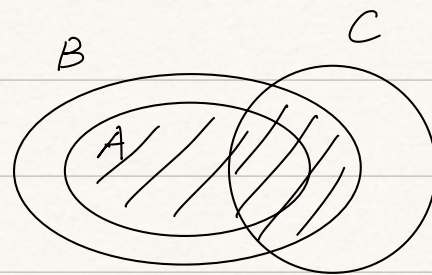1. (a) Assuming that $A \subset B$

$$RHS = B \cap (C \cup A)$$

$$= (B \cap C) \cup (B \cap A)$$

$$= (B \cap C) \cup A = LHS$$

(b) Assuming that i) holds for some subset C,

Suppose $\forall x \in A$,

since $A \subset (B \cap C) \cup A$, we have $x \in (B \cap C) \cup A$

due to $(B \cap C) \cup A = B \cap (C \cup A)$, we also have $x \in B \cap (C \cup A)$

since, $B \cap (C \cup A) \subset B$, $x \in B$

Thus, $\forall x \in A$, $x \in B$. $\Rightarrow A \subset B$

B   C

2. 5 vowel $\{a, e, i, o, u\}$, 21 consonants

$$|\Omega| = A_{26}^2 = 26 \times 25 = 650$$

$$\Omega = \{\{a, b\}, \cdots, \{a, z\}, \{b.a\}. \{b.c\}. \cdots \{b.z\}, \cdots \{y.a\}, \cdots \{y.z\}\}$$

The sample space Ω has 650 possible outcomes and above format.

(a) $P(\text{vowel and consonant}) = \dfrac{5 \times 21 + 21 \times 5}{650} = \dfrac{210}{650} = \dfrac{21}{65}$
    in either order

(b) $P(\text{two vowels}) = \dfrac{5 \times 4}{650} = \dfrac{20}{650} = \dfrac{2}{65}$
    in any order

3. $\quad P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left[\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right]$

$$= 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$

Suppose that $B_N = \bigcup_{n=1}^{N} A_n^c$, then we have $B_n \subset B_{n+1}$ for each n

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} P(B_n) = \lim_{N \to \infty} P\left(\bigcup_{n=1}^{N} A_n^c\right)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$= P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \qquad\qquad = \lim_{N \to \infty} P\left[\left(\bigcap_{n=1}^{N} A_n\right)^c\right]$$

$$= 1 - P\left(\bigcap_{n=1}^{\infty} A_n\right) \underset{\text{equal}}{\rule{3cm}{0.4pt}} = \lim_{N \to \infty}\left[1 - P\left(\bigcap_{n=1}^{N} A_n\right)\right]$$

$$= 1 - \lim_{N \to \infty} P\left(\bigcap_{n=1}^{N} A_n\right)$$

Therefore, $P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N \to \infty} P\left(\bigcap_{n=1}^{N} A_n\right)$

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4. Since, $A = A_1, A_2, \cdots, A_n$ be a partitio of Ω.

Then, we have $A_i \cap A_j = \emptyset$, for all $i \neq j$ and $\bigcup_{i=1}^{n} A_i = \Omega$

All subsets of Ω can be represented by countable unions, intersections

and complements of $A_1, \cdots, A_n$. Since, $A_1 \cdots A_n$ are mutually exclusive,

any subset of Ω formed by operations on A_i, like unions, intersections, and

complements, is either be an individual A_i, a union of some A_i, the

empty set $\emptyset$, or Ω.

Since, $A_1, \cdots, A_n$ is finite and mutually exclusive, the unions of some A_i

will be a finite union, the intersection will be a empty space $\emptyset$, and the compliment

of any A_i in Ω will be $\bigcup_{j \neq i} A_j$.

Therefore, $6(A)$ consists of empty set along with all unions of the form $\bigcup_i A_{k_i}$,

where k_i is a finite sequence of distinct elements from $\{1, 2, \cdots, n\}$

5. (a) Assume $F_i \in I$ is a σ-algebra, where I is an index set.

We define that $F = \bigcap_{i \in I} F_i$

Contains the empty set; since, F_i is a σ-algebra, $\emptyset \in F_i$. Thus, $\emptyset \in \bigcap_{i \in I} F_i = F$

closed under complements: Suppose A is an element of F, $A \in F = \bigcap_{i \in I} F_i$,

which means that $A \in F_i$, for each i.

Since F_i is σ-algebra, $A \in F_i$, $A^c \in F_i$, for each i,

then we have $A^c \in \bigcap_{i \in I} F_i = F$

closed under countable unions: Suppose $\{A_n\}$ is a countable collection of sets where

each $A_n \in F = \bigcap_{i \in I} F_i$, which implies that $A_n \in F_i$ for each i.

Since, F_i is σ-algebra, $\bigcup_{n=1}^{\infty} A_n \in F_i$ for each i. Therefore $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{i \in I} F_i = F$

Therefore, F meets all the criteria of a σ-algebra, F is a σ-algebra.

(b) Assume that F_1 and F_2 are two σ-algebra, consider $A \in F_1 \cup F_2$

Contains the empty set: since $\emptyset \in F_1$, $\emptyset \in F_1$, we have $\emptyset \in F_1 \cup F_2$

Closed under complement: If $A \in F_1$ but $A \notin F_2$, we have $A^c \in F_1$, $A^c \in F_1 \cup F_2$
$(F_2)(F_1)(F_2)$
If $A \in F_1$, $A \in F_2$, we have $A^c \in F_1$, $A^c \in F_2$, $A^c \in F_1 \cup F_2$

Closed under countable unions: Suppose $\{A_i\}$ are sets, such that each $A_i \in F_1 \cup F_2$

This means that each A_i is either in F_1 or F_2.

If A_i are all in F_1, $\bigcup_{i=1}^{\infty} A_i \in F_1$, $\bigcup_{i=1}^{\infty} A_i \in F_1 \cup F_2$

If A_i are all in F_2, $\bigcup_{i=1}^{\infty} A_i \in F_2$, $\bigcup_{i=1}^{\infty} A_i \in F_1 \cup F_2$

However, if some A_i in F_1 and some A_i in F_2, $\bigcup_{i=1}^{\infty} A_i$ might not bleong entirely

to F_1 or F_2, which is conflected to F_1 and F_2 are both σ-algebra.

For example, if $F_1 = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$

$F_2 = \{\emptyset, \{b\}, \{a,c\}, \{a,b,c\}\}$

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{a\}, \{b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

However, $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$

Therefore, the union of two σ-algebras is not a σ-algebra in general.

6. A = No two students have the same birthdays

$$P(A) = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-k+1}{365}$$

$$= \frac{365!}{365^k (365-k)!}$$

If $k = 30$, we have $P(A) = \frac{365!}{365^{30} \times 335!}$

$$= \frac{365 \times 364 \times \cdots \times 336}{365^{30}}$$

$$\approx 0.2937$$

7. A = <u>Alice</u> and <u>Betty</u> have the same birthday

B = <u>Betty</u> and <u>Carol</u> have the same birthday

C = <u>Carol</u> and <u>Alice</u> have the same birthday

$$P(A) = \frac{1}{365}, \quad P(B) = \frac{1}{365}, \quad P(C) = \frac{1}{365}$$

$$\implies P(A \cap B) = \frac{1}{365^2} = P(A)P(B)$$

$$P(B \cap C) = \frac{1}{365^2} = P(B)P(C)$$

$$P(A \cap C) = \frac{1}{365^2} = P(A)P(C)$$

Therefore, events $\{A, B, C\}$ are pairwise independent

Since $P(A \cap B \cap C)$ = Probability that all three have the same birthday $= \frac{1}{365^2}$

$$P(A)P(B)P(C) = \frac{1}{365^3}$$

$$\implies P(A \cap B \cap C) \neq P(A)P(B)P(C)$$

Thus, all three events are not independent

8. (a)   A = Bobby is an only child

B = Bobby has no brother

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B/A)\,P(A)}{P(B)}$$

When Bobby is an only child, he has no brother, which means $P(B/A) = 1$

$$P(A/B) = \frac{P(A)}{P(B)} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2}} = \frac{1}{1 + \frac{1}{2} + \frac{1}{4}} = \frac{4}{7}$$

(b)   A = Bobby is an only child

B = Bobby has no sister

The result and procedure are as same as problem (a)

$$P(A/B) = \frac{P(A)}{P(B)} = \frac{4}{7}$$

9. ($\Rightarrow$)   Assume that A and B are independent

Then we have $P(A \cap B) = P(A)P(B)$

We need to show that any $E \in \mathcal{F}^A$ and $F \in \mathcal{F}^B$, we will have $P(E \cap F) = P(E)$

Since $\mathcal{F}^A = \{\phi, \Omega, A, A^c\}$, $\mathcal{F}^B = \{\phi, \Omega, B, B^c\}$

$P(\phi \cap \phi) = P(\phi)P(\phi) = P(\phi \cap \Omega) = P(\phi)P(\Omega) = P(\phi \cap B) = P(\phi)P(B) = P(\phi \cap B^c) = P(\phi)P(B^c) = 0$

$P(\Omega \cap \phi) = P(\Omega)P(\phi) = 0, \ P(\Omega \cap \Omega) = P(\Omega)P(\Omega) = 1, \ P(\Omega \cap B) = P(\Omega)P(B) = P(B)$

$$P(\Omega \cap B^c) = P(\Omega)P(B^c) = P(B^c)$$

$P(A \cap \phi) = P(A)P(\phi) = 0, \ P(A \cap \Omega) = P(A)P(\Omega) = P(A), \ P(A \cap B) = P(A)P(B)$

$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$

$P(A^c \cap \phi) = P(A)P(\phi) = 0, \ P(A^c \cap \Omega) = P(A^c)P(\Omega) = P(A^c),$

$P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(B)P(A^c)$

$P(A^c \cap B^c) = P(B^c) - P(B^c \cap A) = P(B^c) - P(A)P(B^c) = P(B^c)(1 - P(A)) = P(B^c)P(A^c)$

Therefore, the σ-algebras $\mathcal{F}^A$ and $\mathcal{F}^B$ are independent

($\Longleftarrow$) Assume that $\mathcal{F}^A$ and $\mathcal{F}^B$ are independent

Since $A \in \mathcal{F}^A$, $B \in \mathcal{F}^B$, $P(A \cap B) = P(A)P(B)$

Event A and B are independent.

Therefore, A and B are independent if and only if σ-algebra $\mathcal{F}^A$ and $\mathcal{F}^B$ are independent