0.1 Lecture 6

0.1.1 Topics

- Open Sets
- Closed Sets
- Interior Points
- Closure of a set

Let's recall the Euclidean metric on $\mathbb{C}$:

$$d(z,w) = |\vec{z} - \vec{w}|.$$

For real vector spaces, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ using the map $\mathbb{C} \to \mathbb{R}^2$ with $\alpha + i\beta \to (\alpha, \beta)$. Thus, we can visualize $\mathbb{C}$ using $\mathbb{R}^2$.

Recall that definition of continuity for functions in $\mathbb{R}$.

Definition (continuity in $\mathbb{R}$)**.** Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **continuous** for each $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - f(a)| < \varepsilon.$$

Note that whenever we say $|x - a| < \delta$, we can also say that $x \in (a - \delta, a + \delta)$. Similarly, we have $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

0.1.2 Open Sets

If we want to write the former using set notation, we can denote it as

$$\{x \in \mathbb{R} : d(x,a) < \delta\}$$

which is a more general statement than the one found in the definition above.

Definition (Open Ball)**.** Consider the usual metric $d(z,w) = |z - w|$ in $\mathbb{C}$. Let $\varepsilon > 0$ and $a \in \mathbb{C}$. By an **open ball** centered at a, and radius $\varepsilon > 0$, we mean the set

$$B(a,\varepsilon) = \{z \in \mathbb{C} : d(z,a) < \varepsilon\}.$$

Definition (Open Set)**.** Let $D \subseteq \mathbb{C}$. We say D is **open** if for each $a \in D$, we can find $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq D$.

Example. (i) $\mathbb{C}$ is open.

(ii) Let $\varepsilon > 0$ and $a \in \mathbb{C}$. Then $B(a,\varepsilon)$ is also open; that is, open balls are open.

(iii) Let D_1 and D_2 be open sets. Show that $D_1 \cap D_2$ is open. Let $x \in D_1 \cap D_2$ To show that $D_1 \cap D_2$ is open, we need to find $\delta > 0$ such that $B(x,\delta) \subseteq D_1 \cap D_2$. Since $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$. Since D_1 and D_2 are open sets, we can find $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$B(x,\varepsilon) \subseteq D_1 \text{ and } B(x,\varepsilon') \subseteq D_2, \text{ respectively.}$$

By definition, we see that $d(x,p) < \varepsilon$ for all $p \in D_1$ and $d(x,q) < \varepsilon'$. Then pick $\delta = \min\{d(x,p), d(x,q)\}$. Since x is in both D_1 and D_2, we have that

$$B(x,\delta) \subseteq D_1 \cap D_2.$$

Hence, $D_1 \cap D_2$ is open.

1. The empty set is open.

2. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a collection of open ets, where Λ is an indexing set. Define the set

$$\bigcup_{\lambda\in\Lambda} D_\lambda = \{z \in \mathbb{C} : z \in D_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Show that $\bigcup_{\lambda\in\Lambda} D_\lambda$ is also open.

0.1.3 Interior Points

Definition (Interior Point)**.** Let $D \subseteq \mathbb{C}$ and $a \in D$. We say that a is an **interior point** of D if we can find $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq D$.

Proposition. D is open if and only if each $a \in D$ is an interior point of D.

Example. Consider the following intersection

$$\bigcap_{n=1}^{\infty} B(0, 1/n) \tag{1}$$

Let us compute (1). This is just the singleton $\{0\}$. Show that $\{0\}$ is not open. Note that the intersection of arbitrary many open sets may not be open.

0.1.4 Closed sets

Definition (Closed Set)**.** We say $A \subseteq \mathbb{C}$ **closed** if its complement

$$\mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin A\}$$

is open.

Example. Define a closed ball with center at $a \in \mathbb{C}$ and radius $\varepsilon > 0$ by

$$\overline{B}(a,\varepsilon) = \{z \in \mathbb{C} : |z-a| \le \varepsilon\}.$$

Note that this is not the same thing as the closure of the open ball! Show that $\overline{B}(a,\varepsilon)$ is closed.

Example.
- For any $a \in \mathbb{C}$, we say that the singleton $\{a\}$ is closed.
- If $A_1, A_2, \dots, A_n$ are closed sets. Then

$$\bigcup_{i=1}^{n} A_i \text{ is also closed.}$$

  We can just show that the complement of this set is open.
- Let $\{A_\Lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets. Define

$$\bigcap_{\lambda\in\Lambda} A_\lambda = \{z \in \mathbb{C} : z \in A_\Lambda \ \forall \lambda \in \Lambda\},$$

  then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is closed.

Definition (Boundary Points)**.** Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is a **boundary point** of D if for all $\varepsilon > 0$, $B(a,\varepsilon) \cap D \neq \emptyset$ and $B(a,\varepsilon) \cap (\mathbb{C} \setminus D) \neq \emptyset$; that is, $B(a,\varepsilon)$ intersects D and $\mathbb{C} \setminus D$ non-trivially.

Example. Any point on the circle $\{z \in \mathbb{C} : |z - a| = \varepsilon\}$ is a boundary point of $B(a, \varepsilon)$, where $\varepsilon > 0$.

Example (Boundary Points of Singletons).
- Note that the boundary point of the singleton is just the singleton itself.
- Let $D = \mathbb{C} \setminus \{a\}$. Just like the first item, the boundary points of this set is just $\{a\}$.

Definition (Accumulation Point). Let $D \subseteq \mathbb{C}$. We say $a \in \mathbb{C}$ is an **accumulation point** of D if for any $\varepsilon > 0$,

$$B(a, \varepsilon) \cap (D \setminus \{a\}) \neq \emptyset.$$

Example.
(i) Let $D = \mathbb{C}$. Then $\{0\}$ is an accumulation point of D.
(ii) Let a such that $|a| = 1$. Then a is an accumulation point of $D = \{z : |z| < 1\}$.
(iii) Let $D = \{1, i\}$. Then D has no accumulation points.
(iv) If $D = \{w\}$ for some nonzero $w \in \mathbb{C}$. Thus, $D' = \emptyset$. That is, every singleton set contains no accumulation points. Furthermore, any finite set does not have any accumulation points.
(v) Let $D = \mathbb{R}$. Then the accumulation points of D is just $\mathbb{R}$ itself. That is, any point of $\mathbb{R}$ is an accumulation point.
(vi) Let $D = B(0, 1)$. The accumulation points are contained in the following set:

$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Another way to define the set of accumulation points of D is the following:

$$\{a \in \mathbb{C} : \forall \varepsilon > 0, B(a, \varepsilon) \cap (D \setminus \{a\})\}.$$

Definition (Closure). Let $D \subseteq \mathbb{C}$. We denote the **closure of** D by $\mathbf{D}$ and define $\overline{D}$ as

$$\overline{D} = D \cup \{x \in D : \forall \varepsilon > 0 \; B(x, \varepsilon) \cap D \neq \emptyset\}.$$

The closure is the smallest closed set that contains D!

Lemma.
(i) $\overline{D}$ is closed in $\mathbb{C}$.
(ii) The following statements are equivalent:
  (a) $a \in \mathbb{C}$ is an accumulation point of D
  (b) There is a sequence (z_n) with $z_n \in D$, $z_n \neq a$ for all $n \in \mathbb{Z}_+$ and $z_n \to a$.

Proof. Proof is in the homework! ■

Definition (Bounded). We say $D \subseteq \mathbb{C}$ is **bounded** if there exists $R > 0$ such that $D \subseteq \overline{B}(0, R)$; that is, $|z| \leq R$ for all $z \in D$; that is, $D \subseteq \overline{B}(0, R)$.