Homework 7

1. Consider the following theorem and definition:

   Theorem (Weierstrass extreme value theorem). Let $D \subset \mathbb{R}^n$ be a compact (i.e., closed and bounded) set. Any continuous function f defined on D has a global minimizer in D.

   Definition. A continuous function f defined on $\mathbb{R}^n$ is called *coercive* if

   $$\forall f_0 \in \mathbb{R} \;\; \exists r > 0 : \quad \|x\| > r \Rightarrow f(x) > f_0,$$

   or, in a more short-hand notation, if $\lim_{\|x\|\to\infty} f(x) = +\infty$.

   (a) Show that for a coercive function f, the sublevel sets of f for any given value $f_0 \in \mathbb{R}$, given by
   $$\mathcal{L}^-_{f_0} := \{x \in \mathbb{R}^n : f(x) \leq f_0\}$$
   are bounded sets, i.e., there exists an $r > 0$ so that $\|x\| \leq r$ for all $x \in \mathcal{L}^-_{f_0}$, or, equivalently, that
   $$\mathcal{L}^-_{f_0} \subset B_r(0).$$

   (b) The goal in this subproblem is to use the Weierstrass extreme value theorem to show that a coercive function always has a global minimizer. To this end, fix an arbitrary $x_0 \in \mathbb{R}$ and let $y_0 = f(x_0)$, which is a function value that is attained by f (unless the pick was very lucky, this is probably not the value of the global minimum). Now since the minimization problem seeks the smallest possible value of f, all $x \in \mathbb{R}^n$ producing function values $f(x)$ that are greater than f_0 can be disregarded entirely. This is where coercivity of f can be exploited to limit the search of the global minimum of f over $\mathbb{R}^n$ to a closed and bounded region D, where the Weierstrass extreme value theorem then applies.

   (c) Let $y \in \mathbb{R}^n$. Consider the functional $f(x) = \|x - y\|^2$. Show that f is coercive.
   Hint: The inverse triangle inequality $\|x - y\| \geq |\|x\| - \|y\||$ may be useful.

   (d) Consider a strictly convex function f defined over $\mathbb{R}^n$. Give an example (e.g., in the case $n = 1$) of a strictly convex function that is not coercive.

2. Work out a detailed proof of the Separating Hyperplane Theorem introduced in class. Provide illustrations, a detailed presentation of the rationale in each step, as well as the detailed derivations of the mathematical formulations that are stated in the theorem.

3. Show that the two alternatives in the Key Lemma cannot hold at the same time.

4. Consider the following optimization problem

$$\begin{aligned} \text{minimize} \quad & (x_1-12)^2+(x_2+6)^2 \\ \text{subject to} \quad & x_1^2+x_2^2+3x_1-4.5x_2-6.5 \le 0, \\ & (x_1-9)^2+x_2^2-64 \le 0, \\ & 8x_1+4x_2-20=0. \end{aligned}$$

Apply the Fritz John necessary conditions to this problem.

5. Apply the KKT Theorem to locate all solutions of the following nonlinear program:

$$\begin{aligned} \text{minimize} \quad & e^{-(x_1+x_2)} \\ \text{subject to} \quad & e^{x_1}+e^{x_2} \le 10, \\ & -x_1 \le 0. \end{aligned}$$

6. Use the KKT conditions to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & x^\top Qx + c^\top x \\ \text{subject to} \quad & x^\top Rx \le 1, \\ & e^\top x = 1. \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is invertible but not necessarily positive definite, $R \succ 0$ and $e^\top = (1, 1, \ldots, 1)$ is a row vector with all 1s as entries.

7. Solve the following minimization problem using optimality conditions:

$$\begin{aligned} \text{minimize} \quad & (x_1-16)^2+(x_2-14)^2 \\ \text{subject to} \quad & (x_1-8)^2+(x_2-9)^2-49 \le 0 \\ & -x_1+2 \le 0 \\ & x_1-13 \le 0 \\ & x_1+x_2-24=0. \end{aligned}$$