ESE 520 Probability and Stochastic Processes

HW 1 - Solutions

Total: 40 points

Problem 1. [3p.] Let Ω be a sample space and A, B be arbitrary subsets from Ω.
a) Assuming that $A \subset B$, show that

$$(B \cap C) \cup A = B \cap (C \cup A) \tag{1}$$

for every subset $C \subset \Omega$;
b) Assuming that (1) holds for some subset C, show that $A \subset B$.

Solution: *a) By the distribute law and the fact that $A \cap B = A$, we get that*

$$B \cap (C \cup A) = (B \cap C) \cup (A \cap B) = (B \cap C) \cup A$$

which proves the statement.

b) To show that $A \subset B$, it is enough to show that, for any element $\omega \in A$, it follows that $\omega \in B$. Assuming (1) is true, it follows that $\omega \in (B \cap C) \cup A = B \cap (C \cup A)$ which also means that $\omega \in B$. This proves the statement.

Problem 2. [4p.] A collection of plastic letters, a-z, is mixed in a jair (26 letters in total). Two letters are drawn at random, one after the other. What is the probability of drawing a vowel (a,e,i,o,u) and a consonant in either order? Two vowels in any order? Specify the sample space Ω and probability P.

Solution: *We can assign any letter from the alphabet an integer $1, 2, ..., 26$ with 1 being the first letter a ..... and 26 being the last letter z. The sample space is then*

$$\Omega = \{(i, j) | i, j = 1, 2, ..., 26\}$$

We also assume that all letter are equally likely to be chosen so that $|\Omega| = 26 \times 25 = 650$.

Let A be the event of drawing a vowel and a consonant in any order. Clearly that a vowel can be drawn from 5 available letters and a consonant can be drawn from other 21 available letters. By multiplication rule and taking into account the order in which we select a vowel and a consonant, we obtain that

$$P(A) = 2 \times \frac{5 \times 21}{650} = \frac{21}{65}.$$

Let B be the event of drawing two vowels in any order. By a similar argument,

$$P(B) = \frac{5 \times 4}{650} = \frac{2}{65}.$$

Problem 3. [5p.] Use appropriate properties of probability to show that, for any sequence of events $\{A_n\}\}, n = 1, 2, ...,$ it holds that

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{N\to\infty} P\Big(\bigcap_{n=1}^{N} A_n\Big).$$

Solution: *We will use here the property of probability called "continuity from below or above". First, by De Morgan's law, we have that*

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = 1 - P\Big(\bigcup_{n=1}^{\infty} \bar{A}_n\Big),$$

where $\bar{A}_n$ is the complement of A_n. Let $B_n := \bigcup_{i=1}^{n} \bar{A}_i, n \geq 1$. Clearly

$$\bigcup_{n=1}^{\infty} \bar{A}_n = \bigcup_{n=1}^{\infty} B_n$$

and the sequence of events B_n is increasing in the sense that $B_n \subset B_{n+1}, n = 1, 2, ...$ We have that

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = 1 - P\Big(\bigcup_{n=1}^{\infty} \bar{A}_n\Big) = 1 - P\Big(\bigcup_{n=1}^{\infty} B_n\Big)$$

$$1 - \lim_{N\to\infty} P(B_N) = \lim_{N\to\infty} (1 - P(B_N) = \lim_{N\to\infty} P(\bigcap_{n=1}^{N} A_n)$$

which proves the statement.

Problem 4. [5p.] Let $\mathcal{A} =: A_1, ..., A_n$ be a partition of Ω. Show that $\sigma(\mathcal{A})$ consists of empty set along with all unions of the form $\cup_i A_{k_i}$, where k_i is a finite subsequence of distinct elements from $\{1, 2, ..., , n\}$.

Solution: *By definition, the collection of events $\sigma(\mathcal{A})$ is the smallest σ-algebra that contains all elements of the generating set $\mathcal{A}$ plus all other sets that can be obtain from the sets of $\mathcal{A}$ being countable unions, intersections or complements of them. Thus, to prove the statement it is enough to show the collection consisting from the empty set and all finite unions of the form $\cup_i A_{k_i}$ is a σ-algebra, i.e. it suffices to prove three things:*

i) $\emptyset$ belongs to that collection by definition;

ii) Choose a set $A = \cup_i A_{k_i}$ for some subsequence $\{k_i\}$. Then

$$\Big(\cup_i A_{k_i}\Big)^c = \cup_j A_{m_j},$$

where $\{m_j\}$ is a subsequence $\{1, 2, ..., n\}\} \setminus \{k_i\}$. The right hand-side of the last relation is a union of elements from $\mathcal{A}$ over a finite subsequence of distinct elements from $\{1, 2, ..., n\}$ thus belongs to the collection.

iii) Finally, consider

$$\bigcup_{n=1}^{\infty}\Big(\bigcup_i A_{n,i}\Big).$$

Since the sequence of sets is a partition $A_1, ..., A_n$, it is then clear that the above countable union of sets being finite unions of non-intersecting sets from a fixed partition is noting but a union of only finitely many elements from that partition, i.e. we can write that

$$\bigcup_{n=1}^{\infty}\Big(\bigcup_i A_{n,i}\Big) = \bigcup_j A_{m_j}$$

for some subsequence from $\{1, 2, .., n\}$.

The latter proves that a countable union of elements from the collection belongs to the collection as well. The statement is proven.

Problem 5. [4p.] a) Assume $\mathcal{F}_i \in I$ is a σ-algebra, where I is an index set (countable or uncountable). Show that $\mathcal{F} := \cap_{i\in I}\mathcal{F}_i$ is a σ-algebra as well.

b) Argue that union of two σ-algebras is not a *sigma*-algebra (in general).

Solution: *a) By definition of a σ-algebra, it is enough to show three things:*

i) $\Omega \in \mathcal{F}$: It is clearly satisfied since by the definition of intersection Ω belongs to $\mathcal{F}$ if and only if it belongs to any element from the intersection. However, $\Omega \in \mathcal{F}_i$ for any $i \in I$ since $\mathcal{F}_i$ is a σ-algebra for any $i \in I$;

ii) Choose $A \in \mathcal{F}$. It follows then that $A \in \mathcal{F}_i$ for all $i \in I$. Again, since $\mathcal{F}_i$ is a σ-algebra, it follows that $\bar{A}_i \in \mathcal{F}_i$ for any $i \in I$. We conclude then that $\bar{A} \in \mathcal{F}$ due to the definition of intersection.

iii) Similarly the last property. Choose $A_1, A_2, ...$ from $\mathcal{F}$. It follows then that the sequence $A_1, A_2, ...$ belongs to any σ-algebra $\mathcal{F}_i, i \in I$. We conclude that $\cup_{n=1}^{\infty}A_n \in \mathcal{F}_i$ for all $i \in I$ thus $\cup_{n=1}^{\infty}A_n \in \mathcal{F}$ which concludes the proof.

Problem 6. [3p.] There are k students in a probability class. What is probability of A="No two students have the same birthday"? Give numerical answer for $k = 30$. Assume that there are 365 days in a year.

Solution: *We assume that every student has the same chance to have his/her birthday on any of 365 days in a year, i.e. we have a classical probability model so that for every event A*

$$P(A) = \frac{m}{n},$$

where n=number of all possible outcomes and m=number of outcomes when the event A occurs.

It is then easy to see that in our case

$$P(A) = \frac{365(365-1)\cdots(365-k+1)}{365^k}.$$

For $k = 30$, $P(A) = 0.2937$.

Problem 7. [4p.] Let A="Alice and Betty have the same birthday, B="Betty and Carol have the same birthday", C="Carol and Alice have the same birthday". Show that events $\{A, B, C\}$ are *pairwise independent*. Are all three events independent as well? Assume that there are 365 days in a year.

Solution: *First, using similar argument as in Problem 6, it is clear that*

$$P(A) = P(B) = P(C) = \frac{365 \times 1}{365^2} = \frac{1}{365}.$$

From another side,

$$P(A \cap B) = \frac{365 \times 1 \times 1}{365^3} = \frac{1}{365^2} = P(A)P(B)$$

thus A and B are independent. Similar for pairs A, C and B, C.

However, all three events A, B, C are not independent since $A \cap B = A \cap B \cap C$ so that

$$P(A \cap B \cap C) = \frac{1}{365} \neq \frac{1}{365^3} = P(A)P(B)P(C).$$

Problem 8. [6p.] a) Suppose that the number of children in a family is 1,2, or 3 with probability 1/3 each. Little Bobby has no brothers. What is the probability he is an only child?

b) Suppose that the number of children in a family is 1,2, or 3 with probability 1/3 each. Little Bobby has no sisters. What is the probability he is an only child?

Solution: *a) Let B_1, B_2, B_3 be the events that a family has one, two, or three children, and let A be the event that a family has only one boy. We want to compute $P(B_1|A)$.*

Using the definition of conditional probability,

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)}$$

so that we only need to find numerator and denominator.

For the numerator:

$$P(B_1 \cap A) = P(B_1)P(A|B_1) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}.$$

Similarly, $P(B_2 \cap A) = P(B_2)P(A|B_2) = \frac{1}{3}\frac{2}{4} = \frac{1}{6}$ and

$$P(B_3 \cap A) = P(B_3)P(A|B_3) = \frac{1}{3}\frac{3}{8} = \frac{1}{8}.$$

Finally, since $P(A) = \sum_i P(B_i \cap A)$ we have

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)} = \frac{1/6}{1/6 + 1/6 + 1/8} = \frac{4}{11}.$$

b) Solution is similar with the difference that now we define A="family has no girls". We have then to find again $P(B_1|A)$. Using similar formula (which nothing but Bayes' formula), we obtain that

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)},$$

where $P(A|B_1)P(B_1) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ and

$$P(A) = \sum_{i=1}^{3} P(B_i)\Big[P(A|B_1) + P(A|B_2) + P(A|B_3)\Big] = \frac{1}{3}[1/2 + 1/4 + 1/8] = 7/24.$$

As the result,

$$P(B_1|A) = \frac{1/6}{7/24} = 4/7.$$

Problem 9. [6p.] Let $(\Omega, \mathcal{F}, P)$ be a probability space and A, B are two arbitrary events from $\mathcal{F}$. Show that events A and B are independent if and only if the σ-algebras $\mathcal{F}^A$ and $\mathcal{F}^B$ generated by A and B, respectively, are independent. Recall that, by definition, $\mathcal{F}^A = \{\emptyset, \Omega, A, A^c\}$ and $\mathcal{F}^B = \{\emptyset, \Omega, B, B^c\}$.

Solution: *We know that events A and B are independent if and only if*

$$P(A \cap B) = P(A)P(B). \tag{2}$$

Two σ-algebras $\mathcal{F}^A$ and $\mathcal{F}^B$ are independent if and only if

$$P(C_1 \cap C_2) = P(C_1)P(C_2), \tag{3}$$

where C_1 is any element from $\mathcal{F}^A$ and C_2 is any element from $\mathcal{F}^B$.

The proof in one direction (3) $\to$ (2) *is trivial since A, B is one of the possible pairs for which (3) is assumed to be true so that (2) is included in statement (3) already.*

To prove another direction (2) $\to$ (3)*, one has to prove the independence of all other possible pairs. The independence of some pairs including sets $\emptyset$ and Ω will be also trivial. Non-trivial pairs left are pairs $(A, \bar{B})$, $(\bar{A}, B)$, and $(\bar{A}, \bar{B})$.*

For example, let us prove the independence of events $\bar{A}$ and $\bar{B}$.

Assuming (2) being true, we have that

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \Big[P(A) + P(B) - P(A \cap B)\Big] =$$
$$1 - P(A) - P(B) + P(A)P(B) = P(\bar{A}) - P(B)[1 - P(A)] =$$
$$P(\bar{A}) - P(B)P(\bar{A}) = P(\bar{A})[1 - P(B)] = P(\bar{A})P(\bar{B})$$

which proves the independence of events $\bar{A}$ and $\bar{B}$. The proof for two remaining pairs is similar.