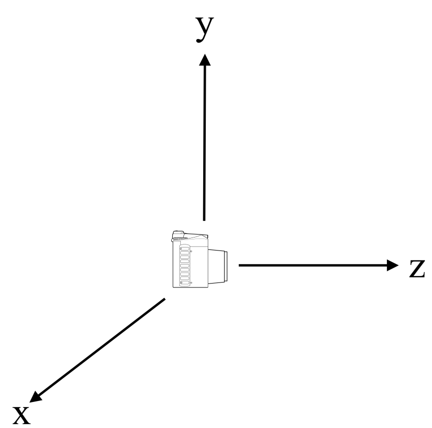
THE CAMERA MATRIX

This problem describes how to use linear algebra to describe the world as seen through a camera.

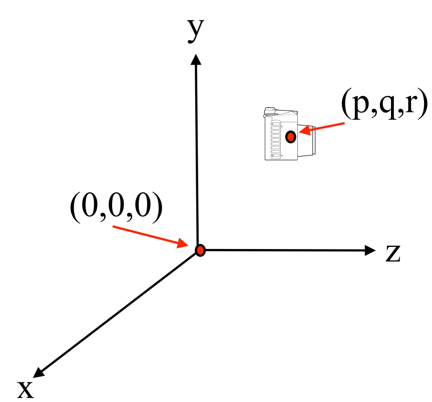
The camera matrix. When we describe coordinates (x, y, z) on 3-dimensional space, we take the y-axis to point up. We start with the camera at the origin of the (x, y, z) coordinate system, pointing along the positive z-axis (so parallel to the ground), and held in its ordinary orientation. The camera will draw the point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ at the point $\begin{bmatrix} x/z \\ y/z \end{bmatrix}$ on the image pane.



The camera starts at the origin, pointing along the z-axis

<u>Problem 1</u> Explain why $\begin{bmatrix} x/z \\ y/z \end{bmatrix}$ is the correct formula.

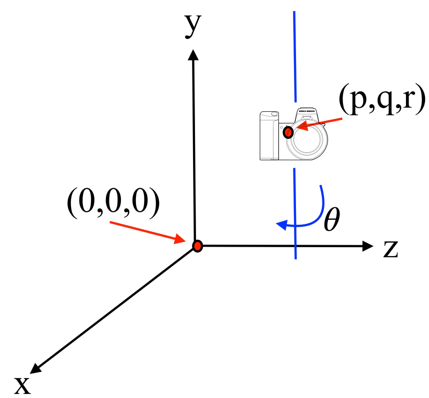
We now move the camera to position $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, while keeping it pointing in the same direction (parallel to the z axis) and held upright.



The camera has been displaced to $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, still pointing parallel to the z-axis

<u>Problem 2</u> Where will the point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be drawn now? (Note that your answer will include the variables p, q, r, x, y and z.)

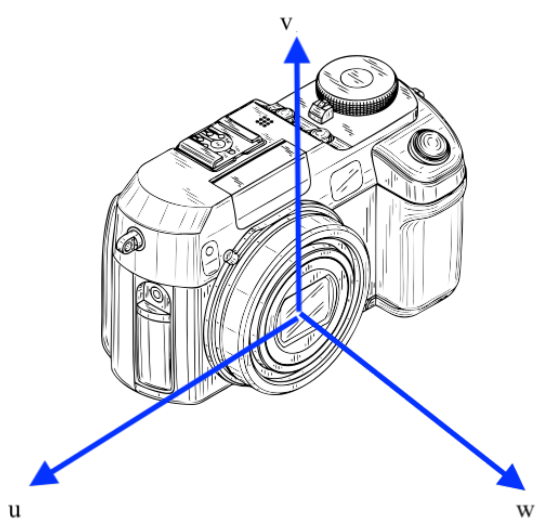
We now keep the camera at position $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and continue to hold it upright, but swivel an angle θ around the vertical axis through the camera position, turning toward the positive x-axis. In other words, we pan horizontally.



The camera has been rotated by θ around a vertical axis

Problem 3 Where will the point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be drawn now? (Note that your answer will include the angle θ, as well as the variables p, q, r, x, y and z.)

We'll now want to distinguish two coordinate systems. The first coordinate system keeps the origin at a fixed point in space, with the y-axis pointing straight up. The second coordinate system, (u, v, w), is relative to the camera. The u and v axes are in the plane of the camera, parallel to its vertical and horizontal orientation respectively, and the w-axis points straight out of the camera.



The u, v, w coordinates are centered on the camera, and relative to its orientation.

So, in the first problem, the two origins were together and $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Problem 4 Express the (u, v, w) coordinates in terms of the (x, y, z) coordinates in the scenarios of problems 2 and 3.

So the point at coordinates $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is always plotted at position $\begin{bmatrix} u/w \\ v/w \end{bmatrix}$.

When you considered translating the camera to position $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, you should have found that the formulas expressing the (u, v, w) coordinates in terms of the (x, y, z) coordinates were not purely linear. To fix this, we introduce the notion of the **camera matrix** P. This is a 3×4 matrix P such that

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

Problem 5 Compute the camera matrices for the two scenarios in the previous problems: The camera translated to $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, and the camera translated to that position and then rotated θ around a vertical axis.

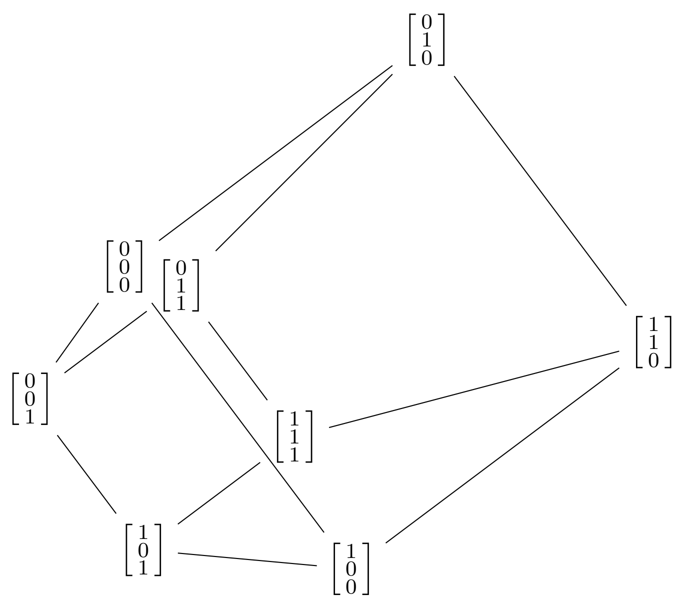
Reconstructing the camera matrix. Suppose we know the coordinates of a number of points in space, and we would like to determine where the camera was placed.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and that the camera plots them in positions

$$\begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0.25 \\ 0.35 \end{bmatrix}, \begin{bmatrix} 1.3 \\ 1.3 \end{bmatrix}, \begin{bmatrix} 0.65 \\ 0.65 \end{bmatrix}, \begin{bmatrix} 1.1 \\ -0.1 \end{bmatrix}, \begin{bmatrix} 0.55 \\ -0.05 \end{bmatrix}, \begin{bmatrix} 1.9 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.95 \\ 0.25 \end{bmatrix}.$$

respectively.



Problem 6 Find the camera matrix. You should need to solve 16 linear equations in 12 variables. Is there a unique solution? Why or why not? You should use software to solve these equations (`Maple`, `MATLAB`, `Mathematica`, `num.py`, etcetera), your solution should make it clear what equations you solved, how you solved them, and what answer you got.

Problem 7 Find the location of the camera in (x, y, z) coordinates.