Homework 5

1. (a) $X \sim Exp(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$

$$F_X(x) = P(X \leq x) = \int_0^x f_X(x)\,dx = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^x = 1 - e^{-\lambda x}$$

$$P(X > t+s) = 1 - P(X \leq t+s) = 1 - (1 - e^{-\lambda(t+s)}) = e^{-\lambda(t+s)}$$

$$P(X > t) = 1 - P(X \leq t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$

$$P(X > s) = 1 - P(X \leq s) = 1 - (1 - e^{-\lambda s}) = e^{-\lambda s}$$

$$P(X > t) \cdot P(X > s) = e^{-\lambda t} \cdot e^{-\lambda s} = e^{-\lambda(t+s)}$$

Thus, $P(X > t+s) = P(X > t) \cdot P(X > s)$ for all $s, t \geq 0$

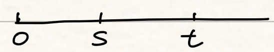
1b) From (a), we have $P(X > t+s) = P(X > t) \cdot P(X > s)$

Since $$P(X > t+s \mid X > s) = \frac{P(X > t+s, X > s)}{P(X > s)} = \frac{P(X > t+s)}{P(X > s)}$$
$$= \frac{P(X > t) \cdot P(X > s)}{P(X > s)} = P(X > t)$$

Thus, $P(X > t+s \mid X > s) = P(X > t)$ for all $s, t \geq 0$

2. $N_t \sim Poisson(\lambda t)$, $P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, $\lambda > 0$, $0 \leq s \leq t$

$$P(N_s = 0, N_t = 1) = P(N_s - N_0 = 0, N_t - N_s = 1)$$



$$= P(N_s - N_0 = 0) \cdot P(N_t - N_s = 1)$$

$$= P(N_s = 0) \cdot P(N_{t-s} = 1)$$

$$= \frac{(\lambda s)^0}{0!} e^{-\lambda s} \cdot \frac{(\lambda(t-s))^1}{1!} e^{-\lambda(t-s)}$$

$$= e^{-\lambda s} \cdot \lambda(t-s) \cdot e^{-\lambda(t-s)} = \lambda(t-s) e^{-\lambda t}$$

3. $N_t \sim Poisson(\lambda t)$, $P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, $\lambda > 0$

$$P(N_t \text{ is odd}) = \sum_{\text{odd } k} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$P(N_t \text{ is even}) = \sum_{\text{even } k} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

since $\sinh(\lambda t) = \frac{e^{\lambda t} - e^{-\lambda t}}{2} = \lambda t + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots = \sum_{\text{odd } k} \frac{(\lambda t)^k}{k!}$

$\cosh(\lambda t) = \frac{e^{\lambda t} + e^{-\lambda t}}{2} = 1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots = \sum_{\text{even } k} \frac{(\lambda t)^k}{k!}$

Thus, $P(N_t \text{ is odd}) = e^{-\lambda t} \sinh(\lambda t)$

$P(N_t \text{ is even}) = e^{-\lambda t} \cosh(\lambda t)$

4. $N_t \sim \text{Poisson}(\lambda t)$, $P(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$, $M_t = \sum_{i=1}^{N_t} Y_i$

$P(M_t = k \mid N_t = n) = P\left(\sum_{i=1}^{n} Y_i = k\right) = \binom{n}{k} p^k (1-p)^{n-k}$

Thus, $M_t \mid N_t = n \sim \text{Binomial}(n, p)$

Since $P(M_t = k) = \sum_{n=k}^{\infty} P(M_t = k \mid N_t = n) \cdot P(N_t = n)$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= p^k e^{-\lambda t} (\lambda t)^k \frac{1}{k!} \sum_{n=k}^{\infty} \frac{((1-p)(\lambda t))^{n-k}}{(n-k)!}$$

$$= \frac{p^k e^{-\lambda t} (\lambda t)^k}{k!} \sum_{n=0}^{\infty} \frac{((1-p)(\lambda t))^n}{n!}$$

$$= \frac{p^k e^{-\lambda t} (\lambda t)^k}{k!} e^{(1-p)\lambda t} = \frac{(p\lambda t)^k e^{-p\lambda t}}{k!}$$

Thus, $M_t \sim \text{Poisson}(\lambda p t)$

5. (W_t), $t \geq 0$ is a Wiener Process, $T > 0$, $V_t = W_{T+t} - W_T$, $t \geq 0$

a) $V_0 = 0$: $V_0 = W_{T+0} - W_T = W_T - W_T = 0$

b) Independent of increments for (V_t): it is enough to choose $0 \leq t_1 < t_2 < t_3 < t_4$ and to prove the independence of increments

$V_{t_2} - V_{t_1} = (W_{T+t_2} - W_T) - (W_{T+t_1} - W_T) = W_{T+t_2} - W_{T+t_1}$

$V_{t_4} - V_3 = (W_{T+t_4} - W_T) - (W_{T+t_3} - W_T) = W_{T+t_4} - W_{T+t_3}$

Since (W_t) is a Wiener process, its increments over non-overlapping intervals are independent. The intervals $[T+t_1, T+t_2]$ and $[T+t_3, T+t_4]$

are non-overlapping, Thus, $V_{t_2}-V_{t_1}$ and $V_{t_4}-V_{t_3}$ are independent

c) For all $s<t$, the increment V_t-V_s has normal distribution as a linear combination of normal random variables and we have that

$$V_t-V_s=(W_{T+t}-W_T)-(W_{T+s}-W_T)=W_{T+t}-W_{T+s}$$

Since (W_t) is a Wiener process, $W_{T+t}-W_{T+s}$ is normally distributed with mean 0 and variance $(T+t)-(T+s)=t-s$

$$E(V_t-V_s)=E(W_{T+t}-W_{T+s})=0$$

$$Var(V_t-V_s)=Var(W_{T+t}-W_{T+s})=t-s$$

Thus, $V_t-V_s\sim N(0,t-s)$

Therefore, we conclude that V_t is a Wiener process.

6. From the lecture, we can get $n(t,y)=\frac{1}{\sqrt{2\pi t}}\exp(-\frac{y^2}{2t})$

$$\frac{\partial n}{\partial t}=-\frac{1}{2\sqrt{2\pi}\,t^{3/2}}\exp(-\frac{y^2}{2t})+\frac{1}{\sqrt{2\pi t}}\frac{y^2}{2t^2}\exp(-\frac{y^2}{2t})$$

$$=\left(-\frac{1}{2t}+\frac{y^2}{2t^2}\right)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{y^2}{2t})$$

$$=\left(-\frac{1}{2t}+\frac{y^2}{2t^2}\right)n(t,y)$$

$$\frac{\partial n}{\partial y}=\frac{1}{\sqrt{2\pi t}}\exp(-\frac{y^2}{2t})\cdot\left(-\frac{y}{t}\right)=-\frac{y}{t}\cdot n(t,y)$$

$$\frac{\partial^2 n}{\partial y^2}=-\frac{1}{t}n(t,y)+\frac{y^2}{t^2}n(t,y)=\left(-\frac{1}{t}+\frac{y^2}{t^2}\right)n(t,y)$$

Thus, $\frac{\partial n}{\partial t}=\frac{1}{2}\frac{\partial^2 n}{\partial y^2}$

7. $W_t=(W_t^1,W_t^2)$ is two-dimensional Wiener process, where W_t^1, W_t^2 are independent one-dimensional Wiener process.

$$|W_t|=\sqrt{(W_t^1)^2+(W_t^2)^2},\quad W_t^1\sim N(0,t),\ W_t^2\sim N(0,t)$$

In polar coordinates, $W_t^1=R\cos\theta$, $W_t^2=R\sin\theta$

$$R = |W_t| = \sqrt{(W_t^1)^2 + (W_t^2)^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{W_t^2}{W_t^1}\right)$$

$$f_{W_t^1, W_t^2}(w_1, w_2) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{w_1^2}{2t}} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{w_2^2}{2t}} = \frac{1}{2\pi t} e^{-\frac{w_1^2+w_2^2}{2t}}$$

$$J = \text{Jacobian matrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|J| = r\cos^2\theta + r\sin^2\theta = r$$

$$f_{R,\theta}(r,\theta) = \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} \cdot r = \frac{r}{2\pi t} \cdot e^{-\frac{r^2}{2t}}, \quad r \geq 0, \ 0 \leq \theta < 2\pi$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi t} e^{-\frac{r^2}{2t}} d\theta = \frac{r}{t} \cdot e^{-\frac{r^2}{2t}}$$

$$P(|W_t| < R) = P(r < R) = \int_0^R f_R(r)\,dr = \int_0^R \frac{r}{t} e^{-\frac{r^2}{2t}} dr$$

$$= -\int_0^R e^{-\frac{r^2}{2t}} d\left(-\frac{r^2}{2t}\right)$$

$$= - \left. e^{-\frac{r^2}{2t}} \right|_0^R = -\left(e^{-\frac{R^2}{2t}} - 1\right)$$

$$= 1 - e^{-\frac{R^2}{2t}}$$

8. $(W_t)_{t \geq 0}$ is a standard Wiener process.

$$W_0 = 0, \quad W_{t_1} \sim N(0, t_1), \quad f_{W_{t_1}}(x_1) = \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{(0-x_1)^2}{2t_1}\right)$$

Since $W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$, we have $W_{t_i} | W_{t_{i-1}} = x_{i-1} \sim N(x_{i-1}, t_i - t_{i-1})$

$$f_{W_{t_i}|W_{t_{i-1}}}(x_i | x_{i-1}) = \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\frac{(x_{i-1} - x_i)^2}{2(t_i - t_{i-1})}\right)$$

According to the independency of increments of W_t, we can get

$$f_{t_1,\cdots,t_n}(x_1, \cdots, x_n) = f_{W_{t_1}}(x_1) \times f_{W_{t_2}|W_{t_1}}(x_2|x_1) \times \cdots \times f_{W_{t_n}|W_{t_{n-1}}}(x_n | x_{n-1})$$

$$= f_{W_{t_1}}(x_1) \cdot \prod_{i=2}^{n} f_{W_{t_i}|W_{t_{i-1}}}(x_i | x_{i-1})$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^{n} \frac{(x_{i-1} - x_i)^2}{2(t_i - t_{i-1})}\right)$$

where $t_0 = 0$, $x_0 = 0$

Thus, we have $f_{t_1,\cdots,t_n}(x_1, \cdots, x_n) = n(t_1; 0, x_1) \times \prod_{i=2}^{n} n(t_i - t_{i-1}; x_{i-1}, x_i)$

$$= n(t_1; 0, x_1) \times n(t_2 - t_1; x_1, x_2) \times \cdots \times n(t_n - t_{n-1}; x_{n-1}, x_n)$$

where $n(\tau; x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(x-y)^2}{2\tau}\right)$