ESE 520 Probability and Stochastic Processes

Lecture 19

"Gaussian processes/Brownian motion continued."

We recall again that it holds

$$E(W_t - W_s)^{2m} = (2m-1) \times ... \times 3 \times 1|t-s|^m$$

if (W_t) is a Brownian motion.

By the Kolmogorov's theorem for continuous sample paths, with $\alpha = 2m, \beta = m-1$ and any γ such that

$$\alpha\gamma < \beta,$$

it holds that there exists a *Hoelder continuous* version of (W_t) with the exponent $\gamma = \frac{m-1}{2m}$ as long as we choose $m \geq 2$.

We recall also that (W_t) is said to be Hoelder continuous of order γ if there is a constant $C > 0$ such that

$$|W_t(w) - W_s(w)| \leq C|t-s|^\gamma$$

for all $s, t \geq 0$.

It follows from the above that (W_t) is Hoelder continuous of order $\frac{1}{2}$ since $\lim_{m\to\infty} \gamma(m) = \lim_{m\to\infty} \frac{m-1}{2m} = \frac{1}{2}$.

In particular, we already know that (W_t) has sample-path continuous version.

In general, it is a known fact in real analysis that:

"If a given function is Hoelder continuous of order $\gamma (0 < \gamma < 1)$, then it is continuous".

Another known fact in real analysis:

"If a given function is differentialbe, then it is Hoelder continuous of order $\gamma = 1$ *(called Lipshitz continuous)"*.

What can be said about the differentiability of paths of Brownian motion?

Theorem. (*Brownian motion paths are nowhere differentiable*) Let $(W_t), t \geq 0$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Then, for any $t \geq 0$, it holds

$$P(w : W_t(w) \text{ is differentiable at } t) = 0. \tag{1}$$

Proof.

Step 1. The trick is to show that, with probability one, for any positive $n \in \mathbb{N}$ there exists $t \in [0, \frac{1}{n^4}]$ such that

$$\frac{|W_t|}{t} > n.$$

This would imply that W_t cannot be differentiable at $t = 0$.

To show that, define

$$A_n := \{w : \frac{|W_t(w)|}{t} > n \text{ for some } t \in [0, \frac{1}{n^4}]\}.$$

We use then the fact that (see Example 3 from Lecture 18)

$$V_t = \frac{1}{n^2} W_{n^4 t}$$

is again a Wiener process.

We can write that

$$P(A_n) \geq P(\frac{|W_{\frac{1}{n^4}}|}{\frac{1}{n^4}} > n) = P(\frac{|V_{\frac{1}{n^4}}|}{\frac{1}{n^4}} > n) = P(|W_1| > \frac{1}{n}).$$

Now we observe that, by construction, the sequence A_n is a decreasing sequence in the sense that $A_{n+1} \subset A_n, n = 1, 2, ...$ By continuity property of probability,

$$\lim_{n \to \infty} P(A_n) = P(\cap_{n=1}^{\infty} A_n).$$

It is also clear that

$$P(|W_1| > \frac{1}{n}) = P(W_1^2 > \frac{1}{n^2}) = \int_{\frac{1}{n^2}}^{\infty} f_{W_1^2}(x)dx \to \int_0^{\infty} f_{W_1^2}(x)dx = 1$$

as $n \to \infty$ where W_1^2 is $\chi^2(1)$ distributed random variable with the pdf concentrated on $(0, \infty)$.

Therefore, we have shown that

$$P(\cap_{n=1}^{\infty} A_n) = 1$$

proving the statement (1) for $t = 0$.

Step 2. We can use Step 1 to show that (W_t) is nowhere differentiable with probability one at any $t \geq 0$.

For that, we use the fact that a process $(V_s), s \geq 0$ defined as

$$V_s := W_{s+t} - W_t, s \geq 0$$

is again a Winer process (exercise to show it!) where $t > 0$ is any fixed number.

By Step 1, $(V_s), s \geq 0$ is not differentiable at $s = 0$ with probability one, that is, the limit

$$\lim_{s \downarrow 0} \frac{V_s}{s}$$

does not exist a.e. It means that the limit

$$\lim_{s \downarrow 0} \frac{W_{s+t} - W_t}{s}$$

does not exist a.e.

In other words, (W_t) is not differentiable a.e. for any $t \geq 0$. $\square$

Remark 1. The fact that the paths of (W_t) are nowhere differentiable a.e. implies that (W_t) has a.e. the paths of *unbounded variation* on any interval $[0, t], t > 0$. Practically it means that the paths of Brownian motion are completely irregular ones having infinite length on any interval $[0, t]$.

We already know that a Brownian motion process (W_t) has *stationary Gaussian increments*. More general, the following is true:

Theorem. (stationary Gaussian processes) A Gaussian process (X_t) is stationary if and only if the following two conditions are satisfied:

1) $E(X_t) = const$ for all $t \geq 0$;

2) $R(s,t) := E(X_s X_t)$ depends on $t - s$ only.

Proof. " $\rightarrow$ " We assume first that (X_t) has stationary FDDs. In particular, it means that random variables X_t and X_{t+h} have the same distribution for all $t \geq 0$ and any $h > 0$. It implies that $E(X_t) = E(X_0) = const$ for all $t \geq 0$.

Also, for all $t_1 < t_2$, the joint distributions of $(X_0, X_{t_2 - t_1})$ and (X_{t_1}, X_{t_2}) are the same implying that

$$cov(X_{t_1}, X_{t_2}) = cov(X_0, X_{t_2 - t_1})$$

depends only on the difference $t_2 - t_1$.

" $\leftarrow$ " Now assume that both statements 1) and 2) of the Theorem are true. Then, the FDDs of the the process (X_t) are completely determined by $\mu(t)$ and $R(s,t)$ (because a Gaussian process) and they will be h-shift invariant thus stationary. $\square$

All processes $(X_t), t \geq 0$ satisfying two conditions 1) and 2) from the Theorem (not necessarily Gaussian processes) form an important class of processes called *WSS-processes*.

Definition 1. (*WSS-processes*) A stochastic process (X_t) satisfying the two conditions 1) and 2) from the Theorem above, is called a *wide-sense stationary process* (WSS-process).

Remark 2. If a process (X_t) is stationary in the usual sense (FDDs are h-shift invariant), then such stationarity is sometimes called the stationarity in "narrower sense". In case of Gaussian processes both concepts coincide as we saw it in the above Theorem, in general - not.

In summary:

"if a process is stationary in the "narrower sense", then it is a WSS-process";

"if a process is a WSS-process, then it is not, in general, stationary in "narrower sense".

Example. (Construction of a *white noise process* as an example of WSS-process)

Let $(W_t), t \geq 0$ be a standard Wiener process. We extend $(W_t), t \geq 0$ to the entire line $\mathbb{R}$ as a sample-path continuous zero-mean Gaussian process $(\tilde{W}_t), t \in \mathbb{R}$ with covariance function

$$R(s,t) := \frac{1}{2}[|t| + |s| - |t-s|].$$

In other words, $\tilde{W}$ can be written as

$$\tilde{W}_t = W_t \mathbf{1}_{[0,\infty)}(t) + W_{-t}\mathbf{1}_{(-\infty,0)}(t).$$

The sample paths of the new process run as path of W on $[0,\infty)$ and backwards on $(-\infty, 0)$.

For $h > 0$, define a new process

$$X_t^h := \frac{1}{h}(W_{t+h} - W_t), t \in \mathbb{R}.$$

Then $(X_t^h), t \in \mathbb{R}$ is a zero-mean Gaussian process with covariance

$$E[X_t^h X_s^h] = \frac{1}{h^2}E[(W_{t+h} - W_t)(W_{s+h} - W_s)] =$$

$$\frac{1}{h^2}[E(W_{t+h})(W_{s+h}) - E(W_{t+h})(W_s) - E(W_t)(W_{s+h}) + E(W_t)(W_s)] =$$

$$\frac{1}{2h^2}[|t+h| + |s+h| - |t-s| - |t+h| - |s| +$$

$$|t+h-s| - |t| - |s+h| + |t-s-h| + |t| + |s| - |t-s|] =$$

$$\frac{1}{2h^2}[|t+h-s| + |t-s-h| - 2|t-s|].$$

It follows that the covarince function of the process (X_t^h) does depend on the difference $t - s$ only so that $(X_t^h), t \in \mathbb{R}$ is a WSS-process for any $h > 0$.

In fact, for any WSS-process the covariance function $R(s,t)$ is a function of *one variable only*: $\tau : t - s$.

We have then for any $\tau \in \mathbb{R}$ and $s \in \mathbb{R}$:

$$R^h(\tau) = E[X^h_{\tau+s} X^h_s] = \frac{1}{2h^2}[|\tau + h| + |\tau - h| - 2|\tau|] =$$

$$= \begin{cases} \frac{2h-2|\tau|}{2h^2}, & |\tau| \le h \\ 0, & |\tau| > h. \end{cases}$$

$$= \frac{1}{h}\max\{0, 1 - \frac{|\tau|}{h}\} = \frac{1}{h}(1 - \frac{|\tau|}{h})^+.$$

Therefore, the process $(X^h_t), t \in \mathbb{R}$ is a well defined WSS process with the covariance function $R^h(\tau)$.

But: What happens if $h \to 0$?

We know: For a.e. $w \in \Omega$, $\lim_{h\to 0} X^h_t(w), t \in \mathbb{R}$ *does not exist* due to the fact that the paths of (W_t) are nowhere differentiable.

At the same time: it is not hard to see that

$$\lim_{h\to 0} R^h(\tau) = \delta_0(\tau)$$

where $\delta_0(\tau)$ is well-known *Dirac δ-function* defined as

$$\delta_0(\tau) = \begin{cases} \infty, & \tau = 0 \\ 0, & \tau \ne 0 \end{cases}$$

such that

$$\int_{-\infty}^{\infty} \delta_0(\tau) d\tau = 1.$$

Now, $\delta_0(\tau)$ is *not a function in usual sense* but only a *distribution function in generalized sense*: it integrates to 1 over $\mathbb{R}$.

The "stochastic process" behind this construction is called *"white noise"*. We write "stochastic process" because (X_t) does not exist in regorous mathematical sense.

Proposition. If f is a continuous function at $\tau = 0$, then

$$\lim_{h\to 0} \int_{-\infty}^{\infty} R^h(\tau) f(\tau) d\tau = f(0) =: < \delta_0, f > .$$

Proof.

$$\lim_{h\to 0} \int_{-\infty}^{\infty} R^h(\tau) f(\tau) d\tau =$$

$$\lim_{h\to 0} (\frac{1}{h} \int_{-h}^{0} (1 + \frac{\tau}{h}) f(\tau) d\tau + \frac{1}{h} \int_{0}^{h} (1 - \frac{\tau}{h}) f(\tau) d\tau) =$$

$$\lim_{h\to 0} (\frac{1}{h} \int_{-h}^{h} f(\tau) d\tau + \frac{1}{h^2} \int_{-h}^{0} \tau f(\tau) d\tau - \frac{1}{h^2} \int_{0}^{h} \tau f(\tau) d\tau)) =$$

$$\lim_{h\to 0} (\frac{f(h) + f(-h)}{1} + \frac{(-h) f(-h)}{2h} - \frac{h f(h)}{2h}) =$$

- using L'Hospital's rule -

$$(2f(0) - \frac{1}{2} f(0) - \frac{1}{2} f(0)) = f(0).$$

In other words, $\delta_0(\tau)$ acts as *linear functional on continuous functions*:

$$< \delta_0, f > = \int_{\mathbb{R}} \delta_0(\tau) f(\tau) d\tau = f(0).$$