Lecture 15 Introduction to stochastic processes

<u>Definition</u> (finite-dimensional distributions) Given a stochastic process $(X_t), t\in I$, the collection of all probability distributions

$$P_{t_1,\cdots,t_n}(B_1,\cdots,B_n) = P(X_{t_1}\in B_1,\cdots,X_{t_n}\in B_n)$$

Borel sets

for all distinct $t_i\in I, n\in N, B_i\in B(R)$ are called finite-dimensional distributions of X (FDD's) 有限维分布，联合分布

The FDD's are natural characteristics of a stochastic process (X_t) and they trivially satisfy the following two consistency properties

(1) Given $\{t_1,\cdots,t_n\}\in I$ and any permutation π of the set $\{t_1,\cdots,t_n\}$ it holds:

$$P_{t_1,\cdots,t_n}(B_1,\cdots,B_n) = P_{\pi(t_1),\cdots,\pi(t_n)}(B_{\pi(t_1)},\cdots,B_{\pi(t_n)})$$

(2) $P_{t_1,\cdots,t_n,t_{n+1}}(B_1,\cdots,B_n,\mathbb{R}) = P_{t_1,\cdots,t_n}(B_1,\cdots,B_n)$

Lecture 16-17 The Poisson Process

$X\sim Poisson(\lambda)$, $P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ $E(X)=\lambda$, $Var(X)=\lambda=E(X^2)-(E(X))^2$

<u>Definition</u> a) A process $(X_t), t\in I$ is said to be stationary if for all $h>0$.

$P_{t_1+h,\cdots,t_n+h}(B_1,\cdots,B_n) = P_{t_1,\cdots,t_n}(B_1,\cdots,B_n)$ for arbitrary $B_i\in B(R)$, $i=1,2,\cdots,n$, $n\in\mathbb{N}$ (all FDD's of (X_t) are shift-invariant)

b) A process X_t is said to have independent increments if for any $t_0<t_1<\cdots<t_m$ it holds $X_{t_1}-X_{t_0}, X_{t_2}-X_{t_1},\cdots,X_{t_m}-X_{t_{m-1}}$ are independent rv's.

<u>Definition</u> A Poisson process (N_t), $t\geq 0$ (continuous time) is an integer valued process satisfying the following properties:

1) $N_0=0$ 2) it has stationary and independent increments

3) $P(N_{\Delta t}=1)=\lambda\Delta t+o(\Delta t)$ and $P(N_{\Delta t}\geq 2)=o(\Delta t)$, where $N_{\Delta t}=N_{t+\Delta t}-N_t$ is the increment of N over a time interval of length Δt.

(N_t) will be a 'counting process' registering how many times a particular event occurred in the time interval $[0,t]$

<u>Theorem</u> (probability distribution of Poisson process). Let (N_t) $t\geq 0$ be a Poisson process. Then. $P(N_t=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$, $k=0,1,2,\cdots$ $N_t\sim Poisson(\lambda t)$

$N_t-N_s\sim Poisson(\lambda(t-s))\sim N_{t-s}$

Lecture 18 Gaussian process, Wiener process/Brownian motion

<u>Def</u> A stochastic process (X_t). $t\in I$ is said to be Gaussian if all its finite-dimensional distributions (FDD's) are Gaussian

$X\sim N(\mu,\sigma^2)$ $f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$

$X=(X_1,\cdots,X_n)\sim N(\mu,\Sigma)$ mean vector, covariance matrix

$$f_X(x)=\frac{1}{\sqrt{(2\pi)^n|\Sigma|}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

- Any Gaussian vector $(X_{t_1},\cdots,X_{t_n})$ is uniquely determined by its mean vector μ and its covariance matrix R $n\times n$

<u>Def</u> Given (X_t) $t\in I$, a) $\mu_t=E(X_t)$, $t\in I$ is called the mean function of the process (X_t), b) $R(s,t)=E[(X_s-\mu_s)(X_t-\mu_t)]$ $= Cov(X_s,X_t)$ for all $s,t\in I$ is called the covariance function of the process (X_t)

Given any function $\mu(t)$, $t\in I$ and any positive semidefinite function $R(s,t)$, $s,t\in I$, there is unique gaussian process (X_t) for which $\mu(t)=E(X_t)$ mean function, $R(s,t)$ covariance function

<u>Def</u> (Wiener process/Bronian motion) The Wiener process is a sample-path continuous Gaussian process with mean 0 ($E(W_t)=0$) and covariance function $R(s,t)=\min(s,t)$. Notion: (W_t), $t\geq 0$

<u>Theorem</u> Let (W_t), $t\geq 0$ be a sample-path continuous stochastic process with W_0-0. Then the following statements are equivalent

1) (W_t) is a Wiener process (mean 0 and covariance function $R(s,t)=s\wedge t$)

2) (W_t) has stationary and independent increments and for all $s<t$ the variable W_t-W_s has $N(0,t-s)$ distribution

3) the FDD's of $(W_{t_1},\cdots,W_{t_n})$ $(0<t_1<\cdots<t_n)$ have the density

$n(t_1;0,x_1)\times n(t_2-t_1;x_1,x_2)\times\cdots\times n(t_n-t_{n-1};x_{n-1},x_n)$

where $n(t;x,y)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2}\frac{(x-y)^2}{t}}$

<u>Def</u> (W_t) $t\in[0,+\infty)$ is a Wiener process 1) $W_0=0$ 2) it has independent increments 3) for all $s<t$, $W_t-W_s\sim N(0,t-s)$

计算 $E(W_t-W_s)=0$, $Var(W_t-W_s)=t-s$

$t_1<t_2<t_3<t_4$ $W_{t_2}-W_{t_1}$, $W_{t_4}-W_{t_3}$

$E(X)=\sum x\cdot P(X=x)$ $E(X)=\int_{-\infty}^{+\infty}xf_X(x)dx$

X, Y independent

$E(aX+bY)=aE(X)+b(Y)$ $E(g(X)h(Y))=E(g(X))E(h(Y))$

$E(g(X))=\sum g(x)\cdot P(X=x)$ $E(g(X))=\int_{-\infty}^{+\infty}g(x)\cdot f_X(x)dx$

$Var(X)=E((X-E(X))^2)=E(X^2)-(E(X))^2$

$Cov(X,Y)=E((X-\mu_X)(Y-\mu_Y))=E(XY)-E(X)E(Y)$

$Var(X\pm Y)=Var(X)+Var(Y)\pm 2Cov(X,Y)$

Euler's Formula: $e^{i\theta}=\cos\theta+i\sin\theta$

Lecture 19 Gaussian process, Brownian motion continued, WSS process

<u>Theorem</u> (stationary Gaussian processes) A Gaussian process (X_t) is stationary if and only if the following two conditions are satisfied

1) $\mu_X(t)=E(X_t)=constant$ for all $t\geq 0$

2) $R(s,t)=cov(X_s,X_t)$ depends only on $t-s$,

i.e. $R(s,t)=Cov(X_s,X_t)=Cov(X_0,X_{t-s})=R(t-s)$

<u>Def (WSS-process)</u> A stochastic process (X_t) satisfying the two conditions 1) $\mu_X(t)=E(X_t)=const$, $t\in I$ and 2) $R(s,t)=R(t-s)$ depends only on $t-s$. from the Theorem above is called wide-sense stationary process (WSS process)

Tech:
① $W_t+W_s\to W_t-W_s+2W_s$
$W_t\to W_t-W_s+W_s$
构造 W_t-W_s 和 W_s, independent

<u>Remark</u> (white noise in discrete time)

Assume we have a sequence of rv's $(X_n)_{n=0,1,\cdots}$ with $E(X_n)=0$, $Var(X_n)=1$.

$$\begin{cases}\mu(n)=E(X_n)=0,\ n=0,1,\cdots\\ R(n,m)=cov(X_n,X_m)=E(X_nX_m)=\begin{cases}0, & n\neq m\\ 1, & n=m\end{cases}\end{cases}\Rightarrow \underset{\delta(\tau)}{\overset{\shortparallel}{R(\tau)}}=\begin{cases}0, & \tau\neq 0\\ 1, & \tau=0\end{cases}$$

(X_n) is called white noise

(White noise) A stochastic process (X_t) is white noise

1) $E(X_t)=0$ $\forall t$ 2) $Var(X_t)=\sigma^2$, $\forall t$ 3) $Cov(X_t,X_s)=\begin{cases}0, & t\neq s\\ \sigma^2, & t=s\end{cases}$

White noise is WSS-process 高斯白噪声是高斯过程，非高斯白噪声不一定

Lecture 20 White noise continued

<u>Def</u> Let (X_t), $t\in R$ be a WSS process with mean $\mu(t)=const$ and covariance $R(\tau)$. Assume that $\int_{-\infty}^{+\infty}|R(\tau)|d\tau<\infty$. Then, the Fourier transform $S(\nu)$ of $R(\tau)$ exists and is called the spectral density of (X_t): $S(\nu)=\int_{-\infty}^{+\infty}R(\tau)e^{-2\pi i\nu\tau}d\tau$

If, additionally, also $\int_{-\infty}^{+\infty}|S(\nu)|d\nu<\infty$, then it can be shown that $R(\tau)$ is continuous and it holds $R(\tau)=\int_{-\infty}^{+\infty}S(\nu)e^{2\pi i\nu\tau}d\nu$. In other words, $R(\tau)$ is obtained as the inverse Fourier transform of $S(\nu)$.

white noise $S(\nu)=1$, $\nu\in R$

Lecture 21 An application of WSS processes

Assume that (X_t) $t\in R$ is a WSS process with mean $\mu(t)$ and covariance function $R(\tau)$.

Let us consider a deterministic linear dynamical system (LDS) described by the following ODE system $\begin{cases}x'(t)=Ax(t)+Bu\\ y(t)=Cx(t)\end{cases}$ where $x(t_0)=x_0$ is the initial state of $x(t)$. In general case: the coefficients A,B,C are matrices; In case of $n=1$, they are just constants. u can be seen as a control parameter The solution of the above first ODE: $x(t)=e^{A(t-t_0)}[x_0+\int_{t_0}^{t}e^{A(t_0-s)}Buds]$

to simplify the situation, assume that $x_0=0\Rightarrow x(t)=\int_{t_0}^{t}e^{A(t-s)}Bud$

and $y(t)=C\int_{t_0}^{t}e^{A(t-s)}\cdot B\cdot u\,ds$. Formally. we use as $u=X_t$ $Y_t=C\int_{t_0}^{t}e^{A(t-s)}BX_s\,ds$.

(Y_t) seen as the response of the system. We also assume here that the linear model considered and the input process X are such that we can let in the last relation $t_0\to-\infty$ and obtain a steady-state response $Y_t=C\int_{-\infty}^{t}e^{A(t-s)}BX_s\,ds$

<u>Define</u> $h(\tau)=\begin{cases}Ce^{A\tau}\cdot B, & \tau\geq 0\\ 0, & \tau<0\end{cases}$ called the impulse-response of the system. Then, $Y_t=\int_{-\infty}^{+\infty}h(t-s)X_s\,ds$

<u>Properties of X_t, Y_t</u>

① (X_t) - WSS process, $\mu_X=E(X_t)=const$

$$E(Y_t)=\int_{-\infty}^{+\infty}h(t-s)E(X_s)ds=\mu\int_{-\infty}^{+\infty}h(t-s)ds\overset{(\tau=t-s)}{=}\mu\int_{-\infty}^{+\infty}h(\tau)d\tau=\mu\underset{}{\overset{L}{\shortparallel}}L$$

$$R_Y(s,t)=E((Y_t-EY_t)(Y_s-EY_s))=E\left(\left(\int_{-\infty}^{+\infty}h(t-\tau)(X_\tau-\mu)\right)\left(\int_{-\infty}^{+\infty}h(s-\sigma)(X_\sigma-\mu)\right)d\tau d\sigma\right)$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t-\tau)h(s-\sigma)\overbrace{E(X_\tau-\mu)(X_\sigma-\mu)}^{R_X(\tau,\sigma)}d\tau d\sigma$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t-\tau)h(s-\sigma)R_X(\tau-\sigma)d\tau d\sigma=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(\tilde{\tau})h(\tilde{s})R_X(t-s-\tilde{\tau}+\tilde{s})d\tilde{s}d\tilde{\tau}$$

$R_Y(s,t)$ depends only on $\tau=t-s$ $\Rightarrow$ (Y_t) is WSS process

<u>Def</u> Given the impulse-response function $h(t)$, its Fourier transform $H(\nu)=\int_{-\infty}^{+\infty}h(t)e^{-2\pi i\nu t}dt$ is called the transfer function

<u>Proposition</u> It holds $S_Y(\nu)=|H(\nu)|^2S_X(\nu)$

In particular, if (X_t) is a whit noise ($S_X(\nu)=1$), then $S_Y(\nu)=|H(\nu)|^2$

② We consider the case of LDS with $C=B=1$, $A=0$, $t_0=0$, we choose $u=X_t=$ white noise $\Rightarrow Y_t=\int_0^t X_s\,ds$. (Y_t) is Gaussian process with zero mean and covariance $R_Y(s,t)=\min(s,t)$ thus a Wiener process

③ We consider the case of LDS with $C=B=1$, $A=-\alpha$, $\alpha>0$

$u=X_t=$ white noise $S_X(\nu)=1$, $R_X(\tau)=\delta_0(\tau)$