ESE 520 Probability and Stochastic Processes

HW 5: due by 11pm CST on Thursday, November 14, 2024

Total: 40 points

Problem 1. a) Show that a random variable X with exponential distribution satisfies the equation

$$P(X > t + s) = P(X > t)P(X > s)$$

for all $s, t \geq 0$.

b) Show that equality in part a) is equivalent to

$$P(X > t + s | X > s) = P(X > t)$$

for all $s, t \geq 0$.

The equality in part b) is known as "lack of memory property".

Problem 2. Let $(N_t), t \geq 0$ be a Poisson process with parameter $\lambda > 0$. Compute

$$P(N_s = 0, N_t = 1)$$

for any $0 \leq s \leq t$.

Problem 3. Let (N_t) be a Poisson process with parameter $\lambda > 0$. Verify that

$$P(N_t = \text{ is odd}) = e^{-\lambda t} \sinh(\lambda t),$$

$$P(N_t = \text{ is even}) = e^{-\lambda t} \cosh(\lambda t).$$

Problem 4. Diners arrive at popular restaurant according to a Poisson process N_t of rate λ. A confused maitre d" seats the ith diner with probability p, and turns the diner away with probability $1 - p$. Let $Y_i = 1$ if the ith diner is seated, and $Y_i = 0$ otherwise. The number of diners seated up to time t is

$$M_t := \sum_{i=1}^{N_t} Y_i.$$

Show that M_t is a Poisson random variable and find its parameter. Assume the Y_i are independent of each other and the Poisson process.

Problem 5. Let $(W_t), t \geq 0$ be a Wiener process and $T > 0$ is a fixed time. Show that

$$V_t := W_{T+t} - W_T, t \geq 0$$

is a Winer process again.

Problem 6. Show that the transition density $n(t, y)$ (see lecture notes for its definition) of the Wiener process (W_t) satisfies the following heat equation

$$\frac{\partial n}{\partial t} = \frac{1}{2} \frac{\partial^2 n}{\partial y^2}.$$

Problem 7. We call $W_t = (W_t^1, ..., W_t^n)$ an *n-dimensional Winer process* if $W_t^1, ..., W_t^n$ are independent one-dimensional Wiener processes. For a two-dimensional Wiener process $W_t = (W_t^1, W_t^2)$ find the probability that $|W_t| < R$, wher $R > 0$ and $|x|$ is the Euclidean norm in $\mathbb{R}^2$.

Hint: Use polar coordinates transformation when calculating the desired probability.

Problem 8. Let $(W_t), t \geq 0$ be a standard Wiener process, and let $f_{t_1,t_2,...,t_n}$ denote the joint density of $(W_{t_1}, W_{t_2}, ..., W_{t_n})$. Find $f_{t_1,t_2,...,t_n}$.