HW#6 – Solutions

ESE-520

Total: 40 points

Pr. 1 [5p] given: (X_t) – WSS with $\mu_X(t) \equiv 0$ & $R_X(\tau)$,

$Y_t := X_t \cdot \cos(2\pi f t + \theta)$, $\theta \sim U[-\pi, \pi]$,

θ is independent from (X_t).

a) $R_Y(s,t) = E(Y_s \cdot Y_t) = E\left(X_s \cdot X_t \cdot \cos(2\pi f s + \theta) \cdot \cos(2\pi f t + \theta)\right)$

= independency = $E(X_s \cdot X_t)\, E\left(\cos(2\pi f s + \theta) \cdot \cos(2\pi f t + \theta)\right)$ = trig. identity = $\frac{1}{2} R_X(t-s)\, E\left(\cos 2\pi f(s-t) + \cos(2\pi f(s+t) + 2\theta)\right) =$

$= \frac{1}{2} R_X(t-s)\left[\cos 2\pi f(s-t) + \underbrace{E \cos(2\pi f(s+t) + 2\theta)}_{=0 \quad (*)}\right] =$

$= \boxed{\frac{1}{2} R_X(t-s) \cos 2\pi f(s-t)}$

b) $E(X_s \cdot Y_t) = E\left(X_s \cdot X_t \cos(2\pi f t + \theta)\right)$ = independency =

$= E(X_s X_t)\, \underbrace{E\left(\cos(2\pi f t + \theta)\right)}_{=0} = 0.$

c) Clear that $\mu_Y(t) = \text{const}$ and $R_Y(s,t)$ depends on $(t-s)$ only $\Rightarrow$ Y is a WSS process

→ More details to $(*)$: $E\left(\cos[2\pi f(s+t) + 2\theta]\right) =$

$= \int_{-\pi}^{\pi} \cos(2\pi f(s+t) + 2\theta)\, d\theta =$

$+ \frac{1}{2} \sin(2\pi f(s+t) + 2\theta)\Big|_{-\pi}^{\pi} = 0.$

Pr. 2 [5p] Given: (X_t) is WSS with $R_X(\tau) = \frac{1}{1+\tau^2}$

Find $S_X(\nu)$.

- We can use here one of previous HW's showing that a characteristic function $\varphi(t) = e^{-\lambda|t|}$, $\lambda > 0$ and a density $f(\tau) = \frac{\lambda/\pi}{\lambda^2+\tau^2}$ relate to each other by Fourier / inverse Fourier transforms.

With this in mind,

$$S_X(\nu) = \int R_X(\tau) e^{-2\pi i \nu \tau} d\tau = \int \frac{1}{1+\tau^2} e^{-2\pi i \nu \tau} d\tau =$$

$$= [t = 2\pi\tau] = \frac{1}{2\pi} \int \frac{(2\pi)^2}{(2\pi)^2 + t^2} e^{-i\nu t} dt =$$

$$= \pi \int \frac{(2\pi)/\pi}{(2\pi)^2+t^2} e^{-i\nu t} dt = [\text{for } \lambda = 2\pi] =$$

$$= \pi \cdot e^{-2\pi|\nu|}$$

- One could use tables of Fourier transforms as well

Pr. 3 [5p]

We have to use here the properties of Dirac $\delta_0(\tau)$ function: $\int_{-\infty}^{\infty} \delta_0(\tau) f(\tau) d\tau = f(0)$ for f - continuous.

a) $R_X(\tau) = \int_{-\infty}^{\infty} S_X(\nu) e^{2\pi i \nu \tau} d\nu = \int_{-\infty}^{\infty} \delta_0(\nu) e^{2\pi i \nu \tau} d\nu$

$= e^{2\pi i 0 \cdot \tau} = 1.$

b) $R_X(\tau) = \int_{-\infty}^{\infty} \delta_0(\nu - \nu_0) e^{2\pi i \nu \tau} d\nu + \int_{-\infty}^{\infty} \delta_0(\nu + \nu_0) e^{2\pi i \nu \tau} d\nu$

$=$

$= e^{2\pi i \nu_0 \tau} + e^{-2\pi i \nu_0 \tau} = 2\cos(2\pi\nu_0\tau)$ by the formula $e^{iz} = \cos z + i\sin z$.

c) $R_X(\tau) = \int_{-\infty}^{\infty} e^{-\nu^2/2} \cdot e^{2\pi i \nu \tau} d\nu = \ldots$ complete square

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\nu - 2\pi i \tau)^2}{2}} e^{-2\pi^2\tau^2} d\nu = \left[\begin{matrix}\nu' = \nu \\ -\pi i \tau\end{matrix}\right]$$

$$= \sqrt{2\pi} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\nu'^2/2} d\nu'\right)}_{=1} \cdot e^{-2\pi^2\tau^2} = \underline{\sqrt{2\pi} \cdot e^{-2\pi^2\tau^2}}$$

d) $R_X(\tau) = \int_{-\infty}^{\infty} e^{-|\nu|} e^{-2\pi i \nu \tau} d\nu = 2\pi \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\nu|} e^{-i\nu(2\pi\tau)} d\nu\right)}_{(*)}$

The expression $(*)$ is by Problem 1 from HW3 equal to $\frac{\lambda/\pi}{\lambda^2 + \tau^2}$ with $\lambda = 1$ & $\tau = 2\pi\tau$ $\Rightarrow$

$$= 2\pi \cdot \frac{1/\pi}{1 + (2\pi\tau)^2} = \frac{2}{1 + (2\pi\tau)^2}$$

Pr. 4 [5p.] Given: $(W_t)_{t \geq 0}$ - Wiener process,

$$Y_t = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} W_{e^{2\lambda t}}, \quad \lambda > 0$$

Calculate $R_Y(t_1, t_2)$

$$R_Y(t_1,t_2) = E(Y_{t_1} Y_{t_2}) = \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} E(W_{e^{2\lambda t_1}} \cdot W_{e^{2\lambda t_2}})$$

$$= \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} \min(e^{2\lambda t_1}, e^{2\lambda t_2})$$

→ For $t_1 < t_2$, this is equal to

$$\frac{e^{-\lambda(t_1+t_2)}}{2\lambda} e^{2\lambda_1 t_1} = \frac{1}{2\lambda} e^{-\lambda(t_2-t_1)}$$

→ For $t_2 < t_1$, we have

$$\frac{e^{-\lambda(t_1+t_2)}}{2\lambda} e^{2\lambda t_2} = \frac{1}{2\lambda} e^{-\lambda(t_1-t_2)}$$

⇒ in general:

$$E(Y_{t_1} Y_{t_2}) = \frac{1}{2\lambda} e^{-\lambda|t_1-t_2|} \quad \otimes.$$

(we assumed $\sigma^2=1$)

Pr. 5 [5p] Given: $(W_t)_{t\geq 0}$ is a Wiener process ($\sigma^2=1$)

$Y_t = e^{W_t}$ Find $R_Y(t_1,t_2) = E(Y_{t_1} Y_{t_2})$ for $t_2 < t_1$

$$E(Y_{t_1} Y_{t_2}) = E(e^{W_{t_1}} \cdot e^{W_{t_2}}) = E\left(e^{W_{t_2}-W_{t_1}} e^{2W_{t_1}}\right) =$$

$$= E\left(e^{W_{t_2}-W_{t_1}} e^{2(W_{t_1}-W_0)}\right) = \text{we use independency of increments of } W =$$

$$= E\left(e^{W_{t_2}-W_{t_1}}\right) E\left(e^{2(W_{t_1}-W_0)}\right) = e^{(t_2-t_1)/2} \cdot e^{4t_1/2} =$$

$$= e^{\frac{t_2+3t_1}{2}},$$ where we used the fact

If $X \sim N(0, \sigma^2)$, then

$$E(e^X) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{x}\cdot e^{-x^2/2\sigma^2}\,dx = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\sigma^2}{\sigma}\right)^2} e^{\frac{\sigma^2}{2}}\,dx$$

$$= \left[\frac{x-\sigma^2}{\sigma} = z\right] = \frac{1}{\sqrt{2\pi}}\, e^{\sigma^2/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz =$$

$$= e^{\sigma^2/2}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz}_{=1} = e^{\sigma^2/2}$$

Pr. 6. [5P] Given: (W_t), $t \geq 0$ is a Wiener process.

- To show that all ~~FDD's of (W_t) are Gaussian~~, we choose $n \in \mathbb{N}$ and $0 < t_1 < \dots < t_n$ and write

$$\underbrace{\begin{bmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ \vdots \\ W_{t_n} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots 0 \\ \vdots & & \ddots & \\ \vdots & \vdots & & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} W_{t_2} - W_{t_1} \\ \\ \vdots \\ \\ W_{t_n} - W_{t_{n-1}} \end{bmatrix}}_{Y}$$

- The vector Y is jointly Gaussian since its joint pdf is Gaussian as the product of 1-dimensional Gaussian pdfs (all increments are independent).

$\Rightarrow$ vector X is a linear transformation of a Gaussian vector hence Gaussian.

Pr. 7 [5p.]

We note that $E[Y_{t+\frac{1}{2}} Y_t] = R_Y(\frac{1}{2})$

Since $S_X(\nu) = 1$, we use the formula

$$S_Y(\nu) = |H(\nu)|^2 S_X(\nu) = |H(\nu)|^2$$

$H(\nu)$ is the Fourier transform of $h(t) = \frac{1}{1+t^2}$

We use then one of previous HW's (or Fourier tables) to obtain

$$H(\nu) = \pi e^{-2\pi|\nu|}$$

Thus, $S_Y(\nu) = \pi^2 \cdot e^{-4\pi|\nu|}$

$R_Y(\tau)$ is then the Fourier inverse of $S_Y(\nu)$ so that (again by previous HW)

$$R_Y(\tau) = \pi \cdot \frac{2}{4+\tau^2}$$

$$\Rightarrow E[Y_{t+\frac{1}{2}} Y_t] = R_Y(\tfrac{1}{2}) = \pi \frac{2}{4+\frac{1}{4}} = \pi \frac{2}{\frac{16+1}{4}}$$

$$= \pi \cdot \frac{8}{17}$$

#8 [5p]

Let $P = (p_{ji})$ be a stochastic matrix, $i, j \in S$

We will use the following fact

$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ for any non-negative sequence (a_{ij}), $i,j = 1, 2, \ldots$

Then $P^2 = P \cdot P = (c_{ji})$, where $c_{ji} = \sum_k p_{jk} p_{ki}$

Thus, for any i:

$$\sum_j c_{ji} = \sum_j \left(\sum_k p_{jk} p_{ki} \right) = \sum_k \left(\sum_j p_{jk} p_{ki} \right)$$

$$= \sum_k \underbrace{\left(\sum_j p_{jk} \right)}_{=1} p_{ki} = \sum_k p_{ki} = 1$$

The general result follows then by induction