MATLAB for SMS
Lesson 6: Wavelets

Historical Perspectives

From Ingrid Daubechies' essay entitled "Where do wavelets come from? – A personal point of view"

> *It is clear that all of this results in a highly personal version of the development of wavelets; similar stories told by a harmonic analyst or an approximation theorist or an electrical engineer would all be quite different from mine or each other.*

In the beginning, Fourier created the heavens and the Earth…

Most historical versions of wavelet theory however, despite their source's perspective, begin with Joseph Fourier. Fourier analysis and the Fourier transform are based on transformation of some function to the frequency domain. Such a transformation involves expressing our function in terms of a sum, or superposition, of periodic functions. These periodic functions, i.e., sine and cosine, form a basis set of functions that are used to reconstruct our original function. Similarly, a wavelet basis is a set of functions that is used to reconstruct an input function in a different domain. In this respect, wavelet transformations do not differ from Fourier transformations at all, but are extensions of the same thought process.

Rather simply, the basis functions used in such a transformation are often more advantageous if they are not periodic. More specifically, if the functions comprising a wavelet basis are nonzero only inside a certain interval, i.e., compactly supported, the basis is scaled and translated to best reconstruct the original functions. This is sometimes called *scale analysis*, and it is in this respect that wavelet transformations differ from the Fourier transformation. Fourier transforms use a fixed interval and vary basis functions, while wavelets use fixed basis functions that are dilated and translated (scaled and shifted) to reconstruct the original function.

Haar and Lévy walks

The first mention of wavelets was in a 1909 dissertation by Hungarian mathematician Alfred Haar. Haar's work wasn't necessarily about wavelets, as "wavelets" would not appear in their current form until the late 1980s. Specifically, Haar focused on orthogonal function systems, and proposed an orthogonal basis, now known as the Haar wavelet basis, in which functions were to be transformed by two basis functions. One basis function is constant on a fixed interval, and is known as the scaling function. The other basis function is a step function that contains exactly one zero-crossing (vanishing moment) over a fixed interval.

The next major contribution to wavelet theory was from French scientist Paul Pierre Lévy. More correctly, Lévy's contribution was less of a contribution and more of a validation. While studying the ins and outs of Brownian motion in the years following Haar's publication, Lévy discovered that a scale-varied Haar basis produced a more accurate representation of Brownian motion than did the Fourier basis. Lévy, being more of a physicist than mathematician, moved on to make large contributions to our understanding of stochastic processes. A *Lévy process* is a type of stochastic random walk.

Morlet, more or less

Contributions to wavelet theory between the 1930s and 1970s were slight. Most importantly, the windowed Fourier transform was developed, with the largest contribution being made by another Hungarian (turned Englishman) named Dennis Gabor. The next major advancement in wavelet theory is considered to be that of Jean Morlet in the late 1970s.

Morlet, a French geophysicist working with windowed Fourier transforms, discovered that fixing frequency and stretching or compressing (scaling) the time window was a more useful approach than varying frequency and fixing scale. In 1981, Morlet worked with Croatian-French physicist Alex Grossman on the idea that a function could be transformed by a wavelet basis and transformed back without loss of information, thereby outlining the wavelet transformation. It is of note that Morlet initially developed his ideas with nothing more than a *handheld calculator*. Said Yves Meyer of Morlet's influence (see course website for the complete essay):

> *Jean Morlet launched a scientific program which already offered fruitful alternatives to Fourier analysis and is now moving beyond wavelets.*

Hello wavelets

In 1986, Stéphane Mallat noticed a publication by Yves Meyer that built on the concepts of Morlet and Grossman. Mallat sought Meyer's consult, and the result of said consult was Mallat's publication of multiresolution analysis (MRA). Mallat's MRA connected wavelet transformations with the field of digital signal processing. Specifically, Mallat developed the wavelet transformation as a multiresolution approximation produced by a pair of digital filters. The scaling and wavelet functions that constitute a wavelet basis are represented by a pair of finite impulse response filters, and the wavelet transformation is computed as the convolution of these filters with the input function. The importance of Mallat's contribution cannot be overstated. Without the fast computational means of wavelet transformation provided by the MRA, wavelets, undoubtedly, would not be the effective and widely used signal processing tools that they are today.

In 1988, a student of Alex Grossman, named Ingrid Daubechies, combined the ideas of Morlet, Grossman, Mallat, and Meyer by developing the first family of wavelets as they are known today. Named the Daubechies wavelets, the family consists of 8 separate wavelet and scaling functions. Each of pair wavelet and scaling functions is orthogonal, continuous, regular, and compactly supported.

Daubechies methods are rigorously presented in her famous text *Ten Lectures on Wavelets*. The wavelet era in signal processing and data compression had officially begun.

Wavelets for "De-noising"

The final piece of our puzzle was provided by David Donoho (in conjunction with Ian Johnstone) in 1994. Although denoising was not new concept, Donoho and Johnstone's 1994 publication entitled "Ideal Spatial Adaptation via Wavelet Shrinkage" outlined a statistically-based approach to denoising signals with wavelets. Donoho further detailed wavelet thresholding in his 1995 publication "De-noising by Soft Thresholding." In these publications, Donoho details the processes of wavelet shrinkage and noise estimation as a means to reduce or remove the part of a signal that is purely noise. It is prudent to mention that the basis of my work involving wavelet denoising is, primarily, a direct extension of the work of Donoho.