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Homework 4 – Solutions

P1 $\quad Y \subset \mathbb{R}^m$ is convex $\overset{def.}{\iff} \forall_{\substack{y_1, y_2 \in Y \\ \alpha \in [0,1]}} \ \alpha y_1 + (1-\alpha) y_2 \in Y.$

Take two arbitrary $\underbrace{x_1, x_2 \in A^{-1}(Y)}$.

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$$\Rightarrow \begin{cases} A x_1 =: y_1 \in Y \\ A x_2 =: y_2 \in Y \end{cases}$$

$$\forall_{\alpha \in [0,1]} : \underbrace{\alpha y_1 + (1-\alpha) y_2}_{\underbrace{\alpha A x_1 + (1-\alpha) A x_2}_{A(\alpha x_1 + (1-\alpha) x_2) \ \text{Linearity}}} \in Y$$

This means

$$\forall_{\alpha \in [0,1]} \quad A(\alpha x_1 + (1-\alpha) x_2) \in Y$$

$$\iff \forall_{\alpha \in [0,1]} \quad \alpha x_1 + (1-\alpha) x_2 \in A^{-1}(Y).$$

Recall that $x_1, x_2 \in A^{-1}(Y)$ were picked arbitrarily. Thus

$$\forall_{\substack{x_1, x_2 \in A^{-1}(Y) \\ \alpha \in [0,1]}} \quad \alpha x_1 + (1-\alpha) x_2 \in A^{-1}(Y)$$

Hence $A^{-1}(Y)$ is convex.

P1 (Cont'd) Let $y_1, y_2 \in A(X)$ be two elements in the image of X under A.

$\exists\, x_1, x_2 \in X$ $\quad$ $Ax_1 = y_1$, $Ax_2 = y_2$.

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Moreover X is convex, so

$$\forall \alpha \in [0,1] \quad \alpha x_1 + (1-\alpha) x_2 \in X.$$

Applying A onto $\alpha x_1 + (1-\alpha) x_2$ yields

$$A(X) \ni A(\alpha x_1 + (1-\alpha) x_2) \underset{\text{Linearity}}{=} \alpha \underbrace{Ax_1}_{y_1} + (1-\alpha) \underbrace{Ax_2}_{=y_2}$$

Thus $\alpha y_1 + (1-\alpha) y_2 \in A(X)$ and so we have shown that $A(X)$ is a convex set.

P2 use $g(\alpha x+(1-\alpha)y) \leq \alpha g(x) + (1-\alpha) g(y)$

with $\underbrace{g(x) = e^{x^2}}$ and $\alpha = \frac{1}{4}$.

this can be shown to be convex via the $(Hg)(x) \succeq 0$ test:

$$g'(x) = 2x e^{x^2}$$
$$g''(x) = 2(2x^2+1) e^{x^2} \geq 0$$

Thus $g(\frac{1}{4}x + \frac{3}{4}y) \leq \frac{1}{4} g(x) + \frac{3}{4} g(y)$

$$\underset{g(x)=e^{x^2}}{\Leftrightarrow} \quad e^{(\frac{1}{4}x+\frac{3}{4}y)^2} \leq \frac{1}{4} e^{x^2} + \frac{3}{4} e^{y^2} \quad \Big| \log(\dots)$$

$$(\frac{1}{4}x + \frac{3}{4}y)^2 \leq \log(\frac{1}{4} e^{x^2} + \frac{3}{4} y^2) \quad \Big| \sqrt{(\dots)}$$

$$\frac{1}{4}x + \frac{3}{4}y^2 \leq \sqrt{\log(\frac{1}{4} e^{x^2} + \frac{3}{4} y^2)}$$

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P3. Similarly to P2 but with 4-point convex combo:

$\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{12}$, $\alpha_4 = \frac{1}{12}$. Note $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$.

$$g\left(\sum_{i=1}^{4} \alpha_i x_i\right) \leq \sum_{i=1}^{4} \alpha_i \, g(x_i)$$

$$\boxed{\begin{array}{l} x_1 = x \\ x_2 = y \\ x_3 = z \\ x_4 = w \end{array}}$$

and $g(u) = u^4$ (convex since $g''(u) = 12x^2 \geq 0$, yields the claim.

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P4: Consider
$$f(x) = \frac{1}{x} + x$$
as a cost functional.

$$f'(x) = -\frac{1}{x^2} + 1$$ critical point at $x = 1$.

$$f''(x) = \frac{2}{x^3} > 0 \quad \forall_{x>0}$$

$\Rightarrow$ $x = 1$ is global minimizer of f with domain $\{x \in \mathbb{R} \mid x > 0\}$.

$$f(1) = \frac{1}{1} + 1 = 2$$

$$\Rightarrow \forall_{x>0} \quad \frac{1}{x} + x \geq 2$$ as claimed.

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P5 $\quad f$ convex $\overset{\text{def.}}{\iff}$ $\forall x_1, x_2 \in C$, $\alpha \in [0,1]$

$$f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Since φ is an increasing function (non-decreasing)

$$a \leq b \implies \varphi(a) \leq \varphi(b)$$

we have with $a := f(\alpha x_1 + (1-\alpha) x_2)$
$\qquad b := \alpha f(x_1) + (1-\alpha) f(x_2)$

two points in dom(φ)

$$\underbrace{\varphi(f(\alpha x_1 + (1-\alpha) x_2))}_{= (\varphi \circ f)(\alpha x_1 + (1-\alpha) x_2)} \leq \underbrace{\varphi(\alpha f(x_1) + (1-\alpha) f(x_2))}_{\leq \alpha \underbrace{\varphi(f(x_1))}_{(\varphi \circ f)(x_1)} + (1-\alpha) \underbrace{\varphi(f(x_2))}_{(\varphi \circ f)(x_2)}}$$

by definition of the composition of two functions

Thus

two points in dom($\varphi \circ f$)

$$\underbrace{(\varphi \circ f)}_{=g}(\alpha x_1 + (1-\alpha) x_2) \leq \alpha \underbrace{(\varphi \circ f)}_{=g}(x_1) + (1-\alpha) \underbrace{(\varphi \circ f)}_{=f}(x_2)$$

Hence $g = \varphi \circ f$ is convex.

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P5 (cont'd):

$$\left.\begin{array}{l} f(x) = \|x\|^2 \text{ convex (obviously, but can be checked, doesn't hurt)} \\ \varphi(t) = e^t \text{ convex and increasing} \\ \Rightarrow g(x) = (\varphi \circ f)(x) = e^{\|x\|^2} \text{ is convex.} \end{array}\right\} (*)$$

Hessian test route:

$$\nabla g(x) = 2x\, e^{\|x\|^2}$$

$$(Hg)(x) = \text{Jacobian}(\nabla g(x)) = \underset{\substack{\uparrow \\ \text{some rules}}}{\dots} = \underline{2e^{\|x\|^2} I + 4xx^T e^{\|x\|^2}}$$

rather hard, or at least harder to see, that this is $\succcurlyeq 0$.

Anyways, it is, but even if we can't see that, we got an answer via the approach (*).

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