

# ESE 520 Probability and Stochastic Processes

## Lecture 21

”An application of WSS processes: White noise representations of signals.”

Assume that  $(X_t), t \in \mathbb{R}$  is a WSS process with mean  $\mu(t)$  and covariance function  $R(\tau)$ .

Let us consider a deterministic *linear dynamical system* described by the following ODE system:

$$x'(t) = Ax(t) + Bu,$$

$$y(t) = Cx(t),$$

where  $x(t_0) = x_0$  is the initial state of  $x(t)$  (which we can assume for simplicity to be zero).

In a general case: the coefficients  $A, B, C$  are matrices; in case of  $n = 1$  they are just constants. Here  $u$  can be seen as a ”control parameter”.

It can be directly verified that the solution of the above first ODE is

$$x(t) = e^{A(t-t_0)}[x_0 + \int_{t_0}^t e^{A(t_0-s)} Buds].$$

If we let the control parameter to be  $u = X_t$ , then we obtain as the ”output” process for the system described

$$Y_t = C \int_{t_0}^t e^{A(t-s)} BX_s ds + x_0 e^{A(t-t_0)} C = [ \text{we use for simplicity } x_0 = 0 ] = C \int_{t_0}^t e^{A(t-s)} BX_s ds.$$

We also assume here that the linear model considered and the input process  $X$  are such that we can let in the last relation  $t_0 \rightarrow -\infty$  and obtain a *steady-state response*

$$Y_t = C \int_{-\infty}^t e^{A(t-s)} BX_s ds.$$

What can we say about the process  $(Y_t)$ ?

Define

$$h(\tau) := \begin{cases} Ce^{A\tau}B, & \tau \geq 0 \\ 0, & \tau < 0. \end{cases}$$

called the *impulse-response* of the system.

Then

$$Y_t = \int_{-\infty}^{\infty} h(t-s)X_s ds.$$

Formally (integrals need to converge), we obtain

$$\begin{aligned} E(Y_t) &= \int_{-\infty}^{\infty} h(t-s)E(X_s)ds = [E(X_s) = \mu = \text{const}] = \\ &\mu \int_{-\infty}^{\infty} h(t-s)ds = [\tau := t-s] = \mu \int_{-\infty}^{\infty} h(\tau)d\tau = \mu L, \end{aligned}$$

where  $\int_{-\infty}^{\infty} h(\tau)d\tau = L$ .

Similarly, we have

$$\begin{aligned} R_Y(s, t) &= E[(Y_t - EY_t)(Y_s - EY_s)] = \\ E[(\int_{-\infty}^{\infty} h(t-\tau)(X_\tau - \mu))(\int_{-\infty}^{\infty} h(s-\sigma)(X_\sigma - \mu))]d\tau d\sigma &= \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\tau)h(s-\sigma)E(X_\tau - \mu)(X_\sigma - \mu)d\tau d\sigma. \end{aligned}$$

Now, by definition of the covariance function,  $R_X(\tau, \sigma) = E(X_\tau - \mu)(X_\sigma - \mu)$ , and since  $X$  is a WSS process,  $R_X(\tau, \sigma)$  depends only of the difference  $\tau - \sigma$  so that we can write  $R_X(\tau, \sigma) = R_X(\tau - \sigma)$  yielding

$$R_Y(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\tau)h(s-\sigma)R_X(\tau-\sigma)d\tau d\sigma.$$

After change of variables  $\tilde{t} = t - \tau, \tilde{s} = s - \sigma$ , we obtain

$$R_Y(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tilde{t})h(\tilde{s})R_X(t-s-\tilde{t}+\tilde{s})d\tilde{s}d\tilde{t} \quad (1)$$

so that  $R_Y(s, t)$  depends on the difference  $\tau := t - s$  only.

In other words, we have shown that given the "input" process  $X$  is a WSS process with covarince function  $R_X$ , the "output" process  $Y$  is again a WSS process with covarince function  $R_Y$  having the form (1).

The relation (1) becomes more simple if we use Fourier transforms.

**Definition 1.** Given the impulse-response function  $h(t)$ , its Fourier transform

$$H(\nu) = \int_{-\infty}^{\infty} h(t)e^{-2\pi i\nu t} dt$$

is called the *transfer function*.

**Proposition.** It holds

$$S_Y(\nu) = |H(\nu)|^2 S_X(\nu). \quad (2)$$

*Proof.* Given that the integrals converge and

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\nu) e^{2\pi i\nu\tau} d\nu,$$

we have that

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t)h(s) \left( \int_{-\infty}^{\infty} S_X(\nu) e^{2\pi i\nu(\tau-s+t)} d\nu \right) ds dt = \\ &= \int_{-\infty}^{\infty} S_X(\nu) e^{2\pi i\nu\tau} \overline{\left( \int_{-\infty}^{\infty} h(t) e^{-2\pi i\nu t} dt \right)} \left( \int_{-\infty}^{\infty} h(s) e^{-2\pi i\nu s} ds \right) d\nu = \\ &= \int_{-\infty}^{\infty} H(\nu) \bar{H}(\nu) S_X(\nu) e^{2\pi i\nu\tau} d\nu = \\ &= \int_{-\infty}^{\infty} |H(\nu)|^2 S_X(\nu) e^{2\pi i\nu\tau} d\nu. \end{aligned}$$

Since

$$R_Y(\tau) = \int_{-\infty}^{\infty} S_Y(\nu) e^{2\pi i\nu\tau} d\nu,$$

the statement (2) follows.  $\square$

In particular, if  $(X_t)$  is a "white noise" ( $S_X(\nu) = 1$ ), then  $S_Y(\nu) = |H(\nu)|^2$ .

**Example 1.** Given the "input" process is "white noise", what is then the "output" process  $(Y_t)$ ? For simplicity, assume that  $t_0 = 0, C = 1, A = 0, B = 1$ .

Formally, we have that

$$Y_t = \int_0^t X_s ds. \quad (3)$$

The expression (3) is in fact only a formal representation since the existence of  $X$  cannot be mathematically verified (we really operate only with its covariance function).

We also recall that  $X$  is a zero mean Gaussian "process" with covariance  $R(\tau) = \delta_0(\tau)$ .

At least formally, however, we can calculate that

$$E(Y_t) = \int_0^t E(X_s) ds = 0$$

for any  $t \geq 0$  and

$$\begin{aligned} cov(Y_s, Y_t) &= E[Y_s Y_t] = E \int_0^t \int_0^s X_u X_v du dv = \int_0^t \int_0^s E(X_u X_v) du dv = \\ &= \int_0^t \left( \int_0^s \delta_0(u - v) du \right) dv = \int_0^t \left( \int_{-\infty}^{\infty} \delta_0(u - v) \mathbf{1}_{[0,s]}(u) du \right) dv. \end{aligned}$$

We evaluate separately

$$\int_{-\infty}^{\infty} \delta_0(u - v) \mathbf{1}_{[0,s]}(u) du = h(v)$$

where

$$h(v) := \begin{cases} 1, & 0 < v < s \\ \frac{1}{2}, & v = 0, s \\ 0, & v \notin (0, s). \end{cases}$$

We also note that the value  $1/2$  is the result of the function  $\mathbf{1}_{[0,s]}(u)$  to be discontinuous at  $u = s$ .

Therefore, we obtain that

$$cov(Y_s, Y_t) = \int_0^t \mathbf{1}_{[0,s]}(v) dv$$

since for the value of the integral values of  $h(v)$  at two points  $v = 0$  and  $v = s$  do not matter.

In summary,

$$\text{cov}(Y_s, Y_t) = \int_0^t \mathbf{1}_{[0,s]}(v) dv = \min(s, t).$$

In other words, it means (at least formally) that  $(Y_t)$  is a Gaussian process with zero mean and covariance  $R_y(s, t) = \min(s, t)$  thus a Wiener process!

The expression (3) can be interpreted that Wiener process is "integrated white noise".

Again, at least formally: "white noise" =  $W'$  ("derivative" of  $W$ ).

**Remark 1.** The name "white noise" comes from the analogy with "white light" consisting from different colors of the same intensity ( $S(\nu) = \text{const}$  for all frequencies  $\nu$ ).

**Example 2.** We consider the linear system of the form

$$x'(t) = -\alpha x(t) + u, y(t) = x(t).$$

In other words, we let  $C = B = 1, A = -\alpha, \alpha > 0$  in the system described above.

Another form for the system would be to write

$$Y_t' = -\alpha Y_t + X_t,$$

where  $X$  is a "white noise" process, or, equivalently

$$dY_t = -\alpha Y_t dt + dW_t, \tag{4}$$

where  $dW_t = W_t' dt$  and  $W$  is a Wiener process.

The expression (4) is called a *stochastic differential equation* and there is a rigorous mathematical theory called *stochastic calculus* which allows to work with equations like (4) in a solid mathematical way. Stochastic

calculus is based, from another side, on the concept of a *stochastic integral*. However, we won't go in that direction in our course.

Nevertheless, we can do some calculations in the example.

First, the impulse response is defined in this case as

$$h(\tau) = e^{-\alpha\tau}, \tau > 0.$$

We can also calculate the transfer function (frequency response):

$$H(\nu) = \int_0^\infty e^{-\alpha t} e^{-2\pi i \nu t} dt = \frac{-1}{\alpha + 2\pi i \nu} e^{-(\alpha + 2\pi i \nu)t} \Big|_0^\infty = \frac{1}{\alpha + 2\pi i \nu}.$$

The spectral density of the output signal is then

$$S_Y(\nu) = |H(\nu)|^2 S_X(\nu) = \frac{1}{\alpha + 2\pi i \nu} \frac{1}{\alpha - 2\pi i \nu} \times 1 = \frac{1}{\alpha^2 + (2\pi \nu)^2}.$$

The covariance function is then calculated as the inverse Fourier transform of  $S_Y$ :

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^\infty \frac{1}{\alpha^2 + (2\pi \nu)^2} e^{2\pi i \nu \tau} d\nu = \\ &= \int_{-\infty}^\infty \frac{\cos(2\pi \nu \tau)}{\alpha^2 + (2\pi \nu)^2} d\nu + i \int_{-\infty}^\infty \frac{\sin(2\pi \nu \tau)}{\alpha^2 + (2\pi \nu)^2} d\nu = \\ &= \int_{-\infty}^\infty \frac{\cos(2\pi \nu \tau)}{\alpha^2 + (2\pi \nu)^2} d\nu. \end{aligned}$$

However, the last integral is not an easy one to calculate...

We can use here the *residual integration theory* from complex analysis. In short, it holds that

$$\int_{-\infty}^\infty \frac{1}{\alpha^2 + (2\pi \nu)^2} e^{2\pi i \nu \tau} d\nu = 2\pi i \operatorname{Res}\left(\frac{1}{\alpha^2 + (2\pi \nu)^2} e^{2\pi i \nu \tau}\right)$$

where the residual values are calculated in the upper complex half plane.

The function

$$f(z) = \frac{1}{\alpha^2 + (2\pi z)^2} e^{2\pi i z \tau}$$

has only one simple pole at  $z = i\frac{\alpha}{2\pi}$  since

$$\frac{1}{\alpha^2 + (2\pi \nu)^2} e^{2\pi i \nu \tau} = \frac{1}{z - i\frac{\alpha}{2\pi}} \frac{1}{(2\pi)^2} \frac{e^{2\pi i z \tau}}{z + i\frac{\alpha}{2\pi}}.$$

It follows then that

$$R_Y(\tau) = 2\pi i \frac{1}{(2\pi)^2} \frac{e^{2\pi i i \frac{\alpha}{2\pi} \tau}}{2i\frac{\alpha}{2\pi}} = \frac{e^{-\alpha\tau}}{2\alpha}, \tau > 0.$$

**Remark 2.** More generally, the function

$$R_Y(\tau) = \frac{1}{2\alpha} e^{-\alpha|\tau|}, \tau \in \mathbb{R}$$

will be also a covariance function of a WSS process called a "random telegraph signal".

*Some details to the above formula:*

$$dY_t = -\alpha Y_t dt + dW_t, y' = -\alpha y + u.$$

Then,  $Y$  is a WSS process with mean 0 and steady-state response

$$Y_t = \int_{-\infty}^t e^{-\alpha(t-s)} dW_s.$$

We can calculate that

$$\begin{aligned} R_Y(\tau) &= E[Y_\tau Y_0] = E\left[\int_{-\infty}^\tau \int_{-\infty}^0 e^{-\alpha(\tau-s)} e^{-\alpha(0-t)} dW_t dW_s\right] = \\ E[Y_\tau Y_0] &= E\left[\int_{-\infty}^\tau \int_{-\infty}^0 e^{-\alpha(\tau-s)} e^{\alpha t} dW_t dW_s\right] = \\ e^{-\alpha\tau} \int_{-\infty}^\tau e^{\alpha s} \left(\int_{-\infty}^0 e^{\alpha t} \delta(t-s) dt\right) ds &= \\ e^{-\alpha\tau} \int_{-\infty}^\tau e^{\alpha s} (e^{\alpha s} \mathbf{1}_{(-\infty, 0]}(s)) ds &= \\ e^{-\alpha\tau} \int_{-\infty}^{\min(\tau, 0)} e^{2\alpha s} ds &= e^{-\alpha\tau} \frac{1}{2\alpha} e^{2\alpha s} \Big|_{-\infty}^{\tau \wedge 0} = \frac{1}{2\alpha} e^{-\alpha|\tau|} \end{aligned}$$

since  $e^{-\alpha\tau} \frac{1}{2\alpha} e^{2\alpha\tau} = \frac{e^{\alpha\tau}}{2\alpha}$  for  $\tau < 0$  and  $e^{-\alpha\tau} \frac{1}{2\alpha} = \frac{e^{-\alpha|\tau|}}{2\alpha}$  for  $\tau > 0$ .

We also note that a "random telegraph signal" process is a WSS process but not a Gaussian process.