continuous on A .

Proof. Recall that f is NOT uniformly continuous if and only if there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ satisfying $d(x, c) < \delta$ but $\tilde{d}(f(x), f(c)) \geq \varepsilon$. If this holds, then we can set $\varepsilon = \varepsilon_0$ such that for any $\delta > 0$, there exists N such that $d(x_N, c_N) < \delta$, but $\tilde{d}(f(x_N), f(c_N)) \geq \varepsilon$. ■

Example. Prove that $f(x) = x^2$ is not uniformly continuous on \mathbb{R} .

Proof. Let $x_n = n$ and $c_n = n + \frac{1}{n}$. We have

$$\lim_{n \rightarrow \infty} |x_n - c_n| = \lim_{n \rightarrow \infty} \left| \frac{-1}{n} \right| = 0.$$

Also, for all n , we have

$$\begin{aligned} |f(x_n) - f(c_n)| &= \left| n^2 - \left(n + \frac{1}{n} \right)^2 \right| \\ &= \left| n^2 - \left(n^2 + 2 + \frac{1}{n^2} \right) \right| \\ &= \left| - \left(2 + \frac{1}{n^2} \right) \right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2. \end{aligned}$$

Hence, we conclude that $f(x) = x^2$ is NOT uniformly continuous on \mathbb{R} . ■

Example. Prove that $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Solution. Use $x = \frac{1}{2n\pi}$ and $c_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Now, observe that $\lim x_n = 0$ and $\lim c_n = 0$ implies $\lim(x_n - c_n) = 0$. Thus, $\lim |x_n - c_n| = 0$. But for all n , we have

$$|f(x_n) - f(c_n)| = \left| \sin(2n\pi) - \sin\left(2n\pi + \frac{\pi}{2}\right) \right| = |0 - 1| = 1.$$

So, f is NOT uniformly continuous. ■

Theorem (Continuous Functions on Compact Sets are Uniformly Continuous). Let $f : A \subseteq (X, d) \rightarrow (Y, \tilde{d})$ is continuous and let A be a compact set. Then f is uniformly continuous on A .

Proof. Assume for sake of contradiction that f is NOT uniformly continuous; that is, there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \tilde{d}(f(x), f(c)) \geq \varepsilon.$$

In particular,

$$\text{For } \delta = 1 \exists x_1, c_1 \in A \text{ satisfying } d(x_1, c_1) < 1 \text{ but } \tilde{d}(f(x_1), f(c_1)) \geq \varepsilon$$

$$\text{For } \exists x_2, c_2 \in A \text{ satisfying } d(x_2, c_2) < \frac{1}{2} \text{ but } \tilde{d}(f(x_2), f(c_2)) \geq \varepsilon$$

$$\text{For } \exists x_3, c_3 \in A \text{ satisfying } d(x_3, c_3) < \frac{1}{3} \text{ but } \tilde{d}(f(x_3), f(c_3)) \geq \varepsilon$$

\vdots

In this way, we will obtain two sequences (x_n) and (c_n) in A such that

$$(i) \quad 0 \leq d(x_n, c_n) < \frac{1}{n} \text{ for all } n. \text{ This implies that } \lim_{n \rightarrow \infty} d(x_n, c_n) = 0.$$

$$(ii) \quad \tilde{d}(f(x_n), f(c_n)) \geq \varepsilon \text{ for all } n.$$

Note that A is compact, so it is sequentially compact. If (x_n) is a sequence in A , then (x_n) contains a subsequence (x_{n_k}) that converges to a point in A . Let $x = \lim_{k \rightarrow \infty} x_{n_k}$. Let (c_{n_k}) be the corresponding subsequence of (c_n) . We have

$$0 \leq d(c_{n_k}, x) \leq d(c_{n_k}, x_{n_k}) + d(x_{n_k}, x).$$

If we let $k \rightarrow \infty$ on the right-hand side of the above inequality, we have $d(c_{n_k}, x_{n_k}) \rightarrow 0$ and $d(x_{n_k}, x) \rightarrow 0$. Using the Squeeze Theorem, we have $\lim_{k \rightarrow \infty} c_{n_k} = x$. Therefore, (x_{n_k}) and (c_{n_k}) are two sequences in A that converge to $x \in A$. Since f is continuous and $x_{n_k} \rightarrow x$, we have

$$f(x_{n_k}) \rightarrow f(x). \tag{1}$$

Similarly, if $c_{n_k} \rightarrow x$, we have

$$f(c_{n_k}) \rightarrow f(x). \tag{2}$$

So, there exists $N_0 \in \mathbb{N}$ such that for all $k > N_0$

$$\forall k > N_0 \quad \tilde{d}(f(x_{n_k}), f(x)) < \frac{\varepsilon}{4} \quad \text{and} \quad \tilde{d}(f(c_{n_k}), f(x)) < \frac{\varepsilon}{4}.$$

As a consequence, for all $k > N_0$, we have

$$\begin{aligned} \tilde{d}(f(x_{n_k}), f(c_{n_k})) &\leq \tilde{d}(f(x_{n_k}), f(x)) + \tilde{d}(f(x), f(c_{n_k})) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

But this contradicts (ii). ■