Homework 4

Problem 1

Suppose $Y = (Y_1, Y_2)^T$, then we have

$$E(X \mid Y_1, Y_2) = \mu_X + R_{XY} R_{YY}^{-1} (Y - \mu_Y)$$

$$R_{XY} = cov(X, Y) = \left(cov(X, Y_1), cov(X, Y_2) \right) = (-1, 1)$$

$$R_{YY} = \begin{pmatrix} \sigma_{Y_1}^2 & cov(Y_1, Y_2) \\ cov(Y_1, Y_2) & \sigma_{Y_2}^2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$$

$$\det(R_{YY}) = 15 - 4 = 11$$

So, we have $R_{YY}^{-1} = \frac{1}{11} \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$

$$R_{XY} \cdot R_{YY}^{-1} = (-1, 1) \cdot \frac{1}{11} \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} = \frac{1}{11}(-3, 1)$$

Thus, $E(X \mid Y_1 = y_1, Y_2 = y_2) = E(X \mid Y_1 = 2, Y_2 = 5)$

$$= 3 + \frac{1}{11}(-3, 1) \begin{pmatrix} 2-3 \\ 5-3 \end{pmatrix} = 3 + \frac{5}{11} = \frac{38}{11}$$

Problem 2

Since $X = R\cos\theta$, $Y = R\sin\theta$, we have $XY = R^2 \sin\theta \cos\theta$

Since X and Y are both $N(0,1)$ and independent, the joint pdf of X and Y

is $f_{XY}(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$

$$D\psi(R, \theta) = \text{Jacobian matrix} = \begin{pmatrix} \frac{\partial R\cos\theta}{\partial R} & \frac{\partial R\cos\theta}{\partial \theta} \\ \frac{\partial R\sin\theta}{\partial R} & \frac{\partial R\sin\theta}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix}$$

$$f_{R,\theta}(r, \theta) = \frac{1}{2\pi} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}} \cdot \left| r\cos^2\theta + r\sin^2\theta \right| = \frac{1}{2\pi} r e^{-\frac{r^2}{2}}$$

This shows that R and θ are independent and $f_\theta(\theta) = \frac{1}{2\pi}$, $0 \le \theta < 2\pi$

$f_R(r) = r e^{-\frac{r^2}{2}}$, $r > 0$, which means $\theta \sim U([0, 2\pi))$, $R \sim Rayleigh(1)$

Thus, $R^2 \sim \chi^2(2)$ and $E(R^2)=2$

Therefore, $E(XY|\theta) = E(R^2 \sin\theta \cos\theta|\theta) = \sin\theta\cos\theta\, E(R^2)$

$$= 2\sin\theta\cos\theta = \sin(2\theta)$$

Problem 3

$X \sim N(0,1)$, $W \sim Laplace(0,\lambda)$ are independent, and $Y = X+W$

$$R_{XY} = cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - E(X)E(Y)$$

$$= E(XY) = E[X(X+W)] = E(X^2) + E(X)E(W) = E(X^2)$$

Since $var(X) = E(X^2) - (E(X))^2 = E(X^2) = 1$

Thus, $R_{XY} = 1$

$$R_{YY} = cov(Y,Y) = E(Y^2) - (E(Y))^2 = E(Y^2) = E[(X+W)^2] = E(X^2+2XW+W^2)$$

$$= E(X^2) + 2E(X)E(W) + E(W^2) = E(X^2) + 2E(X)E(W) + var(W) + (E(W))^2$$

$$= 1 + 0 + 2\lambda^2 + 0 = 2\lambda^2+1$$

In summary, the best estimator is then

$$E(X|Y) = \mu_X + R_{XY}R_{YY}^{-1}(Y-\mu_Y)$$

$$= 0 + \frac{1}{2\lambda^2+1}(Y-0) = \frac{1}{2\lambda^2+1}Y$$

Problem 4

$$f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)}, & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_0^{+\infty} f_{XY}(x,y)\,dy = \int_0^{+\infty} x e^{-x(y+1)}\,dy = x e^{-x}\int_0^{+\infty} e^{-xy}\,dy$$

$$= x e^{-x}\left(-\frac{1}{x} e^{-xy}\Big|_0^{+\infty}\right) = x e^{-x}\left(0+\frac{1}{x}\right) = e^{-x}$$

Thus, $f_X(x) = e^{-x}$, $x>0$

Then, $f_{Y|X}(x,y) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy}$

Thus, $f_{Y|X}(x,y) = x e^{-xy}$, $y>0$

$$E(Y|X=x) = \int_0^{+\infty} y x e^{-xy}\,dy = \int_0^{+\infty} -y\, d e^{-xy}$$

$$= -y e^{-xy}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-xy}\,dy$$

$$= (0-0) - \left(-\frac{1}{x} e^{-xy}\Big|_0^{+\infty}\right) = \frac{1}{x}$$

Thus, $E(Y|X=x) = \frac{1}{x}$ for $x>0$

Problem 5

$N \sim \text{Geometric}(p)$, $P(N=n) = p(1-p)^{n-1}$, for $n=1,2,\cdots$

$X_i \sim Exp(\lambda)$, $f_{X_i}(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

N is independent of the i.i.d. X_i sequence.

Since $Y = \sum_{i=1}^{N} X_i$, $\varphi_Y(t) = E(e^{itY}) = E\left(e^{it\sum_{i=1}^{N} X_i}\right) = E\left(e^{itX_1}\cdot e^{itX_2}\cdots e^{itX_N}\right)$

$$E(e^{itX_1}) = \int_0^{+\infty} e^{itx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^{+\infty} e^{(it-\lambda)x}\,dx = \lambda\left(\frac{1}{it-\lambda} e^{(it-\lambda)x}\Big|_0^{+\infty}\right)$$

$$= \lambda\left(0 - \frac{1}{it-\lambda}\right) = \frac{\lambda}{\lambda - it}$$

Thus, $\varphi_Y(t) = E\left[\left(\frac{\lambda}{\lambda-it}\right)^N\right]$

Since $P(N=n) = (1-p)^{n-1}p$, $n=1,2,\cdots$, then we have

$$\varphi_N(t) = E(e^{itN}) = \sum_{n=1}^{\infty} e^{itn} P(N=n) = \sum_{n=1}^{\infty} e^{itn}(1-p)^{n-1}p$$

$$= pe^{it}\sum_{n=1}^{\infty}(e^{it}(1-p))^{n-1} = pe^{it}\sum_{n=0}^{\infty}(e^{it}(1-p))^{n}$$

Suppose $S=\sum_{n=0}^{\infty}(e^{it}(1-p))^{n}$, we have $e^{it}(1-p)S=\sum_{n=0}^{\infty}(e^{it}(1-p))^{n+1}$

$$S-e^{it}(1-p)S = \sum_{n=0}^{\infty}(e^{it}(1-p))^{n} - \sum_{n=0}^{\infty}(e^{it}(1-p))^{n+1}$$

$$[1-e^{it}(1-p)]S = (e^{it}(1-p))^{0} = 1$$

$$S = \frac{1}{1-e^{it}(1-p)}$$

Thus, $\varphi_N(t) = E(e^{itN}) = \dfrac{pe^{it}}{1-(1-p)e^{it}}$

Since $\varphi_Y(t) = E\left(\left(\frac{\lambda}{\lambda-it}\right)^N\right) = \dfrac{p\cdot\frac{\lambda}{\lambda-it}}{1-(1-p)\frac{\lambda}{\lambda-it}}$

$$= \frac{\lambda p}{\lambda - it - \lambda(1-p)} = \frac{\lambda p}{\lambda p - it}$$

Problem 6

Let X_i be a random variable showing the result of the i-th toss.

And sequence X_i is i.i.d. with head $P(X_i=1)=\frac{1}{2}$ and tails $P(X_i=0)=\frac{1}{2}$

The total number of heads is $S_n = X_1+X_2+\cdots+X_n$, so we have the relative frequency of heads is $\frac{S_n}{n}$

$$E(S_n) = E(X_1+\cdots+X_n) = E(X_1)+\cdots+E(X_n) = \frac{n}{2}$$

$$E\left(\frac{S_n}{n}\right) = \frac{1}{n}E(S_n) = \frac{1}{2}$$

$$Var(X_i) = E(X_i^2)-(E(X_i))^2 = 1^2\cdot P(X_i=1)+0^2\cdot P(X_i=0) - [1\cdot P(X_i=1)+0\cdot P(X_i=0)]^2$$

$$= \frac{1}{2}-\frac{1}{4} = \frac{1}{4}$$

$$Var(S_n) = Var(X_1+\cdots+X_n) = Var(X_1)+\cdots+Var(X_n) = \frac{n}{4}$$

$$Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2}Var(S_n) = \frac{1}{4n}$$

Apply Chebyshev's inequality to $\frac{S_n}{n}$. For any $\varepsilon > 0$

$$P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \frac{Var\left(\frac{S_n}{n}\right)}{\varepsilon^2} = \frac{1}{4n\varepsilon^2}$$

Since $\lim_{n\to\infty} \frac{1}{4n\varepsilon^2} = 0$, then we have $\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \varepsilon\right) = 0$

which means that the number of heads occured relatively to the total number of tosses will converge to $\frac{1}{2}$.

Problem 7

Let X_i be the transmission time for i-th packet, and sequence X_i is i.i.d, with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$.

Suppose the total transmission time is $S_n = X_1 + \cdots + X_n$

$E(S_n) = E(X_1 + \cdots + X_n) = n\mu$, $Var(S_n) = n\sigma^2$

By the central limit theorem, S_n canbe approximated by normal distribution

$S_n \sim N(n\mu, n\sigma^2)$

$$P(S_n \geq 2n\mu) = P\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \geq \frac{2n\mu - n\mu}{\sqrt{n}\sigma}\right) = P\left(Y \geq \frac{\sqrt{n}\mu}{\sigma}\right)$$

where $Y = \frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$

Thus $P(S_n \geq 2n\mu) = P\left(Y \geq \frac{\sqrt{n}\mu}{\sigma}\right) = 1 - P\left(Y \leq \frac{\sqrt{n}\mu}{\sigma}\right) = 1 - \phi\left(\frac{\sqrt{n}\mu}{\sigma}\right)$

where ϕ is the cdf of the standard normal distribution

Problem 8

$$\varphi_{Y_n}(t) = E(e^{itY_n}) = E\left(e^{it\sum_{i=1}^{n}\frac{X_i}{\sqrt{n}}}\right) = E\left(e^{it\frac{X_1}{\sqrt{n}}} \cdot \dots \cdot e^{it\frac{X_n}{\sqrt{n}}}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{it\frac{X_i}{\sqrt{n}}}\right) = \left(E\left(e^{it\frac{X_i}{\sqrt{n}}}\right)\right)^n$$

Since $X_i = \pm 1$ with equal probability, $P(X_i=1)=\frac{1}{2}$, $P(X_i=-1)=\frac{1}{2}$, then

$$E\left(e^{it\frac{X_i}{\sqrt{n}}}\right) = e^{it\frac{1}{\sqrt{n}}} \cdot \frac{1}{2} + e^{it\frac{-1}{\sqrt{n}}} \cdot \frac{1}{2} = \frac{1}{2}\left(e^{i\frac{t}{\sqrt{n}}} + e^{i(-\frac{t}{\sqrt{n}})}\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{t}{\sqrt{n}}\right) + i\sin\left(\frac{t}{\sqrt{n}}\right) + \cos\left(-\frac{t}{\sqrt{n}}\right) + i\sin\left(-\frac{t}{\sqrt{n}}\right)\right)$$

$$= \frac{1}{2} \cdot 2\cos\left(\frac{t}{\sqrt{n}}\right) = \cos\left(\frac{t}{\sqrt{n}}\right) \approx 1 - \frac{t^2}{2n}$$

Thus, $\varphi_{Y_n}(t) = \left(1 - \frac{t^2}{2n}\right)^n$

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = \lim_{n\to\infty}\left(1 - \frac{t^2}{2n}\right)^n = R$$

$$\ln R = \lim_{n\to\infty} \ln\left(\left(1-\frac{t^2}{2n}\right)^n\right) = \lim_{n\to\infty} n \ln\left(1-\frac{t^2}{2n}\right) = \lim_{n\to\infty} n\ln\left(1 - \frac{\frac{t^2}{2}}{n}\right)$$

$$= \lim_{n\to\infty} n\left(-\frac{\frac{t^2}{2}}{n}\right) = -\frac{t^2}{2} \implies R = e^{-\frac{t^2}{2}}$$

Therefore $\lim_{n\to\infty} \varphi_{Y_n}(t) = e^{-\frac{t^2}{2}}$