MINIMIZING LAMINATIONS IN REGULAR COVERS, HOROSPHERICAL ORBIT CLOSURES, AND CIRCLE-VALUED LIPSCHITZ MAPS

SUMMARY OF RESULTS

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In this paper we will study the dynamical behavior of horocyclic and geodesic flows in geometrically infinite hyperbolic manifolds (mostly in dimension 2). These two flows, while geometrically related, exhibit dramatically different dynamical behaviors. Indeed, over a finite area surface Σ, the geodesic flow is "chaotic" and supports a plethora of invariant measures and orbit closures while the horocycle flow is extremely rigid, with all non-periodic orbits dense [Hed36] and equidistributed in $T^1\Sigma$ [Fur73, DS84]. Something similar is true for the geometrically finite case (which in dimension 2 just means finitely-generated fundamental group), see e.g. [Ebe77, Dal00, Bur90, Rob03].

We consider arguably the simplest and most symmetric geometrically infinite setting, that of $\mathbb{Z}$-covers of compact surfaces. Let $G = \mathrm{PSL}_2(\mathbb{R})$ be the group of orientation preserving isometries of real hyperbolic 2-space $\mathbb{H}^2$, let $\Gamma_0 < G$ be a torsion-free uniform lattice and let $\Gamma \lhd \Gamma_0$ with $\Gamma_0/\Gamma \cong \mathbb{Z}$, that is, $\mathbb{H}^2/\Gamma$ is a $\mathbb{Z}$-cover of the compact surface $\mathbb{H}^2/\Gamma_0$.

Let $A = \{a_t : t \in \mathbb{R}\}$ denote the diagonal subgroup generating the geodesic flow on G/Γ, and let N denote the lower unipotent subgroup generating the (stable with respect to A) horocycle flow. We call a horocycle orbit closure $\overline{Nx}$ *non-maximal* if it is not all of G/Γ.

In this setting we:

(1) Study the structure of non-maximal horocycle orbit closures in $\mathbb{Z}$-covers and expose their delicate dependence on the particular geometry of the covered compact surface;
(2) Describe novel constructions of $\mathbb{Z}$-covers with prescribed geometric and dynamical properties; and, in doing so,
(3) Provide the first examples of $\mathbb{Z}$-covers with a full horocycle orbit closure classification, including a description of orbit closures that are neither minimal nor maximal.

While the strongest results in this paper hold solely for $\mathbb{Z}$-covers of compact surfaces, much of the techniques we develop are applicable in greater generality, both to higher-dimensional hyperbolic manifolds as well as maximal horospherical group actions on higher-rank homogeneous spaces.

Remark 1.1. In contrast to the finite area setting, measure rigidity and equidistribution results for the horocycle flow over $\mathbb{Z}$-covers has limited utility. Indeed, non-maximal horocycle orbit closures do not support any locally finite N-invariant measures, since all such measures are AN-quasi-invariant and hence have full support, see [Sar04] as well as [LL22].

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At the heart of our analysis lies a connection to a seemingly unrelated geometric optimization problem of independent interest — tight Lipschitz maps to the circle $\mathbb{R}/\mathbb{Z}$.

Tight circle-valued maps. Given a compact hyperbolic surface Σ_0 and a homotopically nontrivial map $f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$, we are interested in geometric properties of maps realizing the minimum Lipschitz constant in the homotopy class of f.

The homotopy class of f is the same data as a cohomology class $\varphi \in H^1(\Sigma_0, \mathbb{Z})$ recording the degree of the restriction to loops in Σ_0. Evidently, the minimum Lipschitz constant for circle valued maps representing φ is bounded below by

$$\kappa = \sup_\gamma \frac{|\varphi(\gamma)|}{\ell(\gamma)},$$

where the sup runs over all geodesic loops γ and $\ell(\gamma)$ is the length in Σ_0. A map in $[f]$ is called *tight* if its Lipschitz constant is equal to κ.

Theorem (Daskalopoulos-Uhlenbeck [DU20])**.** *There exists a tight map in every non-trivial homotopy class $[f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}]$, whose maximal stretch locus is a geodesic lamination.*

More generally, let us call a Lipschitz map $f : X \to Y$ between non-positively curved Riemannian manifolds *tight* if its Lipschitz constant realizes the lower bound

$$\sup_\gamma \frac{\ell_Y(\gamma)}{\ell_X(\gamma)}.$$

Thurston proved that tight homeomorphisms exist between finite volume hyperbolic surfaces in any homotopy class [Thu98], and that the maximal stretch locus is a geodesic lamination. Motivated by finding a good analytic framework for explaining Thurston's results, Daskalopoulos and Uhlenbeck developed a notion of "∞-harmonic" maps to the circle [DU20] and between hyperbolic surfaces [DU22]. See also [GK17], who studied tight maps and canonical maximally stretched geodesic laminations in the context of equivariant maps between $\mathbb{H}^n$.

To a Lipschitz map $f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$, there is an upper semi-continuous function $\hat{L}_f : T^1\Sigma_0 \to \mathbb{R}_{\geq 0}$ measuring the local Lipschitz constant along lines.

Theorem 1.2. *Let $u_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ be tight. The A-invariant part of the maximally stretched locus $\hat{L}_{u_0}^{-1}(\kappa)$ is tangent to a non-empty geodesic lamination on Σ_0. The chain recurrent part λ_0 is contained in the maximal stretch locus of any tight map in the same homotopy class.*

Quasi-minimizing points. The bridge between horocycle orbit closures and tight maps is given by the notion of a quasi-minimizing ray and a theorem of Eberlein and Dal'bo.

As before, let $\Sigma = \mathbb{H}^2/\Gamma$ be a $\mathbb{Z}$-cover of Σ_0:

Definition 1.3. A point $x \in G/\Gamma \cong T^1\Sigma$ is called *quasi-minimizing* if there exists a constant $c \geq 0$ for which

$$d_{G/\Gamma}(a_t x, x) \geq t - c \quad \text{for all } t \geq 0,$$

where $a_t x$ is the point at distance t along the geodesic emanating from the point and direction x.

The above condition implies that the geodesic ray $(a_t x)_{t\geq 0}$ escapes to infinity in G/Γ at the fastest rate possible, up to an additive constant error. Denote by $\mathcal{Q} \subset G/\Gamma$ the set of all quasi-minimizing points.

The following theorem is of fundamental importance:

Theorem (Eberlein [Ebe77],Dal'bo [Dal00])**.** *$\overline{Nx}$ is non-maximal if and only if x is quasi-minimizing.*

This result in fact holds for any Zariski-dense discrete subgroup $\Gamma < G$, with a suitable interpretation of non-maximal (see [LO22] for a generalized formulation in the higher-rank setting).

An immediate corollary is that all non-maximal horocycle orbit closures are contained in $\mathcal{Q}$. Hence analyzing the set $\mathcal{Q}$ is a good first step to understanding non-maximal orbit closures.

The quotient map $\Gamma_0 \to \Gamma_0/\Gamma \cong \mathbb{Z}$ determines a homotopy class of circle maps, and we let λ_0 be the canonical maximal stretch lamination from Theorem 1.2 for tight maps in this homotopy class. We show the following:

Theorem 1.4. *The geodesic ω-limit set of $\mathcal{Q}$ as projected onto Σ_0 is λ_0.*

In other words,

$$\lambda_0 = \{q \in \Sigma_0 : \exists x \in \mathcal{Q} \text{ and } t_j \to \infty \text{ with } P(a_{t_j} x) \to q\},$$

where $P : G/\Gamma \to \Sigma_0$ is the natural projection from $T^1\Sigma$.

As a corollary we show:

Corollary 1.5. *The set of endpoints in S^1 of quasi-minimizing rays has Hausdorff dimension zero.*

Much of the work in this paper is to investigate the subtle relationship between the structure and dynamics of the geodesic lamination λ_0 and the structure and topology of non-maximal horocycle orbit closures. We uncover a number of aspects of this; for example in certain cases it's possible to give a complete orbit closure classification, see Theorem 1.11 below. Before we proceed with that, let us refine our understanding of tight maps and their stretch laminations.

Constructing tight circle maps with prescribed laminations. Let S_0 be an orientable closed topological surface of genus ≥ 2. The theorem of Daskalapoulos-Uhlenbeck provides a tight circle map $S_0 \to S^1$ for any choice of nonzero $\varphi \in H^1(S_0, \mathbb{Z})$ and hyperbolic metric on S_0. We establish the following complementary result, which allows us to specify the maximal-stretch lamination instead of the hyperbolic metric:

Theorem 1.6. *Let $\varphi \in H^1(S_0, \mathbb{Z})$ and let λ be the support of an oriented measured lamination in S_0. Suppose that φ is Poincaré-dual to an oriented multicurve α, such that α intersects λ transversely with positive orientation and $\alpha \cup \lambda$ binds the surface (all complementary components are disks). Then there exists a hyperbolic metric g on S_0 and a tight circle map $f : (S_0, g) \to S^1$ whose homotopy class is φ and whose maximal stretch lamination contains λ.*

In particular, any measured lamination can occur as the stretch lamination of a suitable tight map, and this is used in Theorem 1.11 below to describe a class of $\mathbb{Z}$-covers for which the orbit closures can be classified.

An interval exchange transformation can provide the data $(\varphi, \alpha, \lambda)$ for use in this theorem. Starting with an interval exchange map $T : S^1 \to S^1$, we suspend it to obtain an annulus $[0,1] \times S^1$ with its two boundary circles identified by T. This gives a surface S with a foliation F inherited from the foliation by horizontal lines in the annulus. The lamination λ is then the "straightening" of F with respect to any hyperbolic structure on S, and a circle $\alpha = S^1 \times \{t\}$ represents the cohomology class φ associated to the map $S \to S^1$ obtained from projection to the $[0,1]$ factor in the annulus (see Figure 1).

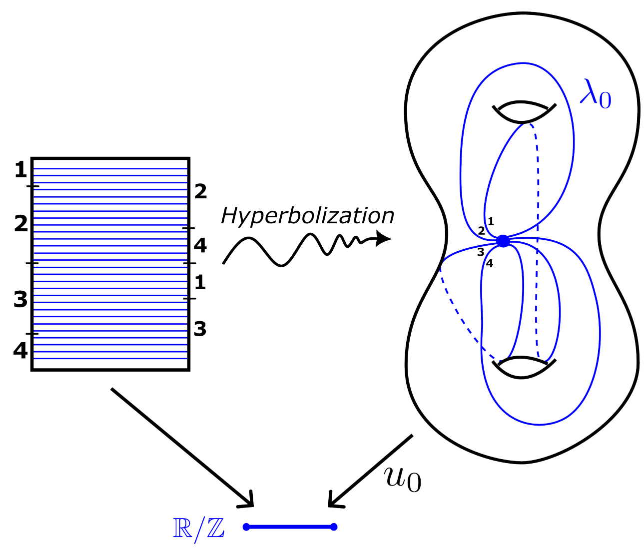

FIGURE 1. Surface constructed from an IET with permutation $\sigma = (1342)$. The corresponding distance minimizing lamination λ_0 lies within the depicted train track.

Borrowing from ideas of Mirzakhani [Mir08], we use recent work of Calderon-Farre [CF21] to "hyperbolize" this construction. Namely, we obtain a *different* hyperbolic structure Σ_0, a measured geodesic lamination λ_0 which is measure-equivalent to λ, and a tight map $u_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ taking the leaves of

λ_0 locally isometrically to the circle. Moreover, the vertical foliation of the annulus by α-parallel curves is converted in Σ_0 to the *orthogeodesic foliation* of λ_0, whose leaves are collapsed by u_0. See Figure 2 for a local picture of this.

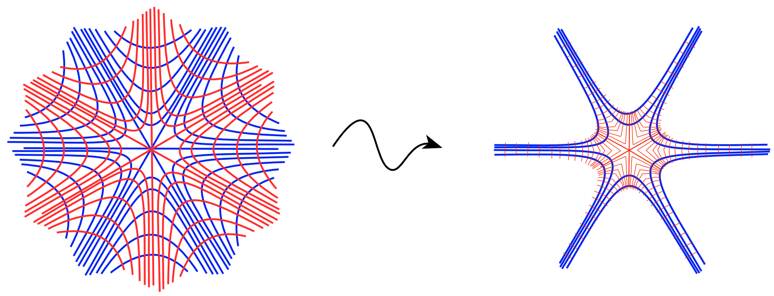

FIGURE 2. Horizontal and vertical singular foliations correspond to geodesic lamination and orthogeodesic singular foliation, respectively.

Uniform Busemann-type functions. Fix $u_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ a tight Lipschitz map, either one constructively provided above or an ∞-harmonic function via the Daskalopolous-Uhlenbeck approach. Lifting u_0 to Σ and rescaling by κ^{-1} one obtains a 1-Lipschitz $\kappa^{-1}\mathbb{Z}$-equivariant map $u : \Sigma \to \mathbb{R}$.

Denoting λ the lift of λ_0 to Σ, we see that λ is contained in the 1-Lipschitz locus of the map u. In particular, all geodesic lines in λ are isometrically embedded copies of $\mathbb{R}$ in Σ. The geodesic lamination $\lambda \subset \Sigma$ contains the "fastest routes" traversing along the $\mathbb{Z}$-cover and the map u indicates how to "collapse" Σ onto these routes.

The surface Σ has two infinite ends, one corresponding to the positive values of u and the other to the negative. Using u one can define a sort of "uniform Busemann function" $\beta_+ : G/\Gamma \to [-\infty, \infty)$ with respect to the positive end, by

$$\beta_+(x) = \lim_{t\to\infty} u(a_t x) - t.$$

This function is N-invariant and upper semicontinuous. Furthermore, a point $x \in G/\Gamma$ is quasi-minimizing and facing the positive end if and only if $\beta_+(x) > -\infty$. The closed N-invariant set

$$\mathcal{H}_+(x) := \beta_+^{-1}\left([\beta_+(x), \infty)\right)$$

can be thought of as a uniform horoball based at the positive end and passing through x. We thus have the following:

Theorem 1.7. *Let Σ be any $\mathbb{Z}$-cover of a compact hyperbolic surface together with a tight Lipschitz map $u : \Sigma \to \mathbb{R}$. All quasi-minimizing points $x \in T^1\Sigma$ facing the positive end of Σ satisfy*

$$\overline{Nx} \subseteq \mathcal{H}_+(x).$$

An analogous statement holds for the negative end of Σ.

Structure of horocycle orbit closures. All horocycle orbit closures satisfy the following two structural properties:

Theorem 1.8. *Let Σ be any $\mathbb{Z}$-cover of a compact hyperbolic surface and let x be any quasi-minimizing point. Then*

(1) *There exists a non-trivial, non-discrete closed subsemigroup Δ_x of $A_{\geq 0} = \{a_t : t \geq 0\}$ under which $\overline{Nx}$ is strictly sub-invariant, that is,*
$$a\overline{Nx} \subsetneq \overline{Nx} \quad \text{for all } a \in \Delta_x \smallsetminus \{e\}.$$
(2) *$\overline{Nx}$ intersects all quasi-minimizing rays escaping through the same end as x. That is, if $y \in T^1\Sigma$ is quasi-minimizing and facing the same end as x then $a_t y \in \overline{Nx}$ for some t.*

An immediate corollary of theorems 1.7 and 1.8(1) is that:

Corollary 1.9. *Every $\mathbb{Z}$-cover of a compact manifold contains uncountably many distinct non-maximal horocycle orbit closures, all of which are not closed N-orbits.*

Remark 1.10. Theorems 1.7 and 1.8 hold for $\mathbb{Z}$-covers of higher dimensional compact hyperbolic manifolds as well[1].

While the full nature of the sub-invariance semigroup Δ_x is still quite mysterious, we show there are examples where $e \in \Delta_x$ is an isolated point and examples where $\Delta_x = A_{\geq 0}$.

We construct a particular class of surfaces having favorable dynamical properties under which a full orbit-closure classification is given:

Theorem 1.11. *If Σ is a $\mathbb{Z}$-cover surface constructed by Theorem 1.6 from a weakly-mixing and minimal IET then all non-maximal horocycle orbit closures in Σ are uniform horoballs. That is, for all $x \in T^1\Sigma$, either Nx is dense or*
$$\overline{Nx} = \mathcal{H}_{\pm}(x).$$

It is worth remarking that in light of Avila-Forni [AF07a], our construction in Theorem 1.6 ensures an abundance of such examples.

Non-rigidity of orbit closures. It is intuitively clear that changing the geometry of Σ_0 could dramatically change the maximal stretch locus of u_0 and hence λ_0. In light of the orbit closure rigidity in the finite volume and geometrically finite settings, and in light of the measure rigidity in the $\mathbb{Z}$-cover setting, it was quite surprising to discover that slight changes to the geometry could dramatically change the topology of non-maximal horocycle orbit closures. To that effect we show:

[1] with suitable adjustments addressing the group of frame rotations M commuting with A in $\mathrm{SO}^+(d,1)$.

Theorem 1.12. *Let S be any $\mathbb{Z}$-cover of an orientable closed surface S_0 of genus ≥ 2. There exist two $\mathbb{Z}$-invariant hyperbolic metrics on S corresponding to discrete groups Γ_1 and Γ_2 for which any two non-maximal orbit closures*

$$\overline{Nx_1} \subsetneq G/\Gamma_1 \quad \textit{and} \quad \overline{Nx_2} \subsetneq G/\Gamma_2$$

are non-homeomorphic. Moreover, these two metrics may be taken to be arbitrarily small deformations of one another.

We remark that the topological obstruction described in this theorem does not arise from the fiber, that is, the orbit closures $\overline{Nx_i}$ are non-homeomorphic even after projecting onto the respective surfaces $\mathbb{H}^2/\Gamma_i$.

References

[AF07a] Artur Avila and Giovanni Forni, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. (2) **165** (2007), no. 2, 637–664. MR 2299743

[AF07b] ______, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. (2) **165** (2007), no. 2, 637–664. MR 2299743

[Bel18] Alexandre Bellis, *On the links between horocyclic and geodesic orbits on geometrically infinite surfaces*, J. Éc. polytech. Math. **5** (2018), 443–454. MR 3808891

[Bur90] Marc Burger, *Horocycle flow on geometrically finite surfaces*, Duke Math. J. **61** (1990), no. 3, 779–803. MR 1084459

[CB88] Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988. MR 964685

[CEG06] R. D. Canary, D. B. A. Epstein, and P. L. Green, *Notes on notes of Thurston*, Fundamentals of hyperbolic geometry: selected expositions, London Math. Soc. Lecture Note Ser., vol. 328, Cambridge Univ. Press, Cambridge, 2006, With a new foreword by Canary, pp. 1–115.

[CF21] Aaron Calderon and James Farre, *Shear-shape cocycles for measured laminations and ergodic theory of the earthquake flow*, 2021.

[Dal00] Françoise Dal'bo, *Topologie du feuilletage fortement stable*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 3, 981–993. MR 1779902

[DS84] S. G. Dani and John Smillie, *Uniform distribution of horocycle orbits for Fuchsian groups*, Duke Math. J. **51** (1984), no. 1, 185–194. MR 744294

[DU20] Georgios Daskalopoulos and Karen Uhlenbeck, *Transverse measures and best lipschitz and least gradient maps*, 2020.

[DU22] ______, *Analytic properties of stretch maps and geodesic laminations*, 2022.

[Ebe77] Patrick Eberlein, *Horocycle flows on certain surfaces without conjugate points*, Trans. Amer. Math. Soc. **233** (1977), 1–36. MR 516501

[FLP12] Albert Fathi, François Laudenbach, and Valentin Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012, Translated from the 1979 French original by Djun M. Kim and Dan Margalit. MR 3053012

[Fur73] Harry Furstenberg, *The unique ergodicity of the horocycle flow*, Recent advances in topological dynamics (Proc. Conf. Topological Dynamics, Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Math., Vol. 318, Springer, Berlin, 1973, pp. 95–115. MR 0393339

[Fur81] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures. MR 603625

[GK17] François Guéritaud and Fanny Kassel, *Maximally stretched laminations on geometrically finite hyperbolic manifolds*, Geom. Topol. **21** (2017), no. 2, 693–840. MR 3626591

[Gla03] Eli Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR 1958753

[GM91] F. P. Gardiner and H. Masur, *Extremal length geometry of Teichmüller space*, Complex Variables Theory Appl. **16** (1991), no. 2-3, 209–237.

[Hed36] Gustav A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. **2** (1936), no. 3, 530–542. MR 1545946

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1920389

[Lev83] Gilbert Levitt, *Foliations and laminations on hyperbolic surfaces*, Topology **22** (1983), no. 2, 119–135. MR 683752

[LL22] Or Landesberg and Elon Lindenstrauss, *On Radon measures invariant under horospherical flows on geometrically infinite quotients*, Int. Math. Res. Not. IMRN (2022), no. 15, 11602–11641. MR 4458560

[LO22] Or Landesberg and Hee Oh, *On denseness of horospheres in higher rank homogeneous spaces*, 2022.

[Mat95] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability. MR 1333890

[Mir08] Maryam Mirzakhani, *Ergodic theory of the earthquake flow*, Int. Math. Res. Not. IMRN (2008), no. 3, Art. ID rnm116, 39. MR 2416997

[NR97] A. Nogueira and D. Rudolph, *Topological weak-mixing of interval exchange maps*, Ergodic Theory Dynam. Systems **17** (1997), no. 5, 1183–1209. MR 1477038

[PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992.

[Rob03] Thomas Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. (N.S.) (2003), no. 95, vi+96. MR 2057305

[Sar04] Omri Sarig, *Invariant Radon measures for horocycle flows on abelian covers*, Invent. Math. **157** (2004), no. 3, 519–551. MR 2092768

[Thu82] W. P. Thurston, *The geometry and topology of 3-manifolds*, 1982, Princeton University Lecture Notes.

[Thu88] ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431.

[Thu98] William P. Thurston, *Minimal stretch maps between hyperbolic surfaces*, 1998.

[ZB04] X. Zhu and F. Bonahon, *The metric space of geodesic laminations on a surface. I*, Geom. Topol. **8** (2004), 539–564.