ESE 520 Probability and Stochastic Processes

HW 3: due by 11pm CST on Thursday, October 10, 2024

Total: 40 points

Problem 1. Find the moment-generating function $\psi(t)$ for $X \sim Laplace(\lambda)$, i.e. X has the density $f(x) = \frac{\lambda}{2}e^{-\lambda|x|}, x \in \mathbb{R}$. Use your result to find $Var(X)$.

Problem 2. Apply the Fourier inversion formula to $\phi_X(t) = e^{-|t|}$ to verify that this is the characteristic function of $Cauchy(\lambda)$ random variable, i.e. it corresponds to the density $f_X(x) = \frac{\lambda/\pi}{\lambda^2+x^2}, x \in \mathbb{R}$.

Problem 3. Let $Z := X + Y$, where X and Y are independent with $X \sim exp(1)$ and $Y \sim Laplace(1)$. Find $cov(X, Z)$ and $Var(Z)$.

Problem 4. Find the marginal density $f_Y(y)$ if

$$f_{XY}(x, y) = \frac{4e^{-(x-y)^2/2}}{y^5\sqrt{2\pi}}, y \geq 1.$$

Problem 5. Let $X := \cos\theta$ and $Y := \sin\theta$, where $\theta \sim U[-\pi, \pi]$. Show that $E[XY] = 0$. Show that $E[X] = E[Y] = 0$. Argue that X and Y cannot be independent. This gives an example of continuous random variables that are uncorrelated, but not independent.

Problem 6. Let X and Y be jointly Gaussian with density $f_{XY}(x, y)$ as given in lecture notes (lecture 10). Find $f_X(x)$.

Problem 7. Evaluate

$$f(x) = \frac{\exp[-\frac{1}{2}(x-\mu)^T R^{-1}(x-\mu)]}{(2\pi)^{n/2}\sqrt{\det R}}$$

with $\mu = 0$ and

$$R = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2, \end{bmatrix}$$

where $\rho| < 1$. Show that your result has the same form as the bivariate normal density in lecture 10.

Problem 8. Let $X_1, ..., X_n$ be i.i.d. $N(\mu, \sigma^2)$ random variables, and denote the average of the X_i by $\bar{X} := \frac{1}{n}\sum_{i=1}^n X_i$. For $j = 1, ..., n$, put $Y_j := X_j - \bar{X}$. Show that $E[Y_j] = 0$ and that $E[\bar{X}Y_j] = 0$ for $j = 1, ..., n$.