ESE 520 Probability and Stochastic Processes

Lecture 18

"Gaussian processes/Brownian motion."

Definition 1. A stochastic process $(X_t), t \in I$ is said to be *Gaussian* if all its finite-dimensional distributions are Gaussian.

Since FDDs determine a stochastic process completely and uniquely (up to a modification) - due to the Theorem of Kolmogorov - the above definition describes the class of Gaussian processes precisely.

We do want to find another, often more convenient way, to describe a Gaussian process.

We already know that any Gaussian vector $(X_{t_1}, ..., X_{t_n})$ is uniquely determined by its *mean vector* and its *covariance matrix*.

The above motivates then the following

Definition 2. a) $\mu_t := E(X_t), t \in I$ is called the *mean function* of the process (X_t);

b) $R(s,t) := E[(X_s - \mu_s)(X_t - \mu_t)]$ for all $s,t \in I$ is called the *covariance function* of the process (X_t).

It follows that if $(X_t), t \in I$ is a Gaussian process and we choose $t_1, ..., t_n \in I$, then $(X_{t_1}, ..., X_{t_n})$ is a Gaussian vector with mean vector $(\mu_{t_1}, ..., \mu_{t_n})^T$ and covariance matrix $(R(t_i, t_j), 1 \leq i, j \leq n$.

We also note that covariance functions are *positive semi-definite* in the sense that for any finite set $(t_1, ..., t_n)$ of distinct times t_i and any complex numbers $\alpha_1, ..., \alpha_n$, it holds

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j R(t_i, t_j) \geq 0, \tag{1}$$

where $\bar{\alpha}_j$ is the complex conjugate of α_j.

As we will see later, it is convenient to treat covariance functions as being complex-valued, the later due to the use of Fourier transforms when working with covariance functions. This explains why the coefficients α_i are assumed to be complex-valued.

Remark 1. The property of covariance function being positive semi-definite extends basically the property of a positive semi-definite matrix (disctere case) to a function (continuous case). The property (1) means that $R(s,t)$ is positive semi-definite if and only if any corresponding matrix generated by a discrete number of values $t_1, ..., t_n$ is positive semi-definite. $R(s,t)$ is said then be *positive definite* if " $\geq$ " is replaced by " $>$ ".

Theorem. Given an arbitrary function $\mu : I \to \mathbb{R}$ and any positive semi-definite function $R : I \times I \to \mathbb{R}$, there exists a unique (up to a modification) Gaussian stochastic process with mean function $\mu(t)$ and covariance function $R(s,t)$.

Proof. It follows basically from Kolmogorov's theorem on the existence of a stochastic process.

Indeed, for any $n \in \mathbb{N}$ and any $t_1, t_2, ..., t_n, t_{n+1} \in I$, we have a mean vector $\mu = (\mu_{t_1}, ..., \mu_{t_n}, \mu_{t_{n+1}}) \in \mathbb{R}^{n+1}$ and $(n+1)\times(n+1)$ covariance matrix $R(t_i, t_j)$ which will uniquely determine a Gaussian vector $(X_{t_1}, ..., X_{t_n}, X_{t_{n+1}})$ with the corresponding FDDs $F_{t_1,...,t_n,t_{n+1}}$. The later will follow basically from the fact that using μ and R, we have a characteristic function of a Gaussian vector which distribution then uniquely determined by inverse Fourier of the characteristic function.

It is then not hard to verify that the family of FDDs of the form $F_{t_1,...,t_n,t_{n+1}}$ will satisfy the consistency conditions (note for that that the marginal cdf of first n components of the vector $(X_{t_1}, ..., X_{t_n}, X_{t_{n+1}})$ is again Gaussian). By Kolmogorov's theorem, there will exist then a stochastic process $(X_t), t \in I$ which will have the given FDDs as its own. By definition, it is then a Gaussian process.

If $R(s,t)$ is a positive definite function, then the corresponding Gaussian process has then non-degenerate FDDs. $\square$

Example 1 (of positive semi-definite functions $R : I \times I \to \mathbb{R}$).

a) Let $\phi_k, k = 1, 2, ...n$ be arbitrary complex-valued functions and $\beta_k > 0$, $k = 1, 2, .., n$. Then

$$R(s,t) := \sum_{k=1}^{n} \beta_k \phi_k(t) \bar{\phi_k}(s)$$

is positive semi-definite.

Proof. For arbitrary distinct $t_1, ..., t_n$ and arbitrary complex numbers $\alpha_1, ..., \alpha_n$ we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j R(t_i, t_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j (\sum_{k=1}^{n} \beta_k \phi_k(t_i) \bar{\phi_k}(t_j)) =$$

$$\sum_{k=1}^{n} \beta_k (\sum_{i=1}^{n} \alpha_i \phi_k(t_i))(\overline{\sum_{j=1}^{n} \alpha_j \phi_k(t_j)}) =$$

$$\sum_{k=1}^{n} \beta_k |\sum_{i=1}^{n} \alpha_i \phi_k(t_i)|^2 \geq 0.$$

b) If $R_k(s,t), k = 1, 2, ...$ is a sequence of positive semi-definite functions which converge point-wise to $R(s,t)$ as $k \to \infty$, then $R(s,t)$ is also positive semi-definite.

c) Define

$$R(s,t) := \min(s,t) \text{ for all } s, t \geq 0. \tag{2}$$

The function in (2) is *positive definite.*

To see that, choose $0 < t_1 < t_2 < ... < t_n$. One has to show that the corresponding matrix $R(t_i, t_j)$ is positive definite. For that, we can use the *Silvester's criterion* which says that a $n \times n$ matrix R is positive definite if and only if all its principal minors $m_k, k = 1, 2, ..., n$ are positive. By definition, principal minors are the determinants of submatrices along the diagonal of the matrix.

For $n = 2$, one has then two different matrices $R(t_1, t_2)$ depending if $t_1 < t_2$ or $t_2 < t_1$.

For $t_1 < t_2$: $m_1 = t_1 > 0$ and $m_2 = t_1 t_2 - t_1^2 = t_1(t_2 - t_1) > 0$;

For $t_2 < t_1$: $m_1 = t_1 > 0$ and $m_2 = t_1 t_2 - t_2^2 = t_2(t_1 - t_2) > 0$;

For general n, the proof can be done accordingly.

It follows then that there exists a unique Gaussian process $(X_t), t \geq 0$ with mean $\mu_t = 0$ and covariance function $R(s,t) = \min(s,t)$ (Note also that $R(0,0) = 0$ so that $X_0 = 0$) satisfying the following properties:

1) (X_t) has *stationary increments*, that is: $X_t - X_s$ has $\mathcal{N}(0, t-s)$ distribution for all $s < t$;

2) (X_t) has *independent increments*;

3) (X_t) has a *sample-path continuous version*;

Proof.

For 1): Random variable $X_t - X_s$ is Gaussian (since a linear combination of two Gaussian random variables) with mean 0 and the variance

$$E(X_t - X_s)^2 =$$

$$EX_t^2 - 2E(X_s X_t) + EX_s^2 = (t,t) - 2(s,t) + R(s,s) = t - 2s + s = t - s$$

for all $s < t$.

For 2): Choose $t_1 < t_2 < ... < t_n$. One has then to show that the increments $X_{t_2} - X_{t_1}, ...X_{t_n} - X_{t_{n-1}}$ are independent.

Since increments are Gaussian, it is then enough to show that they are uncorrelated.

As a particular case, we obtain for $t_1 < t_2 < t_3$:

$$E[(X_{t_2} - X_{t_1})(X_{t_3} - X_{t_2})] =$$

$$EX_{t_2}X_{t_3} - EX_{t_2}^2 - EX_{t_1}X_{t_3} + EX_{t_1}X_{t_2} =$$

$$t_2 - t_2 - t_1 + t_1 = 0.$$

For 3): We recall the property of normal distribution we proved before:

$$E(X_t - X_s)^{2m} = (2m-1)(2m-3) \times ... \times 3 \times 1 \times |t-s|^m, m = 1, 2, ...$$

The existence of a path-continuous version for (X_t) follows then from Kolmogorov's theorem for the existence of path-continuous version for $\alpha = 2m > 0$ and $\beta = m - 1 > 0$ for any $m \geq 2$.

Definition 3. (Wiener process/Brownian motion) The *Wiener process* is a sample-path continuous Gaussian process with mean 0 and covariance function $R(s, t) = \min(s, t)$.

Notation: $(W_t), t \geq 0$.

Another common name for the Wiener process is a *process of Brownian motion*.

Remark 2. a) The history of Brownian motion goes back to 1826 when the British botanist Robert Brown observed under a microscope the irregular motion of a small grain immersed into a fluid. The irregular motion was caused by collisions with molecules of surrounding fluid (approximately 10^{21} collisions per second).

b) The physical explanation of Brownian motion was first provided in 1905 by A. Einstein.

c) One of the first applications of Brownian motion is known to be done by L. Bachelier in his Ph. D. thesis "Theorie de la speculation" (1990), where he used Brownian motion to describe the behaviour of stocks at Paris stock exchange.

d) The mathematical model of Brownian motion was first provided by N. Wiener in 1923.

Theorem (*equivalent characterizations of Brownian motion*). Let $(W_t), t \geq 0$ be a sample-path continuous stochastic process with $W_0 = 0$. Then the following statements are equivalent:

i) (W_t) is a Wiener process (with mean 0 and covariance function $R(s, t) = s \wedge t$);

ii) (W_t) has stationary and independent increments and for all $s < t$ the variable $W_t - W_s$ has $\mathcal{N}(0, t-s)$ distribution;

iii) the FDD's of $(W_{t_1}, ..., W_{t_n})$ (for any $n \in \mathbb{N}$ and $0 < t_1 < t_2 < ... < t_n$) have the density

$$n(t_1; 0, x_1) \times n(t_2 - t_1; x_1, x_2) \times ... \times n(t_n - t_{n-1}; x_{n-1}, x_n),$$

where

$$n(\tau; x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2}\frac{(x-y)^2}{\tau}}.$$

Remark 3. The condition iii) will mean later that the process (W_t) is a *Markov process* with transition function n.

Example 2. Let $(W_t), t \geq 0$ be a Wiener process and define $V_t := -W_t, t \geq 0$. Show that $(V_t), t \geq 0$ is again a Wiener process.

Solution: By the above theorem, it is enough to show three properties:

a) $V_0 = 0$ which is trivial since $V_0 = -W_0 = 0$;

b) independence of increments for (V_t);

For that, it is enough to choose $0 \leq t_1 < t_2 < t_3 < t_4$ and to prove the independence of increments

$$V_{t_2} - V_{t_1} = -(W_{t_2} - W_{t_1})$$

and

$$V_{t_4} - V_{t_3} = -(W_{t_4} - W_{t_3}).$$

However, it is clearly true since the increments $(W_{t_2} - W_{t_1})$ and $(W_{t_4} - W_{t_3})$ are independent by assumption.

c) For all $s < t$, the increment $V_t - V_s$ has normal distribution as a linear combination of normal random variables and we have that

$$E(V_t - V_s) = -E(W_t - W_s) = 0$$

and

$$Var(V_t - V_s) = Var(-(W_t - W_s)) = Var(W_t - W_s) = t - s$$

yielding that $V_t - V_s$ has $\mathcal{N}(0, t-s)$ distribution.

Example 3. (exercise) For any real number $c > 0$, define

$$V_t := \frac{1}{c} W_{c^2 t}, t \geq 0.$$

Show that $(V_t), t \geq 0$ is again a Wiener process.