0.1 Lecture 12

0.1.1 Topics

- Definition of $K-$cell.
- Theorem: If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.
- Theorem: Every $k-$cell is compact.
- Theorem: Suppose $E \subseteq \mathbb{R}^k$. Then
- Connected sets

  E is closed and bounded $\iff$ E is compact $\iff$ Every infinite subset of E has a limit point in E.
- Theorem: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Corollary. If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of compact sets, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Theorem (Nested Interval Property). If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Definition (K-cell). The set $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is called a $k-$cell in $\mathbb{R}^k$.

Example. Let $I = [a_1, a_2] \times [a_2, b_2]$ is a $2-$cell in $\mathbb{R}^2$.

Theorem (Nested Cell Property). If $I_1 \supseteq I_2 \supseteq I_3 \cdots$ is a nested sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(n)}, b_1^{(n)}] \times \cdots \times [a_k^{(n)}, b_k^{(n)}]$$

Also, let

$$\forall n \in \mathbb{N} \text{ and } \forall 1 \leq i \leq k, \text{ we have } A_i^{(n)} = [a_i^{(n)}, b_i^{(n)}]$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$A_i^{(n)} \supseteq A_i^{(n+1)} \; \forall 1 \leq i \leq k.$$

That is,

$$\begin{aligned}
I_1 &= A_1^{(1)} \times \cdots \times A_k^{(1)} \\
I_2 &= A_1^{(2)} \times \cdots \times A_k^{(2)} \\
&\vdots \\
I_n &= A_1^{(n)} \times \cdots A_k^{(n)}.
\end{aligned}$$

Hence, it follows from the nested interval property that there exists

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}$$
$$\exists x_2 \in \bigcap_{i=1}^{\infty} A_2^{(n)}$$
$$\vdots$$
$$\exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}.$$

Thus, by a fact in set theory; that is,

$$(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D).$$

$$\begin{aligned}(x_1, \dots, x_k) &\in \Big[\bigcup_{n=1}^{\infty} A_1^{(n)}\Big] \times \Big[\bigcap_{n=1}^{\infty} A_2^{(n)}\Big] \times \cdots \times \Big[\bigcap_{n=1}^{\infty} A_k^{(n)}\Big] \\ &\subseteq \bigcap_{n=1}^{\infty} [A_1^{(n)} \times \cdots \times A_k^{(n)}] \\ &= \bigcap_{n=1}^{\infty} I_n.\end{aligned}$$

Hence, we see that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

Theorem. Every $k-$cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for $2-$cells. The proof for a general $k-$cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a $2-$cell. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Let

$$\delta = d(a, b) = \|a - b\| = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$$

. Note that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in I, then

$$x_1, y_1 \in [a_1, b_2] \Longrightarrow |x_1 - y_1| \leq |b_1 - a_1|$$
$$x_2, y_2 \in [a_2, b_2] \Longrightarrow |x_2 - y_2| \leq |b_2 - a_2|$$

which implies that

$$\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta.$$

So, $d(x, y) \leq \delta$. Let us assume for contradiction that I is NOT compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of I that does NOT have a finite subcover; that is, $I \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2} \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i].$$

These subintervals determine 4 $2-$cells. There is at least one of these 4 $2-$cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let us call this 2−cell as I_1. Notice that

$$\forall x, y \in I_1 \quad \|x-y\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into 4 2−cells and continue this process inductively. In this manner, we will obtain a sequence of 2−cells

$$I, I_1, I_2, I_3, \ldots$$

such that

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \tag{1}$$

$$\forall x, y \in I_n, \quad \|x-y\| \leq \frac{\delta}{2} \tag{2}$$

$$\forall n \in \mathbb{N} \quad I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in I}$$

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Theorem (Heine-Borel Theorem). Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point of E.

Proof. ■

Remark. Note that in any general metric space, we have $(a) \implies (b)$ is not necessarily true.

Theorem (Bolzano-Weierstrass Theorem). Let $E \subseteq \mathbb{R}^k$ and E is an infinite set and bounded. Then $E' \neq \emptyset$.

Proof. Suppose that E is bounded. Then there exists a k−cell I such that $E \subseteq I$. By Theorem 2.40, we know that I is a compact set. Furthermore, we know that I is limit point compact by Theorem 2.41. So, every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E \neq \emptyset$. ■

Definition (Connected Sets, Disconnected, connected). Let (X, d) be a metric space.

(i) Two sets $A, B \subseteq X$ are aid to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(iii) A set $E \subseteq X$ is said to **disconnected** if it can be written as a union of tow nonempty separated sets A and B; that is, $E = A \cup B$.

(iv) A set $E \subseteq X$ is said to be connected if it is NOT disconnected.

Example ($\mathbb{R}$ with the standard metric). (*) If we have $A = (1, 2)$ and $B = (2, 5)$ are separated, then

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset$$
$$A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.$$

Hence, $E = A \cup B$ is disconnected.

(*) We have $C = (1, 2]$ and $D = (2, 5)$ are disjoint but not separated; that is, we have

$$C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\}$$
$C \cup D = (1, 5)$ is indeed connected.

0.2 Lecture 12