# ESE 520 Probability and Stochastic Processes

## Lecture 15

”Introduction to stochastic processes: general theory and facts.”

Now, after we have discussed important notions of probability as a discipline and studied single random variables and then random vectors, we are ready to take on even more general object of study: a stochastic process.

**Definition 1.** A *stochastic process* $(X_t)_{t\in I}$ will be a family of random variables $X_t$ defined on a probability space $(\Omega, \mathcal{F}, P)$:

$$X_t : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), t \in I.$$

The parameter $t$ could be *discrete*, for example, $t \in \mathbb{N}$. Then $(X_n), n = 1, 2, ...$ is a discrete-time process.

If $t \in I \subset \mathbb{R}$ ($I$ is a continuous set), then $(X_t)_t$ is a continuous-time process. For example, we can choose $I = [0, \infty)$.

By fixed $w \in \Omega$, the function $t \to X_t(w)$ is called a *sample path* of $X$.

Note: $X_t(\omega), t \geq 0, \omega \in \Omega$ is a map that depends on $\omega$ and $t$. By a fixed $t$, $X_t(\omega)$ is a random variable with values in $\mathbb{R}$, By a fixed $\omega$, $X_t(\omega)$ is a real-valued (deterministic) function of time parameter.

As it was the case with single random variables and random vectors, we have to understand first what is *probability distribution* of a stochastic process?

Formally, we have to consider a stochastic process $(X_t), t \in I$ as a map

$$X : \Omega \to \mathbb{R}^I$$

where now $R^I$ is not the Euclidean space $\mathbb{R}$ or $\mathbb{R}^n$ but a space of ”trajectories”/sample paths of the process $X$. Following the same argument we used before, we first have to define on $R^I$ a $\sigma$-algebra of Borel sets and then to see how we can construct on that $\sigma$-algebra some probabilities measures using the crucial result of the measure theory called the Caratheodory theorem. This, however, is a difficult mathematical task - construction of a probability measure on an infinite-dimensional space since $R^I$ is in general such space.

However, thanks to a very important result from the general theory of stochastic processes (Kolmogorov's theorem about finite-dimensional distributions), the description of probability distribution of a stochastic process is basically reduced its so-called finite-dimensional distributions which are defined on finite dimensional spaces $\mathbb{R}^n, n = 1, 2, \ldots$.

Thus, we start with

**Definition 2.** (finite-dimensional distributions) Given a stochastic process $(X_t), t \in I$, the collection of all probability distributions

$$P_{t_1,\ldots,t_n}(B_1, \ldots, B_n) := P(X_{t_1} \in B_1, \ldots X_{t_n} \in B_n)$$

for all distinct $t_i \in I, n \in \mathbb{N}, B_i \in \mathcal{B}(\mathbb{R})$ are called *finite-dimensional distributions* of $X$ (FDD's).

From the probabilistic point of view, the FDD's are natural characteristics of a stochastic process $(X_t)$ and they trivially satisfy the following two *consistency properties*:

i) Given $\{t_1, \ldots, t_n\} \in I$ and any permutation $\pi$ of the set $\{t_1, \ldots, t_n\}$, it holds:

$$P_{t_1,\ldots,t_n}(B_1, \ldots, B_n) = P_{\pi(t_1),\ldots,\pi(t_n)}(B_{\pi(t_1)}, \ldots, B_{\pi(t_n)});$$

ii)

$$P_{t_1,\ldots,t_n,t_{n+1}}(B_1, \ldots, B_n, \mathbb{R}) = P_{t_1,\ldots,t_n}(B_1, \ldots, B_n);$$

The above means that if we are given that a stochastic process $X$ exists on a probability space $(\Omega, \mathcal{F}, P)$, then it has its FDD's and they satisfy the consistency properties.

But: Can we be sure that a particular process $(X_t), t \in I$ exists?

Why is that important?

Often we encounter in probability theory and applications statements like "Let $X$ be a random variable (or a stochastic process) with the cdf $F_X(x)$ or the pdf $f_X(x)$....." In order for us to be able to do something further, we have to be sure that such object exists indeed...

We discussed the existence of a random variable $X$ with a given cdf $F_X$ in the first part of the course (Caratheodory extension theorem). In short, the construction of a corresponding probability space $(\Omega, \mathcal{F}, P)$ and $X(w)$ on it was done like this:

$$\Omega := \mathbb{R}, \mathcal{F} := \mathcal{B}(\mathbb{R}), X(w) := w$$

and

$$P(A) := P_X(B)$$

where $A = X^{-1}(B), B \in \mathcal{B}(\mathbb{R})$ and

$$F_X(x) = P_X((-\infty, x]).$$

What about a stochastic process $X$? It turns out that a similar construction is possible and it is answered in the following

**Theorem** (Kolmogorov's theorem on existence of a stochastic process). Given any family of *consistent* FDD's $P_{t_1,\ldots,t_n}, t_i \in I$ on the index set $I, n \in \mathbb{N}$, there exists (up to an equivalent version) a unique stochastic process $(X_t), t \in I$ that has the given FDD's $P_{t_1,\ldots,t_n}$ as its own FDD's.

**Definition 3.** Let $(X_t)$ and $(Y_t)$ be two processes defined on the same $(\Omega, \mathcal{F}, P)$. One says that $(X_t)$ and $(Y_t)$ are equivalent to each other (versions of each other) if for any $t \in I$ there exists a set $\mathcal{N}_t$ (possibly depending on $t$) so that

$$P(X_t \neq Y_t) = 0$$

on $\mathcal{N}_t$ where $P(\mathcal{N}_t) = 0$.

The construction of a process $(X_t), t \in I$ and $(\Omega, \mathcal{F}, P)$ on which it is defined is similar to the case of a single random variable:

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), F_X).$$

Let

$$R^I := \{x(t) : I \to \mathbb{R}\}$$

be the space of *all real-valued functions* defined on $I$.

The Borel $\sigma$-algebra on $R^I$ is then constructed as

$$\mathcal{B}(R^I) := \sigma(\text{ "all cyllindrical sets"})$$

where the set of all "cylindrical sets" from $R^I$ is defined as the collection of all sets of the from

$$\{x : x(t_1) \in (a_1, b_1] := I_1, ..., x(t_n) \in (a_n, b_n] := I_n\}$$

for all $n \in \mathbb{N}, t_i \in I, i = 1, 2, ..., n$.

Then

$$\Omega := R^I, \mathcal{F} := \mathcal{B}(R^I), X_t(w) := w(t), t \in I$$

where $(X_t), t \in I$ is then the "canonical process" on $\Omega$.

Finally, $P$ is then constructed as a unique probability measure on $\Omega$ so that

$$P_{|\text{"}cyllindrical\ sets\text{"}} = P_{t_1,...,t_n}, t_i \in I, n \in \mathbb{N}.$$

**However:** The $\sigma$-algebra $\mathcal{B}(R^I)$ is not very large!!! There will be not so many "events" in $\mathcal{B}(R^I)$ since any "event" in $\mathcal{B}(R^I)$ is determined only by using finitely many or countably many operations on "cylindrical sets".

**Example 1.** Let $I = [0, \infty)$. The "events" will be

$$\{w \in \Omega : |X_1(w)| > 5\},$$

$$\{w \in \Omega : X_n(w) \in [-1, 1] \text{ for all } n \in \mathbb{N}\},$$

$$\{w \in \Omega : X_n(w) \in [-1, 1] \text{ for all } n \in \mathbf{Q}\}.$$

Not "events" will be:

$$\{w \in \Omega : X_t(w) \text{ is continuous at } t = 1\},$$

$$\{w \in \Omega : \min_{0 \le t \le 1} X_t(w) > 0\},$$

$$\{w \in \Omega : X_t(w) \in [\alpha, \beta] \text{ for all } t \in [a, b]\}.$$

*Practically*: Most if not all "events" we do want to talk about are NOT "events".

What to do? One needs to be more specific about the space $R^I := \{x(t) : I \to \mathbb{R}\}$. In other words, we need some more properties of $x(t)$.

*Important:* **Continuity** (or at least right-continuity) of $x(t)$ makes the bridge!!!

**Definition 4.** A stochastic process $(X_t), t \in I$ is said to have *continuous sample paths* if there exists a set $N \in \mathcal{F}$ with $P(N) = 0$ such that

$$t \to X_t(w)$$

is continuous for all $w \notin N$.

**Example 2.** Let $X$ be continuous. Then

$$\{w \notin N : \min_{t \in [0,1]} X_t(w) \geq a\} = \{w \notin N : \min_{t \in [0,1] \cap \mathbf{Q}} X_t(w) \geq a\}.$$

The next theorem is important by determining if a process $X$ has continuous sample paths.

**Theorem** (Kolmogorov's theorem on the existence of continuous version). Let $(X_t), t \in I$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ and there exists constants $\alpha > 0, \beta > 0$ and $C > 0$ so that

$$E\|X_t - X_s\|^\alpha \leq C|t - s|^{1+\beta}$$

for all $s, t \in I$. Then there exists a sample path continuous version $(Y_t)$ of $(X_t)$.

Moreover, if $\alpha\gamma < \beta$, then $(Y_t)$ can be chosen so that its paths are *Hoelder-continuous* of order $\gamma$:

$$\|X_t - X_s\| \leq C_1|t - s|^\gamma, C_1 > 0.$$