ESE 415: Optimization (FL2024) Instructor: Shen Zeng
https://wustl.instructure.com/courses/132084 **Due:** November 6, 2024

Homework 5

1. Consider the following system of linear equations

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 3 \\ x_1 + 3x_2 &= 3. \end{aligned}$$

   (a) Demonstrate both analytically and graphically that this system of linear equations is inconsistent, i.e., there exists not a single $(x_1, x_2) \in \mathbb{R}^2$ that satisfies the three linear equations all at the same time.

   (b) While the original problem is unsolvable, the problem can be reformulated to be solved in a least-squares sense. Provide the least squares formulation and the solution to the least-squares optimization problem.

2. (**Programming Problem**) Load `least_squares_data.mat` into your Matlab workspace and run the following script to obtain a visualization of the data:

```
plot(0:dt:9.99,y_meas_noise,'r.');
xlabel('Time'); ylabel('Noisy position measurement');
```

   The data has been obtained from measuring the x-position of a moving object with a constant velocity. The measurement process has intorduced noise into the data.

   Use least-squares to obtain the best-approximation of the position-over-time signal which is supposed to be a straight line. What is the best estimate for the velocity? Plot your best-approximation line together with the data points to obtain a visualization.

3. Let $A \in \mathbb{R}^{n\times m}$ be a matrix with m linearly independent columns that constitute a basis for a subspace $X \subset \mathbb{R}^n$ of dimension m. Show that

$$X = \text{range}(A) := \{x \in \mathbb{R}^n : x = Au \text{ for some } u \in \mathbb{R}^m\}.$$

   Remark: This means that for any subspace X of $\mathbb{R}^n$ of dimension m, there exists $A \in \mathbb{R}^{n\times m}$ with m linearly independent columns such that the subspace X can be expressed as the range space of A. This is a useful result that we will make use of.

4. The orthogonal complement of a subspace X of $\mathbb{R}^n$ is defined as

$$X^\perp := \{y \in \mathbb{R}^n : x^\top y = 0 \text{ for all } x \in X\}.$$

(a) Does $X^\perp$ have a vector space structure, i.e., is $X^\perp$ a subspace of $\mathbb{R}^n$?

(b) Show that $(X^\perp)^\perp = X$.

5. Let $A \in \mathbb{R}^{n\times m}$ be a matrix whose columns constitute a basis for a subspace $X \subset \mathbb{R}^n$. The orthogonal projection operator from $\mathbb{R}^n$ to X is given as the $n \times n$-matrix

$$P_X := A(A^\top A)^{-1}A^\top.$$

Consider the subspace $X = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and two different matrices

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Verify that $X = \text{range}(A_1) = \text{range}(A_2)$ and that the resulting P_X in those two different cases are identical, i.e., P_X is indeed independent of the parametrization of X through range(A) (as it should be).

6. Find the point on the intersection of the two planes given by

$$x_1 + x_2 + x_3 = 1 \qquad \text{and} \qquad -x_1 - x_2 + x_3 = 0$$

that is closest to the origin.