100/100

Homework 5 – Solutions

1. (a) 10/10

Analytically: First two eq's have unique sol'n

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ can be "guessed"}$$

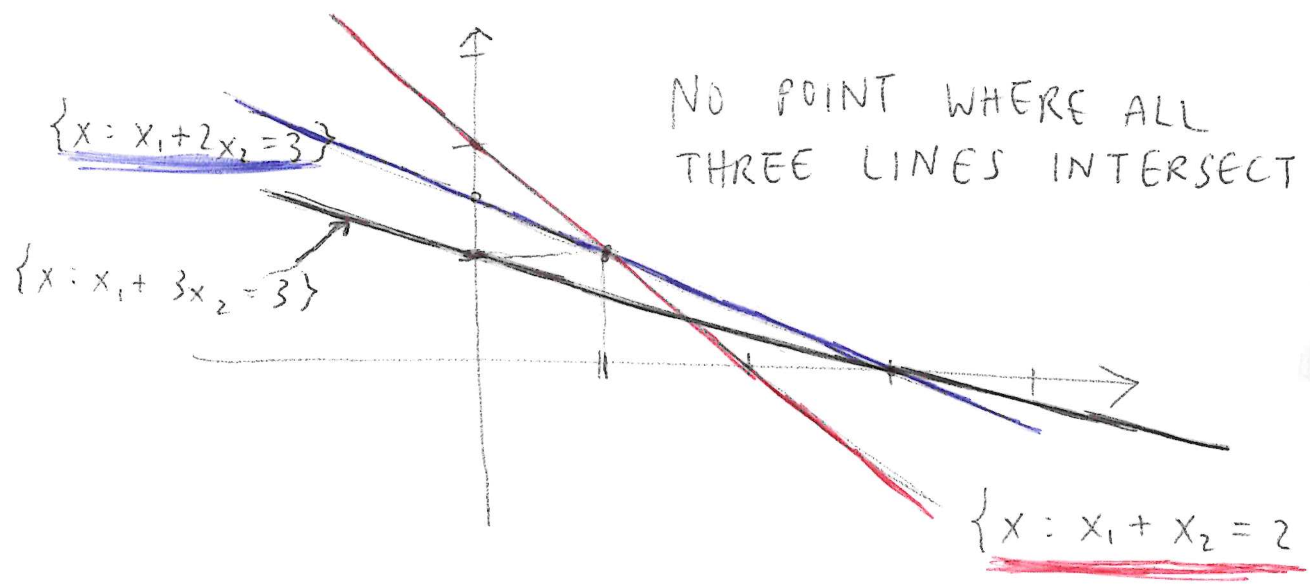
invertible, since linearly independent rows/ columns

But $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ plugged into third eq'n left-hand side yields

$$x_1 + 3x_2 = 1 + 3 = 4 \neq 3 \quad ↯$$

5

Graphically:



NO POINT WHERE ALL THREE LINES INTERSECT

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1 (b): Least squares formulation:

10/10

$$\min_{x \in \mathbb{R}^2} \left\| \underbrace{\begin{pmatrix} 1, & 1 \\ 1, & 2 \\ 1, & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \underbrace{\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}}_{b} \right\|^2$$

Solution:

$$x^* = \underbrace{(A^T A)^{-1} A^T}_{A^\dagger} b \quad \text{as sol'n of } (A^TA)x^* = A^Tb$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 1, & 1, & 1 \\ 1, & 2, & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3, & 6 \\ 6, & 14 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

Eyeballing A^TA and A^Tb, we can actually see:

$$8 = 5/3 \times 3 + 1/2 \times 6 = 5 + 3$$

$$17 = 5/3 \times 6 + 1/2 \times 14 = 10 + 7$$

That is $x^* = \begin{pmatrix} 5/3 \\ 1/2 \end{pmatrix}$. If you can't see it, just solve

$$\begin{pmatrix} 3, & 6 \\ 6, & 14 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 17 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/2 \end{pmatrix}.$$

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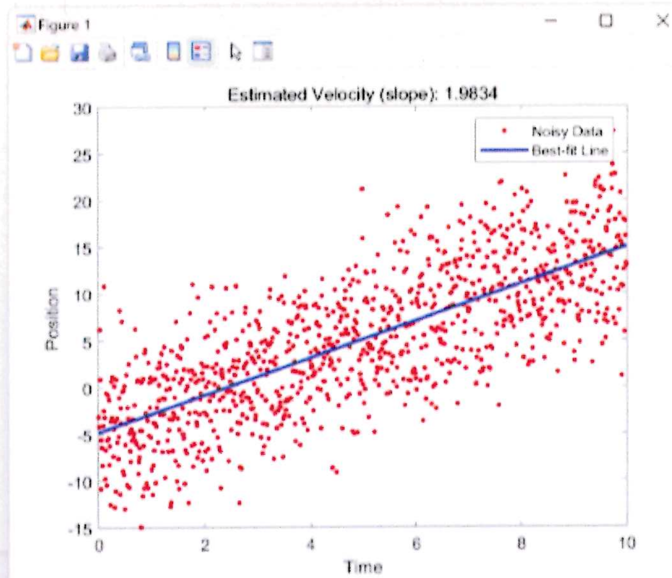
P2

hw05_2.m

```
% load data
load('least_squares_data.mat');  % 'y_meas_noise' and 'dt'

% Generate time vector
t = 0:dt:dt*(length(y_meas_noise)-1);

% Convert y_meas_noise to a column vector
y_meas_noise = y_meas_noise';

% Construct matrix A in the form [t, 1] for linear fitting
A = [t', ones(length(t), 1)];

% Check dimensions of A and y_meas_noise
disp(['Size of A: ', num2str(size(A, 1)), 'x', num2str(size(A, 2))]);
disp(['Size of y_meas_noise: ', num2str(size(y_meas_noise, 1)), 'x', num2str(size(y_meas_noise, 2))]);

% Compute least squares solution for parameters [v, b]
x_ls = A \ y_meas_noise;

% Extract velocity estimate and intercept
v_est = x_ls(1);  % velocity, i.e., slope
b_est = x_ls(2);  % Intercept

% Compute the fitted line
y_fit = v_est * t + b_est;

% Plot the data
figure;
plot(t, y_meas_noise, 'r.', 'MarkerSize', 10);  % Noisy data points
hold on;
plot(t, y_fit, 'b-', 'LineWidth', 2);  % Fitted line
xlabel('Time');
ylabel('Position');
legend('Noisy Data', 'Best-fit Line');
title(['Estimated Velocity (slope): ', num2str(v_est)]);
hold off;
```



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3. The m linearly independent columns of A, let us call them $a_1, \dots, a_m \in \mathbb{R}^n$, form a basis of the linear subspace $X \leq \mathbb{R}^n$.

Thus, for all $x \in X$, $\exists$ coefficients $u_1, u_2, \dots, u_m \in \mathbb{R}$

So that

$$x = u_1 a_1 + u_2 a_2 + \dots + u_m a_m$$

$$= \underbrace{\begin{pmatrix} a_1, & a_2, & \dots, & a_m \end{pmatrix}}_{\in \mathbb{R}^{n \times m}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}}_{\in \mathbb{R}^m}$$

$$= Au.$$

Thus, $x \in \text{range}(A)$. Since $x \in X$ was picked arbitrary, $X = \text{range}(A)$ is confirmed. $\blacksquare$

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4. (a) Take $x_1, x_2 \in X^\perp$.

This means precisely that

$$x_1^T x = 0 \quad \forall x \in X$$

and also $$x_2^T x = 0 \quad \forall x \in X.$$

Clearly $\alpha x_1 + \beta x_2$ with $\alpha, \beta \in \mathbb{R}$ arbitrary

satisfy $$(\alpha x_1 + \beta x_2)^T \underbrace{x}_{\in X} = \alpha \underbrace{x_1^T x}_{=0} + \beta \underbrace{x_2^T x}_{=0} = 0 \quad \forall x \in X$$

So indeed $\alpha x_1 + \beta x_2 \in X^\perp$. $\forall \alpha, \beta \in \mathbb{R}$

In summary: $\underbrace{x_1, x_2 \in X^\perp \Rightarrow \alpha x_1 + \beta x_2 \in X^\perp}_{\text{vector space structure is clearly there!}}$

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4b Proof via $\underbrace{X \subset (X^{\perp})^{\perp}}_{\text{part one}}$ and $\underbrace{(X^{\perp})^{\perp} \subset X}_{\text{part two}}$

Part one: $X \subset (X^{\perp})^{\perp}$, i.e. $x \in X \Rightarrow x \in (X^{\perp})^{\perp}$

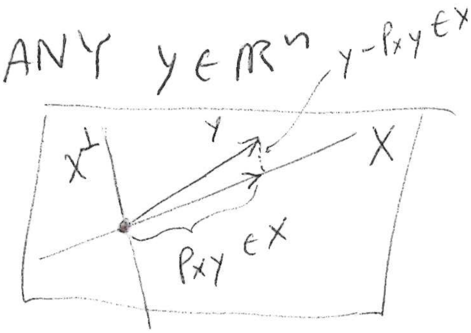
Take $x \in X$. For all $y \in X^{\perp}$, we have (by def. of $X^{\perp}$)

$$x^T y = 0.$$

But this means $x \perp X^{\perp}$, i.e. $x \in (X^{\perp})^{\perp}$.

Part two: $(X^{\perp})^{\perp} \subset X$

From the projection Theorem, for ANY $y \in \mathbb{R}^n$ we have that



$$(y - P_x y) \in X^{\perp}.$$

Now let $y \in (X^{\perp})^{\perp}$, i.e. $y \perp X^{\perp}$, so also

$$y^T (y - P_x y) = 0. \qquad (\mathrm{I})$$

Moreover:

$$\underbrace{(P_x y)}_{\in X}{}^T \underbrace{(y - P_x y)}_{\in X^{\perp}} = 0 \qquad (\mathrm{II})$$

(I) − (II):

$$\underbrace{(y - P_x y)^T (y - P_x y)}_{\|y - P_x y\|^2} = 0 \Rightarrow y = \underbrace{P_x y}_{\in X}.$$

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P5

$$P_X = A_1 (A_1^T A_1)^{-1} A_1^T = \underset{\substack{\uparrow \\ \text{just calculate} \\ \text{it all} \\ \text{(e.g. via MatLab)}}}{\dots} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$\|$

$$P_X = A_2 (A_2^T A_2)^{-1} A_2^T = \dots = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

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$$(1, 1, 1)\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = (0, 0),$$

basis of X
= columns of A_1

i.e. $$(1, 1, 1)\left(\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\alpha + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\beta}_{\text{arbitrary element in } X}\right)$$

$$= (1, 1, 1)\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (0, 0)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

Similarly $$(1, 1, 1)\begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = (0, 0)$$ $\blacksquare$

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P6 Intersection of the two planes

$$P_1 = \{x : x_1 + x_2 + x_3 = 1\}$$

$$P_2 = \{x : -x_1 - x_2 + x_3 = 0\}$$

is given as the solution set of

$$\underbrace{\begin{pmatrix} 1, & 1, & 1 \\ -1, & -1, & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{b}.$$ Define $X = \mathrm{sol}(A, b)$.

Minimum norm solution of $Ax = b$ is what we are looking for. It is given by

$$x^+ = A^T (AA^T)^{-1} b$$

$$= \dots$$

$$= \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$\blacksquare$

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