1. a) Let A_1, A_2, and A_3 be three arbitrary events. What is the formula for

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$
$- [P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_1)]$
$+ P(A_1 \cap A_2 \cap A_3)$: generalization of addition formula

A_i = "We get a 6 on the i-th die", $i = 1, 2, 3$.

b) Suppose we roll three (fair) dice. What is the probability that we get at least one 6?

• We might use the formula in a).

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{6}$

$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{1}{36} = \frac{1}{6^2}$

$P(A_1 \cap A_2 \cap A_3) = \frac{1}{216} = \frac{1}{6^3}$

$\Rightarrow P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{216} = \frac{108 - 18 + 1}{216}$
$= \frac{91}{216}$

2. a) Let X has an exponential distribution with parameter $\lambda = 1$. That is, X has the density $f(x) = e^{-x}, x > 0$ (and zero otherwise). Define $Y = F(X)$ where $F(x)$ is the cdf of X. Show that the new random variable Y has the uniform distribution on the interval $[0, 1]$.

- We use the formula $f_Y(y) = f_X(F^{-1}(y)) \cdot |(F^{-1}(y))'|$
- We find $F(x) = \int_0^x e^{-y} dy = 1 - e^{-x}, \; x \geq 0$

and invert $y = 1 - e^{-x} \Rightarrow x = -\ln(1-y) = F^{-1}(y)$

$\Rightarrow (F^{-1}(y))' = \frac{1}{1-y}$

$\Rightarrow f_Y(y) = e^{-(-\ln(1-y))} \cdot \frac{1}{1-y} = 1$ for $0 < y < 1$

$\Rightarrow f_Y(y) = \begin{cases} 1, & 0 < y < 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow Y \sim U[0,1]$

b) Now, assume that U a uniform distribution on the interval $[0, 1]$ and F^{-1} is the inverse of the cdf F of the exponential distribution from a), that is $F^{-1}(F(x)) = x$. Let $Y = F^{-1}(U)$. Show that Y has the exponential distribution.

- We find $P(Y \leq y) = P(F^{-1}(U) \leq y) =$

$= P(U \leq F(y)) = F(y)$ since $P(U \leq u) = u$

$\Rightarrow$ the cdf of $Y = F(y) \rightarrow Y$ has exponential distribution with $\lambda = 1$.

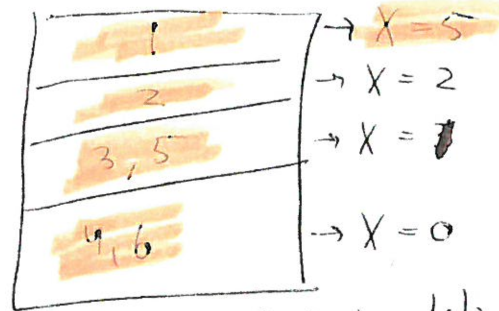
3. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and consider a random variable $X = 2\mathbf{1}_{\{1,2\}} + 3\mathbf{1}_{\{1,3,5\}} - 2\mathbf{1}_{\{3,5\}}$ where

$$\mathbf{1}_A(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A. \end{cases}$$

is an indicator function of a set A.

a) Identify the partition of Ω genetated by X.

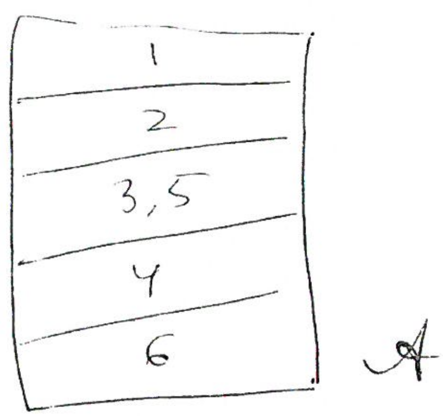
• We find $X(1) = 5$
$X(2) = 2$
$X(3) = 1$
$X(4) = 0$ $\Rightarrow$
$X(5) = 1$
$X(6) = 0$



$\Rightarrow$ partition of Ω

b) Is X a measurable map with respect to the σ-algebra generated by the events from $\mathcal{A} = \{1\}, \{2\}, \{3, 5\}, \{4\}$, and $\{6\}$?

Yes, since any set $X^{-1}(x)$, $x = 0, 2, 3, 5$ can be expressed through the events in $\mathcal{A}$



4. Suppose $X_1, ..., X_n$ are independent random variables each having an exponential distribution with parameter $\lambda > 0$. That is, their pdf has the form $f(x) = \lambda e^{-\lambda x}, x > 0$. Show that $\min\{X_1, ..., X_n\}$ has an exponential distribution with parameter $n\lambda$.

- $F(x) = \int_0^x \lambda \cdot e^{-\lambda y} dy = 1 - e^{-\lambda x}, \quad x > 0$
  $\Downarrow$
- $P(X_i > x) = 1 - F(x) = e^{-\lambda x}$
- $Y = \min\{X_1, ..., X_n\}$

$F_Y(y) = P(\min\{X_1, ..., X_n\} \le y) = 1 - P(\min\{X_1, ..., X_n\} > y)$

$= 1 - P(X_1 > y, X_2 > y, ..., X_n > y) =$ independency $=$

$= 1 - \prod_{i=1}^n P(X_i > y) = 1 - e^{-\lambda n y}$

$\Rightarrow f_Y(y) = F_Y'(y) = (1 - e^{-\lambda n y})' = \begin{cases} \lambda n e^{-\lambda n y}, & y > 0 \\ 0, & y \le 0 \end{cases}$

$\Rightarrow Y$ has exponential distribution with parameter λn.

5. Let (X, Y) has the jointly continuous density

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional densities $f_{X|Y=y}(x,y)$ and $f_{Y|X=x}(x,y)$.

• We find $f_{X|Y=y}(x,y)$ as $f_{X|Y=y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

where $f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y e^{-y}\,dx = x \cdot e^{-y} \Big/_0^y =$

$= y \cdot e^{-y}, \; y > 0$

$\Rightarrow f_{X|Y=y}(x,y) = \dfrac{e^{-y}}{y \cdot e^{-y}} = \dfrac{1}{y}$ for $0 < x < y$

$\Rightarrow X_{|Y=y} \sim U[0,y]$

• We find $f_{Y|X=x}(x,y)$ as $f_{Y|X=x}(x,y) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

where $f_X(x) = \int_x^\infty f(x,y)\,dy = \int_x^\infty e^{-y}\,dy = -e^{-y} \Big/_x^\infty = e^{-x}$

$\Rightarrow f_{Y|X=x}(x,y) = \dfrac{e^{-y}}{e^{-x}} = e^{-(y-x)}$ for $x < y < \infty$

6. Suppose U and V are independent random variables having both uniform distribution on the interval $[0, 1]$. Find $E(U - V)^2$.

$U \sim U[0,1]$

$V \sim U[0,1]$

$\Rightarrow f_{U,V}(u,v) = f_U(u) \cdot f_V(v) = 1 \cdot 1 = \begin{cases} 1, & 0 \leq u \leq 1, \quad 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow E(U-V)^2 = \int_0^1 \int_0^1 (u-v)^2 \cdot f_{U,V}(u,v)\, du\, dv =$

$= \int_0^1 \int_0^1 (u-v)^2\, du\, dv = \int_0^1 \int_0^1 (u^2 - 2uv + v^2)\, du\, dv =$

$= \int_0^1 \left(\frac{u^3}{3} - u^2 \cdot v + u \cdot v^2 \right) \Big|_0^1 dv = \int_0^1 \left(\frac{1}{3} - v + v^2 \right) dv$

$= \left(\frac{v}{3} - \frac{v^2}{2} + \frac{v^3}{3} \right) \Big|_0^1 = \frac{1}{6}.$

7. Let X has a Poisson distribution with parameter $\lambda > 0$, that is

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0, 1, 2, \ldots.$$

Find the variance of X.

- $Var(X) = E(X^2) - (EX)^2$,

- $E(X) = \sum_{x=0}^{\infty} x \cdot P(X=x) = e^{-\lambda}\sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} =$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda \cdot e^{-\lambda} \cdot \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}_{e^{\lambda}} = \lambda,$$

$$E(X^2) = E(X(X-1)) + \underbrace{EX}_{=\lambda}$$

- $E(X(X-1)) = \sum_{x=0}^{\infty} e^{-\lambda} x(x-1) \frac{\lambda^x}{x!} =$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot e^{-\lambda} \cdot \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}_{e^{\lambda}} = \lambda^2$$

$$\Rightarrow Var(X) = E(X(X-1)) + EX - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$