ESE-520

Total: 40 points

HW#4 – Solutions

1. [5p] Two measurements $y_1 = 2$ and $y_2 = 5$ are taken to estimate the value x of a random variable X. Assume that the joint distribution of the full random vector (X, Y_1, Y_2) is Gaussian and that X has expectation $\mu_X = 3$, variance $\sigma_X^2 = 4$, and the following covariances with the random variables Y_1 and Y_2 which describe the first and second measurement, $cov(X, Y_1) = -1$ and $cov(X, Y_2) = +1$. Also suppose the measurements Y_1 and Y_2 have means $\mu_{Y_1} = \mu_{Y_2} = 3$, variances $\sigma_{Y_1}^2 = 3$ and $\sigma_{Y_2}^2 = 5$ and covariance $cov(Y_1, Y_2) = -2$. Find the best mean square estimate for X in terms of these measurements.

$$E(X|Y) = \mu_X + R_{XY} \cdot R_{YY}^{-1}(Y - \mu_Y),$$

$$\mu_X = 3, \quad \mu_Y = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad R_{XY} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}^T,$$

$$R_{YY} = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}, \quad Y = y\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad R_{YY}^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\Rightarrow E(X|Y=y) = 3 + (-1, 1)\frac{1}{11}\begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$$

$$= 3 + \frac{1}{11}(-1, 1)\begin{pmatrix} -1 \\ 4 \end{pmatrix} = 3 + \frac{5}{11} = \boxed{3\frac{5}{11}}.$$

2. [5p] Let (X, Y) be a randomly selected point on $\mathbb{R}^2$ where both coordinates X and Y have $\mathcal{N}(0,1)$ distribution and are independent. Also, the polar coordinates for (X, Y) are (R, θ) and are defined through $X = R\cos\theta, Y = R\sin\theta$. Calculate $E(XY|\theta)$.

- Fix $\theta = \tilde{\theta}$ & calculate

$$E(XY/\theta=\tilde{\theta}) = E(R^2 \cos\theta \sin\theta / \theta = \tilde{\theta}) =$$

$$= \frac{1}{2} E(R^2 \cdot \sin 2\tilde{\theta} \mid \theta = \tilde{\theta}) = \frac{1}{2} \sin 2\tilde{\theta}\, E(R^2 / \theta = \tilde{\theta}) =$$

R & θ are indep

$$= \frac{1}{2} \sin 2\tilde{\theta} \cdot E(R^2)$$

$$E(R^2) = \int_0^\infty r^3 \cdot e^{-r^2/2}\, dr = \underbrace{-r^2 e^{-r^2/2} \Big/_0^\infty}_{=0}$$

$$+ 2\underbrace{\int_0^\infty r e^{-r^2/2}\, dr}_{=1} = 2$$

$$\Rightarrow \boxed{E(XY/\theta) = \sin 2\theta}$$

3. [5p] Given: $X \sim \mathcal{N}(0,1)$, $W \sim Laplace\ (\lambda)$

X & W are independent; $\quad f_W(z) = \frac{\lambda}{2} \cdot e^{-\lambda|z|}$

$Y = X + W$

We know that the best linear estimator of X by Y is given as $E(X|Y) = \mu_X + R_{XY} \cdot R_{YY}^{-1} (Y - \mu_Y)$

- Since $E(X) = E(W) = 0 \Rightarrow E(Y) = 0 \Rightarrow \mu_X = \mu_Y = 0$
- $R_{XY} = E(X \cdot Y) = E(X(X+W)) = E(X^2) + E(X \cdot W) =$

$$= E(X^2) + \underbrace{E(X)}_{=0} \cdot \underbrace{E(W)}_{=0} = 1$$

- $R_{YY} = E(Y^2) = E(X+W)^2 = E(X^2) + 2\underbrace{E(X)}_{=0} E(W) + E(W^2)$

$$= 1 + \frac{2}{\lambda^2} \Rightarrow$$

$$E(X|Y) = 1 \cdot \frac{1}{1 + 2/\lambda^2} \cdot Y = \frac{\lambda^2}{2+\lambda^2} Y$$

8. [5p]

We have that: $\varphi_{Y_n}(t) = E\left[e^{it(X_1 + \cdots + X_n) \cdot \frac{1}{\sqrt{n}}} \right] =$

$$= E\left(\prod_{j=1}^{n} e^{\frac{it}{\sqrt{n}} X_j} \right) = \prod_{j=1}^{n} \left(\frac{1}{2} e^{\frac{-it}{\sqrt{n}}} + \frac{1}{2} e^{\frac{it}{\sqrt{n}}} \right) =$$

$=$ use Euler's formula $= \left(\cos \frac{t}{\sqrt{n}} \right)^n =$ use Taylor's formula for cos $= \left(1 - \frac{t^2}{n} \right)^n + o\left(\frac{t^2}{n} \right)$

$$\xrightarrow[n \to \infty]{} e^{-t^2/2}, \forall t \in \mathbb{R} \quad \boxtimes$$

6.[5p] Consider the experiment "tosssing a coin" where probability of occuring of heads is equal to 1/2. Using the Chebyshev's inequality, show that "in the long run", the number of heads occured relatively to the total number of tosses (relative frequency of heads) will converge to 1/2 (in probability).

$X_i \in \{1,0\}$ = # of heads in toss $i = 1, 2,$

$P(X_i = 1) = \frac{1}{2} = p, \forall i$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ - rel. frequency of heads in n tosses

$E(\bar{X}_n) = p, \; Var(\bar{X}_n) = \frac{p(1-p)}{n} = \frac{1}{4n}$

$\Rightarrow P(|\bar{X}_n - \frac{1}{2}| > \varepsilon) \overset{\text{Chebyshev}}{\leq} \frac{E(\bar{X}_n - \frac{1}{2})^2}{\varepsilon^2} = \frac{Var(\bar{X}_n)}{\varepsilon^2}$

$= \frac{p(1-p)}{n \cdot \varepsilon^2} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow \bar{X}_n \xrightarrow[n \to \infty]{(P)} \frac{1}{2}$

4. [5p] $f_{XY}(x,y) = x \cdot e^{-x(1+y)}, \; x>0, y>0$

$f_X(x) = e^{-x}, \; x>0$: as well

$f_Y(y) = \dfrac{1}{(1+y)^2}, \; y>0$: this fact is shown in lecture notes!

$\cdot \; f_{Y|X=x}(x,y) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{x \cdot e^{-x(1+y)}}{e^{-x}} = x \cdot e^{-xy}$

$\Rightarrow E(Y|X=x) = \int_0^\infty y \cdot f_{Y|X=x}(x,y)\,dy = \int_0^\infty x \cdot y \cdot e^{-xy}\,dy =$

$x>0$

$= x \cdot y \cdot \dfrac{1}{-x} \cdot e^{-xy}\Big/_0^\infty + \int_0^\infty e^{-xy}\,dy = -\dfrac{1}{x} \cdot e^{-xy}\Big/_0^\infty$

$= \dfrac{1}{x}, \; x>0$

5. [5p] $Y = \sum_{i=1}^{N} X_i, \quad N \sim \text{geometric}(p)$
$X_i \sim \exp(\lambda)$

$\cdot \; \varphi_Y(t) = E(e^{itY}) = E\left[e^{it \cdot \sum_{j=1}^{N} X_j}\right] =$ use conditioning on $N = \sum_{n=1}^{\infty}\left[E\left[e^{it\sum_{j=1}^{N} X_j} \mid N=n\right] \cdot P(N=n)\right]$

$= \sum_{n=1}^{\infty} E\left[e^{it\sum_{j=1}^{n} X_j} \mid N=n\right] \cdot P(N=n) =$ use independence

$= \sum_{n=1}^{\infty} E\left[e^{it\sum_{j=1}^{n} X_j}\right] \cdot P(N=n) =$

$= \sum_{n=1}^{\infty} \prod_{j=1}^{n} E(e^{itX_j}) \cdot P(N=n) =$

$$= \sum_{n=1}^{\infty} \left(\varphi_X(t)\right)^n \cdot P(N=n) = \text{use the fact that}$$

$$\varphi_X(t) = \frac{\lambda}{\lambda - it} \quad \& \quad P(N=n) = (1-p)p^{n-1}, \; n = 1, 2, \ldots$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{\lambda}{\lambda - it}\right)^n \cdot (1-p) \cdot p^{n-1} \right) =$$

$$= \frac{(1-p)}{p} \cdot \sum_{n=1}^{\infty} \left(\frac{\lambda p}{\lambda - it}\right)^n = \frac{1-p}{p} \left[\frac{1}{1 - \frac{\lambda p}{\lambda - it}} - 1 \right],$$

where $\left| \frac{\lambda p}{\lambda - it} \right| < 1$

7. [5p]

Let X_i = time to transmit the i-th packet $\Rightarrow$

$E(X_i) = \mu$ & $Var(X_i) = \sigma^2$

$T_n := X_1 + \ldots + X_n$ = total time $\Rightarrow E[T_n] = n\mu$

We have to find (approximate) $P(T_n > 2n\mu)$

$$\Rightarrow P(T_n > 2n\mu) = P\left(\frac{T_n}{n} > 2\mu\right) = P\Big(\underbrace{\frac{X_1 + \ldots + X_n}{n}}_{\overline{X}_n} > 2\mu\Big)$$

$$= P(\overline{X}_n > 2\mu) = P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} > \frac{2\mu - \mu}{\sigma/\sqrt{n}} \right)$$

$$= P\left(Z_n > \frac{\mu}{\sigma/\sqrt{n}} \right) = P\left(Z_n > \frac{\mu \cdot \sqrt{n}}{\sigma} \right)$$

$= Z_n \sim N(0,1)$ approx. $\Rightarrow$

$$P(T_n > 2n\mu) \approx 1 - \Phi\left(\frac{\mu \cdot \sqrt{n}}{\sigma}\right), \; \Phi \text{ is the cdf of } N(0,1)$$