ESE 520 Probability and Stochastic Processes

Lecture 16

"The Poisson process."

One of the first classes of stochastic processes studied intensively over time has been the class of *stationary processes with independent increments*.

Definition 1. a) A process X is said to be *stationary* if for all $h > 0$:

$$P_{t_1+h,...,t_n+h}(B_1, ..., B_n) = P_{t_1,...,t_n}(B_1, ..., B_n)$$

for arbitrary $B_i \in \mathcal{B}(\mathbb{R}), i = 1, 2, ..., n, n \in \mathbb{N}$.

(in other words, the FDDs of the process X are invariant with respect to time-shift)

b) A process X is said to have *independent increments* if for any $t_0 < t_1 < .... < t_m$ it holds:

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_m} - X_{t_{m-1}}$$

are independent random variables.

The random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_m} - X_{t_{m-1}}$ are called the increments of (X_t). Note also that increments must be taken over non-overlapping time intervals where points $t_1 < t_2 < t_3$ produce then two non-overlapping intervals $t_2 - t_1$ and $t_3 - t_2$.

Definition 2. A process $(X_t), t \in I$ is called *a process with stationary and independent increments* if

i) it has independent increments;

ii) its increments are stationary;

One of the most important processes with stationary and independent increments is a *Poisson process*. There are many equivalent definitions of a Poisson process. Here is one of them.

Definition 3. A Poison process $(N_t), t \geq 0$ (continuous time) is an *integer-*

valued process satisfying the following properties:

i) $N_0 = 0$;

ii) it has stationary and independent increments;

iii) $P(N_{\Delta t} = 1) = \lambda \Delta t + o(\Delta t)$ and $P(N_{\Delta t} \geq 2) = o(\Delta t)$;

where $N_{\Delta t} := N_{t+\Delta t} - N_t$ is the increment of N over a time interval of length Δt.

We also recall that a function $\alpha(t)$ is of order $o(t)$ as $t \to 0$ if $\lim_{t\to 0} \frac{\alpha(t)}{t} = 0$.

As we will see, (N_t) will be a "counting process" registering how many times a particular event occurred in the time interval $[0, t]$.

Interpretation of assumptions:

∘ stationary increments: number of events occurring in the time interval $[a, b]$ only depends on the length of the interval $|b - a|$ but not its location $[a, b]$;

∘ independent increments: number of events occurring in non-overlapping intervals are independent;

∘ condition iii): $P(N_{\Delta t} = 1) = \lambda \Delta t + o(\Delta t)$ means that the probability that exactly one event occurs in a small interval of length Δt is approximately proportional to its length where λ is the intensity of occurring of jumps of size one for N. The condition $P(N_{\Delta t} \geq 2) = o(\Delta t)$ means that it is unlikely that more than one event occurs in a small time interval. Combining the two conditions in iii), we can also write that

$$P(N_{\Delta t} = 0) = 1 - \lambda \Delta t + o(\Delta t). \tag{1}$$

Theorem (probability distribution of Poisson process). Let $(N_t), t \geq 0$ be a Poisson process. Then

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, 2, ... \tag{2}$$

Proof. Define

$$P_k(t) := P(N_t = k),\ \ k = 0, 1, 2, ... \tag{3}$$

It follows then that

$$P_0(t+h) = P(N_{t+h} = 0) = P(\{N_t = 0\} \cap \{N_{t+h} - N_t = 0\}) =$$

- we use here the independence of increments -

$$P(N_t = 0)P(N_{t+h} - N_t = 0) = \text{ use that the increments are stationary } =$$

$$P(N_t = 0)P(N_h = 0) = P_0(t)P_0(h).$$

Now, using the last relation and (1), we obtain that

$$\frac{P_0(t+h) - P_0(t)}{h} = \frac{P_0(t)P_0(h) - P_0(t)}{h} = P_0(t)\frac{P_0(h) - 1}{h} =$$

$$-P_0(t)\frac{\lambda h + o(h)}{h}$$

so that by taking the limit $\lim_{h\to 0}$, one gets

$$P_0'(t) = -\lambda P_0(t). \tag{4}$$

The equation (4) is the first order ODE. Being subject to the initial condition

$$P_0(0) = 1,$$

its solution is

$$P_0(t) = e^{-\lambda t}. \tag{5}$$

Thus, we have proven the relation (2) for $k = 0$.

Similarly, we can proceed for any **fixed** $k \geq 1$:

$P_k(t+h) = P(\cup_{i=0}^{k}\{k-i \text{ events occur in } [0,t]\} \cap \{i \text{ events occur in } [t, t+h]\}) =$

$$\sum_{i=0}^{k} P_{k-i}(t)P_i(h) =$$

$$P_k(t)P_0(h) + P_{k-1}(t)P_1(h) + \sum_{i=2}^{k} P_{k-i}(t)P_i(h).$$

Subtracting from both sides in the last relation $P_k(t)$ and dividing by h, yields

$$\frac{P_k(t+h) - P_k(t)}{h} =$$

$$\frac{P_k(t)[P_0(h) - 1]}{h} + \frac{P_{k-1}(t)P_1(h)}{h} + \frac{o(h)}{h}.$$

By taking the $\lim_{h\to 0}$, we obtain the following first order ODE

$$P'_k(t) = -\lambda P_k(t) + \lambda P_{k-1}(t) \tag{6}$$

subject to the initial condition

$$P_k(0) = 0. \tag{7}$$

To solve (6), we let

$$Q_k(t) := P_k(t)e^{\lambda t}$$

where

$$Q_0(t) = 1 \tag{8}$$

(it follows from (5)).

Using (6) and simple differentiation, we obtain

$$Q'_k(t) = P'_k(t)e^{\lambda t} + P_k(t)\lambda e^{\lambda t} = \lambda P_{k-1}(t)e^{\lambda t} = \lambda Q_{k-1}(t).$$

From (8) we get recursively:

$$Q_1(t) = \lambda t$$

$$(Q_1(t) = \lambda t + const)$$

but $const = 0$ since $Q_1(0) = 0$.

In general, we obtain

$$Q_k(t) = \frac{(\lambda t)^k}{k!}$$

and

$$P_k(t) = \frac{(\lambda t)^k}{k!}e^{-\lambda t},\ k = 0, 1, 2, ...$$

□

Definition 4. (counting process) Let T_k, $k = 1, 2, ...$ be a strictly increasing sequence of positive random variables where we set $T_0 = 0$ ($T_0 < T_1 < T_2 < ...$). Define

$$N_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{T_k \leq t\}}, t \geq 0 \tag{9}$$

called the *counting process* associated with the sequence $\{T_k\}$, $k = 1, 2, ...$

$T := \sup_{k \in \mathbb{N}} T_k$ is called the *explosion time* of the process N. If $T = \infty$ a.e., then N is said to be a process *without explosion*.

Theorem. Let N be a counting process without explosion which has *stationary and independent increments*. Then N is the Poisson process, that is

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, t \geq 0, k = 0, 1, 2, ...$$

where $\lambda > 0$ is a constant.

Proof. Let $P_0(t) := P(N_t = 0)$. Then we can write

$$P_0(t) = P(N_t = 0) = P(T_1 > t) = 1 - P(T_1 \leq t) = 1 - F_1(t)$$

where F_1 is the cdf of the random variable T_1. It follows that $P_0(t)$ is a *right-continuous* function.

Since N has stationary and independent increments, we have that

$$P_0(t) = P(N_t = 0) = P(\{N_s = 0\} \cap \{N_t - N_s = 0\}) =$$

$$P(\{N_s = 0\})P(\{N_t - N_s = 0\}) =$$

$$P(N_s = 0)P(N_{t-s} = 0) = P_0(s)P_0(t - s)$$

for all $0 < s < t$.

It follows then that

$$P_0(t) = P_0(\frac{t}{n} + ... + \frac{t}{n}) = (P_0(\frac{t}{n}))^n.$$

In particular, for $t = 1$, it yields

$$P_0(\frac{1}{n}) = (P_0(1)])^{1/n}.$$

More generally, for $r = \frac{m}{n} \in \mathbf{Q}$, one has then

$$P_0(r) = P_0(\frac{m}{n}) = (P_0(\frac{1}{n}))^m = (P_0(1))^{m/n} = (P_0(1))^r.$$

For any $t \geq 0$, choose a sequence $r_i \in \mathbf{Q}$ such that $r_i \to t+$. By the right-continuity of $P_0(t)$, one has then

$$P_0(t) = \lim_{n\to\infty} P_0(r_n) = \lim_{n\to\infty} [P_0(1)]^{r_n} = [P_0(1)]^t.$$

In other words, we have shown that $P_0(t)$ is an *exponential function.*

Note: we can assume that $0 < P_0(1) < 1$. Indeed, if $P_0(1) = 1$, then $P_0(t) = 1$ for all $t \geq 0$ which would contradict to the assumption that T_1 is finite?!

Let $\lambda := -\ln P_0(1)$ so that

$$P_0(t) = e^{-\lambda t}, t \geq 0$$

and Theorem is proven for $k = 0$.

For $k = 1, 2, ...,$ we let $P_k(t) := P(N_t = k)$ and divide $[0, t]$ into 2^n equal subintervals where $n \in \mathbb{N}$.

We can write that

$$P_k(t) = P(\text{ at least one event occurs in each of k intervals } (\frac{jt}{2^n}, \frac{(j+1)t}{2^n}],$$
$$j = 0, 1, ...2^n - 1 \text{ and no event in the remaining intervals}) + q_n,$$

where

$$q_n \leq P(\text{ two or more events occur in at least one of the intervals}$$
$$(\frac{jt}{2^n}, \frac{(j+1)t}{2^n}], j = 0, 1, ...2^n - 1) := P(A_n).$$

Note that $A_{n+1} \subset A_n$ since the partitions refine.

Let $A := \cap_{n=1}^{\infty} A_n$. Then, if $w \in A$, then infinitely many events occur along this sample path, that is $T(w) < t$ a.e.?! since N has no explosions. Therefore, it follows that $P(A) = 0$.

As the result,

$$\lim_{n\to\infty} q_n \leq \lim_{n\to\infty} P(A_n) = P(A) = 0.$$

It can be seen that the first probability on the right side of the expression for $P_k(t)$ is the binomial one with parameters 2^n and $1 - \exp(-\frac{\lambda t}{2^n})$ (by the first step of the proof).

Therefore,

$$P_k(t) = \lim_{n\to\infty} {}_{2^n}C_k (1 - \exp(-\frac{\lambda t}{2^n}))^k (\exp(-\frac{\lambda t}{2^n}))^{2^n - k} =$$

$$\frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, 2, ...$$

due to the known approximation result stating that $Bin(n, p_n) \to \text{Poisson}(\lambda)$ when $np_n \to \lambda$ as $n \to \infty$.

In our case $n = 2^n$ and $p_n = 1 - \exp(-\frac{\lambda t}{2^n})$ so that

$$2^n(1 - \exp(-\frac{\lambda t}{2^n})) = \lambda t - \frac{\lambda^2 t^2}{2^n 2!} + ... + o(n) \to_{n\to\infty} \lambda t.$$

$\square$