ESE 520 Probability and Stochastic Processes

HW 4: due by 11pm CST on Thursday, October 24, 2024

Total: 40 points

Problem 1. Two measurements $y_1 = 2$ and $y_2 = 5$ are taken to estimate the value x of a random variable X. Assume that the joint distribution of the full random vector (X, Y_1, Y_2) is Gaussian and that X has expectation $\mu_X = 3$, variance $\sigma_X^2 = 4$, and the following covariances with the random variables Y_1 and Y_2 which describe the first and second measurement, $cov(X, Y_1) = -1$ and $cov(X, Y_2) = +1$. Also suppose the measurements Y_1 and Y_2 have means $\mu_{Y_1} = \mu_{Y_2} = 3$, variances $\sigma_{Y_1}^2 = 3$ and $\sigma_{Y_2}^2 = 5$ and covariance $cov(Y_1, Y_2) = -2$. Find the best mean square estimate for X in terms of these measurements.

Problem 2. Let (X, Y) be a randomly selected point on $\mathbb{R}^2$ where both coordinates X and Y have $\mathcal{N}(0, 1)$ distribution and are independent. Also, the polar coordinates for (X, Y) are (R, θ) and are defined through $X = R\cos\theta, Y = R\sin\theta$. Calculate $E(XY|\theta)$.

Problem 3. Let $X \sim N(0, 1)$ and $W \sim Laplace(\lambda)$ be independent, and put $Y := X + W$. Find the best linear mean squares estimator of X based on Y.

Problem 4. Let

$$f_{XY}(x, y) = \begin{cases} xe^{-x(y+1)}, x > 0, y > 0 \\ 0, \text{ otherwise} \end{cases}$$

be a joint density of (X, Y). Compute $E[Y|X = x]$ for $x > 0$.

Problem 5. Let N denote the number of primaries in a photomultiplier, and let X_i be the number of secondaries due to the ith primary. Then the total number of secondaries is $Y = \sum_{i=1}^{N} X_i$. Find the characteristic function $\phi_Y(t)$ if $N \sim geometric(p)$ and $X_i \sim exp(\lambda)$. Assume that N is independent of the i.i.d. X_i sequence.

Problem 6. Consider the experiment "tosssing a coin" where probability of occuring of heads is equal to 1/2. Using the Chebyshev's inequality, show that "in the long run", the number of heads occured relatively to the total number of tosses (relative frequency of heads) will converge to 1/2 (in probability).

Problem 7. Packet transmission times on a certain Internet link are i.i.d. with mean μ and standard deviation σ. Suppose n packets are transmitted. Then the total expected transmission time for n packets is $n\mu$. Use the central limit theorem to approximate the probability that the total transmission time for the n packets exceeds twice the expected transmission time.

Problem 8. Let $X_i + \pm 1$ with equal probability. Then the x_i are zero mean and have unit variance. Put

$$Y_n = \sum_{i=1}^{n} \frac{X_i}{\sqrt{n}}.$$

Derive the central limit theorem for this case, i.e., show that $\phi_{Y_n}(t) \to e^{-t^2/2}$. Hint: Use the Taylor series approximation $\cos(x) \approx 1 - x^2/2$.