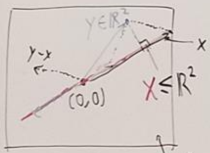
Underdetermined Systems: (Orthogonal) Projections & "Best Approximations"

Two problem types:

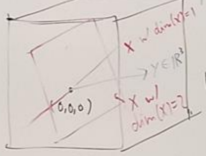
$y \in \mathbb{R}^n$ given

$$\underset{x}{\text{minimize}} \quad \|y - x\|^2$$
$$\text{subject to} \quad x \in X$$

subspace of $\mathbb{R}^n$



$n=2$



$n=3$

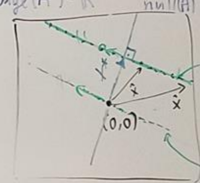
$(x, y \in X \Rightarrow \alpha x + \beta y \in X)$

$$\text{minimize} \quad \|x\|^2$$
$$\text{subject to} \quad x \in X = \underbrace{\{ x \in \mathbb{R}^n : Ax = b \}}_{\text{Solution set of } Ax=b \;\; Sol(A,b)}$$

$[A][x] = [b]$

homogeneous eq. $Ax = 0$

$\text{null}(A) \overset{\perp}{\oplus} \text{range}(A^T) = \mathbb{R}^n$



$Sol(A,b) = \{\hat{x}\} + \text{null}(A)$
$= \{x \in \mathbb{R}^2 : Ax = b\}$

$\text{null}(A)$ ... "parallel" to $Sol(A,b)$

$n=2$

Projections onto Subspaces

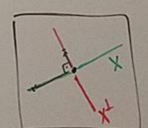
Def.: Let $X \subseteq \mathbb{R}^n$ be a subspace. The orthogonal complement of X is defined as

$$X^\perp := \{ y \in \mathbb{R}^n : \underbrace{x^T y}_{=\langle x, y \rangle} = 0, \; \forall_{x \in X} \}$$

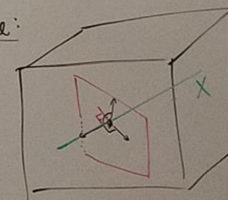
As a direct consequence:

$$\forall_{\substack{x \in X \\ y \in X^\perp}} \quad x^T y = 0$$

Example:



$n=2$



$n=3$

Extremely important result in (Linear) Optimization:

Theorem (Projection Theorem): $X \subseteq \mathbb{R}^n$ subspace. For any $y \in \mathbb{R}^n$ there exists a unique $x^* \in X$ such that

$$\|y - x^*\| \le \|y - x\| \quad \forall x \in X.$$

The point $x^* \in X$ is uniquely characterized by $(y - x^*) \in X^\perp$.

We call x^* the (orthogonal) projection of y onto X.

$$x^* = \arg\min_{x \in X} \|\underbrace{y - x}_{\text{error vector } e := y - x}\|^2$$

Optimal error $e^* = y - x^*$, and $e^* \perp X$.

Proof: $\exists\, A \in \mathbb{R}^{n \times m}$ $X = \mathrm{range}(A)$. Then, observe that

$$\|y - x^*\| \le \|y - x\| \quad \forall x \in X \iff \underbrace{\|y - \overset{= x^*}{\widetilde{Au^*}}\|^2}_{} \le \|y - Au\|^2 \quad \forall u \in \mathbb{R}^m$$

$m \le n$ ↑ $\dim(X)$

$$\min_u \|\underset{\substack{\text{given}\\\text{tall matrix}}}{A}u - \underset{\text{given}}{y}\|^2$$

This is a overdermined LS problem!

Unique sol'n: $u^* = A^\dagger y = (A^T A)^{-1} A^T y$

$$\Rightarrow x^* = AA^\dagger y = \underbrace{A(A^T A)^{-1} A^T}_{=: P_X} y.$$

Next: we show $y - x^* \perp X \iff y - x^* \in X^\perp$. Take an arbitrary $z \in X$. Then:

$y - z \in X^\perp$ is true if and only if $(y - z)^T x = 0 \ \forall x \in X \iff (y - z)^T A u = 0 \ \forall u \in \mathbb{R}^m$

Since we can also express $z = Av$ for some $v \in \mathbb{R}^m$, the above relationships are satisfied if and only if

$$0 = (y - Av)^T A u = y^T A u - v^T A^T A u = \underbrace{(y^T A - v^T A^T A)\, u \quad \forall u \in \mathbb{R}^m}$$

$\Rightarrow A^T y - A^T A v = 0$ NORMAL EQUATION

Thus $(y - z) \in X^\perp$ iff $z = Av$ with v as the unique sol'n of

$$v^* = \underbrace{(A^T A)^{-1} A^T}_{A^\dagger} y$$

$$\Rightarrow y - z \in X^\perp \text{ iff } z = Av^* = AA^\dagger y.$$

Comparing with $x^* = AA^\dagger y$, we can conclude that indeed $z = x^*$. ∎