

ESE 520 Probability and Stochastic Processes

HW2-Solutions; Total: 40 points

Problem 1. [5p] Assume that X_1, X_2, \dots, X_n are independent and identically distributed (i.i.d. for short) random variables each having a Poisson distribution with parameter $\lambda = 2$. Calculate $P(\min(X_1, \dots, X_n) > m)$ and $P(\max(X_1, \dots, X_n) \leq m)$ for a positive integer m .

Solution: We have that

$$\begin{aligned} P(\min(X_1, \dots, X_n) > m) &= P(X_1 > m, X_2 > m, \dots, X_n > m) = [\text{use independence}] = \\ &= P(X_1 > m) \times P(X_2 > m) \times \dots \times P(X_n > m) = \\ &= [1 - P(X_1 \leq m)] \times [1 - P(X_2 \leq m)] \dots \times [1 - P(X_n \leq m)] = \prod_{i=1}^n [1 - P(X_i \leq m)] \\ &= \prod_{i=1}^n [1 - e^{-2} \sum_{i=1}^m \frac{2^i}{i!}]. \end{aligned}$$

Similarly,

$$\begin{aligned} P(\max(X_1, \dots, X_n) \leq m) &= P(X_1 \leq m) \times \dots \times P(X_n \leq m) \\ &= \prod_{i=1}^n [e^{-2} \sum_{i=1}^m \frac{2^i}{i!}]. \end{aligned}$$

Problem 2. [5p.] Assume that the joint distribution of an integer-valued discrete vector (X, Y) is given by the joint pmf (probability mass function)

$$p_{XY}(x, y) = \begin{cases} \frac{(1-p)p^{x-1}x^ye^{-x}}{y!}, & x \geq 1, y \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

- a) Find $p_X(x)$ for $x \geq 1$;
- b) Compute $p_Y(0)$;
- c) Determine whether or not X and Y are independent.

Solution: a) By definition of a marginal distribution, we have that

$$\begin{aligned} p_X(x) &= \sum_{y=0}^{\infty} \frac{(1-p)p^{x-1}x^ye^{-x}}{y!} = (1-p)p^{x-1}e^{-x} \sum_{y=0}^{\infty} \frac{x^y}{y!} \\ &= (1-p)p^{x-1}e^{-x}e^x = (1-p)p^{x-1}, x \geq 1. \end{aligned}$$

b) Again, by definition of a marginal distribution,

$$p_Y(0) = \sum_{x=1}^{\infty} (1-p)p^{x-1}e^{-x} = \frac{1-p}{e} \sum_{x=1}^{\infty} \left(\frac{p}{e}\right)^{x-1} = \frac{1-p}{e} \frac{1}{1-p/e} = \frac{1-p}{e-p}.$$

c) By inspection, we can see that

$$p_X(1)p_Y(0) = \frac{(1-p)^2}{e-p} \neq p_{XY}(1,0) = \frac{1-p}{e}.$$

Thus, X and Y are not independent.

Problem 3. [5p.] Let X be a continuous random variable that has the Weibull distribution, i.e. it has density of the form

$$f(x) = \begin{cases} \lambda p x^{p-1} e^{-\lambda x^p}, & x > 0, \\ 0, & \text{for } x \leq 0, \end{cases}$$

where $\lambda > 0$ and $p > 0$ are parameters.

a) Show that this density integrates to one;

b) Let X_1, \dots, X_n be i.i.d. Weibull (p, λ) random variables. Find the probability that at least one of them exceeds 3.

Solution: a)

$$\begin{aligned} \int_0^\infty \lambda p x^{p-1} e^{-\lambda x^p} dx &= [\text{we make change of variable } y = \lambda x^p, dy = \lambda p x^{p-1} dx] = \\ &= \int_0^\infty e^{-y} dy = 1. \end{aligned}$$

b) We calculate

$$P\left(\bigcup_{i=1}^n \{X_i > 3\}\right) = 1 - P\left(\bigcap_{i=1}^n \{X_i \leq 3\}\right) = 1 - [1 - e^{-\lambda 3^p}]^n.$$

Problem 4. [5p.] For $p > 0$ and $q > 0$, let

$$B(p, q) := \int_0^1 u^{p-1} (1-u)^{q-1} du$$

called the Beta function. Consider

$$f(z) := \frac{1}{B(p, q)} \frac{z^{p-1}}{(1+z)^{p+q}}, z > 0.$$

Show that $f(z)$ is a valid density. Hint: Make the change of variable $t = 1/(1+z)$.

Solution: Using the suggested change of variables, we calculate

$$\begin{aligned} \int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}} dz &= \int_0^1 \left(\frac{1}{t} - 1\right)^{p-1} t^{p+q} \frac{dt}{t^2} = \\ &= \int_0^1 (1-t)^{p-1} t^{q-1} dt = B(q, p) = B(p, q) \end{aligned}$$

since $B(p, q)$ is a symmetric function with respect to its parameters. The statement is verified.

Problem 5. [5p.] For $n = 1, 2, \dots$, let $f_n(x)$ be a probability density function, and let p_n be a sequence of non-negative numbers summing to one, i.e. sequence p_n forms a discrete probability distribution. Show that

$$f(x) := \sum_n p_n f_n(x)$$

is a probability density function.

Solution: First of all, it is clear that $f(x) \geq 0$ for all $x \in \mathbb{R}$. In other words, one has only to show that $f(x)$ integrates to one (or it sums to one in case when values x are discrete).

Let $f_N(x) := \sum_{n=1}^N p_n f_n(x)$, $N = 1, 2, \dots$. Then,

$$\int_{\mathbb{R}} f_N(x) dx = \int_{\mathbb{R}} \sum_{n=1}^N p_n f_n(x) dx = \sum_{n=1}^N p_n \int_{\mathbb{R}} f_n(x) dx = \sum_{n=1}^N p_n \leq 1$$

for all $N = 1, 2, \dots$. We also note that all functions $f_N(x)$ are non-negative and

$$\lim_{N \rightarrow \infty} f_N(x) = f(x), x \in \mathbb{R}.$$

By Monotone Convergence Theorem,

$$\begin{aligned} \int_{\mathbb{R}} f(x) dx &= \int_{\mathbb{R}} \lim_{N \rightarrow \infty} f_N(x) dx = \lim_{N \rightarrow \infty} \int_{\mathbb{R}} f_N(x) dx = \\ \lim_{N \rightarrow \infty} \int_{\mathbb{R}} \sum_{n=1}^N p_n f_n(x) dx &= \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \int_{\mathbb{R}} f_n(x) dx = \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n = 1. \end{aligned}$$

Problem 6. [5p.] Assume X has Weibull (p, λ) distribution as in Problem 3. Show that

$$E[X^n] = \frac{\Gamma(1 + \frac{n}{p})}{\lambda^{n/p}}.$$

Solution: We use here change of variables $y = \lambda x^p, dy = \lambda p x^{p-1} dx$.

$$\begin{aligned} E[X^n] &= \int_0^\infty x^n \lambda p x^{p-1} dx = \int_0^\infty \left[\frac{y}{\lambda}\right]^{1/p} e^{-y} dy = \\ \left(\frac{1}{\lambda}\right)^{n/p} \int_0^\infty y^{[(n/p)+1]-1} e^{-y} dy &= \frac{\Gamma(1 + n/p)}{\lambda^{n/p}}. \end{aligned}$$

Problem 7. [5p.] Let X be exponential with parameter $\lambda = 1$. Show that $Y = \sqrt{X}$ is Reyleigh $(1/\sqrt{2})$.

Solution: We first find the cdf of Y :

$$F_Y(y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2) = 1 - e^{-y^2},$$

where $y > 0$.

Therefore,

$$f_Y(y) = F'_Y(y) = -e^{-y^2}(-2y) = \frac{y}{(1/\sqrt{2})} e^{-[y/(1/\sqrt{2})]^2/2}$$

which is the pdf of Reyleigh distribution with parameter $1/\sqrt{2}$.

Problem 8. [5p.] If X and Y are independent $\exp(\lambda)$ random variables, find $E[\max(X, Y)]$.

Solution: We can use here Problem 1 to conclude that the random variable $Z := \max\{X, Y\}$ will have the cdf $F_Z(z) = [F_X(z)]^2$. Thus,

$$f_Z(z) = 2F_X(z)f_X(z) = 2(1 - e^{-\lambda z})\lambda e^{-\lambda z}, z \geq 0.$$

Therefore,

$$\begin{aligned} E[Z] &= \int_0^\infty z f_Z(z) dz = 2 \int_0^\infty \lambda z e^{-\lambda z} (1 - e^{-\lambda z}) dz = \\ &= 2 \int_0^\infty z \lambda e^{-\lambda z} dz - \int_0^\infty z (2\lambda) e^{-(2\lambda)z} dz = 2 \frac{1}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}. \end{aligned}$$