ESE 520 Probability and Stochastic Processes

Instructor: Vladimir Kurenok

EXAM 2-Practice

Name (Please print):

Solutions

Total: 70 points (each problem is 10 points worth)

Instructions:

1. You must show all work to completely justify your answers in order to receive any credit.
2. You can use One one-sided sheet of paper with your own formulas.

1. Let (X, Y) be a randomly selected point on $\mathbb{R}^2$ where both coordinates X and Y have $\mathcal{N}(0,1)$ distribution and are independent. Also, the polar coordinates for (X, Y) are (R, θ) and are defined through $X = R\cos\theta, Y = R\sin\theta$. Calculate $E(X|\theta)$.

- $E(X / \underset{\uparrow \text{fixed}}{\theta = \bar{\theta}}) = E(R \cdot \cos\theta / \theta = \bar{\theta}) =$

$$= E(R \cdot \underbrace{\cos\bar{\theta}}_{\text{constant}} / \theta = \bar{\theta}) = \cos\bar{\theta} \cdot E(R / \theta = \bar{\theta}) \underset{\text{independence}}{=}$$

$$= \cos\bar{\theta} \cdot E(R)$$

- $E(R) = \int_0^\infty r^2 \cdot e^{-r^2/2} dr = \underbrace{-r \cdot e^{-r^2/2} \Big/_0^\infty}_{=0} + \int_0^\infty e^{-r^2/2} dr =$

$$= \frac{1}{2}\sqrt{2\pi} \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-r^2/2} dr}_{=1} = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow E(X/\theta) = \sqrt{\frac{\pi}{2}} \cdot \cos\theta.$$

2. Let X_1 X_2 be independent random variables each of which has a normal distribution $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, respectively. Define the random variable $Y := X_1 + X_2$. Using characteristic functions, show that Y also has a normal distribution $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Hint: The characteristic function of a normal rv with mean μ and variance σ^2 has the form $\phi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.

- $\varphi_Y(t) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t) = e^{it(\mu_1+\mu_2) - \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$

$\Rightarrow \quad Y \sim \mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

3. Two measurements $y_1 = 2$ and $y_2 = 5$ are taken to estimate the value x of a random variable X. Assume that the joint distribution of the full random vector (X, Y_1, Y_2) is Gaussian and that X has expectation $\mu_X = 3$, variance $\sigma_X^2 = 4$, and the following covariances with the random variables Y_1 and Y_2 which describe the first and second measurement, $cov(X, Y_1) = -1$ and $cov(X, Y_2) = +1$. Also suppose the measurements Y_1 and Y_2 have means $\mu_{Y_1} = \mu_{Y_2} = 3$, variances $\sigma_{Y_1}^2 = 3$ and $\sigma_{Y_2}^2 = 5$ and covariance $cov(Y_1, Y_2) = -2$. Find the best mean square estimate for X in terms of the measurements (y_1, y_2).

$$Y = (Y_1, Y_2)$$

- $E(X|Y) = \mu_X + R_{XY} \cdot R_{YY}^{-1}(Y - \mu_Y)$
- $R_{XY} = (-1, 1)$
- $R_{YY} = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix} \Rightarrow R_{YY}^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$
- $\mu_X = 3$, $y - \mu_Y = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\Rightarrow E(X|Y=y) = 3 + (-1\ 1)\ \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$$

$$= 3 + \frac{1}{11} \cdot (-1\ 1)\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \frac{38}{11}$$

4. Let $X_1, X_2, ..., X_n$ be independent and exponentially distributed (with parameter $\lambda > 0$) random variables. Define $Y := \max\{X_1, ..., X_n\}$. Find the density of Y.

- $X_i \sim f(x) = \lambda \cdot e^{-\lambda x}$, $x>0$, $F_{X_i}(x) = 1 - e^{-\lambda x}$, $x>0$

- $F_Y(y) = P(\max\{X_1, \dots, X_n\} \leq y) =$

$= \prod_{i=1}^{n} P(X_i \leq y) = (1-e^{-\lambda y})^n$, $y>0$

- $f_Y(y) = F_Y'(y) = n\lambda \cdot e^{-\lambda y}(1-e^{-\lambda y})^{n-1}$, $y>0$

5. Assume that a jointly continuous vector (X, Y) has the joint pdf $f(x, y) = \frac{1}{x}, 0 < y < x < 1$ (and equal to zero otherwise). Find $cov(X, Y)$. Hint: Use integration by parts when calculating $E(Y)$.

- $\mathrm{cov}(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$
- $E(X \cdot Y) = \iint xy \cdot f_{XY}(x,y)\,dx\,dy = \iint y\,dx\,dy =$

$$= \int_0^1 \left(\int_0^x y\,dy \right) dx = 1/6$$

- $E(X) = \int_0^1 x \left(\int_0^x f_{XY}(x,y)\,dy \right) dx = \int_0^1 \int_0^x 1\,dy = \int_0^1 x\,dx = 1/2$
- $E(Y) = \int_0^1 y \left(\int_y^1 f(x,y)\,dx \right) dy = \int_0^1 y \left(\int_y^1 \frac{dx}{x} \right) dy =$

$$= \int_0^1 y \ln x \Big|_y^1 dy = \int_0^1 (-y \cdot \ln y)\,dy = \underbrace{-\frac{y^2}{2} \cdot \ln y \Big|_0^1}_{=0} + \int_0^1 \frac{y}{2}\,dy$$

$$= 1/4$$

$$\Rightarrow \mathrm{cov}(X,Y) = 1/6 - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{24}.$$

6. Let X be a continuous rv X with the density $f(x) = \frac{1}{2}e^{-\frac{1}{2}|x|}, x \in \mathbb{R}$. Find $\psi(t) = E(e^{tX})$.

$$\psi(t) = \int_{-\infty}^{\infty} \frac{1}{2} e^{tx} e^{-\frac{1}{2}|x|} dx = \frac{1}{2}\int_{-\infty}^{0} e^{x(\frac{1}{2}+t)} dx +$$

$$+ \frac{1}{2}\int_0^{\infty} e^{-x(\frac{1}{2}-t)} dx = \frac{1}{2}\left(\frac{e^{x(\frac{1}{2}+t)}}{\frac{1}{2}+t}\Big|_{-\infty}^{0} - \frac{e^{-x(\frac{1}{2}-t)}}{\frac{1}{2}-t}\Big|_0^{\infty} \right)$$

$$= \frac{1}{2}\left(\frac{1}{\frac{1}{2}+t} + \frac{1}{\frac{1}{2}-t} \right) = \frac{1/2}{(\frac{1}{2})^2 - t^2} = \frac{1/2}{\frac{1}{4} - t^2},$$

$$-\frac{1}{2} < t < \frac{1}{2}$$

7. With (X, Y) and (R, θ) defined as in Problem 1, show that random variables X and R are *uncorrelated*.

$$\cdot \ \rho_{XR} = \frac{\operatorname{cov}(X, R)}{\sigma_X \cdot \sigma_R}$$

$$\cdot \ \operatorname{cov}(X, R) = E(X \cdot R) - \underbrace{E(X)}_{=0} \cdot E(R) =$$

$$= E(R^2 \cdot \cos\theta) = E(R^2) \cdot E(\cos\theta) = 0$$

$$\text{since } E(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \, d\theta = 0.$$

$\Rightarrow \operatorname{cov}(X, R) = \rho_{X,R} = 0$ thus X & R are uncorrelated.