

Problem 1 (OLS, linear regression)

$$(a) \text{ Sum of Squared Residuals} = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n [y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2$$

$$\frac{\partial}{\partial \hat{\beta}_0} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = -2 \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\implies \sum_{i=1}^n y_i = n \hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^n x_i$$

$$\implies \hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n y_i + \hat{\beta}_1 \cdot \frac{1}{n} \sum_{i=1}^n x_i$$

$$= \bar{y} + \hat{\beta}_1 \bar{x}$$

$$\frac{\partial}{\partial \hat{\beta}_1} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = -2 \sum_{i=1}^n x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\implies \sum_{i=1}^n x_i y_i - \sum_{i=1}^n \hat{\beta}_0 x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

Substitute $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$$\implies \sum_{i=1}^n x_i y_i - \sum_{i=1}^n (\bar{y} - \hat{\beta}_1 \bar{x}) x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

$$\implies \sum_{i=1}^n x_i y_i - \sum_{i=1}^n \bar{y} x_i + \hat{\beta}_1 \sum_{i=1}^n \bar{x} x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2 - \bar{x} \sum_{i=1}^n x_i}$$

$$= \frac{\sum_{i=1}^n x_i y_i - \frac{1}{n} (\sum_{i=1}^n x_i) (\sum_{i=1}^n y_i)}{\sum_{i=1}^n x_i^2 - \frac{1}{n} (\sum_{i=1}^n x_i)^2}$$

$$= \frac{\sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}$$

Thus, $\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}$, $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$, where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$, $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$