Problem 2

(a) $A - \lambda I = \begin{pmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{pmatrix}$

$|A - \lambda I| = (5-\lambda)^2 - 16 = 0$

$\Longrightarrow 5 - \lambda = \pm 4$

$\Longrightarrow \lambda_1 = 1, \ \lambda_2 = 9$

$A\vec{x} = \lambda \vec{x} \Longrightarrow (A - \lambda I)\vec{x} = 0$

For $\lambda_1 = 1$, $(A - \lambda_1 I)\vec{x}_1 = (A - I)\vec{x}_1 = 0$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1' \\ x_1'' \end{pmatrix} = 0$

$4x_1' + 4x_1'' = 0 \Longrightarrow x_1' = -x_2''$

$\Longrightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\lambda_2 = 9$ $(A - 9I)\vec{x}_2 = 0$

$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_2' \\ x_2'' \end{pmatrix} = 0$

$-4x_2' + 4x_2'' = 0 \Longrightarrow x_2' = x_2''$

$\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $A = Q\Lambda Q^T$, where Q is formed by the normalized eigenvectors

$Q = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \quad Q^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$

$Q\Lambda Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = A$

c) Since $\lambda_1 = 1 > 0$, $\lambda_2 = 9 > 0$ and A is a symmetric matrix,

$\Longrightarrow$ the matrix A is positive definite

Problem 4

$$f(\vec{x}) = \vec{a}^T\vec{x} + b \ , \ \vec{a}, \vec{x} \in R^n, \ b \in R$$

$$f(t\vec{x}_1 + (1-t)\vec{x}_2) = \vec{a}^T[t\vec{x}_1 + (1-t)\vec{x}_2] + b = t\vec{a}^T\vec{x}_1 + (1-t)\vec{a}^T\vec{x}_2 + b, \quad \vec{x}_1, \vec{x}_2 \in R$$

$$t f(\vec{x}_1) + (1-t) f(\vec{x}_2) = t(\vec{a}\vec{x}_1 + b) + (1-t)(\vec{a}^T\vec{x}_2 + b) = t\vec{a}\vec{x}_1 + tb + (1-t)\vec{a}^T\vec{x}_2 + (1-t)b$$

$$= t\vec{a}\vec{x}_1 + (1-t)\vec{a}^T\vec{x}_2 + b$$

$$\Longrightarrow f(t\vec{x}_1 + (1-t)\vec{x}_2) = t f(\vec{x}_1) + (1-t) f(\vec{x}_2)$$

Thus, $f(\vec{x}) = \vec{a}^T\vec{x} + b$ is convex

Problem 5

(a) $f(x,y) = x^2 + 2y^2 - 2xy$

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

$$Hf - \lambda I = \begin{pmatrix} 2-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix}$$

$$|Hf - \lambda I| = (2-\lambda)(4-\lambda) - 4 = 8 - 2\lambda - 4\lambda + \lambda^2 - 4 = \lambda^2 - 6\lambda + 4 = 0$$

$$\lambda_1 = \frac{6 + \sqrt{36 - 4\times4}}{2} = 3 + \sqrt{5} \qquad \lambda_2 = 3 - \sqrt{5}$$

Since. $\lambda_1 = 3+\sqrt{5} > 0$, $\lambda_2 = 3 - \sqrt{5} > 0$, Hf is positive definite

(b) $\begin{cases} \frac{\partial f}{\partial x} = 2x - 2y = 0 \\ \frac{\partial f}{\partial y} = 4y - 2x = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ x = 2y \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$f(0.0) = 0$

Since Hf is positive definite. f is strictly convex. Then, $(0,0)$ is the global minimum of f.