Linear Algebra Exercises

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Chapter 1

Vector Spaces

1.1 Vector Spaces

Exercise 1.2.1

Label the following statements as true or false.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

Exercise 1.2.7

Let $S = \{0,1\}$ and $F = \mathbb{R}$. In $\mathscr{F}(S,\mathbb{R}$, show that $f = g$ and where $f(t) = 2t+1$, $g(t) = 1+4t-2t^2$, and $h(t) = 5^t + 1$.

Proof. To show that $f = g$, we have to show that for each $s \in S$ that $f(s) = g(s)$. Since $S = \{0,1\}$, we can just evaluate both f and g for elements in S. Note that

$$f(0) = 2(0) + 1 = 1$$

and likewise,

$$g(0) = 1 + 4(0) - 2(0)^2.$$

Hence, $f(0) = g(0)$. Now let us evaluate both functions f and g at $s = 1$. Hence, we have

$$f(1) = 2(1) + 1 = 3$$

and

$$g(1) = 1 + 4(1) - 2(1)^2 = 3.$$

Thus, we must have $f(s) = g(s)$ for all $s \in S$.

Now, we need to show that $f + g = h$. Like we did above, we have to show that this is the case for all $s \in S$. Note that

$$(f+g)(s) = f(s) + g(s).$$

Hence, we have

$$f(s) + g(s) = 2 + 6s - 2s^2.$$

Evaluating at $s = 0$, we have

$$f(0) + g(0) = 2.$$

and likewise,

$$h(0) = 5^0 + 1 = 2.$$

Hence, $(f+g)(0) = h(0)$. Now let us evaluate $f+g$ at $s=1$

$$f(1)+g(1) = 2+6(1)-2(1)^2 = 6$$

and likewise, we have

$$h(1) = 5^1 + 1 = 6.$$

Hence, we have $(f+g)(1) = h(1)$. Thus, we have that $f+g = h$ for all $s \in S$.

■

Exercise 1.2.8

In any vector space V, we have

$$(a+b)(x+y) = ax+ay+bx+by$$

for any $x, y \in V$ and any $a, b \in F$.

Proof. Observe the following set equalities:

$$\begin{aligned}(a+b)(x+y) &= a(x+y)+b(x+y) && \text{(VS 8)}\\ &= ax+ay+bx+by. && \text{(VS 7)}\end{aligned}$$

Hence, we have that

$$(a+b)(x+y) = ax+ay+bx+by$$

for any $x, y \in V$ and any $a, b \in F$. ■

Exercise 1.2.10

Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Let $f, g \in \mathscr{F}(S,\mathbb{R})$ and let $x \in S$. Using the addition operation defined in Example 3, we have

$$(f+g)(x) = f(x)+g(x).$$

Since $f(x), g(x) \in \mathbb{R}$ where $\mathbb{R}$ is a field, we know that commutativity is preserved. Hence, we have that

$$f(x)+g(x) = g(x)+f(x).$$

By the same reasoning, we have that associativity is also preserved.

Since $\mathbb{R}$ is a field, we also know that there exists an element O in $\mathbb{R}$ such that for every $y \in \mathbb{R}$, we have $y+O = y$. Since $f(y) \in \mathbb{R}$, there exists an f_0 such that $f(y)+f_0 = f(y)$. Hence, the third property is also satisfied. Likewise, the fourth property is also satisfied since $\mathbb{R}$ is a field. Since $\mathbb{R}$ is a field in which multiplicative identities exists and that for every $x \in S$ with $f(x) \in \mathbb{R}$, we know that $1 \cdot f(x) = f(x)$. Hence, property 5 is satisfied.

Let $x \in S$. Since $f(x) \in \mathbb{R}$ and $\mathbb{R}$ is a field, we also know that properties 6 and 7 are satisfied. Now, let $f \in \mathscr{F}(S,\mathbb{R})$ and $x \in S$. Let $a, b \in \mathbb{R}$. Using the operations of addition and scalar multiplication and the fact that $\mathbb{R}$ is a field, we have that

$$(a+b)f(x) = af(x)+bf(x).$$

Hence, we conclude that $V = \mathscr{F}(S,\mathbb{R})$ is a vector space.

■

Exercise 1.2.11

Let $V = \{0\}$ consist of a single vector O and define $O+O = O$ and $cO = O$ for each scalar $c \in F$. Prove that V is a vector space over F.

Proof. Let $x, y \in O$. Since $V = \{0\}$, we know that x and y are both the 0 vector. By using the addition defined on V, we can see that

$$x+y = O+O = y+x.$$

Note that $x+y \in V$ implies that $x+y=O+O$ and likewise $y+z=O+O$ for every $x,y,z \in V$. Hence, we have that

$$x+(y+z)=O+(O+O)=(O+O)+O=(x+y)+z.$$

Since V consists of only the zero vector O, we know that (VS 3) and (VS 4) are satisfied. By the same reasoning, (VS 5) is satisfied because

$$1 \cdot O = O.$$

Let $a,b \in F$. Then (VS 6) and (VS 7) are satisfied because

$$(ab)O = 0 = a \cdot 0 = a(bO) \text{ and } a(O+O) = 0 = O+O = aO+aO \text{ respectively.}$$

Let $a,b \in F$ again. Then we have

$$\begin{aligned}(a+b)O &= 0\\ &= O+O\\ &= aO+bO.\end{aligned}$$

Hence, (VS 8) is satisfied. ■

Exercise 1.2.12

A real-valued function f defined on the real line is called an **even function** if $f(-t)=f(t)$ for all $t \in \mathbb{R}$. Prove that the set of even functions defined on $\mathbb{R}$ with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof. First, let us show that for every $t \in \mathbb{R}$ that $(f+g)(t)$ is also an even function for every pair of even functions f,g and likewise $(cf)(t)$ is an even function for every $c \in \mathbb{R}$. Observe that

$$\begin{aligned}(f+g)(-t) &= f(-t)+g(-t)\\ &= f(t)+g(t)\\ &= (f+g)(t).\end{aligned}$$

Hence, the function $f+g$ is also even. Now observe that

$$\begin{aligned}(cf)(-t) &= cf(-t)\\ &= cf(t)\\ &= (cf)(t).\end{aligned}$$

Hence, cf is also even function.

Now we can show that the set of even functions is a vector space.

(VS 1) Let f,g be a pair of real-valued even functions. Let $t \in \mathbb{R}$ be arbitrary. We need to show that $f+g = g+f$. Since $f(t)$ and $g(t)$ are real numbers, observe that

$$(f+g)(t) = f(t)+g(t) = g(t)+f(t) = (g+f)(t).$$

Hence, we have that $f+g=g+f$.

(VS 2) Let f,g,h be even functions and let $t \in \mathbb{R}$ be arbitrary. We need to show that $f+(g+h)=(f+g)+h$. By the same reasoning we used to prove (VS 1), observe that

$$\begin{aligned}f(t)+((g+h)(t)) &= f(t)+(g(t)+h(t))\\ &= (f(t)+g(t))+h(t)\\ &= ((f+g)(t))+h(t)\end{aligned}$$

Hence, we have that $f+(g+h)+(f+g)+h$.

(VS 3) Let f be an even function. Note that the zero function is an even function. Denote the zero function as f_0. For every $t \in \mathbb{R}$, we have $f_0(x) = O$. We need to show that $f + f_0 = f$. Since $f_0(x)$ is just a real number, we have

$$(f + f_0)(t) = f(t) + f_0(t) = f(t) + O = f(t).$$

(VS 4) Let f be an even function and let f_0 be the zero function defined above. We need to show that there exists a even function g such that $f + g = f_0$. Let $t \in \mathbb{R}$ be arbitrary. Note that

$$(f + g)(t) = f(t) + g(t)$$

Since $f(t)$ and $g(t)$ are real numbers and there exists an additive identity in the real numbers, we have that $f(t) + g(t) = f_0(t)$. Hence, (VS 5) is satisfied.

(VS 5) Let $x \in \mathbb{R}$ and f an even function. Then we immediately have $(1 \cdot f)(t) = 1f(t) = f(t)$.

(VS 6) Let $a, b \in \mathbb{R}$. We need to show that $(ab)f = a(bf)$. Hence, let $t \in \mathbb{R}$. Then we have

$$(abf)(t) = a(bf)(t).$$

(VS 7) Let $a \in \mathbb{R}$ and let f, g be even functions. Let $t \in \mathbb{R}$. We need to show that $a(f + g) = af + ag$. Then we have

$$\begin{aligned} a(f + g)(t) &= a[f(t) + g(t)] \\ &= af(t) + ag(t). \end{aligned}$$

Hence, (VS 7) is satisfied.

(VS 8) Now let $a, b \in \mathbb{R}$ and let f be an even function. We need to show that $(a + b)f = af + bf$. Let $t \in \mathbb{R}$ be arbitrary. Observe that

$$\begin{aligned} (a + b)f(t) &= af(t) + bf(t) \\ &= (af)(t) + (bf)(t) \end{aligned}$$

Hence, (VS 8) is satisfied.

■

Exercise 1.2.13

Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, a_2 b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2).$$

Is V is a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We have that V is not a vector space of $\mathbb{R}$. To see why, let $(2, 1), (4, 2) \in V$ where $x = (2, 1)$ and $y = (4, 2)$. We will show that (VS 1) does not hold; that is, $x + y \neq y + x$. Hence, observe that

$$(2, 1) + (4, 2) = (2 + 1, 2) = (3, 2)$$

and

$$(4, 2) + (2, 1) = (4 + 2, 2) = (6, 2)$$

Hence, we have $x + y \neq y + x$ and so V is **NOT** a vector space. ■

Exercise 1.2.14

Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in C \text{ for } i = 1, 2, \ldots, n)\}$; so V is a vector space over C by Example 1. Is V is a vector space over the field of real numbers with the operations of coordinate-wise addition and multiplication?

Proof. Yes, V where

$$V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$$

is a vector space of $\mathbb{R}$.

(VS 1) Let $x, y \in V$ such that $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$. Since addition is entry-wise in V and each entry in both x and y are elements of $\mathbb{R}$ (where $\mathbb{R}$ is a field), we have $a_i + b_i = b_i + a_i$ for all $i = 1, 2, \ldots, n$. Hence, $x + y = y + x$.

(VS 2) Let $x, y, z \in V$ with x and y as defined as before where z contains entries c_i for all $i = 1, 2, \ldots n$. We can see that the entries of x, y, z are elements of $\mathbb{R}$ so associativity is preserved; that is, $a_i + (b_i + c_i) = (a_i + b_i) + c_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $x + (y + z) = (x + y) + z$.

(VS 3) Since $\mathbb{R}$ contains the zero element 0 and V is the set of n-tuples, there exists an element denoted by O such that this element consisting of entries that only have the zero element 0; that is, $O = (0, 0, \ldots, 0)$. Take $x \in V$. Hence, we have $a_i + 0 = a_i$ for every $i = 1, 2, \ldots, n$. Thus, we must have $x + O = x$.

(VS 4) Let $x \in V$ be arbitrary as defined before. Since every entry in x is an element of $\mathbb{R}$; that is, every $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we know that every entry contains an element c_i such that $a_i + c_i = 0$ for every $i = 1, 2, \ldots, n$. Denote $x' = (c_1, c_2, \ldots, c_n)$. Hence, we have $x + x' = O$.

(VS 5) Let $x \in V$. Every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $1 \cdot a_i = a_i$ which holds for all i. Denote this identity element as I with entries consisting only of 1. Hence, we have $I \cdot x = x$.

(VS 6) Let $e, r \in \mathbb{R}$ and let $x \in V$. For every entry $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $(er)a_i = e(ra_i)$ for all $i = 1, 2, \ldots, n$. By using the operations of scalar multiplication for n-tuples, this tells us that $(er)x = e(rx)$. Hence, (VS 6) is satisfied.

(VS 7) Let $e \in \mathbb{R}$ and let $x, y \in V$. We need to show that $e(x + y) = ex + ey$. Note that $a_i, b_i \in \mathbb{R}$ implies that $e(a_i + b_i) = ea_i + eb_i$. Hence, we have $e(x + y) = ex + ey$.

(VS 8) Let $e, r \in \mathbb{R}$ and $x \in V$ as defined before. We need to show that $(e + r)x = ex + rx$. Since $a_i \in \mathbb{R}$ with $e, r \in \mathbb{R}$, we are guaranteed to have $(e + r)a_i = ea_i + ra_i$ for all $i = 1, 2, \ldots, n$. Hence, we have $(e + r)x = ex + rx$.

■

Exercise 1.2.16

Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over $\mathbb{R}$ by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

Proof(VS 1) Let $A, B \in V$. Since A, B consist of elements $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $A_{ij} + B_{ij} = B_{ij} + A_{ij}$. Hence, $A + B = B + A$.

(VS 2) Let $A, B, C \in V$ with A, B defined as before and C containing real entries C_{ij}. With the same reasoning used to prove (VS 1), we know that $A_{ij} + (B_{ij} + C_{ij}) = (A_{ij} + B_{ij}) + C_{ij}$. Hence, we have $A + (B + C) = (A + B) + C$.

(VS 3) Let $A \in V$ once again. Since the entries of A imply that there exists an element O such that $A_{ij} + 0 = A_{ij}$, we know that $A + O = A$ where O is the **zero matrix** of V.

(VS 4) Since the real entries of A also consists of an element A'_{ij} such that $A_{ij} + A'_{ij} = 0$, this implies that $A + A' = O$ where A' is the additive inverse matrix of V.

(VS 5) Let $x \in V$ as defined as before. Every entry of A, $A_{ij} \in \mathbb{R}$, has the following property: $1 \cdot A_{ij} = A_{ij}$

for all for all $1 \le i \le m$ and for all $1 \le j \le n$. The matrix whose entries consists of only one we can define as the **identity matrix** denoted by I where $I_{ij} = 1$ for all $1 \le i \le m$ and $1 \le j \le n$. Hence, we have $A \cdot I = A$.

(VS 6) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. We need to show that $(rt)A = r(tA)$. Since $A_{ij} \in \mathbb{R}$, entry-wise scalar multiplication implies that $(rt)A_{ij} = r(tA_{ij})$ for all $1 \le i \le m$ and $1 \le j \le n$. Hence, we must have $(rt)A = r(tA)$.

(VS 7) Let $r \in \mathbb{Q}$ and $A, B \in V$ as defined before. Since $A_{ij}, B_{ij} \in \mathbb{R}$, we know that $r(A_{ij} + B_{ij}) = rA_{ij} + rB_{ij}$. Hence, we have $r(A+B) = rA + rB$.

(VS 8) Let $r, t \in \mathbb{Q}$ and let $A \in V$ as defined before. Since $A_{ij} \in \mathbb{R}$ for all i, j, we must have $(r+t)A_{ij} = rA_{ij} + tA_{ij}$. Hence, we have $(r+t)A = rA + tA$.

Hence, V is a vector space over $\mathbb{Q}$. ■

Exercise 1.2.17

Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinate-wise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (ca_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

Proof. We claim that V is not a vector space over F because V fails to satisfy (VS 5). To see why, let $(1,2) \in V$. Using (VS 5), we have

$$1 \cdot (1,2) = (1,0) \neq (1,2).$$

Hence, V cannot be a vector space. ■

Exercise 1.2.18

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $F = \mathbb{R}$ with these operations? Justify your answer?

Proof. We claim that V is not a vector space over $\mathbb{R}$ and we will use (VS 1) to show this. Let $x, y \in V$ be defined by $x = (1,2)$ and $y = (3,4)$. Observe that

$$x + y = (1,2) + (3,4) = (7,14)$$

and

$$y + x = (3,4) + (1,2) = (5,10).$$

Clearly, we have $x + y = (7,14) \neq (5,10) = y + x$ and so (VS 1) does not hold. ■

Exercise 1.2.19

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinate-wise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Proof. We claim that V is not a vector space over $\mathbb{R}$. To see why, consider (VS 8). If we let $(0,1) \in V$ with $c = 2 + 1 = 3$. Observe that

$$(2+1)(0,1) = \left(0, \frac{1}{2+1}\right) = \left(0, \frac{1}{3}\right).$$

Likewise, we have

$$2(0,1) + 1(0,1) = \left(0, \frac{1}{2}\right) + \left(0, 1\right) = \left(0, \frac{3}{2}\right).$$

Notice that $(2+1)(0,1) \neq 2(0,1)+1(0,1)$. Hence, V cannot be a vector space over $\mathbb{R}$. ■

Exercise 1.2.20

Let V denote the set of all real-valued functions f defined on the real line such that $f(1)=0$. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Define V as a vector space with the operations of addition and scalar multiplication defined in Example 3. We must show that V is a vector space.

(VS 1) Let $f,g \in V$. This means that $f(1)=0$ and $g(1)=0$. We need to show that $f+g=g+f$. Since $f(1),g(1) \in \mathbb{R}$ and commutativity holds in $\mathbb{R}$, we can write

$$\begin{aligned}(f+g)(1) &= f(1)+g(1)\\ &= g(1)+f(1)\\ &= (g+f)(1)\end{aligned}$$

Hence, we have $f+g=g+f$.

(VS 2) Let $f,g,h \in V$ then $f(1)=g(1)=h(1)=0$. We need to show that $f+(g+h)=(f+g)+h$. Observe that

$$\begin{aligned}\Big(f+(g+h)\Big)(1) &= f(1)+(g+h)(1)\\ &= f(1)+g(1)+h(1)\\ &= (f+g)(1)+h(1)\\ &= \Big((f+g)+h\Big)(1).\end{aligned}$$

Hence, we have $f+(g+h)=(f+g)+h$.

(VS 3) Let $f \in V$. We need to show that $f+f_0=f$ for some $f_0 \in V$. Since V contains elements of $f \in V$ such that $f(1)=0$, we can choose f_0 such that $f_0(1)=0$. We can show that this is indeed the additive inverse of V by writing the following:

$$(f+f_0)(1)=f(1)+f_0(1)=f(1)+0=f(1).$$

Hence, we have $f+f_0=f$.

(VS 4) Let $f \in V$. We need to find an element $g \in V$ such that $f+g=f_0$ By definition of V, $f(1)=0$. We need to show that $f+g=f_0$ with f_0 defined as before. Choose $g=-f$ as our additive inverse and observe that

$$\begin{aligned}(f+g)(1) &= (f-f)(1)\\ &= f(1)-f(1)\\ &= 0-0\\ &= 0\\ &= f_0(1).\end{aligned}$$

Hence, $g=-f$ an element such that $f+g=f_0$.

(VS 5) Let $f \in V$. By definition of V, we have $f(1)=0$. Since $f(1) \in \mathbb{R}$, we know that $1 \cdot f(1)=f(1)$. We need to show that $1f=f$. Observe that

$$(1f)(1)=1\cdot f(1)=f(1).$$

Hence, (VS 5) holds.

(VS 6) Let $a,b \in \mathbb{R}$ and let $x \in V$. We need to show that $(ab)f=a(bf)$. By using scalar multiplication, we can see that

$$(ab\cdot f)(1)=(ab)f(1)=a(bf(1))=a(b\cdot f)(1).$$

Hence, (VS 6) must hold.

(VS 7) Let $a \in \mathbb{R}$ and let $f, g \in V$. We need to show that $a(f+g) = af + ag$. Observe that

$$\begin{aligned} a(f+g)(1) &= a(f(1)+g(1)) \\ &= af(1) + ag(1) \\ &= (af)(1) + (ag)(1). \end{aligned}$$

Hence, $a(f+g) = af + ag$ and so (VS 7) is satisfied.

(VS 8) Let $a, b \in \mathbb{R}$ and let $f \in V$. Observe that

$$\begin{aligned} (a+b)f(1) &= af(1) + bf(1) \\ &= (af)(1) + (bf)(1) \end{aligned}$$

Hence, $(a+b)f = af + bf$. Thus, V must be a vector space over $\mathbb{R}$. ■

Exercise 1.2.21

Let V and W be vector spaces over a field F. Let

$$Z = \{(v, w) : v \in v \text{ and } w \in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. Let V and W be vector spaces over a field F.

(VS 1) Let $x, y \in Z$ where $x = (v_1, w_1)$ and $y = (v_2, w_2)$. Since V and W are vector spaces, commutativity holds. Since the addition defined on both V and W is entry-wise, we can see that

$$\begin{aligned} x + y &= (v_1, w_1) + (v_2, w_2) \\ &= (v_1 + v_2, w_1 + w_2) \\ &= (v_2 + v_1, w_2 + w_1) \\ &= (v_2, w_2) + (v_1, w_1) \\ &= y + x. \end{aligned}$$

Hence, we can see that (VS 1) is satisfied.

(VS 2) Let $x, y, z \in Z$ with x and y as defined before as well as $z = (z_3, w_3)$. Using the entry-wise addition defined for Z, we can see that

$$\begin{aligned} x + (y+z) &= (v_1, w_1) + \Big((v_2, w_2) + (v_3, w_3)\Big) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (x + y) + z. \end{aligned}$$

(VS 3) Since V and W are vector spaces, we know that there exists an additive identity O_V and O_W respectively. Hence, we have $(O_V, O_W) \in Z$ and denote $O_Z = (O_V, O_W)$. Now, let $x \in Z$ as defined before. Observe that

$$\begin{aligned} x + O_Z &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + O_V, w_1 + O_W) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 4) Observe that V and W contain additive inverses for each $v \in V$ and $w \in W$ respectively. Since $x = (v_1, w_1)$ with $v_1 \in V$ and $w_1 \in W$, there exists an additive inverse $v_1' \in V$ and $w_1' \in W$ such that $v_1 + v_1' = O_V$ and $w_1 + w_1' = O_W$. This implies that $(v_1', w_1') \in Z$ which we will denote by x' such that

$$\begin{aligned} x + x' &= (v_1, w_1) + (O_V, O_W) \\ &= (v_1 + v_1', w_1 + w_1') \\ &= (O_V, O_W) \\ &= O_Z. \end{aligned}$$

(VS 5) Let $x \in Z$. Since V and W are vector spaces (VS 5) implies that $1 \cdot v_1 = v_1$ and $1 \cdot w_1 = w_1$ respectively. Then observe that

$$\begin{aligned} 1 \cdot x &= 1 \cdot (v_1, w_1) \\ &= (1 \cdot v_1, 1 \cdot w_1) \\ &= (v_1, w_1) \\ &= x. \end{aligned}$$

(VS 6) Let $x \in Z$ and $a, b \in F$. Then

$$\begin{aligned} (ab)x &= (ab)(v_1, w_1) \\ &= ((ab)v_1, (ab)w_1) \\ &= (a(bv_1), a(bw_1)) && (V, W \text{ vector space}) \\ &= a(bv_1, bw_1) \\ &= a(bx) \end{aligned}$$

(VS 7) Let $x, y \in Z$ as defined before. Let $a \in F$. Then

$$\begin{aligned} a(x+y) &= a\Big((v_1, w_1) + (v_2, w_2)\Big) \\ &= a\Big((v_1 + v_2, w_1 + w_2)\Big) \\ &= (a(v_1 + v_2), a(w_1 + w_2)) \\ &= (av_1 + av_2, aw_1 + aw_2) \\ &= (av_1, av_2) + (aw_1, aw_2) \\ &= a(v_1, v_2) + a(w_1, w_2) \\ &= ax + ay. \end{aligned}$$

(VS 8) Let $a, b \in F$ and let $x \in V$ as defined before. Since V and W are vector spaces, we know that (VS 8) holds for bot entries $v_1 \in V$ and $w_1 \in W$. Hence, observe that

$$\begin{aligned} (a+b)x &= (a+b)(v_1, w_1) \\ &= \Big((a+b)v_1, (a+b)w_1\Big) \\ &= (av_1 + bv_1, aw_1 + bw_1) \\ &= (av_1, aw_1) + (bv_1, bw_1) \\ &= a(v_1, w_1) + b(v_1, w_1) \\ &= ax + bx. \end{aligned}$$

Hence, Z is a vector space.

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