ESE 520 Probability and Stochastic Processes

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EXAM 3 – Practice

Name (Please print):

Solutions

Total: 70 points (each problem is 10 points worth)

1. You must show all work to completely justify your answers in order to receive any credit.
2. You can use One one-sided sheet of paper with your own formulas.

1. Let $(W_t), t \geq 0$, be a one-dimensional Wiener process. For $t \geq 0$, define new process $Z_t := e^{W_t - \frac{1}{2}t}$. One can show that $E(Z_t) = 1$. Use this result to compute the covariance function of Z_t. Is this process WSS process?

$R_{\cancel{Z}}(t,s) = E(Z_t \cdot Z_s) - 1$,

$E(Z_t \cdot Z_s) = E\left(e^{W_t - t/2} e^{W_s - s/2}\right) = e^{-\frac{t+s}{2}} E\left(e^{W_t} \cdot e^{W_s}\right)$

$= e^{-\frac{t+s}{2}} \cdot E\left(e^{W_t - W_s} e^{2W_s}\right)$ = use independence of increments

$= e^{-\frac{t+s}{2}} E\left(e^{W_t - W_s}\right) \times E\left(e^{2W_s}\right)$

$= e^{-\frac{t+s}{2}} \cdot e^{\frac{t-s}{2}} \cdot e^{2s} = e^{s}$ if $s < t$

In general, $R_Z(s,t) = e^{\min\{s,t\}} - 1$

$\Longrightarrow$ not a WSS process

$E(Z_t) = 1$ implies that $E\left(e^{W_t}\right) = e^{t/2}$

2. Let $(W_t), t \geq 0$, be a one-dimensional Wiener process. For a positive constant α define a new stochastic process

$$X_t := \alpha W_{\frac{t}{\alpha^2}}.$$

Prove that X_t is also a Wiener process.

To prove 3 properties

1) $X_0 = 0$ — clear;

2) independent increments: $t_1 < t_2 < t_3 < t_4$

$$X_{t_2} - X_{t_1} = \alpha \left(W_{\frac{t_2}{\alpha^2}} - W_{\frac{t_1}{\alpha^2}} \right) = \alpha \left(W_{\tau_2} - W_{\tau_1} \right)$$

$$X_{t_4} - X_{t_3} = \alpha \left(W_{\frac{t_4}{\alpha^2}} - W_{\frac{t_3}{\alpha^2}} \right) = \alpha \left(W_{\tau_4} - W_{\tau_3} \right)$$

$\tau = \frac{t}{\alpha^2}$ $\qquad \tau_1 < \tau_2 < \tau_3 < \tau_4$

$\Rightarrow$ independence follows

3) $s < t$: $X_t - X_s = \alpha \left(W_{\frac{t}{\alpha^2}} - W_{\frac{s}{\alpha^2}} \right)$ is normal with mean 0 &

$$\mathrm{Var}(X_t - X_s) = \alpha^2 \left(\frac{t}{\alpha^2} - \frac{s}{\alpha^2} \right) = t - s$$

$$\Rightarrow \quad X_t - X_s \sim \mathcal{N}(0, t-s) \quad \boxtimes$$

$t = 1 = 1$ month

3. Space shuttles are lunched according to a Poisson process. The average time between lunches is 2 months.

a) Find the probability that there are no lunches during a 4 month period.

b) Find the probability that during at least 1 month out of three consecutive months, there are at least two lunches.

- N_t = # of lunches is a Poisson Process with $\lambda = \frac{1}{2}$

a) $P(N_4 = 0) = e^{-4 \times \frac{1}{2}} = e^{-2} \approx 0.1353$

b) $P\left(\bigcup_{i=1}^{3} \{N_i - N_{i-1} \geq 2\}\right) = 1 - P\left(\bigcap_{i=1}^{3} \{N_i - N_{i-1} \leq 1\}\right)$

$= 1 - \prod_{i=1}^{3} P(N_i - N_{i-1} \leq 1) = 1 - \left(e^{-\lambda} + \lambda e^{-\lambda}\right)^3 =$

$= 1 - e^{-3\lambda}(1+\lambda)^3 = 1 - e^{-3/2}\left(\frac{3}{2}\right)^3 = 0.247$

$E(Y_t) = 0$ because $E(X_t) = 0$

4. In a linear dynamical system with input process (X_t) being white noise, the transfer function is given by

$$H(\nu) = 1 - |\nu|^2, |\nu| \leq 1,$$

(and equal to zero for $|nu| > 1$).

Find $E(Y^2)$ (output power of the system), where (Y_t) is the output process.

$E(Y^2) = R_Y(0) = \int_{\mathbb{R}} S_Y(\nu)\,d\nu$, where

$S_Y(\nu) = |H(\nu)|^2 \underbrace{S_X(\nu)}_{=1} = (1 - |\nu|^2)^2 \Rightarrow$

$E(Y^2) = 2\int_0^1 (1 - \nu^2)^2\,d\nu = 2\int_0^1 (1 - 2\nu^2 + \nu^4)\,d\nu =$

$= 2\left(\nu - \frac{2}{3}\nu^3 + \frac{\nu^5}{5}\right)\Big|_0^1 = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$

$\approx 1{,}067$

5. In a communication system, the carrier signal at the receiver is modeled by $X_t = \cos(2\pi v t + \theta), t \in \mathbb{R}$, where θ has uniform distribution $U[-\pi, \pi]$ and v is a parameter. Find the the covariance function $R_X(s,t)$.

- $E(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi v t + \theta)\, d\theta = \frac{1}{2\pi} \sin(2\pi v t + \theta) \Big|_{-\pi}^{\pi} =$

$= \frac{1}{2\pi}\left(\sin(2\pi v t + \pi) - \sin(2\pi v t - \pi)\right) = 0$

- $R_X(s,t) = E(X_t \cdot X_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi v t + \theta) \cdot \cos(2\pi v s + \theta)\, d\theta$

$= \left[\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \sin\beta \right] =$

$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[\cos\left(2\pi v(t+s) + 2\theta\right) + \cos\left(2\pi v(t-s)\right)\right] d\theta$

$= \frac{2\pi}{4\pi} \cos 2\pi v(t-s) + \underbrace{\frac{1}{2}\sin\left(2\pi v(t+s) + 2\theta\right)\Big|_{-\pi}^{\pi}}_{=0} =$

$= \frac{1}{2}\cos 2\pi v(t-s)$

6. Let (W_t) be a Wiener process. For $\lambda > 0$, define

$$Y_t := \frac{e^{-\lambda t}}{\sqrt{\lambda}} W_{e^{2\lambda t}}.$$

Is Y_t a WSS process?

1) $E(Y_t) = \frac{e^{-\lambda t}}{\sqrt{\lambda}} \underbrace{E\left(W_{e^{2\lambda t}}\right)}_{=0} = 0$;

2) $R_Y(t,s) = E(Y_t \cdot Y_s) = \frac{e^{-\lambda(t+s)}}{\lambda} E\left(W_{e^{2\lambda t}} W_{e^{2\lambda s}}\right)$

$\boxed{s<t}$

$= \frac{e^{-\lambda(t+s)}}{\lambda} \cdot \min\{e^{2\lambda t}, e^{2\lambda s}\} = \frac{1}{\lambda} e^{-\lambda(t+s)} \cdot e^{2\lambda s}$

$= \frac{1}{\lambda} e^{-\lambda(t-s)}$ depends on $(t-s)$ only

$\Rightarrow (Y_t)$ is a WSS process

7. Let $(W_t), t \geq 0$ be a Wiener process. For $0 < s < t$, calculate $E\Big((W_t - W_s)^4 W_t^2\Big)$.

We use independence of increments: $s < t$

$$E\left[(W_t - W_s)^4 \cdot W_t^2\right] = E\left[(W_t - W_s)^4 \cdot (W_t - W_s + W_s)^2\right] =$$

$$= E\,(W_t - W_s)^6 + 2E\left[(W_t - W_s)^5 \times W_s\right] + E\left[W_s^2 \cdot (W_t - W_s)^4\right]$$

$m = 3$

$$= 5 \times 3 \times 1\,(t-s)^3 + 2\,\underbrace{E(W_t - W_s)^5 \times E(W_s)}_{=0}$$

$$+ E(W_s)^2 \times E(W_t - W_s)^4$$

$$= 15(t-s)^3 + 3s \cdot (t-s)^2$$