Homework 6

1. (a)

$Y_t = X_t \cos(2\pi f t + \theta)$ where $\theta \sim U(-\pi, \pi)$ and θ is independent of the process (X_t)

$R_Y(\tau) = E(Y_t Y_{t+\tau}) = E\left(X_t \cos(2\pi f t+\theta)\, X_{t+\tau} \cos(2\pi f(t+\tau)+\theta)\right)$

$= E\left(X_t X_{t+\tau} \cos(2\pi f t+\theta) \cos(2\pi f(t+\tau)+\theta)\right)$

$= E\left(X_t X_{t+\tau} \frac{1}{2}[\cos(4\pi f t + 2\pi f\tau + 2\theta) + \cos(2\pi f \tau)]\right)$

$= E(X_t X_{t+\tau})\, E\left(\frac{1}{2}[\cos(4\pi f t + 2\pi f \tau + 2\theta) + \cos(2\pi f\tau)]\right)$

$= E(X_t X_{t+\tau})\, \frac{1}{2} \cos(2\pi f \tau)$

$= \frac{1}{2} \cos(2\pi f \tau)\, R(\tau)$

(b)

$R_{XY}(\tau) = E(X_t Y_{t+\tau}) = E\left(X_t X_{t+\tau} \cos(2\pi f(t+\tau)+\theta)\right)$

$= E(X_t X_{t+\tau})\, E\left(\cos(2\pi f(t+\tau)+\theta)\right) \quad = 0$

(c)

① $E(Y_t) = E\left(X_t \cos(2\pi f t+\theta)\right) = E(X_t)\, E\left(\cos(2\pi f t+\theta)\right) = 0$

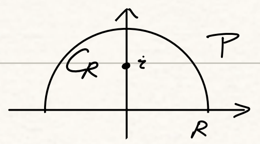
② Since $R_Y(\tau) = \frac{1}{2} R(\tau) \cos(2\pi f \tau)$ which depends only on τ

Thus, (Y_t) is WSS process.

2.

$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-2\pi i f \tau} d\tau = \int_{-\infty}^{+\infty} \frac{1}{1+\tau^2} e^{-2\pi i f \tau} d\tau = \int_P \frac{1}{1+\tau^2} e^{-i 2\pi f \tau} d\tau$

$= \int_{C_R} \frac{1}{1+\tau^2} e^{-i2\pi f\tau} d\tau + \int_{-R}^{R} \frac{1}{1+\tau^2} e^{-i 2\pi f \tau} d\tau$



$R \to \infty$

$f(\tau) = \frac{e^{-i2\pi f\tau}}{1+\tau^2} \quad 1+\tau^2 = 0 \Rightarrow \tau = \pm i$

$\int_{C_R} \frac{e^{-i2\pi f \tau}}{1+\tau^2} d\tau \overset{\tau = Re^{i\theta}}{=} \int_0^{\pi} \frac{e^{-i2\pi f Re^{i\theta}}}{1+(Re^{i\theta})^2} iRe^{i\theta} d\theta \le \int_0^{\pi} \frac{iRe^{i\theta}}{1+(Re^{i\theta})^2} d\theta \le \int_0^{\pi} \frac{1}{R} d\theta = \frac{\pi}{R}$

when $R \to +\infty$, $\int_{C_R} \frac{e^{-i2\pi f \tau}}{1+\tau^2} d\tau \to 0$

$$\int_P f(\tau)\,d\tau = 2\pi i \sum Res(f(\tau), \tau_k) = 2\pi i\, Res(f(\tau), \tau_k = i)$$

$$= 2\pi i \lim_{\tau\to i} (\tau - i) \frac{e^{-i2\pi f\tau}}{1+\tau^2} = 2\pi i \lim_{\tau \to i} \frac{e^{-i2\pi f\tau}}{\tau + i} = 2\pi i \frac{e^{-i2\pi f i}}{2i} = \pi e^{2\pi f}$$

Thus, $S_x(f) = \pi e^{2\pi |f|}$

3. $$R(\tau) = \int_{-\infty}^{+\infty} S(v)\, e^{2\pi i v\tau}\, dv$$

(a) $\delta(\tau) \longrightarrow S(v) = \delta(v)$

since $\delta(v) = \begin{cases} 0, & v \neq 0 \\ \infty, & v = 0 \end{cases}$ $\int_{-\infty}^{+\infty} \delta(v)\, dv = 1$,

$$R(\tau) = \int_{-\infty}^{+\infty} S(v) e^{2\pi i v\tau} dv = \int_{-\infty}^{+\infty} \delta(v) e^{2\pi i v\tau} dv = e^{2\pi i \cdot 0 \cdot \tau} = 1$$

(b) $\delta(\tau - \tau_0) + \delta(\tau + \tau_0) \longrightarrow S(v) = \delta(v - v_0) + \delta(v - v_0)$

$$R(\tau) = \int_{-\infty}^{+\infty} S(v)\, e^{2\pi i v\tau} dv = \int_{-\infty}^{+\infty} (\delta(v - v_0) + \delta(v + v_0))\, e^{2\pi i v\tau} dv$$

$$= \int_{-\infty}^{+\infty} \delta(v - v_0) e^{2\pi i v\tau} dv + \int_{-\infty}^{+\infty} \delta(v + v_0) e^{2\pi i v\tau} dv = e^{2\pi i v_0 \tau} + e^{-2\pi i v_0 \tau}$$

$(e^{i\theta} = \cos\theta + i\sin\theta)$ $= \cos(2\pi v_0 \tau) + i\sin(2\pi v_0\tau) + \cos(-2\pi v_0\tau) + i\sin(-2\pi v_0 \tau)$

$$= 2\cos(2\pi v_0 \tau) + i\sin(2\pi v_0 \tau) - i\sin(2\pi v_0\tau)$$

$$= 2\cos(2\pi v_0 \tau)$$

(c) $e^{-\frac{\tau^2}{2}} \longrightarrow S(v) = e^{-\frac{v^2}{2}}$

$$R(\tau) = \int_{-\infty}^{+\infty} S(v)\, e^{-2\pi i v\tau} dv = \int_{-\infty}^{+\infty} e^{-\frac{v^2}{2}} e^{-2\pi i v\tau} dv = \int_{-\infty}^{+\infty} e^{-\frac{v^2}{2} - 2\pi i v \tau} dv$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{v^2 - 4\pi i v\tau}{2}} dv = \int_{-\infty}^{+\infty} e^{-\frac{(v - 2\pi i\tau)^2 + 4\pi^2\tau^2}{2}} dv$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi^2\tau^2} e^{-\frac{(v-2\pi i \tau)^2}{2}} dv = e^{-2\pi^2\tau^2} \int_{-\infty}^{+\infty} e^{-\frac{(v - 2\pi i\tau)^2}{2}} dv$$

$(z = v - 2\pi i\tau)$ $= e^{-2\pi^2\tau^2} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}\, e^{-2\pi^2\tau^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz$

$$= \sqrt{2\pi}\, e^{-2\pi^2\tau^2}$$

(d) $e^{-|\tau|} \longrightarrow S(v) = e^{-|v|} = \begin{cases} e^{-v}, & v \geq 0 \\ e^{v}, & v < 0 \end{cases}$

$$R(\tau) = \int_{-\infty}^{+\infty} S(v)\, e^{-2\pi i v\tau} dv = \int_{-\infty}^{0} e^{v} e^{-2\pi i v\tau} dv + \int_0^{+\infty} e^{-v} e^{-2\pi i v\tau} dv$$

$$= \int_{-\infty}^{0} e^{v(1 - 2\pi i\tau)} dv + \int_0^{+\infty} e^{v(-1-2\pi i\tau)} dv$$

$$= \frac{1}{1-2\pi i t} e^{v(1-2\pi i t)}\Big|_{-\infty}^{0} + \frac{1}{-1-2\pi i t} e^{v(-1-2\pi i t)}\Big|_{0}^{+\infty}$$

$$= \frac{1}{1-2\pi i t}(1-0) + \frac{1}{-1-2\pi i t}(0-1)$$

$$= \frac{1}{1-2\pi i t} + \frac{1}{1+2\pi i t} = \frac{2}{1-(2\pi i t)^2} = \frac{2}{1+4\pi^2 t^2}$$

4

$$R_Y(t_1,t_2) = E(Y_{t_1} Y_{t_2}) = E\left(\frac{e^{-\lambda t_1}}{\sqrt{2\lambda}} W_{e^{2\lambda t_1}} \frac{e^{-\lambda t_2}}{\sqrt{2\lambda}} W_{e^{2\lambda t_2}}\right) = \frac{1}{2\lambda} E\left(e^{-\lambda(t_1+t_2)} W_{e^{2\lambda t_1}} W_{e^{2\lambda t_2}}\right)$$

$$= \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} E\left(W_{e^{2\lambda t_1}} W_{e^{2\lambda t_2}}\right) = \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} \min(e^{2\lambda t_1}, e^{2\lambda t_2})$$

$$= \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} e^{2\lambda \min(t_1,t_2)}$$

Without loss of generality, assume that $t_1 \le t_2$, then $\min(t_1,t_2) = t_1$

$$R_Y(t_1,t_2) = \frac{1}{2\lambda} e^{-\lambda(t_1+t_2)} e^{2\lambda t_1} = \frac{1}{2\lambda} e^{\lambda(t_1-t_2)} = \frac{1}{2\lambda} e^{-\lambda|t_1-t_2|}$$

Thus, $R_Y(t_1,t_2) = \frac{1}{2\lambda} e^{-\lambda|t_1-t_2|}$

5.

$t_2 > t_1$

$$R_Y(t_1,t_2) = E(Y_{t_1} Y_{t_2}) = E(e^{W_{t_1}} e^{W_{t_2}}) = E(e^{W_{t_1}+W_{t_2}})$$

$$= E(e^{2W_{t_1}} \cdot e^{W_{t_2}-W_{t_1}}) = E(e^{2W_{t_1}}) E(e^{W_{t_2}-W_{t_1}})$$

Since $W_{t_1} \sim N(0,t_1)$, $W_{t_2}-W_{t_1} \sim N(0, t_2-t_1)$

Thus, $$R_Y(t_1,t_2) = \int_{-\infty}^{+\infty} e^{2x} \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x^2}{2t_1}} dx \cdot \int_{-\infty}^{+\infty} e^{x} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{x^2}{2(t_2-t_1)}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{(x-2t_1)^2}{2t_1}+2t_1} dx \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x-(t_2-t_1))^2}{2(t_2-t_1)}+\frac{t_2-t_1}{2}} dx$$

$$= e^{2t_1} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{(x-2t_1)^2}{2t_1}} dx \cdot e^{\frac{t_2-t_1}{2}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x-(t_2-t_1))^2}{2(t_2-t_1)}} dx$$

$$= e^{2t_1} \cdot e^{\frac{t_2-t_1}{2}} = e^{\frac{3}{2}t_1+\frac{1}{2}t_2}$$

6.

Since W_t is Wiener process, we have $W_0 = 0$, it has independent increments and

$W_t - W_s \sim N(0, t-s)$, where $t > s$.

$E(W_t) = E(W_t - W_0) = 0$

$Cov(W_s, W_t) = E(W_s W_t) - E(W_s)E(W_t) = E(W_s W_t) = \min(s,t)$

Consider a finite set of time point $t_1 < t_2 < \cdots < t_n$, The increments of the Wiener process are independent

$$W_{t_1}, W_{t_2} - W_{t_1}, \cdots W_{t_n} - W_{t_{n-1}}$$

are independent, and each increment is normally distributed

We can write $(W_{t_1}, \cdots, W_{t_n})$ as a linear combination of independent normal random variables. Since a linear combination of independent normal random variables is itself normal, the joint distribution of $(W_{t_1}, \cdots, W_{t_n})$ is multivariate normal.

The mean vector $E(W_{t_i}) = 0$, $i = 1, \cdots, n$

The covariance matrix Σ has entries, $\Sigma_{ij} = Cov(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$

Since the process satisfies the mean and covariance structure, and the finite-dimensional distributions are multivariate normal, the process (W_t) is Gaussian.

7.

$h(t) = \frac{1}{1+t^2}$, $Y_t = \int_{-\infty}^{+\infty} h(t-s) X_s \, ds$

$$E(Y_{t+\frac{1}{2}} Y_t) = E\left(\int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-s) X_s \, ds \cdot \int_{-\infty}^{+\infty} h(t-s) X_s \, ds\right)$$

$$= E\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-u) \cdot h(t-v) X_u X_v \, du dv\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-u) \cdot h(t-v) E(X_u X_v) \, du dv$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-u) \cdot h(t-v) \delta(u-v) \, du dv$$

$$= \int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-u) \int_{-\infty}^{+\infty} h(t-v)\delta(u-v)\, dv\, du$$

$$= \int_{-\infty}^{+\infty} h(t+\tfrac{1}{2}-u)\, h(t-u)\, du$$

$$= \int_{-\infty}^{+\infty} \frac{1}{1+(t+\frac{1}{2}-u)^2} \frac{1}{1+(t-u)^2}\, du \qquad \text{let } v = u-t,\ du = dv$$

$$= \int_{-\infty}^{+\infty} \frac{1}{1+(\frac{1}{2}-v)^2} \frac{1}{1+v^2}\, dv = \int_{-\infty}^{+\infty} \frac{1}{(v^2+1)(v^2-v+\frac{5}{4})}\, dv$$

$$\text{let } H(z) = \frac{1}{(1+z^2)(\frac{5}{4}-z+z^2)}$$

$$1+z^2=0 \implies z = \pm i \quad ; \quad z^2-z+\tfrac{5}{4}=0 \implies z = \frac{1\pm 2i}{2}$$

For $z=i$, $\mathrm{Res}(H(z), z=i) = \lim_{z\to i} \dfrac{z-i}{(1+z^2)(\frac{5}{4}-z+z^2)} = \lim_{z\to i} \dfrac{1}{(z+i)(z^2-z+\frac{5}{4})} = \dfrac{1}{2i(\frac{1}{4}-i)}$

$$= \frac{1}{\frac{1}{2}i+2} = \frac{2}{i+4}$$

For $z=\frac{1+2i}{2}$, $\mathrm{Res}(H(z), z=\frac{1+2i}{2}) = \lim_{z\to\frac{1+2i}{2}} \dfrac{z-\frac{1+2i}{2}}{(1+z^2)(z-\frac{1+2i}{2})(z-\frac{1-2i}{2})} = \lim_{z\to\frac{1+2i}{2}} \dfrac{1}{(1+z^2)(z-\frac{1-2i}{2})}$

$$= \frac{1}{(1+\frac{1+4i-4}{4})(\frac{1+2i}{2}-\frac{1-2i}{2})} = \frac{1}{\frac{1}{4}+i+2i} = \frac{4}{1+12i}$$

$$\int_{-\infty}^{+\infty} \frac{1}{(v^2+1)(v^2-v+\frac{5}{4})}\, dv = 2\pi i \left(\frac{2}{i+4} + \frac{4}{1+12i} \right)$$

Thus, $E(Y_{t+\frac{1}{2}} Y_t) = 2\pi i \left(\dfrac{2}{i+4} + \dfrac{4}{1+12i} \right)$

8.

$P = (p_{ji})$, $i, j \in S$ is a stochastic matrix, where S is a finite or a countable set.

Thus, by definition of stochastic matrix, $p_{ji} \geq 0$ for all $i, j \in S$.

Each element of P^n, $P^n_{ji} = \sum_{k_1, \dots, k_{n-1} \in S} P_{jk_1} P_{k_1 k_2} \cdots P_{k_{n-1} i} \geq 0$

For matrix P, $\sum_{j\in S} P_{ji} = 1$ for all $i \in S$

$$\sum_{j\in S} P^n_{ji} = \sum_{j\in S}\sum_{k\in S} P^{n-1}_{jk} P_{ki}$$

Since P^{n-1} is assumed to be a stochastic matrix, we know that

$$\sum_{j\in S} P^n_{ji} = \sum_{j\in S}\sum_{k\in S} P^{n-1}_{jk} P_{ki} = \sum_{k\in S}\sum_{j\in S} P^{n-1}_{jk} P_{ki} = \sum_{k\in S} P_{ki} = 1$$

Therefore, P^n is a stochastic matrix