ESE 520 Probability and Stochastic Processes

Lecture 17

"The Poisson process continued."

What can be said about the relation of a Poisson process $(N_t), t \geq 0$ and the sequence of random variables $\{T_k\}, k = 1, 2, ...$ from the point of view of their probability distributions?

First, we can write

$F_1(t) = P(T_1 \leq t) = 1 - P(T_1 > t) = 1 - P(N_t = 0) = 1 - P_0(t) = 1 - e^{-\lambda t}$

so that the pdf of T_1 is given by

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Therefore, T_1 has the *exponential distribution* with parameter λ.

For any $k = 1, 2, ...$, we have that

$$F_k(t) = P(T_k \leq t) = P(N_t > k - 1) =$$

$$1 - P(N_t \leq k - 1) = 1 - \sum_{j=0}^{k-1} P(N_t = j) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!} e^{-\lambda t}.$$

The density of T_k is then found as

$$f_k(t) = F_k'(t) = -(\sum_{j=0}^{k-1} j\lambda \frac{(\lambda t)^{j-1}}{j!})e^{-\lambda t} + \lambda \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!} e^{-\lambda t} =$$

$$\lambda e^{-\lambda t}[-\sum_{j=1}^{k-1} \frac{(\lambda t)^{j-1}}{(j-1)!} + \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}] =$$

$$\lambda e^{-\lambda t}[-\sum_{i=0}^{k-2} \frac{(\lambda t)^i}{(i)!} + \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}] =$$

$$\lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}, k = 1, 2, ... \tag{1}$$

The pdf $f_k(t)$ found in (1) is called the *Erlang's probability distribution*, with notation Erlang $(k, \lambda), k = 1, 2, \dots$

We see that Erlang (k, λ) is nothing but the Gamma probability distribution $\Gamma(\alpha, \beta)$ (in notation we used before), where the first parameter $\alpha = k$ is an integer and the second parameter $\beta = \frac{1}{\lambda}$.

Using general properties of Gamma distribution, we have that

$$E(T_k) = \frac{k}{\lambda},$$

and

$$Var(T_k) = \frac{k}{\lambda^2}.$$

Moreover, for any $t > 0$, we can formally calculate that

$$P(T_k > s + t | T_{k-1} = s) = P(N_{s+t} = k - 1 | N_s = k - 1) =$$

$$P(N_{s+t} - N_s = 0 | N_s = k - 1) = P(N_t = 0) = e^{-\lambda t}$$

since N has independent increments.

It implies that

$$P(T_k - T_{k-1} > t | T_{k-1} = s) = e^{-\lambda t}$$

meaning that *random variable $T_k - T_{k-1}$ is independent of earlier arrivals and has the same distribution as T_1.*

Thus, by letting

$$\tau_k := T_k - T_{k-1},$$

we have shown that **random variables** $\tau_k, k = 1, 2, \dots$ **are i.i.d. random variables each of which has exponential distribution with parameter** $\lambda > 0$.

Also, it holds

$$T_k = \sum_{i=1}^{k} \tau_i \text{ with } \tau_0 = 0.$$

Example 1. Let $(N_t), t \geq 0$ be a Poisson process with parameter $\lambda > 0$. Show that

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \text{ a.e.}$$

Solution: Choose $n \in \mathbb{N}$ and write

$$N_n = N_1 + (N_2 - N_1) + \ldots + (N_n - N_{n-1}) = \sum_{i=1}^{n} X_i,$$

where $X_i := N_i - N_{i-1}$.

By independence of increments, all $X_1, \ldots, X_n$ are independent random variables with

$$E(X_i) = \lambda, i = 1, 2, \ldots, n.$$

By strong law of large numbers,

$$\frac{1}{n} N_n \to \lambda \text{ as } n \to \infty \text{ a.e.}$$

Then, for any $t > 1$, we find $n \in \mathbb{N}$ such that $n \leq t \leq n+1$ and

$$N_n \leq N_t \leq N_{n+1}$$

implying

$$\frac{n}{n+1} \frac{N_n}{n} \leq \frac{N_n}{n+1} \leq \frac{N_t}{n+1} \leq \frac{N_{n+1}}{n+1}.$$

Since $\lim_{n\to\infty} \frac{N_n}{n} = \lim_{n\to\infty} \frac{N_{n+1}}{n+1}$, it follows that $\lim_{t\to\infty} \frac{N_t}{t} = \lambda$ a.e.

Alternative characterization of Poisson process $(N_t), t \geq 0$:

Let $\tau_1, \tau_2, \ldots$ be a sequence of independent random variables each of which having the same exponential distribution with parameter $\lambda > 0$.

For example, τ_i could be times between the emissions of radioactive particles or times between calls made at a telephone tower.

Define

$$T_n := \tau_1 + \ldots + \tau_n \text{ with } \tau_0 = 0, n = 0, 1, \ldots$$

T_n can be interpreted as time of n-th emission (the n-th call).

Let

$$N_t := \max_n \{n : T_n \leq t\}, t \geq 0.$$

N_t is then the number of calls made up to time t (on the interval $[0, t]$). Alternatively,

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{\{T_k \le t\}}.$$

Theorem. It holds that

$$P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, n = 0, 1, 2, ...$$

Proof. We observe that

$$\{N_t < n\} = \{T_n > t\}.$$

Also, since (N_t) is an integer-valued process, we can write

$$P(N_t = n) = P(N_t < n + 1) - P(N_t < n) =$$

$$P(T_{n+1} > t) - P(T_n > t).$$

It is then enough to prove that

$$P(T_n > t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}, n = 1, 2, ... \tag{2}$$

We prove (2) by induction.
For $n = 1$:

$$P(T_1 > t) = P(\tau_1 > t) = e^{-\lambda t}.$$

Assume that (2) is true for some n. Then

$$P(T_{n+1} > t) = P(T_n + \tau_{n+1} > t) =$$

$$P(\tau_{n+1} > t) + P(T_n > t - \tau_{n+1}, t \ge \tau_{n+1} > 0) =$$

$$e^{-\lambda t} + \int_0^t P(T_n > t - s) f_{\tau_{n+1}}(s) ds \tilde{=},$$

where the last formula is the result of using the *conditioning formula* on the variable τ_{n+1}.

Remark 1. (the conditioning formula)

a) If X and Y are discrete random variables, the we can write

$$P(X=x)=P(X=x,Y\in\mathbb{R})=\sum_y P(X=x,Y=y)=$$

$$\sum_y P(X=x|Y=y)P(Y=y)=\sum_y P(X=x|Y=y)f_Y(y).$$

b) Similarly, for continuous random variables X and Y:

$$P(X\in B)=....=\int P(X\in B|Y=y)f_Y(y)dy.$$

We continue with the calculations:

$$\tilde{=}e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}\sum_{k=0}^{n-1}\frac{[\lambda(t-s)]^k}{k!}\lambda e^{-\lambda s}ds=$$

$$e^{-\lambda t}+e^{-\lambda t}\sum_{k=0}^{n-1}\frac{\lambda^{k+1}}{k!}\int_0^t(t-s)^k ds=$$

$$e^{-\lambda t}+e^{-\lambda t}\sum_{k=0}^{n-1}\frac{\lambda^{k+1}}{k!}\frac{t^{k+1}}{k+1}=$$

$$e^{-\lambda t}+e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^{k+1}}{(k+1)!}=$$

$$e^{-\lambda t}\sum_{k=0}^{n}\frac{(\lambda t)^k}{(k)!}.$$

It is also not hard to see that N will be a process with stationary and independent increments. □

Example 2. Let (N_t) be a Poisson process with $\lambda=2$.
a) Find $P(N_5=10)$. By definition of N,

$$P(N_5=10)=e^{-\lambda t}\frac{(\lambda t)^{10}}{10!}=[\lambda=2,t=5]=e^{-10}\frac{10^{10}}{10!}.$$

b) Find the probability that $N_i-N_{i-1}=2$ for all $i=1,...,5$.

We have:

$$P(N_i - N_{i-1} = 2, 1 = 1, ..., 5) = P(\cap_{i=1}^5 \{N_i - N_{i-1} = 2\}) =$$

$$\prod_{i=1}^{5} (\frac{2^2}{2!} e^{-2}) = (\frac{2^2}{2!} e^{-2})^5.$$

Example 3. Customers arrive at a store according to a Poisson process of rate λ. What is the expected time until the nth customer arrives? What is the expected time between customers?

Solution: We use the property of the Poisson process following from its construction:

$$T_n = \tau_1 + ... + \tau_n,$$

where T_n is then the time of arrival of nth customer and τ_i is the time between arrivals of two corresponding customers.

We know that

$$E(\tau_i) = \frac{1}{\lambda}$$

and

$$E(T_n) = nE(\tau_i) = \frac{n}{\lambda}.$$

We also know that, in fact, T_n has Erlang (n, λ) distribution.