HW#3 - Solutions ESE-520

Total: 40 p.

Pr.2 [5p]. Given: $\varphi_X(v) = e^{-\lambda|v|}$.
Verify that $X \sim \text{Cauchy}(\lambda)$, $\lambda > 0$.

- By the inversion formula,

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixv}\varphi_X(v)\,dv =$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{\lambda v} e^{-ixv}\,dv + \int_{0}^{\infty} e^{-\lambda v} e^{-ixv}\,dv\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{v(\lambda - ix)}\,dv + \int_0^{\infty} e^{-v(\lambda+ix)}\,dv\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{\lambda - ix} e^{v(\lambda - ix)}\Big|_{-\infty}^{0} - \frac{1}{\lambda + ix} e^{-v(\lambda+ix)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{\lambda - ix} + \frac{1}{\lambda + ix}\right] = \frac{1}{2\pi}\left[\frac{2\lambda}{\lambda^2 + x^2}\right] = \frac{\lambda/\pi}{\lambda^2 + x^2}$$

$\Rightarrow X \sim \text{Cauchy}(\lambda)$.

Pr.3 [4p] $Z := X + Y$ → independent rv's
$X \sim \exp(1)$
$Y \sim \text{Laplace}(1)$

Find $\text{cov}(X, Z)$ & $\text{Var}(Z)$.

- We recall that: $E(X) = \frac{1}{\lambda}$ $\Rightarrow$ $E(X) = 1$

$E(Y) = 0$

$\Rightarrow E(Z) = E(X) + E(Y) = 1$

- By definition, $\mathrm{cov}(X,Z) = E(X\cdot Z) - E(X)\cdot E(Z)$
$= E(X\cdot Z) - 1$
- Also: $E(X\cdot Z) = E(X(X+Y)) = E(X^2) - E(X\cdot Y) =$
$= E(X^2) - E(X)\cdot E(Y) =$ by independency $= E(X^2)$ so that $E(X\cdot Z) = E(X^2) = Var(X) + (E(X))^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \underline{\underline{2}}$

$\Rightarrow \mathrm{cov}(X,Z) = 2 - 1 = \underline{\underline{1}}$.

- Moreover: $Var(Z) = Var(X+Y) \underset{\substack{\uparrow \\ \text{independency}}}{=} Var(X) + Var(Y) = 1+2 = \underline{\underline{3}}$

Since $Var(Y) = \frac{2}{\lambda^2} = 2$.

Pr. 1 [5p]

By definition of $\psi_X(s) = E[e^{sX}]$, we have

$$\psi_X(s) = \int_{-\infty}^{\infty} e^{sx}\frac{\lambda}{2}e^{-\lambda|x|}dx = \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{sx}\cdot e^{\lambda x}dx + \int_0^\infty e^{sx}\cdot e^{-\lambda x}dx\right]$$

$$= \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{x(\lambda+s)}dx + \int_0^\infty e^{-x(\lambda-s)}dx\right] =$$

$$= \frac{\lambda}{2}\left[\frac{-1}{\lambda-s}e^{-x(\lambda-s)}\Big|_0^\infty + \frac{1}{\lambda+s}e^{x(\lambda+s)}\Big|_{-\infty}^{0}\right]$$

$$= \frac{\lambda}{2}\left[\frac{1}{\lambda-s} + \frac{1}{\lambda+s}\right] = \frac{\lambda^2}{\lambda^2-s^2}$$

where we require $\lambda+s>0$ & $\lambda-s>0$

$\Longleftrightarrow \boxed{-\lambda < s < \lambda}$

for both integrals to exist.

• Then : $\psi'_X(s) = \dfrac{2s\lambda^2}{(\lambda^2 - s^2)^2}$ & $E(X) = \psi'_X(0) = 0$

Additionally:

$$\psi''_X(s) = 2\lambda^2 \frac{(\lambda^2 - s^2)^2 + 4s^2(\lambda^2 - s^2)}{(\lambda^2 - s^2)^4}$$ so that

$$Var(X) = E(X^2) - (E(X))^2 = E(X^2) = \psi''_X(0) = \frac{2}{\lambda^2}.$$

Pr. 4 [4p]. Given: $f_{XY}(x,y) = \dfrac{4 e^{-\frac{(x-y)^2}{2}}}{y^5 \cdot \sqrt{2\pi}}$, $y \geq 1$

Find $f_Y(y)$.

• Recall : $\dfrac{1}{\sqrt{2\pi}} \cdot \displaystyle\int_{\mathbb{R}} e^{-\frac{(x-\mu)^2}{2}} dx = 1$

Since $f(x) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2}}$ is the pdf of $N(\mu, 1)$

• Then, by definition of $f_Y(y)$:

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx = \int_{\mathbb{R}} \frac{4}{y^5} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-y)^2}{2}} dx =$$

$$= \frac{4}{y^5} \cdot \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-y)^2}{2}} dx}_{=1} = \frac{4}{y^5}, \; y \geq 1.$$

Pr. 5 [6 p.] Given: $X := \cos\theta$, $Y := \sin\theta$, $\theta \sim U[-\pi, \pi]$

Show that: i) $E(XY) = 0$ & $E(X) = E(Y) = 0$

ii) X & Y are dependent r.v.'s

- $E(X \cdot Y) = E[\cos\theta \cdot \sin\theta] = \frac{1}{2} E[\sin 2\theta] = \frac{1}{4\pi} \int_{-\pi}^{\pi} \sin 2\theta \, d\theta =$

$= -\frac{1}{8\pi} \cdot \cos 2\theta \Big/_{-\pi}^{\pi} = -\frac{1}{8\pi} [\cos 2\pi - \cos 2\pi] = 0.$

- $E(X) = E[\cos\theta] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\theta \, d\theta = \frac{1}{2\pi} \sin\theta \Big/_{-\pi}^{\pi} = 0$

- $E(Y) = E[\sin\theta] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\theta \, d\theta = \frac{1}{2\pi} [-\cos\theta] \Big/_{-\pi}^{\pi} = 0$

- $X^2 + Y^2 = \cos^2\theta + \sin^2\theta = 1$ $\forall$ X, Y - values

$\Rightarrow$ X & Y are functionally dependent thus cannot be probabilistically independent.

- Another, a more direct proof would be to use the result from Problem 35 in Ch. 5 where was shown that

$$f_X(x) = \frac{1}{\pi \cdot \sqrt{1-x^2}}, \quad |x| < 1$$

$$f_Y(y) = \frac{1}{\pi \sqrt{1-y^2}}, \quad |y| < 1$$

- Assume that X & Y are independent. Then $\Rightarrow$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

and $\forall$ $|x| < 1$ & $|y| < 1$: $f_{XY}(x,y) \geq f_{XY}(0,0) = \frac{1}{\pi^2}$

Define $S := \{(x,y) : x^2+y^2 \le \frac{1}{2}\}$

Then $P((X,Y) \in S) = 0$ and

$$P((X,Y)\in S) = \iint_S f_{XY}(x,y)\,dx\,dy \geq \iint_S \frac{1}{\pi^2}\,dx\,dy$$

$$= \frac{1}{\pi^2}\iint_S dx\,dy = \frac{1}{\pi^2}\cdot \text{Area}\, S > 0 \;?!$$

Pr. 7 [5 p.] $n=2$, $m=(\mu_1,\mu_2)=(0,0)$

$$R = \begin{bmatrix} \sigma_1^2 & \sigma_1\cdot\sigma_2\cdot\rho \\ \sigma_1\cdot\sigma_2\cdot\rho & \sigma_1^2 \end{bmatrix}, \quad X=(X_1,X_2)$$

$$R = \begin{bmatrix} Var(X_1) & cov(X_1,X_2) \\ cov(X_2,X_1) & Var(X_2) \end{bmatrix}$$

- $\det R = \begin{vmatrix} \sigma_1^2 & \sigma_1\cdot\sigma_2\cdot\rho \\ \sigma_1\cdot\sigma_2\cdot\rho & \sigma_2^2 \end{vmatrix} = \sigma_1^2\cdot\sigma_2^2(1-\rho^2)$

- $\sqrt{\det R} = \sigma_1\cdot\sigma_2\cdot\sqrt{1-\rho^2}$,

$$R^{-1} = \frac{1}{\det R}\begin{bmatrix} \sigma_1^2 & \sigma_1\cdot\sigma_2\cdot\rho \\ \sigma_1\cdot\sigma_2\cdot\rho & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1^2(1-\rho^2)} & \frac{-\rho}{\sigma_1\cdot\sigma_2\cdot(1-\rho^2)} \\ \frac{-\rho}{\sigma_1\cdot\sigma_2\cdot(1-\rho^2)} & \frac{1}{\sigma_2^2(1-\rho^2)} \end{bmatrix}$$

- $[x,y]\cdot R^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = [x,y]\cdot\begin{bmatrix} \frac{x}{\sigma_1^2(1-\rho^2)} - \frac{\rho y}{\sigma_1\cdot\sigma_2(1-\rho^2)} \\ \frac{-\rho x}{\sigma_1\cdot\sigma_2(1-\rho^2)} + \frac{y}{\sigma_2^2(1-\rho^2)} \end{bmatrix} =$

$$= \frac{x^2}{\sigma_1^2(1-\rho^2)} - \frac{\rho x y}{\sigma_1 \cdot \sigma_2(1-\rho^2)} - \frac{\rho x y}{\sigma_1 \cdot \sigma_2(1-\rho^2)} + \frac{y^2}{\sigma_2^2(1-\rho^2)} =$$

$$= \frac{1}{1-\rho^2}\left[\frac{x^2}{\sigma_1^2} - 2\rho\frac{xy}{\sigma_1 \cdot \sigma_2} + \frac{y^2}{\sigma_1^2}\right]$$

so that the result follows.

<u>Pr 8</u> [5p] given: $X_i \sim N(m, \sigma^2)$, $i = \overline{1,n}$ — i.i.d.

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$; $Y_j = X_j - \bar{X}$

Show that: i) $E[Y_j] = 0$

ii) $E(\bar{X} \cdot Y_j) = 0$, $j = \overline{1,n}$

• $E[Y_j] = E[X_j - \bar{X}] = m - \frac{1}{n}\sum_{i=1}^{n} E(X_i) = m - m = 0$

• $E(\bar{X} \cdot Y_j) = E(\bar{X}(X_j - \bar{X})) = E(\bar{X} \cdot X_j) - E(\bar{X})^2$

$\rightarrow E(\bar{X})^2 = E(\bar{X} \cdot \bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i X_j) =$

$= \frac{1}{n^2}\left[\sum_{i=1}^{n} E(X_i^2) + \sum_{i \neq j} E(X_i X_j)\right]$

$= \frac{1}{n^2}\left[\sum_{i=1}^{n}(\sigma^2 + m^2) + \sum_{i \neq j} m^2\right] =$

$= \frac{1}{n^2}\left[n(\sigma^2 + m^2) + n(n-1) \cdot m^2\right] = \frac{1}{n^2}\left[n\sigma^2 + n^2 m^2\right] =$

$$= \frac{6^2}{n} + m^2.$$

$$\to E(\bar{X} \cdot X_j) = \frac{1}{n} \sum_{i=1}^{n} E(X_i \cdot X_j) = \frac{1}{n}\left[(6^2+m^2) + (n-1)m^2\right]$$

$$= \frac{6^2}{n} + m^2$$

$$\Rightarrow E(\bar{X} \cdot Y_j) = 0 \quad \forall j$$

Pr. 6 [6p]

$$f_{XY}(x,y) = \frac{1}{2\pi 6_1 6^2 \sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[-\frac{1}{1-\rho^2}\left(\frac{(x-\mu_1)^2}{6_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{6_1 6_2} + \frac{(y-\mu_2)^2}{6_2^2}\right)\right.},$$

$$\psi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Using from L10 the notation

$$\psi_\rho(u,v) = \psi(u) \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right),$$

we have that

$$f_{xy}(x,y) = \frac{1}{6_1 6_2} \cdot \psi_\rho\left(\frac{x-\mu_1}{6_1}, \frac{y-\mu_2}{6_2}\right) =$$

$$= \frac{1}{6_1 \cdot 6_2} \psi\left(\frac{x-\mu_1}{6_1}\right) \cdot \frac{1}{\sqrt{1-\rho^2}} \psi\left(\frac{\frac{y-\mu_2}{6_2} - \rho\frac{x-\mu_1}{6_1}}{\sqrt{1-\rho^2}}\right)$$

$$\Longrightarrow f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy = \frac{1}{6_1}\psi\left(\frac{x-\mu_1}{6_1}\right) \cdot \int_{\mathbb{R}} \frac{1}{6_2\sqrt{1-\rho^2}} \psi\left(\frac{\frac{y-\mu_2}{6_2} - \rho\frac{x-\mu_1}{6_1}}{\sqrt{1-\rho^2}}\right) dy$$

$$= \left[z := \frac{y-\mu_2}{6_2},\ dz = \frac{dy}{6_2} \right] =$$

$$= \frac{1}{\sigma_1} \varphi\left(\frac{x-\mu_1}{\sigma_1}\right) \cdot \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{z - \rho \frac{x-\mu_1}{\sigma_1}}{\sqrt{1-\rho^2}}\right) dz}_{\text{valid density thus } = 1}$$

$$= \frac{1}{\sigma_1} \varphi\left(\frac{x-\mu_1}{\sigma_1}\right) = \frac{1}{\sqrt{2\pi} \cdot \sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

$\Rightarrow f_X(x)$ is $N(\mu_1, \sigma_1^2)$ density