

# Problem 1 (OLS, linear regression)

$$(a) \text{ Sum of Squared Residuals} = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n [y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2$$

$$\frac{\partial}{\partial \hat{\beta}_0} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = -2 \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\Rightarrow \sum_{i=1}^n y_i = n \hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^n x_i$$

$$\Rightarrow \hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n y_i + \hat{\beta}_1 \cdot \frac{1}{n} \sum_{i=1}^n x_i$$

$$= \bar{y} + \hat{\beta}_1 \bar{x}$$

$$\frac{\partial}{\partial \hat{\beta}_1} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = -2 \sum_{i=1}^n x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\Rightarrow \sum_{i=1}^n x_i y_i - \sum_{i=1}^n \hat{\beta}_0 x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

$$\text{Substitute } \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\Rightarrow \sum_{i=1}^n x_i y_i - \sum_{i=1}^n (\bar{y} - \hat{\beta}_1 \bar{x}) x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

$$\Rightarrow \sum_{i=1}^n x_i y_i - \sum_{i=1}^n \bar{y} x_i + \hat{\beta}_1 \sum_{i=1}^n \bar{x} x_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2 - \bar{x} \sum_{i=1}^n x_i}$$

$$= \frac{\sum_{i=1}^n x_i y_i - \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n y_i \right)}{\sum_{i=1}^n x_i^2 - \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2}$$

$$= \frac{\sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}$$

$$\text{Thus, } \hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}, \quad \text{where } \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$