ESE 520 Probability and Stochastic Processes

HW 1: due by 11pm CST on Tuesday, September 10, 2024

Total: 40 points

Problem 1. Let Ω be a sample space and A, B be arbitrary subsets from Ω.

a) Assuming that $A \subset B$, show that

$$(B \cap C) \cup A = B \cap (C \cup A) \tag{1}$$

for every subset $C \subset \Omega$;

b) Assuming that (1) holds for some subset C, show that $A \subset B$.

Problem 2. A collection of plastic letters, a-z, is mixed in a jair (26 letters in total). Two letters are drawn at random, one after the other. What is the probability of drawing a vowel (a,e,i,o,u) and a consonant in either order? Two vowels in any order? Specify the sample space Ω and probability P.

Problem 3. Use appropriate properties of probability to show that, for any sequence of events $\{A_n\}\}, n = 1, 2, ...,$ it holds that

$$P\Big(\cap_{n=1}^{\infty} A_n\Big) = \lim_{N\to\infty} P\Big(\cap_{n=1}^{N} A_n\Big).$$

Problem 4. Let $\mathcal{A} =: A_1, ..., A_n$ be a partition of Ω. Show that $\sigma(\mathcal{A})$ consists of empty set along with all unions of the form $\cup_i A_{k_i}$, where k_i is a finite subsequence of distinct elements from $\{1, 2, ..., , n\}$.

Problem 5. a) Assume $\mathcal{F}_i \in I$ is a σ-algebra, where I is an index set (countable or uncountable). Show that $\mathcal{F} := \cap_{i\in I}\mathcal{F}_i$ is an σ-algebra as well.

b) Argue that union of two σ-algebras is not a σ-algebra (in general).

Problem 6. There are k students in a probability class. What is probability of A="No two students have the same birthday"? Give numerical answer for $k = 30$. Assume that there are 365 days in a year.

Problem 7. Let A="Alice and Betty have the same birthday, B="Betty and Carol have the same birthday", C="Carol and Alice have the same birthday". Show that events $\{A, B, C\}$ are *pairwise independent*. Are all three events independent as well? Assume that there are 365 days in a year.

Problem 8. a) Suppose that the number of children in a family is 1,2, or 3 with probability 1/3 each. Little Bobby has no brothers. What is the probability he is an only child?

b) Suppose that the number of children in a family is 1,2, or 3 with probability 1/3 each. Little Bobby has no sisters. What is the probability he is an only child?

Problem 9. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A, B are two arbitrary events from $\mathcal{F}$. Show that events A and B are independent if and only if the σ-algebras $\mathcal{F}^A$ and $\mathcal{F}^B$ generated by A and B, respectively, are independent. Recall that, by definition, $\mathcal{F}^A = \{\emptyset, \Omega, A, A^c\}$ and $\mathcal{F}^B = \{\emptyset, \Omega, B, B^c\}$.