Optimization Homework 2

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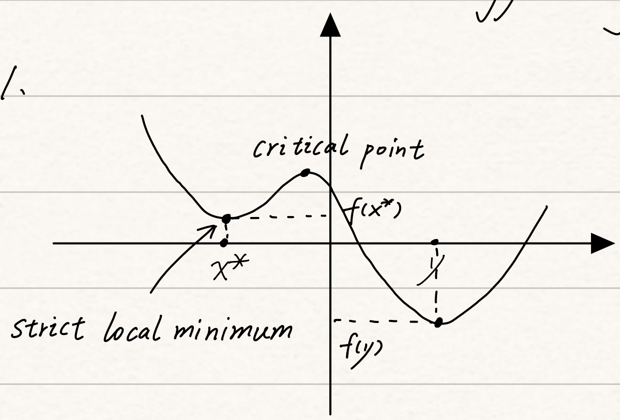
1.



Rolle's theorem states that if a function f is continuous on closed interval $[a,b]$, and differentiable on the open interval (a,b) such that $f(a)=f(b)$, then $f'(x)=0$ for some x with $a\le x\le b$.

Intermediate Value theorem states that if f is a continuous function whose domain contains the interval $[a,b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.

Since x^* is a strict local minimum, $\exists \delta>0$, $\forall x\in(x^*-\delta,x^*)\cup(x^*,x^*+\delta)$, $f(x^*)<f(x)$

then we will have $x_1\in(x^*,x^*+\delta)$. s.t. $f(x_1)>f(x^*)$

Since f is differentiable and continuous over R, $f(y)<f(x^*)<f(x_1)$, $x^*<x_1<y$

based on Intermediate Value Theorem, it takes on any given value between $f(x_1)$ and $f(y)$. And there must exist a point $a\in[x_1,y]$, s.t. $f(a)=f(x^*)$

According to Rolle's Theorem, there exist a critical point $c\in[x^*,a]$, $f'(c)=0$

Therefore, f must possess another critical point.

2. (a) $f(x,y)=e^{3x}+y^3-3e^xy$

$$\frac{\partial f}{\partial x}=3e^{3x}-3e^xy \quad , \quad \frac{\partial f}{\partial y}=3y^2-3e^x$$

$\Rightarrow \frac{\partial f}{\partial x} = 3e^{3x} - 3e^{x}y = 0 \quad \Rightarrow \quad y = e^{2x}$

$\Rightarrow \frac{\partial f}{\partial y} = 3y^2 - 3e^{x} = 0 \Rightarrow y^2 = e^{x} \Rightarrow e^{4x} = e^{x} \Rightarrow x = 0$

$y = e^{2x} = e^{2\times 0} = 1$

$\Rightarrow$ f has exactly one critical point $(0,1)$

(b) $\frac{\partial^2 f}{\partial x^2} = 9e^{3x} - 3ye^{x}$, $\frac{\partial^2 f}{\partial y^2} = 6y$

$\frac{\partial^2 f}{\partial x \partial y} = -3e^{x}$, $\frac{\partial^2 f}{\partial y \partial x} = -3e^{x}$

$$\Rightarrow Hf(0,1) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,1) & \frac{\partial^2 f}{\partial x \partial y}(0,1) \\ \frac{\partial^2 f}{\partial y \partial x}(0,1) & \frac{\partial^2 f}{\partial y^2}(0,1) \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$\Rightarrow \det(Hf(0,1)) = 36 - 9 = 27 > 0$, $6 > 0$

$\mathrm{trace}(Hf(0,1)) = 6 + 6 = 12 > 0$

The Hessian matrix is positive definite, indicating that $(0,1)$ is a local minimizer.

Since, $f(0,1) = 1 + 1 - 3 = -1$,

if $y = -e^{2x}$ $f(x,y) = f(x, -e^{2x}) = e^{3x} - e^{6x} + 3e^{3x} = 4e^{3x} - e^{6x}$

when $x \to \infty$, $f(x, -e^{2x}) \to -\infty < -1$

Therefore, the critical point is a local minimizer but not a global minimizer of f.

3. (a) $f(x,y) = x^2 - 5xy^2 + 5y^4$

$\frac{\partial f}{\partial x} = 2x - 5y^2$, $\frac{\partial f}{\partial y} = -10xy + 20y^3$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} 2x - 5y^2 = 0 \\ -10xy + 20y^3 = 0 \end{cases} \Rightarrow \begin{cases} 2x = 5y^2 \\ y(x - 2y^2) = 0 \end{cases} \Rightarrow y^3 = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the critical point of f is $(0,0)$

(b) $f(x,0) = x^2$

$f'(x,0) = 2x = 0 \implies x = 0.$, $f(0,0) = 0$

when $x<0$ or $x>0$, $f(x,0) > 0$,

Thus, $\bar{x} = 0$ is a strict global minimum of $x \to f(x,0)$

$f(0,y) = 5y^4$

$f'(0,y) = 20y^3 = 0 \implies y = 0$, $f(0,0) = 0$

when $y<0$ or $y>0$, $f(0,y) > 0$

Thus, $\bar{y} = 0$ is a strict global minimum of $y \to f(0,y)$

(C) $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = -10y$,

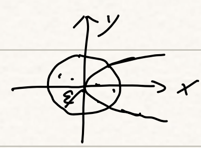
$\frac{\partial^2 f}{\partial y^2} = -10x + 60y^2$, $\frac{\partial^2 f}{\partial y \partial x} = -10y$

$$Hf(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\det(Hf(0,0)) = 0$

The above solution means that the above test is inconclusive

Suppose $x = 2y^2$, we can have



$$f(2y^2, y) = 4y^4 - 10y^4 + 5y^4 = -y^4 \leq 0$$

$(x,y) \to (0.0)$ along the line $x = 2y^2$, $f(x,y) \to 0^-$

$\forall \varepsilon > 0$, $\exists (\check{x}, \check{y}) \in B_\varepsilon(0,0)$, s.t. $f(\check{x}, \check{y}) < 0$

Therefore, $(\bar{x}, \bar{y}) = (0,0)$ is not a local minimum of f.

4. For $x_a = (\frac{1}{2}, \frac{1}{2})$

constraint 1: $1 - x_1^2 - x_2^2 = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} > 0$

constraint 2: $\sqrt{2} - x_1 - x_2 = \sqrt{2} - \frac{1}{2} - \frac{1}{2} = \sqrt{2} - 1 > 0$

constraint 3: $x_2 = \frac{1}{2} > 0$

Thus, x_a is a feasible point and in the interior of the feasible set.

For $x_b = (1, 0)$

constraint 1: $1 - 1^2 - 0^2 = 0$

constraint 2: $\sqrt{2} - 1 - 0 = \sqrt{2} - 1 > 0$

constraint 3: $0 = 0$

Thus, x_b is a feasible point and on the boundary of the feasible set

For $x_c = (-1, 0)$

constraint 1: $1 - 1 - 0 = 0$

constraint 2: $\sqrt{2} - (-1) - 0 = \sqrt{2} + 1 > 0$

constraint 3: $0 = 0$

Thus, x_c is a feasible point and on the boundary of the feasible set

For $x_d = (-\frac{1}{2}, 0)$

constraint 1: $1 - \frac{1}{4} - 0 = \frac{3}{4} > 0$

constraint 2: $\sqrt{2} - (-\frac{1}{2}) - 0 = \sqrt{2} + \frac{1}{2} > 0$

constraint 3: $0 = 0$

Thus, x_d is a feasible point and on the boundary of the feasible set

For $x_e = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

constraint 1: $1 - \frac{1}{2} - \frac{1}{2} = 0$

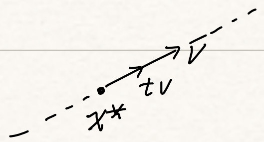
constraint 2: $\sqrt{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \sqrt{2} - \frac{2}{\sqrt{2}} = \sqrt{2} - \sqrt{2} = 0$

constraint 3: $\frac{1}{\sqrt{2}} > 0$

Thus, x_e is a feasible point and on the boundary of the feasible set

5. (a) For $n=2$, $h(t) = f(x^* + tv)$, $x^* \in R^2$ and $v \in R^2$, $f: R^2 \to R$

The function $h(t) = f(x^* + tv)$ describes how the value of f changes as you move from x^* along the line defined by v



(b) $h'(t) = \frac{\partial f(x^* + tv)}{\partial t} = v^T \cdot \nabla f(x^* + tv)$

$h''(t) = \frac{\partial [v \cdot \nabla f(x^* + tv)]}{\partial t} = v^T H f(x^* + tv) \cdot v$

$h'(0) = v^T \cdot \nabla f(x^*)$

$h''(0) = v^T \cdot Hf(x^*) \cdot v$

(c) Since $(\nabla f)(x^*) = 0$, x^* is a stationary point

Since $v^T (Hf)(x^*) v < 0$, the second-order Taylor expansion formula of $f(x^* + tv)$ at x^* is

$$f(x^* + tv) \approx f(x^*) + t v^T \nabla f(x^*) + \frac{1}{2} t^2 v^T Hf(x^*) v$$

$$\approx f(x^*) + \frac{1}{2} t^2 \underbrace{v^T Hf(x^*) v}_{<0}$$

$$\Longrightarrow f(x^*) > f(x^* + tv)$$

This implies that moving along v starting from x^* leads to decrease in the value of f

Therefore, x^* is not a minimizer of f.

(d) Since x^* is a local minimizer of a function $f \in C^2(R^n, R)$,

x^* must be a stationary point, which means that $(\nabla f)(x^*) = 0$

$$f(x^*+\varepsilon v) \approx f(x^*) + \varepsilon v^T \nabla f(x^*) + \frac{1}{2}\varepsilon^2 v^T Hf(x^*) v$$

$$\approx f(x^*) + \frac{1}{2}\varepsilon^2 v^T Hf(x^*) v$$

For x^* to be minimizer, we require $f(x^*) < f(x^*+\varepsilon v)$ for all ε and any v, which means that $\frac{1}{2}\varepsilon^2 v^T Hf(x^*) v \geq 0$

$v^T Hf(x^*) v \geq 0$ for all v. It means that $Hf(x^*) \geq 0$

6. (a) $$x^T A x = x^T U \Lambda U^T x$$

$$= (x^T U) \Lambda (U^T x)$$

let $y = U^T x$, $y^T = x^T U$, then we have

$$x^T A x = y^T \Lambda y$$

since $y = U^T x$, we have

$$\|y\|^2 = \|U^T x\|^2 = (U^T x)^T (U^T x) = x^T U U^T x = x^T x = \|x\|^2$$

Thus, $\|y\| = \|x\|$

(b) $$y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2 \geq \sum_{i=1}^{n} \lambda_{min} y_i^2 = \lambda_{min} \sum_{i=1}^{n} y_i^2 = \lambda_{min} \|y\|^2$$

$$x^T A x = y^T \Lambda y \geq \lambda_{min} \|y\|^2 = \lambda_{min} \|x\|^2$$