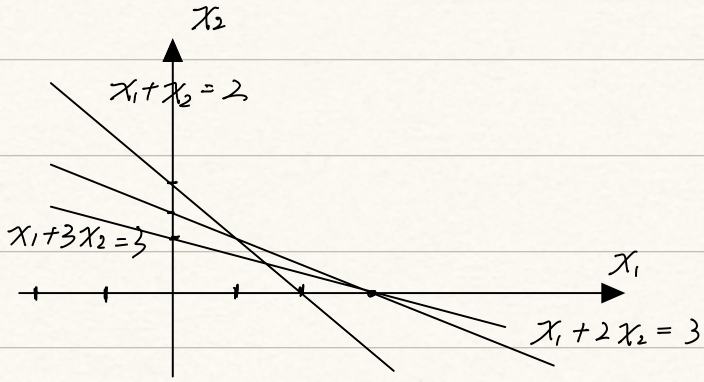
Homework 5

1. (a) $\begin{cases} x_1 + x_2 = 2 \\ x_1 + 2x_2 = 3 \\ x_1 + 3x_2 = 3 \end{cases}$

From last two equations, we can get $\begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases}$

Since $x_1 + x_2 = 3 \neq 2$.



Thus, there exists not a single $(x_1, x_2) \in R^2$ that satisfies the three linear equations all at the same time

(b) $\begin{cases} x_1 + x_2 = 2 \\ x_1 + 2x_2 = 3 \\ x_1 + 3x_2 = 3 \end{cases} \Longrightarrow Ax = b$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$$\min_x \|Ax - b\|^2$$

$$A^TA\, x^* = A^Tb \Longrightarrow x^* = (A^TA)^{-1}A^Tb$$

Since, $A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$, $\det(A^TA) = 42 - 36 = 6$

$$(A^TA)^{-1} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$$(A^TA)^{-1}A^Tb = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 10 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{2} \end{pmatrix}$$

Thus, $x^* = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{2} \end{pmatrix}$

3. According to the question, $A \in R^{n \times m}$ is a matrix with m linearly independent columns that constitute a basis for a subspace $X \subset R^n$ of dimension m, and $range(A) = \{x \in R^n : x = Au \text{ for some } u \in R^m\}$

$\forall x \in range(A)$, $\exists u \in R^m$ s.t. $x = Au$, which means that x is a linear combination of the columns of A, where the coefficients of this combination are given by the components of u. Thus, $x \in X$

Therefore, $range(A) \subseteq X$

$\forall x \in X$, where x is the linear combination of a basis of X. Since the columns of A form a basis for X. Thus, x can be expressed as Au for some $u \in R^m$. Thus, $x \in range(A)$

Therefore, $X \subseteq range(A)$

Thus, $X = range(A)$

4. (a) $X^{\perp} = \{y \in R^n : x^T y = 0 \text{ for all } x \in X\}$

$0 \in R^n$, $x^T 0 = 0$ for all $x \in X$, Thus, $0 \in X^{\perp}$

Suppose $y_1, y_2 \in X^{\perp}$. Then, for any $x \in X$, we have $x^T y_1 = 0$, $x^T y_2 = 0$

$x^T(y_1 + y_2) = x^T y_1 + x^T y_2 = 0 + 0 = 0$, Thus, $y_1 + y_2 \in X^{\perp}$

Suppose $y \in X^{\perp}$ and α is a scalar, $x^T y = 0$. For any $x \in X$, we have

$x^T(\alpha y) = \alpha x^T y = 0$. Thus, $\alpha y \in X^{\perp}$

Therefore, $X^{\perp}$ have a vector space structure and is a subspace of R^n

(b) Prove that $X \subset (X^{\perp})^{\perp}$

$\forall x \in X$, $X^{\perp} = \{y \in R^n : x^T y = 0 \text{ for all } x \in X\}$, we know that

$x^T y = 0$, $\forall y \in X^{\perp}$

This means that $x \in (X^{\perp})^{\perp}$. This implies that $X \subset (X^{\perp})^{\perp}$

Prove that $(X^{\perp})^{\perp} \subset X$

From the projection theorem, we know that for any $y \in R^n$, we have that

$$(y - x^*) \in X^{\perp} \quad \text{where } x^* = P_X y$$

Consider any $y \in (X^{\perp})^{\perp}$, then we must have $y^T(y - P_X y) = 0$

Note that we also have $y - P_X y \in X^{\perp}$ and $P_X y \in X$

Thus, $(P_X y)^{\perp}(y - P_X y) = 0$

$$0 = y^T(y - P_X y) - (P_X y)^{\perp}(y - P_X y) = \|y - P_X y\|_2^2 \Longrightarrow y = P_X y \in X$$

Hence, any vector in $(X^{\perp})^{\perp}$ is also in X, which means that $(X^{\perp})^{\perp} \subset X$

Therefore, $(X^{\perp})^{\perp} = X$

5. $X = \{x \in R^3 : x_1 + x_2 + x_3 = 0\}$. This is a plane through the origin in R^3

range$(A) = \{x \in R^n : x = Au$ for some $u \in R^m\}$. where u is coefficient vector for column vector of A

$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, column vector $\vec{a}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{a}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $a_{11} + a_{12} + a_{13} = 0$, $a_{21} + a_{22} + a_{23} = 0$

Since both columns of A_1 satisfy the equation $x_1 + x_2 + x_3 = 0$. This means $\vec{a}_1$ and $\vec{a}_2$ are on the plane X. Also, $\vec{a}_1$ and $\vec{a}_2$ are linearly independent. Thus, range$(A_1) = X$

$A_2 = \begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, column vector $\vec{a}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{a}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $a_{11} + a_{12} + a_{13} = 0$, $a_{21} + a_{22} + a_{23} = 0$

Since both columns of A_2 satisfy the equation $x_1 + x_2 + x_3 = 0$. This means $\vec{a}_1$ and $\vec{a}_2$ are on the plane X. Also, $\vec{a}_1$ and $\vec{a}_2$ are linearly independent. Thus, range$(A_2) = X$

Therefore, $X = \text{range}(A_1) = \text{range}(A_2)$

$P_X = A(A^TA)^{-1}A^T$

Suppose $P_{X1} = A_1(A_1^TA_1)^{-1}A_1^T$

$$A_1^TA_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A_1^TA_1)^{-1} = \frac{1}{4-1}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$P_{x_1} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Suppose $P_{x_2} = A_2(A_2^TA_2)^{-1}A_2^T$

$$A_2^TA_2 = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$$

$$(A_2^TA_2)^{-1} = \frac{1}{12-9}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$$

$$P_{x_2} = A_2(A_2^TA_2)^{-1}A_2^T = \begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ 3 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Thus, $P_{x_1} = P_{x_2}$

Therefore, the resulting P_x in those two different cases are identical.

6. The line can be expressed as an underdetermined linear system $Ax = b$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftarrow \begin{cases} x_1 + x_2 + x_3 = 1 \\ -x_1 - x_2 + x_3 = 0 \end{cases}$$

$$x^* = A^T(AA^T)^{-1}b$$

$$AA^T = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

$$(AA^T)^{-1} = \frac{1}{9-1}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$x^* = A^T(AA^T)^{-1}b = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2 & -2 \\ 2 & -2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$