ESE 415: Optimization (FL2024) Instructor: Shen Zeng
https://wustl.instructure.com/courses/132084 **Due:** September 9, 2024

Homework 2

1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and consider the following situation: The function f possesses a strict local minimum $x^\star$ which is not a global minimum, i.e., there exists a $y \in \mathbb{R}$ (assume without loss of generality that $x^\star < y$) for which $f(y) < f(x^\star)$. Provide a sketch of the situation and show that f must possess another critical point. The following results may be helpful: Intermediate Value Theorem, Rolle's Theorem.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = e^{3x} + y^3 - 3e^x y.$$

   (a) Show that f has exactly one critical point.

   (b) Show that the critical point is a local minimizer but not a global minimizer of f.

   (This shows that the property found in the foregoing problem is in fact a property that is unique to the one-dimensional setting.)

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}, \ \ (x, y) \mapsto x^2 - 5xy^2 + 5y^4$.

   (a) Determine all critical points of f.

   (b) Show that $\bar{x} = 0$ is a strict global minimum of $x \mapsto f(x, 0)$ and that $\bar{y} = 0$ is a strict global minimum of $y \mapsto f(0, y)$.

   (c) Show that $(\bar{x}, \bar{y}) = (0, 0)$ is not a local minimum of f.

4. Consider the feasible region defined by the constraints

$$\begin{aligned} 1 - x_1^2 - x_2^2 &\geq 0, \\ \sqrt{2} - x_1 - x_2 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

   For each of the following points, determine whether the point is a feasible point or not, and, if feasible, whether it is in the interior of the feasible set or on the boundary:

   $x_a = (1/2, 1/2), \quad x_b = (1, 0), \quad x_c = (-1, 0), \quad x_d = (-1/2, 0), \quad x_e = (1/\sqrt{2}, 1/\sqrt{2}).$

5. Consider $f : \mathbb{R}^n \to \mathbb{R}$ and consider $x^\star \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Define a new function

$$h : \mathbb{R} \to \mathbb{R}, \qquad h(t) = f(x^\star + tv).$$

   (a) Provide an illustration for $n = 2$ that shows the relation between f and h.

   (b) Calculate $h'(t)$ and $h''(t)$ and evaluate both derivatives at $t = 0$.

   (c) Assume that $(\nabla f)(x^\star) = 0$ and that the Hessian of f at $x^\star$ has the property that $v^\top (Hf)(x^\star) v < 0$. Can $x^\star$ be a (local) minimizer of f under these circumstances?

   (d) The so-called second-order necessary condition for optimality states that for $x^\star$ to be a local minimizer of a function $f \in C^2(\mathbb{R}^n, \mathbb{R})$, both $(\nabla f)(x^\star) = 0$ and $(Hf)(x^\star) \succcurlyeq 0$ have to hold. Provide a proof of the second-order necessary condition using your insights from the previous subproblems.

6. Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$ that is positive semi-definite. It is known that all eigenvalues of A are real and non-negative: $\lambda_i \geq 0$ for all $i = 1, \ldots, n$. It is further known that as a symmetric matrix A, there exists an orthogonal matrix U (meaning $U^\top U = I$) so that $A = U \Lambda U^\top$, where Λ is a diagonal matrix containing the eigenvalues of A on the diagonal.

   (a) Show that for any $x \in \mathbb{R}^n$, we have that $x^\top A x = y^\top \Lambda y$, where $y = U^\top x$.
   What can be said about the relation between the norm of x and the norm of y?

   (b) Let $\lambda_{\min} := \min\{\lambda_1, \ldots, \lambda_n\}$. Show that $y^\top \Lambda y \geq \lambda_{\min} \|y\|^2$. Combine this result with the insights of the previous subproblem to establish that $x^\top A x \geq \lambda_{\min} \|x\|^2$.