ESE 520 Probability and Stochastic Processes

Lecture 20

"White noise continued."

To be able to work with white noise process (even though we cannot prove the existence of it by Kolmogorov's theorem since we do not have the corresponding family of FFDs we can work with), we need to introduce some other useful characteristics of WSS processes that will allow us to operate with the white noise process as well.

Definition 1. Let $(X_t), t \in \mathbb{R}$ be a WSS process with mean $\mu(t) = const$ and covariance $R(\tau)$. Assume that

$$\int_{-\infty}^{\infty} |R(\tau)| d\tau < \infty.$$

Then, the *Fourier transform* $S(\nu)$ of $R(\tau)$ exists and is called the *spectral density* of (X_t):

$$S(\nu) := \int_{-\infty}^{\infty} R(\tau) e^{-2\pi i\nu\tau} d\tau. \tag{1}$$

If, additionally, also

$$\int_{-\infty}^{\infty} |S(\nu)| d\nu < \infty,$$

then it can be shown that $R(\tau)$ is continuous and it holds

$$R(\tau) = \int_{-\infty}^{\infty} S(\nu) e^{2\pi i\nu\tau} d\nu. \tag{2}$$

In other words, $R(\tau)$ is obtained as the *inverse Fourier transform* of $S(\nu)$.

Remark 1. a) Usually, one uses the Fourier transform in the form

$$S(\nu) = \int_{-\infty}^{\infty} R(\tau) e^{-i\nu\tau} d\tau$$

and then the inverse Fourier transform is in the form

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\nu) e^{i\nu\tau} d\nu.$$

The forms (1) and (2) represent just another possibility to define Fourier and inverse Fourier transforms such that there is no factor $\frac{1}{2\pi}$ appearing in any of the integrals used.

b) We will assume further that $R(\tau)$ and $S(\nu)$ are always absolutely integrable.

Interpretation of spectral density:

Let X_t be the current flowing through a unit resistor at time t. Then EX_t^2 represents the average power of the current at time t (using the formula $W = I^2R$). From another side,

$$EX_t^2 = E(X_t X_t) = R(0) \text{ since } \tau = t - t = 0,$$

and

$$R(0) = \int_{-\infty}^{\infty} S(\nu) d\nu.$$

Therefore, the *spectral density $S(\nu)$ measures then the average power at frequency ν* (power is split over frequencies).

Now: Let us calculate the spectral density of the white noise process.

From earlier considerations, we know that the WSS process (X_t^h) has the covariance function

$$R^h(\tau) = \frac{1}{h}(1 - \frac{|\tau|}{h})^+.$$

Using the facts that

$$e^{-2\pi i\nu\tau} = \cos(2\pi\nu\tau) - i\sin(2\pi\nu\tau))$$

and $\frac{1}{h}(1 - \frac{|\tau|}{h})^+$ is an even function in τ, we obtain

$$S^h(\nu) = \int_{-\infty}^{\infty} \frac{1}{h}(1 - \frac{|\tau|}{h})^+ e^{-2\pi i\nu\tau} d\tau = \int_{-h}^{h} \frac{1}{h}(1 - \frac{|\tau|}{h}) e^{-2\pi i\nu\tau} d\tau =$$

$$\frac{2}{h}\int_0^h (1 - \frac{\tau}{h})\cos(2\pi\nu\tau) d\tau =$$

- integration by parts -

$$\frac{2}{2\pi\nu h}(1-\frac{\tau}{h})\sin(2\pi\nu\tau)|_0^h + \frac{2}{h^2}\int_0^h \frac{\sin(2\pi\nu\tau)}{2\pi\nu}d\tau =$$

$$\frac{-2}{(2\pi\nu h)^2}\cos(2\pi\nu\tau)|_0^h = \frac{2(1-\cos(2\pi\nu h))}{(2\pi\nu h)^2}.$$

By L'Hospital's rule,

$$\lim_{h\to 0} S^h(\nu) = \lim_{h\to 0}\frac{2(1-\cos(2\pi\nu h))}{(2\pi\nu h)^2} =$$

$$\lim_{h\to 0}\frac{\sin(2\pi\nu h))}{(2\pi\nu h)} = 1 \text{ for all } \nu \in \mathbb{R}.$$

Therefore, we have shown that the limiting process of the sequence (X_t^h) as $h \to 0$ - which we called a "white noise" process as a zero mean, WSS process with the covariance function $R(\tau) = \delta_0(\tau)$ - has the spectral density (over the entire spectrum)

$$S(\nu) = 1, \nu \in \mathbb{R}.$$

Remark 2. a) As it was noticed before, the Dirac delta function $\delta_0(\tau)$ is not a function in the usual sense. It is an example of a *generalized function*. Generalized functions are also called (generalized) distributions. Distributions are introduced as *linear functionals* acting on an appropriate space of so-called test functions. Usually, one uses for the space of test functions the *Schwarz space* $S(\mathbb{R})$ of functions. In this case distributions are called *tempered distributions*. It is well known that tempered distributions have Fourier transforms.

b) One practical motivation to introduce distributions would be to make non-differentiable functions (in usual sense of standard calculus) to be differentiable but then in a generalized sense. The derivatives of such functions become then distributions.

As an illustration, let $X(\omega) = 1$ be a constant random variable. Clearly, the cdf of X is $F(x) = 0$ for all $x < 1$ and $F(x) = 1$ for all $x \geq 1$. We also find

$$f(x) = F'(x) = 0 \text{ for all } x \neq 1.$$

At the point $x = 1$, derivative of $F(x)$ does not exists in the usual sense but it will exist in the generalized sense.

More precisely, it holds that

$$F'(1) = \delta_1(x)$$

so that

$$\int_{\mathbb{R}} \delta_1(x)dx = 1.$$

Now, some additional examples.

Example 1. Let $q(t), t \in \mathbb{R}$ be a periodic function with period T_0, where $T_0 > 0$. Let T has $U[0, T_0]$ distribution and define $X_t := q(t + T)$. Is (X_t) a WSS process?

Solution: By definition, we have to check two things:

a) If $\mu(t)$ is a constant?

We calculate

$$\mu(t) = E(X_t) = \int_{\mathbb{R}} q(t+x)f(x)dx = \frac{1}{T_0}\int_0^{T_0} q(t+x)dx = [t + x = \tau]$$

$$\frac{1}{T_0}\int_t^{t+T_0} q(\tau)d\tau = \frac{1}{T_0}\int_0^{T_0} q(\tau)d\tau$$

since q is a periodic function with period T_0 and, therefore, the integral from q has the same value over any interval of length T_0.

The last relation means that $\mu(t)$ does not depend on t thus is a constant.

b) Does $R(s, t)$ depend only on $t - s$?

We calculate:

$$R(s,t) = E(X_s X_t) = Eq(s+T)q(t+T) = \frac{1}{T_0}\int_0^{T_0} q(s+x)q(t+x)dx =$$

$$[\tau = t + x] = \frac{1}{T_0}\int_t^{t+T_0} q(s+\tau-t)q(\tau)d\tau = \text{ similar argument as before}$$

$$\frac{1}{T_0}\int_0^{T_0} q(s - t + \tau)q(\tau)d\tau.$$

We see that $R(s,t)$ depends only on the difference $t - s$ thus (X_t) is a WSS process.

Example 2. Let (X_t) be a WSS process with $R_X(\tau) = e^{-\frac{\tau^2}{2}}$. Find $S_X(\nu)$.
Solution: We can calculate directly

$$S_X(\nu) = \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2}} e^{-2\pi i\nu\tau} d\tau = [\text{ completing the square }]$$

$$e^{-2\pi^2\nu^2} \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\tau+2\pi i\nu)^2}{2}} d\tau = [z = \tau + 2\pi i\nu]$$

$$\sqrt{2\pi} e^{-2\pi^2\nu^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz =$$

$$\sqrt{2\pi} e^{-2\pi^2\nu^2}.$$