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THE MINIMUM COVERING SPHERE PROBLEM*

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The minimum covering sphere problem, with applications in location theory, is that of finding the sphere of smallest radius which encloses a set of points in E^n. For a finite set of points, it is shown that the Wolfe dual is equivalent to a particular quadratic programming problem and that converse duality holds. A finite decomposition algorithm, based on the Simplex method of quadratic programming, is developed for which computer storage requirements are independent of the number of points and computing time is approximately linear in the number of points.

1. Historical Sketch and Applications

The problem of covering a finite set of points in the plane with the smallest possible circle was originally posed in 1857 by J. J. Sylvester [15]. Later (1860) he gave a graphical solution procedure attributed to Pierce [16] which was rediscovered 25 years later by Chrystal [2]. A modern account of their technique may be found in Rademacher and Toeplitz [14]. For the problem in the plane Hearn [5] has a new procedure which is more efficient (on a computer) than the early one. Mathematical interest in the n-dimensional problem has largely centered around Jung's inequality [9], [10] which relates the diameter of the given set (finite or infinite) to the radius of the minimum covering sphere. A mathematical programming proof of Jung's inequality is given in Fritz John's pioneering paper [8]. Blumenthal and Wahlin discuss the problem's history in more detail [1].

In recent years both Isaacs [7] and Francis [4] have discussed aspects of the problem in a locational context, since it is equivalent to finding the point which minimizes the maximum distance to the given set of points. Also Lawson [12] shows the equivalence of the E^3 problem and that of finding the smallest radar beam (a right circular cone) to cover specified targets. Lawson and Zukhovitskiy and Avdeyeva [19] give asymptotic algorithms to solve the problem; the proposed procedure is finite.

2. Primal Problem

Let a_i, $i = 1, \cdots, m$, be n-dimensional vectors representing the given points and let x be a point in E^n. Then $(a_i - x)^t(a_i - x)$ represents the square of the Euclidean distance between a_i and x (t denotes transpose). Letting the scalar s be the square of the radius of a sphere centered at the point x, we have the following formulation of the minimum covering sphere problem

$$\text{(1)} \qquad \begin{array}{l} \min_{(s,x)}\ s \\ \text{s.t. } s \geqq (a_i - x)^t(a_i - x), \qquad i = 1, \cdots, m. \end{array}$$

This problem will be referred to as the primal problem. It is a convex programming problem which clearly satisfies the constraint qualification.

The Kuhn-Tucker conditions [11], which are both necessary and sufficient for (1), assure the existence of multipliers v_i^*, $i = 1, \cdots, m$, such that

$$\text{(2)} \qquad \textstyle\sum_i v_i^* = 1,$$

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$$\sum_i v_i^*(a_i - x^*) = 0, \tag{3}$$

$$v_i^*(s^* - (a_i - x^*)^t(a_i - x^*)) = 0, \qquad i = 1, \cdots, m, \tag{4}$$

$$v_i^* \geqq 0, \qquad i = 1, \cdots, m, \tag{5}$$

$$s^* - (a_i - x^*)^t(a_i - x^*) \geqq 0, \qquad i = 1, \cdots, m. \tag{6}$$

(Unless otherwise noted all summations in this paper are from 1 to m.) Conditions (2)–(5) indicate that the center x^* will be a convex combination of those points which lie on the surface of the optimal sphere, since $v_i^* = 0$ for any point a_i which is strictly interior to the sphere. Furthermore a theorem by Carathéodory [13] states that if a point is a convex combination of points in E^n then $n + 1$ or fewer points suffice to express the convex combination. This allows us to state

LEMMA 1. *The center x^* of the minimum covering sphere can be expressed as a convex combination of at most $n + 1$ of the given points.*

The existence of (s^*, x^*) is clear and since the minimand of

$$\min_{(s,x)} \max_i (a_i - x)^t(a_i - x)$$

is strictly convex we may state

LEMMA 2. *The minimum covering sphere (s^*, x^*) exists and is unique.*

3. Dual Problem

The Wolfe dual [18] for the primal problem is

$$\max_{(s,x,v)} s + \sum_i v_i[(a_i - x)^t(a_i - x) - s], \tag{7}$$

$$\text{s.t. } \sum_i v_i = 1, \tag{8}$$

$$\sum_i v_i(a_i - x) = 0, \tag{9}$$

$$v_i \geqq 0, \qquad i = 1, \cdots, m. \tag{10}$$

Wolfe's duality results state that the solution of the primal problem insures the existence of a solution to the dual problem such that the optimal primal and dual objective functions are equal. In general, however, converse duality does not hold; that is, solving the dual does not necessarily solve the primal. For the minimum covering sphere problem however converse duality does hold. Before proving converse duality we establish another dual problem.

THEOREM 1. *The Wolfe dual (7)–(10) is equivalent to the concave quadratic programming problem*

$$\max_v \sum_i v_i(a_i^t a_i) - v^t(A^tA)v \tag{11}$$

$$s.t. \sum_i v_i = 1, \tag{12}$$

$$v_i \geqq 0, \qquad i = 1, \cdots, m, \tag{13}$$

with

$$x = \sum_i v_i a_i, \tag{14}$$

and

$$s = \sum_i v_i(a_i - x)^t(a_i - x), \tag{15}$$

where A has columns a_i.

Proof. We call (11)–(13) the QP dual.[1] Equation (14) is the same as (9). Since the Wolfe dual objective function (7) is independent of s for any feasible v (see (8)), we may choose s arbitrarily. The choice given (15) will insure that solving the QP dual solves the primal. To complete the proof we need only show that (11) follows from (7): Using (8) and (9) we have

$$\begin{aligned}\textstyle\sum_i v_i(a_i - x)^t(a_i - x) &= \textstyle\sum_i v_i(a_i{}^t a_i) - 2\sum_i v_i a_i{}^t \sum_j v_j a_j \\ &\quad + \textstyle(\sum_i v_i)(\sum_j v_j a_j{}^t \sum_k v_k a_k) \\ &= \textstyle\sum_i v_i(a_i{}^t a_i) - \sum_i v_i a_i{}^t \sum_j v_j a_j \\ &= \textstyle\sum_i v_i(a_i{}^t a_i) - v^t A^t A v.\end{aligned}$$

We now prove converse duality.

Theorem 2. *If (s^*, x^*, v^*) solves the QP dual problem then (s^*, x^*) solves the primal problem.*

Proof. Huard [6] has shown converse duality for certain "partially linear" problems, of which (1) is a special case. We give a proof here to emphasize the role of the multipliers of the QP dual.

We have only to show that (s^*, x^*, v^*) satisfy Kuhn-Tucker conditions of the primal problem. Conditions (2), (3) and (5) are satisfied by (12), (14) and (13) respectively. The Kuhn-Tucker conditions for the QP dual guarantee the existence of an unconstrained multiplier w^* and nonnegative multipliers u_k^* such that

$$a_k{}^t a_k - 2\textstyle\sum_i v_i^* a_k{}^t a_i + w^* + u_k^* = 0, \qquad k = 1, \cdots, m, \tag{16}$$

$$v_k^* u_k^* = 0, \qquad k = 1, \cdots, m, \tag{17}$$

$$u_k^* \geqq 0, \qquad k = 1, \cdots, m. \tag{18}$$

Performing the summation in (16) and adding and subtracting $x^{*t}x^*$ leads to

$$(x^* - a_k)^t(x^* - a_k) + w^* - x^{*t}x^* + u_k^* = 0. \tag{19}$$

Or, after multiplying by v_k^*, summing over k and using (15) we obtain $w^* = -s^* + x^{*t}x^*$ and from (19)

$$u_k^* = s^* - (x^* - a)^t(x^* - a), \qquad k = 1, \cdots, m.$$

This result immediately shows that (s^*, x^*) satisfy primal Kuhn-Tucker conditions (4) and (6), since (17) and (18) hold. This completes the proof of Theorem 2.

We have proven converse duality by showing that multipliers u_k for the dual problem correspond to the slack variables of the primal problem. That is, u_k^* is the difference between the square of the radius of the optimal sphere and the square of the distance from the center to the given point a_k.

With converse duality established, we can obtain the solution of the primal problem by solving the QP dual and determining the primal variables from (14) and (15). The Simplex method of quadratic programming [3], [17] will solve the QP dual in a finite number of steps. However the basis size is $m + 1$ so if m is large computer storage requirements become prohibitive. In what follows we present a decomposition algorithm for the solution of the QP dual.

[1] The authors express their appreciation to the referees who pointed out that (11) is simpler than the objective function of the QP dual originally proposed.

4. Decomposition Algorithm

We will decompose the problem into a finite sequence of smaller problems each one involving $n + 2$ of the given points. The motive of the decomposition is simple. For any subset of $n + 2$ points there exist a primal subproblem analogous to (1) and a corresponding QP dual subproblem involving $n + 2$ variables analogous to (11)–(13). We can obtain the solution to the subproblem by solving the QP dual. If the optimal sphere for this subproblem encloses all m points it is optimal for the primal problem. If this is not the case then there is a point outside the sphere. Since the subproblem has been solved over $n + 2$ points, from Lemma 1 we know that at least one of the points is not needed in the convex combination expressing the center of the sphere. The point outside the sphere can replace such a point, and the next subproblem is solved with the new set of $n + 2$ points.

Since converse duality holds for the subproblems, we can solve a sequence of QP dual subproblems of dimension $n + 2$. We require that the solution to the subproblem satisfy Lemma 1 (i.e., the center of the optimal sphere is a convex combination of at most $n + 1$ points). In terms of the QP dual, this requires that a solution to the subproblem be found in which no more than $n + 1$ of the v_i are positive. The Simplex method of quadratic programming operates upon the linear system (12) and (16) which can be written (for $m = n + 2$) as

$$\begin{bmatrix} 1^t \\ -2\bar{A}^t\bar{A} \end{bmatrix} v + \begin{bmatrix} 0^t \\ I \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ a_1^{\,t} a_1 \\ \vdots \\ a_{n+2}^{\,t} a_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{20}$$

where $\bar{A}$ has $n + 2$ columns a_i, and 0 and 1 represent either scalars or vectors of zeros or ones.

Lemma 3. *The optimal solution of any subproblem such as (20) will have at least one v_i nonbasic.*

Proof. The Simplex method of quadratic programming obtains a *basic* solution to (20) for which (13), (17) and (18) hold. In (20) the bracketed matrix which multiplies v has $n + 3$ rows and $n + 2$ columns. Since $\bar{A}$ is $n \times (n + 2)$, the last $n + 2$ rows have rank $\leqq n$ hence the rank of the whole matrix does not exceed $n + 1$. Therefore all v_i cannot be basic in the optimal solution of a subproblem.

Now we prove that subproblems are never repeated. Let (s_p^*, x_p^*) be the optimal sphere for the pth subproblem.

Lemma 4. $s_{p+1}^* > s_p^*$ *for all* $p \geqq 1$.

Proof. Assume that the sphere (s_p^*, x_p^*) does not cover the point a_e. In the solution of the pth QP dual subproblem certain of the v_i will be positive (no more than $n + 1$) corresponding to points a_i which define the optimal sphere (s_p^*, x_p^*). Let $K_p = [i \mid v_i^* > 0]$ in the pth subproblem. Then (s_p^*, x_p^*) is the minimum covering sphere for the points a_i, $i \in K_p$. The minimum covering sphere for the $(p + 1)$th subproblem, (s_{p+1}^*, x_{p+1}^*), covers the points a_i, $i \in K_p$, and in addition the new point a_e outside the (s_p^*, x_p^*) sphere. To assume that $s_{p+1}^* \leqq s_p^*$ contradicts the unique optimality of (s_p^*, x_p^*), the minimum covering sphere for a_i, $i \in K_p$.

Since the objective function strictly increases with the solution to each new subproblem, we have only to note that there are finitely many possibilities for defining spheres in terms of $n + 1$ or fewer points. This proves

THEOREM 3. *The decomposition algorithm is finite.*

The outline of the algorithm may now be stated:

1. Given $a_1, \cdots, a_m$, select $n + 2$ of the a_i.
2. Solve the QP dual subproblem.
3. If the sphere determined by this solution encloses all m points, stop. Otherwise replace a point not needed in the optimal solution by a point outside the sphere and go to 2.

5. Computational Aspects

When applying the decomposition it is desirable to use the optimal tableau for the pth subproblem to obtain a feasible tableau of the $(p + 1)$th subproblem. In this section we show how this is accomplished in a manner similar to updating a basis inverse in the revised Simplex method of linear programming.

To initialize the first subproblem we add an artificial variable y to (12). Then the tableau form of (20) is

(21)

Basic Variables	Value	u	w	v	y
y	1	0^t	0	1^t	1
u	$-\bar{a}$	I	1	$-2\bar{A}^t\bar{A}$	0

where $\bar{a} = (a_1{}^t a_1, \cdots, a_{n+2}^t a_{n+2})^t$.

Following Van de Panne and Whinston [17] the first two pivots introduce some v_i to replace y and then w to replace the u_i which is complementary to the v_i in the basis. Once w becomes basic, it remains so. Details of the rules by which u_i and v_i enter and leave the basis are given in [3] and [17]. The optimal tableau will have $u_i^* \geqq 0$, $v_i^* \geqq 0$ and $u_i^* v_i^* = 0$ $(i = 1, \cdots, n + 2)$.

Assume now that we have solved some subproblem p and obtained the optimal sphere for the particular subset of $n + 2$ points. If some point a_e is not enclosed by this sphere we wish to solve the subproblem $p + 1$ with a_e replacing the a_i associated with some nonbasic v_i which is guaranteed by Lemma 3. Assume without loss of generality that a_e replaces a_{n+2}.

Let $z^* = (w^*, v_1^*, \cdots, v_r^*, u_{r+1}^*, \cdots, u_{n+2}^*)$ be the vector of optimal basic variables for subproblem p (where $r \leqq n + 1$). That is, if B_p is the optimal basis matrix then

$$z^* = B_p^{-1}\begin{bmatrix} 1 \\ -\bar{a} \end{bmatrix}.$$

THEOREM 4. *A feasible basis vector for subproblem* $p + 1$ *is given by* z *where*

$$z = (w^*, v_1^*, \cdots, v_r^*, u_{r+1}^*, \cdots, u_{n+1}^*, u_e),$$

$$u_e = s^* - (x^* - a_e)^t(x^* - a_e) < 0,$$

and (s^*, x^*) *is the optimal sphere from subproblem* p.

PROOF. For subproblem $p + 1$ an initial tableau would differ from (21) only in the u_{n+2} row and the v_{n+2} column. The u_{n+2} column is unchanged. Accordingly a feasible basis matrix B_{p+1} differs only by a single row from B_p. In particular

$$B_k = \begin{bmatrix} B_0 & 0 \\ B_{00} & \\ b_k{}^t & I \end{bmatrix}, \qquad k = p, p + 1,$$

where B_0 is an $(r + 1) \times (r + 1)$ nonsingular matrix, B_{00} is $(n - r + 1) \times (r + 1)$, I is $(n - r + 2) \times (n - r + 2)$ and

$$b_p{}' = (1, -2a_1{}'a_{n+2}, \cdots, -2a_r{}'a_{n+2}),$$

$$b'_{p+1} = (1, -2a_1{}'a_e, \cdots, -2a_r{}'a_e).$$

Furthermore B_{p+1}^{-1} differs from B_p^{-1} only in the last row:

$$B_k^{-1} = \begin{bmatrix} B_0^{-1} & 0 \\ -B_{00}B_0^{-1} & \\ -b_k{}'B_0^{-1} & I \end{bmatrix}, \qquad k = p, p + 1 .$$

Therefore the basic variables $w, v_1, \cdots, v_r, u_{r+1}, \cdots, u_{n+1}$ retain their values in the new basis for problem $p + 1$. (The procedure for obtaining B_{p+1}^{-1} from B_p^{-1} is similar to the revised Simplex method of linear programming; however here the two matrices differ by a row rather than a column.) Now

$$u_e = (-b'_{p+1}B_0^{-1}, 0', 1) \begin{bmatrix} 1 \\ -a_1{}'a_1 \\ \vdots \\ -a_e{}'a_e \end{bmatrix}$$

$$= -b'_{p+1}B_0^{-1} \begin{bmatrix} 1 \\ -a_1{}'a_1 \\ \vdots \\ -a_r{}'a_r \end{bmatrix} - a_e{}'a_e$$

$$= -b'_{p+1} \begin{bmatrix} w^* \\ v_1{}^* \\ \vdots \\ v_r{}^* \end{bmatrix} - a_e{}'a_e$$

$$= -w^* + 2 \textstyle\sum_{k=1}^{r} v_k{}^* a_k{}' a_e - a_e{}'a_e$$

$$= -w^* + 2x^* a_e - a_e{}'a_e .$$

Or since $w^* = -s^* + x^{*\prime}x^*$ from the proof of Theorem 2,

$$u_e = s^* - (x^* - a_e)'(x^* - a_e)$$

which completes the proof.

We can now give a detailed outline of the procedure:

1. Given $a_1, \cdots, a_m$, select $n + 2$ of the a_i. Go to 2.
2. Label the a_i as $a_1, \cdots, a_{n+2}$. Set up the initial Simplex tableau (21). Go to 3.
3. Solve the QP dual subproblem by the Simplex method. Call the optimal basis B_p. If the sphere defined by this solution encloses all a_i then stop. Otherwise call an outside point a_e and go to 4.
4. Obtain from B_p^{-1} a starting basis inverse B_{p+1}^{-1} for the next subproblem with v_e replacing some v_i which is nonbasic in the current solution. Generate a feasible tableau for the next subproblem by multiplying its initial tableau by B_{p+1}^{-1}. Go to 3.

6. Example

As an example of Theorem 4 we consider a simple problem of five points in the plane:

$$a_1{}^t = (1, 0) \quad a_3{}^t = (2, 0) \quad a_5{}^t = (3, 2).$$

$$a_2{}^t = (4, 0) \quad a_4{}^t = (2, 1)$$

For the first subproblem we use a_1, a_2, a_3 and a_4. The tableau corresponding to (21) is

Basic Variable	Value	u_1	u_2	u_3	u_4	w	v_1	v_2	v_3	v_4	y
y	1	0	0	0	0	0	1	1	1	1	1
u_1	-1	1	0	0	0	1	-2	-8	-4	-4	0
u_2	-16	0	1	0	0	1	-8	-32	-16	-16	0
u_3	-4	0	0	1	0	1	-4	-16	-8	-8	0
u_4	-5	0	0	0	1	1	-4	-16	-8	-10	0

The first pivot has v_1 replace y in the basis and the second has w replace u_1. Then after v_2 replaces u_2 we have the optimal tableau for this subproblem:

Basic Variable	Value	u_1	u_2	u_3	u_4	w	v_1	v_2	v_3	v_4	y
v_1	$\frac{1}{2}$	$-\frac{1}{18}$	$\frac{1}{18}$	0	0	0	1	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
w	4	$\frac{4}{3}$	$-\frac{1}{3}$	0	0	1	0	0	0	0	0
v_2	$\frac{1}{2}$	$\frac{1}{18}$	$-\frac{1}{18}$	0	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
u_3	2	$-\frac{2}{3}$	$-\frac{1}{3}$	1	0	0	0	0	0	0	0
u_4	1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	1	0	0	0	0	-2	0

This solution represents a circle through the points a_1 and a_2, centered at (0, 2.5). From the above tableau we have

$$B_1^{-1} = \begin{array}{c} \begin{matrix} v_1 & \quad w & \quad v_2 & u_3 & u_4 \end{matrix} \\ \left[\begin{array}{ccc|cc} \frac{4}{3} & -\frac{1}{18} & \frac{1}{18} & 0 & 0 \\ 0 & \frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{18} & -\frac{1}{18} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ \hline 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \end{array} = \begin{bmatrix} B_0^{-1} & 0 \\ -B_{00}B_0^{-1} & \\ -b_1{}^t B_0^{-1} & I \end{bmatrix}$$

where B_1 is given by columns of the initial tableau. Similarly for the next subproblem (with a_5 replacing a_4) an initial tableau is

Basic Variables	Value	u_1	u_2	u_3	u_5	w	v_1	v_2	v_3	v_5	y
y	1	0	0	0	0	0	1	1	1	1	1
u_1	-1	1	0	0	0	1	-2	-8	-4	-6	0
u_2	-16	0	1	0	0	1	-8	-32	-16	-24	0
u_3	-4	0	0	1	0	1	-4	-16	-8	-12	0
u_5	-13	0	0	0	1	1	-6	-24	-12	-26	0

So $b_2{}^t = (-6, 1, -24)$ and using B_1^{-1} we obtain

$$
\begin{array}{cc}
 & \begin{array}{ccccc} v_1 & w & v_3 & u_3 & u_5 \end{array} \\
B_2^{-1} = & \begin{bmatrix}
\frac{4}{3} & -\frac{1}{18} & \frac{1}{18} & 0 & 0 \\
0 & \frac{4}{3} & -\frac{1}{3} & 0 & 0 \\
-\frac{1}{3} & \frac{1}{18} & -\frac{1}{18} & 0 & 0 \\
0 & -\frac{2}{3} & -\frac{1}{3} & 1 & 0 \\
0 & -\frac{1}{3} & -\frac{2}{3} & 0 & 1
\end{bmatrix}
\end{array}
$$

which when multiplied by the preceding tableau gives

Basic Variables	Value	u_1	u_2	u_3	u_5	w	v_1	v_2	v_3	v_5	y
v_1	$\frac{1}{2}$	$-\frac{1}{18}$	$\frac{1}{18}$	0	0	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
w	4	$\frac{4}{3}$	$-\frac{1}{3}$	0	0	1	0	0	0	0	0
v_2	$\frac{1}{2}$	$\frac{1}{18}$	$-\frac{1}{18}$	0	0	0	0	1	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
u_3	2	$-\frac{2}{3}$	$-\frac{1}{3}$	1	0	0	0	0	0	0	0
u_5	-2	$-\frac{1}{3}$	$-\frac{2}{3}$	0	1	0	0	0	0	-8	0

One further pivot on the encircled element leads to the solution, $v_1^* = \frac{5}{12}$, $w = 4$, $v_2^* = \frac{1}{3}$, $u_3^* = 2$ and $v_5^* = \frac{1}{4}$. This represents a circle through a_1, a_2, and a_5 centered at $x^* = (\frac{5}{2}, \frac{1}{2})$ with $s^* = \frac{5}{2}$. The solution is easily checked graphically.

7. Conclusions

Although our example involved covering a set of points in the plane we emphasize that a more efficient procedure is given in [5].[2] Indeed using Lemma 1 and the sufficiency of (2)–(6) many small problems can be solved by hand. In E^n a good approximation (or possibly the solution) may be found by letting x^* be the midpoint of the line segment joining the two most distant a_i and letting $s^* = \max_i (a_i - x^*)^t(a_i - x^*)$. In a

[2] See also J. Elzinga and D. W. Hearn, "Geometrical Solutions for Some Minimax Location Problems," *Transportation Science* (to appear).

companion paper we consider the problem of covering a set of points with k spheres where $k \geqq 2$ for which the algorithm discussed here is a subproblem.

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