Recap: $\underset{x}{\text{minimize}}\ f(x)$ / $\text{maximize}\ g(x)$

TODAY $f: \mathbb{R} \to \mathbb{R}$

(subject to $x \in X$) ← constraint set

Examples:

minimize ... costs

distance

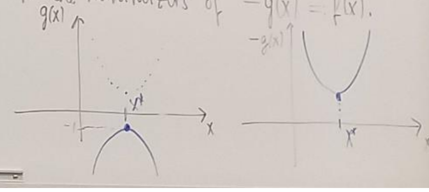
time

labor/effort/energy

waste

risk

Sidenote: We focus on the minimization formulation, as the maximizers of a given function $g(x)$ are equal to the minimizers of $-g(x) = f(x)$.

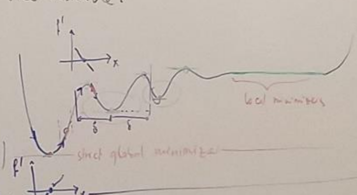


Goal today: Gentle introduction via consideration of $f: \mathbb{R} \to \mathbb{R}$, function of one variable.

Definition: Let $I \subseteq \mathbb{R}$ and consider $f: I \to \underline{\mathbb{R}^1}$ (functional).

A point $x^* \in I$ is called:

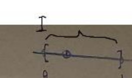
1. a global minimizer of f on I if: $\forall x \in I \quad f(x^*) \leq f(x)$
2. a strict global min. of f on I if: $\forall x \in I,\ x \neq x^* \quad f(x^*) < f(x)$
3. a local minimizer of f: $\exists \delta > 0 \quad \forall x \in \underbrace{(x^*-\delta, x^*+\delta)}_{B_\delta(x^*)} \subset I \quad f(x^*) \leq f(x)$
4. a strict local min. of f: $\exists \delta > 0 \quad \forall x \in (x^*-\delta, x^*+\delta) \subset I,\ x \neq x^* \quad f(x^*) < f(x)$



5. A point $x^* \in I$ is called a stationary point if $f'(x^*) = 0$ (assuming that f is differentiable)

Theorem (First-order necessary condition) **Suppose f is continuously differentiable on I.**

If a point x^* in the interior of I is a local minimizer of f, then $f'(x^*) = 0$.



$\text{int}(I) = (a,b) = \{x : a < x < b\}$

$\partial I = \{a, b\}$

Proof: Recall: $f'(x) = \lim_{x \to x^*} \frac{\overbrace{f(x) - f(x^*)}^{\geq 0}}{x - x^*}$

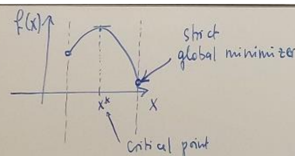
Since x^* is a local minimizer of f: $\exists \delta > 0 \quad \forall x \in (x^*-\delta, x^*+\delta) \quad \underbrace{f(x^*) \leq f(x)}_{f(x) - f(x^*) \geq 0}$

Case 1: $x < x^* \Leftrightarrow x - x^* < 0 \longrightarrow f'(x^*-) \leq 0$

Case 2: $x > x^* \Leftrightarrow x - x^* > 0 \longrightarrow f'(x^*+) \geq 0$

By continuity of f': $f'(x^*-) = f'(x^*+) = 0$.

Example:



$\Rightarrow$ always check boundary points after checking the stationary pt's.

Next: Sufficient conditions for when stationary points are in fact minimizers $\rightsquigarrow$ Taylor expansion (HW1)

Theorem (Second-order sufficient conditions): Suppose f, f', f'' are continuous on I ($f \in C^2(I, \mathbb{R})$) and $x^* \in I$ is a stationary point of f.

1. If $f''(x) \geq 0 \ \forall x \in I$, then x^* is a global minimizer of f on I.
2. If $f''(x) > 0 \ \forall x \in I, x \neq x^*$, then x^* is a strict global minimizer of f on I.
3. $f''(x^*) > 0$, then x^* is strict local minimizer of f.

Proof of 2nd-order suff. conditions: If $x \in I$, $x \neq x^*$, Taylor's formula yields

$$f(x) = f(x^*) + \underbrace{f'(x^*)}_{=0}(x - x^*) + \frac{1}{2} f''(z)(x - x^*)^2 \quad \text{with } z \text{ between } x \text{ and } x^*.$$

1. $f''(x) \geq 0 \ \forall x \in I$ implies $f''(z) \geq 0$ also. Moreover, x^* is a stationary point $\Leftrightarrow f'(x^*) = 0$

$$\rightsquigarrow f(x) = f(x^*) + \frac{1}{2} \underbrace{f''(z)}_{\geq 0} \underbrace{(x - x^*)^2}_{\geq 0}$$

$$f(x) = f(x^*) + \text{"something} \geq 0\text{"} \Rightarrow f(x^*) \leq f(x) \ \forall x \in I.$$

2. $x \neq x^*$ means $(x - x^*)^2 > 0$. Moreover $f''(z) > 0$ (assuming $z \neq x^*$)

$$\rightsquigarrow f(x) = f(x^*) + \text{"something} > 0\text{"} \Rightarrow f(x^*) < f(x) \ \forall x \in I, x \neq x^*.$$

3. If $f''(x^*) > 0$, the continuity of f'' implies that $\exists \delta > 0 \ f''(x) > 0 \ \forall x \in (x^* - \delta, x^* + \delta) \subset I \Rightarrow f(x) > f(x^*) \ \forall x \in (x^* - \delta, x^* + \delta) \subset I$.

Example: $f(x) = 3x^4 - 4x^3 + 1$.

- find stationary pt's: $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1) \rightsquigarrow x = 0 \ \& \ x = 1$ stationary points
- analyze stat. pt's: $f''(x) = 36x^2 - 24x = 12x(3x - 2)$

Case 1: $x = 1$: $f''(1) = 12 > 0 \Rightarrow$ strict local min.

Case 2: $x = 0$: $f''(0) = 0 \rightsquigarrow$ Then our 2nd order suff. condition doesn't provide information.