ESE 415: Optimization (FL2024) Instructor: Shen Zeng
https://wustl.instructure.com/courses/132084 **Due:** October 21, 2024

# Homework 4

1. Let $A \in \mathbb{R}^{m \times n}$ and let $Y \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^n$ be sets. Recall the sets

   - $A^{-1}(Y) := \{x \in \mathbb{R}^n : Ax \in Y\}$,
   - $A(X) := \{Ax : x \in X\}$,

   called the pre-image of $Y$ under $A$, and the image of $X$ under $A$, respectively.

   Show that the pre-image of a convex set $Y$ under $A$ and the image of a convex set $X$ under $A$ are both convex sets again.

2. Consider the following inequality for all positive numbers $x$ and $y$

   $$\frac{x}{4} + \frac{3y}{4} \leq \sqrt{\log\left(\frac{e^{x^2}}{4} + \frac{3}{4}e^{y^2}\right)}.$$

   Show the validity of this inequality with an argument that uses the convexity

   $$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$$

   of an appropriate function $g$ and a suitable choice for $\alpha$.

3. Consider the inequality

   $$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4.$$

   Show the validity of this inequality with a convexity-based argument.

4. It is known that the inequality

   $$\frac{1}{x} + x \geq 2$$

   holds for all $x > 0$. Use an approach pertaining to the methods of optimization to show the validity of the inequality.

5. Show the following result: If $f$ is a convex function (with a convex domain) and if $\varphi$ is an increasing convex function, then the composite function $g = \varphi \circ f$ is convex.

   Use the result to show the convexity of the function $g(x) = e^{\|x\|^2}$. Discuss whether the Hessian test would have yielded a conclusive result.