

Week 1: Lecture Notes

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0.1 Lecture 1

0.1.1 Why do we need complex analysis?

We need it for:

- Solving polynomials with either coefficients in either \mathbb{R} or \mathbb{C} . For example, the polynomial $x^2 + 1$ does not have a solution in \mathbb{R} , but it does have a solution \mathbb{C} .
- Solving real integrals that may be difficult to deal with using standard techniques developed in \mathbb{R} ; that is, something like

$$\int_0^\infty \frac{\sin x}{x} dx.$$

- Solving problems in physics, particularly, in the Quantum Field Theory.

0.1.2 What is the goal?

Let's recall some facts about the real number system \mathbb{R} .

- (i) $(\mathbb{R}, +, \cdot)$ is a field.
- (ii) We have an order relation on \mathbb{R} .
- (iii) For all $x \in \mathbb{R}$, $x^2 + 1 > 0$. Hence, the polynomial equation $x^2 + 1 = 0$ does not have any solutions in \mathbb{R} .

Our goal is to find the "smallest" field \mathbb{C} such that

- (i) \mathbb{R} is "contained" in \mathbb{C} .
- (ii) For any polynomial $f \in \mathbb{C}$, there exists a solution for f in \mathbb{C} .

Let's assume for a moment that we CAN solve the equation $x^2 + 1 = 0$. Then we define the following set

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Note that in this set, we are using the properties of \mathbb{R} as a vector space, and using the operations defined on that vector space to define the operations of \mathbb{C} . Recall from Linear Algebra that ζ is just the span of the basis vectors 1 and i . In other words, we have

$$\zeta = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\} = \text{span}_{\mathbb{R}}\{1, i\}.$$

Now, let's define the two operations, addition and multiplication, on \mathbb{C} .

Definition (Operations on \mathbb{C}). Let $z, w \in \mathbb{C}$ and set $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We define the two operations, addition $+$ and multiplication \cdot , in the following way:

- Addition:

$$\begin{aligned} z + w &= (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) \\ &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2). \end{aligned}$$

- Multiplication:

$$\begin{aligned} z \cdot w &= (\alpha_1 + i\beta_1) \cdot (\alpha_2 + i\beta_2) \\ &= (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \beta_1\alpha_2). \end{aligned}$$

With these operations, we can say that \mathbb{C} forms a field.

Proposition. The defined operations of \mathbb{C} form a field.

Proof. To do. ■

Lemma (Existence of a Square Root). Let $\alpha + i\beta \in \mathbb{C}$. Then there exists $\gamma + i\delta \in \mathbb{C}$ such that $(\gamma + i\delta)^2 = \alpha + i\beta$.

Proof. To do. ■

0.2 Lecture 2

0.2.1 Topics

- Discuss the complex plane \mathbb{C} as a working model for complex numbers.
- Discuss the Euclidean Topology on \mathbb{C} .
- Discuss polar representation of a complex number.

0.3 Complex Plane \mathbb{C} as a working model for Complex Numbers

Lemma. Let F be a field containing \mathbb{R} and the equation $x^2 + 1 = 0$ contains a solution $i \in F$.

(i) Let

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} is a **subfield** of F .

(ii) Let F be another field containing \mathbb{R} and containing a solution i' of $x^2 + 1 = 0$ and

$$\mathbb{C}' = \{\alpha + \beta i' : \alpha, \beta \in \mathbb{R}\}.$$

Then \mathbb{C} and \mathbb{C}' are isomorphic as fields.

The second part of this lemma is simply saying that \mathbb{C} is a unique subfield of F .

0.3.1 A model for \mathbb{C}

Let $\mathbb{C} = \mathbb{R}^2$. Then define addition $+$ and multiplication \cdot as

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1),$$

respectively.

Lemma. The complex numbers \mathbb{C} have the following properties

(i) $(\mathbb{C}, +, \cdot)$ is a field.

(ii) Let $\mathcal{J} : \mathbb{R} \rightarrow \mathbb{C}$ by $x \rightarrow (x, 0)$. Then \mathcal{J} is a subfield and forms an isomorphism between \mathbb{R} and $\mathcal{J}(\mathbb{R})$.

Proof. Rough outline of proof:

- (i) Prove that $\mathcal{J}(\mathbb{R})$ is a subfield.
 - (ii) Prove that $\mathcal{J}(\mathbb{R})$ is a field homomorphism.
 - (iii) Prove that $\mathcal{J}(\mathbb{R})$ is an isomorphism; that is, show that \mathcal{J} is a bijective map.
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Definition (Constructing a solution for $x^2 + 1 = 0$). Define $i = (0, 1)$ as our imaginary number in \mathbb{C} and let $i^2 = (-1, 0)$.

Proposition. Given $\alpha, \beta \in \mathbb{R}$, show that $(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, \beta)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha, 0) + (0, 1)(\beta, 0) = (\alpha, 0) + (\beta, 0) = (\alpha, \beta).$$

■

Definition (Real and Imaginary part of Complex Number). Let $z = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Then

$$\Re(z) = \alpha \text{ and } \Im(z) = \beta,$$

are the **real and imaginary** of z , respectively. If $\Im(z) = 0$, z is a real number, and if $\Re(z) = 0$, then we call z **purely imaginary**.

Definition (Complex Conjugate). Let $z = \alpha + i\beta$ be a complex number. Its complex conjugate is defined as $\bar{z} = \alpha - i\beta$.

Geometrically, this is viewed as a reflection of the ordered pair $z = (\alpha, \beta)$ over the x -axis.

Proposition. For any $z, w \in \mathbb{C}$, we have the following properties:

- (i) $\bar{\bar{z}} = z$.
- (ii) $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
- (iii) $\overline{zw} = \bar{z} \cdot \bar{w}$.
- (iv) $\Re(z) = \frac{1}{2}(z + \bar{z})$.
- (v) $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
- (vi) $z\bar{z} = \alpha^2 + \beta^2 \geq 0$.

Proof. (i)

■

Definition (Modulus of a Complex Number). Let $z \in \mathbb{C}$. We define the **modulus** $|z| = \sqrt{z\bar{z}}$.

Proposition (More Properties of Complex Numbers). Given $z, w \in \mathbb{C}$, we have the following properties:

- (i) $|z| = 0$ if and only if $z = 0$.
- (ii) $|zw| = |z||w|$.
- (iii) $|\Re(z)| \leq |z|$.
- (iv) $|\Im(z)| \leq |z|$.
- (v) $|z + w| \leq |z| + |w|$.
- (vi) For any $z \in \mathbb{C}$ and $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

0.3.2 Viewing \mathbb{C} as a Metric Space

In \mathbb{C} , the metric we will be using is $d(z, w) = |z - w|$.

Definition (Metric Space). For $z, w \in \mathbb{C}$, we call \mathbb{C} a metric space if it satisfies the following properties:

- (i) $d(z, w) = d(w, z)$.
- (ii) $d(z, w) = 0$ if and only if $z = w$ and $d(z, w) > 0$ if and only if $z \neq w$.
- (iii) For any $z, w, u \in \mathbb{C}$, we have

$$d(z, w) \leq d(z, u) + d(u, w).$$

Now we have \mathbb{C} is a metric space endowed with the metric $d(z, w) = |z - w|$.