0.1 Lecture 10

Theorem (E is open relative to Y). Let (X, d) be a metric space and $E \subseteq Y \subseteq X$ and $Y \neq \emptyset$. E is open relative to Y if and only if there exists an open set $G_0 \subseteq X$ such that $E = G \cap Y$.

Proof. ($\Longrightarrow$) Assume that E is open relative to Y. Our goal is to show that there exists an open set $G \subseteq X$ such that $E = G \cap Y$. Since E is open relative to Y, every $a \in E$, we have that a is an interior point of E; that is, there exists $\varepsilon_0 > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$. Hence, for all $a \in E$, there exists $\varepsilon_a > 0$ such that $N_{\varepsilon_a}(a) \cap Y \subseteq E$. Let $G = \bigcup_{a \in E} N_{\varepsilon_a}(a)$. Clearly, G is open in X since

(1) For all $a \in E$, $N_{\varepsilon_a}(a)$ is a neighborhood and so it is open in X.

(2) A union of open sets is open in any metric space.

In what follows, we will prove that $E = G \cap Y$. Note that

$$G \cap Y = \Big(\bigcup_{a \in E} N_{\varepsilon_a}(a)\Big) \cap Y = \bigcup_{a \in E} \Big(N_{\varepsilon_a} \cap Y\Big) \subseteq \bigcup_{a \in E} E = E.$$

Suppose $b \in E$. We have $b \in N_{\varepsilon_b}(b)$ and thus $b \in G$. Furthermore, $b \in E$ implies $b \in Y$ since $E \subseteq Y$. Thus, we see that $E \subseteq G \cap Y$.

($\Longleftarrow$) Assume that there exists $G \subseteq X$ such that $E = G \cap Y$. We want to show that E is open relative to Y. Our goal is to show that for all $a \in E$, there exists $\varepsilon > 0$ such that $N^Y_{\varepsilon_a}(a) \subseteq E$.

So, let $a \in E$ be given. Our goal is to find $\varepsilon > 0$ such that

$$N_\varepsilon(a) \cap Y \subseteq E.$$

By assumption, we have

$$\begin{aligned} a \in E = G \cap Y &\Longrightarrow a \in G \\ &\Longrightarrow_{G \text{ is open}} \exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \subseteq G. \end{aligned}$$

Hence, we see that

$$N_\varepsilon(a) \cap Y \subseteq G \cap Y = E.$$

■

Theorem. (i) If $E \subseteq \mathbb{R}$ is bounded above, then $\sup E \in \overline{E}$.

(ii) If $E \subseteq \mathbb{R}$ is bounded below, then $\inf E \in \overline{E}$.

Proof. Here we will prove (1) and the proof of (2) is completely analogous.

Since E is bounded above, we have that $\sup E$ exists and is a real number by the least upper bound property of $\mathbb{R}$. Let $\alpha = \sup E$. Our goal is to show that $\alpha \in \overline{E}$; that is, we want to show that for all $\varepsilon > 0$,

$$N_\varepsilon(\alpha) \cap E \neq \emptyset. \qquad \text{(Exercise 11 of HW4)}$$

Let $\varepsilon > 0$ be given. Since $\alpha = \sup E$, we know that there exists $x \in E$ such that $\alpha - \varepsilon < x$. Hence, there exists $x \in E$ such that

$$\alpha - \varepsilon < x \leq \alpha < \alpha + \varepsilon.$$

Hence, $\alpha - \varepsilon < x < \alpha + \varepsilon$; that is, $x \in N_\varepsilon(\alpha)$. Therefore, $N_\varepsilon(\alpha) \cap E \neq \emptyset$. ■

Definition (Open Covers). Let (X, d) be a metric space and $E \subseteq X$. A collection of sets $\{O_\alpha\}_{\alpha \in \Lambda}$ is said to be an **open cover** of E if

(i) for every $\alpha \in \Lambda$, O_α is open in X.

(ii) $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$.

Example. Consider $(\mathbb{R}, |\bullet|)$ and the subset $E = [0, \infty)$. The collection $\{E_n\}_{n\in\mathbb{N}}$ defined by for all $n \in \mathbb{N}$, we have

$$E_n = \left(\frac{-1}{n}, n\right)$$

is an open cover of E. The reason is as follows:

(i) In homework 4, we proved that every open interval is an open set with the standard metric on $\mathbb{R}$.

(ii) $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Indeed, let $a \in E$. If $a = 0$, then a belongs E_n for all $n \in \mathbb{N}$. So, we have

$$a \in \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, if $a \neq 0$, then $\frac{1}{a} > 0$. By the Archimedean Property of $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a}$. So, $a < n$. Clearly,

$$\frac{-1}{n} < 0 < a < n.$$

Hence, $a \in E_n = \left(\frac{-1}{n}, n\right)$. Consequently,

$$a \in \bigcup_{m=1}^{\infty} E_m.$$

Definition (Compactness). Let (X, d) be a metric space and $K \subseteq X$. We say that K is **compact** if every open cover of K has a finite subcover; that is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover of K, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$K \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Example. Let (X, d) be a metric space and $E \subseteq X$. If E is finite, then E is compact. The reason is as follows:

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover. Our goal is to show that this open cover has a finite subcover; that is, we can choose $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$

If $E = \emptyset$, there is nothing to prove. Otherwise, $E \neq \emptyset$, we can denote the elements of E by $x_1, \ldots, x_n$. So,

$$E = \{x_1, \ldots, x_n\}.$$

We have

$$x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

and so continue this process until you have reached the nth element of E. Hence, we have

$$x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}.$$

Thus, we have that

$$E \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$$

and so E is compact.

Example. Let $(\mathbb{R}, |\cdot|)$ and

$$E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Prove that E is compact (In general, if $a_n \to a$ in $\mathbb{R}$, then the set $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact).

Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover. Notice that $0 \in E$ and $E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha$ implies that

$$O \in \bigcup_{\alpha\in\Lambda} O_\alpha \Longrightarrow \exists \alpha_0 \in \Lambda \text{ such that } O \in O_{\alpha_0}.$$

So, if $O \in O_{\alpha_0}$ and O_{α_0} is open, we have

$$\exists \varepsilon > 0 \text{ such that } (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}.$$

By the Archimedean Property of $\mathbb{R}$, we see that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. So, for all $n \geq m$, we have $\frac{1}{n} < \varepsilon$. Hence, for all $n \geq m$, we have $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_{\alpha_0}$. Note that $1 \in E$ so there exists $\alpha_1 \in \Lambda$ such that $1 \in O_{\alpha_1}$. Similarly, $\frac{1}{2} \in E$ implies that there exists $\alpha_2 \in \Lambda$ such that $\frac{1}{2} \in O_{\alpha_2}$. Continue this process until, we have

$$\frac{1}{m-1} \in E \Longrightarrow \exists \alpha_m \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}}.$$

Thus, we conclude that

$$E \subseteq \bigcup_{n=0}^{\infty} O_{\alpha_n}$$

and so E is compact.

Remark. If X itself is compact, we say that (X, d) is a compact metric space; that is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that

$$X = \bigcup_{\alpha\in\Lambda} O_\alpha$$

then there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$X = \bigcup_{i=1}^{n} O_{\alpha_i}.$$

Theorem (Compactness implies Closed)**.** Let (X, d) be a metric space and $K \subseteq X$ is compact. Then K is compact.

Proof. It is enough to show that K^c is open; that is, we need to find $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq K^c$ for every $x \in K^c$. That is, we want to show that

$$\exists \varepsilon > 0 \text{ such that } N_\varepsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\Longrightarrow a \notin K \\ &\Longrightarrow \forall x \in K \ d(x,a) > 0 \end{aligned}$$

For all $x \in K$, let $\varepsilon_x = \frac{1}{4}d(x,a)$. Clearly, we have

$$\forall x \in K \ N_{\varepsilon_x} \cap N_{\varepsilon_x}(a) = \emptyset.$$

Note that $\{N_{\varepsilon_x}(x)\}_{x \in K}$ is an open cover for K. Since K is compact, there is a finite subcover. That is, there exists $x_1, \ldots x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_{\varepsilon_{x_i}}(x_i)$$

and of course

$$N_{\varepsilon_{x_1}}(x_1) \cap N_{\varepsilon_{x_1}}(a) = \emptyset.$$

If we continue this process until the nth step, we have that

$$N_{\varepsilon_{x_n}}(x) \cap N_{\varepsilon_{x_n}}(a) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}\}$. Clearly, $N_\varepsilon(a) \subseteq N_{\varepsilon_{x_i}}(a) \forall 1 \leq i \leq n$. Hence, we have

$$N_{\varepsilon_{x_1}} \cap N_\varepsilon(a) = \emptyset$$

and similarly for the nth step, we have

$$N_{\varepsilon_{x_n}}(x) \cap N_\varepsilon(a) = \emptyset.$$

Therefore, we have

$$N_\varepsilon(a) \cap \left[N_{\varepsilon_{x_1}} \cup \cdots \cup N_{\varepsilon_{x_n}}\right] = \emptyset.$$

So,

$$N_\varepsilon(a) \cap K = \emptyset.$$

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0.2 Lecture 11

0.2.1 Topics

- Metric subspace
- Theorem 2.35
- Theorem 2.33
- Theorem 2.37
- Theorem 2.36

Theorem. Closed subsets of compact sets are compact.

Proof. ■

Corollary. If F is closed and K is compact, then $F \cap K$ is compact.

Proof. ■

Theorem. Suppose $K \subseteq Y \subseteq X$. We have E is compact if and only if K is compact relative to Y.

Proof. ■

Theorem. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. ■

Remark. Let (X, d) be a metric space and $K \subseteq X$. The following statements are equivalent:

(i) K is compact.

(ii) Every infinite subset of K has a limit point in K.

(iii) Every sequence in K has a subsequence that converges to a point in K.

Corollary. If $K_1 \supseteq K_2 \supseteq \ldots$ is a sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. ■

Theorem. Let $\{K_\alpha\}$ a collection of compact subsets of X. If the intersection of any finite subcollection is nonempty, then

$$\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset.$$

Proof. ■

Theorem (Nested Interval Property). If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. ■