Complexity Theory

Poster by Dr. Kai Sauerwald (Artificial Intelligence Group, FernUniversität in Hagen)
based on a lecture by Prof. Dr. Schwentick (LogiDAC, TU Dortmund University)

TM-HALT — in **RE**

SAT-FO-STRING — in **R**

Input: A first-order formula φ for strings
Decision: Is φ satisfiable?

2-PLAYERCORRIDORTILING — **EXP-C**

Input: Tilings T and integer k (unary)
Decision: Does player one has a winning strategy in the tiling game?

QBF — **PSPACE-C**

Input: Quantified Boolean formula φ (in prenex normal form and CNF)
Decision: Is φ satisfiable?

CORRIDORTILING — **PSPACE-C**

Input: Tilings T and integer k (unary)
Decision: Is there a $k \times l$ tiling for T?

SAT-UNSAT — **DP-C**

Input: Boolean formulas φ_1, φ_2 in CNF
Decision: Is φ_1 satisfiable and φ_2 unsatisfiable?

SAT — **NP-C**

Input: Boolean formula φ in CNF
Decision: Is φ satisfiable?

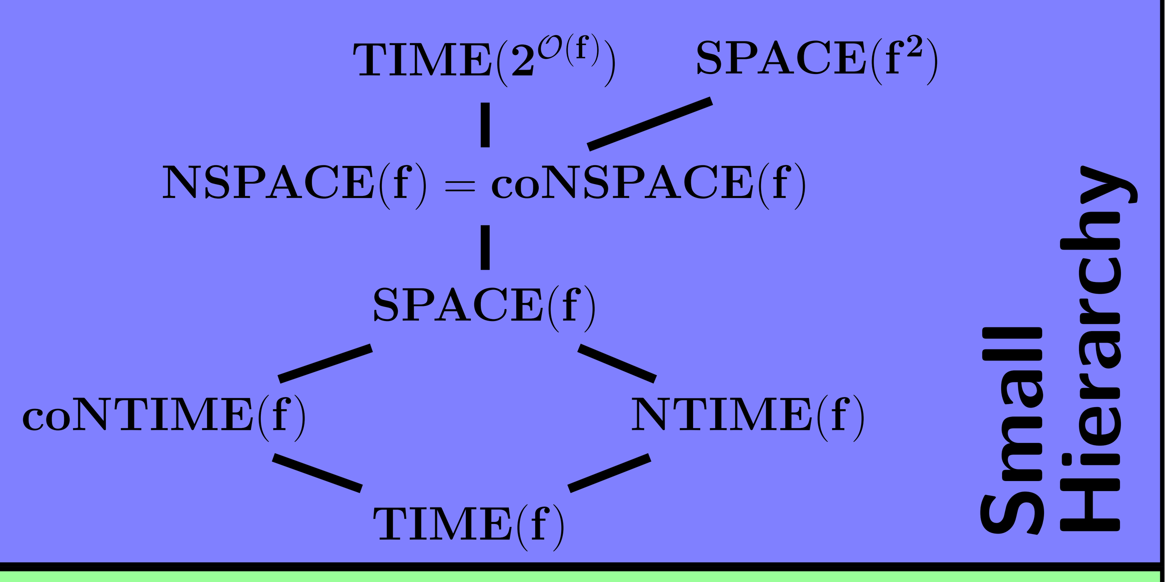
TILING — **NP-C**

Input: Tilings T und integers c, r (unary)
Decision: Is there a $c \times r$ tiling for T?

TSP — **NP-C**

Input: Weighted Graph G and $k \in \mathbb{N}$
Decision: Is there a circle in G with a weight smaller or equal to k that visits all nodes?

For $f(n) \geq \log(n)$



HORNSAT — **P-C**

Input: Finite set of Horn-clauses K
Decision: Is K satisfiable?

DETERMINANT — in $\mathbf{NC^2}$

ACYCLICCS — **LOGCFL-C**

Input: Variables V, domain U, constraints C, such that (V, C) is acyclic
Decision: Is there a solution $\sigma \subseteq V \times U$ such that $\sigma(\mathcal{C})$ is true?

REACH — **NL-C**

Input: Graph $G = (V, E)$ and $s, t \in V$
Decision: Is there a path from s to t in G?

UREACH — **L-C**

Input: Undirected graph $G{=}(V, E)$ and $s, t \in V$
Decision: Is there a path from s to t in G?

FORMULAVALUEPROBLEM — $\mathbf{NC^1}$**-C**

Input: Variable free Boolean formula B
Decision: Does B evaluate to true?

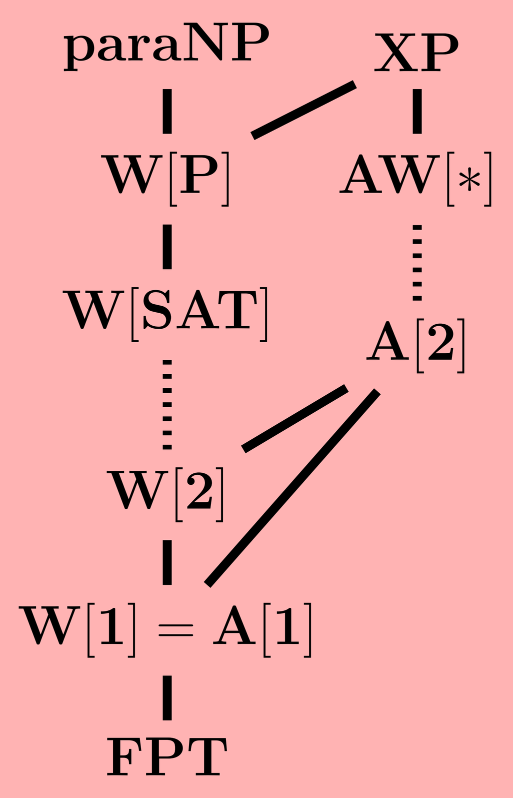
MULTIPLICATION — $\mathbf{TC^0}$**-C**

ADDITION — in $\mathbf{AC^0}$

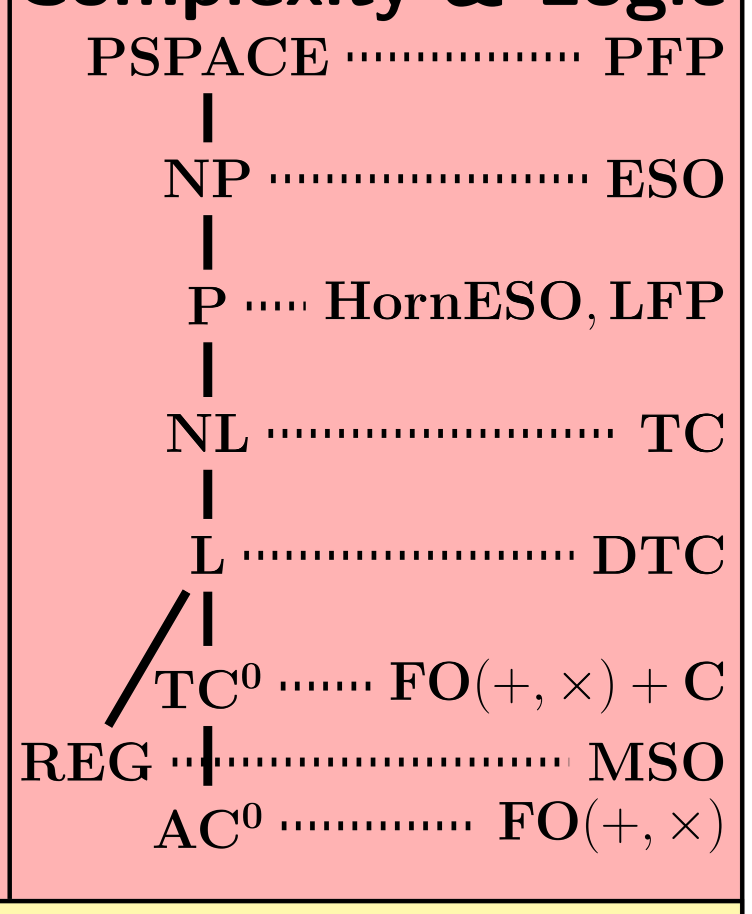
Parameterized Problems

A tuple (Q, κ) is a parameterized problem, if $Q \subseteq \Sigma^*$ and $\kappa : \Sigma^* \to \mathbb{N}$ is computable in polynomial time.

Descriptive Complexity & Parametrization



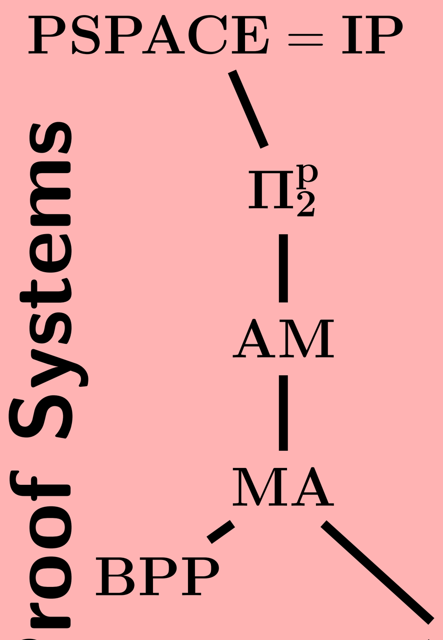
Complexity & Logic



Complexity Classes and Logic

A logic $\mathcal{L}$ has complexity $\mathcal{C}$, if for each sentence $\varphi \in \mathcal{L}$ it holds $\{\, \mathrm{enc}(\mathfrak{A}) \mid \mathfrak{A} \models \varphi \,\} \in \mathcal{C}$.

Proof Systems



PCP *(probabilistically checkable proof)*

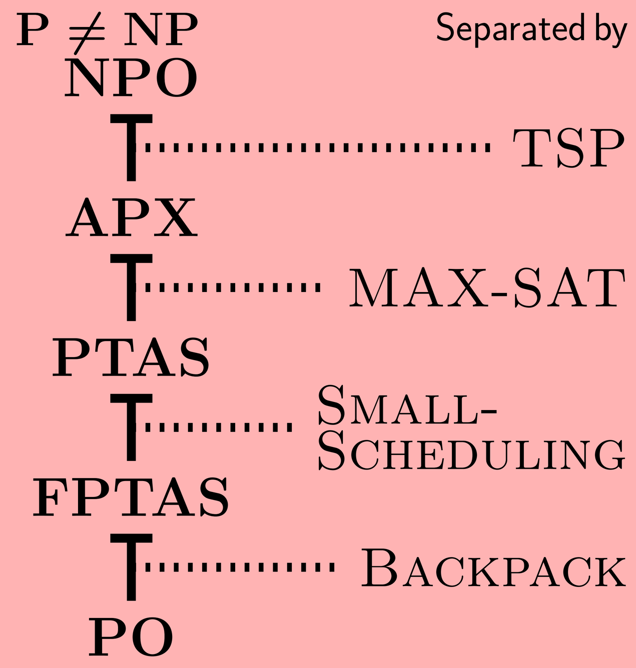
$PCP(\mathcal{F}, \mathcal{G})$ is the class of languages L such there is a verifier V that

- uses $\leq r(n)$ random bits, $(r \in \mathcal{F})$
- reads $\leq q(n)$ bits of the proof, $(q \in \mathcal{G})$
- accepts for a proof π everything from L and does not accept any $x \notin L$ with probability $> \frac{1}{2}$.

$\mathbf{NP = PCP(\log, poly) = PCP(0, poly) = PCP(\log, 1)}$

Optimization

For $P \neq NP$ — Separated by



Class operators

For every $\mathcal{C}$ which is closed under Turing-reduction holds:

$$\mathbf{BP} \cdot \mathbf{BP} \cdot \mathcal{C} \subseteq \mathbf{BP} \cdot \mathcal{C}$$
$$\oplus \cdot \oplus \cdot \mathcal{C} \subseteq \oplus\mathcal{C}$$
$$\oplus \cdot \mathbf{BP} \cdot \mathcal{C} \subseteq \mathbf{BP} \cdot \oplus \cdot \mathcal{C}$$
$$\exists \cdot \mathcal{C} \subseteq \mathbf{BP} \cdot \oplus \cdot \mathcal{C}$$
$$\exists \cdot \mathbf{BP} \cdot \mathcal{C} \subseteq \mathbf{BP} \cdot \exists \cdot \mathcal{C}$$
$$\exists \cdot \mathbf{BP} \cdot \mathbf{P} = \mathbf{MA}$$
$$\mathbf{BP} \cdot \exists \cdot \mathbf{P} = \mathbf{AM}$$

Theorem — Chandra, Kozen, Stockmeyer

(a) For $S(n) = \Omega(\log n)$ holds:
$$\mathbf{ASPACE(S) \subseteq TIME(2^{\mathcal{O}(S)})}$$
(b) For $T(n) = \Omega(n)$ holds:
$$\mathbf{TIME(T) \subseteq ASPACE(\log T)}$$

Immerman–Szelepcsényi theorem

For space constructible S with $S(n) \geq \log n$ holds:
$$\mathbf{NSPACE(S) = coNSPACE(S)}$$

Savitch's theorem — Savitch

For space constructible S with $S(n) \geq \log n$ holds:
$$\mathbf{NSPACE(S) \subseteq SPACE(S^2)}$$

Space hierarchy theorem — Stearns, Szepietowski

For time constructible f, g with $f(n) \in \Omega(\log n)$ and $g(n) \in \omega(f(n))$ holds:
$$\mathbf{SPACE(f) \subsetneq SPACE(g)}$$
$$\mathbf{NSPACE(f) \subsetneq NSPACE(g)}$$

Time hierarchy theorem — Hartmanis, Stearns

For time constructible f, g with $f(n) \in \Omega(n)$ and $g(n) \in \omega(f(n) \cdot \log(f(n)))$ holds:
$$\mathbf{TIME(f) \subsetneq TIME(g)}$$