Problem 1

$X_1, \cdots, X_n$ are independent and $X_i \sim \text{Poisson}(2)$, $i = 1, \cdots, n$

$$P(\min(X_1, \cdots, X_n) > m) = P(X_1 > m) \cdots P(X_n > m) = \prod_{i=1}^{n} P(X_i > m)$$

$$= P(X_1 > m)^n = [1 - P(X_1 \le m)]^n$$

Since $P(X_1 = k) = \frac{e^{-2} 2^k}{k!}$, $P(X_1 \le m) = e^{-2} \sum_{i=0}^{m} \frac{2^i}{i!}$

$$P(\min(X_1, \cdots, X_n) > m) = [1 - e^{-2} \sum_{i=0}^{m} \frac{2^i}{i!}]^n$$

$$P(\max(X_1, \cdots, X_n) \le m) = P(X_1 \le m) \cdots P(X_n \le m) = P(X_1 \le m)^n$$

$$= \left(e^{-2} \sum_{i=0}^{m} \frac{2^i}{i!}\right)^n$$

Problem 2

(a) $$P_{XY}(x,y) = \begin{cases} \frac{(1-p) p^{x-1} x^y e^{-x}}{y!}, & x \ge 1, y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$P_X(x) = \sum_{y=0}^{\infty} P_{XY}(x,y) = \sum_{y=0}^{\infty} \frac{(1-p) p^{x-1} x^y e^{-x}}{y!} = (1-p) p^{x-1} e^{-x} \sum_{y=0}^{\infty} \frac{x^y}{y!}$$

$$= (1-p) p^{x-1} e^{-x} \cdot e^{x} = (1-p) p^{x-1}$$

Therefore, $$P_X(x) = \begin{cases} (1-p) p^{x-1}, & x \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $$P_Y(y) = \sum_{x=1}^{\infty} P_{XY}(x,y) = \sum_{x=1}^{\infty} \frac{(1-p) p^{x-1} x^y e^{-x}}{y!}, \quad y \ge 0$$

$$P_Y(0) = \sum_{x=1}^{\infty} \frac{(1-p) p^{x-1} \cdot 1 \cdot e^{-x}}{1} = \frac{1-p}{p} \sum_{x=1}^{\infty} p^x \cdot e^{-x} = \frac{1-p}{p} \sum_{x=1}^{\infty} (pe^{-1})^x$$

$$= \frac{1-p}{p} \left(\frac{1}{1-pe^{-1}} - 1 \right) = \frac{1-p}{p} \cdot \frac{pe^{-1}}{1-pe^{-1}} = \frac{(1-p)e^{-1}}{1-pe^{-1}}$$

(c) $$P_X(x) P_Y(y) = (1-p) p^{x-1} \cdot \frac{1-p}{y!} \sum_{x=1}^{\infty} p^{x-1} x^y e^{-x}$$

$$= \frac{(1-p)^2 p^{x-1}}{y!} \sum_{x=1}^{\infty} p^{x-1} x^y e^{-x}, \quad x \ge 1, y \ge 0$$

$$\ne P_{XY}(x,y)$$

Thus, X and Y are not independent.

Problem 3

(a) $f(x) = \begin{cases} \lambda p x^{p-1} e^{-\lambda x^p}, & x>0 \\ 0, & x \le 0 \end{cases}$, where $\lambda > 0$ and $p > 0$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{+\infty} f(x)dx = \int_0^{+\infty} \lambda p x^{p-1} e^{-\lambda x^p} dx$$

let $u = \lambda x^p$, $\frac{du}{dx} = \lambda p x^{p-1}$, $du = \lambda p x^{p-1} dx$

$$\int_0^{+\infty} \lambda p x^{p-1} e^{-\lambda x^p} dx = \int_0^{+\infty} e^{-u} du = -e^{-u}\Big|_{u=+\infty} - (-e^{-u})\Big|_{u=0} = 1$$

Thus, this density integrates to 1

(b) $P(\text{at least one } X_i > 3) = 1 - P(\text{none of } X_i > 3)$

$= 1 - P(\text{all } X_i \le 3) = 1 - P(X_1 \le 3) \cdots P(X_n \le 3)$

$= 1 - P(X_1 \le 3)^n$

since $u = \lambda x^p$, $u(0) = 0$, $u(3) = \lambda 3^p$

$$P(X_1 \le 3) = \int_0^3 \lambda p x^{p-1} e^{-\lambda x^p} dx = \int_0^{\lambda 3^p} e^{-u} du = -e^{-u}\Big|_{u=\lambda 3^p} - (-e^{-u})\Big|_{u=0}$$
$$= 1 - e^{-\lambda 3^p}$$

$$P(\text{at least one } X_i > 3) = 1 - P(X_1 \le 3)^n = 1 - (1 - e^{-\lambda 3^p})^n$$

Problem 4

A probability density function f must satisfy $f(x) \ge 0$ for all x and $\int_{-\infty}^{+\infty} f(x)dx = 1$

Since $f(z) = \frac{1}{B(p,q)} \cdot \frac{z^{p-1}}{(1+z)^{p+q}}$, $z > 0$, $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1} du$, $p>0$, $q>0$

$u \in (0,1)$, $u^{p-1} > 0$, $(1-u)^{q-1} > 0$, $B(p,q) > 0$, $z^{p-1} > 0$, $(1+z)^{p+q} > 0$

$\Longrightarrow f(z) \ge 0$ for all z

$$\int_{-\infty}^{+\infty} f(z)dz = \int_0^{+\infty} f(z)dz = \frac{1}{B(p,q)} \int_0^{+\infty} \frac{z^{p-1}}{(1+z)^{p+q}} dz$$

let $t = \frac{1}{1+z}$, $z = \frac{1-t}{t}$, $\frac{dz}{dt} = -\frac{1}{t^2}$, $dz = -\frac{1}{t^2}dt$

when $z=0$, $t=1$, when $z=+\infty$, $t=0$

$$\int_0^{+\infty} \frac{z^{p-1}}{(1+z)^{p+q}} dz = \int_1^0 \frac{\left(\frac{1-t}{t}\right)^{p-1}}{\left(1+\frac{1-t}{t}\right)^{p+q}} \left(-\frac{1}{t^2}\right) dt$$

$$= \int_0^1 t^{p+q} \frac{(1-t)^{p-1}}{t^{p-1}} \frac{1}{t^2} dt = \int_0^1 t^{q-1}(1-t)^{p-1} dt$$

let $v = 1-t$, $t = 1-v$ $\frac{dt}{dv} = -1$, $dt = -dv$

when $t=0$, $v=1$, when $t=1$, $v=0$

$$\int_0^1 t^{q-1}(1-t)^{p-1} dt = \int_1^0 (1-v)^{q-1} v^{p-1}(-1)\, dv = \int_0^1 v^{p-1}(1-v)^{q-1} dv = B(p,q)$$

Thus, $\int_{-\infty}^{+\infty} f(z)\, dz = \frac{1}{B(p,q)} \int_0^{+\infty} \frac{z^{p-1}}{(1+z)^{p+q}} dz = \frac{B(p,q)}{B(p,q)} = 1$

Therefore, $f(z)$ is a valid density

Problem 5

Since $f_n(x)$ is a probability density function, $f_n(x) \geq 0$ for all x,

Since, $p_n \geq 0$ for all n, $f(x) = \sum_n p_n f_n(x) \geq 0$ for all x

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{+\infty} \sum_n p_n f_n(x)\, dx = \sum_n p_n \int_{-\infty}^{+\infty} f_n(x) dx = \sum_n p_n = 1$$

Therefore, $f(x) = \sum_n p_n f_n(x)$ is a probability density function

Problem 6

Since $X \sim Weibull(p, \lambda)$, then $f(x) = \begin{cases} \lambda p x^{p-1} e^{-\lambda x^p}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

$$E(X^n) = \int_{-\infty}^{+\infty} x^n f(x) dx = \int_0^{+\infty} x^n \lambda p x^{p-1} e^{-\lambda x^p} dx$$

let $u = \lambda x^p$, $\frac{du}{dx} = \lambda p x^{p-1}$, $du = \lambda p x^{p-1} dx$, $x = \left(\frac{u}{\lambda}\right)^{\frac{1}{p}}$

$$E(X^n) = \int_0^{+\infty} x^n e^{-u} du = \int_0^{+\infty} \left(\frac{u}{\lambda}\right)^{\frac{n}{p}} e^{-u} du$$

Since $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$, $\Gamma\left(1+\frac{n}{p}\right) = \int_0^{+\infty} t^{\frac{n}{p}} e^{-t} dt$

$$\text{Thus, } E(X^n) = \frac{\Gamma(1+\frac{n}{p})}{\lambda^{n/p}}$$

Problem 7

$X \sim Exp(1)$, $f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $F_X(x) = \begin{cases} 1-e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

We want to show that $Y = \sqrt{X} \sim Reyleigh(\frac{1}{\sqrt{2}})$, $f(y) = 2x e^{-x^2}$, $x \geq 0$

So, we have $F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2)$

$$= F_X(y^2) = 1 - e^{-y^2}, \quad y \geq 0$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = 2y e^{-y^2}, \quad y \geq 0$$

Therefore, $Y = \sqrt{X} \sim Reyleigh(\frac{1}{\sqrt{2}})$

Problem 8

$X, Y \sim Exp(\lambda)$, X and Y are independent, $F_X(x) = 1 - e^{-\lambda x}$, $F_Y(y) = 1 - e^{-\lambda y}$
$x \in [0, +\infty)$

let $U = \max(X, Y)$

Thus, $F_U(u) = P(U \leq u) = P(\max(X,Y) \leq u) = P(X \leq u \text{ and } Y \leq u)$

$$= P(X \leq u) \cdot P(Y \leq u) = (1 - e^{-\lambda u})^2$$

$$f_U(u) = \frac{dF_U(u)}{du} = \frac{dF_U(u)}{dv} \cdot \frac{dv}{du} = 2(1-e^{-\lambda u}) \lambda e^{-\lambda u}$$

$(v = 1 - e^{-\lambda u})$

$$= 2\lambda e^{-\lambda u}(1 - e^{-\lambda u})$$

$$E(U) = \int_{-\infty}^{+\infty} u f_U(u) du = \int_0^{+\infty} 2u\lambda e^{-\lambda u}(1 - e^{-\lambda u}) du$$

$$= 2 \int_0^{+\infty} \lambda u e^{-\lambda u} - \lambda u e^{-2\lambda u} \, du$$

$$= 2\int_0^{+\infty} \lambda u e^{-\lambda u} du - 2\int_0^{+\infty} \lambda u e^{-2\lambda u} du$$

let $v = \lambda u$, $\frac{dv}{du} = \lambda$, $dv = \lambda du$

$$E(U) = 2\int_0^{+\infty} \frac{v}{\lambda} e^{-v} dv - 2\int_0^{+\infty} \frac{v}{\lambda} e^{-2v} dv$$

$$= \frac{2}{\lambda}\int_0^{+\infty} v e^{-v} dv - \frac{2}{\lambda}\int_0^{+\infty} v e^{-2v} dv$$

$$= \frac{2}{\lambda}\left(-\int_0^{+\infty} v \, d(e^{-v})\right) - \frac{2}{\lambda}\left(\frac{1}{-2}\int_0^{+\infty} v \, d(e^{-2v})\right)$$

$$= -\frac{2}{\lambda}\left(ve^{-v}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-v} dv\right) + \frac{1}{\lambda}\left(ve^{-2v}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-2v} dv\right)$$

$$= -\frac{2}{\lambda}\left(0 + \int_0^{+\infty} -e^{-v} dv \right) + \frac{1}{\lambda}\left(0 + \frac{1}{2}\int_0^{+\infty} -2e^{-2v} dv \right)$$

$$= -\frac{2}{\lambda} \, e^{-v}\Big|_0^{+\infty} + \frac{1}{2\lambda} \, e^{-2v}\Big|_0^{+\infty}$$

$$= -\frac{2}{\lambda}(0-1) + \frac{1}{2\lambda}(0-1)$$

$$= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$$

Therefore, $E[\max(X,Y)] = \frac{3}{2\lambda}$