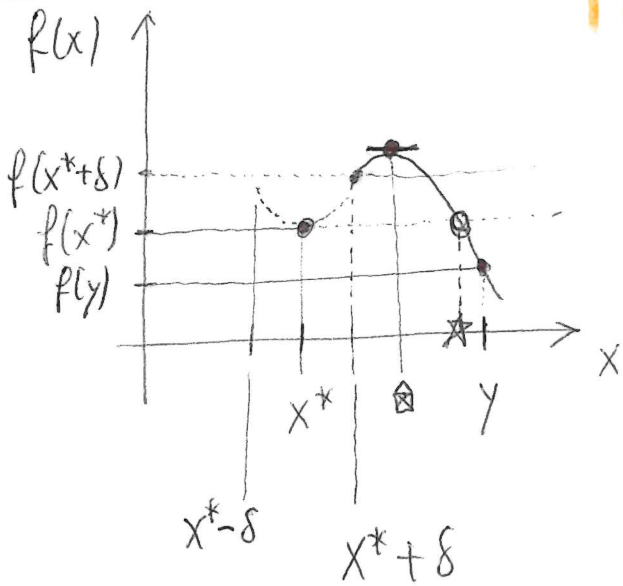
Homework 2 – Solutions

115/100 (15 bonus points)

Problem 1

5/15



Sketch

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Since x^* is a strict local minimizer, there is a $\delta > 0$ such that $f(x^*+\delta) > f(x^*)$.

In the above figure, we have drawn the graph so that at a $y > x^*$, $f(y) < f(x^*)$.
Furthermore, we can assume δ is chosen so that also $x^*+\delta < y$.

Applying the Intermediate Value Theorem with $a = x^*+\delta$ and $b = y$

$$f(a) = f(x^*+\delta) > f(x^*) \quad \text{and} \quad f(b) = f(y) < f(x^*),$$

there will be a point between $a = x^*+\delta$ and $b = y$ for which the value $f(x^*)$ is attained: ☆. 5

Now from Rolle's Theorem, there exist a point: ⌂ 5
for which $f'(⌂) = f(☆) - f(x^*) = 0$. ⇒ Critical pt!

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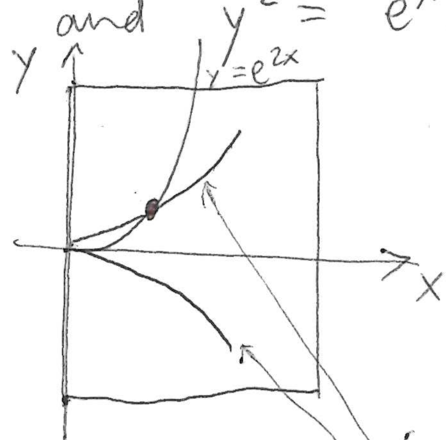
2. $f(x,y) = e^{3x} + y^3 - 3e^x y$

(a) $(\nabla f)(x,y) = \begin{pmatrix} 3e^{3x} - 3ye^x \\ 3y^2 - 3e^x \end{pmatrix}$ 5

For $(\nabla f)(x,y)$ to vanish, we need:

$\underbrace{e^{3x} = \cancel{y}e^x}_{y = e^{2x}}$ and $y^2 = e^x$ ← $y = \pm\sqrt{e^x}$

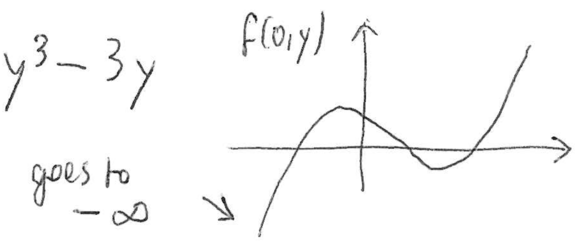


only one intersection, only one critical point $(x,y) \in \mathbb{R}^2$ so that $(\nabla f)(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. 5

Alternatively: Instead of graphical solution, an algebraic one will also work perfectly.

(b) It is somewhat natural to examine the case

$\underline{x = 0}$: $f(0,y) = 1 + y^3 - 3y$



Therefore, whatever value the function takes on at the critical point, there will be a smaller one by picking y negative enough while fixing $x = 0$. 5

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3. $f(x,y) = x^2 - 5xy^2 + 5y^4$

(a) $(\nabla f)(x,y) = \begin{pmatrix} 2x - 5y^2 \\ -10xy + 20y^3 \end{pmatrix}$

Critical points (x^*, y^*) satisfy

$$\begin{cases} 2x^* - 5y^{*2} = 0 & (1) \\ -10x^*y^* + 20y^{*3} = 0 & (2) \end{cases}$$

from (1): $x^* = \frac{5}{2} y^{*2}$, which when plugged into (2)

$-25y^{*3} + 20y^{*3} = 0 \Rightarrow -5y^{*3} = 0.$

back into (1): $x^* = 0$.

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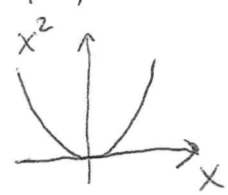
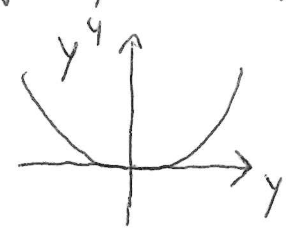
$\Rightarrow$ (0,0) only critical point.

(b) $x \mapsto f(x,0) = x^2$, so $\bar{x} = 0$ clearly strict global min.



$y \mapsto f(0,y) = 5y^4$, so $\bar{y} = 0$ clearly strict global min



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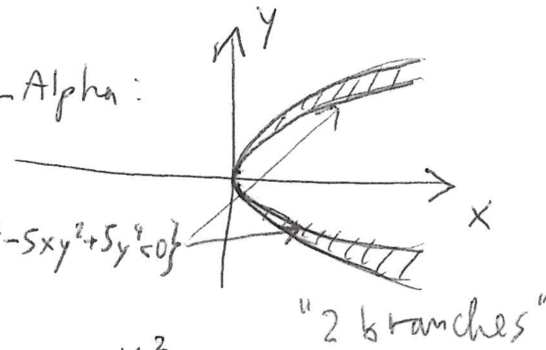
3(c): Consider $\underbrace{x^2 - 5xy^2 + 5y^4 < 0}$ as $f(0,0) = 0$

need to find such solutions arbitrarily close to $(0,0)$!

$\forall \varepsilon > 0 \ \exists_{(\check{x},\check{y})} \in B_\varepsilon(0,0) \quad f(\check{x},\check{y}) < 0$

$x^2 - 5xy^2 + 5y^4 < 0 \longleftarrow$ Wolfram Alpha:

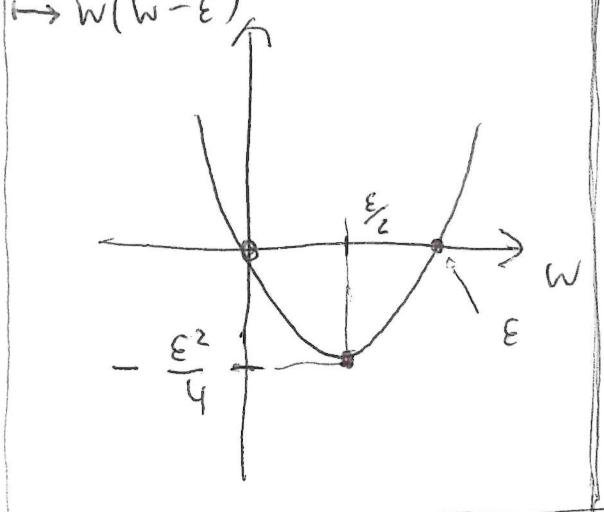


$\Longleftrightarrow \quad y^4 - xy^2 < -\frac{x^2}{5}$

set $x := \varepsilon$, then: $\quad y^4 - \varepsilon y^2 < -\frac{x^2}{5}$

with $w := y^2$ $\quad w^2 - \varepsilon w < -\frac{\varepsilon^2}{5}$

The function $w \mapsto w(w-\varepsilon)$



$\underbrace{w(w-\varepsilon) < -\frac{\varepsilon^2}{5}}_{w = +\frac{\varepsilon}{2}}$

$\Rightarrow y = \pm\sqrt{\frac{\varepsilon}{2}}$

$\Rightarrow (\check{x},\check{y}) = \left(\varepsilon, \pm\sqrt{\frac{\varepsilon}{2}}\right)$

with ε arbitrarily small)

all yield $f(\check{x},\check{y}) < 0$.

Hence $(0,0)$ not a local min.

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$f(0,0) = 0$

since $\exists_{(\check{x},\check{y})}$ arbitrarily close with $f(\check{x},\check{y}) < \overbrace{0}^{=f(0,0)}$

25/25

4. Consider the feasible region defined by the constraints

$$\begin{aligned} 1 - x_1^2 - x_2^2 &\geq 0, \\ \sqrt{2} - x_1 - x_2 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

For each of the following points, determine whether the point is feasible or infeasible, and (if it is feasible) whether it is interior to or on the boundary of each of the constraints:

$$x_a = (\frac{1}{2}, \frac{1}{2})^T, \quad x_b = (1, 0)^T, \quad x_c = (-1, 0)^T, \quad x_d = (-\frac{1}{2}, 0)^T, \quad x_e = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T.$$

Solution:
To determine the points above are feasible or not, we just simply plug each point in the relations above and see if they make sense. The feasible region is an upper-half disk of radius 1 centered at the origin.

For $x_a = (\frac{1}{2}, \frac{1}{2})^T$,

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$$\left.\begin{aligned} 1 - (1/2)^2 - (1/2)^2 = 1/2 &\geq 0, \\ \sqrt{2} - 1/2 - 1/2 \simeq 0.4142 &\geq 0, \\ 1/2 &\geq 0, \end{aligned}\right\} \text{feasible, interior.}$$

For $x_b = (1, 0)^T$,

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$$\left.\begin{aligned} 1 - (1)^2 - (0)^2 = 0 &\geq 0, \\ \sqrt{2} - 1 - 0 \simeq 0.4142 &\geq 0, \\ 0 &\geq 0, \end{aligned}\right\} \text{feasible, on the boundary.}$$

For $x_c = (-1, 0)^T$,

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$$\left.\begin{aligned} 1 - (-1)^2 - (0)^2 = 0 &\geq 0, \\ \sqrt{2} - (-1) - 0 \simeq 2.4142 &\geq 0, \\ 0 &\geq 0, \end{aligned}\right\} \text{feasible, on the boundary.}$$

For $x_d = (-\frac{1}{2}, 0)^T$,

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$$\left.\begin{aligned} 1 - (-1/2)^2 - (0)^2 = 3/4 &\geq 0, \\ \sqrt{2} - (-1/2) - 0 \simeq 1.9142 &\geq 0, \\ 0 &\geq 0, \end{aligned}\right\} \text{feasible, on the boundary.}$$

For $x_e = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$,

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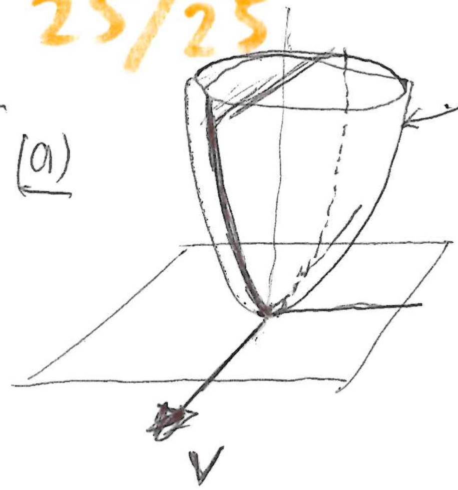
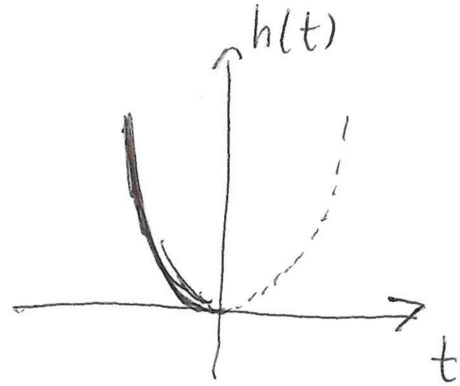
$$\left.\begin{aligned} 1 - (1/2) - (1/2) = 0 &\geq 0, \\ \sqrt{2} - (1/\sqrt{2}) - (1/\sqrt{2}) = 0 &\geq 0, \\ (1/\sqrt{2}) &\geq 0, \end{aligned}\right\} \text{feasible, on the boundary.}$$

5. 25/25

(a)



$f(x_1, x_2) = x_1^2 + x_2^2$



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(b) $\frac{d}{dt} f(x^* + tv) = (\nabla f)(x^* + tv)^T v$

$$\frac{d}{dt} \underbrace{(\nabla f)(x^* + tv)}_{= (g_1, g_2, \dots, g_n)}{}^T v = \frac{d}{dt}\left(v_1 g_1 + v_2 g_2 + \dots + v_n g_n \right)$$

$$= v_1 \frac{d}{dt} \underbrace{g_1}_{\left(\frac{\partial f}{\partial x_1}\right)(x^* + tv)} + \dots + v_n \frac{d}{dt} g_n$$

$$= v_1 \begin{pmatrix} \partial x_1^2 f \\ \partial x_2 \partial x_1 f \\ \partial x_3 \partial x_1 f \\ \vdots \\ \partial x_n \partial x_1 f \end{pmatrix}^T v + \dots + v_n \begin{pmatrix} \partial x_1 \partial x_n f \\ \partial x_2 \partial x_n f \\ \partial x_3 \partial x_n f \\ \vdots \\ \partial x_n \partial x_n f \end{pmatrix}^T v$$

$$= \left(v_1 \begin{pmatrix} \partial x_1^2 f \\ \partial x_2 \partial x_1 f \\ \vdots \\ \partial x_n \partial x_1 f \end{pmatrix}^T + \dots + v_n \begin{pmatrix} \partial x_1 \partial x_n f \\ \partial x_2 \partial x_n f \\ \vdots \\ \partial x_n^2 f \end{pmatrix} \right) v$$

$$= v^T \begin{pmatrix} \partial x_1^2 f & \partial x_1 \partial x_2 f & & \cdot \\ & & \ddots & \\ \cdot & & & \ddots \\ & \cdot & & \end{pmatrix} v$$

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Hessian matrix

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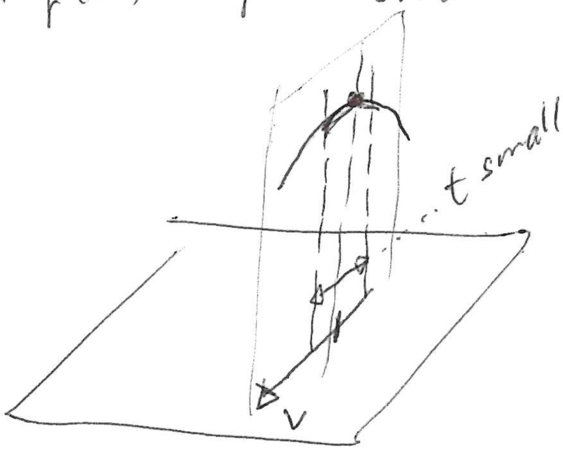
(c): with $h(t) = f(x^* + tv)$

Taylor: $h(t) = \underbrace{h(0)}_{f(x^*)} + \underbrace{\cancel{h'(0)}}_{=0} t + \underbrace{h''(0)}_{v^T(Hf)(x^*)v < 0} t^2 + o(t^2)$

$\underbrace{h(t)}_{= f(x^*+tv)} < f(x^*)$ for small enough t



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(d) The insights of (c) show that for x^* to be a local min. of f, $(\nabla f)(x^*) = 0$ and $\underbrace{v^T (Hf)(x^*) v \geq 0 \quad \forall v \in \mathbb{R}^n}_{(Hf)(x^*) \succeq 0.}$ has to hold.

This proves the second-order NECESSARY condition.

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6. 15/15

(a): $x^T A x = \overbrace{x^T U}^{=y^T} \Lambda \underbrace{U^T x}_{=:y} = y^T \Lambda y$

and $y^T y = x^T \underbrace{U U^T}_{=I} x = x^T x$

$\Rightarrow \|y\| = \|x\|$ 5

(b): $y^T \Lambda y = (y_1, y_2, \dots, y_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$= \sum_{i=1}^{n} \lambda_i y_i^2$

$\geq \sum_{i=1}^{n} \lambda_{min} y_i^2 = \lambda_{min} \left(\sum_{i=1}^{n} y_i^2 \right) = \lambda_{min} \|y\|^2$

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So: $\underbrace{y^T \Lambda y}_{=x^T A x} \geq \lambda_{min} \underbrace{\|y\|^2}_{=\|x\|^2}$

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$x^T A x \geq \lambda_{min} \|x\|^2$