Problem 1.

$$X \sim \text{Laplace}(\lambda), \qquad f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad x \in \mathbb{R}$$

$$\psi(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} f(x)\,dx = \int_{-\infty}^{+\infty} e^{tx} \frac{\lambda}{2} e^{-\lambda|x|}\,dx$$

$$= \frac{\lambda}{2} \int_{-\infty}^{+\infty} e^{tx} e^{-\lambda|x|}\,dx$$

$$= \frac{\lambda}{2} \int_{-\infty}^{0} e^{tx} e^{\lambda x}\,dx + \frac{\lambda}{2} \int_{0}^{+\infty} e^{tx} e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{2} \int_{-\infty}^{0} e^{(t+\lambda)x}\,dx + \frac{\lambda}{2} \int_{0}^{+\infty} e^{(t-\lambda)x}\,dx$$

$$= \frac{\lambda}{2} \frac{1}{t+\lambda} e^{(t+\lambda)x}\Big|_{-\infty}^{0} + \frac{\lambda}{2} \frac{1}{t-\lambda} e^{(t-\lambda)x}\Big|_{0}^{+\infty}$$

$$\implies \begin{cases} \text{for } t < \lambda \;, \quad \psi(t) = \frac{\lambda}{2}\left(\frac{1}{t+\lambda} - \frac{1}{t-\lambda}\right) \\[2mm] \text{for } t \geq \lambda \;, \quad \text{integral divergence, } \psi(t) \text{ does not exist} \end{cases}$$

$$\implies \text{for } t \in (-\lambda, \lambda)\;, \quad \psi(t) = \frac{\lambda}{2} \cdot \frac{t-\lambda-t-\lambda}{(t+\lambda)(t-\lambda)} = \frac{\lambda^2}{\lambda^2-t^2}$$

Since, $Var(X) = E(X^2) - \big(E(X)\big)^2$

$$\psi'(t) = E(Xe^{tX}) = +2t\,\frac{\lambda^2}{(\lambda^2-t^2)^2} = \frac{2t\lambda^2}{(\lambda^2-t^2)^2}$$

$$E(X) = \psi'(0) = 0$$

$$\psi''(t) = E(X^2 e^{tX}) = \frac{2\lambda^2(\lambda^2-t^2)^2 - 2t\lambda^2 \cdot 2 \cdot(-2t)(\lambda^2-t^2)}{(\lambda^2-t^2)^4}$$

$$= \frac{2\lambda^2(\lambda^2-t^2)^2 + 8t^2\lambda^2(\lambda^2-t^2)}{(\lambda^2-t^2)^4} = \frac{2\lambda^4 - 2\lambda^2 t^2 + 8\lambda^2 t^2}{(\lambda^2-t^2)^3}$$

$$= \frac{2\lambda^4 + 6\lambda^2 t^2}{(\lambda^2-t^2)^3}$$

$$E(X^2) = \psi''(0) = \frac{2\lambda^4}{\lambda^6} = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - \big(E(X)\big)^2 = \frac{2}{\lambda^2}$$

Problem 2

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \, \varphi_X(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \, e^{-|t|} \, dt$$

$$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{-itx} e^{t} dt + \int_{0}^{+\infty} e^{-itx} e^{-t} dt \right)$$

$$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{t(1-ix)} dt + \int_{0}^{+\infty} e^{t(-1-ix)} dt \right)$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-ix} e^{t(1-ix)} \Big|_{-\infty}^{0} + \frac{1}{-1-ix} e^{t(-1-ix)} \Big|_{0}^{+\infty} \right)$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-ix} + \frac{1}{1+ix} \right) = \frac{1}{2\pi} \cdot \frac{1-ix+1+ix}{(1+ix)(1-ix)} = \frac{1}{\pi(1+x^2)}$$

Thus, $f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \implies X \sim Cauchy(1)$, $\lambda = 1$

Therefore, $\varphi_X(t) = e^{-|t|}$ is the characteristic function of Cauchy

distribuction with $\lambda = 1$

Problem 3

$$Z = X + Y, \quad X \sim Exp(1), \quad Y \sim Laplace(1)$$

$$Cov(X, Z) = Cov(X, X+Y) = Cov(X, X) + Cov(X, Y)$$

$$Cov(X, X) = Var(X),$$

Since X and Y are independent, $Cov(X, Y) = 0$

Thus, $Cov(X, Z) = Var(X) = E(X^2) - (E(X))^2$

$$E(X) = \int_{0}^{+\infty} x f_X(x) \, dx = \int_{0}^{+\infty} x e^{-x} \, dx = \int_{0}^{+\infty} -x \, d e^{-x}$$

$$= -x e^{-x} \Big|_{0}^{+\infty} - \int_{0}^{+\infty} e^{-x} d(-x) = 0 + 1 = 1$$

$$E(X^2) = \int_{0}^{+\infty} x^2 f_X(x) \, dx = \int_{0}^{+\infty} x^2 e^{-x} \, dx = \int_{0}^{+\infty} -x^2 \, d e^{-x}$$

$$= -x^2 e^{-x} \Big|_{0}^{+\infty} - \int_{0}^{+\infty} e^{-x} d(-x^2) = + \int_{0}^{+\infty} 2x e^{-x} \, dx = 2$$

Thus, $\text{cov}(X, Z) = \text{Var}(X) = 2 - 1^2 = 1$

$\text{Var}(Z) = \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

According Problem 1, $\text{Var}(Y) = 2$.

Thus, $\text{Var}(Z) = 1 + 2 = 3$

Problem 4

$$f_{XY}(x,y) = \frac{4 e^{-(x-y)^2/2}}{y^5 \sqrt{2\pi}}, \quad y \geq 1$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dx = \int_{-\infty}^{+\infty} \frac{4 e^{-(x-y)^2/2}}{y^5 \sqrt{2\pi}}\, dx$$

$$= \frac{4}{y^5} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}}\, dx = \frac{4}{y^5} \cdot 1 = \frac{4}{y^5}$$

Thus, $f_Y(y) = \frac{4}{y^5}, \quad y \geq 1$

Problem 5

$X = \cos\theta, \quad Y = \sin\theta, \quad \theta \sim U[-\pi, \pi] \quad f_\theta(\theta) = \frac{1}{2\pi}$

$$E(XY) = E(\cos\theta \sin\theta) = \int_{-\pi}^{\pi} \cos\theta \sin\theta \frac{1}{2\pi}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \sin(2\theta)\, d\theta$$

$$= \frac{1}{4\pi} \cdot 0 = 0$$

$$E(X) = E(\cos\theta) = \int_{-\pi}^{\pi} \cos\theta \frac{1}{2\pi}\, d\theta = \frac{1}{2\pi} \sin\theta \Big|_{-\pi}^{\pi} = 0$$

$$E(Y) = E(\sin\theta) = \int_{-\pi}^{\pi} \sin\theta \frac{1}{2\pi}\, d\theta = 0$$

Since $X = \cos\theta, \ Y = \sin\theta, \quad X^2 + Y^2 = 1$, we have $Y = \pm\sqrt{1-X^2} = g(X)$

Therefore, X and Y cannot be independent

Problem 6.

$$f_{XY}(x,y) = \frac{1}{2\pi\,\sigma_x\,\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_Y)}{\sigma_x\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right)$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$

Since, we have $\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_Y)}{\sigma_x\sigma_Y} = \left(\frac{y-\mu_Y}{\sigma_Y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2 - \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2$

$$f_X(x) = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot\sigma_x}\frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2\right]\right)dy$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma_x}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_x}(x-\mu_x)}{\sigma_Y}\right)^2\right)dy$$

$\left(\text{let } z = \frac{y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_x}(x-\mu_x)}{\sigma_Y\sqrt{1-\rho^2}}, \quad \frac{dz}{dy} = \frac{1}{\sigma_Y\sqrt{1-\rho^2}}, \quad dz = \frac{1}{\sigma_Y\sqrt{1-\rho^2}}dy\right)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma_x}\cdot\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)$$

Thus, $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)$

Problem 7

$$f(x) = \frac{\exp\left[-\frac{1}{2}(x-\mu)^T R^{-1}(x-\mu)\right]}{(2\pi)^{n/2}\sqrt{\det R}} \quad \text{with } \mu=0 \text{ and } R = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix} \quad \text{where } |\rho|<1$$

$$f(x) = \frac{\exp\left(-\frac{1}{2}x^T R^{-1}x\right)}{2\pi\sqrt{\det R}}$$

$$\det R = \sigma_1^2\sigma_2^2 - \sigma_1^2\sigma_2^2\rho^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$R^{-1} = \frac{1}{\det R}\begin{pmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{pmatrix} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{pmatrix}$$

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x^T R^{-1}x = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}(x_1, x_2)\begin{pmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}(\sigma_2^2 x_1^2 - 2\sigma_1\sigma_2\rho x_1 x_2 + \sigma_1^2 x_2^2)$$

$$f(x) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}(\sigma_2^2 x_1^2 - 2\sigma_1\sigma_2\rho x_1 x_2 + \sigma_1^2 x_2^2)\right]$$

$$= \frac{1}{2\pi\,\sigma_1\,\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1}{\sigma_1}\right)\left(\frac{x_2}{\sigma_2}\right) + \left(\frac{x_2}{\sigma_2}\right)^2\right)\right]$$

Therefore, it has the same form as the bivariate normal density

Problem 8

$$E(Y_j) = E(X_j - \bar{X}) = E(X_j) - E(\bar{X}) = E(X_j) - \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$= \mu - \frac{1}{n} \cdot n\mu = \mu - \mu = 0$$

$$E(\bar{X}\, Y_j) = E\left(\bar{X}(X_j - \bar{X})\right) = E\left(X_j\bar{X} - \bar{X}^2\right) = E(X_j\bar{X}) - E(\bar{X}^2)$$

$$= E\left(X_j \frac{1}{n}\sum_{i=1}^{n} X_i\right) - E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i X_j) - \frac{1}{n^2} E\left(\sum_{i=1}^{n}\sum_{k=1}^{n} X_i X_k\right)$$

$$= \frac{1}{n}\sum_{\substack{i=1 \\ i \neq j}}^{n} E(X_i)E(X_j) + \frac{1}{n}E(X_j^2) - \frac{1}{n^2}\sum_{i=1}^{n}E(X_i^2) - \frac{1}{n^2}\sum_{\substack{i=1 \\ i \neq k}}^{n}\sum_{k=1}^{n} E(X_i)E(X_k)$$

$$= \frac{1}{n}\cdot(n-1)\mu^2 + \frac{1}{n}(\sigma^2 - \mu^2) - \frac{1}{n^2}(\sigma^2 - \mu^2) - \frac{1}{n^2} n(n-1)\mu^2$$

$$= 0$$