Math 230A Lecture Notes

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Contents

Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Goals of Course

- The goal of this course is to explore and generalize many of concepts that we learned in our calculus classes.
- Examples of such concepts are
  - Limits
  - Continuity
  - Sequence convergence
  - Differentiability
  - Integration

  and their results will all be rigorously proven and generalized.

1.1.2 The Structure of the Real Numbers

The set $\mathbb{R}$ is NOT just a boring collection of elements. $\mathbb{R}$ is a set equipped with four defining properties.

- $\mathbb{R}$ is a **field**.
- $\mathbb{R}$ is an **ordered field**.
- $\mathbb{R}$ is a unique ordered field that **least upper bound property**.
- $\mathbb{R}$ contains a metric which is a notion that describes length and distance.
- $\mathbb{R}$ is a normed space and a metric space (these two are not equivalent).

1.1.3 The First Defining Property

The set of real numbers is a field.

Definition (Fields)**.** A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms, respectively:

(A1) For all $x, y \in F$, we have $x + y \in F$.

(A2) For all $x, y \in F$, we have $x + y = y + x$.

(A3) For all $x, y, z \in F$, we have $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that for any $x \in F$, $x + 0 = x$.

(A5) If $x \in F$, then there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) For all $x, y \in F$, we have $xy \in F$.

(M2) For all $x, y \in F$, we have $xy = yx$.

(M3) For all $x, y, z \in F$, we have $(xy)z = x(yz)$.

(M4) For all $x \in F$, there exists an element $1 \neq 0$ such that $x \cdot 1 = x$.

(M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$.

1.1.4 The Second Defining Property

Definition (Ordered Fields)**.** An **ordered field** is a field F equipped with a relation, $<$, with the following properties

(i) If $x \in F$ and $y \in F$, then one and only one of the statements is true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) (**Transitive Property**) If $x, y, z \in F$ and $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(iv) If $x, y \in F$, and $x > 0$ and $y > 0$, then $xy > 0$.

Remark. We say that x is positive if $x > 0$, and negative if $x < 0$. Furthermore, $x \leq y$ is equivalent to $x = y$ or $x < y$.

The first two defining properties alone of $\mathbb{R}$ do not uniquely specify it. For example, $\mathbb{Q}$ is another field that satisfies the first two properties of $\mathbb{R}$.

Definition (Upper Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\beta \in F$ such that for all $x \in A$, $x \leq \beta$ for all $x \in A$. We call β an **upper bound of** A.

Remark. We call the collection of upper bounds of A by $\mathrm{UP}(A)$. If $\mathrm{UP}(A) \neq \emptyset$, then we say that A **is bounded above**.

Similarly, we define the lower bounds of a set.

Definition (Lower Bounds)**.** Suppose F is an ordered field, and $A \subseteq F$. If there exists $\alpha \in A$ such that for all $x \in A$, $x \geq \alpha$, then α is called the **lower bound of** A.

Remark. Similarly, we denote the set of lower bounds of A by $\mathrm{LO}(A)$. We say that A is bounded below if $\mathrm{LO}(A) \neq \emptyset$.

Example. Suppose we have $A = [0, 1)$. We have

$$\begin{aligned} \mathrm{UP}(A) &= [1, \infty) \\ \mathrm{LO}(A) &= (-\infty, 0]. \end{aligned}$$

1.2 Lecture 2

1.2.1 Review of Least Upper Bound Property

Definition (Supremum)**.** Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\beta \in F$ such that

(i) $\beta \in \mathrm{UP}(A)$

(ii) If $\gamma \in F$ and $\gamma < \beta$, then $\gamma \notin \mathrm{UP}(A)$.

We call β the **least upper bound** of A or the **supremum** of A. We denote the supremum of A as $\beta = \sup A$.

Remark. When we say THE supremum, we are implicitly stating that the supremum of A is unique.

Definition (Infimum)**.** Suppose F is an ordered field, and $A \subseteq F$. Suppose there exists $\alpha \in F$ such that

(i) $\alpha \in \mathrm{LO}(A)$

(ii) If $\gamma \in F$ and $\gamma > \alpha$, then $\gamma \notin \mathrm{LO}(A)$.

We call α the **greatest upper bound** of A or the **infimum** of A, and write $\alpha = \inf A$.

Definition (Least Upper Bound Property)**.** An ordered field F is said to have the **least-upper-bound property** if the following is true:

Every nonempty set A in F that is bounded above has a least upper bound in F.

That is, if $A \neq \emptyset$ and $\mathrm{UP}(A) \neq \emptyset$, then $\sup(A)$ exists.

Theorem. There is exactly one ordered field that has the least-upper-bound bound property. The set $\mathbb{R}$ is the unique ordered field that contains $\mathbb{Q}$ as a subfield.

This is equivalent to saying that:

- $\mathbb{R}$ is dedekind complete
- $\mathbb{R}$ satisfies the Axiom of Completeness.

Remark. Note that $\mathbb{Q}$ being an ordered field does not immediately imply that $\mathbb{Q}$ has the LUBP.

Definition (Maximums and Infimums)**.** Let $A \subseteq \mathbb{R}$.

- If $\sup A \in A$, then we call, $\sup A$, the **maximum of** A and we denote this by $\max A$.
- If $\inf A \in A$, we call, $\inf A$, the **minimum of** A and we denote this by $\min A$.

Lemma (Useful Fact for Supremum)**.** Let $A \subseteq \mathbb{R}$. Then $\beta = \sup A$ if and only if

(i) $\beta \in \mathrm{UP}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a > \beta - \varepsilon$.

Remark. We can restate property (ii) above as "for all $\varepsilon > 0$, $\beta - \varepsilon \notin \mathrm{UP}(A)$".

Lemma (Useful Fact for Infimums)**.** Let $A \subseteq \mathbb{R}$. Then $\alpha = \inf A$ if and only if

(i) $\alpha \in \mathrm{LO}(A)$ and

(ii) For all $\varepsilon > 0$, there exists $a \in A$ such that $a < \alpha + \varepsilon$.

Remark. Similarly, we can restate property (ii) as "for all $\varepsilon > 0$, $\alpha + \varepsilon \notin \mathrm{LO}(A)$".

Theorem (Greatest Lower Bound Property of $\mathbb{R}$)**.** Every nonempty subset A of $\mathbb{R}$ that is bounded below has a **greatest upper bound in** $\mathbb{R}$.

Another way to say this is the following:

$$\text{If } A \neq \emptyset \text{ and } \mathrm{LO}(A) \neq \emptyset, \text{ then } \inf A \text{ exists in } \mathbb{R}.$$

1.2.2 Consequences of Least Upper Bound Property

Theorem (Archimedean Property)**.** If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.

Proof. Let $A = \{nx : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ since $1 \cdot x \in A$. Suppose for sake of contradiction that for all $n \in \mathbb{Z}^+$, $nx \leq y$. This means that y is an upper bound of A. Let $\beta = \sup A$. By the first useful fact, we have that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\beta - \varepsilon < nx$. Let $\varepsilon = x$. Then we find that

$$\beta < nx + \varepsilon = nx + x = x(n+1) \implies \beta < x(n+1).$$

But this tells us that $x(n+1) \in A$ ($x \in A$ and $n+1 \in \mathbb{N}$) and that β is NOT an upper bound which is a contradiction. Thus, it must be the case that $nx > y$ for some $n \in \mathbb{Z}^+$. ■

Remark. The well ordering property of $\mathbb{N}$ can be proven as a consequence of nonempty sets of natural numbers containing a minimum.

Corollary. Let A be a nonempty subset of $\mathbb{R}$ that consists of only integers.

(i) If A is bounded above, then $\sup(A) \in A$.

(ii) If A is bounded below, then $\inf(A) \in A$.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$ with $x < y$, there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Our goal is to find a $p \in \mathbb{Q}$ such that

$$x < p < y$$

with $p = \frac{m}{n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; that is, find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$nx < m < ny.$$

First, notice that $x < y$. This implies that $y - x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x \iff x < y - \frac{1}{n}. \tag{1}$$

Choose $m \in \mathbb{Z}$ such that m to be the minimum element greater than nx; that is, choose $m \in \mathbb{Z}$ such that

$$m - 1 \le nx < m. \tag{2}$$

Let $A = \{k \in \mathbb{Z} : k > nx\}$ which is nonempty by the Archimedean Property. Furthermore, nx is a lower bound for A. By the Well-ordering property, A contains a minimum. Thus, $m = \min A$. Hence,we have

$$nx < m \implies x < \frac{m}{n}. \tag{3}$$

Using the left-hand side of (2) and the inequality found in (1), we can write

$$\begin{aligned} m - 1 \le nx \implies m \le nx + 1 &< n\Big(y - \frac{1}{n}\Big) + 1 \\ &= ny - 1 + 1 \\ &= ny. \end{aligned}$$

Thus, we see that

$$m < ny. \tag{4}$$

With (3) and (4), we can conclude that

$$x < \frac{m}{n} < y \iff x < p < y.$$

■

Chapter 2

Week 2

2.1 Lecture 3

2.1.1 Topics

- Review, Existence of Roots.
- Function, injective, and surjective.
- Equivalent Sets
- Finite, Infinite, Countable, At most countable.

2.1.2 Review, Existence of Roots

Proposition. There is no rational number whose square is 2.

Theorem. There is a unique positive real number α satisfying $\alpha^2 = 2$.

Proof. (i) **Uniqueness:** Suppose there are two of them α_1 and α_2. Prove that both $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ lead to a contradiction. Thus, $\alpha_1 = \alpha_2$.

(ii) **Existence:** Show that A is nonempty and bounded above. Let $\alpha = \sup A$. Prove that both $\alpha^2 > 2$ and $\alpha^2 < 2$ leads to a contradiction. Thus, $\alpha^2 = 2$.

■

Remark. A similar argument can be used to prove that if $x > 0$ and $m \in \mathbb{N}$, then t there exists a unique positive real number α such that $\alpha^m = x$. We write

$$\alpha = \sqrt[m]{x} \text{ and } \alpha = x^{1/m}.$$

2.1.3 Functions, Injective, and Surjective

There are two definitions for functions. The former is the most common way it is defined and the latter is the more rigorous and more "correct" definition.

Definition (Usual Way of Defining Functions)**.** Let A and B be two sets. A **function** from A to B denoted by $f : A \to B$, is a rule that assigns each element $x \in A$ a unique element $f(x) \in B$.

In the definition above, what do we mean by "rule" and "assigning"? Notice how these words are not very mathematically precise.

Definition (The Correct Way of Defining Functions)**.** Let A and B be two sets. A function from A to B is a triple (f, A, B) where f is a **relation** form A to B satisfying

(i) $\text{Dom}(f) = A$

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. (In this case, A is called the **domain** of f and B is called the **codomain** of f)

Example. Let $A = \emptyset$ and B be any set. Clearly, $\emptyset \times B = \emptyset$. So, the only function from $A = \emptyset$ to B is the empty function $(f, \emptyset, B)$.

- The empty function is one-to-one.
- The empty function is onto only when $B = \emptyset$.

Definition (Image, Range, Onto (Surjective))**.** Consider a function $f : A \to B$. Let $E \subseteq A$. Define the **image** of f as the set

$$f(E) = \{f(x) : x \in E\} = \{y \in B : y = f(x) \text{ for some } x \in E\}.$$

Define the **range** of f as

$$f(A) = \{\text{the collection of all the outputs of } f\}.$$

If $f(A) = B$, then we say f is **Onto (Surjective)**.

Definition (Preimage)**.** Consider a function $f : A \to B$. Let $D \subseteq B$. Then the **preimage** of D under f is denoted by

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

Definition (One-to-One (Injective))**.** Consider a function $f : A \to B$. We call f one-to-one if any of the following equivalent conditions hold:

(i) For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(iii) For all $y \in B$, the set $f^{-1}(\{y\})$ consists at most one element of A.

2.1.4 Equivalent Sets

Definition. Let A and B be two sets. We say that A and B have the same cardinal number, and we write $A \sim B$, if there is a function $f : A \to B$ that is both injective and surjective.

Remark.
- An injective and surjective mapping is a bijective mapping.

- A and B have the same cardinal number

$$\begin{aligned}
&= A \text{ and } B \text{ have the same cardinality} \\
&= A \text{ and } B \text{ can be put in the } \textbf{one-to-one correspondence} \\
&= \operatorname{card} A = \operatorname{card} B \\
&= A \text{ and } B \text{ are equivalent} \\
&= A \text{ and } B \text{ are equipotent}
\end{aligned}$$

Example. Consider $\{1,2,3\} \sim \{a,b,c\}$. Indeed, the function $f : \{1,2,3\} \to \{a,b,c\}$ defined by

$$f(1) = a, f(2) = b, f(3) = c$$

is a bijection.

Example. $\mathbb{N} \sim \{2,4,6,\dots\}$. Indeed, the function $f : \mathbb{N} \to \{2,4,6,\dots\}$ defined by

$$f(n) = 2n$$

is a bijection.

Example. $\mathbb{N} \sim \mathbb{Z}$. Indeed, $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

Example. $(-\infty,\infty) \sim (0,\infty)$. Indeed, $f(x) = e^x$ is a bijection between $(-\infty,\infty)$ and $(0,\infty)$.

Example. $(0,\infty) \sim (0,1)$. Indeed, the function $f : (0,\infty) \to (0,1)$ defined by

$$f(x) = \frac{x}{x+1}$$

is a bijection.

Example. $[0,1) \sim (0,1)$. Indeed, the function $f : [0,1) \to (0,1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for } n \geq 2 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

Definition ($\sim$ is an equivalence relation)**.** Let A and B be two sets. Note that

(i) $A \sim A$ ($\sim$ is reflexive)

(ii) If $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive).

Observe the following notation

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}$$
$$\mathbb{N} = \{1, 2, 3, \dots\}$$

2.1.5 Finite, Infinite, Countable, At most countable

Definition (Finite, Infinite, Countable, At most countable)**.** Let A be any set.

(a) We say that A is **finite** if $A \neq \emptyset$ or $A \sim \mathbb{N}_n$ for some natural number n.

  (*) When $A \sim \mathbb{N}_n$, we say A has n elements and we write $\text{card}(A) = n$.

  (*) Also, we set $\text{card}(\emptyset) = 0$.

(b) The set A is said to be **infinite** if it is not finite.

(c) The set A is said to be **countable** if $A \sim \mathbb{N}$; that is, there exists $g : \mathbb{N} \to A$ is a bijection where $A = \{g(1), g(2), g(3), \dots\}$.

(d) The set A is said to be **uncountable** if it is neither countable or finite.

(e) The set A is said to be **at most countable** if it is either finite or countable.

Remark. Previously, we shared $\mathbb{Z} \sim \mathbb{N}$. Thus, $\mathbb{Z}$ is countable. (Also, note that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, nevertheless, $\mathbb{N} \sim \mathbb{Z}$)

Below are some Basic Theorems

(i) Every countable set is infinite (There is no bijection $\mathbb{N}_n \to \mathbb{N}$).

(ii) Suppose $A \sim B$. Then

$$A \text{ is finite} \Longleftrightarrow B \text{ is finite}$$
$$A \text{ is countable} \Longleftrightarrow B \text{ is countable}$$
$$A \text{ is uncountable} \Longleftrightarrow B \text{ is uncountable}$$

(iii) The union of two finite sets is finite. If A is infinite and B is infinite, then $A \setminus B$ is infinite.

(iv) If A **is at most countable**, then there exists a $1-1$ function $f : A \to \mathbb{N}$.

Chapter 3

Week 3

3.1 Lecture 4

3.1.1 Topics

(1) Sequences

(2) Infinite subset of a countable set is countable.

3.1.2 Sequences

Definition (Sequence). We call a **sequence**, we mean a function f on the set $\mathbb{N}$.

- We can let $x_n = f(n)$. Then it is customary to denote the sequence f by $(x_n)_{n\geq 1}$ or $x_1, x_2, \ldots$.
- Note that $x_1, x_2, \ldots$ need not be distinct.
- If for all $n \in \mathbb{N}$, $x_n \in A$, then we say $(x_n)_{n\geq 1}$ is a sequence in A.
- Sometimes it is convenient to replace $\mathbb{N}$ in the definition above with $\{0, 1, 2, \ldots\}$ or $\{-1, 0, 1, 2, \ldots\}$.

3.1.3 Infinite subset of a countable set is countable

Theorem. Every infinite subset of a countable set is countable.

Proof. Let A be a countable set. Let $E \subseteq A$ and E is infinite. Our goal is to show that E is countable. Since A is countable, there exists a bijective function $g : \mathbb{N} \to A$, so

$$A = \{g(n) : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$$

with $x_n = g(n)$ for all $n \in \mathbb{N}$. Now, let us construct the sequence $n_1, n_2, \ldots$ as follows:

(1) Let n_1 be the smallest positive integer such that $x_{n_1} \in E$.

(2) Let n_2 be the smallest positive integer greater than n_1 such that $x_{n_2} \in E$.

$\vdots$

(k) Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Observe that the set

$$\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}$$

is both nonempty (since E is infinite) and bounded below (by the well ordering property of natural numbers). Thus, we know that

$$n_k = \min\{m \in \mathbb{N} : m > n_{k-1} \wedge x_m \in E\}.$$

Now, define the function $f : \mathbb{N} \to E$ as follows:

$$f(k) = x_{n_k}.$$

We claim that this is a bijective map. If we can prove this fact, then surely E is countable. We need to prove two things:

(1) f is injective and

(2) f is surjective.

Starting with (1), suppose $k_1 \neq k_2$. Then we have $n_{k_1} \neq n_{k_2}$ and thus, $x_{n_{k_1}} \neq x_{n_{k_2}}$. Hence, we see that $f(k_1) \neq f(k_2)$. Therefore, f is injective which proves (1).

With (2), let $b \in E$. Since $E \subseteq A$, we must have $b \in A$ and thus there exists an $m \in \mathbb{N}$ such that $b = x_m$. Hence, there exists $1 \leq k \leq m$ such that $x_{n_k} = x_m = b$. Thus, $f(k) = b$ which shows that f is surjective and so (2) is satisfied.

Therefore, f must be a bijective map and we can now conclude that E is countable. ■

Remark. Consider the contrapositive of the theorem above:

Let $E \subseteq A$. If E is not countable, then A is not countable.

When we say that a set is NOT countable, we do not necessarily mean that the set is uncountable. However, in the case that E is an infinite set, we can say that E being not countable is equivalent to saying that E is uncountable. So, only in the case in which E is infinite then the contrapositive is

Let $E \subseteq A$ be infinite. If E is uncountable, then A is uncountable. By the theorem we just proved, we see that $f(A) \subseteq S$ and S being countable implies that $f(A)$ must be at most countable. Thus, A must be at most countable.

Corollary. Let A be any set and let S be a countable set. If there exists an injective mapping $f : A \to S$, then A is at most countable.

Proof. Let A be any set and let S be a countable set. Suppose there exists an injective mapping $f : A \to S$. We can restrict the codomain S of f to its range $f(A)$. Because f is injective, the mapping $f : A \to f(A)$ will be bijective. Thus, we have that $A \sim f(A)$. ■

Example ($\mathbb{N} \times \mathbb{N}$ is countable). (1) The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(x, y) = 2^x 3^y$$

is injective. By the corollary we have just proved, we see that $\mathbb{N} \times \mathbb{N}$ is at most countable.

(2) Notice that $g : \mathbb{N} \to \{1\} \times \mathbb{N}$ is a bijection $g(a) = (1, a)$. Hence, $\{1\} \times \mathbb{N}$ is countable. Now, we have that

(i) $\{1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$

(ii) $\{1\} \times \mathbb{N}$ is countable implies that $\{1\} \times \mathbb{N}$ is infinite.

Thus, (i) and (ii) imply that $\mathbb{N} \times \mathbb{N}$ is countable.

Proving that a set A is countable can be done by finding a subset B that is countable which forces A to be countable.

Example ($\mathbb{Q}$ is countable). (1) The function $f : \mathbb{Q} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2(2^p 3^q) & \text{if } x = \frac{p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 2(2^p 3^q) + 1 & \text{if } x = \frac{-p}{q}\ p \in \mathbb{N}, q \in \mathbb{N}, (p,q) = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is an injective mapping. So, $\mathbb{Q}$ is at most countable.

(2) Observe that $\mathbb{N} \subseteq \mathbb{Q}$ and $\mathbb{N}$ being a countable (and thus $\mathbb{N}$ is infinite) set implies that $\mathbb{Q}$ is infinite. Thus, $\mathbb{Q}$ is is countable.

3.1.4 Countable union of at most countable sets is at most countable

Theorem. Countable union of at most countable sets is at most countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of at most countable sets; that is, for each $n \in \mathbb{N}$, A_n is at most countable. Our goal is to show that

$$K = \bigcup_{n \in \mathbb{N}} A_n \text{ is at most countable.}$$

To this end, it suffices to show that there exists an injective map $f : K \to \mathbb{N} \times \mathbb{N}$. Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\vdots \\ B_{n+1} &= A_{n+1} \setminus \Big(\bigcup_{k=1}^{n} A_k \Big). \end{aligned}$$

We leave as an exercise to show that

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \text{ are pairwise disjoint.}$$

Note that for all $n \in \mathbb{N}$, we have $B_n \subseteq A_n$ and A_n is at most countable. Then for each $n \in \mathbb{N}$. B_n must be at most countable. So, for each $n \in \mathbb{N}$, there exists an injective mapping $f_n : B_n \to \mathbb{N}$. Our goal is to show that

$$\bigcup_{n=1}^{\infty} B_n \text{ is at most countable.}$$

To this end, we define the function

$$f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N} \times \mathbb{N}$$

as follows:

For each $x \in \bigcup_{n=1}^{\infty} B_n$, there is exactly one $n \in \mathbb{N}$ such that $x \in B_n$; let us denote this n by n_x.

Thus, we define
$$f(x) = (n_x, f_{n_x}(x)).$$
This function is injective because
$$\begin{aligned} f(x) = f(y) &\implies (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y)) \\ &\implies n_x = n_y \ \wedge \ f_{n_x}(x) = f_{n_y}(y) \\ &\implies f_{n_k}(x) = f_{n_x}(y) \\ &\implies x = y && \text{(Since } f_{n_x} \text{ is injective)} \end{aligned}$$
By the corollary, we see that the set
$$\bigcup_{n=1}^{\infty} B_n$$
is at most countable. ■

Corollary. A countable union of countable sets is countable.

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. By the previous Theorem, we see that $\bigcup_{n\in\mathbb{N}} A_n$ is at most countable. Note that $A_1 \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and that A_1 is countable. Thus, A_1 must be infinite and so
$$\bigcup_{n\in\mathbb{N}} A_n \text{ is infinite.}$$
Thus, we see that
$$\bigcup_{n\in\mathbb{N}} A_n$$
must be countable. ■

Corollary. If A and B are at most countable, then $A \cup B$ at most countable.

Proof. Let $A_1 = A$ and $A_2 = B$ where $A_3 = A_4 = \cdots = A_n = \emptyset$. Then $A \cup B$ is at most countable by the previous Theorem. ■

Theorem. If A is countable, then $A \times A$ is countable.

Proof. (1) Note that $A \times A = \bigcup_{b\in A}(\{b\} \times A)$ for each $b \in A$.

(2) The function $f : A \to \{b\} \times A$ defined by $f(x) = (b, x)$ is bijective. So, $A \sim \{b\} \times A$. Hence, $\{b\} \times A$ is countable.

Since a countable union of countable sets is countable, we see that $A \times A$ must be countable by (1) and (2). ■

3.2 Lecture 5

3.2.1 Topics

- Summary of last lecture
- Collection of all binary sequences is uncountable.
- Preliminary Remarks

- Inequalities

3.2.2 Summary of Last Lecture

1. How to prove A is at most countable:
   - Show that $A \subseteq S$ where S is countable.
   - Find an injective function $f : A \to S$ where S is countable.
2. How to prove A is infinite:
   - Prove that A has a countable subset.
   - Find an injective function $f : \mathbb{N} \to A$.
   - Find an injective function $f : A \to A$ that is not onto.
   - Find a proper subset $B \subsetneq A$ such that $A \sim B$.
3. How to prove that $A \sim B$.
   - find a bijective function $f : A \to B$.
   - (Shroder-Bernstein)
     - Find an injective function $g : A \to B$
     - Find an injective function $h : B \to A$.
4. A countable union of countable sets is countable **AND**
5. A finite product of countable sets is countable.

Theorem (D)**.** Let A be the set of all sequences whose terms are the digits 0 and 1 (that is A is the collection of all binary sequences). This set A is uncountable.

Proof. First, notice that A is infinite. Let $h : \mathbb{N} \to A$ be the function defined by

for all $h(n) =$ The binary sequence whose nth term is the digit 1 and all other terms are zero.

Clearly, h is an injective map. Hence, $h : \mathbb{N} \to h(\mathbb{N})$ is bijective. We have $\mathbb{N} \sim h(\mathbb{N})$, and so $h(\mathbb{N})$ is infinite. Note that $h(\mathbb{N}) \subseteq A$. Thus, A is infinite

Suppose for sake of contradiction that A is NOT uncountable. Since A is infinite and not uncountable, this assumption tells us that A must be countable. Thus, there exists a bijective map $f : \mathbb{N} \to A$. So, we can write

$$A = \{f(n) : n \in \mathbb{N}\}.$$

This means that for each $n \in \mathbb{N}$, $f(n)$ is a binary sequence. Let

$$\begin{aligned}
f(1) &= (a_1^1, a_2^1, \dots) \\
f(2) &= (a_1^2, a_2^2, \dots) \\
f(3) &= (a_1^3, a_2^3, \dots) \\
&\vdots \\
f(n) &= (a_1^n, a_2^n, \dots) \\
&\vdots
\end{aligned}$$

The goal is to construct a binary sequence $(b_1, b_2, \dots)$ that is not in this list! This will contradict the fact that A contains all the binary sequences.

Define

$$b_1 = \begin{cases} 0 & \text{if } a_1^1 = 1 \\ 1 & \text{if } a_1^1 = 0 \end{cases}$$

Similarly, for the second term, we see that

$$b_2 = \begin{cases} 0 & \text{if } a_2^2 = 1 \\ 1 & \text{if } a_2^2 = 0 \end{cases}$$

and so on. More generally,

$$\text{for all } i \in \mathbb{N}\ b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}.$$

Clearly, this sequence (b_n) is not the same as any of the sequences of the list above; that is, for all $i \in \mathbb{N}$, $b_i \neq f(i)$. ■

3.2.3 Preliminary Remarks

- $\mathbb{R}$ is NOT just an ordered field, it has more extra structures.
- In $\mathbb{R}$, there is a standard notion of length (size, norm) and a corresponding notion of distance.
- Given $a \in \mathbb{R}$, the **size** of a is defined to be $|a|$. But note that the **absolute value of** a is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0. \end{cases}$$

- Given two numbers a and b in $\mathbb{R}$, the **standard distance** between a and b is

$$\operatorname{dist}(a, b) = |a - b|.$$

- The notion of distance plays an essential role in the development of calculus.
- When there is a notion of distance, then it will be possible to make sense of statements such as *as h gets close to zero....*
- Note that size, length, or norm are all equivalent in describing the size of the difference between two numbers/elements.

Theorem (Basic Properties of Standard Size in $\mathbb{R}$)**.** Let $a \in \mathbb{R}$. Then

(i) $|a| \geq 0$

(ii) $|a| = 0$ if and only if $a = 0$.

(iii) $|\alpha a| = |\alpha||a|$ for all $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.

(iv) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Remark. Note that (iv) is equivalent to saying that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This is an immediate consequence of the following inequalities:

(i) Every $a \in \mathbb{R}$, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Is it possible to generalize the notion of "size"? **YES!**

Definition. Let V be a (real) vector space. A function from $\|\cdot\| : V \to \mathbb{R}$ is called a **norm** on V if it satisfies the following properties:

(i) For all $x \in V$, $\|x\| \geq 0$

(ii) For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$.

(iv) For all $x, y \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

A vector space V equipped with notion of norm is called a normed space. (Sometimes, we write $(V, \|\cdot\|)$ is a normed space). So, call $d(x, y) = \|x - y\|$.

Theorem (Basic Properties of Standard Distance in $\mathbb{R}$)**.** For all $a, b \in \mathbb{R}$.

(i) $\text{dist}(a, b) \geq 0$

(ii) $\text{dist}(a, b) = 0$ if and only if $a = b$.

(iii) $\text{dist}(a, b) = \text{dist}(b, a)$.

(iv) For all $a, b, c \in \mathbb{R}$, we have $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

3.2.4 Metric Spaces

Definition (Metric Spaces)**.** Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is said to be a **distance function** or a **metric** if it satisfies the following properties:

(i) For all $x, y \in X$, $d(x, y) \geq 0$.

(ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(iii) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(iv) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X equipped with a metric d is called a **metric space**. (Sometimes we write (X, d) is a metric space)

Remark. X does **NOT** need to be a vector space. Furthermore, d is a more general notion of size than $\|\cdot\|$.

Example. $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |x - y|.$$

(or $d(x, y) = \alpha|x - y|$ where $\alpha > 0$ is a fixed real number)

Chapter 4

Week 4

4.1 Lecture 6

4.1.1 A few examples of Metrics

Example. Consider $(\mathbb{R}, d)$ where $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

This is a metric on $(\mathbb{R}, d)$. **Prove this on homework!**

Remark. If (X, D) is a metric space, then (X, D) is also a metric space where

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

We can define a metric that will always be less than or equal to 1.

Example (Taxi Cab Metric). Consider $(\mathbb{R}^2, d)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ is defined by

$$d((a, b), (x, y)) = |a - x| + |b - y|$$

We want to show that this is a metric.

(i) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have $d((a, b), (x, y)) = |a-x| + |b-y| \geq 0$ by property of the absolute value $|\cdot|$.

(ii) For all $(a, b), (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} d((a, b), (x, y)) = 0 &\iff |a - x| + |b - y| = 0 \\ &\iff |a - x| = 0 \text{ and } |b - y| = 0 \\ &\iff a - x = 0 \text{ and } b - y = 0 \\ &\iff (a, b) = (x, y). \end{aligned}$$

(iii) For all $(a, b), (x, y) \in \mathbb{R}^2$.

$$d((a, b), (x, y)) = |a - x| + |b - y| = |x - a| + |y - b| = d((x, y), (a, b)).$$

(iv) For all $(a,b),(x,y),(t,s)\in\mathbb{R}^2$, we want to show that

$$d((a,b),(x,y))\le d((a,b),(t,s))+d((t,s),(x,y)).$$

We have

$$\begin{aligned}d((a,b),(t,s))+d((t,s),(x,y))&=|a-t|+|b-s|+|t-x|+|s-y|\\&=(|a-t|+|t-x|)+(|b-s|+|s-y|)\\&\ge|a-x|+|b-y|\\&=d((a,b),(x,y))\end{aligned}$$

Example (Discrete Metric)**.** Consider $X\to$ any nonempty set. Consider $d:X\times X\to[0,\infty)$,

$$d(x,y)=\begin{cases}1 & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}.$$

We want to show that this is a metric on $(X,\text{any non-empty set})$.

(i) For all $x,y\in X$, we have $d(x,y)\ge0$ because either the $d(x,y)=1$ or $d(x,y)=0$.

(ii) Let $x,y\in X$. Clearly, we have $x=y$ if and only if $d(x,y)=0$ by definition of $d(x,y)$.

(iii) Let $x,y\in X$. Clearly, we have $d(x,y)=d(y,x)$ by definition.

(iv) Let $x,y,z\in X$. We want to show that

$$d(x,y)\le d(x,z)+d(z,y)$$

Let us consider two cases:

(1) $x=y$. Thus, both sides of the triangle inequality clearly hold since $d(x,y)=0$.

(2) $x\neq y$. In this case, we have $d(x,y)=1$. So, we need to show that

$$d(x,z)+d(z,y)\ge1.$$

Since $x\neq y$, at least one of the statements $z\neq y$ or $z\neq x$ is true. If $z\neq x$, then $d(z,x)=1$ and so

$$d(x,z)+d(z,y)=1+d(z,y)\ge1.$$

If $z\neq y$, then $d(z,y)=1$, and so

$$d(x,z)+d(z,y)=d(x,z)+1\ge1.$$

Example. Consider $(V,\|\cdot\|)\to$ any normed vector space with the metric $d:V\times V\to[0,\infty)$ and $d(x,y)=\|x-y\|$. We wan to show that this is a metric on V.

(i) Let $x,y\in V$. By the property of the norm, we have $d(x,y)=\|x-y\|\ge0$.

(ii) Let $x,y\in V$. By the 2nd property of the norm, we have

$$\begin{aligned}d(x,y)=0&\iff\|x-y\|=0\\&\iff x-y=0\\&\iff x=y.\end{aligned}$$

(iii) Let $x, y \in V$. We have

$$d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x).$$

(iv) Let $x, y, z \in V$. We want to show that

$$d(x,y) \leq d(x,z) + d(z,y).$$

We have,

$$\begin{aligned} d(x,z) + d(z,y) &= \|x-z\| + \|z-y\| \\ &\geq \|(x-z)+(z-y)\| \\ &= \|x-y\| \\ &= d(x,y). \end{aligned}$$

This example illustrates how every time we have a norm, we immediately have a metric space. But the other around is not necessarily true!

Example (Standard Distance in $\mathbb{R}^n$)**.** Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we have

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

Note that if we define for all $x \in \mathbb{R}^n$, we have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

Then

$$d(x,y) = \|x-y\|_2.$$

Proving that the above is a norm is enough to show that the original in question is a metric.

Let $x, y \in \mathbb{R}^n$.

(i) We have

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \geq 0.$$

(ii) We have

$$\begin{aligned} \|x_2\|_2 = 0 &\iff \sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0 \\ &\iff |x_1|^2 + \cdots + |x_n|^2 = 0 \\ &\iff |x_1| = 0, |x_2| = 0, \ldots, |x_n| = 0 \\ &\iff x_1 = 0, x_2 = 0, \ldots, x_n = 0. \end{aligned}$$

Thus, we have $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|\alpha x\|_2 &= \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2(x_1^2 + \cdots + x_n^2)} \\ &= |\alpha|\sqrt{x_1^2 + \cdots + x_n^2} \\ &= |\alpha|\|x\|_2. \end{aligned}$$

(iv) Now, we want to show that

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

That is, we want to show that

$$\sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} \leq \sqrt{x_1^2+\cdots+x_n^2} + \sqrt{y_1^2+\cdots+y_n^2}.$$

We will show this later!

Example. Consider $(\mathbb{R}^n, d)$ where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by for all $x, y \in \mathbb{R}^n$, we have

$$d_p(x,y) = \Big[|x_1-y_1|^p + \cdots + |x_n-y_n|^p\Big]^{\frac{1}{p}}.$$

4.1.2 Inequalities

In what follows we will list several key inequalities that can be used in proving that, for example, a given expression is a metric or a norm.

(1) The triangle inequality for the standard norm in $\mathbb{R}$. We have for all $x, y \in \mathbb{R}$, we have

$$|x+y| \leq |x| + |y|.$$

More generally, we have for all $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\Big|\sum_{i=1}^n x_i\Big| \leq \sum_{i=1}^n |x_i|.$$

(2) Reverse triangle inequality for the standard norm in $\mathbb{R}$; that is, we have for all $x, y \in \mathbb{R}$,

$$||x|-|y|| \leq |x-y|.$$

(3) For all $a, b \geq 0$ and for all $\rho > 0$, we have

$$ab \leq \frac{1}{2}\Big(\rho a^2 + \frac{1}{\rho}b^2\Big).$$

Note that we are dealing with real numbers here!

(4) Cauchy-Schwarz Inequality. For al $x, y \in \mathbb{R}^n$, we have

$$|x_1y_1+\cdots+x_ny_n| \leq \Big(\sqrt{x_1^2+\cdots+x_n^2}\Big)\Big(\sqrt{y_1^2+\cdots+y_n^2}\Big). \tag{*}$$

4.2 Lecture 7

4.2.1 Topics

- Review of Inequalities
- Neighborhood of a point
- Limit point of a set, isolated point of a set
- Closed set
- Interior point of a set

- Open set
- Bounded set
- Closure
- Dense

4.2.2 Inequalities

4.2.3 Minkowski

We wan to show the triangle inequality for $\|\cdot\|_2$ in $\mathbb{R}^n$; that is, we want to show that

Proposition. Let $\|\cdot\|_2$ be a norm in $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$, $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.

Proof. We see that $x \cdot y \leq |x \cdot y| \leq \|x\|_2 \|y\|_2$ by the Cauchy Cauchy-Schwarz Inequality. Thus, we have

$$\begin{aligned}
\|x+y\|_2^2 &= (x+y)\cdot(x+y) \\
&= x\cdot x + x\cdot y + y \cdot x + y\cdot y \\
&= \|x\|_2^2 + 2x\cdot y + \|y\|_2^2 \\
&\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
&= (\|x\|_2 + \|y\|_2)^2.
\end{aligned}$$

Hence, we have

$$\|x+y\|_2^2 \leq (\|x\|_2 + \|y\|_2)^2.$$

Therefore, we have

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

■

4.2.4 Minkowski for General p

For general p, we have Holder's Inequality which is

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

for any fixed real number $p \geq 1$ and for any $x, y \in \mathbb{R}^n$.

4.2.5 $(x+1)^n$

Recall that for all $a, b \in \mathbb{R}$, for all $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

So, in particular, if $x \geq 0$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\
&\geq 1 + nx.
\end{aligned}$$

Hence, we have for all $x \geq 0$ and for all $n \in \mathbb{N}$, we have

$$(x+1)^n \geq 1 + nx.$$

4.2.6 $p-$means

Let $x_1, \ldots, x_n$ be positive real numbers. Let $p \in \mathbb{N} \cup \{0\}$. By the $p-$mean of $x_1, \ldots, x_n$ denoted by $A_p(x_1, \ldots, x_n)$, we mean

$$A_p(x_1, \ldots, x_n) = \begin{cases} \frac{\sqrt[p]{x_1^p + \cdots + x_n s^p} n}{} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 \ldots x_n} & \text{if } p = 0. \end{cases}.$$

For example, if $p = 1$, we have

$$A_1(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

which is the Arithmetic Mean. If $p = 2$, we have

$$A_2(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

If $p = 0$, we have

$$A_0(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$$

which is the geometric mean. It can be shown that

$$A_0(x_1, \ldots, x_n) \leq A_1(x_1, \ldots, x_n) \leq A_2(x_1, \ldots, x_n) \leq \ldots .$$

In particular, we have $A_0 \leq A_1$; that is,

$$\sqrt[n]{x_1 \ldots x_n} \leq \frac{x_1 + \cdots + x_n}{n}. \tag{AM-GM Inequality}$$

We can prove this inequality using differentiation. Without differentiation, we can prove the same inequality using **Cauchy Induction**.

4.2.7 Jensen's Inequality

Suppose $f : (a, b) \to \mathbb{R}$ is a convex function ($f"(x) \geq 0$ for all $x \in (a, b)$). Let $x_1, \ldots, x_n$ be points in (a, b). Let $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

We want to show this inequality holds for $n = 2$; that is,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

where $\lambda_1 + \lambda_2 = 1$. Note that

$$f((1 - \lambda_2) x_1 + \lambda_2 x_2) \leq (1 - \lambda_2) f(x_2) + \lambda_2 f(x_2).$$

Remark. If we want to have an expression that defines a function that gives us any number in between two points $e < h$, we can have

$$f(\lambda) = (1 - \lambda) e + \lambda h$$

for any $0 \leq \lambda \leq 1$.

4.2.8 Neighborhood of a point

Definition (Neighborhood)**.** Let (X, d) be a metric space. Let $p \in X$. For any $\varepsilon > 0$, we call

$$N_\varepsilon(p) = \{x \in X : d(p, x) < \varepsilon\}.$$

the **neighborhood of p of radius ε**.

Example. Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. The neighborhood of any $p \in \mathbb{R}$ with radius $\varepsilon > 0$ is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Note that $|x - p| < \varepsilon$ is the same thing as $p - \varepsilon < x < p + \varepsilon$ or that $x \in (p - \varepsilon, p + \varepsilon)$.

Example. Let $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$. Let $(a, b) \in \mathbb{R}^2$ with $\varepsilon > 0$. Then

$$\begin{aligned} N_\varepsilon((a, b)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y)(a, b)) < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \varepsilon\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \varepsilon^2.\} \end{aligned}$$

Thus, $N_\varepsilon((a, b))$ consists of the points inside the circle of radius ε centered in (a, b).

Example. $(\mathbb{R}^2, d)$ with $d((a, b), (x, y)) = |a - x| + |b - y|$ and let $\varepsilon = 1$. We have

$$\begin{aligned} N_1((0, 0)) &= \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x - 0| + |y - 0| < 1\} \\ &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \end{aligned}$$

If we graph this out in $\mathbb{R}^2$, then the shape of the neighborhood will take on a rhombus.

Example. Let $(\mathbb{R}, d)$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $p \in \mathbb{R}$. Let $\varepsilon > 0$. Let us consider two cases:

(1) Let $\varepsilon \leq 1$. Note that if

$$d(x, p) < \varepsilon \leq 1,$$

then $d(x, p) < 1$, and so $d(x, p) = 0$. Hence, $x = p$. Then the neighborhood is

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \{p\}.$$

(2) Let $\varepsilon > 1$. Clearly, for all $x \in \mathbb{R}$, we have $d(x, p) \leq 1 < \varepsilon$. So,

$$N_\varepsilon(p) = \{x \in \mathbb{R} : d(x, p) < \varepsilon\} = \mathbb{R}.$$

4.2.9 Limit Points

Definition (Limit Points, Isolated Points). Let (X, d) is a metric space with $E \subseteq X$. Then we call

(1) A point $p \in X$ is said to be a **limit point of** E if for all $\varepsilon > 0$,

$$N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset.$$

(2) The **collection of all the limit points of** E is denoted by E'; that is,

$$E' = \{p \in X : \text{for all } \varepsilon > 0, N_\varepsilon(p) \cap (E \setminus \{p\}) \neq \emptyset\}.$$

(3) A point $p \in E$ is said to be an **isolated point of** E if p is NOT a limit point; that is, $p \in E$ but $p \notin E'$. Another way of saying this is $E \setminus E'$.

(4) If p is NOT a limit point, we have $p \notin E'$ if and only if there exists $\varepsilon > 0$ such that

$$N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset.$$

Remark. The statement for all $\varepsilon > 0$, $N_\varepsilon(p)$ is equivalent to for all $N_\varepsilon(p)$.

Example. Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$. Note that $0 \notin E$. Also, recall that $0 \in E'$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(0) \cap (E \setminus \{0\}) \neq \emptyset$. If $0 \in E'$, we just need to show that for all $\varepsilon > 0$, $N_\varepsilon(0) \cap E \neq \emptyset$; that is, we need to show that

$$\text{for all } \varepsilon > 0,\ (-\varepsilon, \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Clearly, $1/m \in (-\varepsilon, \varepsilon) \cap E$.

Example. Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$ and

$$E = (1, 2) \cup \{5\}.$$

Prove that 5 is an isolated point. Since $5 \in E$, it is enough to show that 5 is not in E'. Recall that $5 \in E^{'}$ if and only if there exists an $\varepsilon > 0$ such that

$$N_\varepsilon(5) \cap (E \setminus \{5\}) = \emptyset.$$

Noticing that $E \setminus \{5\} = (1, 2)$, we can write $5 \notin E'$ if and only if there exists an $\varepsilon > 0$ such that $(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = \emptyset$. Clearly, $\varepsilon = 1$ does the job and we are done; that is,

$$(5 - \varepsilon, 5 + \varepsilon) \cap (1, 2) = (4, 6) \cap (1, 2) = \emptyset.$$

Example. Let $(\mathbb{R}^2, d)$ and $d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2}$ and

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

What is E'?

$$E' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4.\}$$

For example, if (a, b) is such that $a^2 + b^2 > 4$, then $(a, b) \notin E'$. Let $\delta = \frac{1}{2}(\sqrt{a^2 + b^2} - 2)$. Clearly, $N_\delta \cap (E \setminus \{p\}) = \emptyset$.

Definition (Closed Set). Let (X, d) be a metric space, $E \subseteq X$. We say that E is **closed** if every limit point of E is contained within E; that is,

$$E \text{ is closed} \iff E' \subseteq E.$$

Example. Let $(\mathbb{R}, d)$, $d(x, y) = |x - y|$ and $E = \{1, 2, 3\}$.

(i) What is E'?

Claim: $E' = \emptyset$. Let $p \in \mathbb{R}$. Our goal is to show that p is not in E'. That is, we want to show that there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset$ where $N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon)$.

We may consider the following cases:

(1) If $p < 1$. Let $\varepsilon = \frac{1-p}{2}$ works.

(2) If $p > 3$, we have $\varepsilon = \frac{p-3}{2}$ works.

(3) If $p \in \{1, 2, 3\}$, then $\varepsilon = \frac{1}{4}$ works.

(4) If $1 < p < 2$, then let $\varepsilon = \frac{1}{2}\min\{p - 1, 2 - p\}$ works.

(5) If $2 < p < 3$, then $\varepsilon = \frac{1}{2}\{\min p - 2, 3 - p\}$ works.

(ii) Is E closed? Since $E' = \emptyset$, we have $E' \subseteq E$ and so E is closed.

Remark. Any finite set is closed!

Definition (Interior Point). Let (X, d) is a metric space and let $E \subseteq X$. We say that a point $p \in E$ is said to be an **interior point of** E if there exists a neighborhood $N_\varepsilon(x)$ such that $N_\varepsilon(x) \subseteq E$. The collection of all interior points of E is called the **interior of** E and is denoted by the set:

$$E^\circ = \{x \in E : \exists\ N_\varepsilon(x) \subseteq E\}.$$

Remark. Note that by definition we know that the interior of E is always contained within E; that is, $E^\circ \subseteq E$. Also,

$$p \in E^\circ \text{ if and only if there exists a neighborhood } N_\delta(p) \text{ such that } N_\delta(p) \subseteq E.$$

Example. Let $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$ with $E = (1, 3]$. What is $E^\circ =$?. We claim that $E^\circ = (1, 3)$.

Let $p \in (1, 3)$. We want to show that

(1) If $p \in (1, 3)$, then p is an interior point.

(2) If $p = 3$, then $p \notin E^\circ$.

We proceed by showing each case above:

(1) It suffices to show that there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$. Clearly, choose $\delta = \frac{1}{2}\min\{p - 1, 3 - p\}$ and we are done.

(2) Suppose $p = 3$. It suffices to show that

$$\forall\ \varepsilon > 0,\ N_\varepsilon(3) \not\subseteq E.$$

That is, we want to show that

$$\forall\, \varepsilon > 0, (3-\varepsilon, 3+\varepsilon) \cap E^c \neq \emptyset.$$

Clearly, for all $\varepsilon > 0$, we have $3 + \frac{\varepsilon}{2} \in (3-\varepsilon, 3+\varepsilon)$. Thus, $3 + \frac{\varepsilon}{2} \in E^c$. Hence, we have

$$(3-\varepsilon, 3+\varepsilon) \cap E^c \neq \emptyset.$$

To show that boundary points are not interior points, it suffices to show that intersection with each neighborhood and the complement of the set is question is nonmepty.

Example. Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = \{1,2,3\}$. What is $E^\circ = ?$. We claim that $E^\circ = \emptyset$. The reason is as follows: Let $p \in \{1,2,3\}$; that is, for all $\varepsilon > 0$, we have $N_\varepsilon(p) \not\subseteq E$ but $N_\varepsilon(p) = (p-\varepsilon, p+\varepsilon)$ has infinitely many points. We proved that if $p \in E$, then $p \notin E^\circ$. So, $E^\circ = \emptyset$.

Definition (Open Sets)**.** Let (X,d) be a metric space and $E \subseteq X$. We say that E is **open** if every point of E is an interior point of E; that is,

$$E \text{ is open} \iff E \subseteq E^\circ.$$

Remark. We know that, for any set, $E^\circ \subseteq E$. So, we can rewrite our definition as follows:

$$E \text{ is open} \iff E = E^\circ.$$

That is, the other inclusion holds!

Example. Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = \{1,2,3\}$. Is E open? Note that $E^\circ = \emptyset$. So, $E^\circ \neq E$ and so E is NOT open.

Example. Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = (1,4)$. Prove that E is open. It suffices to show that every point $p \in E$ is an interior point. Let $p \in E$. That is, we want to show that there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$. If we choose $\delta = \frac{1}{2}\{p-1, 4-p\}$ does the job and we are done.

Definition (Bounded Sets)**.** Let (X,d) be a metric space and $E \subseteq X$. We say that E is **bounded** if there exists $\varepsilon > 0$ and $q \in X$ such that $E \subseteq N_\varepsilon(q)$.

Example. Let $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$ and $E = [0,\infty)$. Is E bounded? **NO!** This is because for all $q \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$[0,\infty) \not\subseteq (q-\varepsilon, q+\varepsilon).$$

Example. Let $(\mathbb{R}, d)$ with the discrete metric and $E = [0,\infty)$. Is E bounded? **YES!** For example, we have

$$E \subseteq N_{10}(0) = \mathbb{R}.$$

Definition (Closure)**.** Let (X,d) be a metric space and $E \subseteq X$. The **closure** of E, denoted by $\overline{E}$, is defined as follows:

$$\overline{E} = E \cup E'$$

that is, the closure of E is the union of the isolated points and limit points.

Example. Let $(\mathbb{R}, d)$ and $d(x, y) = |x - y|$. What is $\overline{Q}$? Show within the homework that $\mathbb{Q}' = \mathbb{R}$. So,

Definition (Dense)**.** Let (X, d) metric space and $E \subseteq X$. We say that E is dense in X if $\overline{E} = X$. (that is, every point of X is either in E or is a limit point of E).

Example. $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$.

Chapter 5

Week 5

5.0.1 Topics

(1) Every neigborhood is an open set.

(2) If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

(3) Theorem: E is open $\Longleftrightarrow$ E^C is closed.

(4) Theorem: arbitrary union of open sets is open, Finite intersection of open sets is open.

(5) Theorem: $\overline{E}$ is the smallest closed subset of X that contains E.

Theorem. Let (X, d) be a metric space and let $p \in X$ and $\varepsilon > 0$. Every neighborhood is an open set; that is, $N_\varepsilon(p)$ is an open set.

Proof. Our goal is to show that every point of $N_\varepsilon(p)$ is an interior point of $N_\varepsilon(p)$. Let $q \in N_\varepsilon(p)$. We need to show that there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$. Let $\delta = \frac{\varepsilon - d(p,q)}{2}$. We claim that $N_\delta(q)$ is a subset of $N_\varepsilon(p)$. Indeed, if $x \in N_\delta(q)$, then

$$d(q, x) < \delta \Longrightarrow d(q, x) < \varepsilon - d(p, q)$$

and so

$$d(p, q) + d(q, x) < \varepsilon \Longleftrightarrow d(p, x) < \varepsilon. \qquad \text{(triangle inequality)}$$

Thus, $x \in N_\varepsilon(p)$.

■

Theorem. Let (X, d) be a metric space and $E \subseteq X$. If $p \in E'$, then every neighborhood of p contains infinitely many points of E

Proof. Suppose for sake of contradiction that there exists a neighborhood of p that contains finitely many points of E; that is,

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \cap E \text{ is a finite set.}$$

Since $N_\varepsilon(p) \cap (E \setminus \{p\}) \subseteq N_\varepsilon(p) \cap E$, we can immediately conclude that $N_\varepsilon(p) \cap (E \setminus \{p\})$ is finite also. Furthermore, $N_\varepsilon(p) \cap (E \setminus \{p\})$ is nonempty since $p \in E'$. Let us denote the elements of $N_\varepsilon(p) \cap (E \setminus \{p\})$ by $x_1, \ldots, x_n$. Our goal is to find a $\delta > 0$ such that the neighborhood with radius $\delta > 0$, we will not contain any of the finite points we introduced. For each $i \in \{1, \ldots, n\}$, $d(p, x_i) > 0$,

$$\delta = \min\{d(p, x_i) : i \in 1, \ldots, n\} > 0.$$

Clearly, $N_{\delta/2}(p) \cap (E \setminus \{p\}) = \emptyset$. But this contradicts our assumption that p is NOT a limit point of E. ■

Corollary. A finite set has no limit points; that is, if E is finite, then $E' = \emptyset$.

Proof. This is just the contrapositive of the theorem above. ■

Theorem. Let (X, d) be a metric space and $E \subseteq X$. We have that E is open if and only if E^c closed.

Proof. ($\Longrightarrow$) Assume that E is open. We want to show that E^c is closed; that is, every limit point of E^c is contained in E^c. Let p be a limit point of (E^c). Assume for sake of contradiction that $p \notin E^c$. Then $p \in E$. Since E is open, p is an interior point of E. Thus, there exists $\delta > 0$ such that $N_\delta(p) \subseteq E$; that is, there exists $\delta > 0$ such that $N_\delta(p) \cap E^c = \emptyset$. Furthermore, we have that

$$N_\delta(p) \cap (E^c \setminus \{p\}) = \emptyset.$$

But this tells us that p is not a limit point which is a contradiction.

($\Longleftarrow$) Assume that E^c is closed. We want to show that E is open; that is, every $x \in E$ is an interior point. Let $p \in E$. Assume for sake of contradiction that $p \notin E^\circ$. Then for all $\delta > 0$, $N_\delta(p) \nsubseteq E$. Hence, for all $N_\varepsilon(p) \cap E^c \neq \emptyset$. Therefore,

$$\forall \delta > 0 \ N_\delta(p) \cap (E^c \setminus \{p\}) \neq \emptyset.$$

That is, p is a limit point of E^c. But by assumption, E^c is closed. Thus, $p \in E^c$. But this contradicts the assumption that $p \in E$. ■

Theorem. Let (X, d) be a metric space. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets where Λ is an index set (can be finite or infinite). Then

$$\bigcup_{\alpha \in \Lambda} A_\alpha$$

is an open set.

Proof. Our goal is to show that every point of $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ is an interior point. Let $p \in A$. Thus, there exists $\alpha \in \Lambda$ such that $p \in A_\alpha$. Since A_α is open, so there exists $\delta > 0$ such that $N_\delta(p) \subseteq A_\alpha$. But note that

$$A_{\alpha_0} \subseteq A.$$

Hence,

$$N_\delta(p) \subseteq A$$

and thus p is an interior point of A.

■

Theorem. Let $A_1, \ldots, A_n$ be open sets in the metric space (X, d). Then

$$\bigcap_{k=1}^{n} A_k \text{ is open.}$$

Proof. Our goal is to show that every point of $\bigcap_{k=1}^{n} A_k$ is an interior point of $\bigcap_{k=1}^{n} A_k$. Let $p \in \bigcap_{k=1}^{n} A_k$. Then for all $1 \leq k \leq n$ such that $p \in A_k$. Since A_k is open for all k, we know that for all $1 \leq k \leq n$, there exists $\delta_k > 0$ such that $N_{\delta_k}(p) \subseteq A_k$. Let $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. Then we have that

$$N_\delta(p) \subseteq N_{\delta_k}(p) \subseteq A_k.$$

Consequently, we have

$$N_\delta(p) \subseteq \bigcap_{k=1}^{n} A_k.$$

Hence, p is an interior point of the intersection $\bigcap_{k=1}^{n} A_k$. ■

Theorem. Let (X, d) be a metric space and $E \subseteq X$.

(1) $\overline{E}$ is a closed set.

(2) E is closed if and only if $E = \overline{E}$.

(3) If $E \subseteq F$ and F is closed, then $\overline{E} \subseteq F$.

Proof. (1) Our goal is to show that $(\overline{E})^c$ is open. We need to show that every point of $(\overline{E})^c$ is an interior point of $(\overline{E})^c$. Let $p \in (\overline{E})^c$. We have

$$\begin{aligned} p \in (\overline{E})^c &\Longrightarrow p \notin \overline{E} \\ &\Longrightarrow p \notin (E \cup E') \\ &\Longrightarrow p \notin E \ \wedge \ p \notin E'. \end{aligned}$$

Note that

$$\begin{aligned} p \notin E' &\Longrightarrow \exists \varepsilon > 0 \ N_\varepsilon(p) \cap (E \setminus \{p\}) = \emptyset \\ &\Longrightarrow \exists \varepsilon > 0 \ N_\varepsilon(p) \cap E = \emptyset. \end{aligned} \tag{1}$$

In what follows, we will show that $N_\varepsilon(p) \cap E' = \emptyset$. So, we have

$$\begin{aligned} &N_\varepsilon(p) \cap (E \cup E') = \emptyset \\ &\Longrightarrow N_\varepsilon(p) \cap \overline{E} = \emptyset \\ &\Longrightarrow N_\varepsilon(p) \subseteq (\overline{E})^c. \end{aligned}$$

Thus, we have that p is an interior point of $(\overline{E})^c$. It remains to show that $N_\varepsilon(p) \cap E' = \emptyset$. Assume for sake of contradiction that $N_\varepsilon(p) \cap E' \neq \emptyset$. Let $q \in N_\varepsilon(p) \cap E'$. Then we have $q \in N_\varepsilon(p)$ and $q \in E'$. Because $N_\varepsilon(p)$ is an open set, there exists $\delta > 0$ such that $N_\delta(q) \subseteq N_\varepsilon(p)$ and that $N_\delta(q) \cap (E \setminus \{q\}) \neq \emptyset$, respectively. But note that since $N_\delta(q) \subseteq N_\varepsilon(p)$ and $E \setminus \{q\} \subseteq E$ implies that

$$N_\varepsilon(p) \cap E \neq \emptyset$$

which contradicts (1).

■

Definition (Metric Subspace)**.** Let (X, d) be a metric space and let Y be a nonempty subset of X. Then $(Y, d|_{Y \times Y})$ is called a **metric subspace** of (X, d).

Remark. Consider the metric subspace $(Y, d|_{Y\times Y}$ of (X,d). Let $p \in Y$ and $\varepsilon > 0$. Then we define

$$\begin{aligned} N_\varepsilon^Y(p) = \{y \in Y : d^Y(y,p) < \varepsilon\} &= \{y \in Y : d(y,p) < \varepsilon\} \\ &= N_\varepsilon(p) \cap Y. \end{aligned}$$

Chapter 6

Week 6

6.1 Lecture 10

Theorem (E is open relative to Y)**.** Let (X, d) be a metric space and $E \subseteq Y \subseteq X$ and $Y \neq \emptyset$. E is open relative to Y if and only if there exists an open set $G_0 \subseteq X$ such that $E = G \cap Y$.

Proof. ($\Longrightarrow$) Assume that E is open relative to Y. Our goal is to show that there exists an open set $G \subseteq X$ such that $E = G \cap Y$. Since E is open relative to Y, every $a \in E$, we have that a is an interior point of E; that is, there exists $\varepsilon_0 > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$. Hence, for all $a \in E$, there exists $\varepsilon_a > 0$ such that $N_{\varepsilon_a}(a) \cap Y \subseteq E$. Let $G = \bigcup_{a \in E} N_{\varepsilon_a}(a)$. Clearly, G is open in X since

(1) For all $a \in E$, $N_{\varepsilon_a}(a)$ is a neighborhood and so it is open in X.

(2) A union of open sets is open in any metric space.

In what follows, we will prove that $E = G \cap Y$. Note that

$$G \cap Y = \Big(\bigcup_{a \in E} N_{\varepsilon_a}(a)\Big) \cap Y = \bigcup_{a \in E} \Big(N_{\varepsilon_a} \cap Y\Big) \subseteq \bigcup_{a \in E} E = E.$$

Suppose $b \in E$. We have $b \in N_{\varepsilon_b}(b)$ and thus $b \in G$. Furthermore, $b \in E$ implies $b \in Y$ since $E \subseteq Y$. Thus, we see that $E \subseteq G \cap Y$.

($\Longleftarrow$) Assume that there exists $G \subseteq X$ such that $E = G \cap Y$. We want to show that E is open relative to Y. Our goal is to show that for all $a \in E$, there exists $\varepsilon > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$.

So, let $a \in E$ be given. Our goal is to find $\varepsilon > 0$ such that

$$N_\varepsilon(a) \cap Y \subseteq E.$$

By assumption, we have

$$\begin{aligned} a \in E = G \cap Y &\Longrightarrow a \in G \\ &\Longrightarrow_{G \text{ is open}} \exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \subseteq G. \end{aligned}$$

Hence, we see that

$$N_\varepsilon(a) \cap Y \subseteq G \cap Y = E.$$

■

Theorem. (i) If $E \subseteq \mathbb{R}$ is bounded above, then $\sup E \in \overline{E}$.

(ii) If $E \subseteq \mathbb{R}$ is bounded below, then $\inf E \in \overline{E}$.

Proof. Here we will prove (1) and the proof of (2) is completely analogous.

Since E is bounded above, we have that $\sup E$ exists and is a real number by the least upper bound property of $\mathbb{R}$. Let $\alpha = \sup E$. Our goal is to show that $\alpha \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,

$$N_\varepsilon(\alpha) \cap E \neq \emptyset. \qquad \text{(Exercise 11 of HW4)}$$

Let $\varepsilon > 0$ be given. Since $\alpha = \sup E$, we know that there exists $x \in E$ such that $\alpha - \varepsilon < x$. Hence, there exists $x \in E$ such that

$$\alpha - \varepsilon < x \leq \alpha < \alpha + \varepsilon.$$

Hence, $\alpha - \varepsilon < x < \alpha + \varepsilon$; that is, $x \in N_\varepsilon(\alpha)$. Therefore, $N_\varepsilon(\alpha) \cap E \neq \emptyset$. ■

Definition (Open Covers)**.** Let (X, d) be a metric space and $E \subseteq X$. A collection of sets $\{O_\alpha\}_{\alpha \in \Lambda}$ is said to be an **open cover** of E if

(i) for every $\alpha \in \Lambda$, O_α is open in X.

(ii) $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$.

Example. Consider $(\mathbb{R}, |\bullet|)$ and the subset $E = [0, \infty)$. The collection $\{E_n\}_{n \in \mathbb{N}}$ defined by for all $n \in \mathbb{N}$, we have

$$E_n = \left(\frac{-1}{n}, n\right)$$

is an open cover of E. The reason is as follows:

(i) In homework 4, we proved that every open interval is an open set with the standard metric on $\mathbb{R}$.

(ii) $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Indeed, let $a \in E$. If $a = 0$, then a belongs E_n for all $n \in \mathbb{N}$. So, we have

$$a \in \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, if $a \neq 0$, then $\frac{1}{a} > 0$. By the Archimedean Property of $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a}$. So, $a < n$. Clearly,

$$\frac{-1}{n} < 0 < a < n.$$

Hence, $a \in E_n = \left(\frac{-1}{n}, n\right)$. Consequently,

$$a \in \bigcup_{m=1}^{\infty} E_m.$$

Definition (Compactness)**.** Let (X, d) be a metric space and $K \subseteq X$. We say that K is **compact** if every open cover of K has a finite subcover; that is, for every open cover $\{O_\alpha\}_{\alpha \in \Lambda}$ of K, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Example. Let (X, d) be a metric space and $E \subseteq X$. If E is finite, then E is compact. The reason is as follows:

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover. Our goal is to show that this open cover has a finite subcover; that is, we can choose $\alpha_1, \dots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

If $E = \emptyset$, there is nothing to prove. Otherwise, $E \neq \emptyset$, we can denote the elements of E by $x_1, \dots, x_n$. So,

$$E = \{x_1, \dots, x_n\}.$$

We have

$$x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

and so continue this process until you have reached the nth element of E. Hence, we have

$$x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}.$$

Thus, we have that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$$

and so E is compact.

Example. Let $(\mathbb{R}, |\cdot|)$ and

$$E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Prove that E is compact (In general, if $a_n \to a$ in $\mathbb{R}$, then the set $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact).

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover. Notice that $0 \in E$ and $E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha$ implies that

$$O \in \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } O \in O_{\alpha_0}.$$

So, if $O \in O_{\alpha_0}$ and O_{α_0} is open, we have

$$\exists \varepsilon > 0 \text{ such that } (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}.$$

By the Archimedean Property of $\mathbb{R}$, we see that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. So, for all $n \geq m$, we have $\frac{1}{n} < \varepsilon$. Hence, for all $n \geq m$, we have $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}$. Note that $1 \in E$ so there exists $\alpha_1 \in \Lambda$ such that $1 \in O_{\alpha_1}$. Similarly, $\frac{1}{2} \in E$ implies that there exists $\alpha_2 \in \Lambda$ such that $\frac{1}{2} \in O_{\alpha_2}$. Continue this process until, we have

$$\frac{1}{m-1} \in E \implies \exists \alpha_m \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}}.$$

Thus, we conclude that

$$E \subseteq \bigcup_{n=0}^{\infty} O_{\alpha_n}$$

and so E is compact.

Remark. If X itself is compact, we say that (X, d) is a compact metric space; that is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any collection of open sets such that

$$X = \bigcup_{\alpha \in \Lambda} O_\alpha$$

then there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$X = \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Theorem (Compactness implies Closed)**.** Let (X, d) be a metric space and $E \subseteq K$ is closed and K is compact. Then E is compact.

Proof. It is enough to show that K^c is open; that is, we need to find $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq K^c$ for every $x \in K^c$. That is, we want to show that

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\Longrightarrow a \notin K \\ &\Longrightarrow \forall x \in K \ d(x, a) > 0 \end{aligned}$$

For all $x \in K$, let $\varepsilon_x = \frac{1}{4} d(x, a)$. Clearly, we have

$$\forall x \in K \ N_{\varepsilon_x} \cap N_{\varepsilon_x}(a) = \emptyset.$$

Note that $\{N_{\varepsilon_x}(x)\}_{x \in K}$ is an open cover for K. Since K is compact, there is a finite subcover. That is, there exists $x_1, \dots x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{x_i}}(x_i)$$

and of course

$$N_{\varepsilon_{x_1}}(x_1) \cap N_{\varepsilon_{x_1}}(a) = \emptyset.$$

If we continue this process until the nth step, we have that

$$N_{\varepsilon_{x_n}}(x) \cap N_{\varepsilon_{x_n}}(a) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon_{x_1}, \dots, \varepsilon_{x_n}\}$. Clearly, $N_\varepsilon(a) \subseteq N_{\varepsilon_{x_i}}(a) \forall 1 \leq i \leq n$. Hence, we have

$$N_{\varepsilon_{x_1}} \cap N_\varepsilon(a) = \emptyset$$

and similarly for the nth step, we have

$$N_{\varepsilon_{x_n}}(x) \cap N_\varepsilon(a) = \emptyset.$$

Therefore, we have

$$N_\varepsilon(a) \cap \left[N_{\varepsilon_{x_1}} \cup \cdots \cup N_{\varepsilon_{x_n}}\right] = \emptyset.$$

So,

$$N_\varepsilon(a) \cap K = \emptyset.$$

■

6.2 Lecture 11

6.2.1 Topics

- Metric subspace
- Theorem 2.35
- Theorem 2.33
- Theorem 2.37
- Theorem 2.36

Theorem. Let (X, d) be a metric space and let $K \subseteq X$ be a compact set, and $E \subseteq K$ is closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Our goal is to show that this cover contains a finite subcover. Note that, E is closed by assumption and so E^c is open. Thus, we have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cup E^c.$$

Thus, E^c together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set K. Since K is compact, this open cover contains a finite subcover. So, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$K \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

Considering that $E \subseteq K$, we can write

$$E \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

So, we have $O_{\alpha_1}, \ldots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. ■

Corollary. If F is closed and K is compact, then $F \cap K$ is compact.

Proof. Note $F \cap K$ is a closed subset of the compact set K. Since K is closed and F is closed along with $F \cap K \subseteq K$. Thus, K contains a finite subcover of $F \cap K$. Thus, $F \cap K$ is compact. ■

Theorem. Let (X, d) be a metric space and Suppose $K \subseteq Y \subseteq X$. We have K is compact relative to X

if and only if K is compact relative to Y.

Proof. ($\Longleftarrow$) Assume that K is compact relative to Y. Our goal is to show that K is compact relative to X. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X that covers K. Our goal is to show that this cover has a finite subcover. Note that the set $K = K \cap Y$ since $K \subseteq Y$. Hence, we have

$$K = K \cap Y \subseteq \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y = \bigcup_{\alpha\in\Lambda}(O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y). So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in (Y, d^Y) that covers E. Since K is compact relative to Y, there exists a finite subcover; that is, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n}(O_{\alpha_i} \cap Y) \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

($\Longrightarrow$) Assume that K is compact relative to X. Our goal is to show that K is compact relative to Y. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in (Y, d^Y) that covers K. Our goal is to show that this cover contains a finite subcover. It follows from Theorem 2.30 that for all $\alpha \in \Lambda$, there exists an open set $O_\alpha \subseteq X$ such that $G_\alpha = O_\alpha \cap Y$. We have

$$K \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda}(O_\alpha \cap Y) = \Big(\bigcup_{\alpha\in\Lambda} O_\alpha\Big) \cap Y \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$

Thus, $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for K i n the metric space (X, d). Since K is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Hence, we see that

$$\begin{aligned} K = K \cap Y \subseteq \Big(\bigcup_{i=1}^{n} O_{\alpha_i}\Big) \cap Y &= \bigcup_{i=1}^{n}(O_{\alpha_i} \cap Y) \\ &= \bigcup_{i=1}^{n} G_{\alpha_i}. \end{aligned}$$

That is, this is a finite subcover we were looking for and we are done. ∎

Note that the property of compactness gets preserved in X, but when we are referring to open and closed sets these properties do not get preserved in X.

Theorem. Let (X, d) be a metric space. If E is an infinite subset of a compact set K, then E has a limit point in K. (In fact, these two statements are equivalent, but we will only prove the forwards direction)

Proof. Let (X, d) be a metric space. Assume that $K \subseteq X$ is compact and E is infinite. We will show that E has a limit point in K. Assume for sake of contradiction that E has no limit point in K; that is, every limit point in K will never be a limit point of E. In other words, for all $a \in E$, $a \notin E'$. Furthermore, for all $b \in K \setminus E$, we have $b \notin E'$. Therefore, for all $a \in E$, there exists $\varepsilon_a > 0$ such that

$$N_{\varepsilon_a} \cap (E \setminus \{a\}) = \emptyset. \tag{1}$$

Likewise, for all $b \in K \setminus E$, there exists $\delta_b > 0$ such that

$$N_{\delta_b} \cap (E \setminus \{b\}) = \emptyset \Longleftrightarrow N_{\delta_b} \cap E = \emptyset. \tag{2}$$

Notice that (1) implies that

$$N_{\varepsilon_a} \cap E = \{a\}.$$

Clearly, K is a subset of $\bigcup_{a \in E} N_{\varepsilon_a}(a) \cup \left(\bigcup_{b \in K \setminus E} N_{\delta_b}(b)\right)$. Since K is compact, there exists $a_1, \ldots, a_n \in E$ and $b_1, \ldots, b_m \in K \setminus E$ such that

$$K \subseteq \left(\bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i)\right) \cup \left(\bigcup_{i=1}^{m} N_{\delta_{b_i}}(b_i)\right).$$

Since for all $b \in K \setminus E$ and the fact that $N_{\delta_b}(b) \cap E = \emptyset$, we have

$$E \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i).$$

Hence,

$$\begin{aligned} E &= E \cap \left[\bigcup_{i=1}^{n} N_{\varepsilon_{a_i}}(a_i)\right] \\ &= \bigcup_{i=1}^{n} (E \cap N_{\varepsilon_{a_i}}(a_i)) \\ &= \bigcup_{i=1}^{n} \{a_i\} \\ &= \{a_i : 1 \leq i \leq n\}. \end{aligned}$$

But this implies that E is finite which is a contradiction of our assumption that E is infinite. ■

Remark. Let (X, d) be a metric space and $K \subseteq X$. The following statements are equivalent:

(i) K is compact.

(ii) Every infinite subset of K has a limit point in K (limit point compactness).

(iii) Every sequence in K has a subsequence that converges to a point in K (sequential compactness).

Example. Consider the set of intervals $A_i = [i, \infty)$, then

$$\bigcap_{i=1}^{\infty} A_i = \emptyset.$$

Corollary. If $K_1 \supseteq K_2 \supseteq \ldots$ is a sequence of nonempty compact sets, and every finite intersection of these sets is nonempty, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. Suppose for sake of contradiction that the intersection

$$\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset.$$

Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \Big(\bigcap_{\alpha \in \Lambda} K_\alpha \Big) = \emptyset.$$

So,

$$K_{\alpha_0} \subseteq \Big(\bigcup_{\alpha \in \Lambda} K_\alpha \Big)^c \Longrightarrow K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} K_\alpha^c.$$

Note that K_α^c is open because K_α is closed. So, $\{K_\alpha^c\}$ is an open cover of K_{α_0}. Since K_{α_0} is compact, there exists $\alpha_1, \ldots, \alpha_n$ such that

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^n K_{\alpha_i}^c \subseteq \Big(\bigcap_{i=1}^n K_{\alpha_i} \Big)^c.$$

So, we have

$$K_{\alpha_0} \cap \Big(\bigcap_{i=1}^n K_{\alpha_i} \Big) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty.

■

Theorem. Let $\{K_\alpha\}$ a collection of compact subsets of X. If the intersection of any finite subcollection is nonempty, then

$$\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset.$$

Proof. ■

Theorem (Nested Interval Property)**.** If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^\infty I_n$ is nonempty.

Proof. ■

Chapter 7

Week 7

7.1 Lecture 12

7.1.1 Topics

- Definition of $K-$cell.
- Theorem: If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ is a sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.
- Theorem: Every $k-$cell is compact.
- Theorem: Suppose $E \subseteq \mathbb{R}^k$. Then
- Connected sets

  E is closed and bounded $\iff$ E is compact $\iff$ Every infinite subset of E has a limit point in E.
- Theorem: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Corollary. If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ is a sequence of compact sets, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Theorem (Nested Interval Property)**.** If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Definition (K-cell)**.** The set $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is called a $k-$cell in $\mathbb{R}^k$.

Example. Let $I = [a_1, a_2] \times [a_2, b_2]$ is a $2-$cell in $\mathbb{R}^2$.

Theorem (Nested Cell Property)**.** If $I_1 \supseteq I_2 \supseteq I_3 \cdots$ is a nested sequence of $k-$cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(n)}, b_1^{(n)}] \times \cdots \times [a_k^{(n)}, b_k^{(n)}]$$

Also, let

$$\forall n \in \mathbb{N} \text{ and } \forall 1 \leq i \leq k, \text{ we have } A_i^{(n)} = [a_i^{(n)}, b_i^{(n)}]$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$A_i^{(n)} \supseteq A_i^{(n+1)} \ \forall 1 \leq i \leq k.$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \cdots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \cdots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \cdots A_k^{(n)}. \end{aligned}$$

Hence, it follows from the nested interval property that there exists

$$\begin{aligned} \exists x_1 &\in \bigcap_{n=1}^{\infty} A_1^{(n)} \\ \exists x_2 &\in \bigcap_{i=1}^{\infty} A_2^{(n)} \\ &\vdots \\ \exists x_k &\in \bigcap_{n=1}^{\infty} A_k^{(n)}. \end{aligned}$$

Thus, by a fact in set theory; that is,

$$(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D).$$

$$\begin{aligned} (x_1, \ldots, x_k) &\in \Big[\bigcup_{n=1}^{\infty} A_1^{(n)}\Big] \times \Big[\bigcap_{n=1}^{\infty} A_2^{(n)}\Big] \times \cdots \times \Big[\bigcap_{n=1}^{\infty} A_k^{(n)}\Big] \\ &\subseteq \bigcap_{n=1}^{\infty} [A_1^{(n)} \times \cdots \times A_k^{(n)}] \\ &= \bigcap_{n=1}^{\infty} I_n. \end{aligned}$$

Hence, we see that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

■

Theorem. Every $k-$cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for $2-$cells. The proof for a general $k-$cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a $2-$cell. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Let

$$\delta = d(a, b) = \|a - b\| = \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}$$

. Note that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in I, then

$$x_1, y_1 \in [a_1, b_2] \implies |x_1 - y_1| \leq |b_1 - a_1|$$
$$x_2, y_2 \in [a_2, b_2] \implies |x_2 - y_2| \leq |b_2 - a_2|$$

which implies that

$$\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta.$$

So, $d(x, y) \leq \delta$. Let us assume for contradiction that I is NOT compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of I that does NOT have a finite subcover; that is, $I \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2} \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i].$$

These subintervals determine 4 $2-$cells. There is at least one of these 4 $2-$cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha \in \Lambda}$. Let us call this $2-$cell as I_1. Notice that

$$\forall x, y \in I_1 \quad \|x - y\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into 4 $2-$cells and continue this process inductively. In this manner, we will obtain a sequence of $2-$cells

$$I, I_1, I_2, I_3, \ldots$$

such that

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \tag{1}$$

$$\forall x, y \in I_n, \quad \|x - y\| \leq \frac{\delta}{2} \tag{2}$$

$$\forall n \in \mathbb{N} \quad I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha \in I}$$

■

Theorem (Heine-Borel Theorem). Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point of E.

Proof. We will show that $(a) \implies (b) \implies (c) \implies (a)$.

$((a) \implies (b))$ Assume that E is closed and bounded. Our goal is to show that E is compact. Since E is bounded, there exists a $K-$cell, I, such that $E \subseteq I$. Note that by Theorem 2.40, we see that I is compact. By Theorem 2.3.5, E is compact.

$((b) \implies (c))$ Assume that E is compact. Our goal is to show that E is limit point compact; that is, every infinite subset of E has a limit point in E. See proof from last week.

$((c) \implies (a))$ Assume that every infinite subset of E has a limit point in E. Our goal is to show that E is closed and bounded. Suppose for sake of contradiction that E is NOT bounded and NOT closed.

Suppose that E is NOT bounded. In what follows, we will construct a sequence of points $x_1, x_2, \ldots$ in

E. Since E is not bounded, we know that

$$\begin{aligned}
E \not\subseteq N_1(0) &\implies \exists x_1 \in E \text{ such that } d(x_1, 0) = \|x_1\|_2 \geq 1. \\
E \not\subseteq N_2(0) &\implies \exists x_2 \in E \text{ such that } d(x_2, 0) = \|x_2\|_2 \geq 2 \\
E \not\subseteq N_3(0) &\implies \exists x_3 \in E \text{ such that } d(x_3, 0) = \|x_3\|_2 \geq 3 \\
&\vdots \\
E \not\subseteq N_n(0) &\implies \exists x_n \in E \text{ such that } d(x_n, 0) = \|x_n\|_2 \geq n \\
&\vdots
\end{aligned}$$

That is, we have a sequence of points $S = \{x_n : n \in \mathbb{N}\}$ in E with the property that

$$d(x_n, 0) = \|x_n\|_2 \geq n.$$

Note that S is an infinite set; indeed, if S were finite, then

$$S = \{a_1, \ldots, a_m\}.$$

Now, let

$$r = \max\{d(a_n, 0) : 1 \leq n \leq m\}$$

and let $n \in \mathbb{N}$ be such that $n > r + 1$ (By the Archimedean Property). Since $\|x_n\|_2 \geq n > r + 1$, we can conclude that none of the $a_1, \ldots, a_m$ is x_n and this contradicts how S was constructed.

Now, we will show that E is closed. Assume for contradiction that E is NOT closed; that is, $E' \not\subseteq E$. That is, there exists $y_0 \in \mathbb{R}^k$ such that $y_0 \in E'$ but $y_0 \in E$. We will construct a sequence of points $y_1, y_2, \ldots$ in E as follows:

$$\begin{aligned}
y_0 \in E' &\implies N_1(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_1 - y_0\| < 1 \\
y_0 \in E' &\implies N_{\frac{1}{2}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_2 - y_0\| < \frac{1}{2} \\
&\vdots \\
y_0 \in E' &\implies N_{\frac{1}{m}}(y_0) \cap (E \setminus \{y_0\}) \text{ is infinite such that } \|y_m - y_0\| < \frac{1}{m} \\
&\vdots
\end{aligned}$$

Let $T = \{y_1, y_2, y_3, \ldots\}$ and note that $T \subseteq E$. Note T is infinite (by construction $y_1, y_2, y_3, \ldots$ are distinct elements. We claim that if $z \neq y_0$, then $z \in T'$. To this end, we have for all $n \in \mathbb{N}$

$$d(y_0, z) = \|y_0 - z\|_2 \leq \|y_0 - y_n\|_2 + \|y_n - z\|_2$$

which implies further that

$$\|y_n - z\|_2 \geq \|y_0 - z\|_2 - \|y_n - y_0\|_2 > \|y_0 - z\|_2 - \frac{1}{n}.$$

Hence, for all $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{1}{2}\|y_0 - z\|_2$, we have

$$\begin{aligned} d(y_n, z) &> \|y_0 - z\|_2 - \frac{1}{n} \\ &> \|y_0 - z\|_2 - \frac{1}{2}\|y_0 - z\|_2 \\ &= \frac{1}{2}\|y_0 - z\|_2. \end{aligned}$$

So, for all but finitely many n, we see that

$$d(y_n, z) > \frac{1}{2}\|y_0 - z\|_2.$$

Hence, if we let $\varepsilon = \frac{1}{4}\|y_0 - z\|_2$, then $N_\varepsilon(z) \cap T$ is a finite set which proves $z \notin T'$. But this implies that the only possible limit point of T is y_0, but $y_0 \notin E$. Hence, T is an infinite subset of E with not limit point in E. This contradicts our hypothesis that every infinite subset of E has a limit point of E. ■

Remark. Note that in any general metric space, we have $(a) \implies (b)$ is not necessarily true.

Theorem (Bolzano-Weierstrass Theorem)**.** Let $E \subseteq \mathbb{R}^k$ and E is an infinite set and bounded. Then $E' \neq \emptyset$.

Proof. Suppose that E is bounded. Then there exists a $k-$cell I such that $E \subseteq I$. By Theorem 2.40, we know that I is a compact set. Furthermore, we know that I is limit point compact by Theorem 2.41. So, every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E \neq \emptyset$. ■

7.2 Lecture 13

7.2.1 Topics

- Separated sets, disconnected sets, connected set.
- Theorem: $E \subseteq \mathbb{R}$ is connected if and only if $x, y \in E$ and $z \in (x, y)$ implies $z \in E$.
- Perfect Sets
- Theorem: $P \subseteq \mathbb{R}^k$ is nonempty perfect implies P is uncountable.
- The Cantor Set

Definition (Connected Sets, Disconnected, connected)**.** Let (X, d) be a metric space.

(i) Two sets $A, B \subseteq X$ are aid to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

(iii) A set $E \subseteq X$ is said to **disconnected** if it can be written as a union of tow nonempty separated sets A and B; that is, $E = A \cup B$.

(iv) A set $E \subseteq X$ is said to be connected if it is NOT disconnected.

Example (ℝ with the standard metric). (*) If we have $A = (1, 2)$ and $B = (2, 5)$ are separated, then

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset$$
$$A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.$$

Hence, $E = A \cup B$ is disconnected.

(*) We have $C = (1, 2]$ and $D = (2, 5)$ are disjoint but not separated; that is, we have

$$C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\}$$
$$C \cup D = (1, 5) \text{ is indeed connected.}$$

Theorem. Let $E \subseteq \mathbb{R}$. E is connected if and only if E contains the following property

$$\text{If } x, y \in E \text{ and } x < z < y, \text{ then } z \in E.$$

Proof. ■

Remark (Proposition 3.3.5, "Differential Calculus on Normed Space", Cartan). Let U be an open set in a normed (real) vector space. The following conditions are equivalent:

(a) U is connected

(b) U is path connected (any two points in U can be connected by a path in U)

Remark (Chapter 4, "Introduction" to Topological Manifolds", John Lee). In any metric space,

$$\text{path connected} \implies \text{connected}.$$

However, the converse is not always true!

Definition (Perfect Set). Let (X, d) be a metric space. Let $E \subseteq X$. The following are equivalent definitions:

(i) E is said to be **perfect** if $E' = E$

(ii) E is said to be **perfect** if $E' \subseteq E$ and $E \subseteq E'$.

(iii) E is said to be perfect if E is closed and every point of E is a limit point of E.

(iv) E is said to be perfect if E is closed and E does not have any isolated points.

Example.

- $E = [0, 1] \implies E' = [0, 1]$. Thus, we have $E = E'$ and so E is perfect.
- $E = [0, 1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is NOT perfect.
- $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \implies E' = \{0\} \implies E \neq E'$, so E is not perfect.
- Is E' from the last example perfect? Indeed, $E' = \{0\} \implies (E')' = \emptyset$. Thus, $E' \neq (E')' \implies E'$ is NOT perfect.
- $E = \emptyset$ and $E' = \emptyset$. Thus, $E = E'$ so E is perfect.

Theorem. Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Our proof of the theorem above will use the following two lemmas:

Lemma. Let (X, d) be a metric space. Let $E \subseteq X$ be perfect. If V is any open set in X such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

Proof. Let $q \in V \cap E$. Thus, $q \in V$ and $q \in E$. Then $q \in V$ implies that there exists $\delta > 0$ such that $N_\delta(q) \subseteq V$ and $q \in E$ implies $q \in E'$. As a consequence of these two results, we see that $N_\delta(q) \cap E$ is an infinite set. Thus, $V \cap E$ is an infinite set (Here, we are using the fact that an open set intersected with a perfect set is infinite whenever the intersection is nonempty). ■

Lemma. Let $q \in \mathbb{R}^k$. Let $r > 0$. Then $\overline{N_r(q)} = C_r(q)$.

Proof. Note that since $P' = P$ and $P \neq \emptyset$, we have $P' \neq \emptyset$. Thus, P is infinite. Assume for contradiction that P is countable. Let's denote the distinct elements of P by $x_1, x_2, x_3, \ldots$; that is, we can denote

$$P = \{x_1, x_2, x_3, \ldots\}.$$

In what follows, we will construct a sequence of neighborhoods $V_1, B_2, V_3, \ldots$ such that

(i) For all $n \in \mathbb{N}$, we have $\overline{V_{n+1}} \subseteq V_n$.

(ii) For all $n \in \mathbb{N}$, $x_n \notin \overline{V_{n+1}}$.

(iii) For all $n \in \mathbb{N}$, $V_n \cap P \neq \emptyset$.

Let's assume that we have constructed these neighborhoods (Construction of neighborhoods). Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset.$$

Note that

(I) $\overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n}$ so $\overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P$ and so $K_{n+1} \subseteq K_n$ for each n.

(II) Since $\overline{V_n}$ is a closed and bounded set in $\mathbb{R}^k$, we have that $\overline{V_n}$ is compact. Furthermore, P being a perfect set implies that P is a closed set. As a consequence of these two facts, we can conclude that $K_n = \overline{V_n} \cap P$ is compact.

Using facts (I) and (II), we can conclude that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset \tag{*}$$

by Theorem 2.3.6. Recall that for all n, we have $K_n \subseteq P$, and so we have

$$\bigcap_{n=1}^{\infty} K_n \subseteq P.$$

In what follows, we will contradict (*). Let $b \in P$ be arbitrary. Then $b = x_m$ for some $m \in \mathbb{N}$. By property (ii), we see $x_m \notin \overline{V_{m+1}}$ and so $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$. This tells us that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

■

Remark (On the construction of neighborhoods $V_1, V_2, \ldots$). Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$ (because $x_1 \in V_1$ and $c_1 \in P$). Our goal is to construct an open neighborhood V_2 such that

(i) $\overline{V_2} \subseteq V_1$,

(ii) $x_1 \notin \overline{V_2}$,

(iii) $V_2 \cap P \neq \emptyset$.

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$. By the first lemma found above, there exists $y_1 \in V_1 \cap P$ such that $y_1 \neq x_1$. Since V_1 is open and $y_1 \in V_1$, there exists $\delta_1 > 0$ such that $N_{\delta_1}(y_1) \subseteq V_1$.

$$\text{Let } r_2 = \tfrac{1}{2}\min\{d(x_1, y_1), \delta_1\}.$$

Let $V_2 = N_{r_2}(y_1)$. We claim V_2 has all the desired properties above. Indeed, we see that

(i) Observe that

$$\begin{aligned}\overline{V_2} = \overline{N_{r_2}(y_1)} &= \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\} \\ &\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1) \\ &\subseteq V_1.\end{aligned}$$

(ii) Notice that $d(x_1, y_1) > r_2$ implies that

$$x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}.$$

(iii) Since $y_1 \in V_2$ and $y_1 \in P$, we clearly have that $V_2 \cap P \neq \emptyset$.

We can construct $V_3, V_4, \ldots$ in a similar manner.

As a consequence of the theorem above, we have

Corollary. The interval $[0, 1]$ is uncountable.

7.2.2 The Cantor Set

The construction of the Cantor set is typically done in stages.

Step 0:

Let $E_0 = [0, 1]$.

Step 1:

Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, that is, remove the middle third, and define

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

The middle third, in this case, will be calculated by the following

$$\left(\frac{1}{3}, \frac{2}{3}\right) = \left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right).$$

Step 2:

Take each of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and remove the middle third of each of those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Likewise, we can remove the middle third move explicitly by calculating the following

$$\left(\frac{3(0)+1}{3^2}, \frac{3(0)+2}{3^2}\right) \text{ and } \left(\frac{3(2)+1}{3^2}, \frac{3(2)+2}{3^2}\right).$$

Continue in this manner until we obtain a sequence of compact sets:

$$E_1, E_2, E_3, E_4, \ldots$$

with the following properties

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \ \cdots$

(2) For each $n \in \mathbb{N}$, E_n is the union of 2^n intervals of length $\frac{1}{3^n}$.

The set $P = \bigcap_{n=1}^{\infty} E_n$ is called the **Cantor Set**.

Remark. Notice that in order to obtain E_n, we remove intervals of the form $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ from E_{n-1}; that is, k is such that $0 \leq k$ and $3k+2 < 3^n$.

Theorem (Properties of the Cantor Set). Let P denote the Cantor set. Then

(1) P is compact

(2) P is nonempty

(3) P contains no segment

(4) P is perfect (and so it is uncountable)

(5) P has measure zero.

Proof. (1) Note that P is an intersection of compact sets. Hence, P is compact (see hw5).

(2) It follows from Theorem 2.3.6 that the intersection of a sequence of nested nonempty compact sets is nonempty. (In fact, the endpoint of each interval that appears at any state belong to P)

(3) Our goal is to show that P does NOT contain any set of the form (α, β) (where $0 \leq \alpha, \beta \leq 1$). Note that, by the construction of P, the intervals of the form:

$$I_{k,n} = \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \ n \in \mathbb{N}, 3k+2 < 3^n$$

have no intersection with P. However, (α, β) contains at least one of $I_{k,n}$'s. Indeed, (α, β) contains $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$. That is,

■

Chapter 8

Week 8

8.1 Lecture 14

8.1.1 Topics

- Convergence of a sequence
- Bounded Sequence
- Theorem: $x_n \to x \Longleftrightarrow$ every neighborhood of x contains x_n for all but at most finitely many n.
- Theorem: A sequence cannot have more than one limit.
- Theorem: Convergent sequence $\Longrightarrow$ Bounded sequence.
- If $x \in E'$, there is a sequence (x_n) of distinct points in $E \setminus \{x\}$ such that $x_n \to x$.

Definition (Convergence of Sequence)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. We say that (x_n) converges to a limit $x \in X$ if for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \varepsilon$.

Definition (Bounded Sequence)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. We say that (x_n) is **bounded** if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X; that is,

$$(x_n) \text{ is bounded if and only if there exists } q \in X, \text{ there exists } r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q).$$

Theorem (An Equivalent Characterization of Convergence)**.** Let (X, d) be a metric space. We say that $(x_n) \to x$ if and only if for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains x_n for all but at most finitely many n.

Proof. We have $(x_n) \to x$ if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $d(x_n, x) < \varepsilon$. This holds if and only if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \in N_\varepsilon(x)$. This holds if and only if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $N_\varepsilon(x)$ contains x_n for all $n > N$. This is equivalent to saying that for all $\varepsilon > 0$, $N_\varepsilon(x)$ contains x_n for all but at most finitely many n. ∎

Theorem (Uniqueness of Limit)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. If $x_n \to x$ in X and $x_n \to \tilde{x}$, then $x = \tilde{x}$.

Proof. In order to show that $\tilde{x} = d$, it suffices to show that $d(\tilde{x}, x) = 0$. That is, it suffices to show that for all $\varepsilon > 0$, $d(x, \tilde{x}) < \varepsilon$. Let $\varepsilon > 0$ be given. Since $x_n \to x$, there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$d(x_n, x) < \frac{\varepsilon}{2}.$$

Similarly, $x_n \to \tilde{x}$ implies that there exists an $N_2 \in \mathbb{N}$ such that for any $n > N_2$, we have

$$d(x_n, \tilde{x}) < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. So, for any $n > N$, we have

$$d(x, \tilde{x}) \leq d(x, x_n) + d(x_n, \tilde{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we have $x = \tilde{x}$. ∎

Theorem (Convergent $\implies$ Bounded)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. If $x_n \to x$ in X, then (x_n) is bounded.

Proof. By the definition of convergence with ($\varepsilon = 1$), we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_1(x).$$

Now, let

$$r = \max\{1, d(x_1, x), d(x_2, x), \ldots, d(x_N, x)\} + 1.$$

Then, clearly, we have

$$\forall n \in \mathbb{N} \ d(x_n, x) < r.$$

Hence,

$$\forall n \in \mathbb{N} \ x_n \in N_r(x).$$

∎

Corollary (Contrapositive)**.** If (x_n) is NOT bounded, then (x_n) does NOT converge.

Theorem (Limit Point is a Limit of a Sequence)**.** Let (X, d) be a metric space. Let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence $x_1, x_2, \ldots$ of distinct points in $E \setminus \{x\}$ that converges to x.

Proof. Since $x \in E'$, we have

$$\forall \varepsilon > 0 \ \ N_\varepsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular, we have

For $\varepsilon = 1$ $\exists x_1 \in E \setminus \{x\}$ such that $d(x_1, x) < 1$

For $\varepsilon = \dfrac{1}{2}$ $\exists x_2 \in E \setminus \{x\}$ such that $x_2 \neq x_1$ and $d(x_2, x) < \dfrac{1}{2}$

For $\varepsilon = \dfrac{1}{3}$ $\exists x_3 \in E \setminus \{x\}$ such that $x_3 \neq x_1, x_2$ and $d(x_3, x) < \dfrac{1}{3}$

$\vdots$

For $\varepsilon = \dfrac{1}{n}$ $\exists x_n \in E \setminus \{x\}$ such that $x_n \neq x_1, x_2, \ldots, x_{n-1}$ and $d(x_n, x) < \dfrac{1}{n}$

$\vdots$

In this way, we claim that the sequence $x_1, x_2, x_3, \ldots$ of distinct points in $E \setminus \{x\}$ converges to x.

Indeed, let $\varepsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that for any $n > N$, $d(x_n, x) < \varepsilon$. Choose N such that $\frac{1}{N} < \varepsilon$. Then for all $n > N$, we have

$$d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

as desired. ■

8.2 Lecture 15

Definition (Subsequence)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X and let

$$n_1 < n_2 < n_3 < \cdots$$

be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ is called a **subsequence** of $(x_1, x_2, x_3, \ldots)$ and is denoted by (x_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem. Let (X, d) be a metric space. Let (x_n) be a sequence in X. If $\lim\limits_{n\to\infty} x_n = x$, then every subsequence of (x_n) converges to x.

Proof. Let (x_{n_k}) be a subsequence of (x_n). Our goal is to show that $\lim\limits_{k\to\infty} x_{n_k} = x$. That is, we want to show that

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \text{ such that } \forall k > N \;\; d(x_{n_k}, x) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Our goal is to find an $N \in \mathbb{N}$ such that

$$\text{if } k > N \text{ then } d(x_{n_k}, x) < \varepsilon. \tag{I}$$

Since $x_n \to x$, we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N} \;\; d(x_n, x) < \varepsilon. \tag{II}$$

We claim that this $\hat{N}$ can be used as the same N we were looking for. Indeed, if we let $N = \hat{N}$, then for any $k > N$, we can conclude that $n_k \geq k > N$ and so, by (II) we can write $d(x_{n_k}, x) < \varepsilon$. ■

Corollary. (1) If a subsequence of (x_n) does NOT converge to x, then (x_n) does NOT converge to x.

(2) If (x_n) has a pair of subsequences converging to different limits, then (x_n) does not converge.

Theorem. Let (X, d) be a metric space. Let (x_n) be a sequence in X. The subsequential limits of (x_n) form a closed set in X.

Proof. Let $E = \{b \in X : b \text{ is a limit of a subsequence of } (x_n)\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$, that is, we will show that there is a subsequence of (x_n) that converges to some $a \in X$. We may considering two cases:

(1) $\forall n \in \mathbb{N}$, we have $x_n = a$. In this case, (x_n) and any subsequence of (x_n) converges to a. So, $a \in E$.

(2) $\exists n_1 \in \mathbb{N}$ such that $x_{n_1} \neq a$. In this case, let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, we have $N_{\frac{\delta}{2^2}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^3}.$$

Since $n \in E$, b is a limit of a subsequence of (x_n), so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

By the triangle inequality, we have that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b, a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n) such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}.$$

From this construction, we can see that $x_{n_k} \to a$. Thus, $a \in E$.

■

Theorem (Compactness implies Sequential Compactness)**.** Let (X, d) be a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let (x_n) be a sequence is the compact metric space X. Let $E = \{x_1, x_2, \dots\}$. If E is finite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < \cdots$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \cdots = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ converges (it converges to x).

If E is infinite, then since X is compact by Theorem 2.3.7, E has a limit point $x \in X$ (that is, there exists $x \in X$ such that $x \in E'$). Thus, we have

$$\forall \varepsilon > 0 \quad N_\varepsilon(x) \cap (E \setminus \{x\}) \text{ is infinite}.$$

In particular,

$$\begin{aligned}
&\text{For } \varepsilon = 1,\ \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1\\
&\text{For } \varepsilon = \frac{1}{2},\ \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2}\\
&\text{For } \varepsilon = \frac{1}{3},\ \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3}\\
&\vdots\\
&\text{For } \varepsilon = \frac{1}{m},\ \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m}\\
&\vdots
\end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, \ldots$ of (x_n) that converges (it converges to x). ■

Corollary. Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}^k$. Then we have

$$\exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q).$$

Note that $N_r(q)$ is a bounded set and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_r(q)}$ is a compact subset of $\mathbb{R}^k$. So, we have $\overline{N_r(q)}$ is a compact metric space and (x_n) is a sequence in $\overline{N_r(q)}$ implies, by the previous theorem, there exists a subsequence (x_{n_k}) of (x_n) that converges in the metric space $\overline{N_r(q)}$. Since the distance $\overline{N_r(q)}$ is the same as distance function in $\mathbb{R}^k$, we can conclude that (x_{n_k}) converges in $\mathbb{R}^k$ as well. ■

Definition (Cauchy Sequence)**.** Let (X, d) be a metric space. A sequence (x_n) in X is said to be a **Cauchy Sequence** if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N\ d(x_n, x_m) < \varepsilon.$$

Theorem (Convergent implies Cauchy)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. Then

$$(x_n) \text{ converges} \implies (x_n) \text{ is Cauchy.}$$

Proof. Suppose $(x_n) \to x$ for some $x \in X$. Our goal is to show that (x_n) is a Cauchy sequence. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N\ d(x_n, x_m) < \varepsilon.$$

By assumption, there exists an $\hat{N} \in \mathbb{N}$ such that for any $n > \hat{N}$, we have

$$d(x_n, x) < \frac{\varepsilon}{2}. \tag{1}$$

Similarly, for any $m > \hat{N}$

$$d(x_m, x) < \frac{\varepsilon}{2}. \tag{2}$$

We claim that $\hat{N}$ can be used as the N we were looking for. Set $N = \hat{N}$ and suppose for any $n, m > N$, we have

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

Remark. The converse of the theorem above does not necessarily hold. For example, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. In $\mathbb{Q}$, it is not true that every Cauchy sequence is convergent. For example, let (q_n) be a sequence in $\mathbb{Q}$ such that $q_n \to \sqrt{2}$. That is,

$$\begin{aligned} q_n \to \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q}. \end{aligned}$$

However, we know that (q_n) does not converge in $\mathbb{Q}$.

Definition (Complete Metric Space)**.** A metric space in which every Cauchy sequence is convergent is called a **complete metric space**.

Chapter 9

Week 9

9.1 Lecture 16

9.1.1 Topics

- Diameter of a set
- Theorem: $\operatorname{diam} \overline{E} = \operatorname{diam} E$
- Theorem: Nested sequence of nonempty compact sets K_n and if $\operatorname{diam} K_n \to 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.
- Theorem: Every compact metric space is complete.
- Theorem: $\mathbb{R}^k$ is a complete metric space.

Definition (Diameter of a Set)**.** Let (X, d) be a metric space. Let E be a nonmempty set in X. The diameter of E, denoted by $\operatorname{diam} E$, is defined as follows:

$$\operatorname{diam} E = \sup\{d(a, b) : a, d \in E\}.$$

Remark. Note that if $\emptyset \neq A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Thus, we have

$$\sup\{d(a, b) : a, b \in A\} \leq \sup\{d(a, b) : a, b \in B\}.$$

That is,

$$\operatorname{diam} A \leq \operatorname{diam} B.$$

Theorem (Interesting Observation)**.** Let (X, d) be a metric space. Let (x_n) be a sequence in X. For all $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+1}, \dots\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n \to \infty} \operatorname{diam} E_n = 0.$$

Proof. ($\Longrightarrow$) Assume that (x_n) is a Cauchy sequence. Note that

$$\begin{aligned} E_1 &= \{x_2, x_3, x_4, x_5, \dots\} \\ E_2 &= \{x_3, x_4, x_5, \dots\} \\ E_3 &= \{x_4, x_5, \dots\} \\ E_4 &= \dots \\ &\vdots \end{aligned}$$

Clearly, we have

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq \cdots.$$

So, we have

$$\operatorname{diam} E_1 \geq \operatorname{diam} E_2 \geq \operatorname{diam} E_3 \geq \cdots.$$

Our goal is to show that

$$\forall\, \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \ \underbrace{|\operatorname{diam} E_n - 0|}_{\operatorname{diam} E_n \geq 0} < \varepsilon. \tag{*}$$

So, it suffices to show that $\operatorname{diam} E_n < \varepsilon$. To this end, let $\varepsilon > 0$ be given. Since (x_n) is Cauchy, there exists $\hat{N} \in \mathbb{N}$ such that

$$\forall n, m > \hat{N}\ \ d(x_n, x_m) < \frac{\varepsilon}{2}.$$

We claim that this $\hat{N}$ is the same N we were looking for. Letting $N = \hat{N}$, we have

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}.$$

Hence, we have

$$\forall a, b \in E_{\hat{N}}\ \ d(a,b) < \frac{\varepsilon}{2}.$$

Thus, we have

$$\operatorname{diam} E_{\hat{N}} = \sup\{d(a,b) : a, b \in E_{\hat{N}}\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

If $n > \hat{N}$, then

$$\operatorname{diam} E_n \leq \operatorname{diam} E_{\hat{N}} < \varepsilon$$

as desired.

($\Longleftarrow$) Assume that $\lim_{n\to\infty} \operatorname{diam} E_n = 0$. Our goal is to show that (x_n) is Cauchy; that is,

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N\ d(x_n, x_m) < \varepsilon. \tag{*}$$

Since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, for this given ε, there exists $\hat{N}$ such that

$$\forall n > \hat{N}\quad \operatorname{diam} E_n < \varepsilon.$$

In particular, we have $\operatorname{diam} E_{\hat{N}+1} < \varepsilon$. Now, we claim that $N = \hat{N} + 1$ can be used as the same N we were looking for. Indeed, if we let $N = \hat{N} = 1$, we have

$$\forall n, m > \hat{N} + 1,\ x_n, x_m \in E_{\hat{N}+1}$$

and thus

$$d(x_n, x_m) \leq \operatorname{diam} E_{\hat{N}+1} < \varepsilon.$$

■

Theorem. Let (X, d) be a metric space. Let E be a nonempty subset of X. Then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

Proof. Note that, since $E \subseteq \overline{E}$, we have $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$. In what follows, we will prove that $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$ by showing that

$$\forall \varepsilon > 0 \quad \operatorname{diam} \overline{E} \leq \operatorname{diam} E + \varepsilon.$$

Let $\varepsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \operatorname{diam} E + \varepsilon.$$

To this end, it suffices to show that $\operatorname{diam} E + \varepsilon$ is an upper bound for $\{d(a, b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$\begin{aligned} a \in \overline{E} &\Longrightarrow N_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset \Longrightarrow \exists x \in E \text{ such that } d(x, a) < \frac{\varepsilon}{2} \\ b \in \overline{E} &\Longrightarrow N_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset \Longrightarrow \exists y \in E \text{ such that } d(y, b) < \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\varepsilon}{2} + d(x, y) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \operatorname{diam} E + \frac{\varepsilon}{2} \\ &= \varepsilon + \operatorname{diam} E \end{aligned}$$

which is our desired result. ∎

Theorem. Let (X, d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a nested sequence of nonempty compact sets where $\lim_{n\to\infty} \operatorname{diam} K_n = 0$. Then

$$\bigcap_{n=1}^{\infty} K_n \text{ consists of exactly one point.}$$

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.3.6, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose that $a, b \in K$ and we will prove $a = b$. In order to show that $a = b$, we will prove that $d(a, b) = 0$. Showing this is equivalent to showing that

$$\forall \varepsilon > 0 \quad d(a, b) < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n > N \quad \operatorname{diam} K_n < \varepsilon.$$

In particular, $\operatorname{diam} K_{N+1} < \varepsilon$. Now, we have

$$a \in \bigcap_{n=1}^{\infty} \Longrightarrow a \in K_{N+1}$$

and
$$b \in \bigcap_{n=1}^{\infty} K_n \Longrightarrow b \in K_{N+1}$$
which implies further that
$$d(a,b) \leq \operatorname{diam} K_{N+1} < \varepsilon$$
which is our desired result. ■

Theorem (Compact Space $\Longrightarrow$ Complete Space)**.** Any compact metric space is complete.

Proof. Let (X,d) be a compact metric space. Let (x_n) be a Cauchy sequence in X. Our goal is to show that (x_n) converges in X. For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3} \dots\}$. We know that

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

(2) (x_n) is Cauchy $\Longrightarrow \lim_{n\to\infty} \operatorname{diam} E_n = 0$.

It follows from (1) that
$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \cdots \tag{†}$$
Since closed subsets of a compact metric space are compact, we know that (†) is a nested sequence of nonempty compact sets. Since $\operatorname{diam} E_n = \operatorname{diam} \overline{E_n}$, it follows from (2) that $\lim_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$. Hence, we know by the previous theorem that $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let us denote this point as a where
$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}.$$
In what follows, we will prove that $\lim_{n\to\infty} x_n = a$. To this end, it suffices to show that
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ d(x_n, a) < \varepsilon. \tag{*}$$
Let $\varepsilon > 0$ be give. Since $\lim_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$, for this given ε, there exists $\hat{N}$ such that
$$\forall n > \hat{N} \quad \operatorname{diam} \overline{E_n} < \varepsilon.$$
We claim that $\hat{N}+1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}+1$, then (*) holds. The reason is as follows: If $n > \hat{N}+1$, then
$$x_n \in E_{\hat{N}+1} \Longrightarrow x_n \in \overline{E_{\hat{N}+1}}. \tag{I}$$
Furthermore,
$$a \in \bigcap_{n=1}^{\infty} \overline{E_n} \Longrightarrow a \in \overline{E_{\hat{N}+1}}. \tag{II}$$
Thus, (I) and (II) imply that
$$d(x_n, a) \leq \operatorname{diam} \overline{E_{\hat{N}+1}} < \varepsilon.$$
■

Theorem ($\mathbb{R}^k$ is complete)**.** $\mathbb{R}^k$ is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^k$. By homework 7, we see that (x_n) must be bounded; that is,
$$\exists p \in \mathbb{R}^k, \varepsilon > 0 \text{ such that } \forall n \in \mathbb{N} \ x_n \in N_\varepsilon(p).$$

Note that $\overline{N_\varepsilon(p)}$ is a closed and bounded set in $\mathbb{R}^k$, and so it must be compact by the Heine-Borel Theorem. Thus, if $\overline{N_\varepsilon(p)}$ is a compact metric space and (x_n) is a Cauchy sequence in $\overline{N_\varepsilon(p)}$, we have that (x_n) converges to a point $x \in \overline{N_\varepsilon(p)}$ by the previous theorem. Since the metric in $\overline{N_\varepsilon(p)}$ is exactly the same as the metric in $\mathbb{R}^k$, we can conclude that $x_n \to x$ in $\mathbb{R}^k$.

■

9.2 Lecture 17

9.2.1 Topics

- Algebraic Limit Theorem (for sequences of numbers)
- Divergent sequences
- Monotone sequences
- Monotone Convergence Theorem
- Extended real numbers
- Properties of limit in $\overline{\mathbb{R}}$ (Extended Real Numbers)

Theorem (Algebraic Limit Theorem)**.** Suppose (a_n) and (b_n) are sequences of real numbers, and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$

(ii) n $\lim_{n\to\infty}(ca_n) = ca$ for any real number c.

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided that $b \neq 0$.

Proof. *Proof left to the reader.* ■

Definition. Consider $\mathbb{R}$ with its standard metric. Let (x_n) be a sequence of real numbers. If (x_n) does not converge, we say (x_n) **diverges**.

Divergence can come in three different forms:

(i) (x_n) becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n > M.$$

From this, we say that $x_n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$.

(ii) $(-x_n)$ becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ \underbrace{-x_n > M}_{x_n < -M}.$$

From this, we say that $x_n \to -\infty$ or $\lim_{n\to\infty} x_n = -\infty$.

(iii) (x_n) is NOT convergent and it does NOT diverge to ∞ nor $-\infty$. For example,

$$(x_n) = ((-1)^n) = (-1, +1, -1, +1, \dots)$$

or

$$(x_n) = (n(-1)^n) = (-1, 2, -3, 4, -5, 6, \dots).$$

Remark (Strategies to Prove Divergence)**.** Let (a_n) be a sequence of real numbers.

(1) If (a_n) is unbounded, then (a_n) diverges.

(2) If (a_n) has a pair of subsequences converging to different limits, then (a_n) diverges.

(3) Let $a \in \mathbb{R}$ be fixed but arbitrary. Show that the assumption $a_n \to a$ leads to a contradiction.

Definition (Increasing, Decreasing, Monotone)**.** Consider $\mathbb{R}$ with the standard metric.

(i) (a_n) is said to be **increasing** if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$.

(ii) (a_n) is said to be **decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

(iii) (a_n) is said to be **monotone** if it is either increasing or decreasing or both.

(iv) (a_n) is said to be **strictly increasing** if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

(v) (a_n) is said to be **strictly decreasing** if for all $n \in \mathbb{N}$, $a_{n+1} < a_n$.

Example. (i) $1, -1, 1, -1, 1, -1, \ldots$ is neither an increasing nor decreasing sequence.

(ii) $2, 4, 6, 8, 10, \ldots$ is strictly increasing.

(iii) $6, 6, 7, 7, 8, 8, \ldots$ is an increasing sequence but not strictly increasing.

(iv) $-2, -4, -6, -8, -10, -12, \ldots$ is a strictly decreasing sequence.

Theorem (Monotone Convergence Theorem)**.** Consider $\mathbb{R}$ with its standard metric.

(i) If (a_n) is increasing and bounded, then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$.

(ii) If (a_n) is decreasing and bounded, then (a_n) converges to $\inf\{a_n : n \in \mathbb{N}\}$.

(iii) If (a_n) is increasing and unbounded, then $a_n \to \infty$.

(iv) If (a_n) is decreasing and unbounded, then $a_n \to -\infty$.

Proof. Here will prove item (i). The rest of the parts follow analogously.

Suppose that (a_n) is increasing $(\forall n \; a_n \leq a_{n+1})$ and (a_n) is a bounded sequence. Our goal is to show that $a_n \to s$ where

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$

Notice that since $\{a_n : n \in \mathbb{N}\}$ is a bounded set, we know that $\{a_n : n \in \mathbb{N}\}$ must exists in $\mathbb{R}$. Our goal is to show that

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n > N \; |a_n - s| < \varepsilon. \qquad (*)$$

Let $\varepsilon > 0$ be given. To show (*), it suffices to show that

$$\text{if } n > N, \text{ then } s - \varepsilon < a_n < s + \varepsilon.$$

Since $s = \sup\{a_n : n \in \mathbb{N}\}$, we see that $s - \varepsilon$ must NOT be an upper bound of $\{a_n : n \in \mathbb{N}\}$. Hence, there exists an element in $\{a_n : n \in \mathbb{N}\}$ that is larger than $s - \varepsilon$. That is, there exists $\hat{N} \in \mathbb{N}$ such that $a_{\hat{N}} > s - \varepsilon$. We claim that this $\hat{N}$ can be used as the N that were looking for. Indeed, if we let $N = \hat{N}$, then

(1) if $n > \hat{N}$, then $a_n \geq a_{\hat{N}} > s - \varepsilon$

(2) if $n > \hat{N}$, then $a_n \leq \sup\{a_n : n \in \mathbb{N}\} = s < s + \varepsilon$.

Thus, (1) and (2) imply that if $n > \hat{N}$, then $s - \varepsilon < a_n < s + \varepsilon$ as desired. ■

Remark. Consider $\mathbb{R}$ with its standard metric.

(*) Boundedness does not generally imply convergence. An example of such a bounded sequence that does not converge is $a_n = (-1)^n$.

(*) Monotone does not necessarily imply convergence. An example of such a sequence is $a_n = n$.

(*) Convergence does not generally imply being monotone; that is, consider $a_n = \frac{(-1)^n}{n}$ which converges to 0 but does not have a monotone behavior.

Example. Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Use induction to show that $a_n \leq 2$ for every n. We claim that $\forall n \geq 1$, $a_n \leq 2$. Let our base case be $n = 1$. Then $a_1 = 1 \leq 2$. Suppose for our induction hypothesis that the claim holds for $n = k (k \geq 1)$, that is, suppose that $a_k \leq 2$. Our goal is to show that the claim holds for $n = k+1$, that is, prove that $a_{k+1} \leq 2$. Hence, we have

$$a_{k+1} = \frac{1}{2}a_k + 1 \underbrace{\leq}_{a_k \leq 2} \frac{1}{2}(2) + 1 = 1 + 1 = 2.$$

(ii) Use induction to show that (a_n) is an increasing sequence. Our claim that for all n, $a_n \leq a_{n+1}$. Let our base case be $n = 1$. Then $a_1 = 1$ and

$$a_2 = \frac{1}{2}a_1 + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2}.$$

Hence, this implies that $a_1 \leq a_2$. Suppose that the claim for $n = k + 1$, that is, prove that $a_{k+1} \leq a_{k+2}$. Now, we have

$$\begin{aligned} a_k \leq a_{k+1} &\implies \frac{1}{2}a_k \leq \frac{1}{2}a_{k+1} \\ &\implies \frac{1}{2}a_k + 1 \leq \frac{1}{2}a_{k+1} + 1 \\ &\implies a_{k+1} \leq a_{k+2}. \end{aligned}$$

(iii) We will show that (i) and (ii) will imply that (a_n) converges. We have

(1) $\forall n$, $1 = a_1 \leq a_n \leq 2 \implies (a_n)$ is bounded,

(2) (a_n) is increasing.

By the Monotone Convergence Theorem, we see that (a_n) converges.

(iv) Now, we will show that $\lim_{n\to\infty} a_n = 2$. Let $A = \lim_{n\to\infty} a_n$. We have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[\frac{1}{2}a_n + 1\right] &\implies \lim_{n\to\infty} a_{n+1} = \frac{1}{2}(\lim_{n\to\infty} a_n) + 1 \\ &\implies A = \frac{1}{2}A + 1 \\ &\implies \frac{1}{2}A = 1 \\ &\implies A = 2. \end{aligned}$$

9.2.2 Extended Real Numbers

The set of extended real number, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols $+\infty, -\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

(*) $\overline{\mathbb{R}}$ is equipped with an order. We preserve the original order in $\mathbb{R}$ and we define

$$\forall x \in \mathbb{R} \quad -\infty < x < \infty.$$

(*) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{aligned}
&\forall x \in \overline{\mathbb{R}} \text{ with } x > 0 \; x \cdot (+\infty) = +\infty \quad x \cdot (-\infty) = -\infty \\
&\forall x \in \overline{\mathbb{R}} \text{ with } x < 0, x \cdot (+\infty) = -\infty \; x \cdot (-\infty) = +\infty \\
&\forall x \in \mathbb{R}, \; x + \infty = +\infty \\
&\forall x \in \mathbb{R}, \; x - \infty = -\infty.
\end{aligned}$$

As for the last convention, we define

$$+\infty + \infty = +\infty \quad -\infty - \infty = -\infty.$$

Furthermore, we have

$$\forall x \in \mathbb{R} \quad \frac{x}{+\infty} = 0 \quad \frac{x}{-\infty} = 0.$$

Please notice that we did <u>not</u> define the following

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \ldots, 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0.$$

(*) $\sup A = \infty \iff$ either $\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and A is not bounded above in $\mathbb{R} \cup \{-\infty\}$.

(*) $\inf A = -\infty \iff -\infty$ or $A \subseteq \mathbb{R} \cup \{\infty\}$ and A is not bounded below in $\mathbb{R} \cup \{\infty\}$.

(*) $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Remark. Let (a_n) bea sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

(i) $\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that $\forall n > N \; |a_n - a| < \varepsilon$

(ii) $\lim_{n\to\infty} a_n = \infty \iff \forall M > 0 \; \exists N \in \mathbb{N}$ such that $\forall n > N \; a_n > M$

(iii) $\lim_{n\to\infty} a_n = -\infty \iff \forall M > 0 \; \exists N \in \mathbb{N}$ such that $\forall n > N \; -a_n > M$.

Theorem (Algebraic Limit Theorem for $\overline{\mathbb{R}}$)**.** Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $c \in \mathbb{R}$, then $ca_n \to ca$

(ii) $a_n + b_n \to a + b$ (provided that $\infty - \infty$ does not appear)

(iii) $a_n b_n \to ab$ (provided that $(\pm\infty) \cdot 0$ or $0 \cdot (\pm\infty)$ do not appear).

(iv) If $a = \infty$, then $\frac{1}{a_n} \to 0$. If $a = -\infty$, then $\frac{1}{a_n} \to 0$.

(v) If $a_n \to 0$ and $a_n > 0$, then $\frac{1}{a_n} \to \infty$. If $a_n \to 0$ and $a_n < 0$, then $\frac{1}{a_n} \to -\infty$.

Theorem (Order Limit Theorem for $\overline{\mathbb{R}}$)**.** Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $a_n \leq b_n$, then $a \leq b$.

(ii) If $a_n \leq e_n$ and $a_n \to \infty$, then $e_n \to \infty$.

(iii) If $e_n \leq a_n$ and $a_n \to -\infty$, then $e_n \to -\infty$.

Theorem (Monotone Convergence Theorem in $\overline{\mathbb{R}}$)**.** Let (a_n) be a sequence in $\overline{\mathbb{R}}$.

(i) If (a_n) is increasing, then $a_n \to \sup\{a_n : n \in \mathbb{N}\}$.

(ii) If (a_n) is decreasing, then $a_n \to \inf\{a_n : n \in \mathbb{N}\}$.

Note that if $a_n \to \infty$, then $\frac{1}{a_n} \to 0$, however, $\frac{1}{a_n} \to 0$ does not imply that $a_n \to \infty$. For example, consider $(a_n) = (-n)$ or $(a_n) = ((-1)^n n)$.

Remark. (i) $\overline{\mathbb{R}}$ can be equipped with the following metric:

Let $f : \overline{\mathbb{R}} \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ be defined by

$$f(x) = \begin{cases} \frac{-\pi}{2} & x = \infty \\ \arctan(x) & -\infty < x < \infty \\ \frac{\pi}{2} & x = \infty \end{cases}$$

Define $\overline{d}(x, y) = |f(x) - f(y)|\ \forall x, y \in \overline{\mathbb{R}}$. The closure of $\mathbb{R}$ in $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(ii) One can show that if (a_n) is a sequence in $\mathbb{R}$, then

$$a_n \to a \in \overline{\mathbb{R}} \iff \text{The sequence } a_n \text{ converges to } a \text{ in the metric space } (\overline{\mathbb{R}}, \overline{d}).$$

(iii) The closure of $\mathbb{R}$ in the metric space $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

(iv) Every set in $(\overline{\mathbb{R}}, \overline{d})$ is bounded:

$$\forall x, y \in \overline{\mathbb{R}} \quad \overline{d}(x, y) \leq \pi.$$

Chapter 10

Week 10

10.1 Lecture 18-19

10.1.1 Topics

- lim sup and lim inf (Two equivalent characterizations)
- Theorem: $\liminf a_n \leq \limsup a_n$.
- Theorem: $\lim_{n\to\infty} a_n$ exists in $\overline{\mathbb{R}}$ if and only if $\liminf a_n = \limsup a_n \in \overline{\mathbb{R}}$.
- Theorem: $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ provided that the right-hand side is not $\infty - \infty$.
- Some special sequences

10.1.2 First Characterization of Limsup and Liminf

Let (x_n) be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \text{there exists a subsequence } (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \to x\}.$$

We define,

$$\limsup x_n = \sup S$$
$$\liminf x_n = \inf S.$$

10.1.3 Second Characterization of Limsup and Liminf

Let (x_n) be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \geq n\}$. Clearly, we have

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \ .$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \cdots \qquad \text{(A decreasing sequence in } \overline{\mathbb{R}}\text{)}$$

and similarly, we have

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \dots \ . \qquad \text{(An increasng sequence in } \overline{\mathbb{R}}\text{)}$$

By the Monotone Convergence Theorem (in $\overline{\mathbb{R}}$), we know that $\lim_{n\to\infty} \sup F_n$ and $\lim_{n\to\infty} \inf F_n$ exists in $\overline{\mathbb{R}}$. We define

$$\limsup x_n = \lim_{n\to\infty} \sup F_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf F_n.$$

That is, we have

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf_n(\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup_n(\inf F_n).$$

Take note of the following notation:

$$\limsup x_n = \lim_{n\to\infty} \sup x_n = \overline{\lim} x_n$$
$$\liminf x_n = \lim_{n\to\infty} \inf x_n = \underline{\lim} x_n.$$

Example. (i) $x_n = (-1)^n$

Notice that

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \sup\{-1, 1\} = \lim_{n\to\infty} 1 = 1$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 1\} = \lim_{n\to\infty} -1 = -1.$$

(ii) Consider $(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots)$

Then we have

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \{-1, 2, 3\} = \lim_{n\to\infty} 3 = 3$$
$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} = \lim_{n\to\infty} \inf\{-1, 2, 3\} = \lim_{n\to\infty} -1 = -1.$$

(iii) Consider $a_n = n$

$$\begin{aligned}\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\} &= \lim_{n\to\infty} \sup\{n, n+1, n+2, \dots\} \\ &= \lim_{n\to\infty} n = \infty.\end{aligned}$$

and similarly, we have

$$\begin{aligned}\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\} &= \lim_{n\to\infty} \inf\{n, n+1, n+2, \dots\} \\ &= \lim_{n\to\infty} n = \infty.\end{aligned}$$

Remark. (i) $\liminf x_n = \sup_n \inf\{x_k : k \geq n\}$

(ii) $\limsup x_n = \inf_n \sup\{x_k : k \geq n\}$

Theorem. Let (a_n) be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n.$$

Proof. Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}.$$

Since we already proved that the limits of both sides exists (in $\overline{\mathbb{R}}$), it follows from the order limit theorem (in $\overline{\mathbb{R}}$) that

$$\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

That is, we have

$$\liminf a_n \leq \limsup a_n.$$

■

Theorem. Let (a_n) be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n \text{ exists in } \overline{\mathbb{R}} \text{ if and only if } \limsup a_n = \liminf a_n.$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.

Proof. ($\Longleftarrow$) Let $A = \limsup a_n = \liminf a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\lim a_n = A$. We may consider three cases; that is,

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), note that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}.$$

Since $\lim\limits_{n\to\infty} \sup\{a_k : k \geq n\} = \lim\limits_{n\to\infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that $\lim\limits_{n\to\infty} a_n = A$.

For (2) ($A = \infty$), we have for all $n \in \mathbb{N}$ that $\inf\{a_k : k \geq n\} \leq a_n$ and $\lim\limits_{n\to\infty} \inf\{a_k : k \geq n\} = \infty$ implies $\lim\limits_{n\to\infty} a_n = \infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

For (3) ($A = -\infty$), we know that for all $n \in \mathbb{N}$ that $a_n \leq \sup\{a_k : k \geq n\}$ and $\lim\limits_{n\to\infty} \sup\{a_k : k \geq n\} = -\infty$ implies that $\lim\limits_{n\to\infty} a_n = -\infty$ by the Order Limit Theorem in $\overline{\mathbb{R}}$.

($\Longrightarrow$) Let $A = \lim\limits_{n\to\infty} a_n$ with $A \in \overline{\mathbb{R}}$. In what follows, we will show that $\limsup a_n = A$ and $\liminf a_n = A$. We may consider three cases:

(1) $A \in \mathbb{R}$

(2) $A = \infty$

(3) $A = -\infty$

For (1), suppose that $A \in \mathbb{R}$. Our goal is to show that

$$A \leq \liminf a_n \text{ and } \limsup a_n \leq A,$$

and so

$$A \leq \liminf a_n \leq \limsup a_n \leq A.$$

Thus, it suffices to show that for all $\varepsilon > 0$

$$A - \varepsilon \leq \liminf a_n \text{ and } \limsup a_n \leq A + \varepsilon.$$

To this end, let $\varepsilon > 0$ be given. Since $a_n \to A$, there exists an $N \in \mathbb{N}$ such that

$$\forall n > N \quad |a_n - A| < \varepsilon;$$

that is,

$$\forall n > N \quad A - \varepsilon < a_n < A + \varepsilon.$$

Now, observe that

$$\begin{aligned}
\forall n > N \quad a_n < A + \varepsilon &\implies A + \varepsilon \text{ is an upper bound of } \{a_k : k \geq n\} \\
&\implies \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \varepsilon \\
&\implies \lim_{n\to\infty} \sup\{a_k : k \geq n\} \leq \lim_{n\to\infty} (A + \varepsilon) \qquad \text{(Order Limit Theorem)} \\
&\implies \limsup a_n \leq A + \varepsilon
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
\forall n > N \quad A - \varepsilon < a_n &\implies A - \varepsilon \text{ is a lower bound of} \{a_k : k \geq n\} \\
&\implies \forall n > N \ \inf\{a_k : k \geq n\} \geq A - \varepsilon \\
&\implies \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} A - \varepsilon \\
&\implies \liminf a_n \geq A - \varepsilon.
\end{aligned}$$

Now, suppose (2). Our goal is to show that $\liminf a_n = \infty$ so that $\liminf a_n \leq \limsup a_n$ will imply that $\limsup a_n = \infty$. In order to show that $\liminf a_n = \infty$, it suffices to show that

$$\forall M > 0 \quad M \leq \liminf a_n.$$

To this end, let $M > 0$ be given. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
\forall n > N \quad a_n > M &\implies \forall n > N \ \inf\{a_k : k \geq n\} \geq M \\
&\implies \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} M \\
&\implies \liminf a_n \geq M.
\end{aligned}$$

Note that an analogous process to the above is used to prove (3). ■

Theorem. Let (a_n) and (b_n) be the two sequences of real numbers. Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that the right-hand side is not of the form $\infty - \infty$ or $-\infty + \infty$.

Proof. First note that, by our assumption, $\limsup a_n + \liminf a_n$ is not of the form $\infty - \infty$ or $(-\infty + \infty)$, there exists n_0 such that

$$\forall n \geq n_0 \quad \sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\} \text{ is not of the form } \infty - \infty \text{ or } -\infty + \infty$$

For each $n \geq n_0$, we have

$$\begin{aligned}
\forall k \geq n \quad a_k &\leq \sup\{a_\ell : \ell \geq n\} \\
\forall k \geq n \quad b_k &\leq \sup\{b_m : m \geq n\}.
\end{aligned}$$

Thus, we have

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Therefore,

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_\ell : \ell \geq n\} + \sup\{b_m : m \geq n\}.$$

Now, label $R_n = \sup\{a_k + b_k : k \geq n\}$, $L_n = \sup\{a_\ell : \ell \geq n\}$ and $S_n = \sup\{b_m : m \geq n\}$. From the above, we can see that $\lim_{n\to\infty} R_n$, $\lim_{n\to\infty} L_n$, and $\lim_{n\to\infty} S_n$ all exists in $\overline{\mathbb{R}}$. Since $\lim_{n\to\infty} L_n + \lim_{n\to\infty} S_n$ is not of the form $\infty - \infty$, it follows form the Algebraic Limit Theorem that $\lim_{n\to\infty}(L_n + S_n)$ exists and is equal to that of $\lim L_n + \lim S_n$. By the Order Limit Theorem, we see that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

■

Theorem ((e)). If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. Clearly, if x = 0, then the claim holds. So, let's assume $x \in (-1, 1)$ and $x \neq 0$. Our goal is to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ |x^n - 0| < \varepsilon.$$

That is, we need to show, given the setup above, that $|x^n| < \varepsilon$. Since $0 < |x| < 1$, there exists $y > 0$ such that $|x| = \frac{1}{1+y}$. Note that

$$|x|^n < \varepsilon \iff \frac{1}{(1+y)^n} < \varepsilon.$$

Using the Binomial Theorem ($(1+y)^n \geq 1 + ny$), we can see that

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}.$$

Therefore, in order to ensure that $|x|^n < \varepsilon$, we just need to choose n large enough so that $\frac{1}{ny} < \varepsilon$. To this end, it suffices to choose n larger than $\frac{1}{\varepsilon y}$; that is, we can take $N = \frac{1}{\varepsilon y}$ and the result follows. ■

Theorem ((b)). If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Proof. If $p = 1$, the claim obviously holds. If $p \neq 1$, we may consider two cases.

For the first case, assume that $p > 1$. Then let $x_n = \sqrt[n]{p} - 1$. It suffices to show that $\lim_{n\to\infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also, we have

$$\begin{aligned} \sqrt[n]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n}. \end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim_{n\to\infty} x_n = 0$.

Now, suppose that $0 < p < 1$. Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by the previous case, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1 \iff \lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = 1.$$

■

Theorem ((c)). $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Proof. Let $x_n = \sqrt[n]{n} - 1$. Observe that, by the binomial formula, we have for all $n \geq 2$,

$$\begin{aligned}\sqrt[n]{n} = 1 + x_n \implies n = (1+x_n)^n &\geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}.\end{aligned}$$

Thus, we have

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \to 0$ and so $\sqrt[n]{n} \to 1$. ■

Chapter 11

Week 11

11.1 Lecture 20-21

11.1.1 Topics

- Infinite series (Basic definitions)
- Telescoping Series, Geometric Series
- Algebraic Limit Theorem for Series
- Divergence Test
- Cauchy Criterion for Series
- Absolute Convergence Test

Consider the following expression:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\ .$$

How can we make sense of the infinite sum above? More generally, let (a_n) be a sequence of real numbers. Then what does the following expression mean?

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots\ ?$$

Definition (Infinite Series). Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X.

(*) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1+x_2+x_3+\cdots$$

is called an **infinite series**.

(*) $x_1, x_2, \ldots$ are called the **terms** of this infinite series.

(*) The corresponding sequence of **partial sums** is defined by

$$\forall m\in\mathbb{N}\quad s_m = \text{(finite) sum of the first } m \text{ terms of the series;}$$

that is,

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + x_2 + \cdots + x_m \\ &\vdots \end{aligned}$$

1. We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty} x_n = L$) if $\lim\limits_{n\to\infty} s_m = L$.
2. We say that the infinite series **diverges**, if (s_m) diverges.
3. If $X = \mathbb{R}$ and $s_m \to \infty$, we write $\sum_{n=1}^{\infty} x_n = \infty$.

(*) If $X = \mathbb{R}$ and $s_m \to -\infty$, we write $\sum_{n=1}^{\infty} x_n = -\infty$.

Remark (1). Given an infinite series $\sum_{n=1}^{\infty} x_n$, it is important to keep a clear distinction between

(a) the sequence of terms: $(x_1, x_2, x_3, \dots)$

(b) the sequence of partial sums: $(s_1, s_2, s_3, \dots)$.

Remark (2). We may sometimes consider infinite series where the summation begins with $n = 0$ or $n = n_0$ for some integer n_0 different from 1.

As we shall see, some of our theorems apply specifically to series in $\mathbb{R}$ or to series with terms in $[0, \infty)$. Also, in our examples, we will primarily focus on series in $\mathbb{R}$; however, we will also consider encounter highly useful theorems that hold in more general normed spaces.

In most cases it is difficult (or even impossible) to find a simple formula for the partial sum s_m. However, there are two types of series for which we can easily find a simple formula for the partial sums. These two types are:

(1) Telescoping Series

(2) Geometric Series

11.1.2 Telescoping Series

Example. Consider the following series

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Notice that $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned}
s_1 &= 1 - \frac{1}{2} \\
s_2 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) = 1 - \frac{1}{3} \\
s_3 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) = 1 - \frac{1}{4} \\
&\vdots \\
s_m &= \sum_{n=1}^{m} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = \Big(\sum_{n=1}^{m} \frac{1}{n}\Big) - \Big(\sum_{n=1}^{m} \frac{1}{n+1}\Big) \\
&= 1 - \frac{1}{m+1}.
\end{aligned}$$

Clearly, we see that

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \Big[1 - \frac{1}{m+1}\Big] = 1.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

In general, a telescoping series is an infinite series where partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n - y_{n+1})$ is a telescoping series; that is,

$$\begin{aligned}
s_m = \sum_{n=1}^{m}(y_n - y_{n+1}) = \Big(\sum_{n=1}^{m} y_n\Big) - \Big(\sum_{n=1}^{m} y_{n+1}\Big) &= [y_1 + y_2 + \cdots + y_m] - [y_2 + y_3 + \cdots + y_{m+1}] \\
&= y_1 - y_m.
\end{aligned}$$

11.1.3 Geometric Series

Let k be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \cdots$ is called a **geometric series** with common ration "r". For example,

$$\sum_{n=1}^{\infty} \Big(\frac{1}{2}\Big)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ is a geometric series with common ratio } \frac{1}{2}.$$

Another example is that

$$\sum_{n=1}^{\infty} \frac{7^n}{29^n} \text{ is a geometric series with common ratio } \frac{7}{29}.$$

A non-example is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We can easily find a formula for the mth partial sum of $\sum_{n=k}^{\infty} r^k$ where

$$\begin{aligned}
s_1 &= r^k \\
s_2 &= r^k + r^{k+1} \\
s_3 &= r^k + r^{k+1} + r^{k+2} \\
&\vdots \\
s_m &= r^k + r^{k+1} + \cdots + r^{k+m-1} \qquad (*)
\end{aligned}$$

Now, if $r = 1$, we have

$$s_m = \underbrace{1 + 1 + \cdots + 1}_{m\text{summands}} = m.$$

If $r \neq 1$, then multiply both sides of (*) by r:

$$rs_m = r^{k+1} + r^{k+2} + \cdots + r^{k+m}. \tag{**}$$

Subtracting (**) from (*), we get

$$s_m - rs_m = r^k - r^{k+m}.$$

Since $r \neq 1$, we have

$$s_m = \frac{r^k - r^{k+m}}{1-r} = \frac{r^k(1-r^m)}{1-r}.$$

Note that

(i) If $|r| < 1$, then $\lim r^m = 0$.

(ii) If $|r| > 1$ or $r = -1$, then $\lim\limits_{n\to\infty} r^m$ does not exists.

Hence, we have

$$\lim_{m\to\infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{DNE} & \text{if} |r| \geq 1. \end{cases}$$

Thus,

$$\sum_{n=k}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{diverges} & \text{if} |r| \geq 1. \end{cases}$$

Example.

- $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

  Observe that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^1}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

- $\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n$

  Observe that

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)} = \frac{1}{8}.$$

Theorem. Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) and (b_n) be two sequence in X. Suppose that

$$\sum_{n=1}^{\infty} a_n = A \quad (A \in X), \quad \sum_{n=1}^{\infty} b_n = B \quad (B \in X).$$

Then

(i) For any scalar λ, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$.

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$.

Proof. Can easily be proven via the Algebraic Limit Theorem for Sequences. ■

Theorem (Divergence Test)**.** Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $s_n = x_1 + \cdots + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n\to\infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Note that $\lim s_n = L$ and $\lim s_{n-1} = L$. Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (s_n - s_{n-1}) = L - L = 0$$

by the Algebraic Limit Theorem for normed spaces. ■

Remark. Note that the divergence test is just the contrapositive of the above.

Example.
- $\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim_{n\to\infty} (-1)^n$ does not exist.
- $\sum_{n=1}^{\infty} \frac{3n+1}{7n-4}$ diverges because $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

From the above statements, we can now see make two key observations:

- If $\lim_{n\to\infty} x_n = 0$, then $\sum_{n=1}^{\infty} x_n$ may or may not converge.
- If $\lim_{n\to\infty} x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ diverges.

As for the first observation above, we see that $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^2}$ converges.

Theorem (Cauchy Criterion)**.** Let $(X, \|\cdot\|)$ be a complete normed space. Let (x_n) be a sequence in X. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges } \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=1}^{n} x_k\| < \varepsilon.$$

Proof. Let $s_n = x_1 + \cdots + x_k$. Assuming that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\|$$

where $n > m$ and from the fact that

$$\begin{aligned} s_n - s_m &= (x_1 + \cdots + x_m + \cdots + x_n) - (x_1 + \cdots + x_m) \\ &= \sum_{k=m+1}^{n} s_k. \end{aligned}$$

Then we have

$$\begin{aligned}\sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges}\\ &\iff (s_k) \text{ is Cauchy}\\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \varepsilon\\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=m+1}^{n} x_k\| < \varepsilon\end{aligned}$$

as desired. ■

From here, we will refer to complete normed spaces as Banach spaces.

Theorem (Absolute Convergence Test)**.** Let $(X, \|\cdot\|)$ be a Banach Space. Let (x_n) be a sequence in X. If $\underbrace{\sum_{n=1}^{\infty} \|x_n\|}_{\text{a sum in } \mathbb{R}}$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for series, it suffices to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \left\| \sum_{k=m+1}^{n} x_k \right\| < \varepsilon. \tag{*}$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for series that there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{n} \|x_k\| \right| < \varepsilon.$$

We claim that $\hat{N}$ is the same N we were looking for. Hence, if $n > m > \hat{N}$, then we have

$$\left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| = \left| \sum_{k=m+1}^{n} \|x_k\| \right| < \varepsilon$$

as desired. ■

Please take note of the following observations:

(1) If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges (in Banach spaces).

(2) If $\sum_{n=1}^{\infty} \|x_n\|$ diverges, then $\sum_{n=1}^{\infty} x_n$ may converge or diverge.

From (2), we shall see (in the next lecture) that

(1) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ diverges but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

(2) $\sum_{n=1}^{\infty} |(-1)^n|$ diverges, also $\sum_{n=1}^{\infty} (-1)^n$ diverges (by the divergence test).

Definition (Absolute Convergence and Conditional Convergence)**.** We say that a series $\sum x_n$ **absolutely converges** if $\sum \|x_n\|$ converges and $\sum x_n$ converges. We say that $\sum x_n$ **conditionally converges** if $\sum \|x_n\|$ diverges but $\sum x_n$ converges.

Example (Conditionally Convergent). Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We see that this series is conditionally convergent since

$$\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \to 0$$

which tell us that the above series diverges by the divergence test. But the above series converges via the Leibniz Test (As we shall see in the next lecture).

11.2 Lecture 21-22

11.2.1 Topics

- Cauchy Condensation Test
- Comparison Test
- More on lim sup and lim inf
- Root Test
- Ratio Test
- Dirichlet's Test

Theorem (Cauchy Condensation Test). Assume $a_n \geq 0$ for all n, and (a_n) is a decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \text{ converges.}$$

Proof. Let $s_m = a_1 + \cdots + a_m$ and $t_m = a_1 + 2a_2 + 4a_4 + \cdots + 2^{m-1}a_{2^{m-1}}$. Using the fact that (a_n) is a decreasing sequence, we can see that

$$\begin{aligned}
s_{2^k} &= a_1 = a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1}a_{2^k} \\
&= a_1 + \frac{1}{2}\Big[t_{k+1} - a_1\Big] \\
&= a_1 + \frac{1}{2}t_{k+1} - \frac{1}{2}a_1 \\
&= \frac{1}{2}(a_1 + t_{k+1}) \\
&\geq \frac{1}{2}t_{k+1}.
\end{aligned}$$

Thus, we have

$$s_{2^k} \geq \frac{1}{2}t_{k+1}. \tag{*}$$

Similarly, we have

$$\begin{aligned}
s_{2^k-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1}) \\
&\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^{k-1}}) \\
&= a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1}a_{2^{k-1}} \\
&= t_k.
\end{aligned}$$

Thus, we have that

$$s_{2^k-1} \leq t_k. \tag{**}$$

($\Longleftarrow$) Assume that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). Our goal is to show that $\sum_{n=1}^{\infty} a_n$ converges; that is, (s_m) converges. Note that since $a_n \geq 0$, both (s_m) and (t_m) are increasing sequences. It follows from the Monotone Convergence Theorem that in order to prove (s_n) converges, it suffices to show that (s_m) is bounded.

Since (t_m) converges, we have that (t_m) is bounded. Hence, there exists $R > 0$ such that $t_m \leq R$ for all $m \in \mathbb{N}$. In what follows, we will show that R is an upper bound for (s_m) as well. Indeed, let $m \in \mathbb{N}$ be given. Choose k large enough so that $m < 2^k - 1$, then

$$s_m \leq s^{2^k-1} \leq t_k \leq R.$$

Thus, for all $m \in \mathbb{N}$, $0 \leq s_m \leq R$. Hence, (s_m) is bounded and so (s_m) converges by MCT.

($\Longrightarrow$) Assume that $\sum_{n=1}^{\infty} a_n$ ((s_m) converges). Our goal is to show that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges; that is, (t_m) converges.

We will prove the contrapositive: we will show that if (t_m) diverges then (s_m) diverges. Suppose (t_m) is divergent. Let $R >)$ be given. We will show that there is a term in the nonnegative sequence (s_m) that is larger than R. Since (t_m) diverges and (t_m) is an increasing sequence (where $t_m \geq 0$), we see that (t_m) cannot be bounded above by the Monotone Convergence Theorem. Hence, there exists $k \in \mathbb{N}$ such that $t_{k+1} > 2R$. Now, we have

$$s_{2^k} \geq \frac{1}{2} t_{k+1} ? \frac{1}{2}(2R) = R.$$

Thus, (s_m) is not bounded. ■

11.2.2 Applications of the Cauchy-Condensation Test

Example (P-series Test). Let $p > 0$. One can show that the sequence $\left(a_n = \frac{1}{n^p}\right)$ is a decreasing nonnegative sequence. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1.$$

Proof. Using the Cauchy Condensation test, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \text{ converges} \\
&\iff \left|\frac{1}{2^{p-1}}\right| < 1 \\
&\iff 1 < 2^{p-1} \\
&\iff 0 < p - 1 \\
&\iff 1 < p.
\end{aligned}$$

In the fourth equivalence statement, we used the fact $\sum_{n=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^n$ is a geometric series with common ratio $\frac{1}{2^{p-1}}$. ■

Example. Let $p > 0$. One can show that the sequence $\left(a_n = \frac{1}{n(\ln n)^p}\right)_{n\geq 2}$ is a decreasing nonnegative sequence. Prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \Longleftrightarrow p > 1.$$

Proof. By the Cauchy Condensation Test, we see that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} &\Longleftrightarrow \sum_{n=1}^{\infty} 2^n \frac{1}{2^n(\ln(2^n))^p} \text{ converges}\\ &\Longleftrightarrow \sum_{n=1}^{\infty} \frac{1}{(n\ln 2)^p} \text{ converges}\\ &\Longleftrightarrow \frac{1}{(\ln 2)^p}\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}\\ &\Longleftrightarrow p > 1.\end{aligned}$$

■

Theorem (Comparison Test)**.** Assume there exists an integer n_0 such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Notice that (ii) is just the contrapositive of (i). So, it suffices to show (i).

By the Cauchy Criterion for convergence of series, it is enough to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N\ \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Our goal is to find an N such that (*) holds. Since $\sum_{n=1}^{\infty} b_n$ converges, it follows from the Cauchy Criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N}\ \left|\sum_{k=m+1} b_k\right| < \varepsilon.$$

Let $N = \max\{n_0, \hat{N}\}$. If $k \geq n_0$ where $a_k, b_k \geq 0$, we see that

$$\left|\sum_{k=m+1}^{n} a_k\right| = \sum_{k=m+1}^{n} a_k \text{ and } \left|\sum_{k=m+1}^{n} a_k\right| = \sum_{k=m+1}^{n} b_k. \tag{1}$$

Furthermore, if $k \geq n_0$, we have $a_k \leq b_k$, we have

$$\sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k. \tag{2}$$

If $n > m > N$, we see that (1) and (2) imply that

$$\left|\sum_{k=m+1}^{n} a_k\right| = \sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k = \left|\sum_{k=m+1}^{n} b_k\right| < \varepsilon.$$

■

Example. (*) Does $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converge?

Indeed, for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{1}{n+5^n} \leq \frac{1}{5^n}.$$

Note that $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges (because it is a geometric series). Thus, the comparison test implies that $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converges.

(*) Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Indeed, we see that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longrightarrow \lim a_n = 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $0 \leq a_n < 1$. Thus,

$$\forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n.$$

By the comparison test, we can conclude that $\sum_{n=1}^{\infty} a_n^2$ converges.

Remark (Some useful properties)**.** Let (a_n) be a sequence of real numbers. Suppose $\lim_{n\to\infty} a_n = A \in \mathbb{R}$.

(i) If $A < \beta$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n < \beta$.

(ii) If $\alpha < A$, then there exists N such that for all $n > N$, $\alpha < a_n$.

Theorem. Let (a_n) be a sequence of real numbers.

(i) Suppose $\beta \in \mathbb{R}$ is such that $\limsup a_n < \beta$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \quad a_n < \beta.$$

(ii) Suppose $\alpha \in \mathbb{R}$ is such that $\liminf a_n > \alpha$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \quad a_n > \alpha.$$

Proof. Here we will prove (i). Since $\limsup a_n < \beta$, we have $\limsup a_n \neq \infty$. We may consider two cases:

(1) Suppose $\limsup a_n = -\infty$. Since $\liminf a_n \leq \limsup a_n$, we can conclude that $\liminf a_n = -\infty$. Therefore, $\lim a_n = -\infty$. The claim immediately follows from the definition of $a_n \to -\infty$.

(2) Suppose $A = \limsup a_n$. Let $A = \limsup a_n$ and $r = \frac{\beta - A}{2}$. Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = A$, there exists N such that

$$\forall n > N \quad \sup\{a_k : k \geq n\} < A + r.$$

In particular, we have

$$\forall n > N \quad \sup\{a_k : k \geq n\} < \beta.$$

Therefore, we have (noticing that $a_n \leq \sup\{a_k : k \geq n\}$),

$$\forall n > N \quad a_n < \beta.$$

Note that the proof of (ii) is completely analogous to the proof of (i). ■

Theorem. Let (a_n) be a sequence of real numbers.

(i) Suppose $\limsup a_n > \beta$. Then, for infinitely many k, we have $a_k > \beta$. That is,

$$\forall n \in \mathbb{N} \ \ \exists k \geq n \text{ such that } a_k > \beta.$$

(ii) Suppose $\liminf a_n < \alpha$. Then, for infinitely many k, $a_k < \alpha$. That is,

$$\forall n \in \mathbb{N} \ \ \exists k \geq n \text{ such that } a_k < \alpha.$$

Proof. Here we will prove (i) (the proof for (ii) is completely analogous). Assume for contradiction that only for finitely many k, $a_k > \beta$. Then there exists an $N \in \mathbb{N}$ such that for all $k > N$, we have $a_k \leq \beta$. Therefore,

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that $\limsup a_k > \beta$. ■

Theorem (Root Test (Version 1))**.** Let (a_n) be a sequence of real numbers. Let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem (Root Test (Version 2))**.** Let (a_n) be a sequence of real numbers. Assume that following limit exists:

$$\alpha = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\alpha < \beta < 1$. We have $\limsup \sqrt[n]{|a_n|} < \beta$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $\sqrt[n]{|a_n|} < \beta$. Hence, we have for all $n > N$, $0 \leq |a_n| < \beta^n$ and $\sum_{n=1}^{\infty} \beta^n$ converges (it is a geometric series with common ratio $0 < \beta < 1$). As a consequence, we see that $\sum_{n=1}^{\infty} |a_n|$ converges by the Comparison Test.

(ii) Choose a number β such that $1 < \beta < \alpha$. We have $\beta < \limsup \sqrt[n]{|a_n|}$. By Useful Theorem 2, we have for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \exists k \geq n \text{ such that } \sqrt[k]{|a_k|} > \beta &\implies |a_k| > \beta^k \\ &\implies \sup\{|a_m| : m \geq n\} > \beta^k. \end{aligned}$$

Since $k \geq n$, we have $\beta^k \geq \beta^n$, and so

$$\forall n \in \mathbb{N} \sup\{|a_m| : m \geq n\} > \beta^n.$$

Since $\lim_{n\to\infty} \beta^n = \infty$ ($\beta > 1$), it follows from the order limit theorem (for $\overline{\mathbb{R}}$) that $\lim_{n\to\infty} \sup\{|a_m| : m \geq n\} = \infty$. So, $\limsup |a_n| = \infty$. This tells us that $\lim a_n \neq 0$ (Explanation of Fact). So, $\sum a_n$ diverges by the Divergence Test.

■

Remark. This is just the contrapositive of the following fact:

If $\lim a_n = 0$, then $\lim |a_n| = 0$, so $\limsup |a_n| = 0$.

Theorem (Ratio Test (Version 1))**.** Let (a_n) be a sequence of real numbers.

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\left|\frac{a_{m+1}}{a_m}\right| \geq 1$ for all $n \geq n_0$ (some integer n_0, then $\sum_{n=1}^{\infty} a_n$ diverges).

(iii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem (Ratio Test (Version 2))**.** Let (a_n) be a sequence of real numbers. Assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\rho < \beta < 1$. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \Longrightarrow \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

Thus, we have

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ |a_{N+3}| &< \beta |a_{N+2}| < \beta^3 |a_N| \\ &\vdots \end{aligned}$$

So, for all $n \in \mathbb{N}$, $|a_{N+n}| < \beta^n |a_N|$. Now, notice that

$$\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n \quad \text{converges.}$$

Now, notice that $\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n$ converges (since it is a geometric series with common ratio $0 < \beta < 1$). It follows from the Comparison Test that $\sum_{n=1}^{\infty} |a_{N+n}|$ converges. Considering that $\sum_{n=1}^{\infty} |a_{N+n}| = \sum_{n=N+1}^{\infty} |a_n|$, we can conclude that $\sum_{n=N+1}^{\infty} |a_n|$ converges. This immediately implies that $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) Choose a number β such that $1 < \beta < \rho$. Then we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \Longrightarrow \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| > \beta.$$

So, we have

$$|a_{N+1}| > \beta |a_N|$$
$$|a_{N+2}| > \beta |a_{N+1}| > \beta^2 |a_N|$$
$$|a_{N+3}| > \beta |a_{N+2}| > \beta^3 |a_N|$$
$$\vdots$$

Thus, for each $n \in \mathbb{N}$, $|a_{N+n}| > \beta^n |a_N|$. Since $\beta > 1$, $\lim_{n\to\infty} \beta^n |a_N| = \infty$. So, $\lim_{n\to\infty} |a_{N+n}| = \infty$. Therefore, $\lim_{n\to\infty} a_{N+n} \neq 0$. Thus, $\lim_{n\to\infty} a_n \neq 0$ (because $(a_{N+n})_{n\geq 1}$) is a subsequence of $(a_n)_{n\geq 1}$. So, $\sum_{n=1}^{\infty} a_n$ diverges by the Divergence Test.

■

Example. Let $R \neq 0$ be a fixed number. Prove that the series $\sum_{n=1}^{\infty} \frac{R^n}{n!}$ converges. Indeed, we have

$$\begin{aligned}
\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} \right| &= \lim_{n\to\infty} \left| \frac{R^{n+1} n!}{R^n (n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{R}{n+1} \right| \\
&= |R| \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0.
\end{aligned}$$

Thus, if $\rho = 0 < 1$, then $\sum_{n=1}^{\infty} \frac{R^n}{n!}$ is absolutely convergent. As a consequence, we have

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0.$$

Remark. If (a_n) is a sequence and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\lim a_n = 0$.

Theorem (Dirichlet's Test)**.** Let (b_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ be bounded, is a decreasing sequence of nonnegative numbers ($b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$), and $\lim_{n\to\infty} b_n = 0$. Then we have $\sum_{n=1}^{\infty} a_n b_n$ converges.

Corollary (Leibniz Test)**.** Let (b_n) be a sequence in $\mathbb{R}$. Suppose $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ and $\lim_{n\to\infty} b_n = 0$. Then we have $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

Consider the infinite sum

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots \tag{*}$$

and the following questions:

(1) What is (s_n)?

(2) What is $\lim_{n\to\infty} s_n$?

Define the sequence of partial sums for the series in (*) (s_n).

Consider the following partial sums

$$
\begin{aligned}
s_1 &= 1 \\
\\
s_2 &= 1 - 1 = 0 \\
s_3 &= 1 - 1 + \frac{1}{2} = \frac{1}{2} \\
s_4 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} = 0 \\
s_5 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3} \\
&\vdots
\end{aligned}
$$

which establishes (1). Looking at the even subsequence of (s_n), we can see that for all $k \in \mathbb{N}$ $s_{2k} = 0$ and the odd subsequence $s_{2k-1} = \frac{1}{k}$, respectively. Clearly, we can see from these subsequences that

$$s_{2k} \to 0 \quad \text{and} \quad s_{2k-1} \to 0$$

as $k \to \infty$. Hence, we can see that $s_n \to 0$ which establishes (2).

11.2.3 Rearrangements

Consider the following rearrangement of (*)

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \cdots = \lim_{n \to \infty} s_n = \ln(2).$$

Consider the sequence of the partial sums of the above:

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= \frac{3}{2} \\
s_3 &= \frac{1}{2} \\
&\vdots \\
s_{2 \times 10^2 + 2} &\approx 0.6939 \\
s_{3 \times 10^4 + 2} &\approx 0.6932 \\
s_{3 \times 10^6 + 2} &\approx 0.6931 \\
&\vdots
\end{aligned}
$$

Theorem. If a series converges absolutely, then for any $L \in \mathbb{R}$, there exists some rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to L.

Chapter 12

Week 12

12.1 Topics

- Limits and Continuity
- Limits of Functions
- Sequential Criterion for Limits of Functions
- Algebraic Limit Theorem for Functions
- Continuous Functions
- Characterization of Continuity via Sequences
- Algebraic Continuity Theorem
- A composition of Continuous Functions is Continuous

12.2 Limits and Continuity

One of the most important concepts in Calculus is the limit of a function. Our goal is to understand what is meant by:

(1) $\lim_{x \to c} f(x) = L$.

(2) "$f(x)$ is continuous at c"

12.3 Limits of Functions

The usual set up for discussing continuity of functions is that we need to have

(1) Two metric spaces, (X, d) and $(Y, \tilde{d})$, for the domain and codomain of the function, respectively.

(2) Let E be some nonempty subset of X.

(3) Let c be some limit point of E; that is, $c \in E'$.

(4) Let $f : E \to Y$.

Recall that $c \in E'$ means that there exists a sequence in $E \setminus \{c\}$ such that $x_n \to c$. In particular, c does not necessarily belong to E. Also, it is not the case that every point in E necessarily belongs to E'.

Some common descriptions made about limits of functions can be made using the following statements:

- $f(x)$ approaches L as x approaches c ($x \in E$ and $x \neq c$).
- For every $\varepsilon > 0$, no matter how small it is, $\tilde{d}(f(x), L) < \varepsilon$ provided that $d(x, c)$ is sufficiently small (i.e $x \in E$ and $x \neq c$).

Definition (Limit of a Function)**.** Let (X, d) and $(Y, \tilde{d})$ be metric spaces. Let $\emptyset \neq E \subseteq X$. Let $c \in E'$. Let $f : E \to Y$. We say that $\lim_{x \to c} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < d(x, c) < \delta \text{ (with } x \in E\text{), then } \tilde{d}(f(x), L) < \varepsilon.$$

Remark. Just like in our definition of sequential limits and how our position $N$is dependent upon our choice of ε, we see a similar notion here when it comes to our $\delta > 0$ being dependent upon our choice of ε.

The reason we require $x \in E$ is to ensure that $f(x)$ makes sense. That is, we may not need to state the condition that $x \in E$ provided that there is no confusion about the setup.

Furthermore, we require the condition that $d_x(x, c) > 0$ because we need only concern ourselves with what happens as the points of our domain approach the limit point c, not necessarily what happens at the point c. In this way, we can see that $f(c)$ could be changed to anything (or could not even exist) and the limit would still be unaffected.

Here are some equivalent statements we can make about limits of functions

(1) $\lim_{x \to c} f(x) = L$

(2) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $0 < d(x, c) < \delta$ (with $c \in E$), then $\tilde{d}(f(x), L) < \varepsilon$.

(3) $\forall \varepsilon > 0, \exists > 0$ such that $\forall x \in E \setminus \{c\}$ satisfying $d(x, c) < \delta$, we have $\tilde{d}(f(x), L) < \varepsilon$.

(4) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$, $\tilde{d}(f(x), L) < \varepsilon$.

(5) $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$, $f(x) \in N_\varepsilon^Y(L)$.

(6) Given any $\varepsilon-$neighborhood $N_\varepsilon^Y(L)$ of L, there exists a $\delta-$neighborhood $N_\delta^X(c)$ of c such that the image of the part of $N_\delta^C(c)$ contained in $E \setminus \{c\}$ is contained in $N_\varepsilon^Y(L)$.

Keep in mind that, given $\varepsilon > 0$, the choice of δ is NOT unique. If $\delta = r$ where (r is a fixed number) works, then any positive number less than r will also work. That is, given any $\varepsilon > 0$, we can always assume that the δ we are looking for is less than 1.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 5$ for all $x \in \mathbb{R}$. Prove that $\lim_{x \to 3} f(x) = 11$.

Our goal is to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|f(x) - 11| < \varepsilon$. Let $\delta = \frac{\varepsilon}{2}$. For any x with $0 < |x - 3| < \delta$, we have

$$|f(x) - 11| = |(2x + 5) - 11| = 2|x - 3| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

as desired.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. Prove that $\lim_{x \to 2} f(x) = 4$.

Our goal is to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 4| < \varepsilon$. Let $\varepsilon > 0$ be given. Recall that, we may assume that the δ we are looking for is not greater than 1. Let

$\delta = \min\{1, \frac{3}{5}\}$. For any x with $0 < |x - c| < \delta$, we have

$$|f(x) - 4| = |x^2 - 4| = |x - 2||x + 2| < 5|x - 2| < 5\Big(\frac{\varepsilon}{5}\Big) = \varepsilon$$

as desired.

Example (Limits with Discrete Metric). Let $f : \mathbb{R} \to (\mathbb{R}, \tilde{d})$ defined by $f(x) = x^2$. Prove that $\lim_{x\to 2} f(x)$ does not exist.

Proof. Assume for sake of contradiction that, $\lim_{x\to 2} f(x) = L$. So,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta, \text{ then } \tilde{d}(f(x), L) < \varepsilon.$$

In particular, for $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\tilde{d}(f(x), L) < \frac{1}{2}$. Note that

$$\tilde{d}(f(x), L) < \frac{1}{2} \Longrightarrow \tilde{d}(f(x), L) = 0 \Longrightarrow f(x) = L.$$

Hence, there exists $\delta > 0$ such that if $2 - \delta < x < 2 + \delta$, then $x^2 = L$. Obviously, it is not the case that for all $x \in (2 - \delta, 2 + \delta)$ that x^2 is equal to the fixed number 1. ■

Theorem (Sequential Criterion for Limits of Functions). Let (X, d) and $(Y, \tilde{d})$ be metric spaces, let E be a nonempty set in X, let $c \in E'$, and $f : E \to Y$. The following statements are equivalent:

(i) $\lim_{x\to c} f(x) = L$.

(ii) For all sequences (a_n) in $E \setminus \{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$.

Proof. ($(i) \Longrightarrow (ii)$) Let (a_n) be a sequence in $E \setminus \{c\}$ such that $a_n \to c$. Our goal is to show that $f(a_n) \to L$. That is, we want to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ \tilde{d}(f(a_n), L) < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$, we see that

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E \setminus \{c\})\ f(x) \in N_\varepsilon^Y(L). \tag{1}$$

Since $\lim a_n = c$, we also see that

$$\exists \hat{N} \in \mathbb{N} \text{ such that } \forall n > \hat{N}\ a_n \in N_\delta^X(c). \tag{2}$$

Our claim is the $\hat{N}$ can be used as the same N we were looking for. Indeed, if we let $N = \hat{N}$ such that for any $n > N$, we can see that from (2) that $a_n \in N_\delta^X(c)$ and $a_n \in E\setminus\{c\}$ imply that $a_n \in N_\delta^X(c)\cap(E\setminus\{c\})$. By part (1), we can see that $f(a_n) \in N_\varepsilon^Y(L)$.

($(ii) \Longrightarrow (i)$) Suppose for all sequences (a_n) in $E \setminus \{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$. Our goal is to show that $\lim_{x\to c} f(x) = L$. Suppose for sake of contradiction that $\lim_{x\to c} \neq L$; that is,

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0\ \exists x \in N_\delta^X(c) \cap (E \setminus \{c\}) \text{ such that } f(x) \notin N_\varepsilon^Y(L).$$

So,

$$\begin{aligned}
\delta = 1 \ &\exists x_1 \in E \setminus \{c\} \text{ such that } d(x_1, c) < 1 \text{ but } \tilde{d}(f(x_1), L) \geq \varepsilon \\
\delta = \frac{1}{2} \ &\exists x_2 \in E \setminus \{c\} \text{ such that } d(x_2, c) < 1 \text{ but } \tilde{d}(f(x_2), L) \geq \varepsilon \\
\delta = \frac{1}{3} \ &\exists x_3 \in E \setminus \{c\} \text{ satisfying } d(x_3, c) < \frac{1}{3} \text{ but } \tilde{d}(f(x_3), L) \geq \varepsilon \\
&\vdots
\end{aligned}$$

In this way we will obtain a sequence (x_n) in $E \setminus \{c\}$ such that $x_n \to c$ but for which $\tilde{d}(f(x_n), L) \geq \varepsilon$, and so $f(x_n) \not\to L$. This contradicts our assumption.

■

Remark. The previous theorem gives us a nice way of establishing that certain limits do NOT exist. If you can find two sequences (a_n) in $E \setminus \{c\}$ and (b_n) in $E \setminus \{c\}$ such that $\lim a_n = c$ and $\lim b_n = c$ but $\lim f(a_n) \neq \lim f(b_n)$, then you can conclude that $\lim\limits_{x \to c} f(x)$ does NOT exist.

Example. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = \sin \frac{1}{x}$. Prove that $\lim\limits_{x \to 0} f(x)$ doe NOT exist.

Proof. Let $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Clearly, (a_n) and (b_n) are sequences in $\mathbb{R} \setminus \{0\}$ and $\lim\limits_{n \to \infty} a_n = 0$ and $\lim\limits_{n \to \infty} b_n = 0$. However,

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \sin \frac{1}{a_n} = \lim_{n \to \infty} \sin(2n\pi) = \lim_{n \to \infty} 0 = 0$$

and

$$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} \sin \frac{1}{b_n} = \lim_{n \to \infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = \lim_{n \to \infty} \sin \frac{\pi}{2} = \lim_{n \to \infty} 1 = 1.$$

So, $\lim f(a_n) \neq \lim f(b_n)$. Therefore, $\lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist. ■

Theorem. Let (X, d) be a metric space. Let $\emptyset \neq E \subseteq X$. Let $c \in E'$. Let $f, g : E \to \mathbb{R}$. Assume that

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M.$$

Then

(i) $\forall k \in \mathbb{R}\ \lim\limits_{x \to c}(kf(x)) = kL$,

(ii) $\lim\limits_{x \to c}(f(x) + g(x)) = L + M$,

(iii) $\lim\limits_{x \to c} f(x)g(x) = LM$

(iv) $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$.

Proof. All these items follow immediately from the Algebraic Limit Theorem for sequences and the sequential criterion for limits of functions. ■

12.4 Continuity

(1) Let $E \subseteq X$. First, we will explain what it means to say that $f : E \to Y$ is continuous at a point $c \in E' \cap E$.

(2) Then we will explain what it means to say that $f : E \to Y$ is continuous at a point $c \in E$.

(3) Finally, we will explain what it means to say that $f : E \to Y$ is continuous.

In what follows, we will present a three definitions of continuity that will best represent the three situations above.

Definition (Version 1 of Continuity). Let (X, d) and $(Y, \tilde{d})$ be metric spaces. Let $E \subseteq X$. Let $c \in E'$. Let $f : E \to Y$. We say that f is continuous at c if all the following three conditions hold:

(1) $c \in E$ (f is defined at c),

(2) $\lim\limits_{x \to c} f(x)$ exists,

(3) $\lim\limits_{x \to c} f(x) = f(c)$.

The definition above is mostly seen in introductory Calculus books.

Definition (Version 2 of Continuity). Let (X, d) and $(Y, \tilde{d})$ be two metric spaces. Let E be a nonempty set in X. Let $c \in E$. Let $f : E \to Y$. We say f is continuous at c if any of the following equivalent statements holds:

(1) $\lim\limits_{x \to c} f(x) = f(c)$.

(2) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $d(x, c) < \delta$ (with $x \in E$), then $\tilde{d}(f(x), f(c)) < \varepsilon$.

(3) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E$, $f(x) \in N_\varepsilon^Y(f(c))$.

(4) For every $\varepsilon-$neighborhood $N_\varepsilon^Y(f(c))$ of $f(c)$, there exists a $\delta-$neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\varepsilon^Y(f(c))$.

Remark. (i) It is a direct consequence of definition 2 that if c is an isolated point of E, then f is continuous at c.

(ii) As we discussed, if $c \in E \cap E'$, then definition 2 is equivalent to saying that $\lim\limits_{x \to c} f(x) = f(c)$.

Definition (Version 3 of Continuity). Let (X, d) be a metric space and let $E \subseteq X$. Suppose $f : E \to Y$. We say that f is **continuous** if it is continuous at every point of E.

Theorem (Characterization of Continuity via Sequences). Let $f : E \subseteq X \to Y$. Let $c \in E$. The following two statements are equivalent:

(i) f is continuous at c

(ii) For all sequences (a_n) in E satisfying $a_n \to c$, we have $f(a_n) \to f(c)$.

Proof. ($(i) \implies (ii)$) Let (a_n) be a sequence in E such that $a_n t \to c$. Our goal is to show that $f(a_n) \to f(c)$, that is, we want to show that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ \tilde{d}(f(a_n), f(c)) < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. Since f is continuous at c,

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap E \ f(x) \in N_\varepsilon^Y(f(c)). \tag{1}$$

Furthermore, we have

$$a_n \to c \Longrightarrow \exists \hat{N} \text{ such that } \forall n > \hat{N} \ a_n \in N_\delta^X(c). \tag{2}$$

We claim that we can use $\hat{N}$ as the same N we were looking for. Indeed, if $n > \hat{N}$, then (2) implies $a_n \in N_\delta^X(c)$. Additionally, since $a_n \in E$, we have $a_n \in N_\delta^X(c) \cap E$. With (1), we have $f(a_n) \in N_\varepsilon^Y(f(c))$.

($(ii) \Longrightarrow (i)$) Assume that for all sequences (a_n) in E such that $a_n \to c$, we have $f(a_n) \to f(c)$. Our goal is to show that f is continuous at c. We may consider two cases.

(1) Suppose $c \in E \setminus E'$; that is, c is an isolated point of E. Indeed, it follows immediately that f is continuous at any isolated point of its domain.

(2) Suppose $c \in E'$. It suffices to show that $\lim_{x \to c} f(x) = f(c)$. By the Sequential Criterion for Limits of functions, it suffices to show that

$$\text{if } (a_n) \text{ is a sequence in } E \setminus \{c\} \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c).$$

But this is a direct consequence of the assumption that

$$\text{if } (a_n) \text{ is a sequence in } E \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c).$$

■

Note that if f is continuous at c and

$$a_1 \ a_2 \ a_3 \cdots \to c,$$

then

$$f(a_1) \ f(a_2) \ f(a_3) \ \cdots \to f(c)$$

that is, if f is continuous at c, then

$$\lim_{n \to \infty} f(a_n) = f(\lim_{n \to \infty} a_n).$$

The above is saying that we can bring a limit inside a continuous function.

For example, we will see in Homework 10 that if $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is continuous. So, we have

$$\lim_{n \to \infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n \to \infty} \frac{1}{n}\right) = \sin 0 = 0.$$

Note that if we can even find one sequence (a_n) in E such that $a_n \to c$ but $f(a_n) \not\to f(c)$, that shows f is NOT continuous at c.

Example (Dirichlet Function)**.** Prove that the Dirichlet Function

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous everywhere.

Proof. Let $c \in \mathbb{R}$. We will show that f is discontinuous at c. We will consider two cases:

(1) Suppose $c \in \mathbb{R} \setminus \mathbb{Q}$; that is, $f(c) = 0$. Let (q_n) be a sequence of rational numbers such that $q_n \to c$. Note that

$$\forall n \ \ q_n \in \mathbb{Q} \Longrightarrow \forall n f(q_n) = 1 \Longrightarrow \lim_{n \to \infty} f(q_n) = \lim_{n \to \infty} 1 = 1.$$

Therefore, we have

$$q_n \to c \wedge f(q_n) \not\to f(c) = 0 \Longrightarrow \ f \text{ is NOT continuous at } c.$$

(2) Suppose $c \in \mathbb{Q}$; that is, $f(c) = 1$. Let (r_n) be sequence of irrational numbers such that $r_n \to c$. Note that

$$\begin{aligned}\forall n\ r_n \in \mathbb{R} \setminus \mathbb{Q} &\Longrightarrow \forall n\ f(r_n) = 0 \\ &\Longrightarrow \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0.\end{aligned}$$

Therefore, we have

$$r_n \to c \wedge f(r_n) \not\to f(c) \Longrightarrow f \text{ is continuous at } c.$$

■

Example. Prove that $f : (\mathbb{R}, d) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous everywhere.

Proof. Let $c \in \mathbb{R}$. Our goal is to show that f is continuous at c, that is, we want to show

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } d(x,c) < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Regardless of the expression of f, the above holds with $\delta = \frac{1}{2}$. Indeed, if $d(x,c) < \frac{1}{2}$, then $d(x,c) = 0$, so $x = c$, and therefore, $|f(x) - f(c)| = |f(c) - f(c)| = |0| = 0 < \varepsilon$ as desired. ■

Example. Let $(X, \|\cdot\|)$ be a normed space. Prove that

$$\|\cdot\| : X \to \mathbb{R}$$

is continuous.

Proof. Let $c \in X$. We will prove that $\|\cdot\|$ is continuous at c, that is, we will show that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } \|x - c\| < \delta \text{ then } |\|x\| - \|c\|| < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. By the reverse triangle inequality, we can see that

$$\Big| \|x\| - \|c\| \Big| \leq \|x - c\|$$

implies that (*) holds for example with $\delta = \varepsilon$. ■

Corollary. If $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.

Example. Let (X, d) be a metric space. Let $p \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(p, x)$. Prove that f is continuous.

Proof. Let $c \in X$. Our goal is to show that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } d(x,c) < \delta \text{ then } |d(p,x) - d(p,c)| < \varepsilon. \tag{*}$$

Let $\varepsilon > 0$ be given. Note that (*) follows immediately from the inequality

$$|d(p,x) - d(p,c)| \leq d(x,c) \iff -d(x,c) \leq d(p,x) - d(p,c) \leq d(x,c)$$

that (*) holds with $\delta = \varepsilon$. ■

Corollary. If $x_n \to x$ in (X,d), and $p \in X$, then $d(p,x_n) \to d(p,x)$ ($\lim_{n\to\infty} d(p,x_n) = d(p, \lim_{n\to\infty} x_n)$).

Example (Continuity with Linear Operators)**.** Consider $C[0,1] = \{f : [0,1] \to \mathbb{R} : \quad f \text{ is continuous}\}$ equipped with the norm

$$\|f\|_\infty = \max_{0\leq x\leq 1} |f(x)|.$$

Prove that $T : C[0,1] \to \mathbb{R}$ defined by $T(f) = f(1/2)$ is continuous.

Proof. Let $g \in C[0,1]$. Our goal is to show that T is continuous at g. To this end, it suffices to show that if $g_n \to g$ (in the normed space $(C[0,1], \|\cdot\|_\infty)$), then $T(g_n) \to T(g)$ in $\mathbb{R}$. Applying the Squeeze Theorem to

$$0 \leq |g_n(1/2) - g(1/2)| \leq \max |g_n - g(x)|,$$

we can see that

$$|g_n(1/2) - g(1/2)| \to 0$$

and so, we have

$$\begin{aligned} g_n \to g &\implies \|g_n - g\|_\infty \to 0 \text{ as } n \to \infty \\ &\implies \max_{0\leq x\leq 1} |g_n(x) - g(x)| \to \ 0 \text{ as } n \to \infty \\ &\implies |g_n(1/2) - g(1/2)| \to 0 \text{ as } n \to \infty \\ &\implies g_n(1/2) \to g(1/2) \text{ in } \mathbb{R} \\ &\implies T(g_n) \to T(g) \text{ in } \mathbb{R}. \end{aligned}$$

■

Theorem (Algebraic Conitnuity Theorem)**.** Assume $f : E \subseteq (X,d) \to \mathbb{R}$ and $g : E \subseteq (X,d) \to \mathbb{R}$ are continuous at $c \in E$. Then

(i) $k \cdot f(x)$ is continuous at c for all $k \in \mathbb{R}$

(ii) $f(x) + g(x)$ is continuous at c

(iii) $f(x)g(x)$ is continuous at c

(iv) $\frac{f(x)}{g(x)}$ is continuous at c provided that the denominator is not zero.

Proof. These items are direct consequences of the ALT for sequences and the characterization of continuity via sequences. For example, let's prove (iii).

By characterization of continuity via sequences, it suffices to show that if (a_n) is a sequence in E such that $a_n \to c$, then

$$f(a_n)g(a_n) \to f(c)g(c).$$

Let (a_n) be such a sequence. Since f is continuous at c and $a_n \to c$, we have $f(a_n) \to f(c)$. Similarly, g is continuous at c and $a_n \to c$ implies that $g(a_n) \to g(c)$. Using these two facts, we can use ALT for sequences of real numbers to conclude that

$$f(a_n)g(a_n) \to f(c)g(c)$$

as desired. ■

Theorem (Composition of Continuous Functions is Continuous). Let $(X, d), (Y, \tilde{d}), (Z, \overline{d})$ are metric spaces. Assume that $A, B \neq \emptyset$ are subsets of X and Y, respectively. Let $f : A \to Y$ and $g : B \to Z$ and let $f(A)$ be contained in $\mathbb{R}$, f is continuous at $c \in A$, and g is continuous at $f(c) \in B$. Then $g \circ f : A \to Z$ is continuous at $c \in A$.

Proof. It suffices to show that if (a_n) is a sequence in A such that $a_n \to c$, then $(g \circ f)(a_n) \to (g \circ f)(c)$. Let (a_n) be such a sequence. Since f is continuous at c and $a_n \to c$, we have $f(a_n) \to f(c)$. Similarly, g being continuous at $f(c)$ and $f(a_n) \to f(c)$ implies that $g(f(a_n)) \to g(f(c))$. So, we have $(g \circ f)(a_n) \to (g \circ f)(c)$ as desired. ■

Example. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous, then $\max\{f, g\}$ and $\min\{f, g\}$ are also continuous. The following are explicit formulas of the max and min functions:

(1) $\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$.

(2) $\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$.

Example. (1) If E is a metric subspace of X, then

$$i : E \to X \quad , \quad i(x) = x$$

is continuous (think of the sequential criterion for continuity).

(2) If $f : x \to Y$ is continuous and $E \subseteq X$, then $f|_E : E \to X$ is continuous.

(3) If $f : X \to Y$ is continuous and Y is a metric subspace of Z, then $f : X \to Z$ is continuous.

Chapter 13

Week 13

13.1 Topics

(1) Topological Continuity

(2) Continuity and compactness

(3) Continuity and connectedness

(4) Continuity and the inverse

(5) Uniform Continuity

So far we have learnt two equivalent descriptions of the concept of continuity for functions $f : (X, d) \to (Y, \tilde{d})$:

(1) f is continuous if and only if

$$\forall c \in X \; \forall \varepsilon > 0 \; \exists \delta_{\varepsilon,c} > 0 \text{ such that if } d(x, c) < \delta_{\varepsilon,c} \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

(2) f is continuous if and only if $\forall c \in X$, $a_n \to c$ implies $f(a_n) \to f(c)$.

In undergraduate analysis, most of the sequences we study are sequences in $\mathbb{R}$. In terms of convergence, we say that

$$a_n \to c \iff \forall \; \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n > N \;\; |a_n - c| < \varepsilon.$$

For graduate analysis, convergence can be viewed more generally; that is, we have

$$a_n \to c \iff \forall N_\varepsilon(c) \; \exists N \in \mathbb{N} \text{ such that} \forall n > N \; a_n \in N_\varepsilon(c).$$

Theorem (Topological Characterization Continuity)**.** Let (X, d) and $(Y, \tilde{d})$ be metric spaces and let $f : X \to Y$. The following statements are equivalent:

(i) f is continuous

(ii) For every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X.

Proof. $((i) \implies (ii))$ Suppose f is continuous. Our goal is to show that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X. Let B be an open set in Y. Our goal is to show that $f^{-1}(B)$ is open in X. That is, we want to show that every point of $f^{-1}(B)$ is an interior point. Let $p \in f^{-1}(B)$. We need to find an $\delta > 0$ such that $N_\delta^X(p) \subseteq f^{-1}(B)$.

Since f is continuous at p, there exists $\hat{\delta} > 0$ such that

$$\forall x \in N_{\hat{\delta}}^X(p) \quad f(x) \in N_\varepsilon^Y(f(p)) \subseteq B.$$

Clearly, we have $N_{\hat{\delta}}^X(p) \subseteq f^{-1}(B)$, so we can use this $\hat{\delta}$ as the δ we were looking for.

($(ii) \implies (i)$) Assume that for all open sets $B \subseteq Y$, we have $f^{-1}(B)$ is open in X. Our goal is to show that f is continuous. We need to show that f is continuous at every point of X. Let $c \in X$. We will prove that f is continuous at c. That is,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if} x \in N_\delta^X(c) \text{ then } f(x) \in N_\varepsilon^Y(f(c)).$$

Let $\varepsilon > 0$ be given. We need to show that there exists $\delta > 0$ such that

$$N_\delta^X(c) \subseteq f^{-1}(N_\varepsilon^Y(f(c))). \tag{*}$$

Since $N_\varepsilon^Y(f(c))$ is open in Y, it follows from the assumption that $f^{-1}(N_\varepsilon^Y(f(c)))$ is open in X. Since $f^{-1}(N_\varepsilon^Y(f(c)))$ is open in X and $c \in f^{-1}(N_\varepsilon^Y(f(c)))$, we have c is an interior point of $f^{-1}(N_\varepsilon^Y(fc))$. Hence, there exists $\delta > 0$ such that $N_\delta^X(c) \subseteq f^{-1}(N_\varepsilon^Y(f(c)))$ as desired. ■

Note that continuous functions does NOT necessarily preserve the following properties:

- open
- closed
- bounded

Some examples of maps that do not preserve the properties above are

(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Note that $E = (-2, 2)$ is open but $f(E) = [0, 4)$ is not.

(2) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$. Note that $E = [0, \infty)$ is closed but $f(E) = (0, 1]$ is not.

(3) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Note that $E = (0, 1)$ is a bounded set but $f(E) = (1, \infty)$ is not.

Theorem. Let (X, d) and $(Y, \tilde{d})$ are metric spaces and let $f : X \to Y$ be continuous and let E be a compact set in X. Then $f(E)$ is compact in Y.

Proof. Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of $F(E)$. Our goal is to show that this open cover has a finite subcover. We have

$$f(E) \subseteq \bigcup_{\alpha \in I} O_\alpha.$$

So,

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\Big(\bigcup_{\alpha \in I} O_\alpha\Big) = \bigcup_{\alpha \in E} f^{-1}(O_\alpha).$$

Since $E \subseteq f^{-1}(f(E))$ and $f^{-1}\Big(\bigcup_{\alpha \in I} O_\alpha\Big) = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$, we can conclude that

$$E \subseteq \bigcup_{\alpha \in I} f^{-1}(O_\alpha).$$

Now, we have $f : X \to Y$ is continuous and for each $\alpha \in I$, O_α is open in Y implies that for all $\alpha \in I$, we have $f^{-1}(O_\alpha)$ is open in X. Thus, $\{f^{-1}(O_\alpha)\}_{\alpha \in I}$ is an open cover for E. Since E is compact, we have

$$\exists \alpha_1, \dots, \alpha_n \in I \text{ such that } E \subseteq \bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i}).$$

Consequently, we have

$$\begin{aligned} f(E) &\subseteq f\Big(\bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i})\Big) \\ &= \bigcup_{i=1}^{n} f(f^{-1}(O_{\alpha_i})) \\ &\subseteq \bigcup_{i=1}^{n} O_{\alpha_i}. \end{aligned}$$

Thus, we have $\{O_{\alpha_i}\}_{1\le i\le n}$ is a finite subcover for $f(E)$. ∎

Theorem (Extreme Value Theorem)**.** Let (X, d) be a compact metric space.

(i) If $f : (X, d) \to (Y, \tilde{d})$ is continuous, then $f(X)$ is a closed and bounded set in Y.

(ii) If $f : (X, d) \to \mathbb{R}$ is continuous, then f attains a maximum values and a minimum value. More precisely, $M = \sup_{x\in X} f(x)$

and $m = \inf_{x\in X} f(x)$ exists, and there exists points $a \in X$ and $b \in X$ such that $f(a) = M$ and $f(b) = m$.

Proof. (i) By the previous theorem, we can see that $f(X)$ must be compact in Y. As we know, every compact set in any metric space is closed and bounded.

(ii) By part (i), $f(X)$ is closed and bounded subset of $\mathbb{R}$. Since $f(X)$ is a bounded set in $\mathbb{R}$, $M = \sup f(X) = \sup_{x\in X} f(x)$ and $m = \inf f(X) = \inf_{x\in X} f(x)$ exists. Recall from Theorem 2.28 that $M \in \overline{F(X)}$ and $m \in \overline{f(X)}$. Since $\overline{f(X)} = f(X)$, we can conclude that $M \in f(X)$ and $m \in f(X)$. That is, there exists $a \in X$ such that $f(a) = M$ and there exists $b \in X$ such that $f(b) = m$.

∎

Theorem (Preservation of Connectedness)**.** Let (X, d) and $(Y, \tilde{d})$ be metric spaces and $f : X \to Y$ be a continuous map, and let $E \subseteq X$ be a connected set. Then $f(E)$ is connected in Y.

Proof. Assume for contradiction that $f(E)$ is NOT connected. Thus, we can write $f(E)$ as a union of two (nonempty) separated sets A and B:

$$f(E) = A \cup B,\ \overline{A} \cap B = \emptyset,\ A \cap \overline{B} = \emptyset.$$

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. In what follows, we will show that G and H form a separation for the set E, which contradicts the assumption that E is connected. We will show that

(1) G and H are both nonempty

(2) $\overline{G} \cap H = \emptyset$ (and similarly, $G \cap \overline{H} = \emptyset$)

(3) $E = G \cup H$

To show (1), we will prove that $f(G) = A$ (similarly, show that $f(H) = B$). We have

(1)

$$\begin{aligned} f(G) = f(E \cap f^{-1}(A)) &\subseteq f(E) \cap f(f^{-1}(A)) \\ &\subseteq f(E) \cap A && (f(E) = A \cup B) \\ &= A. \end{aligned}$$

Then we have $f(G) \subseteq A$. Now, we will show that $A \subseteq f(G)$. Let $y \in A$. Then $y \in f(E)$ and so there exists

$$x \in E \text{ such that } f(x) = y. \tag{*}$$

Then we have

$$f(x) = y \in A \Longrightarrow x \in f^{-1}(A). \tag{**}$$

Then (*) and (**) imply that

$$\begin{aligned} x \in E \cap f^{-1}(A) &\Longrightarrow f(x) \in f(E \cap f^{-1}(A)) = f(G) \\ &\Longrightarrow y \in f(G). \end{aligned}$$

Thus, $A \subseteq f(G)$.

(2) Observe that

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap [f^{-1}(A) \cap f^{-1}(B)] \\ &= E \cap [f^{-1}(A \cup B)] \\ &= E \cap [f^{-1}(f(E))] && (f(E) = A \cup B) \\ &= E && (E \cap f^{-1}(f(E))) \end{aligned}$$

(3) To this end, it is enough to show that $f(\overline{G}) \cap f(H) = \emptyset$. Note that $f(H) = B$. So, our goal is to show that $f(\overline{G}) \cap B = \emptyset$. Since $\overline{A} \cap B$ is empty, and so it suffices to show that $f(\overline{G}) \subseteq \overline{A}$. Notice that

$$G = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}).$$

Since f is continuous and $\overline{A}$ is a closed set in Y, we can see that $f^{-1}(\overline{A})$ is closed in X. Thus, we can write

$$G \subseteq f^{-1}(\overline{A}) \Longrightarrow \overline{G} \subseteq \overline{f^{-1}(\overline{A})} = f^{-1}(\overline{A}).$$

Therefore, we have

$$f(\overline{G}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}.$$

A similar argument shows that $f(\overline{H}) \cap f(G) = \emptyset$.

■

Theorem (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous map and let $f(a) \neq f(b)$. Let $L \in \mathbb{R}$ be such that $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = L$.

Proof. Since $f : [a, b] \to \mathbb{R}$ is continuous and $[a, b]$ is connected. Then, by the preservation of connectedness, we have $f([a, b])$ is connected in $\mathbb{R}$. This implies that $f([a, b])$ is either a singleton or an interval I in $\mathbb{R}$. Since $f(a) \neq f(b)$, we know that $f([a, b])$ cannot be a singleton. That is, $f([a, b])$ is an interval I in $\mathbb{R}$. Note that $f(a), f(b) \in I$ and L is between $f(a)$ and $f(b)$. So, $L \in f([a, b])$. Thus, there exists $c \in [a, b]$ such that $f(c) = L$. But neither $f(a) \neq L$ nor $f(b) \neq L$, so there exists $c \in (a, b)$ such that $f(c) = L$. ■

Note that if $f : X \to Y$ is continuous and bijective =, it is NOT necessarily true that $f^{-1} : Y \to X$ is continuous.

Example. $f : (-1,0] \cup [1,2] \to [0,4]$ given by $f(x) = x^2$ is continuous and bijective. However, $f^{-1} : [0,4] \to (-1,0] \cup [1,2]$ is NOT continuous. Indeed, we see that $[0,4]$ is connected, but $f^{-1}([0,4]) = (-1,0] \cup [1,2]$ is NOT.

Theorem (Continuous and Bijective maps on Compact Sets)**.** Let (X,d) and $(Y,\tilde{d})$ are metric spaces, X is compact, and $f : X \to Y$ is continuous and bijective. Then $f^{-1} : Y \to X$ is continuous.

Proof. It suffices to show that for every open set $B \subseteq X$ $B \subseteq X$, $(f^{-1})^{-1}(B)$ is open in Y. That is, we will show that $f(B)$ is open in Y. Let B be an open set in X. Thus, B^c is closed in X. Since X is a compact set and $B^c \subseteq X$, B^c is compact in X. Since continuity preserves compactness, we have $f(B^c)$ is compact in Y. Since compact sets are closed and bounded, $f(B^c)$ is closed in Y. Now, $[f(B^c)]^c$ is open in Y. Since $[f(B^c)]^c$ is open in Y. Since f is bijective, we have

$$f(B^c) = [f(B)]^c.$$

Thus,

$$f(B) = [(f(B))^c]^c = [f(B^c)]^c$$

is open in Y. ■

Remark (Invariance of Domain)**.** Let $U \subseteq \mathbb{R}^n$ is open and $V \subseteq \mathbb{R}$. Since $f : U \to V$ is continuous and bijective, then V is open and $f^{-1} : V \to U$ is continuous.

13.2 Uniform Continuity

Let A be a proper subset of (X,d).

Consider the following three questions:

(1) Given a continuous function $f : A \to \mathbb{R}$, is it possible to extend f to a continuous function that is defined on all of X.

(2) Under what conditions can we be certain that a continuous function $f : A \to \mathbb{R}$ has a continuous extension to all of X.

Theorem (A special case of Tietze Extension Theorem)**.** Let (X,d) be a metric space. Let A be a nonempty closed set in X. If $f : A \to \mathbb{R}$ is continuous, then f has a continuous extension to all of X.

Theorem. Let (X,d) be a metric space. Let A be a nonempty set in X. If $f : A \to \mathbb{R}$ is **uniformly continuous on** A, then f can be extended to a continuous function $\overline{f} : \overline{A} \to \mathbb{R}$.

Recall that we say that $f : A \subseteq (X,d) \to (Y,\tilde{d})$ is continuous at $c \in A$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x,c) < \delta \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

We say that $f : A \subseteq (X,d) \to (Y,\tilde{d})$ is continuous on A, we have

$$\forall c \in A \ \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } (\forall x \in A) \text{ if } d(x,c) < \delta_{\varepsilon,c} \text{ then } d(f(\widetilde{x), f}(c)) < \varepsilon.$$

In general, the δ in the above statement depends on both ε and c. If the function the desirable property that given an arbitrary $\varepsilon > 0$, one can find a single δ_ε that holds (or stays constant) for all points $c \in A$, then we say that f is uniformly continuous on A.

Definition (Uniform Continuity). Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$ be a function. We say that f is **uniformly continuous on** A if

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 \text{ such that } \forall x, y \in A \text{ if } d(x, c) < \delta_\varepsilon, \text{ then } \tilde{d}(f(x), f(c)) < \varepsilon.$$

Remark. We say that f is NOT uniformly continuous on A if

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0\ \exists x, y \in A \text{ satisfying } d(x, y) < \delta \text{ but } \tilde{d}(f(x), f(y)).$$

Clearly, from the above, we can see that uniform continuity is a stronger property than continuity in the sense that if $f : A \to Y$ is uniformly continuous, then $f : A \to Y$ is continuous. Note that it is not really productive to talk about uniform continuity at a specific point. Uniform continuity is always discussed in the context to a particular domain. For example, we'll see that $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$. However, $f(x) = x^2$ is uniformly continuous on $[0, 1]$. This implies determining whether a function is uniform continuous is a matter of analyzing the domain on which the function is defined on.

Example. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is uniformly continuous on $\mathbb{R}$.

Our goal is to show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \in \mathbb{R} \text{ if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Clearly, we can take $\delta = \frac{\varepsilon}{2}$ (or any positive number less than $\frac{\varepsilon}{2}$).

The definition of uniform continuity directly implies that if some function $f : A \to Y$ is uniformly continuous on A, then it is also uniformly continuous on $B \subseteq A$.

Note that tells us that not all continuous functions are uniformly continuous. The following theorem gives a simple criterion for proving the absence of uniform continuity.

Theorem. Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$. If we can find $\varepsilon_0 > 0$ and two sequences (x_n) and (c_n) in A such that

$$d(x_n, c_n) \to 0 \text{ and } \forall n \tilde{d}(f(x_n), f(c_n)) \geq \varepsilon_0.$$

Then f is NOT uniformly continuous on A.

Proof. Recall that f is NOT uniformly continuous if and only if there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ satisfying $d(x, c) < \delta$ but $\tilde{d}(f(x), f(c)) \geq \varepsilon$. If this holds, then we can set $\varepsilon = \varepsilon_0$ such that for any $\delta > 0$, there exists N such that $d(x_N, c_N) < \delta$, but $\tilde{d}(f(x_n), f(c_n)) \geq \varepsilon$. ■

Example. Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. Let $x_n = n$ and $c_n = n + \frac{1}{n}$. We have

$$\lim_{n\to\infty} |x_n - c_n| = \lim_{n\to\infty} \left|\frac{-1}{n}\right| = 0.$$

Also, for all n, we have

$$\begin{aligned} |f(x_n) - f(c_n)| &= \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| \\ &= \left|n^2 - \left(n^2 + 2 + \frac{1}{n^2}\right)\right| \\ &= \left|-\left(2 + \frac{1}{n^2}\right)\right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2. \end{aligned}$$

Hence, we conclude that $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$. ■

Example. Prove that $f(x) = \sin\frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Solution. Use $x = \frac{1}{2n\pi}$ and $c_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Now, observe that $\lim x_n = 0$ and $\lim c_n = 0$ implies $\lim(x_n - c_n) = 0$. Thus, $\lim |x_n - c_n| = 0$. But for all n, we have

$$|f(x_n) - f(c_n)| = |\sin(2n\pi) - \sin\left(2n\pi + \frac{\pi}{2}\right)| = |0 - 1| = 1.$$

So, f is NOT uniformly continuous. ■

Theorem (Continuous Functions on Compact Sets are Uniformly Continuous)**.** Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$ is continuous and let A be a compact set. Then f is uniformly continuous on A.

Proof. Assume for sake of contradiction that f is NOT uniformly continuous; that is, there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \; \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \tilde{d}(f(x), f(c)) \geq \varepsilon.$$

In particular,

$$\begin{aligned} &\text{For} \delta = 1 \exists x_1, c_1 \in A \text{ satisfying } d(x_1, c_1) < 1 \text{ but } \tilde{d}(f(x_1), f(c_1)) \geq \varepsilon \\ &\text{For} \exists x_2, c_2 \in A \text{ satisfying } d(x_2, c_2) < \frac{1}{2} \text{ but } \tilde{d}(f(x_2), f(c_2)) \geq \varepsilon \\ &\text{For} \exists x_3, c_3 \in A \text{ satisfying } d(x_3, c_3) < \frac{1}{3} \text{ but } \tilde{d}(f(x_3), f(c_3)) \geq \varepsilon \\ &\vdots \end{aligned}$$

In this way, we will obtain two sequences (x_n) and (c_n) in A such that

(i) $0 \leq d(x_n, c_n) < \frac{1}{n}$ for all n. This implies that $\lim\limits_{n\to\infty} d(x_n, c_n) = 0$.

(ii) $\tilde{d}(f(x_n), f(c_n)) \geq \varepsilon$ for all n.

Note that A is compact, so it is sequentially compact. If (x_n) is a sequence in A, then (x_n) contains a subsequence (x_{n_k}) that converges to a point in A. Let $x = \lim\limits_{k\to\infty} x_{n_k}$. Let (c_{n_k}) be the corresponding subsequence of (c_n). We have

$$0 \leq d(c_{n_k}, x) \leq d(c_{n_k}, x_{n_k}) + d(x_{n_k}, x).$$

If we let $k \to \infty$ on the right-hand side of the above inequality, we have $d(c_{n_k}, x_{n_k}) \to 0$ and $d(x_{n_k}, x) \to 0$. Using the Squeeze Theorem, we have $\lim_{k\to\infty} c_{n_k} = x$. Therefore, (x_{n_k}) and (c_{n_k}) are two sequences in A that converge to $x \in A$. Since f is continuous and $x_{n_k} \to x$, we have

$$f(x_{n_k}) \to f(x). \tag{1}$$

Similarly, if $c_{n_k} \to x$, we have

$$f(c_{n_k}) \to f(x). \tag{2}$$

So, there exists $N_0 \in \mathbb{N}$ such that for all $k > N_0$

$$\forall k > N_0 \quad \tilde{d}(f(x_{n_k}), f(x)) < \frac{\varepsilon}{4} \text{ and } \tilde{d}(f(c_{n_k}), f(x)) < \frac{\varepsilon}{4}.$$

As a consequence, for all $k > N_0$, we have

$$\begin{aligned}\tilde{d}(f(x_{n_k}), f(c_{n_k})) &\leq \tilde{d}(f(x_{n_k}), f(x)) + \tilde{d}(f(x), f(c_{n_k})) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon.\end{aligned}$$

But this contradicts (ii). ■

Chapter 14

Week 14

14.1 Topics

1. Types of discontinuities (discontinuity of the first kind, discontinuity of the second kind)
2. Limits of monotone functions
3. Limits involving infinity

Definition (Right-hand limit, left-hand limit). Let $f : (a, b) \to (Y, \tilde{d})$.

(i) Let $a \leq c < b$. We write
$$\lim_{x \to c^+} f(x) = L$$
if any of the following equivalent conditions holds:

(1) For all sequences (x_n) in (c, b) satisfying $x_n \to c$, we have $f(x_n) \to L$.

(2) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $c < x < c + \delta$ then $\tilde{d}(f(x), L) < \varepsilon$.

(ii) Let $a < c \leq b$. We write
$$\lim_{x \to c^-} f(x) = L$$
if any of the following equivalent conditions hold.

(1) For all sequences (x_n) in (a, c) such that $x_n \to c$ we have $f(x_n) \to L$.

(2) For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $c - \delta < x < c$ then $\tilde{d}(f(x), L) < \varepsilon$.

Definition (Classification of Discontinuities). Let $f : (a, b) \to (Y, \tilde{d})$. Let $c \in (a, b)$. Suppose that f is discontinuous at c.

(i) f is said to have a discontinuity of the first kind, or a simple discontinuity, at c if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist. We outline the two possible scenarios below:

(1) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$. This is a removable discontinuity.

(2) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$. This is a Jump discontinuity.

(ii) f is said to have a discontinuity of the second kind at c if at least one of $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ does not exist.

A quick example is $f : (-1, 1) \to \mathbb{R}$ and $f(x) = \frac{1}{x}$ with $x \neq 0$ and $f(0) = 0$.

Recall that

(1) $f : (a, b) \to \mathbb{R}$ is said to be **increasing** on (a, b) if
$$x_1 < x_2 \implies f(x_1) \leq f(x_2).$$

(2) $f : (a, b) \to \mathbb{R}$ is said to be **decreasing** on (a, b) if
$$x_1 < x_2 \implies f(x_1) \geq f(x_2).$$

(3) $f : (a, b) \to \mathbb{R}$ is said to be **monotone** on (a, b) if it is either increasing or decreasing.

Theorem (One-sided Limits of Monotone Functions)**.** Let $f : (a, b) \to \mathbb{R}$ be an increasing function. Then at every point $c \in (a, b)$, the one-sided limits exist, and

(i) $\lim\limits_{x \to c^-} f(x) = \sup_{a<x<c} f(x) \leq f(c)$.

(ii) $\lim\limits_{x \to c^+} f(x) = \inf_{c<x<b} f(x) \geq f(c)$

(iii) If $a < c < d < b$, then
$$\lim_{x \to c^+} f(x) \leq \lim_{x \to c^-} f(x).$$

(A similar statement holds for decreasing functions)

Let $f : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and $L \in \mathbb{R}$.

(1) $\lim\limits_{x \to +\infty} f(x) = L \iff \forall \varepsilon > 0\ \exists M > 0$ such that $\forall x > M\ |f(x) - L| < \varepsilon$.

(2) $\lim\limits_{x \to -\infty} f(x) = L \iff \forall \varepsilon > 0\ \exists M > 0\ \forall x > M\ |f(x) - L| < \varepsilon$.

(3) $\lim\limits_{x \to \infty} f(x) = \infty \iff \forall R > 0\ \exists M > 0$ such that $\forall x > M\ f(x) > R$.

(4) $\lim\limits_{x \to -\infty} f(x) = \infty \iff \forall R > 0\ \exists M > 0$ such that $\forall x < -M\ f(x) > R$.

(5) $\lim\limits_{x \to +\infty} f(x) = -\infty \iff \forall R > 0\ \exists M > 0$ such that $\forall x > M\ f(x) < -R$

(6) $\lim\limits_{x \to -\infty} f(x) = -\infty \iff \forall R > 0\ \exists M > 0$ such that $\forall x < -M\ f(x) < -R$

(7) $\lim\limits_{x \to c} f(x) = +\infty \iff \forall R > 0\ \exists \delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > R$.

(8) $\lim\limits_{x \to c} f(x) = -\infty \iff \forall R > 0\ \exists \delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) < -R$.