Homework 1

1. Indicate whether you satisfy the course requirements

   - Introduction to Computer Science or equivalent,
   - Matrix Algebra or equivalent,
   - Engineering Mathematics A or equivalent.

   If you have not taken WashU courses CSE 131, Math 309, ESE 318 for the requirement, list the equivalent courses you have taken (as well as the institution you have taken the courses at). If you do not satisfy the course requirements, either indicate that you have already gotten instructor approval or use your answer to seek approval from the instructor (you will see the instructor decision once you receive the graded submission).

2. Consider the following fictitious example. There are 5000 units of wood and 1500 units of labor available. Let $x_1, \dots, x_4$ represent the number of different furniture:

   $$x_1 \text{ for nightstands, } x_2 \text{ for chairs, } x_3 \text{ for bookshelves, } x_4 \text{ for dining tables,}$$

   selling for \$100, \$150, \$200, \$400, respectively. Building a nightstand shall require 10 units of wood and 2 units of labor. Building a chair shall require 12 units of wood and 4 units of labor. Building a bookshelf shall require 25 units of wood and 8 units of labor. Building a dining table shall require 20 units of wood and 12 units of labor.

   Assuming that every assembled furniture is going to be bought up by the market for the suggested prices (actual demand of the buyers can be neglected), how should the quantities $x_1, \dots, x_4$ be chosen so as to maximize the sale profit while staying within the limitations set by the resources wood and labor? Formulate an optimization problem. What class of optimization problems does your problem fall into?

3. Consider N points (x_i, y_i), $i = 1, \dots, N$, in a two-dimensional plane. The problem we will consider is that of finding a circle of minimal radius that contains all N points. Formulate the problem as a constrained optimization problem with linear cost functional and quadratic constraints. Describe the solution / solution method.

4. Recall Taylor's Theorem in one variable, which states the following.

Taylor's Theorem in one variable. Let $I \subset \mathbb{R}$ be an open interval and let $x^\star \in I$. Let $f : I \to \mathbb{R}$ be $n+1$ times continuously differentiable. Then, for any $x \in I$

$$f(x) = f(x^\star) + f'(x^\star)(x - x^\star) + \frac{f''(x^\star)}{2!}(x - x^\star)^2 + \cdots + \frac{f^{(n)}(x^\star)}{n!}(x - x^\star)^n + R_n(x)$$

with

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - x^\star)^{n+1}$$

where z is a point between x and $x^\star$.

(a) Consider the case $n = 1$ and show that if $f'(x^\star) = 0$ and furthermore $f''(x) > 0$ for all $x \in I$, then $x^\star$ is a strict global minimizer of f on the open interval I.

(b) Apply the above result on the example of $f(x) = e^{x^2}$.

5. Draw the graph of a function $f : \mathbb{R} \to \mathbb{R}$ that has a strict global minimum, two strict local minima and infinitely many nonstrict local minima.

6. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (xz + y^2, \sin(xyz) + z).$$

(a) Compute the Jacobian matrix of f at the point $(x, y, z) = (0, -1, 1)$.

(b) Are there directions in which the directional derivative of f at $(0, -1, 1)$ is zero?