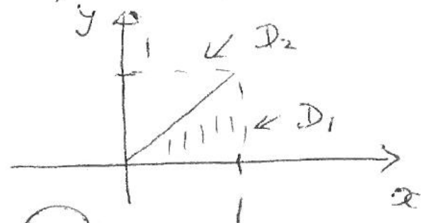
1. Let X and Y be two independent random variables, both uniformly distributed on the interval $[0, 1]$. Calculate $E|X - Y|$.

- We find $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = 1$, $(x,y) \in (0,1) \times (0,1)$

$$|X-Y| = \begin{cases} X-Y & \text{if } Y < X \\ Y-X & \text{if } X < Y \end{cases}$$



$$\Rightarrow E|X-Y| = E(X-Y)\big|_{D_1} + E(Y-X)\big|_{D_2} = \frac{1}{3}$$

- $$E(X-Y)\big|_{D_1} = \iint_{D_1} (x-y)\,dx\,dy = \int_0^1 \int_0^x (x-y)\,dy\,dx = \int_0^1 \left(xy - \frac{y^2}{2}\right)\Big|_0^x dx$$

$$= \int_0^1 x^2/2\,dx = 1/6$$

- By symmetry, $E(Y-X)\big|_{D_2} = \frac{1}{6}$

2. Let X and Y be two continuous random variables with joint pdf $f_{XY}(x,y)$. Let $U = XY$. Prove that f_U, the pdf of the random variable U, is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(\frac{u}{v}, v)|\frac{1}{v}|dv. \qquad (*)$$

Hint: Use the transformation $(x,y) \to (u,v)$ defined as $u = xy, v = y$ to find first the joint pdf $f_{UV}(u,v)$ of the random vector (U,V).

$$(X,Y) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} (U,V) \qquad \begin{cases} u = xy \\ v = y \end{cases} \iff \begin{cases} x = \frac{u}{v} \\ y = v \end{cases} \leftarrow \psi$$

• We find the Jacobian: $\det D\psi$:

$$\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$ $\Rightarrow$ we use the formula

$$f_{U,V}(u,v) = f_{X,Y}(\psi(u,v)) \cdot |\det D\psi(u,v)|$$

to get $f_{U,V}(u,v) = f_{XY}(\frac{u}{v}, v) \cdot |\frac{1}{v}|$

so that $f_U(u) = \int f_{U,V}(u,v)\, dv$ = formula $(*)$

$$\overbrace{\phantom{P(A\cap\bar{B}) = P(A)P(\bar{B})}}^{*}$$

3. a) Let A and B be two arbitrary events. Assume that $P(A \cap \bar{B}) = P(A)P(\bar{B})$. Show that $P(A \cap B)$ $= P(A) \cdot P(B)$. Here $\bar{B}$ denotes the complement of B.

- $A = A \cap \Omega = A \cap (B \cup \bar{B}) = \overline{A \cap B \cup A \cap \bar{B}}$

disjoint

- $P(A) = P(A \cap B) + P(A \cap \bar{B})$,

$$P(A \cap B) = P(A) - P(A \cap \bar{B}) \underset{*}{=} P(A)(1 - P(\bar{B}))$$
$$= P(A) \cdot P(B).$$

b) What is the form of a jointly continuous Gaussian vector of dimension n?

$$f(x_1, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^{n/2} \cdot \det R}} \cdot \exp\left[-\frac{1}{2}(x-\mu)^T \cdot R^{-1}(x-\mu)\right]$$

where $\mu \in \mathbb{R}^n$ and R is a $n \times n$ positive definite matrix

$$X = (X_1, \dots, X_n) \Rightarrow P(X \in B) = \int_B f(x)\,dx$$
$$B \in \mathcal{B}(\mathbb{R}^n)$$

4. Suppose that X_1 and X_2 are two independent random variables each having an exponential distribution with parameter $\lambda = 2$. That is, their pdfs have the form $f(x_i) = 2e^{-2x_i}, x_i > 0, i = 1, 2$.

a) Find the pdf of Y defined as $Y = \max\{X_1, X_2\}$.

- cdf of X_i is $F(x_i) = \int_0^{x_i} 2 \cdot e^{-2y}\, dy = 1 - e^{-2x_i}$
- $F_Y(y) = P(\max\{X_1, X_2\} \le y) = P(X_1 \le y, X_2 \le y)$

$\underset{\text{indep.}}{=} P(X_1 \le y) \cdot P(X_2 \le y) = F_{X_1}(y) \cdot F_{X_2}(y) =$

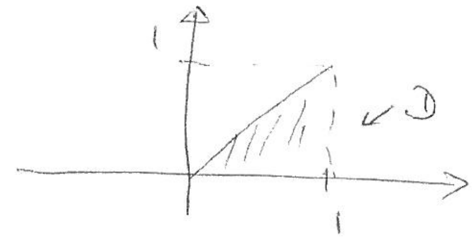
$= (1 - e^{-2y})^2 \Rightarrow$

$f_Y(y) = F_Y'(y) = 4e^{-2y}(1 - e^{-2y}), \; y > 0$

since $F_Y(y)$ is differentiable.

b) Find $P(Y > 1)$. $= 1 - P(Y \le 1) = 1 - \int_0^1 4e^{-2y}(1 - e^{-2y})\, dy =$

$= 1 - (1 - e^{-2y})^2 \Big|_0^1 = 1 - (1 - e^{-2})^2.$



5. Let (X, Y) has the jointly continuous density

$$f(x,y) = \begin{cases} 8xy, & 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

1) $f \geq 0$; ✓

a) Verify that $f(x, y)$ is indeed a correct density.

$$2) \iint_D 8xy\,dx\,dy = \int_0^1 \left(\int_0^x 8xy\,dy \right) dx = \int_0^1 (4xy^2) \Big|_0^x dx =$$

$$= \int_0^1 4x^3\,dx = X^4 \Big|_0^1 = 1 \quad ✓$$

b) Find the conditional densities $f_{X|Y=y}(x, y)$ and $f_{Y|X=x}(x, y)$.

$$\cdot\ f_X(x) = \int_0^x 8xy\,dy = 4xy^2 \Big|_0^x = 4x^3, \quad 0 < X < 1$$

$$\cdot\ f_Y(y) = \int_y^1 8xy\,dx = 4yx^2 \Big|_y^1 = 4y(1-y^2), \quad 0 < Y < 1$$

$$\Rightarrow f_{X|Y}(x,y) = \frac{8xy}{4y(1-y^2)} = \frac{2x}{(1-y^2)} = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(x,y) = \frac{8xy}{4x^3} = \frac{2y}{x^2} = \frac{f_{XY}(x,y)}{f_X(x)}$$

c) Are X and Y independent? No, since $f_{XY} \neq f_X \cdot f_Y$.

6. Show that if $A_1, A_2, ...$ is an *expanding* sequence of events, that is,

$$A_1 \subset A_2 \subset .....,$$

then

$$P(A_1 \cup A_2 \cup ....) = \lim_{n \to \infty} P(A_n).$$

∘ We use properties of probability.

Define $B_n := A_n \setminus A_{n-1}$, $n = 1, 2 \ldots$, $A_0 = \emptyset$ $\Rightarrow$

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ and B_n are disjoint $\Rightarrow$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \text{additivity of } P = \sum_{n=1}^{\infty} P(B_n) =$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} P(B_k) = \lim_{n \to \infty} \underbrace{\sum_{k=1}^{\infty} P(A_k \setminus A_{k-1})}_{P(A_n) - P(A_0)} =$$

$$= \lim_{n \to \infty} P(A_n). \quad \boxtimes.$$

7. Let X be a random variable with the standard normal distribution, That is, its pdf is given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in \mathbb{R}$.

a) Show that $\int_{\mathbb{R}} f(x)dx = 1$.

Let $I = \int_{\mathbb{R}} f(x)\,dx$ where $I > 0$.

We show that $I^2 = 1$.

Use polar coordinates:

$$I^2 = \left(\int_{\mathbb{R}} f(x)\,dx\right)\left(\int_{\mathbb{R}} f(y)\,dy\right) = \iint_{\mathbb{R}^2} \frac{1}{2\pi} \cdot e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy =$$

$$= \left\{\begin{matrix} x = r \cdot \cos\varphi \\ y = r \cdot \sin\varphi \end{matrix}\right] = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{-r^2/2} \cdot \underbrace{r}_{\text{Jacobian}}\,d\varphi\,dr =$$

$= 1$ since $\int_0^\infty r \cdot e^{-r^2/2}\,dr = 1$

b) Calculate $E(X^3) = \int_{\mathbb{R}} x^3 \cdot f(x)\,dx = 0$

since $\underset{\text{odd}}{x^3} \cdot \underset{\text{even}}{f(x)}$ is an odd function