1 Lecture 20-21

1.1 Topics

- Infinite series (Basic definitions)
- Telescoping Series, Geometric Series
- Algebraic Limit Theorem for Series
- Divergence Test
- Cauchy Criterion for Series
- Absolute Convergence Test

Consider the following expression:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots .$$

How can we make sense of the infinite sum above? More generally, let (a_n) be a sequence of real numbers. Then what does the following expression mean?

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ?$$

Definition (Infinite Series). Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X.

(*) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots$$

is called an **infinite series**.

(*) $x_1, x_2, \ldots$ are called the **terms** of this infinite series.

(*) The corresponding sequence of **partial sums** is defined by

$$\forall m \in \mathbb{N} \quad s_m = \text{(finite) sum of the first } m \text{ terms of the series;}$$

that is,

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + x_2 + \cdots + x_m \\ &\vdots \end{aligned}$$

1. We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty} x_n = L$) if $\lim\limits_{n\to\infty} s_m = L$.
2. We say that the infinite series **diverges**, if (s_m) diverges.
3. If $X = \mathbb{R}$ and $s_m \to \infty$, we write $\sum_{n=1}^{\infty} x_n = \infty$.

(*) If $X = \mathbb{R}$ and $s_m \to -\infty$, we write $\sum_{n=1}^{\infty} x_n = -\infty$.

Remark (1). Given an infinite series $\sum_{n=1}^{\infty} x_n$, it is important to keep a clear distinction between

(a) the sequence of terms: $(x_1, x_2, x_3, \ldots)$

(b) the sequence of partial sums: $(s_1, s_2, s_3, \dots)$.

Remark (2). We may sometimes consider infinite series where the summation begins with $n = 0$ or $n = n_0$ for some integer n_0 different from 1.

As we shall see, some of our theorems apply specifically to series in $\mathbb{R}$ or to series with terms in $[0, \infty)$. Also, in our examples, we will primarily focus on series in $\mathbb{R}$; however, we will also consider encounter highly useful theorems that hold in more general normed spaces.

In most cases it is difficult (or even impossible) to find a simple formula for the partial sum s_m. However, there are two types of series for which we can easily find a simple formula for the partial sums. These two types are:

(1) Telescoping Series

(2) Geometric Series

1.2 Telescoping Series

Example. Consider the following series

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Notice that $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned}
s_1 &= 1 - \frac{1}{2} \\
s_2 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) = 1 - \frac{1}{3} \\
s_3 &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) = 1 - \frac{1}{4} \\
&\vdots \\
s_m &= \sum_{n=1}^{m} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = \Big(\sum_{n=1}^{m} \frac{1}{n}\Big) - \Big(\sum_{n=1}^{m} \frac{1}{n+1}\Big) \\
&= 1 - \frac{1}{m+1}.
\end{aligned}$$

Clearly, we see that

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \Big[1 - \frac{1}{m+1}\Big] = 1.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

In general, a telescoping series is an infinite series where partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n - y_{n+1})$ is a telescoping series; that is,

$$\begin{aligned}
s_m = \sum_{n=1}^{m}(y_n - y_{n+1}) = \Big(\sum_{n=1}^{m} y_n\Big) - \Big(\sum_{n=1}^{m} y_{n+1}\Big) &= [y_1 + y_2 + \cdots + y_m] - [y_2 + y_3 + \cdots + y_{m+1}] \\
&= y_1 - y_m.
\end{aligned}$$

1.3 Geometric Series

Let k be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \cdots$ is called a **geometric series** with common ration "r". For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ is a geometric series with common ratio } \frac{1}{2}.$$

Another example is that

$$\sum_{n=1}^{\infty} \frac{7^n}{29^n} \text{ is a geometric series with common ratio } \frac{7}{29}.$$

A non-example is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We can easily find a formula for the mth partial sum of $\sum_{n=k}^{\infty} r^k$ where

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k + r^{k+1} \\ s_3 &= r^k + r^{k+1} + r^{k+2} \\ &\vdots \\ s_m &= r^k + r^{k+1} + \cdots + r^{k+m-1} \end{aligned} \tag{*}$$

Now, if $r = 1$, we have

$$s_m = \underbrace{1 + 1 + \cdots + 1}_{m\text{summands}} = m.$$

If $r \neq 1$, then multiply both sides of (*) by r:

$$rs_m = r^{k+1} + r^{k+2} + \cdots + r^{k+m}. \tag{**}$$

Subtracting (**) from (*), we get

$$s_m - rs_m = r^k - r^{k+m}.$$

Since $r \neq 1$, we have

$$s_m = \frac{r^k - r^{k+m}}{1-r} = \frac{r^k(1-r^m)}{1-r}.$$

Note that

(i) If $|r| < 1$, then $\lim r^m = 0$.

(ii) If $|r| > 1$ or $r = -1$, then $\lim_{n\to\infty} r^m$ does not exists.

Hence, we have

$$\lim_{m\to\infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{DNE} & \text{if} |r| \geq 1. \end{cases}$$

Thus,

$$\sum_{n=k}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if} |r| < 1 \\ \text{diverges} & \text{if} |r| \geq 1. \end{cases}$$

Example.
- $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Observe that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^1}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

- $\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n$

  Observe that
$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)} = \frac{1}{8}.$$

Theorem. Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) and (b_n) be two sequence in X. Suppose that
$$\sum_{n=1}^{\infty} a_n = A \quad (A \in X), \quad \sum_{n=1}^{\infty} b_n = B \quad (B \in X).$$
Then

(i) For any scalar λ, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$.

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$.

Proof. Can easily be proven via the Algebraic Limit Theorem for Sequences. ■

Theorem (Divergence Test)**.** Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $s_n = x_1 + \cdots + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that
$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n\to\infty} s_n = L.$$
Also, note that
$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$
Note that $\lim s_n = L$ and $\lim s_{n-1} = L$. Therefore,
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (s_n - s_{n-1}) = L - L = 0$$
by the Algebraic Limit Theorem for normed spaces. ■

Remark. Note that the divergence test is just the contrapositive of the above.

Example.

- $\sum_{n=1}^{\infty}(-1)^n$ diverges because $\lim_{n\to\infty}(-1)^n$ does not exist.
- $\sum_{n=1}^{\infty} \frac{3n+1}{7n-4}$ diverges because $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

From the above statements, we can now see make two key observations:

- If $\lim_{n\to\infty} x_n = 0$, then $\sum_{n=1}^{\infty} x_n$ may or may not converge.
- If $\lim_{n\to\infty} x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ diverges.

As for the first observation above, we see that $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^2}$ converges.

Theorem (Cauchy Criterion)**.** Let $(X, \|\cdot\|)$ be a complete normed space. Let (x_n) be a sequence in

X. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges } \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=1}^{n} x_k\| < \varepsilon.$$

Proof. Let $s_n = x_1 + \cdots + x_k$. Assuming that

$$\|s_n - s_m\| = \Big\| \sum_{k=m+1}^{n} x_k \Big\|$$

where $n > m$ and from the fact that

$$\begin{aligned} s_n - s_m &= (x_1 + \cdots + x_m + \cdots + x_n) - (x_1 + \cdots + x_m) \\ &= \sum_{k=m+1}^{n} s_k. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges} \\ &\iff (s_k) \text{ is Cauchy} \\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \quad \|\sum_{k=m+1}^{n} x_k\| < \varepsilon \end{aligned}$$

as desired. ■

From here, we will refer to complete normed spaces as Banach spaces.

Theorem (Absolute Convergence Test). Let $(X, \|\cdot\|)$ be a Banach Space. Let (x_n) be a sequence in X. If $\underbrace{\sum_{n=1}^{\infty} \|x_n\|}_{\text{a sum in } \mathbb{R}}$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for series, it suffices to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N \Big\| \sum_{k=m+1}^{n} x_k \Big\| < \varepsilon. \tag{*}$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for series that there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \Big| \sum_{k=m+1}^{n} \|x_k\| \Big| < \varepsilon.$$

We claim that $\hat{N}$ is the same N we were looking for. Hence, if $n > m > \hat{N}$, then we have

$$\Big\| \sum_{k=m+1}^{n} x_k \Big\| \leq \sum_{k=m+1}^{n} \|x_k\| = \Big| \sum_{k=m+1}^{n} \|x_k\| \Big| < \varepsilon$$

as desired. ■

Please take note of the following observations:

(1) If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges (in Banach spaces).

(2) If $\sum_{n=1}^{\infty} \|x_n\|$ diverges, then $\sum_{n=1}^{\infty} x_n$ may converge or diverge.

From (2), we shall see (in the next lecture) that

(1) $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right|$ diverges but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

(2) $\sum_{n=1}^{\infty} |(-1)^n|$ diverges, also $\sum_{n=1}^{\infty}(-1)^n$ diverges (by the divergence test).

Definition (Absolute Convergence and Conditional Convergence). We say that a series $\sum x_n$ **absolutely converges** if $\sum \|x_n\|$ converges and $\sum x_n$ converges. We say that $\sum x_n$ **conditionally converges** if $\sum \|x_n\|$ diverges but $\sum x_n$ converges.

Example (Conditionally Convergent). Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We see that this series is conditionally convergent since

$$\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \to 0$$

which tell us that the above series diverges by the divergence test. But the above series converges via the Leibniz Test (As we shall see in the next lecture).

2 Lecture 21-22

2.1 Topics

- Cauchy Condensation Test
- Comparison Test
- More on lim sup and lim inf
- Root Test
- Ratio Test
- Dirichlet's Test

Theorem (Cauchy Condensation Test). Assume $a_n \geq 0$ for all n, and (a_n) is a decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \text{ converges.}$$

Proof. Let $s_m = a_1 + \cdots + a_m$ and $t_m = a_1 + 2a_2 + 4a_4 + \cdots + 2^{m-1}a_{2^{m-1}}$. Using the fact that (a_n) is a decreasing sequence, we can see that

$$\begin{aligned}
s_{2^k} &= a_1 = a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1}a_{2^k} \\
&= a_1 + \frac{1}{2}\Big[t_{k+1} - a_1\Big] \\
&= a_1 + \frac{1}{2}t_{k+1} - \frac{1}{2}a_1 \\
&= \frac{1}{2}(a_1 + t_{k+1}) \\
&\geq \frac{1}{2}t_{k+1}.
\end{aligned}$$

Thus, we have

$$s_{2^k} \geq \frac{1}{2} t_{k+1}. \tag{*}$$

Similarly, we have

$$\begin{aligned} s_{2^k-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1}) \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^{k-1}}) \\ &= a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1} a_{2^{k-1}} \\ &= t_k. \end{aligned}$$

Thus, we have that

$$s_{2^k-1} \leq t_k. \tag{**}$$

($\Longleftarrow$) Assume that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). Our goal is to show that $\sum_{n=1}^{\infty} a_n$ converges; that is, (s_m) converges. Note that since $a_n \geq 0$, both (s_m) and (t_m) are increasing sequences. It follows from the Monotone Convergence Theorem that in order to prove (s_n) converges, it suffices to show that (s_m) is bounded.

Since (t_m) converges, we have that (t_m) is bounded. Hence, there exists $R > 0$ such that $t_m \leq R$ for all $m \in \mathbb{N}$. In what follows, we will show that R is an upper bound for (s_m) as well. Indeed, let $m \in \mathbb{N}$ be given. Choose k large enough so that $m < 2^k - 1$, then

$$s_m \leq s^{2^k-1} \leq t_k \leq R.$$

Thus, for all $m \in \mathbb{N}$, $0 \leq s_m \leq R$. Hence, (s_m) is bounded and so (s_m) converges by MCT.

($\Longrightarrow$) Assume that $\sum_{n=1}^{\infty} a_n$ ((s_m) converges). Our goal is to show that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges; that is, (t_m) converges.

We will prove the contrapositive: we will show that if (t_m) diverges then (s_m) diverges. Suppose (t_m) is divergent. Let $R >)$ be given. We will show that there is a term in the nonnegative sequence (s_m) that is larger than R. Since (t_m) diverges and (t_m) is an increasing sequence (where $t_m \geq 0$), we see that (t_m) cannot be bounded above by the Monotone Convergence Theorem. Hence, there exists $k \in \mathbb{N}$ such that $t_{k+1} > 2R$. Now, we have

$$s_{2^k} \geq \frac{1}{2} t_{k+1} ? \frac{1}{2}(2R) = R.$$

Thus, (s_m) is not bounded. ■

2.2 Applications of the Cauchy-Condensation Test

Example (P-series Test). Let $p > 0$. One can show that the sequence $\left(a_n = \frac{1}{n^p}\right)$ is a decreasing nonnegative sequence. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1.$$

Proof. Using the Cauchy Condensation test, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges}\\ &\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges}\\ &\iff \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \text{ converges}\\ &\iff \left|\frac{1}{2^{p-1}}\right| < 1\\ &\iff 1 < 2^{p-1}\\ &\iff 0 < p-1\\ &\iff 1 < p.\end{aligned}$$

In the fourth equivalence statement, we used the fact $\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ is a geometric series with common ratio $\frac{1}{2^{p-1}}$. ■

Example. Let $p > 0$. One can show that the sequence $\left(a_n = \frac{1}{n(\ln n)^p}\right)_{n\geq 2}$ is a decreasing nonnegative sequence. Prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \iff p > 1.$$

Proof. By the Cauchy Condensation Test, we see that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} &\iff \sum_{n=1}^{\infty} 2^n \frac{1}{2^n(\ln(2^n))^p} \text{ converges}\\ &\iff \sum_{n=1}^{\infty} \frac{1}{(n\ln 2)^p} \text{ converges}\\ &\iff \frac{1}{(\ln 2)^p}\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}\\ &\iff p > 1.\end{aligned}$$

■

Theorem (Comparison Test). Assume there exists an integer n_0 such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Notice that (ii) is just the contrapositive of (i). So, it suffices to show (i).

By the Cauchy Criterion for convergence of series, it is enough to show that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > m > N\ \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon. \tag{*}$$

To this end, let $\varepsilon > 0$ be given. Our goal is to find an N such that (*) holds. Since $\sum_{n=1}^{\infty} b_n$

converges, it follows from the Cauchy Criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N} \quad \Big| \sum_{k=m+1} b_k \Big| < \varepsilon.$$

Let $N = \max\{n_0, \hat{N}\}$. If $k \geq n_0$ where $a_k, b_k \geq 0$, we see that

$$\Big| \sum_{k=m+1}^{n} a_k \Big| = \sum_{k=m+1}^{n} a_k \text{ and } \Big| \sum_{k=m+1}^{n} a_k \Big| = \sum_{k=m+1}^{n} b_k. \tag{1}$$

Furthermore, if $k \geq n_0$, we have $a_k \leq b_k$, we have

$$\sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k. \tag{2}$$

If $n > m > N$, we see that (1) and (2) imply that

$$\Big| \sum_{k=m+1}^{n} a_k \Big| = \sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k = \Big| \sum_{k=m+1}^{n} b_k \Big| < \varepsilon.$$

■

Example. (*) Does $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converge?

Indeed, for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{1}{n+5^n} \leq \frac{1}{5^n}.$$

Note that $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges (because it is a geometric series). Thus, the comparison test implies that $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converges.

(*) Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Indeed, we see that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longrightarrow \lim a_n = 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $0 \leq a_n < 1$. Thus,

$$\forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n.$$

By the comparison test, we can conclude that $\sum_{n=1}^{\infty} a_n^2$ converges.

Remark (Some useful properties)**.** Let (a_n) be a sequence of real numbers. Suppose $\lim_{n\to\infty} a_n = A \in \mathbb{R}$.

(i) If $A < \beta$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n < \beta$.

(ii) If $\alpha < A$, then there exists N such that for all $n > N$, $\alpha < a_n$.

Theorem. Let (a_n) be a sequence of real numbers.

(i) Suppose $\beta \in \mathbb{R}$ is such that $\limsup a_n < \beta$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \quad a_n < \beta.$$

(ii) Suppose $\alpha \in \mathbb{R}$ is such that $\liminf a_n > \alpha$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \quad a_n > \alpha.$$

Proof. Here we will prove (i). Since $\limsup a_n < \beta$, we have $\limsup a_n \neq \infty$. We may consider two cases:

(1) Suppose $\limsup a_n = -\infty$. Since $\liminf a_n \leq \limsup a_n$, we can conclude that $\liminf a_n = -\infty$. Therefore, $\lim a_n = -\infty$. The claim immediately follows from the definition of $a_n \to -\infty$.

(2) Suppose $A = \limsup a_n$. Let $A = \limsup a_n$ and $r = \frac{\beta - A}{2}$. Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = A$, there exists N such that

$$\forall n > N \quad \sup\{a_k : k \geq n\} < A + r.$$

In particular, we have

$$\forall n > N \quad \sup\{a_k : k \geq n\} < \beta.$$

Therefore, we have (noticing that $a_n \leq \sup\{a_k : k \geq n\}$),

$$\forall n > N \quad a_n < \beta.$$

Note that the proof of (ii) is completely analogous to the proof of (i). ∎

Theorem. Let (a_n) be a sequence of real numbers.

(i) Suppose $\limsup a_n > \beta$. Then, for infinitely many k, we have $a_k > \beta$. That is,

$$\forall n \in \mathbb{N} \ \ \exists k \geq n \text{ such that } a_k > \beta.$$

(ii) Suppose $\liminf a_n < \alpha$. Then, for infinitely many k, $a_k < \alpha$. That is,

$$\forall n \in \mathbb{N} \ \ \exists k \geq n \text{ such that } a_k < \alpha.$$

Proof. Here we will prove (i) (the proof for (ii) is completely analogous). Assume for contradiction that only for finitely many k, $a_k > \beta$. Then there exists an $N \in \mathbb{N}$ such that for all $k > N$, we have $a_k \leq \beta$. Therefore,

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that $\limsup a_k > \beta$. ∎

Theorem (Root Test (Version 1))**.** Let (a_n) be a sequence of real numbers. Let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem (Root Test (Version 2))**.** Let (a_n) be a sequence of real numbers. Assume that following limit exists:

$$\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\alpha < \beta < 1$. We have $\limsup \sqrt[n]{|a_n|} < \beta$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $\sqrt[n]{|a_n|} < \beta$. Hence, we have for all $n > N$, $0 \leq |a_n| < \beta^n$ and $\sum_{n=1}^{\infty} \beta^n$ converges (it is a geometric series with common ratio $0 < \beta < 1$). As a consequence, we see that $\sum_{n=1}^{\infty} |a_n|$ converges by the Comparison Test.

(ii) Choose a number β such that $1 < \beta < \alpha$. We have $\beta < \limsup \sqrt[n]{|a_n|}$. By Useful Theorem 2, we have for all $n \in \mathbb{N}$, we have

$$\exists k \geq n \text{ such that } \sqrt[k]{|a_k|} > \beta \implies |a_k| > \beta^k$$
$$\implies \sup\{|a_m| : m \geq n\} > \beta^k.$$

Since $k \geq n$, we have $\beta^k \geq \beta^n$, and so

$$\forall n \in \mathbb{N} \sup\{|a_m| : m \geq n\} > \beta^n.$$

Since $\lim_{n\to\infty} \beta^n = \infty$ ($\beta > 1$), it follows from the order limit theorem (for $\overline{\mathbb{R}}$) that $\lim_{n\to\infty} \sup\{|a_m| : m \geq n\} = \infty$. So, $\limsup |a_n| = \infty$. This tells us that $\lim a_n \neq 0$ (Explanation of Fact). So, $\sum a_n$ diverges by the Divergence Test.

■

Remark. This is just the contrapositive of the following fact:

$$\text{If } \lim a_n = 0, \text{ then } \lim |a_n| = 0, \text{ so } \limsup |a_n| = 0.$$

Theorem (Ratio Test (Version 1))**.** Let (a_n) be a sequence of real numbers.

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\left|\frac{a_{m+1}}{a_m}\right| \geq 1$ for all $n \geq n_0$ (some integer n_0, then $\sum_{n=1}^{\infty} a_n$ diverges).

(iii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem (Ratio Test (Version 2))**.** Let (a_n) be a sequence of real numbers. Assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\rho < \beta < 1$. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \implies \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

Thus, we have

$$\begin{aligned} |a_{N+1}| &< \beta|a_N| \\ |a_{N+2}| &< \beta|a_{N+1}| < \beta^2|a_N| \\ |a_{N+3}| &< \beta|a_{N+2}| < \beta^3|a_N| \\ &\vdots \end{aligned}$$

So, for all $n \in \mathbb{N}$, $|a_{N+n}| < \beta^n|a_N|$. Now, notice that

$$\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n \quad \text{converges.}$$

Now, notice that $\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n$ converges (since it is a geometric series with common ratio $0 < \beta < 1$). It follows from the Comparison Test that $\sum_{n=1}^{\infty} |a_{N+n}|$ converges.

Considering that $\sum_{n=1}^{\infty}|a_{N+n}| = \sum_{n=N+1}^{\infty}|a_n|$, we can conclude that $\sum_{n=N+1}^{\infty}|a_n|$ converges. This immediately implies that $\sum_{n=1}^{\infty}|a_n|$ converges.

(ii) Choose a number β such that $1 < \beta < \rho$. Then we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \Longrightarrow \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| > \beta.$$

So, we have

$$\begin{aligned}
&|a_{N+1}| > \beta|a_N| \\
&|a_{N+2}| > \beta|a_{N+1}| > \beta^2|a_N| \\
&|a_{N+3}| > \beta|a_{N+2}| > \beta^3|a_N| \\
&\quad\vdots
\end{aligned}$$

Thus, for each $n \in \mathbb{N}$, $|a_{N+n}| > \beta^n|a_N|$. Since $\beta > 1$, $\lim\limits_{n\to\infty}\beta^n|a_N| = \infty$. So, $\lim\limits_{n\to\infty}|a_{N+n}| = \infty$. Therefore, $\lim\limits_{n\to\infty}a_{N+n} \neq 0$. Thus, $\lim\limits_{n\to\infty}a_n \neq 0$ (because $(a_{N+n})_{n\geq 1}$) is a subsequence of $(a_n)_{n\geq 1}$. So, $\sum_{n=1}^{\infty}a_n$ diverges by the Divergence Test.

∎

Example. Let $R \neq 0$ be a fixed number. Prove that the series $\sum_{n=1}^{\infty}\frac{R^n}{n!}$ converges. Indeed, we have

$$\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}}\right| &= \lim_{n\to\infty}\left|\frac{R^{n+1}n!}{R^n(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{R}{n+1}\right| \\
&= |R|\lim_{n\to\infty}\frac{1}{n+1} \\
&= 0.
\end{aligned}$$

Thus, if $\rho = 0 < 1$, then $\sum_{n=1}^{\infty}\frac{R^n}{n!}$ is absolutely convergent. As a consequence, we have

$$\lim_{n\to\infty}\frac{R^n}{n!} = 0.$$

Remark. If (a_n) is a sequence and $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\lim a_n = 0$.

Theorem (Dirichlet's Test)**.** Let (b_n) be the sequence of partial sums of $\sum_{n=1}^{\infty}a_n$ be bounded, is a decreasing sequence of nonnegative numbers ($b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$), and $\lim\limits_{n\to\infty}b_n = 0$. Then we have $\sum_{n=1}^{\infty}a_nb_n$ converges.

Corollary (Leibniz Test)**.** Let (b_n) be a sequence in $\mathbb{R}$. Suppose $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ and $\lim\limits_{n\to\infty}b_n = 0$. Then we have $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges.

Consider the infinite sum

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots \tag{*}$$

and the following questions:

(1) What is (s_n)?

(2) What is $\lim\limits_{n\to\infty}s_n$?

Define the sequence of partial sums for the series in (*) (s_n).

Consider the following partial sums

$$
\begin{aligned}
s_1 &= 1 \\
\\
s_2 &= 1 - 1 = 0 \\
s_3 &= 1 - 1 + \frac{1}{2} = \frac{1}{2} \\
s_4 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} = 0 \\
s_5 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3} \\
&\vdots
\end{aligned}
$$

which establishes (1). Looking at the even subsequence of (s_n), we can see that for all $k \in \mathbb{N}$ $s_{2k} = 0$ and the odd subsequence $s_{2k-1} = \frac{1}{k}$, respectively. Clearly, we can see from these subsequences that

$$s_{2k} \to 0 \quad \text{and} \quad s_{2k-1} \to 0$$

as $k \to \infty$. Hence, we can see that $s_n \to 0$ which establishes (2).

2.3 Rearrangements

Consider the following rearrangement of (*)

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \cdots = \lim_{n \to \infty} s_n = \ln(2).$$

Consider the sequence of the partial sums of the above:

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= \frac{3}{2} \\
s_3 &= \frac{1}{2} \\
&\vdots \\
s_{2 \times 10^2 + 2} &\approx 0.6939 \\
s_{3 \times 10^4 + 2} &\approx 0.6932 \\
s_{3 \times 10^6 + 2} &\approx 0.6931 \\
&\vdots
\end{aligned}
$$

Theorem. If a series converges absolutely, then for any $L \in \mathbb{R}$, there exists some rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to L.