0.1 Lecture 1

0.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

0.1.2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

0.1.3 The First Defining Property

The set of real numbers is a field.

Definition (Fields)**.** A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

0.1.4 The Second Defining Property

Definition (Ordered Fields)**.** An **ordered field** is a field F equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

Remark. We say that x is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

Definition (Upper Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call β an **upper bound of** A.

Remark. We call the collection of upper bounds of A by $\mathrm{UP}(A)$. If $\mathrm{UP}(A) \neq \emptyset$, then we say that A **is bounded above**.

Similarly, we define the lower bounds of a set.

Definition (Lower Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then α is called the **lower bound of** A.

Remark. Similarly, we denote the set of lower bounds of A by $\mathrm{LO}(A)$. We say that A is bounded below if $\mathrm{LO}(A) \neq \emptyset$.

Example 0.1.1. Suppose we have $A = [0, 1)$. We have

$$\begin{aligned} \mathrm{UP}(A) &= [1, \infty) \\ \mathrm{LO}(A) &= (-\infty, 0]. \end{aligned}$$

0.2 Lecture 2

0.2.1 Review of Least Upper Bound Property

Definition (Supremum)**.** Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\beta \in F$ such that

(i) $\beta \in \mathrm{UP}(A)$

(ii) If $\gamma \in F$ and $\gamma < \beta$, then $\gamma \notin \mathrm{UP}(A)$.

We call β the **least upper bound** of A or the **supremum** of A. We denote the supremum of A as $\beta = \sup A$.

Remark. When we say THE supremum, we are implicitly stating that the supremum of A is unique.

Definition (Infimum). Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\alpha \in F$ such that

(i) $\alpha \in \mathrm{LO}(A)$

(ii) If $\gamma \in F$ and $\gamma > \alpha$, then $\gamma \notin \mathrm{LO}(A)$.

We call α the **greatest upper bound** of A or the **infimum** of A, and write $\alpha = \inf A$.

Definition (Least Upper Bound Property). An ordered field F is said to have the **least-upper-bound property** if the following is true:

Every nonempty set A in F that is bounded above has a least upper bound in F.

That is, if $A \neq \emptyset$ and $\mathrm{UP}(A) \neq \emptyset$, then $\sup(A)$ exists.

Theorem. There is exactly one ordered field that has the least-upper-bound bound property. The set $\mathbb{R}$ is the unique ordered field that contains $\mathbb{Q}$ as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$ is dedekind complete
- $\mathbb{R}$ satisfies the Axiom of Completeness.

Remark. Note that $\mathbb{Q}$ being an ordered field does not immediately imply that $\mathbb{Q}$ has the LUBP.

Definition (Maximums and Infimums). Let $A \subseteq \mathbb{R}$.

- If $\sup A \in A$, then we call, $\sup A$, the **maximum of** A and we denote this by $\max A$.
- If $\inf A \in A$, we call, $\inf A$, the **minimum of** A and we denote this by $\min A$.

Lemma (Useful Fact for Supremum). Let $A \subseteq \mathbb{R}$. Then $\beta = \sup A$ if and only if

(i) $\beta \in \mathrm{UP}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a > \beta - \varepsilon$.

Remark. We can restate property (ii) above as "for all $\varepsilon > 0$, $\beta - \varepsilon \notin \mathrm{UP}(A)$".

Lemma (Useful Fact for Infimums). Let $A \subseteq \mathbb{R}$. Then $\alpha = \inf A$ if and only if

(i) $\alpha \in \mathrm{LO}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a < \alpha + \varepsilon$.

Remark. Similarly, we can restate property (ii) as "for all $\varepsilon > 0$, $\alpha + \varepsilon \notin \mathrm{LO}(A)$".

Theorem (Greatest Lower Bound Property of $\mathbb{R}$). Every nonempty subset A of $\mathbb{R}$ that is bounded below has a **greatest upper bound in** $\mathbb{R}$.

Another way to say this is the following:

$$\text{If } A \neq \emptyset \text{ and } \mathrm{LO}(A) \neq \emptyset, \text{ then } \inf A \text{ exists in } \mathbb{R}.$$

0.2.2 Consequences of Least Upper Bound Property

Theorem (Archimedean Property). If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.

Proof. Let $A = \{nx : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ since $1 \cdot x \in A$. Suppose for sake of contradiction that for all $n \in \mathbb{Z}^+$, $nx \leq y$. This means that y is an upper bound of A. Let $\beta = \sup A$. By the first useful fact, we have that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\beta - \varepsilon < nx$. Let $\varepsilon = x$. Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \Rightarrow \beta < x(n+1).$$

But this tells us that $x(n+1) \in A$ ($x \in A$ and $n+1 \in \mathbb{N}$) and that β is NOT an upper bound which is a contradiction. Thus, it must be the case that $nx > y$ for some $n \in \mathbb{Z}^+$. ■

Remark. The well ordering property of $\mathbb{N}$ can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

Corollary. Let A be a nonempty subset of $\mathbb{R}$ that consists of only integers.

(i) If A is bounded above, then $\sup(A) \in A$.

(ii) If A is bounded below, then $\inf(A) \in A$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$). Let $x, y \in \mathbb{R}$ with $x < y$, there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Our goal is to find a $p \in \mathbb{Q}$ such that

$$x < p < y$$

with $p = \frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; that is, find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$nx < m < ny.$$

First, notice that $x < y$. This implies that $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \tag{1}$$

Choose $m \in \mathbb{Z}$ such that m to be the minimum element greater than nx; that is, choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m. \tag{2}$$

Let $A = \{k \in \mathbb{Z} : k > nx\}$ which is nonempty by the Archimedean Property. Furthermore, nx is a lower bound for A. By the Well-ordering property, A contains a minimum. Thus, $m = \min A$. Hence,we have

$$nx < m \Rightarrow x < \frac{m}{n}. \tag{3}$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \leq nx \Longrightarrow m \leq nx + 1 &< n\Big(y - \frac{1}{n}\Big) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \tag{4}$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \iff x < p < y.$$

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