Quiz#5 of ESE 417 FL2024: (10/29/2024)

Consider the following constrained optimization problem (the **primal problem**):

$$\min_{x_1,x_2}\{x_1^2 + x_1x_2 + x_2^2\} \;\; s.t. \quad x_2 - 1 \geq 0$$

(1) Find the Lagrangian function of the **primal problem**

(2) Find the objective function of the **Lagrangian dual problem**

(3) Show and solve the **Lagrangian dual problem**

(4) Use the solution of the **Lagrangian dual problem** to find the solution of the **primal problem**

(1) $\mathcal{L}(x_1, x_2, \lambda_1) = x_1^2 + x_1x_2 + x_2^2 - \lambda_1(x_2 - 1)$

$$\nabla\mathcal{L}(x_1, x_2, \lambda_1) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 - \lambda_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_2 = -2x_1 \\ x_1 + 2x_2 = -3x_1 = \lambda_1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -\frac{1}{3}\lambda_1 \\ x_2 = \frac{2}{3}\lambda_1 \end{cases}$$

(2) $q(\lambda_1) = \frac{1}{9}\lambda_1^2 - \frac{2}{9}\lambda_1^2 + \frac{4}{9}\lambda_1^2 - \lambda_1\left(\frac{2}{3}\lambda_1 - 1\right)$

$= \frac{1}{3}\lambda_1^2 - \frac{2}{3}\lambda_1^2 + \lambda_1 = -\frac{1}{3}\lambda_1^2 + \lambda_1$

(3) $\max_{\lambda_1 \geq 0} \left(-\frac{1}{3}\lambda_1^2 + \lambda_1\right)$

$\frac{d}{d\lambda_1}\left(-\frac{1}{3}\lambda_1^2 + \lambda_1\right) = -\frac{2}{3}\lambda_1 + 1 = 0 \Rightarrow \lambda_1 = \frac{3}{2}$

since $\lambda_1 \geq 0$, Thus, $\lambda_1 = \frac{3}{2}$

(4) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\lambda_1 \\ \frac{2}{3}\lambda_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$