HW#5 - Solutions

ESE-520

Total : 40p.

Pr. 1. [5p]

a) given : $X \sim \exp.(\lambda) \Rightarrow f_X(x) = \lambda \cdot e^{-\lambda x}, \; x > 0$

$$\Rightarrow F_X(x) = \int_0^x \lambda e^{-\lambda y} dy = -e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x}, \quad x > 0 \text{ (0 therwise)}$$

$$\Rightarrow P(X > t+s) = 1 - F_X(t+s) = e^{-\lambda(t+s)} = e^{-\lambda t} \cdot e^{-\lambda s} =$$

$$= P(X > t) \cdot P(X > s) \text{ for all } s, t \geq 0 ;$$

b) The equivalency of statements

$$P(X > t+s) = P(X > t) \cdot P(X > s)$$

and

$$P(X > t+s \mid X > s) = P(X > t)$$

follows from the fact that

$$\frac{P(X > t+s)}{P(X > s)} = \frac{P(X > t+s, X > s)}{P(X > s)} = P(X > t+s \mid X > s)$$

Since $\{X > t+s\} \subseteq \{X > s\}$ for all $s, t \geq 0$.

Pr. 2 [5p] Given: (N_t), $t \geq 0$ is a Poisson process

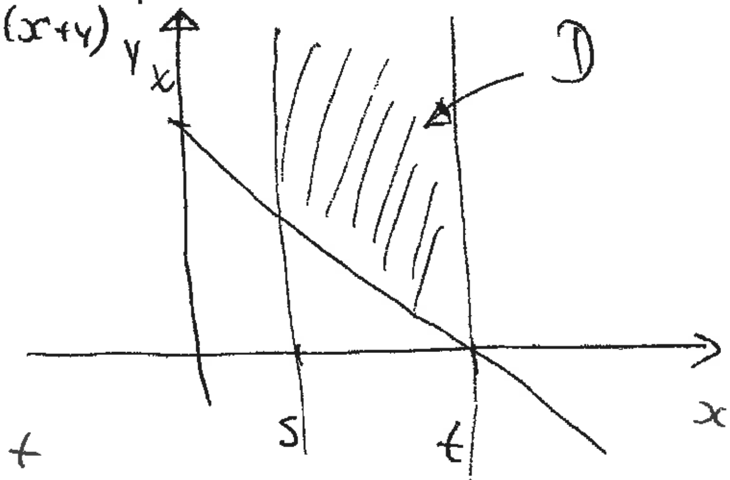
Calculate: $P(N_s = 0, N_t = 1)$ for any $0 \leq s \leq t$.

$\rightarrow P(N_s = 0, N_t = 1) = P(0 \leq s < T_1 \leq t < T_1 + T_2)$

where $T_1, T_2 \sim \exp(\lambda)$, $\lambda > 0$ & independent.

$f_{T_1}(x) = \lambda \cdot e^{-x}$, $f_{T_2}(y) = \lambda \cdot e^{-y}$, $x > 0, y > 0 \Rightarrow$

$f_{T_1,T_2}(x,y) = \lambda^2 \cdot e^{-\lambda(x+y)}$



$\Rightarrow P(0 \leq s < T_1 \leq t < T_1 + T_2) =$

$$= \iint_D \lambda^2 \cdot e^{-\lambda(x+y)} \, dx\,dy =$$

$$= \int_s^t \int_{t-x}^{\infty} \lambda^2 e^{-\lambda(x+y)} \, dy\,dx = \int_s^t -\lambda \, e^{-\lambda(x+y)} \Big|_{t-x}^{\infty} dx =$$

$$= \int_s^t \lambda \, e^{-\lambda(x+t-x)} \, dx = \int_s^t \lambda \cdot e^{-\lambda t} \, dx = \boxed{\lambda(t-s) \cdot e^{-\lambda t}}$$

Pr. 3 [5p] Given: (N_t), $t \geq 0$ is a Poisson process with parameter $\lambda > 0$.

$\rightarrow$ We recall: $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

- By definition & properties of (N_t):

$$P(N_t \text{ is odd}) = \sum_{n=0}^{\infty} P(N_t = 2n+1) =$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2n+1}}{(2n+1)!} = e^{-\lambda t} \cdot \sinh(\lambda t),$$

$$P(N_t \text{ is even}) = \sum_{n=0}^{\infty} P(N_t = 2n) =$$

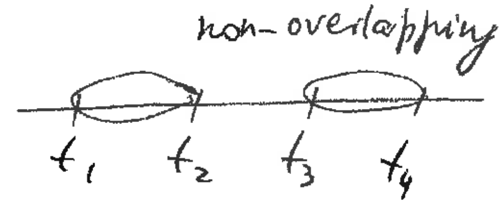
$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2n}}{(2n)!} = e^{-\lambda t} \cosh(\lambda t). \quad \boxtimes$$

Pr.5 [5p] Given: (W_t), $t \geq 0$ is a Wiener process.

<u>Show</u>: $V_t := W_{T+t} - W_T$, $t \geq 0$ is a Wiener process

<u>We have</u>:

i) $V_0 = W_T - W_T = 0$;



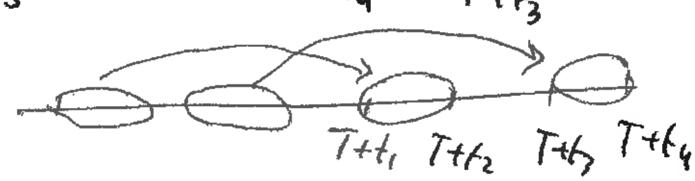
ii) For any $t_1 < t_2 < t_3 < t_4$,

consider rv's:

$$V_{t_2} - V_{t_1} = W_{T+t_2} - W_T - (W_{T+t_1} - W_T) = W_{T+t_2} - W_{T+t_1}$$

$$V_{t_4} - V_{t_3} = W_{T+t_4} - W_T - (W_{T+t_3} - W_T) = W_{T+t_4} - W_{T+t_3}$$

It is clear that (V_t) has also independent increments.



iii) For any $s < t$:

$$V_t - V_s = W_{T+t} - W_T - (W_{T+s} - W_T) = W_{T+t} - W_{T+s}$$

$\rightarrow$

$V_t - V_s$ has normal distribution &

$$E(V_t - V_s) = E(W_{T+t}) - E(W_{T+s}) = 0,$$

$$Var(V_t - V_s) = Var(W_{T+t} - W_{T+s}) = (T+t) - (T+s)$$
$$= t - s$$

$$\Rightarrow \quad V_t - V_s \sim N(0, t-s)$$

$\Rightarrow (V_t)$ is a Wiener process.

Pr. 6 [5p] Given: $h(t,x,y) = \dfrac{1}{\sqrt{2\pi \cdot t}} \cdot e^{-\frac{(y-x)^2}{2t}}$

→ Pure differentiation exercise:

$$\frac{\partial h}{\partial t} = \frac{y^2 - 2yx + x^2 - t}{2t} \cdot h(t,x,y)$$

$$\frac{\partial h}{\partial y} = \frac{x-y}{t} \cdot h(t,x,y) \qquad \Rightarrow$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{y^2 - 2yx + x^2 - t}{t} \cdot h(t,x,y)$$

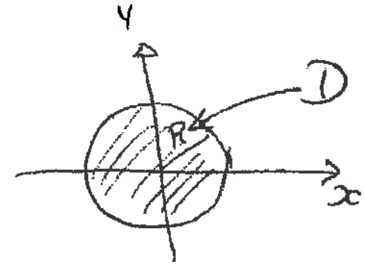
$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial y^2} \quad \boxtimes.$$

Pr. 7 [5p] Given: W_t^1 & W_t^2 are independent BM's and $|W_t| = \sqrt{(W_t^1)^2 + (W_t^2)^2}$.

→ The joint pdf of the vector (W_t) is

$$f_{W_t}(x,y) = f_{W_t^1}(x) \cdot f_{W_t^2}(y) = \frac{1}{2\pi t} \cdot e^{-\frac{(x^2+y^2)}{2t}}, \quad x, y \in \mathbb{R}.$$

$$\Rightarrow P(|W_t|<R) = \iint\limits_{\{(x,y):\underbrace{\sqrt{x^2+y^2}<R}_{D}\}} \frac{1}{2\pi t} \cdot e^{-\frac{(x^2+y^2)}{2t}} \, dx\, dy$$



= we use polar coordinates =

$$= \frac{1}{2\pi t} \cdot \int_0^R \cdot \int_0^{2\pi} r \cdot e^{-r^2/2t} \, d\theta \, dr =$$

$$= \frac{1}{t} \int_0^R r \cdot e^{-r^2/2t} \, dr = -e^{-r^2/2t} \Big|_0^R = \boxed{1 - e^{-R^2/2t}}$$

Pr. 4 [5p]

We find the characteristic function of M_t

$$\varphi_{M_t}(s) = E\left[e^{isM_t}\right] = E\left[e^{is\sum_{k=1}^{N_t} Y_k}\right] =$$

$$= \sum_{n=0}^{\infty} E\left[e^{is\sum_{k=1}^{N_t} Y_k} \,\middle|\, N_t = n\right] \cdot P(N_t = n) =$$

$$= \sum_{n=0}^{\infty} E\left[e^{is\sum_{k=1}^{n} Y_k} \,\middle|\, N_t = n\right] \cdot P(N_t = n)$$

= use independence =

$$= \sum_{n=0}^{\infty} E\left[e^{is\sum_{k=1}^{n} Y_k}\right] \cdot P(N_t = n) =$$

$$= \sum_{n=0}^{\infty} \prod_{k=1}^{n} E\left(e^{isY_k}\right) \cdot P(N_t = n)$$

$$= \sum_{n=0}^{\infty} \left(e^{is}p + (1-p)\right)^n \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!} =$$

$$= e^{-\lambda t} \cdot e^{[e^{is}p + (1-p)]\lambda t} = e^{is p\lambda t - p\lambda t}$$

$$\Rightarrow \quad M_t \sim \text{Poisson}(p\lambda t)$$

Prob. 8. [5pt]

Let $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{pmatrix}$ & $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{pmatrix}$

By independence of increments,

$$f_X(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{x_i^2}{2(t_i - t_{i-1})}}$$

with $t_0 = 0$.

Let $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & & & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} \Rightarrow Y = AX$ &

$\det A = 1$

By transformation formula,

$$f_Y(y_1, \dots, y_n) = f_X(A^{-1}y) \frac{1}{|\det A|}$$

Since

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 - w_0 \\ w_2 - w_1 \\ \vdots \\ w_n - w_{n-1} \end{pmatrix} = f_X(A^{-1}y), \Rightarrow$$

$$f_X(w_1, \dots, w_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{(w_i - w_{i-1})^2}{2(t_i - t_{i-1})}}$$