ESE 520 Probability and Stochastic Processes

HW 2: due by 11pm CST on Thursday, September 19, 2024

Total: 40 points

Problem 1. Assume that $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d. for short) random variables each having a Poisson distribution with parameter $\lambda = 2$. Calculate $P(\min(X_1, ..., X_n) > m)$ and $P(\max(X_1, ..., X_n) \leq m)$ for a positive integer m.

Problem 2. Assume that the joint distribution of an integer-valued discrete vector (X, Y) is given by the joint pmf (probability mass function)

$$p_{XY}(x, y) = \begin{cases} \frac{(1-p)p^{x-1}x^y e^{-x}}{y!}, x \geq 1, y \geq 0, \\ 0, \text{ othetrwise} \end{cases}$$

a) Find $p_X(x)$ for $x \geq 1$;

b) Compute $p_Y(0)$;

c) Determine whether or not X and Y are independent.

Problem 3. Let X be a continuous random variable that has the Weibull distribution, i.e. it has density of the form

$$f(x) = \begin{cases} \lambda p x^{p-1} e^{-\lambda x^p}, \; x > 0, \\ 0, \text{ for } x \leq 0, \end{cases}$$

where $\lambda > 0$ and $p > 0$ are parameters.

a) Show that this density integrates to one;

b) Let $X_1, ..., X_n$ be i.i.d. Weibull (p, λ) random variables. Find the probability that at least one of them exceeds 3.

Problem 4. For $p > 0$ and $q > 0$, let

$$B(p, q) := \int_0^1 u^{p-1}(1-u)^{q-1} du$$

called the Beta function. Consider

$$f(z) := \frac{1}{B(p, q)} \frac{z^{p-1}}{(1+z)^{p+q}}, z > 0.$$

Show that $f(z)$ is a valid density. Hint: Make the change of variable $t = 1/(1+z)$.

Problem 5. For $n = 1, 2, ...$, let $f_n(x)$ be a probability density function, and let p_n be a sequence of non-negative numbers summing to one, i.e. sequence p_n forms a discrete probability distribution. Show that

$$f(x) := \sum_n p_n f_n(x)$$

is a probability density function.

Problem 6. Assume X has Weibull (p, λ) distribution as in Problem 3. Show that

$$E[X^n] = \frac{\Gamma(1 + \frac{n}{p})}{\lambda^{n/p}}.$$

Problem 7. Let X be exponential with parameter $\lambda = 1$. Show that $Y = \sqrt{X}$ is Reyleigh $(1/\sqrt{2})$.

Problem 8. If X and Y are independent $exp(\lambda)$ random variables, find $E[\max(X, Y)]$.