ESE 520 Probability and Stochastic Processes

Instructor: Vladimir Kurenok

EXAM 2-Sample

Name (Please print):

Solutions

Instructions:

1. You must show all work to completely justify your answers in order to receive any credit.
2. You can use One one-sided sheet of paper with your own formulas.

1. a) Assume X has a normal distribution with mean μ and variance σ^2. Find the moment-generating function of X.

- $X \sim N(\mu, \sigma^2) \Rightarrow X = \sigma \cdot Y + \mu$ with $Y \sim N(0,1)$
- $\psi_Y(t) = E e^{tY} = \int_{\mathbb{R}} e^{ty} \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2} dy = e^{t^2/2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-t)^2}{2}} dy}_{=1}$

$= e^{t^2/2}, \quad t \in \mathbb{R}$

- $\psi_X(t) = E\left(e^{t(\mu + \sigma Y)}\right) = e^{\mu t} \cdot E e^{t\sigma Y} = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$

$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$

b) Use part a) to find $E(X)$;

$\psi_X'(t) = (\mu + t\sigma^2) \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}}$

& $E(X) = \psi_X'(0) = \mu.$

2. Let X and Y be two random variables having jointly continuous distribution with the joint pdf given as

$$f(x,y) = \begin{cases} 8xy, & 0 < y < x < 1 \\ 0, & \text{otherwise} . \end{cases}$$

Find the best linear mean-square estimate of Y by X.

- $E(Y|X) = \mu_Y + \frac{cov(X,Y)}{Var(X)} \cdot (X - \mu_X)$,
- $f_X(x) = \int_0^x 8xy\,dy = 4xy^2 /_0^x = 4x^3$;
- $\mu_X = E(X) = \int_0^1 4x^4 dx = \frac{4}{5} x^5 /_0^1 = \frac{4}{5}$;
- $f_Y(y) = \int_y^1 8xy\,dx = 4x^2 y /_y^1 = 4(y - y^3)$;
- $\mu_Y = E(Y) = \int_0^1 4(y^2 - y^4) dy = 4\left(\frac{y^3}{3} - \frac{y^5}{5}\right) /_0^1 = \frac{8}{15}$;
- $E(X^2) = \int_0^1 4x^5 dx = 2/3$;
- $Var(X) = E(X^2) - (E(X))^2 = 2/3 - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$;
- $E(X \cdot Y) = \int_0^1 \left(\int_0^x 8x^2 y^2 dy\right) dx = 8\int_0^1 x^2 \frac{x^3}{3} /_0^x dx$

$$= \frac{8}{3} \int_0^1 x^5 dx = \frac{8}{3 \cdot 6} = \frac{4}{9};$$

- $cov(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{4}{9} - \frac{4}{5} \cdot \frac{8}{15} = \frac{4 \cdot 75 - 9 \cdot 32}{9 \cdot 75}$
- $\frac{cov(X,Y)}{Var(X)} = \frac{4 \cdot 75 - 9 \cdot 32}{9 \cdot 75} \cdot \frac{75}{2} = \frac{2 \cdot 75 - 9 \cdot 16}{9} = \frac{2}{3}$
- $E(Y|X) = \frac{8}{15} + \frac{2}{3}\left(X - \frac{4}{5}\right)$

3. Suppose X and Y have joint density e^{-x} for $0 < y < x$. Find $E(X|Y)$ and $Var(X|Y)$.

- $f_Y(y) = \int_y^{\infty} e^{-x} dx = e^{-y}, \quad y > 0$
- $E(X|Y=y) = \int_y^{\infty} x \cdot \frac{e^{-x}}{e^{-y}} dx = e^{y}\left(-x \cdot e^{-x}\Big|_y^{\infty} + \int_y^{\infty} e^{-x} dx\right)$

$= e^{y}\left(y \cdot e^{-y} - e^{-x}\Big|_y^{\infty}\right) = y+1$

- $E(X^2|Y=y) = \int_y^{\infty} x^2 \cdot e^{y} \cdot e^{-x} dx = e^{y}\left(-x^2 \cdot e^{-x}\Big|_y^{\infty} + \right.$

$\left. + 2 \underbrace{\int_y^{\infty} x \cdot e^{-x} dx}_{\text{calculated before}}\right) = e^{y}\left(y^2 e^{-y} + 2\left(y e^{-y} + e^{-y}\right)\right)$

$= y^2 + 2(y+1)$

- $Var(X|Y) = E(X^2|y) - \left(E(X|y)\right)^2 =$

$= y^2 + 2(y+1) - (y+1)^2 = 1$

4. Suppose you play roulette and bet $1 on black each time. Your net winnings on the ith play, X_i, are 1 with probability 1/3 and -1 with probability 2/3. What is the probability your net winnings are ≥ 0 after 100 plays?

Use the Central Limit Theorem and the enclosed normal table to find the approximate probability.

- CLT: $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ appr., $\bar{X}_n := \frac{X_1 + \cdots + X_n}{n}$

with $E(X_i) = 1 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3} = -\frac{1}{3} = \mu$ &

$Var(X_i) = E(X_i^2) - (E(X_i))^2 = 1 - \frac{1}{9} = \frac{8}{9} = \sigma^2$

- $P(\sum_{i=0}^{100} X_i \geq 0) = P(\bar{X}_{100} \geq 0) = P(\bar{X}_{100} - \mu \geq \frac{1}{3})$

$= P\left(\frac{\bar{X}_{100} - \mu}{\sigma/\sqrt{100}} \geq \frac{1/3}{\sqrt{8}/30} \right) = P(Z \geq \frac{10}{\sqrt{8}})$

$\approx P(Z \geq 3.5) \approx 0$

5. Suppose U, V, W are independent and have mean μ and variance σ^2. Find $\rho(U + V, V + W)$.

- $\rho(U+V, V+W) = \dfrac{\mathrm{cov}(U+V, V+W)}{\sqrt{\mathrm{Var}(U+V)\cdot \mathrm{Var}(V+W)}}$

- $\mathrm{cov}(U+V, V+W) = E(U+V)(V+W) - E(U+V)\cdot E(V+W)$

$= E(U\cdot V) + EV^2 + E(U\cdot W) + E(V\cdot W) - ([\underbrace{E(U)}_{\mu} + \underbrace{E(V)}_{\mu}][\underbrace{(EV)}_{\mu} + \underbrace{EW}_{\mu}])$

= independence

$= (EU)(EV) + EV^2 + (EU)(EW) + (EV)(EW) - 4\mu^2$

$= \mu^2 + \sigma^2 + \mu^2 + \mu^2 + \mu^2 - 4\mu^2 = \sigma^2$

- $\mathrm{Var}(U+V) = \mathrm{Var}(V+W) = 2\sigma^2$

$\Rightarrow \rho = \dfrac{\sigma^2}{\sqrt{2\sigma^2\, 2\sigma^2}} = \dfrac{1}{2}.$

6. Suppose X and Y are independent exponential random variables with the same parameter $\lambda > 0$. Find the density of $\min\{X, Y\}$.

$$Z = \min\{X, Y) \qquad f(x) = \lambda \cdot e^{-\lambda x}, \; x > 0$$

- $F_Z(\cancel{z}) = 1 - P(Z \geq z) =$

$$= 1 - P(\min\{X, Y\} \geq z) = 1 - P(X \geq z) \cdot P(Y \geq z) =$$

$$= 1 - (1 - F_X(z))^2 = 1 - (1 - e^{-\lambda z})^2$$

- $f_Z(z) = F_Z'(z) = 2(1 - e^{-\lambda z}) \cdot \lambda e^{-\lambda z}, \; z > 0$

7. Let X_1 X_2 be independent random variables each of which has a normal distribution $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, respectively. Define the random variable $Y := X_1 + X_2$. Using characteristic functions, show that Y also has a normal distribution $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Hint: The characteristic function of a normal rv with mean μ and variance σ^2 has the form $\phi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.

- $\varphi_Y(t) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t) = e^{i(\mu_1 + \mu_2)t - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \cdot t^2}$

- $\Rightarrow Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$