Homework 9 Extra Credit

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Lemma. Let $(X, \|\cdot\|)$ be a normed space and let (x_n) and (y_n) be two sequences in X where $y_n = x_{n+1} - x_n$. If $\sum_{n=1}^{\infty} y_n$ converges, then (x_n) converges.

Proof. Let $y_n = x_{n+1} - x_n$ and suppose $\sum_{n=1}^{\infty} y_n$ converges. Define

$$s_n = \sum_{k=1}^{n} y_k = \sum_{k=1}^{n} (x_{k+1} - x_k).$$

Since $\sum_{n=1}^{\infty} y_n$ converges, the sequence of partial sums (s_n) of $\sum_{n=1}^{\infty} y_n$ must converge to some $s \in X$. Our goal is to show that (x_n) converges. Observe that

$$\begin{aligned} x_1 + \sum_{k=1}^{n-1} y_k &= x_1 + \sum_{k=1}^{n-1} (x_{k+1} - x_k) \\ &= x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) \\ &= x_n. \end{aligned}$$

Hence, we have for all $n \in \mathbb{N}$ that

$$x_n = x_1 + \sum_{k=1}^{n-1} y_k.$$

Since $\sum_{n=1}^{\infty} y_n$ converges, we can use the algebraic limit theorem on the equation above to conclude that

$$\lim_{n\to\infty} x_n = x_1 + s.$$

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Problem 1. Prove that a normed space $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series is convergent.

Proof. ($\Longrightarrow$) Note that this direction was proven during class.

($\Longleftarrow$) Suppose every absolutely convergent series converges. Our goal is to show that X is a Banach Space; that is, we need to show that X is a complete normed space. Hence, we need to show that every Cauchy sequence in X is convergent. To this end, let (x_n) be a Cauchy sequence in X. Our strategy is to find a subsequence (x_{n_k}) of (x_n) that converges to some $x \in X$.

Since (x_n) is a Cauchy Sequence, there exists an $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$\|x_n - x_m\| < \varepsilon.$$

We claim that for all $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that for all $n > m > m_k$,

$$\|x_n - x_m\| < \frac{1}{2^{k-1}}.$$

Indeed, using the fact that (x_n) is Cauchy, we have

$$\begin{aligned}
&\text{For } \varepsilon = 1 \ \exists m_1 \in \mathbb{N} \text{ such that } \forall n, m > m_1,\ \|x_n - x_m\| < 1\\
&\text{For } \varepsilon = \frac{1}{2^1} \ \exists m_2 \in \mathbb{N} \text{ such that } \forall n, m > m_2,\ \|x_n - x_m\| < \frac{1}{2^1}\\
&\text{For } \varepsilon = \frac{1}{2^2} \ \exists m_3 \in \mathbb{N} \text{ such that } \forall n, m > m_3,\ \|x_n - x_m\| < \frac{1}{2^2}\\
&\vdots\\
&\text{For } \varepsilon = \frac{1}{2^{k-1}} \ \exists m_k \in \mathbb{N} \text{ such that } \forall n, m > m_k,\ \|x_n - x_m\| < \frac{1}{2^{k-1}}\\
&\vdots
\end{aligned}$$

More generally, we see that for any $k \in \mathbb{N}$, there exists m_k such that for any $n > m > m_k$

$$\|x_n - x_m\| < \frac{1}{2^{k-1}}. \tag{*}$$

In what follows, we will construct (x_{n_k}) of (x_n). For every $k \in \mathbb{N}$, choose $n_k > m_k$ defined by $n_k = m_k + 1$ such that $n_k > m_k$. Similarly, for all $k \in \mathbb{N}$, we can choose $n_{k+1} > n_k$ where $n_{k+1} = n_k + 1$. Clearly, we see that for any $k \in \mathbb{N}$, we have $n_{k+1} > n_k > m_k$. By (*), we can see that

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^{k-1}}.$$

Now, observe that

$$0 < \underbrace{\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\|}_{\text{This is a series in } \mathbb{R}} < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1.$$

Note that the above holds because of (*) and the fact that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1$ is a geometric series. Hence, we see that the series above converges absolutely. By assumption, we must have

$$\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}) \text{ converges.}$$

From the lemma above, we can see that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in X$. Using this result along with the result found in exercise 18 from homework 8, we can say that (x_n) must converge to x as well. Thus, we conclude that X is a Banach Space. ∎