1. The joint density of a two-dimensional, non-degenerate Gaussian random vector (X,Y) is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{3}} e^{-\frac{1}{6}[4(x+1)^2-2(x+1)(y-2)+(y-2)^2]}$$

What are the means, variances and covariance of random variables X and Y?

- $f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \cdot \det R}} e^{-\frac{1}{2}[(\mathbf{x}-\mu)^T \cdot R^{-1} \cdot (\mathbf{x}-\mu)]}$, $\mathbf{x}=(x,y)$

- For the exponent alone:

$$-\frac{1}{2}(\mathbf{x}-\mu)^T R^{-1} \cdot (\mathbf{x}-\mu) = -\frac{1}{2}\left[(x+1, y-2) \underbrace{\begin{pmatrix} 4/3 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{R^{-1}} \begin{pmatrix} x+1 \\ y-2 \end{pmatrix}\right]$$

$\Rightarrow \underline{\mu_X = -1}, \ \underline{\mu_Y = 2}$

$$\Rightarrow R = (R^{-1})^{-1} = \frac{1}{4/9 - 1/9} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix} = 3 \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix} \Rightarrow$$

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \Rightarrow \underline{\sigma_X^2 = 1}, \ \underline{\sigma_Y^2 = 4}, \ \underline{cov(X,Y) = 1}$$

2. Let Y be a $\mathcal{N}(0,1)$ random variable and $X := e^Y$ Find the expected value of X.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$E(X) = \int_{\mathbb{R}} e^y f_Y(y)\,dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^y \cdot e^{-y^2/2}\,dy =$$

$$= e^{1/2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(y-1)^2}{2}}\,dy}_{=1} = \boxed{e^{1/2}}$$

3. Two measurements $y_1 = 2$ and y_2 5 are taken to estimate the value x of a random variable X. Assume that the joint distribution of the full random vector (X, Y_1, Y_2) is Gaussian and that X has expectation $\mu_X = 3$, variance $\sigma_X^2 = 4$, and the following covariances with the random variables Y_1 and Y_2 which describe the first and second measurement, $cov(X, Y_1) = -1$ and $cov(X, Y_2) = +1$. Also suppose the measurements Y_1 and Y_2 have means $\mu_{Y_1} = \mu_{Y_2} = 3$, variances $\sigma_{Y_1}^2 = 3$ and $\sigma_{Y_2}^2 = 5$ and covariance $cov(Y_1, Y_2) = -2$. Find the best mean square estimate for X in terms of these measurements.

- We use the formula

$$E(X \mid Y) = \mu_X + R_{XY} \cdot R_{YY}^{-1} (Y - \mu_Y)$$

where $\mu_X = 3$, $\mu_Y = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $R_{XY} = (-1, 1)$

$$R_{YY} = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad R_{YY}^{-1} = \frac{1}{11} \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\Rightarrow E\left(X \,\middle|\, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) = 3 + (-1, 1) \cdot \frac{1}{11} \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= 3 + \frac{1}{11} (-1, 1) \begin{pmatrix} -1 \\ 4 \end{pmatrix} = 3 + \frac{5}{11} = \frac{38}{11}$$

4. Let X be a continuous random variable with pdf $f(x) = \frac{\lambda}{2}e^{-\lambda|x|}, x \in \mathbb{R}$ (Laplace distribution). Find the moment-generating function $\psi_X(t)$. Use it to find $Var(X)$.

$$\psi(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{\lambda}{2} \cdot e^{-\lambda|x|} dx = \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{tx} \cdot e^{\lambda x} dx + \int_{0}^{\infty} e^{tx} \cdot e^{-\lambda x} dx \right]$$

$$= \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{x(\lambda+t)} dx + \int_0^{\infty} e^{-x(\lambda-t)} dx \right] =$$

$$t < -\lambda \qquad \lambda > t$$

$$= \frac{\lambda}{2}\left[\frac{1}{\lambda+t} e^{x(\lambda+t)} \Big|_{-\infty}^{0} - \frac{1}{\lambda-t} e^{-x(\lambda-t)} \Big|_{0}^{\infty} \right]$$

$$= \frac{\lambda}{2}\left[\frac{1}{\lambda+t} + \frac{1}{\lambda-t} \right] = \boxed{\frac{\lambda^2}{\lambda^2-t^2}}$$

$$\psi_X'(t) = \frac{2t\lambda^2}{(\lambda^2-t^2)^2} \Rightarrow \psi_X'(0) = 0 = \underline{E(X)}$$

$$\psi_X''(t) = 2\lambda^2 \frac{(\lambda^2-t^2)^2 + 4t^2(\lambda^2-t^2)}{(\lambda^2-t^2)^2} \Rightarrow \psi_X''(0) = \frac{2}{\lambda^2} = E(X^2)$$

$$\Rightarrow Var(X) = E(X^2) - (E(X))^2 = \boxed{\frac{2}{\lambda^2}}$$

5. Suppose U, V, W are independent and have mean μ and variance σ^2. Find $\rho(U+V, V+W)$.

$$\rho(U+V, V+W) = \frac{\operatorname{cov}(U+V, V+W)}{\sqrt{Var(U+V) \cdot Var(V+W)}}$$

$$\operatorname{cov}(U+V, V+W) = E(U+V)(V+W) - (E(U+V)) \cdot (E(V+W))$$

$$= E(UV) + EV^2 + E(U \cdot W) + E(VW) - ((\underbrace{EU}_{\mu}) + (\underbrace{EV}_{\mu}))((\underbrace{EV}_{\mu}) + (\underbrace{EU}_{\mu}))$$

$$= \text{independence} = (EU)(EV) + EV^2 + (EU)(EW) + (EV)(EW) - 4\mu^2 =$$

$$= \mu^2 + \underbrace{\sigma^2 + \mu^2}_{EV^2} + \mu^2 + \mu^2 - 4\mu^2 = \sigma^2$$

$$Var(U+V) = 2\sigma^2$$
$$Var(V+W) = 2\sigma^2$$ by independence!

$$\Rightarrow \rho = \frac{\sigma^2}{\sqrt{2\sigma^2 \cdot 2\sigma^2}} = \frac{1}{2}.$$

6. Suppose X and Y have joint density $(2m+6)x^m y$ for $0 < y < x < 1$. Find $E(Y|X)$.

$$f_X(x) = \int_0^x (2m+6)\, x^m y\, dy = (2m+6)\,\frac{x^{m+2}}{2},$$

$$f_{Y|X=x}(x,y) = \frac{f_{XY}(x,y)}{f_Y(x)} = \frac{2y}{x^2},$$

$$E(Y|X=x) = \int_0^x y\,\frac{2}{x^2}\,dy = \frac{2}{3}x$$

$$E(Y|X) = \tfrac{2}{3}X.$$

7. Let X be an exponential random variable with parameter $\lambda > 0$.

a) Compute the characteristic function of X.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$\varphi(t) = E(e^{itX}) = \lambda \int_0^\infty e^{itx} \cdot e^{-\lambda x} dx =$$

$$= \lambda \int_0^\infty e^{x(it-\lambda)} dx = \frac{\lambda}{it-\lambda} e^{x(it-\lambda)} \Big/_0^\infty =$$

$$\frac{\lambda}{\lambda - it}, \quad t \in \mathbb{R}$$

b) Use part a) to find $Var(X)$.

$$E(X) = \frac{1}{i} \varphi'(0) = \frac{1}{i} \frac{\lambda i}{(\lambda - it)^2} \Big/ t=0$$

$$= \boxed{\frac{1}{\lambda}}$$

$$E(X^2) = \varphi''(0) = -\frac{2\lambda i^2}{(\lambda - it)^3} \Big/ t=0 = \boxed{\frac{2}{\lambda^2}}$$

$$\Rightarrow Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$