Homework 4

1. $A \in R^{m \times n}, Y \subset R^m, X \subset R^n$

$A^{-1}(Y) = \{x \in R^n : Ax \in Y\}$, $A(X) = \{Ax : x \in X\}$

(1) Since Y is convex set, $\forall y_1, y_2 \in Y$, $\forall \alpha \in [0,1]$, $\alpha y_1 + (1-\alpha) y_2 \in Y$

Suppose , $x_1, x_2 \in A^{-1}(Y)$, $Ax_1, Ax_2 \in Y$.

Consider $\lambda x_1 + (1-\lambda) x_2$, $\lambda \in [0,1]$

$$A(\lambda x_1 + (1-\lambda) x_2) = \lambda A x_1 + (1-\lambda) A x_2$$

Since $Ax_1, Ax_2 \in Y$ and Y is convex, it follows that

$$\lambda A x_1 + (1-\lambda) A x_2 \in Y$$

Thus, $\lambda x_1 + (1-\lambda) x_2 \in A^{-1}(Y)$, which means that $A^{-1}(Y)$ is convex

(2) Since X is convex, and let $y_1, y_2 \in A(X)$,

$\exists x_1, x_2 \in X$ such that $y_1 = Ax_1$, $y_2 = Ax_2$

And for $\forall \lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda) x_2 \in X$

$$A(\lambda x_1 + (1-\lambda) x_2) = \lambda A x_1 + (1-\lambda) A x_2 = \lambda y_1 + (1-\lambda) y_2$$

Thus, $\lambda y_1 + (1-\lambda) y_2 \in A(X)$, which means $A(X)$ is convex

2. Let $g(x) = e^{x^2}$, This is a convex function because

$g'(x) = 2x e^{x^2}$, $g''(x) = e^{x^2}(2+4t^2) > 0$

Let $\alpha = \frac{1}{4}$, $1-\alpha = \frac{3}{4}$

Based on $g(\alpha x + (1-\alpha) y) \leq \alpha g(x) + (1-\alpha) g(y)$, we have

$$e^{(\frac{x}{4} + \frac{3}{4} y)^2} \leq \frac{1}{4} e^{x^2} + \frac{3}{4} e^{y^2}$$

$$(\frac{x}{4} + \frac{3}{4} y)^2 \leq \log(\frac{1}{4} e^{x^2} + \frac{3}{4} e^{y^2})$$

$$\frac{x}{4}+\frac{3}{4}y \leq \sqrt{\log\left(\frac{1}{4}e^{x^2}+\frac{3}{4}e^{y^2}\right)}$$

3. Based on the theorem, we can get an inequality

$$g(\alpha_1 x+\alpha_2 y+\alpha_3 z+\alpha_4 w) \leq \alpha_1 g(x)+\alpha_2 g(y)+\alpha_3 g(z)+\alpha_4 g(w)$$

where $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$

Let $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{1}{3}$, $\alpha_3=\frac{1}{12}$, $\alpha_4=\frac{1}{12}$, and $g(x)=x^4$ which is a convex function, then we will have

$$g\left(\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{12}z+\tfrac{1}{12}w\right) \leq \tfrac{1}{2}g(x)+\tfrac{1}{3}g(y)+\tfrac{1}{12}g(z)+\tfrac{1}{12}g(w)$$

$$\left(\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{12}z+\tfrac{1}{12}w\right)^4 \leq \tfrac{1}{2}x^4+\tfrac{1}{3}y^4+\tfrac{1}{12}z^4+\tfrac{1}{12}w^4$$

4. Define the function $f(x)=\frac{1}{x}+x$, $x>0$

$f'(x)=-\frac{1}{x^2}+1$, $\quad f'(x)=0 \Rightarrow x=1$

$f''(x)=\frac{2}{x^3}>0$

This means that f is convex and $x=1$ is global minimum for $f(x)$

$f(1)=\frac{1}{1}+1=2$

Thus, $\frac{1}{x}+x \geq 2$ for all $x>0$

5. Suppose that f is a convex function with a convex domain C and φ is an increasing convex function.

$$\forall x_1, x_2 \in C,\ \forall \lambda \in [0,1],\ f(\lambda x_1+(1-\lambda)x_2) \leq \lambda f(x_1)+(1-\lambda)f(x_2)$$

Since φ is increasing convex function, we have

$$\varphi(f(\lambda x_1+(1-\lambda)x_2)) \leq \varphi(\lambda f(x_1)+(1-\lambda)f(x_2))$$

$$\varphi(f(\lambda x_1+(1-\lambda)x_2)) \leq \lambda\varphi(f(x_1))+(1-\lambda)\varphi(f(x_2))$$

$$g(\lambda x_1+(1-\lambda)x_2) \leq \lambda g(x_1)+(1-\lambda)g(x_2)$$

Thus, $g = \varphi\circ f$ is convex.

Suppose $g(x) = e^{\|x\|^2}$.

$f(x) = \|x\|^2$ is convex function, because

$f(x) = x_1^2+\cdots+x_n^2$, $Hf(x) = 2I$ which is positive definite

$\varphi(x) = e^x$ is increasing convex function

Thus, $g = \varphi\circ f(x) = e^{\|x\|^2}$ is convex function

Since $g(x) = e^{\|x\|^2}$, $\nabla g(x) = 2xe^{\|x\|^2}$;

$$Hg(x) = 2e^{\|x\|^2}I + 4xx^Te^{\|x\|^2}$$

$2e^{\|x\|^2}I$ is positive definite because $e^{\|x\|^2}>0$ and $4xx^Te^{\|x\|^2}$ is positive semidefinite because $4e^{\|x\|^2}>0$ and xx^T is rank-1 positive semidefinite matrix

Thus, $Hg(x)$ is positive definite and $g(x)$ is convex