MINIMIZING LAMINATIONS IN REGULAR COVERS, HOROSPHERICAL ORBIT CLOSURES, AND CIRCLE-VALUED LIPSCHITZ MAPS

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Abstract. We expose a connection between distance minimizing laminations and horospherical orbit closures in $\mathbb{Z}$-covers of compact hyperbolic manifolds. For surfaces, we provide novel constructions of $\mathbb{Z}$-covers with prescribed geometric and dynamical properties, in which an explicit description of all horocycle orbit closures is given. We further show that even the slightest of perturbations to the hyperbolic metric on a $\mathbb{Z}$-cover can lead to drastic topological changes to horocycle orbit closures.

Contents

1. Introduction

In this paper we will study the dynamical behavior of horocyclic and geodesic flows in geometrically infinite hyperbolic manifolds (mostly in dimension 2). These two flows, while geometrically related, exhibit dramatically different dynamical behaviors. Indeed, over a finite area surface Σ, the geodesic flow is "chaotic" and supports a plethora of invariant measures and orbit closures while the horocycle flow is extremely rigid, with all non-periodic orbits dense [Hed36] and equidistributed in $T^1\Sigma$ [Fur73, DS84]. Something similar is true for the geometrically finite case (which in dimension 2 just means finitely-generated fundamental group), see e.g. [Ebe77, Dal00, Bur90, Rob03].

We consider arguably the simplest and most symmetric geometrically infinite setting, that of $\mathbb{Z}$-covers of compact surfaces. Let $G = \mathrm{PSL}_2(\mathbb{R})$ be the group of orientation preserving isometries of real hyperbolic 2-space $\mathbb{H}^2$, let $\Gamma_0 < G$ be a torsion-free uniform lattice and let $\Gamma \lhd \Gamma_0$ with $\Gamma_0/\Gamma \cong \mathbb{Z}$, that is, $\mathbb{H}^2/\Gamma$ is a $\mathbb{Z}$-cover of the compact surface $\mathbb{H}^2/\Gamma_0$.

Let $A = \{a_t : t \in \mathbb{R}\}$ denote the diagonal subgroup generating the geodesic flow on G/Γ, and let N denote the lower unipotent subgroup generating the (stable with respect to A) horocycle flow. We call a horocycle orbit closure $\overline{Nx}$ *non-maximal* if it is not all of G/Γ.

In this setting we:

(1) Study the structure of non-maximal horocycle orbit closures in $\mathbb{Z}$-covers and expose their delicate dependence on the particular geometry of the covered compact surface;
(2) Describe novel constructions of $\mathbb{Z}$-covers with prescribed geometric and dynamical properties; and, in doing so,
(3) Provide the first examples of $\mathbb{Z}$-covers with a full horocycle orbit closure classification, including a description of orbit closures that are neither minimal nor maximal.

While the strongest results in this paper hold solely for $\mathbb{Z}$-covers of compact surfaces, much of the techniques we develop are applicable in greater generality, both to higher-dimensional hyperbolic manifolds as well as maximal horospherical group actions on higher-rank homogeneous spaces.

Remark 1.1. In contrast to the finite area setting, measure rigidity and equidistribution results for the horocycle flow over $\mathbb{Z}$-covers has limited utility. Indeed, non-maximal horocycle orbit closures do not support any locally finite N-invariant measures, since all such measures are AN-quasi-invariant and hence have full support, see [Sar04] as well as [LL22].

At the heart of our analysis lies a connection to a seemingly unrelated geometric optimization problem of independent interest — tight Lipschitz maps to the circle $\mathbb{R}/\mathbb{Z}$.

Tight circle-valued maps. Given a compact hyperbolic surface Σ_0 and a homotopically nontrivial map $f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$, we are interested in geometric properties of maps realizing the minimum Lipschitz constant in the homotopy class of f.

The homotopy class of f is the same data as a cohomology class $\varphi \in H^1(\Sigma_0, \mathbb{Z})$ recording the degree of the restriction to loops in Σ_0. Evidently, the minimum Lipschitz constant for circle valued maps representing φ is bounded below by

$$\kappa = \sup_{\gamma} \frac{|\varphi(\gamma)|}{\ell(\gamma)},$$

where the sup runs over all geodesic loops γ and $\ell(\gamma)$ is the length in Σ_0. A map in $[f]$ is called *tight* if its Lipschitz constant is equal to κ.

Theorem (Daskalopoulos-Uhlenbeck [DU20]). *There exists a tight map in every non-trivial homotopy class $[f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}]$, whose maximal stretch locus is a geodesic lamination.*

More generally, let us call a Lipschitz map $f : X \to Y$ between non-positively curved Riemannian manifolds *tight* if its Lipschitz constant realizes the lower bound

$$\sup_{\gamma} \frac{\ell_Y(\gamma)}{\ell_X(\gamma)}.$$

Thurston proved that tight homeomorphisms exist between finite volume hyperbolic surfaces in any homotopy class [Thu98b], and that the maximal stretch locus is a geodesic lamination. Motivated by finding a good analytic framework for explaining Thurston's results, Daskalopoulos and Uhlenbeck developed a notion of "∞-harmonic" maps to the circle [DU20] and between hyperbolic surfaces [DU22]. See also [GK17], who studied tight maps and canonical maximally stretched geodesic laminations in the context of equivariant maps between $\mathbb{H}^n$.

To a Lipschitz map $f : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$, there is an upper semi-continuous function $L_f : T^1\Sigma_0 \to \mathbb{R}_{\geq 0}$ measuring the local Lipschitz constant along lines.

Theorem 1.2. *Let $\tau_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ be tight. The A-invariant part of the maximally stretched locus $L_{\tau_0}^{-1}(\kappa)$ is tangent to a non-empty geodesic lamination on Σ_0. The chain recurrent part λ_0 is contained in the maximal stretch locus of any tight map in the same homotopy class.*

Quasi-minimizing points. The bridge between horocycle orbit closures and tight maps is given by the notion of a quasi-minimizing ray and a theorem of Eberlein and Dal'bo.

As before, let $\Sigma = \mathbb{H}^2/\Gamma$ be a $\mathbb{Z}$-cover of Σ_0:

Definition 1.3. A point $x \in G/\Gamma \cong T^1\Sigma$ is called *quasi-minimizing* if there exists a constant $c \geq 0$ for which

$$d_{G/\Gamma}(a_t x, x) \geq t - c \quad \text{for all } t \geq 0,$$

where $a_t x$ is the point at distance t along the geodesic emanating from the point and direction x.

The above condition implies that the geodesic ray $(a_t x)_{t \geq 0}$ escapes to infinity in G/Γ at the fastest rate possible, up to an additive constant error. Denote by $\mathcal{Q} \subset G/\Gamma$ the set of all quasi-minimizing points.

The following theorem is of fundamental importance:

Theorem (Eberlein [Ebe77],Dal'bo [Dal00]). *$\overline{Nx}$ is non-maximal if and only if x is quasi-minimizing.*

This result in fact holds for any Zariski-dense discrete subgroup $\Gamma < G$, with a suitable interpretation of non-maximal (see [LO22] for a generalized formulation in the higher-rank setting).

An immediate corollary is that all non-maximal horocycle orbit closures are contained in $\mathcal{Q}$. Hence analyzing the set $\mathcal{Q}$ is a good first step to understanding non-maximal orbit closures.

The quotient map $\Gamma_0 \to \Gamma_0/\Gamma \cong \mathbb{Z}$ determines a homotopy class of circle maps, and we let λ_0 be the canonical maximal stretch lamination from Theorem 1.2 for tight maps in this homotopy class. We show the following:

Theorem 1.4. *The geodesic ω-limit set of $\mathcal{Q}$ as projected onto Σ_0 is λ_0.*

In other words,

$$\lambda_0 = \{q \in \Sigma_0 : \exists x \in \mathcal{Q} \text{ and } t_j \to \infty \text{ with } P(a_{t_j}x) \to q\},$$

where $P : G/\Gamma \to \Sigma_0$ is the natural projection from $T^1\Sigma$.

As a corollary we show:

Corollary 1.5. *The set of endpoints in S^1 of quasi-minimizing rays has Hausdorff dimension zero.*

Remark 1.6. The cardinality of this set of endpoints can be either countable or uncountable, as follows from the examples provided in Section 5.

Much of the work in this paper is to investigate the subtle relationship between the structure and dynamics of the geodesic lamination λ_0 and the structure and topology of non-maximal horocycle orbit closures. We uncover a number of aspects of this; for example in certain cases it's possible to give a complete orbit closure classification, see Theorem 1.12 below. Before we proceed with that, let us refine our understanding of tight maps and their stretch laminations.

Constructing tight circle maps with prescribed laminations. Let S_0 be an orientable closed topological surface of genus ≥ 2. The theorem of Daskalapoulos-Uhlenbeck provides a tight circle map $S_0 \to S^1$ for any choice of nonzero $\varphi \in H^1(S_0, \mathbb{Z})$ and hyperbolic metric on S_0. We establish the following complementary result, which allows us to specify the maximal-stretch lamination instead of the hyperbolic metric:

Theorem 1.7. *Let $\varphi \in H^1(S_0, \mathbb{Z})$ and let λ be the support of an oriented measured lamination in S_0. Suppose that φ is Poincaré-dual to an oriented multicurve α, such that α intersects λ transversely with positive orientation and $\alpha \cup \lambda$ binds the surface (all complementary components are compact disks). Then there exists a hyperbolic metric g on S_0 and a tight circle map $f : (S_0, g) \to S^1$ whose homotopy class is φ and whose maximal stretch lamination is equal to λ.*

In particular, any measured lamination can occur as the stretch lamination of a suitable tight map, and this is used in Theorem 1.12 below to describe a class of $\mathbb{Z}$-covers for which the horocycle orbit closures can be classified.

An interval exchange transformation can provide the data $(\varphi, \alpha, \lambda)$ for use in Theorem 1.7. Starting with an interval exchange map $T : S^1 \to S^1$, we suspend it to obtain an annulus $[0, 1] \times S^1$ with its two boundary circles identified by T. This gives a surface S with a foliation F inherited from the foliation by horizontal lines in the annulus. The lamination λ is then the "straightening" of F with respect to any hyperbolic structure on S, and a circle $\alpha = \{t\} \times S^1$ represents the cohomology class φ associated to the map $S \to S^1$ obtained from projection to the $[0, 1]$ factor in the annulus (see Figure 1).

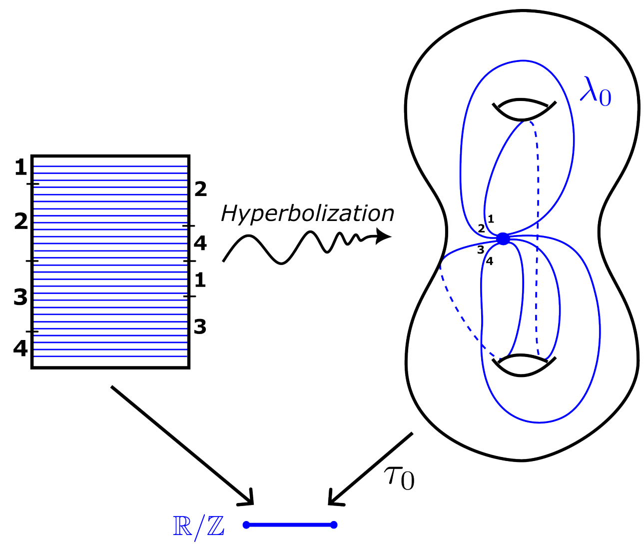

FIGURE 1. Surface constructed from an IET with permutation $\sigma = (1342)$. The corresponding distance minimizing lamination λ_0 lies within the depicted train track.

Borrowing from ideas of Mirzakhani [Mir08], we use recent work of Calderon-Farre [CF21] to "hyperbolize" this construction. Namely, we obtain a *different* hyperbolic structure Σ_0, a measured geodesic lamination λ_0 which is measure-equivalent to λ, and a tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ taking the leaves of λ_0 locally isometrically to the circle. Moreover, the vertical foliation of the annulus by α-parallel curves is converted in Σ_0 to the *orthogeodesic foliation* of λ_0, whose leaves are collapsed by τ_0. See Figure 2 for a local picture of this.

Uniform Busemann-type functions. Fix $\tau_0 : \Sigma_0 \to \mathbb{R}/\mathbb{Z}$ a tight Lipschitz map, either one constructively provided above or an ∞-harmonic function via the Daskalopolous-Uhlenbeck approach. Lifting τ_0 to Σ and rescaling by κ^{-1} one obtains a 1-Lipschitz $\kappa^{-1}\mathbb{Z}$-equivariant map $\tau : \Sigma \to \mathbb{R}$.

Denoting λ the lift of λ_0 to Σ, we see that λ is contained in the 1-Lipschitz locus of the map τ. In particular, all geodesic lines in λ are isometrically

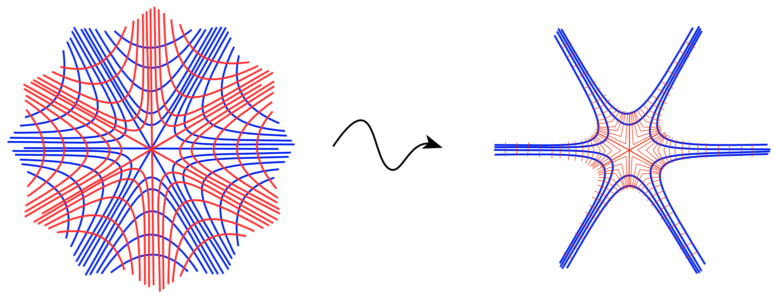

FIGURE 2. Horizontal and vertical singular foliations correspond to geodesic lamination and orthogeodesic singular foliation, respectively.

embedded copies of $\mathbb{R}$ in Σ. The geodesic lamination $\lambda \subset \Sigma$ contains the "fastest routes" traversing along the $\mathbb{Z}$-cover and the map τ indicates how to "collapse" Σ onto these routes.

The surface Σ has two infinite ends, one corresponding to the positive values of τ and the other to the negative. Using τ one can define a sort of "uniform Busemann function" $\beta_+ : G/\Gamma \to [-\infty, \infty)$ with respect to the positive end, by

$$\beta_+(x) = \lim_{t\to\infty} \tau(a_t x) - t.$$

This function is N-invariant and upper semicontinuous. Furthermore, a point $x \in G/\Gamma$ is quasi-minimizing and facing the positive end if and only if $\beta_+(x) > -\infty$. The closed N-invariant set

$$\mathcal{H}_+(x) := \beta_+^{-1}\left([\beta_+(x), \infty)\right)$$

can be thought of as a uniform horoball based at the positive end and passing through x. We thus have the following:

Theorem 1.8. *Let Σ be any $\mathbb{Z}$-cover of a compact hyperbolic surface together with a tight Lipschitz map $\tau : \Sigma \to \mathbb{R}$. All quasi-minimizing points $x \in T^1\Sigma$ facing the positive end of Σ satisfy*

$$\overline{Nx} \subsetneq \mathcal{H}_+(x).$$

An analogous statement holds for the negative end of Σ.

Structure of horocycle orbit closures. All horocycle orbit closures satisfy the following two structural properties:

Theorem 1.9. *Let Σ be any $\mathbb{Z}$-cover of a compact hyperbolic surface and let $x \in T^1\Sigma$ be any quasi-minimizing point. Then*

(1) There exists a non-trivial, non-discrete closed subsemigroup Δ_x of $A_{\geq 0} = \{a_t : t \geq 0\}$ under which $\overline{Nx}$ is strictly sub-invariant, that is,

$$a\overline{Nx} \subsetneq \overline{Nx} \quad \textit{for all } a \in \Delta_x \smallsetminus \{e\}.$$

(2) $\overline{Nx}$ intersects all quasi-minimizing rays escaping through the same end as x. That is, if $y \in T^1\Sigma$ is quasi-minimizing and facing the same end as x then $a_t y \in \overline{Nx}$ for some $t \geq 0$.

While the full nature of the semigroup Δ_x is a bit mysterious, we develop both geometric and algebraic tools to study it; see Section 7. We show there are examples where $e \in \Delta_x$ is an isolated point and examples where $\Delta_x = A_{\geq 0}$ (Theorem 9.3 and Corollary 7.23). An immediate corollary of Theorems 1.8 and 1.9(1) is that:

Corollary 1.10. *Every $\mathbb{Z}$-cover of a compact surface contains uncountably many distinct non-maximal horocycle orbit closures, all of which are not closed N-orbits. Moreover, on such surfaces no horocycle orbit closure is minimal.*

Remark 1.11. Theorems 1.8 and 1.9 hold for $\mathbb{Z}$-covers of higher dimensional compact hyperbolic manifolds as well[1]. In the case of $d = 2$, partial results, applicable also to this setting, were obtained in [Bel18].

We construct a particular class of surfaces having favorable dynamical properties under which a full orbit-closure classification is given:

Theorem 1.12. *If Σ is a $\mathbb{Z}$-cover surface constructed by Theorem 1.7 from a weakly-mixing and minimal IET then all non-maximal horocycle orbit closures in Σ are uniform horoballs. That is, for all $x \in T^1\Sigma$, either Nx is dense or*

$$\overline{Nx} = \mathcal{H}_{\pm}(x).$$

It is worth remarking that in light of Avila-Forni [AF07a], our construction in Theorem 1.7 ensures an abundance of such examples.

Non-rigidity of orbit closures. It is intuitively clear that changing the geometry of Σ_0 could dramatically change the maximal stretch locus of τ_0 and hence λ_0. In light of the orbit closure rigidity in the finite volume and geometrically finite settings, and in light of the measure rigidity in the $\mathbb{Z}$-cover setting, it was quite surprising to discover that slight changes to the geometry could dramatically change the topology of non-maximal horocycle orbit closures. To that effect we show:

Theorem 1.13. *Let S be any $\mathbb{Z}$-cover of an orientable closed surface S_0 of genus ≥ 2. There exist two $\mathbb{Z}$-invariant hyperbolic metrics on S corresponding to discrete groups Γ_1 and Γ_2 for which any two non-maximal orbit closures*

$$\overline{Nx_1} \subsetneq G/\Gamma_1 \quad \textit{and} \quad \overline{Nx_2} \subsetneq G/\Gamma_2$$

are non-homeomorphic. Moreover, these two metrics may be taken to be arbitrarily small deformations of one another.

[1] with suitable adjustments addressing the group of frame rotations M commuting with A in $\mathrm{SO}^+(d, 1)$.

We remark that the topological obstruction described in this theorem does not arise from the fiber, that is, the orbit closures $\overline{Nx_i}$ are non-homeomorphic even after projecting onto the respective surfaces $\mathbb{H}^2/\Gamma_i$.

1.1. **Questions and context.** Many questions remain open. The techniques we develop are applicable to studying non-maximal horospherical orbit closures in other geometrically infinite hyperbolic manifolds, but the input to our analysis requires some dynamical, geometrical, and topological information about the quasi-minimizing locus.

There is a natural class of examples in dimension 3 consisting of $\mathbb{Z}$-covers of closed manifolds that fiber over the circle. These examples are geometrically infinite but have finitely generated fundamental group. Although they have been well studied, e.g., [Thu88, Thu98a, CT07, Min10, BCM12], the authors do not know of any hyperbolic 3-manifold fibering over the circle, where $\mathcal{Q}_\omega$ has been identified in the corresponding $\mathbb{Z}$-cover.

A natural lamination to consider in such a manifold is the geodesic tightening of the orbits of its pseudo-Anosov suspension flow. One of the main results claimed in [LM18] seems to imply that this lamination is in fact $\mathcal{Q}_\omega$; unfortunately, there are errors in the proof that are not easily fixable.

Even for $\mathbb{Z}$-covers in dimension 2 we do not yet have a complete classification of horocycle orbit closures, and our results so far suggest that the possibilities are rich.

It would also be interesting to understand quasi-periodic surfaces that are not covers of compact surfaces, as well as regular covers with other deck groups.

1.2. **Organization of paper.** In section 2 we present our notation and a short background on horospherical orbit closures and geodesic laminations. Section 3 addresses tight Lipschitz maps and their maximal stretch loci, containing proof of Theorems 1.2 & 1.4. A proof of Corollary 1.5 is given in section 4. In section 5 we give our construction of $\mathbb{Z}$-covers from IETs and prove Theorem 1.7. Section 6 discusses "uniform Busemann" functions and their connection to horospherical orbit closures. In sections 7&8 we study the structure of horospherical orbit closures, concluding in particular the content of Theorem 1.9. In the final section, 9, we fully describe the horocycle orbit closures in $\mathbb{Z}$-covers constructed by weakly-mixing and minimal IETs, showing Theorem 1.8 and concluding Theorem 1.13.

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2. Preliminaries

2.1. **Setting and Notation.** We fix the following notations throughout this paper. Set $G_d = \mathrm{SO}^+(d,1)$ the group of orientation preserving isometries of real hyperbolic d-space, $\mathbb{H}^d$. In the case of $d = 2$ we will omit the subscript and denote $G = G_2$. Equip G_d with a right G_d-invariant Riemannian metric.

The group G_d acts freely and transitively on $F\mathbb{H}^d$, the frame bundle of $\mathbb{H}^d$, and hence we can identify $G_d \cong F\mathbb{H}^d$. Consider the following subgroups with their corresponding left-multiplication action on G_d:

- $K \cong \mathrm{SO}(d)$ — stabilizer subgroup of a point $o \in \mathbb{H}^d$, inducing $K\backslash G_d \cong \mathbb{H}^d$.
- $A = \{a_t : t \in \mathbb{R}\}$ — one-parameter subgroup of diagonal elements corresponding to the geodesic frame flow on $F\mathbb{H}^d$.
- $M = Z_K(A) \cong \mathrm{SO}(d-1)$ — the centralizer of A in K, corresponding to rotations of frames in $F\mathbb{H}^d$ about the geodesic flow direction. We may hence identify $M\backslash G_d$ with $T^1\mathbb{H}^d$, the unit tangent bundle of $\mathbb{H}^d$.
- $N = \{n \in G_d : a_t n a_{-t} \to e \text{ as } t \to +\infty\} \cong \mathbb{R}^{d-1}$ — the contracted horospherical subgroup corresponding to a flow along the stable foliation for the geodesic flow.
- $U = \{u \in G_d : a_{-t} n a_t \to e \text{ as } t \to +\infty\} \cong \mathbb{R}^{d-1}$ — the (opposite) expanded horospherical subgroup.

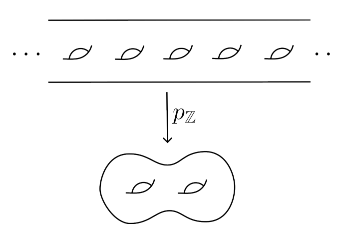
We may assume the left-G_d-invariant metric is also right-K-invariant. Given a discrete subgroup $\Gamma' < G_d$, we denote by $d_{G_d/\Gamma'}$ the induced metric on G_d/Γ' (we will typically omit the subscript and refer to d).

Let $\Gamma_0 < G_d$ be a uniform torsion-free lattice and let $\Gamma \lhd \Gamma_0$ be a normal subgroup with $\Gamma_0/\Gamma \cong \mathbb{Z}$. We denote $\Sigma_0 = K\backslash G_d/\Gamma_0$, a compact hyperbolic d-manifold and $\Sigma = K\backslash G_d/\Gamma$ its respective $\mathbb{Z}$-cover. Their unit tangent bundles are

$$T^1\Sigma = M\backslash G_d/\Gamma \quad \text{and} \quad T^1\Sigma_0 = M\backslash G_d/\Gamma_0$$

We have the following commuting diagram of projections:

$$\begin{tikzcd} G_d/\Gamma \arrow[r, "p_M"] \arrow[d, "p_{\mathbb{Z}}"'] \arrow[rr, bend left, "p_K"] & T^1\Sigma \arrow[r, "p_K"] \arrow[d, "p_{\mathbb{Z}}"'] & \Sigma \arrow[d, "p_{\mathbb{Z}}"] \\ G_d/\Gamma_0 \arrow[r, "p_M"] \arrow[rr, bend right, "p_K"'] & T^1\Sigma_0 \arrow[r, "p_K"] & \Sigma_0 \end{tikzcd} \tag{2.1}$$



where $p_{\mathbb{Z}}(\cdot\,\Gamma) = \cdot\,\Gamma_0$ are the cover maps, and where $p_M(x) = Mx$ and $p_K(x) = Kx$ are the bundle projections.

2.2. **Horospherical orbit closures.** As described in the introduction, whenever $\Gamma' < G_d$ is a lattice the horospherical flow on G_d/Γ' is extremely rigid.

In this setting all horospheres are either dense in G_d/Γ' or periodic (bounding a cusp). This extreme rigidity is understood today as part of a much broader phenomenon of unipotent group action rigidity on finite volume homogeneous spaces, as follows from the works of Hedlund, Furstenberg, Veech, Dani, Margulis, and ultimately Ratner [Rat91].

Consider $\Gamma' < G_d$ a general discrete subgroup, not necessarily a lattice. Let $\Lambda_{\Gamma'} \subseteq S^{d-1} = \partial\mathbb{H}^d$ be the limit set of Γ', that is, the unique closed Γ'-minimal subset of S^{d-1}. Consider the subspace

$$\mathcal{E}_{\Gamma'} = \{g\Gamma' : g^+ \in \Lambda_{\Gamma'}\} \subset G/\Gamma',$$

where $g^+ \in S^{d-1}$ is the forward endpoint of the geodesic in $F\mathbb{H}^d$ emanating from the frame $g \in G_d$. Algebraically, S^{d-1} is identified with the Furstenberg boundary of G_d, i.e. $S^{d-1} \cong P\backslash G_d$ where $P = MAN$, in which case $g^+ = Pg$. The set $\mathcal{E}_{\Gamma'}$ is the non-wandering set for the N-action, see e.g. [Dal11].

Definition 2.1. A limit point $\xi \in S^{d-1}$ is called horospherical w.r.t. Γ' if $\Gamma'.o$ intersects all horoballs based at ξ for some (and any) point $o \in \mathbb{H}^d$.

Recall the notion of a quasi-minimizing point from definition 1.3. We have the following:

Lemma 2.2. *The point $g\Gamma'$ is quasi-minimizing if and only if g^+ is non horospherical.*

Proof. For all $C \geq 0$ the family of balls $B(p_K(a_t g), t - C)$ in $\mathbb{H}^d$ of radius $t - C$ around $p_K(a_t g)$ converges to a horoball tangent to g^+ as $t \to \infty$. This horoball is disjoint from $\Gamma'.p_K(g)$ if and only if g is quasi-minimizing with corresponding constant C. □

A limit point g^+ is called conical if the geodesic ray $(a_t g\Gamma')_{t\geq 0}$ emanating from $g\Gamma'$ recurs infinitely often to a compact set in G_d/Γ'. The lemma above immediately implies that all conical limit points are horospherical (with the converse clearly false in general). In dimensions $d = 2, 3$ all parabolic fixed points are non-horospherical. Surprisingly, this is not true for $d \geq 4$ as demonstrated in [Apa85, Sta97].

The following theorem gives a necessary and sufficient condition for denseness of a horosphere in $\mathcal{E}_{\Gamma'}$:

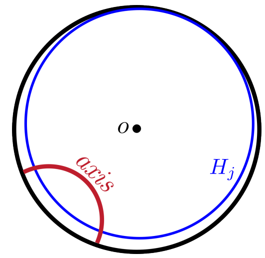
Theorem ([Ebe77],[Dal00],[MS19, Corollary 3.2])**.** *If $\Gamma' < G_d$ is Zariski-dense then $\overline{Ng\Gamma'} = \mathcal{E}_{\Gamma'}$ if and only if g^+ is horospherical.*

As this result is a fundamental ingredient in our analysis, we provide a short sketch of its proof. For technical simplicity we will restrict our discussion to the unit tangent bundle $M\backslash G_d/\Gamma'$. We also refer the reader to [Dal11, Theorem 3.1] for a different flavored elementary proof.

Sketch of proof. Assume g^+ is horospherical, therefore $p_K(g)\Gamma'$ enters arbitrarily deep inside the horoball in $\mathbb{H}^d$ based at g^+ and passing through $o = p_K(g)$. Equivalently, there is a sequence of Γ'-copies $H_j = p_K(Ng)\gamma_j$

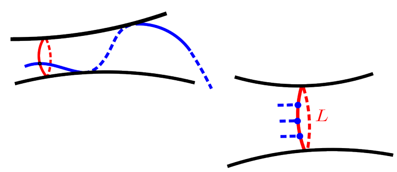
of the horosphere emanating from g whose distance from $p_K(g)$ tends to infinity. This implies that H_j converges uniformly to $\partial\mathbb{H}^d$, e.g. in the ball model of hyperbolic d-space.

The axis of any loxodromic element in Γ' intersects $\partial\mathbb{H}^d$ at a right angle, implying the horospheres H_j tend to do the same. All such axes are compact in $\mathbb{H}^d/\Gamma'$ hence the sequence of intersection points of any such axis with the horosphere $p_K(Ng\Gamma')$ has an accumulation point with angle tending to $\pi/2$. In other words, $\overline{Ng\Gamma'}$ contains frames on any closed geodesic in G_d/Γ'[2].



The length spectrum of any Zariski-dense discrete group is "non-arithmetic" [Dal99, Kim06], that is, the group generated by the set of lengths of all closed geodesics in G_d/Γ' is dense in $\mathbb{R}$. Consider some closed geodesic L of length l_1 in G_d/Γ', and let $\varepsilon > 0$. By non-arithmeticity, there exists a closed geodesic in G_d/Γ' of length l_2 satisfying that $\langle l_1, l_2\rangle$ has covolume$\leq \varepsilon$ in $\mathbb{R}$. Let $y \in \overline{Ng\Gamma'}$ be some point on this second closed geodesic of length τ_2.

The limit point y^+ is clearly conical and hence horospherical, implying that there exists a point $z \in \overline{Ny} \subseteq \overline{Ng\Gamma'}$ contained in L. The N-orbit closure of y is a_{τ_2}-invariant up to a rotation in M, therefore the set of intersection points of $\overline{Ny}$ with L is ε-dense in L (again up to rotation). As L and ε were arbitrary, we conclude that $\overline{MNg\Gamma'}$ contains all closed geodesics and hence $\mathcal{E}_{\Gamma'} \subset \overline{MNg\Gamma'}$. We leave the remaining details to the reader. □



Remark 2.3.

(1) To show denseness in the frame bundle one uses a stronger form of non-arithmeticity, taking into account the M components of the loxodromic elements in Γ', see [GR07] and [MS19].
(2) For a higher rank generalization of the notion of a horospherical limit point and the above theorem regarding orbit closures, see [LO22].

A group Γ' is called geometrically finite if all of its limit points in $\Lambda_{\Gamma'}$ are either conical (and hence horospherical) or bounded parabolic, see e.g. [Rat06, Theorem 12.4.5] or [Bow93] for other quivalent definitions and a more complete discussion. In such a case all horospheres are either dense in $\mathcal{E}_{\Gamma'}$ or closed.

It is worth noting that a discrete group $\Gamma' < G_d$ always has non-horospherical limit points unless Γ' is convex cocompact (or cocompact). Indeed, if $\mathcal{D}_z(\Gamma')$ denotes the Dirichlet domain of Γ' in $\mathbb{H}^d$ based at z, then the set $\overline{\mathcal{D}_z(\Gamma')}\cap\Lambda_{\Gamma'}$ consists of non-horospherical limit points, whenever non-empty (this set is empty only when Γ' is convex cocompact), see e.g. [Dal11, Corollary 4.10].

[2]This argument goes back to Hedlund [Hed36, Theorem 2.3].

In this paper we consider normal subgroups of lattices $\Gamma \lhd \Gamma_0$, which satisfy $\Lambda_\Gamma = \Lambda_{\Gamma_0} = S^{d-1}$ (see e.g. [Rat06, Theorem 12.2.14]) and consequently $\mathcal{E}_\Gamma = G_d/\Gamma$. Therefore the theorem above may be applied, as done in the introduction, to show that a point has a non-dense horospherical orbit in G_d/Γ if and only if it is quasi-minimizing.

We note that thus far, apart from $\mathcal{E}_\Gamma$ and closed MN-orbits, no other horospherical orbit closures have been fully described in the literature, see e.g. [Bel18, Mat16, Kul04, DS00, Apa85, Sta97, CM10, Led97][3] for relevant constructions and partial results in this direction.

2.3. **Bruhat decomposition for geometers.** Let $g, h \in G_d$ be two points which we think of as frames in $\mathbb{H}^d$. Figure 3 indicates a way of moving from g to h using horospherical and diagonal flows:

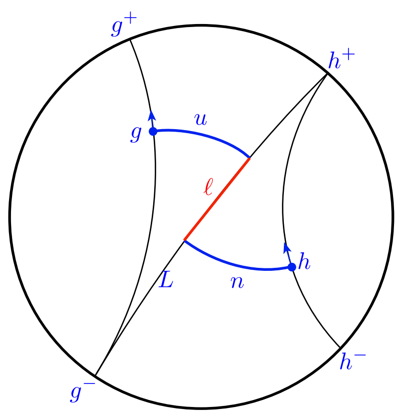


FIGURE 3. We slide g to h by applying an unstable horospherical element $u \in U$, a slide $\ell \in MA$ along L, and a stable horospherical element $n \in N$.

Recall that N preserves the forward-endpoint g^+ of a frame g, and U preserves the backward endpoint g^-.

Supposing that $g^- \neq h^+$, let L denote the directed geodesic joining g^- to h^+. There is a unique element $u \in U$ that takes the A-orbit of g to L, and then a unique element n taking L to the A-orbit of h. The discrepancy between ug and $n^{-1}h$ is given by an element ℓ of MA, describing translation along and rotation around L. This gives us a decomposition

$$h = n\ell ug$$

or in other words we express hg^{-1} in the product $NMAU$.

To deal with the case $g^- = h^+$, we fix ω an involution that reverses the first vector of the baseframe, or equivalently satisfies $a_{-t} = \omega a_t \omega$ (a representative of the generator of the Weyl group). Thus $(\omega g)^+ = g^- = h^+$, which means h can be written as $n\ell\omega g$, where $n\ell \in NMA$. What we have

[3] Note that Prop. 3 in [Led97, Led98] is false, see [Bel18] and §9 for counterexamples.

seen is how to write hg^{-1}, or any element of G_d, in one of two forms. This gives the *Bruhat decomposition*[4],

$$G_d = NMAU \cup NMA\omega. \tag{2.2}$$

We further record the fact that the set $NMAU \subset G_d$ is open, and Zariski-open. The multiplication map $N \times MA \times U \to NMAU$ is a diffeomorphism (see e.g. [Kna02, Lemma 6.44]) and an isomorphism of varieties (see e.g. [Spr09, Lemma 8.3.6(ii)]).

2.4. **Geodesic laminations.** Let M be a hyperbolic d-manifold.

Definition 2.4. A *geodesic lamination* $\lambda \subset M$ is a non-empty closed subset of M that is a disjoint union of complete simple geodesics called its *leaves*. Additionally, we require that λ is equipped with a continuous local product structure, i.e., we have the data of a covering $\{U_i\}$ of a neighborhood of λ in M and charts $\phi_i : U_i \to \mathbb{R} \times \mathbb{R}^{d-1}$ where $\phi_i(\lambda \cap U_i) = \mathbb{R} \times B$ and $B \subset \mathbb{R}^{d-1}$. Moreover, the transitions are required to be of the form $\phi_{ij}(x, y) = (f_{ij}(x, y), g_{ij}(y))$, for $y \in B$.

In the language of homogeneous spaces $\lambda \subset K\backslash G_d/\Gamma$ is a geodesic lamination if it is a projection of a closed A-invariant set in $M\backslash G_d/\Gamma \cong T^1\Sigma$ where each fiber contains exactly two points.

Although we will be working with geometrically infinite manifolds, in this paper, they typically cover a closed (or finite volume) manifold M_0, and our geodesic laminations are invariant under the deck group of the cover, hence define a geodesic lamination $\lambda_0 \subset M_0$. As such, we now specialize to the case that M is closed.

A lamination is *connected* if it is connected as a topological space and *minimal* if it has no non-trivial sublaminations, i.e. proper, non-empty subsets which are themselves laminations. Alternatively, a lamination is minimal if each of its leaves is dense. A lamination is *orientable* if it admits a continuous orientation.

When $d = 2$, there is a good deal of additional structure for geodesic laminations. So assuming that $M = S$, a closed oriented surface of genus $g \geq 2$, the structure theory for geodesic laminations tells us that the local product structure is unique, and λ can be decomposed uniquely as a union of minimal sublaminations $\lambda_1, ..., \lambda_m$ and isolated leaves $\ell_1, ..., \ell_k$ that accumulate (or spiral) onto the minimal components with bounds $0 \leq k \leq 6g - 6$ and $1 \leq m \leq 3g - 3$.

An argument using the Poincaré-Hopf index formula for a line field on S constructed from a geodesic lamination proves that the area of λ is always 0, hence the geodesic completion of the complement of λ in S is a (possibly non-compact, disconnected) negatively curved surface with totally geodesic boundary and area equal to the area of M. We sometimes refer to the

[4]The classical form of the Bruhat decomposition is $G_d = P\omega P \cup P$ with $P = MAN$. Equation (2.2) is given by multiplication by ω.

surface obtained by geodesic completion of the complement of λ with respect to some negatively curved metric as being obtained by *cutting open along* λ. A lamination is *filling* if the surface obtained by cutting open along λ is a union of ideal polygons. A lamination is *maximal* if it is not a proper sublamination of any other geodesic lamination; equivalently, the cut surface is a union of $4g-4$ ideal triangles.

Geodesic laminations were introduced by Bill Thurston in [Thu82, Thu88] and have become an important tool in various problems in Teichmüller theory, low dimensional geometry topology and dynamics. A comprehensive introduction to the structure theory for geodesic laminations in dimension 2 can be found in [CB88]; see also [CEG06, Chapter 4].

We will later make use of the following notion of recurrence in a geodesic lamination:

Definition 2.5 (see [Thu98b])**.** A point p in a geodesic lamination λ is called *chain recurrent* if for any ε there exists an ε-trajectory of λ through p. That is, there exists a closed unit speed path through p such that for any interval of length 1 on the path there is an interval of length 1 on some leaf of λ such that the two paths remain within ε distance from one another in the C^1 sense.

If one point on a leaf $\ell \subset \lambda$ is chain recurrent, then every point of ℓ is chain recurrent, and chain recurrence is a clearly a closed condition. Thus the subset $\lambda^{\mathrm{cr}} \subset \lambda$ of chain recurrent points is a sublamination, called the *chain recurrent part* of λ.

3. Tight Lipschitz functions and maximal stretch laminations

Recall that a Lipschitz map $f : \Sigma_0 \to \mathbb{R}/c'\mathbb{Z}$ representing a cohomology class $\varphi \in H^1(\Sigma_0, c'\mathbb{Z})$ is called *tight* if the Lipschitz constant of f is equal to

$$\kappa = \sup_\gamma \frac{|\varphi(\gamma)|}{\ell(\gamma)},$$

where the supremum is taken over geodesic curves γ in Σ_0 and we think of $|\varphi(\gamma)|$ as the geodesic length of $f(\gamma)$ in the circle $\mathbb{R}/c'\mathbb{Z}$. Using either the results of Daskalopoulos-Uhlenbeck [DU20] or Guéritaud-Kassel [GK17], we know that given a cohomology class $\varphi \in H^1(\Sigma_0, c'\mathbb{Z})$, there is a tight Lipschitz map $f : \Sigma_0 \to \mathbb{R}/c'\mathbb{Z}$ inducing φ on π_1.

We will be especially interested in 1-Lipschitz tight maps. Let $c = \kappa^{-1}c'$ so that post-composition of f with an affine map of the circle $\mathbb{R}/c'\mathbb{Z} \to \mathbb{R}/c\mathbb{Z}$ yields a new map

$$\tau_0 : \Sigma_0 \to c\mathbb{Z}$$

representing the cohomology class $\kappa^{-1}\varphi$ that is 1-Lipschitz and tight. In this section, we establish some properties for arbitrary tight 1-Lipschitz maps (not just those coming from any particular construction).

Define a function

$$L : G_d/\Gamma_0 \to \mathbb{R}$$

that records the local Lipschitz constant of τ_0 restricted to the geodesic tangent to a given frame. That is, for $x \in G_d/\Gamma_0$ we have

$$L(x) = \lim_{r \searrow 0} \sup_{s,t \in (-r,r)} \frac{|\tau_0(p_K(a_s x)) - \tau_0(p_K(a_t x))|}{|s-t|}. \tag{3.1}$$

Clearly, L is M-invariant and takes the same value on antipodal frames, i.e., if x and y are related by $p_M(x) = \pm p_M(y) \in M\backslash G_d/\Gamma_0 \cong T^1\Sigma_0$, then $L(x) = L(y)$.

It is not difficult to see that L is the pointwise limit of a decreasing sequence of continuous functions, and is therefore upper semi-continuous. Moreover, $L(x)$ is bounded above pointwise by the usual local Lipschitz constant of τ_0 at $p_K(x)$.

Proposition 3.1. *Let $\tau_0 : \Sigma_0$ be 1-Lipschitz and tight, and let $\mathcal{L}_0$ be the A-invariant part of $L^{-1}(1)$. Then $p_K(\mathcal{L}_0)$ is a non-empty geodesic lamination on Σ_0.*

Proof. First we show that the A-invariant part of $L^{-1}(1)$ is non-empty, and then we show that its projection to Σ_0 is a geodesic lamination. Note that the A-invariant part of $L^{-1}(1)$ is automatically MA-invariant (by M-invariance of L). To show that $L^{-1}(1)$ is non-empty, the idea is to take a sequence of curves $\gamma_n \subset \Sigma_0$ which are mapped locally almost isometrically to $\mathbb{R}/c\mathbb{Z}$ and show that the entire A orbit of a generic set of the limiting directions are then mapped locally isometrically.

More precisely, we consider a sequence of closed geodesic curves $\gamma_n \subset \Sigma_0$ such that

$$\lim_{n\to\infty} \frac{|\varphi(\gamma_n)|}{\ell(\gamma_n)} = 1.$$

Now take the MA-invariant probability measure μ_n on G_d/Γ_0 supported on all frames x for which $p_M(x)$ is tangent to γ_n and symmetric under taking antipodes.

Since G_d/Γ_0 is compact, we can extract an MA-invariant probability measure μ_0 that is a weak-$*$ limit point of $\{\mu_n\}$. Up to passing to a subsequence with the same name, we can assume that $\mu_n \to \mu_0$. Our goal is to show that for μ_0-a.e. $x \in G_d/\Gamma_0$, that $\tau_0|_{Ax}$ is locally isometric, hence belongs to the A-invariant part of $L^{-1}(1)$.

Observe that for any fixed $x \in G_d/\Gamma_0$, the function $t \mapsto \tau_0(p_K(a_t x))$ is locally 1-Lipschitz, and is therefore differentiable Lebesgue almost everywhere. Moreover, the absolute value of the derivative agrees with $L(a_t x)$ where it is defined. Thus given $\varepsilon > 0$, for all n large enough, we have

$$\int L \, d\mu_n = \frac{1}{\ell(\gamma_n)} \int_0^{\ell(\gamma_n)} |d\tau_0(\gamma_n'(t))| dt \geq \frac{|\varphi(\gamma_n)|}{\ell(\gamma_n)} \geq 1 - \varepsilon.$$

In particular,

$$\mu_n(\{L \geq 1 - \varepsilon^{1/2}\}) \geq 1 - \varepsilon^{1/2}.$$

The set $\{L \geq 1 - \varepsilon^{1/2}\}$ is closed (and compact) due to upper semi-continuity of L, so that weak-* convergence gives that

$$\mu_0(\{L \geq 1 - \varepsilon^{1/2}\}) \geq \limsup \mu_n(\{L \geq 1 - \varepsilon^{1/2}\}) \geq 1 - \varepsilon^{1/2}$$

holds for all ε. Thus $\mu_0(L^{-1}(1)) = 1$.

Let $E = L^{-1}(1)$. Then using flow invariance of μ_0, we have

$$1 = \int_{G_d/\Gamma_0} \chi_E(x) d\mu_0(x) = \frac{1}{2T} \int_{-T}^{T} \int_{G_d/\Gamma_0} \chi_E(a_t x) d\mu_0(x)\ dt.$$

Using Fubini, we have

$$2T = \int_{G_d/\Gamma_0} \left(\int_{-T}^{T} \chi_E(a_t x) \right) d\mu_0(x),$$

so that for μ_0-a.e. x, $a_t x \in E$ for Lebesgue almost every $t \in [-T, T]$. Let x be a point which is in the (full measure) intersection over a sequence of T going to infinty. Then the A orbit of x goes locally isometrically to $\mathbb{R}/c\mathbb{Z}$ via τ_0.

Now that we have proved that $\mathcal{L}_0$ is non-empty, we aim to show that its projection to Σ_0 is in fact a geodesic lamination. By upper semi-continuity, $L^{-1}(1)$ is closed, and hence also $\mathcal{L}_0$. It therefore suffices to prove that if $x, y \in \mathcal{L}_0$ satisfy $p_K(x) = p_K(y)$, then $p_K(Ax) = p_K(Ay)$.

It will be convenient to work with a lift $\tau : \Sigma \to \mathbb{R}$ of τ_0. Let $\mathcal{L} = p_{\mathbb{Z}}^{-1}(\mathcal{L}_0)$. For sake of contradiction, suppose there are $x, y \in \mathcal{L}$ where $p_K(x) = p_K(y)$ but $p_K(Ax)$ meets $p_K(Ay)$ at a definite angle (different from 0 or π).

We have already observed that for $s > 0$,

$$\int_0^s L(p_{\mathbb{Z}}(a_t x))\ dt$$

computes length of $\tau(a_{[0,s]}x)$ in $\mathbb{R}$. Since $x \in \mathcal{L}$, this is just s.

Without loss of generality, we assume that for $s > 0$, τ is increasing along $p_K(a_s x)$ and decreasing along $p_K(a_s y)$ so that the distance between $\tau(p_K(a_{-s}y))$ and $\tau(p_K(a_s x))$ is $2s$. Since $p_K(Ax)$ and $p_K(Ay)$ meet at a definite angle, the length of the shortest geodesic segment connecting $p_K(a_{-s}y)$ to $p_K(a_s x)$ is strictly smaller than $2s$. But τ maps this segment to a segment of length $2s$. This contradicts the fact that τ is 1-Lipschitz and completes the proof that $p_K(\mathcal{L}_0)$ is a non-empty geodesic lamination. $\square$

We draw the following useful corollary:

Lemma 3.2. *Let $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ be 1-Lipschitz and tight, and let $\tau : \Sigma \to \mathbb{R}$ be any lift. There exists a constant $C_\tau > 0$ satisfying for all $y, z \in G_d/\Gamma$*

$$d(p_K(y), p_K(z)) - C_\tau \leq |\tau(p_K(y)) - \tau(p_K(z))| \leq d(p_K(y), p_K(z)). \tag{3.2}$$

Proof. The upper bound in (3.2) follows from τ being 1-Lipschitz on Σ. Let us prove the lower bound. Given $k \in \mathbb{Z} \cong \Gamma_0/\Gamma$ denote by $k.\square$ the isometric action of the deck transformation on Σ. For any k and $x \in G_d/\Gamma$

we have $\tau(p_K(k.x)) = \tau(p_K(x)) + ck$, and $F = \tau^{-1}([0, c)$ is a precompact fundamental domain for the $\mathbb{Z}$-action on Σ. By Proposition 3.1, there exists $x \in \mathcal{L} \subset G_d/\Gamma$ satisfying $\tau(p_K(a_t x)) = \tau(p_K(x)) + t$ for all t. We will use the values of τ along Ax as a kind of ruler for coarsely measuring distances in Σ.

Now, given $y, z \in G_d/\Gamma$ there are t_y and t_z satisfying $\tau(p_K(a_{t_y}x)) = \tau(p_K(y))$ and $\tau(p_K(a_{t_z}x)) = \tau(p_K(z))$. Since $p_K(a_{t_y}x)$ and $p_K(y)$ lie in the same τ level set, they lie in the same $\mathbb{Z}$-translate of F, and so their distance in G_d/Γ is bounded by $\mathrm{diam}(F)$, and similarly for $p_K(a_{t_z}x)$ and $p_K(z)$. Thus

$$\begin{aligned} d(p_K(y), p_K(z)) \leq\ & d(p_K(y), p_K(a_{t_y}x)) + d(p_K(a_{t_y}x), p_K(a_{t_z}x)) \\ & + d(p_K(a_{t_z}x), p_K(z)) \leq \\ \leq\ & |t_y - t_z| + 2\mathrm{diam}(F) \\ =\ & |\tau(p_K(y)) - \tau(p_K(z))| + 2\mathrm{diam}(F). \end{aligned}$$

Set $C_\tau = 2\mathrm{diam}(F)$ and the claim follows. $\square$

Recall our notation of

$$\mathcal{Q} = \{x \in G_d/\Gamma : (a_t x)_{t \geq 0} \text{ is quasi-minimizing}\}$$

and

$$\mathcal{Q}_\pm = \{x \in \mathcal{Q} : \lim_{t \to \infty} \tau(a_t x) = \pm\infty\}.$$

That is, $\mathcal{Q}_+$ is the set of all points in G_d/Γ whose geodesic trajectory as $t \to +\infty$ is quasi-minimizing and escaping through the positive end of Σ.

Definition 3.3. We define the *ω-limit set mod $\mathbb{Z}$* of an A-invariant set $F \subset G_d/\Gamma$ by

$$\omega\text{-}\lim_{\mathbb{Z}}(F) = p_{\mathbb{Z}}^{-1}\left(\{y \in G_d/\Gamma_0 : \exists x \in p_{\mathbb{Z}}(F) \text{ and } t_j \to \infty \text{ s.t. } a_{t_j}x \to y\}\right).$$

That is, $\omega\text{-}\lim_{\mathbb{Z}}(F)$ contains the lifts of accumulation points in G_d/Γ_0 of geodesic trajectories in $p_{\mathbb{Z}}(F)$.

In the sequel, we will abuse notation and denote by τ the composition $\tau \circ p_K$. Denote $\mathcal{Q}_\omega = \omega\text{-}\lim_{\mathbb{Z}}(\mathcal{Q})$, the ω-limit set of quasi-minimizing rays in G_d/Γ. Let $\mathcal{L}_0^{\mathrm{cr}}$ denote the chain recurrent part of $\mathcal{L}_0$. Then $\mathcal{Q}_\omega$ is related to $\mathcal{L}_0^{\mathrm{cr}}$ as follows.

Theorem 3.4 (Definition of λ_0)**.** *The projection $p_K(p_{\mathbb{Z}}(\mathcal{Q}_\omega))$, denoted by λ_0, is a chain recurrent geodesic lamination on Σ_0 contained in $p_K(\mathcal{L}_0^{\mathrm{cr}})$.*

Furthermore, every A-orbit in $\mathcal{Q}_\omega$ is mapped isometrically to $\mathbb{R}$ by τ and is bi-minimizing, i.e., isometrically embedded in G_d/Γ.

Proof. First we show that $p_{\mathbb{Z}}(\mathcal{Q}_\omega) \subset L^{-1}(1)$. Fix $y \in \mathcal{Q}_\omega$ and let x be a point in $\mathcal{Q}$ with $a_{t_j}p_{\mathbb{Z}}(x) \to p_{\mathbb{Z}}(y)$ for some $t_j \to \infty$. Assume for sake of contradiction that $p_{\mathbb{Z}}(y) \notin L^{-1}(1)$. In other words, we have

$$L(p_{\mathbb{Z}}(y)) < 1 - \varepsilon$$

for some $\varepsilon > 0$. The function L is upper semi-continuous and therefore $L^{-1}\left([0, 1-\frac{\varepsilon}{2})\right)$ is an open set in G_d/Γ_0 containing $p_{\mathbb{Z}}(y)$. In particular, any point $z \in G_d/\Gamma$ close enough to any $\mathbb{Z}$ translate of y has $L(z) < 1-\varepsilon$.

Since $a_{t_j} p_{\mathbb{Z}}(x) \to p_{\mathbb{Z}}(y)$, there exists $\delta > 0$ such that for all large j the 2δ-geodesic segment around $a_{t_j}x$ satisfies

$$L(p_{\mathbb{Z}}(A_{[t_j-\delta, t_j+\delta]}x)) < 1 - \varepsilon/2.$$

This implies (see fig. 4) that for all large j

$$|\tau(a_{t_j+\delta}x) - \tau(a_{t_j-\delta}x)| \leq 2\delta\left(1 - \frac{\varepsilon}{2}\right).$$

We may assume the above holds for all $j \geq 1$. Then we have

$$\begin{aligned} |\tau(a_{t_j+\delta}x) - \tau(x)| &\leq |\tau(a_{t_j+\delta}x) - \tau(a_{t_j-\delta}x)| + |\tau(a_{t_j-\delta}x) - \tau(a_{t_{j-1}+\delta}x)| + \\ &\quad ... + |\tau(a_{t_1+\delta}x) - \tau(a_{t_1-\delta}x)| + |\tau(a_{t_1-\delta}x) - \tau(x)| \leq \\ &\leq 2\delta\left(1-\frac{\varepsilon}{2}\right) + [t_j - (t_{j-1}+\delta)] + \\ &\quad ... + 2\delta\left(1-\frac{\varepsilon}{2}\right) + [t_1 - \delta] \leq \\ &\leq t - j\cdot\frac{\delta\varepsilon}{2}. \end{aligned}$$

On the other hand, lemma 3.2 implies

$$d(a_{t_j+\delta}x, x) - C_\tau \leq |\tau(a_{t_j+\delta}x) - \tau(x)| \leq t - j\cdot\frac{\delta\varepsilon}{2}$$

for some $C_\tau > 0$ and all $j \geq 1$, which contradicts our assumption that x is quasi-minimizing. Hence we conclude that $y \in L^{-1}(1)$. Since $\mathcal{Q}_\omega$ is A-invariant, we deduce that $p_{\mathbb{Z}}(\mathcal{Q}_\omega) \subset \mathcal{L}_0$.

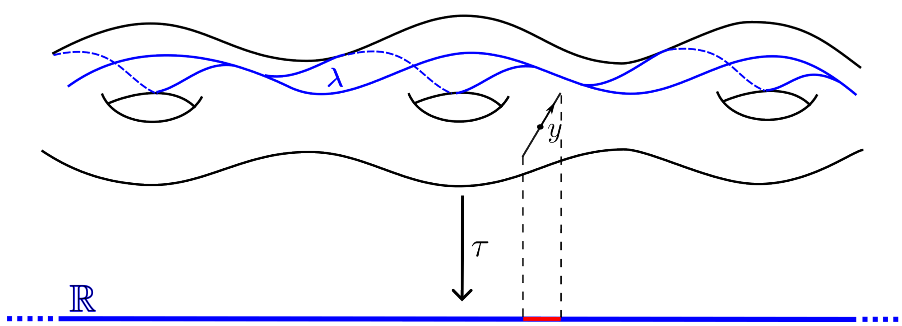


FIGURE 4. The ω-limit set of $\mathcal{Q}$ is Liphschitz maximzing.

Chain recurrence follows from the fact that for a sequence $\varepsilon_n \to 0$, we can use the Closing Lemma to find a sequence of nearly geodesic closed ε_n-trajectories of the lamination $p_K(p_{\mathbb{Z}}(\mathcal{Q}))$ in Σ_0 made out of long segments of $\{p_K(a_t p_{\mathbb{Z}}(x)) : t \in \mathbb{R}_{\geq 0}\}$ joining subsequent recurrences near $p_K(p_{\mathbb{Z}}(y))$. $\square$

In dimension $d = 2$ we can say more, completing the proof of Theorem 1.4:

Proposition 3.5. *If $d = 2$, then $p_{\mathbb{Z}}(\mathcal{Q}_\omega) = \mathcal{L}_0^{\mathrm{cr}}$.*

Proof. When $d = 2$, we can use the structure theory for geodesic laminations to construct a quasi-minimizing ray that accumulates onto any point of $\mathcal{L}_0^{\mathrm{cr}}$. From this we will conclude that $\mathcal{L}_0^{\mathrm{cr}} \subset p_{\mathbb{Z}}(\mathcal{Q}_\omega)$.

Let $\mu = p_K(\mathcal{L}_0^{\mathrm{cr}})$ and μ_ω be the ω-limit set (of μ). The classification theory tells us that μ_ω consists of finitely many minimal components, and $\mu \setminus \mu_\omega$ consists of finitely many isolated leaves; each spirals onto a minimal component in its future and onto another in its past.

Clearly, for any $x \in G_d/\Gamma$ with $p_{\mathbb{Z}}(x) \in \mathcal{L}_0^{\mathrm{cr}}$ and $p_K(p_{\mathbb{Z}}(x)) \in \mu_\omega$, then Ax is bi-minimizing and $p_{\mathbb{Z}}(Ax)$ accumulates onto $p_{\mathbb{Z}}(x)$. This shows that the ω-limit set of $\mathcal{L}_0$ is contained in $p_{\mathbb{Z}}(\mathcal{Q}_\omega)$.

If $p \in \mu \setminus \mu_\omega$ lies on an isolated leaf, then chain recurrence of μ tells us we can find a closed loop γ_0 through p of the form

$$\gamma_0 = g_1 \cdot j_1 \cdot g_2 \cdot j_2 \cdot \ldots \cdot j_{k-1} \cdot g_k,$$

where g_i are segments of isolated leaves ℓ_i of μ that spiral onto minimal components μ_i; the j_i are small segments that that join g_i to g_{i+1} that are C^1-close to a leaf of μ_i.

Let ε_n be a summable sequence of small positive numbers. We can create an ε_n-trajectory γ_n of λ_0 through p by modifying γ_0 as follows. Since g_i accumulates onto μ_i in its forward direction and g_{i+1} accumulates onto μ_i in its backward direction, we can extend the future of the former and the past of the later so that their endpoints are joined by segment of size at most ε_n that is ε_n C^1-close to a leaf of μ_i. We can concatenate the γ_n's to form an infinite path $r = \gamma_1 \cdot \gamma_2 \cdot \ldots$. Let $\tilde{r}$ be a lift of r to $\mathbb{H}^2$ and let $\tilde{r}^*$ be the geodesic ray with the same initial point that is asymptotic to $\tilde{r}$. By stability of geodesics in hyperbolic space, there is a constant c such that

$$d_{\mathbb{H}^2}(\tilde{r}(t), \tilde{r}^*(t)) \leq c \text{ for all } t.$$

Moreover, if $r(t_n)$ belongs to γ_n, then there is a $t_n^* \in [t_n - c, t_n + c]$ such that

$$d_{\mathbb{H}^2}(\tilde{r}^*(t_n^*), \tilde{r}(t_n)) \leq c\varepsilon_n.$$

In particular, the projection r^* of $\tilde{r}^*$ accumulates onto p in Σ_0.

Abusing notation, we let r and r^* also denote lifts to Σ. Then since τ is isometric along the leaves of μ and 1-Lipschitz, we have

$$d_\Sigma(r(t), r(0)) \geq \tau(r(t)) - \tau(r(0)) \geq t - \sum_{j \leq n} \varepsilon_j, \text{ when } r(t) \text{ belongs to } \gamma_n \subset r.$$

The triangle inequality then provides

$$d_\Sigma(r^*(0), r^*(t)) \geq \tau(r^*(t)) - \tau(r^*(0)) \geq t - \sum_{j \leq n} \varepsilon_j - c.$$

Since $\sum_{j=1}^\infty \varepsilon_j$ is finite, we conclude that r^* is quasi-minimizing. Together with theorem 3.4 the proposition follows. ∎

Remark 3.6. Although we do not have a structure theorem for geodesic laminations in dimensions $d \geq 3$, it is not difficult to see that our argument can be adapted to prove $\mathcal{L}_0^{\mathrm{cr}} = p_{\mathbb{Z}}(\mathcal{Q}_\omega)$ whenever the complement of the set of minimal components of $p_K(\mathcal{L}_0^{\mathrm{cr}})$ consists of at most countably many geodesics.

Thurston studied the set of points realizing the maximum local Lipschitz constant for tight maps between finite area complete hyperbolic surfaces. He proved that the *maximal stretch locus* comprises a geodesic lamination and that the chain recurrent part is contained in the maximum stretch locus of any tight map between the same surfaces [Thu98b, Theorem 8.2]. As a corollary of the Proposition 3.5, we deduce the following analogue of Thurston's result.

Corollary 3.7. *To a given cohomology class $\varphi \in H^1(S; c\mathbb{Z})$ and hyperbolic structure Σ_0, the geodesic lamination $p_K(\mathcal{L}_0^{\mathrm{cr}})$ constructed from any tight-Lipschitz map $f : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ inducing φ on π_1, depends only on φ.*

Proof. By Proposition 3.5, $\mathcal{L}_0^{\mathrm{cr}} = p_{\mathbb{Z}}(\mathcal{Q}_\omega)$, which only depends on the $\mathbb{Z}$-cover Σ of Σ_0 corresponding to φ and not the tight Lipschitz function $f : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$. □

3.1. **Overview.** We end this section with a brief summary of notations and inclusions for future reference. Let $\Sigma \to \Sigma_0$ be a $\mathbb{Z}$-cover of a compact hyperbolic manifold to which we associated the following:

Notation	Definition
$\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$	1-Lipschitz tight map corresponding to $\Sigma \to \Sigma_0$
$\tau : \Sigma \to \mathbb{R}$	1-Lipschitz $\mathbb{Z}$-equivariant lift of τ_0
$\mathcal{L}_0$	maximal A-invariant subset of $L^{-1}(1) \subset G_d/\Gamma_0$
$\mathcal{L}$	lift of $\mathcal{L}_0$ to G_d/Γ, i.e. $p_{\mathbb{Z}}^{-1}(\mathcal{L}_0)$
$\mathcal{L}_0^{\mathrm{cr}}$, $\mathcal{L}^{\mathrm{cr}}$	chain recurrent part of $\mathcal{L}_0$ and its lift to G_d/Γ

All the sets above implicitly depend on the choice of a tight map u_0. We further consider the following canonical loci, which are independent of the tight map:

Notation	Definition
$\mathcal{Q}$	set of quasi-minimizing points in G_d/Γ
$\mathcal{Q}_\omega$	ω-limit set mod $\mathbb{Z}$ of $\mathcal{Q}$ in G_d/Γ
λ_0	geodesic lamination in Σ_0 equal to $p_K(p_{\mathbb{Z}}(\mathcal{Q}_\omega))$
λ	lift of λ_0 to Σ, i.e. $p_{\mathbb{Z}}^{-1}(\lambda_0)$ (also $p_K(\mathcal{Q}_\omega)$)

In the sequel, the canonical laminations λ_0 and λ will be referred to as the *minimizing* laminations. Note the following relationship between the canonical objects.

$$\mathcal{Q}_\omega \xrightarrow{p_K} \lambda \xrightarrow{p_{\mathbb{Z}}} \lambda_0 \ . \tag{3.3}$$

Recall that $x \in G_d/\Gamma$ is *bi-minimizing* if $t \mapsto a_t x$ is an isometric embedding. There is a sequence of inclusions:

$$\mathcal{Q}_\omega \subset \mathcal{L}^{\mathrm{cr}} \subset \mathcal{L} \subset \{\text{bi-minimizing points}\} \subset \mathcal{Q} \subset G_d/\Gamma. \tag{3.4}$$

In the case of $d = 2$, the first inclusion is an equality, that is $\mathcal{Q}_\omega = \mathcal{L}^{\mathrm{cr}}$ which is independent of the choice of tight function. The remaining inclusions could otherwise be strict.

4. The non-horospherical limit set in $d = 2$

In this section we show that the subset $\Lambda_{\mathrm{nh}} \subset \partial\mathbb{H}^2$ of non-horospherical limit points is of Hausdorff dimension zero. Since the ω-limit set of $\mathcal{Q}$ is the lift of a geodesic lamination in Σ_0, the statement is a consequence of the following theorem:

Theorem 4.1. *Let $S = \mathbb{H}^2/\Gamma$ be a finite area hyperbolic surface and λ a geodesic lamination. Let $\omega(\lambda)$ be the set of endpoints in the circle at infinity of rays whose ω-limit sets are in λ. Then the Hausdorff dimension of $\omega(\lambda)$ is 0.*

Note that the dimension of the set of endpoints of λ itself is 0 by Birman-Series [BS85] (and similarly Bonahon-Zhu [ZB04]). This is an extension of those ideas, though we note that $\omega(\lambda)$ is a larger set.

Before we start, recall some structural facts about laminations and train tracks:

For ε smaller than the injectivity radius of S, an *ε-track* for a lamination λ is an ε-neighborhood of λ, divided into a finite number of rectangles or "branches", whose "long" edges run along λ and whose short edges have length $O(\varepsilon)$. Branches are foliated by arcs ("ties") parallel to the short edges and transverse to λ, and are attached to each other along the short edges. Collapsing the ties we get a 1-complex whose vertices are images of connected unions of short edges. A train route is a path running through the track transverse to the ties. Each train route is $O(\varepsilon)$-close to a geodesic which is also transverse to the ties. We think of a train route as combinatorial – two routes are the same if they traverse the same branches in the same order. In particular we say a train route is *embedded* if it passes through each vertex at most once. We say a train route is *almost embedded* if it passes through each branch at most twice, with opposite orientations.

A train route is a cycle if it begins and ends in the same branch. When ε is sufficiently small, no two cycles in an ε-track can be homotopic.

A geodesic lamination λ in a finite area hyperbolic surface can have a finite number of closed leaves (possibly none). Let λ_c denote the union of these leaves, and let r denote the number of components of λ_c.

Lemma 4.2. *There is a bound m_0 such that, for any train track in S, there are at most m_0 almost embedded train routes.*

Proof. There is a finite number of homeomorphism types of train tracks on a given surface. □

Lemma 4.3. *Given λ and $K > 0$ there exists ε so that for any ε-track for λ, any cyclic train route of length bounded by K is homotopic to a component of λ_c.*

Proof. Fix K and consider a sequence $\varepsilon_i \to 0$, and ε_i-tracks σ_i. Suppose for each i we have a train route β_i in σ_i of length at most K. After taking a subsequence, β_i must converge to a closed geodesic, and on the other hand the ε_i neighborhoods converge to λ. Hence eventually β_i is homotopic to one of the components of λ_c. □

Lemma 4.4. *There exist c_0, d such that, for any $K > 0$, if ε is small enough then any ε-track for λ has at most $c_0 K^d$ train routes of length bounded by K.*

Proof. Recall that r is the number of components of λ_c. As a warm-up consider the case that λ_c is empty ($r = 0$). Hence fixing K, for sufficiently small ε every train route of length K is almost embedded – if it were not then a sub-route would give a cycle, contradicting Lemma 4.3. Thus, there are at most m_0 of them by Lemma 4.2 and we are done with $d = 0$.

For $r > 0$, let $\gamma_1, \ldots, \gamma_r$ be the components of λ_c. There is exactly one closed train route $\hat{\gamma}_i$ homotopic to each γ_i, since no two closed train routes are homotopic.

Fixing K, choose ε as given by Lemma 4.3. Possibly choosing ε even smaller we can arrange that the $\hat{\gamma}_i$ are embedded and disjoint. If a train route α of length at most K traverses an edge of $\hat{\gamma}_i$ twice with the same orientation, then between those two traversals it must remain in $\hat{\gamma}_i$, for otherwise there would be a cycle of length at most K which is not homotopic to any of the γ_j.

It follows that we can remove an integer number of traversals of each $\hat{\gamma}_i$ from α, leaving an almost embedded train route α_0. The number of such α_0, then, is bounded by m_0. For each one of them we can "splice" back in a power of each $\hat{\gamma}_i$ at a suitable place, but their lengths can add to at most K. The number d of places to splice is bounded by twice the maximal number of edges in any track on S (because each edge can appear at most once with each orientation) and a power p of $\hat{\gamma}_i$ spliced in contributes at least p times the length of γ_i. Hence there is a bound of the form $c(K/b)^d$ on the number of ways to do this, where b is the minimal length of a component of λ_c.

Adding over all m_0 possible α_0's we obtain a bound of the desired form on the number of all train routes of length bounded by K. □

Lemma 4.5. *Given $s > 0$ there exist $\varepsilon > 0$ and c_1 such that, for any ε-track of λ, the number $N(L)$ of train routes of length at most L is bounded by*

$$N(L) \leq c_1 e^{sL}.$$

Proof. Fix $K > 0$ and let $\varepsilon = \varepsilon(K, \lambda)$ and $d \geq 0$ be given by Lemma 4.4. For $L >> K$, any train route can be divided into $\lceil L/K \rceil$ segments of length at most K. Since there are at most $c_0 K^d$ possibilities for each of these, the total number of routes of length at most L is at most

$$(c_0 K^d)^{\lceil L/K \rceil}.$$

Since $K^{1/K} \to 1$ as $K \to \infty$, for sufficiently large K this is bounded by an expression of the form $c_1 b^L$, for b as close as we like to 1. This gives the desired bound. □

Proof of Theorem 4.1. Fixing a basepoint x_0 of $\mathbb{H}^2$, the set $\omega = \omega(\lambda)$ determines a set $R\omega$ of rays emanating from x_0 and landing at the points of ω.

Fix $\delta > 0$ and let $\mathcal{H}_\delta$ denote δ-dimensional Hausdorff content. We must prove $\mathcal{H}_\delta(\omega) = 0$.

Let ε be the value given by Lemma 4.5 for $s = \delta/2$, and let τ be an ε-track for λ. Every ray in $R\omega$ is eventually carried in the preimage track $\tilde{\tau}$. Let $\omega_n \subset \omega$ denote the subset whose rays are carried in $\tilde{\tau}$ after time n. Thus $\omega = \cup_n \omega_n$ and it suffices to prove $\mathcal{H}_\delta(\omega_n) = 0$ for each n.

Any ray in $R\omega_n$ passes, at time n, through some branch in $\tilde{\tau}$. There are finitely many such branches meeting the circle of radius n; let $m_1 = m_1(n)$ denote their number. Fixing such a branch e, all the rays of $R\omega_n$ passing through e continue further along $\tilde{\tau}$ in an infinite train route. For $L > 0$, consider the initial train routes of length L beginning in e.

Any such route projects to a train route of τ beginning at the image of e. Two different train routes upstairs project to different routes downstairs (all the routes begin together, and as soon as two diverge their images do as well). Thus by lemma 4.5 there are at most $c_1 e^{sL}$ such routes.

These routes provide a covering of $\omega_n(e)$, the subset corresponding to rays that pass through e at time n. Two rays with the same initial length-L route stay at distance ε apart for at least length L (in fact $L + n$), which means their endpoints are at most $O(e^{-L})$ apart on the circle. Thus, we have a covering of $\omega_n(e)$ by $c_1 e^{sL}$ intervals of length $O(e^{-L})$. Summing over all the m_1 branches, we find

$$\mathcal{H}_\delta(\omega_n) < m_1 c_1 e^{sL} e^{-\delta L}.$$

Since $s < \delta$, this goes to 0 as $L \to \infty$. This completes the proof that $\mathcal{H}_\delta(\omega_n) = 0$, so that $\mathcal{H}_\delta(\omega) = 0$ as well for all $\delta > 0$ and so $\dim_{\mathrm{H}}(\omega) = 0$. □

Remark 4.6. In fact a stronger claim holds, that is, *the upper packing dimension of $\omega(\lambda)$ is zero.* Recall the definition of the upper packing dimension of a set E in a metric space is

$$\overline{\dim}_{\mathrm{p}} E = \inf \left\{ \sup_{j \geq 1} \overline{\dim}_{\mathrm{box}}(E_j) : E \subseteq \bigcup_{j=1}^{\infty} E_j \right\},$$

where $\overline{\dim}_{\mathrm{box}}$ denotes the upper box (or Minkowski) dimension and the infimum is defined over all countable covers of E by bounded sets, see e.g. [Mat95, §5.9].

Indeed, using the notations in the proof of theorem 4.1, we have for all $\delta > 0$ a decomposition of the set $\omega(\lambda)$ into countably many subsets ω_n. For any $L > 0$, let $N(\omega_n, e^{-L})$ denote the minimal number of sets of diameter at most e^{-L} needed to cover ω_n. Then we have shown

$$N(\omega_n, e^{-L}) \leq m_1 c_1 e^{\delta L},$$

and hence

$$\overline{\dim}_{\mathrm{box}}(\omega_n) = \limsup_{L\to\infty} \frac{\log N(\omega_n, e^{-L})}{-\log(e^{-L})} \leq \delta.$$

Therefore, by definition

$$\overline{\dim}_{\mathrm{p}}\omega(\lambda) \leq \sup_{n\geq 1} \overline{\dim}_{\mathrm{box}}(\omega_n) \leq \delta$$

for all $\delta > 0$, implying $\overline{\dim}_{\mathrm{p}}\omega(\lambda) = 0$.

Corollary 4.7. *For $d = 2$, the set of quasi-minimizing points $\mathcal{Q}$ in G/Γ has Hausdorff dimension 2. In particular, for any $x \in \mathcal{Q}$*

$$1 \leq \dim_{\mathrm{H}} \overline{Nx} \leq 2.$$

Proof. Let $\tilde{\mathcal{Q}}$ denote the lift of $\mathcal{Q}$ to G. The smooth covering map ensures that $\dim_{\mathrm{H}} \mathcal{Q} = \dim_{\mathrm{H}} \tilde{\mathcal{Q}}$ hence it suffices to consider the latter. Since $\tilde{\mathcal{Q}}$ is AN-invariant we may present it under the Iwasawa decomposition as a product set $\tilde{\mathcal{Q}} = NAK_{\mathrm{nh}}$ where $K_{\mathrm{nh}} \subset K$ (recall that the multiplication map $N \times A \times K \to G$ is a diffeomorphism, see e.g. [Kna02, Theorem 6.46]). Furthermore, K_{nh} is bi-Lipschitz equivalent to the non-horospherical limit set $\Lambda_{\mathrm{nh}} \subset \partial\mathbb{H}^2$. Therefore

$$\dim_{\mathrm{H}} \tilde{\mathcal{Q}} = \dim_{\mathrm{H}} NA + \dim_{\mathrm{H}} \Lambda_{\mathrm{nh}} = 2 + 0,$$ [5]

by Theorem 4.1.

The second claim follows directly from the inclusions

$$Nx \subset \overline{Nx} \subset \mathcal{Q}$$

for all $x \in \mathcal{Q}$. (For the last inclusion see the remark after the Eberlein-Dal'bo theorem in Section 1.) □

5. Constructing minimizing laminations in $d = 2$

Let S_0 be an oriented closed (topological) surface of genus $g \geq 2$. In this section, we use methods specific to dimension $d = 2$ to construct minimizing laminations for $\mathbb{Z}$-covers of closed hyperbolic surfaces with prescribed geometric and dynamical properties.

[5] Note that both factors in the product above have their Hausdorff dimension equal to their packing dimension, in particular implying the product formula.

Our main construction mimics a construction from Hodge theory that produces from a rational class $\varphi \in H^1(S_0, \mathbb{R})$ and a conformal structure X on S_0, two orthogonal oriented measured foliations and a distinguished representative of φ as a map $S_0 \to S^1$ obtained by collapsing the leaves of one of the foliations. Our goal is to "hyperbolize" such a construction; the foliation that is not collapsed will play the role of minimizing lamination from Section 3, while the weighted curves comprising the non-singular leaves of the collapsed foliation can be identified with the Poincaré dual of the cohomology class.

5.1. **Background on surface theory.** We rely on a dictionary between certain kinds of geometric, dynamical, and topological objects on surfaces. As such, we recall some background on measured geodesic laminations, singular measured foliations, interval exchange transformations, and holomorphic differentials.

Measured laminations. Endow S_0 with an auxiliary negatively curved metric, let $\lambda_0 \subset S_0$ be a minimal geodesic lamination and let k be a transversal, i.e., a C^1 arc meeting λ_0 transversally. Giving k an orientation gives a local orientation to the leaves of λ_0, and following the leaves of λ_0 induces a *homeomorphic* Poincaré first return map

$$P : k \cap \lambda_0 \to k \cap \lambda_0.$$

If k' is another transversal and k is isotopic to k' preserving the transverse intersections with λ_0, then the corresponding dynamical systems are conjugated by the isotopy, and the invariant ergodic measures are in bijective correspondence.

This discussion leads to the notion of a *transverse measure* μ_0 supported on λ_0, which is, to each transversal k, the assignment of a finite positive Borel measure $\mu_0(k)$ supported on $k \cap \lambda_0$ that is invariant under transverse isotopy and natural with respect to inclusion. A geodesic lamination λ_0 equipped with a transverse measure μ_0 is a *measured lamination*. A measured lamination can also be thought of as an A-invariant finite Borel measure μ_0 on T^1S_0 whose support is the natural lift of a geodesic lamination λ_0. Since the geodesic foliation of T^1S_0 does not depend on a choice of negatively curved metric, the space of measured laminations with its weak-$*$ topology only depends on the topology of S_0. Denote this space by $\mathcal{ML}(S_0)$. See [Thu82, Thu88, PH92, CB88, FLP12] for a development of the theory of measured laminations.

Measured foliations. A (singular) measured foliation on a surface S_0 is a C^1 foliation $\mathcal{F}_0$ of $S_0 \setminus Z$, where Z is a finite set (called singular points) equipped with a transverse measure ν_0 on arcs transverse to $\mathcal{F}_0$. The transverse measure is required to be invariant under holonomy and every singularity is modeled on a standard k-pronged singularity; see [Thu88, FLP12] for details and further development. Isotopic measured foliations are viewed

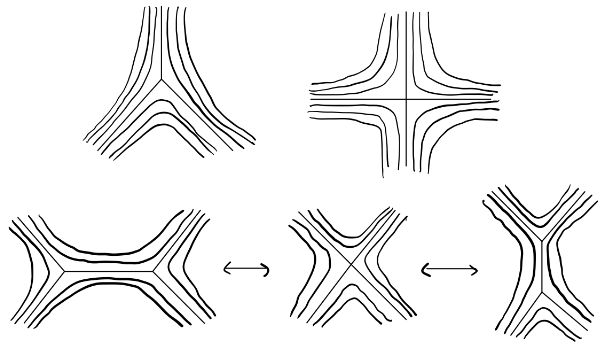


FIGURE 5. 3- and 4-pronged singularities and Whitehead moves.

as identical. A measured foliation is orientable if there is a continuous orientation of the non-singular leaves.

The space $\mathcal{MF}(S_0)$ of *Whitehead equivalence classes* of singular measured foliations (see Figure 5) is equipped with a topology coming from the geometric intersection number with homotopy classes of simple closed curves on S_0. A theorem of Thurston asserts that $\mathcal{MF}(S_0)$ is a $6g - g$ dimensional manifold. We remark that Whitehead equivalence does not in general preserve orientability of a foliation.

Equivalence of measured foliations and laminations. With respect to our auxiliary negatively curved metric on S_0, we can pull each non-singular leaf of a measured foliation $(\mathcal{F}_0, \nu_0)$ tight in the universal cover to obtain a geodesic. The closure is a geodesic lamination invariant under the group of covering transformations that projects to a geodesic lamination λ_0 on S_0, and λ_0 carries a measure of full support μ_0 obtained in a natural way from ν_0.

This procedure defines a natural homeomorphism $\mathcal{MF}(S_0) \to \mathcal{ML}(S_0)$ [Lev83], so that we may pass between measure equivalence classes of measured foliations and the corresponding measured lamination at will. In the sequel, we will often abuse notation and write $\lambda_0 \in \mathcal{ML}(S_0)$ or $\lambda_0 \in \mathcal{MF}(S_0)$ to refer to both the underlying geodesic lamination/equivalence class of foliation and to the transverse measure.

Singular flat structures. A *half-translation structure* on S_0 consists of the following data. A finite set Z (of singularities), and an atlas of smooth charts on $S_0 \setminus Z$ to the Euclidean plane $\mathbb{E}^2$, where the transition maps are either translations or translations with π-rotation. The transitions preserve the Euclidean metric and horizontal/vertical direction(s), and the metric completion makes every singular point a cone point with angle an integer multiple of π. We further require that the cone angle at every singular point is at least 2π.

A half-translation structure has a *vertical foliation* $(\mathcal{V}, |dx|)$ and an orthogonal *horizontal foliation* $(\mathcal{H}, |dy|)$; the singular points of the two foliations coincide, as do the the number of prongs at each singular point. This pair of singular measured foliations *binds* the surface, i.e., there is an $\varepsilon > 0$ such that for all simple closed curves γ on S_0,

$$\int_\gamma |dx| + \int_\gamma |dy| > \varepsilon. \tag{5.1}$$

There is a natural (but somewhat finicky) topology on the collection of isotopy classes of half-translation structures that identifies it with an object from complex analysis, which we now recall. The Teichmüller space $\mathcal{T}(S_0)$ is a set of isotopy classes of *marked complex structures* on an oriented surface S_0. By the Uniformization Theorem, $\mathcal{T}(S_0)$ is also identified with the set of isotopy classes of hyperbolic metrics on S_0. A *holomorphic quadratic differential* q on a marked Riemann surface $X \in \mathcal{T}(S_0)$ is a holomorphic $(2,0)$-form. We can path integrate a square root of q to obtain holomorphic charts to $\mathbb{C} \cong \mathbb{E}^2$, where the transitions are of the form $z \mapsto \pm z + c$, for $c \in \mathbb{C}$. That is, q is the same data as a half-translation structure.

Let $\mathcal{QT}(S_0)$ denote the complex vector bundle of holomorphic quadratic differentials over $\mathcal{T}(S_0)$, identified with the space of isotopy classes of singular flat structures. An important theorem of Gardiner and Masur asserts that a half-translation structure is determined uniquely by the Whitehead equivalence classes of its vertical and horizontal measured foliations, and that every pair of measured foliations that bind (in the sense of (5.1)) can be realized as the vertical and horizontal foliations of a half-translation structure [GM91]. Moreover, this identification is homeomorphic: If $\Delta \subset \mathcal{MF}(S_0) \times \mathcal{MF}(S_0)$ denotes the set of pairs that do not bind, then

$$\mathcal{QT}(S_0) \cong \mathcal{MF}(S_0) \times \mathcal{MF}(S_0) \setminus \Delta.$$

If both the (isotopy classes of) the vertical and horizontal foliations of a quadratic differential q can be oriented so that their intersections are everywhere positive, then q is the square of a holomorphic 1-form (Abelian differential).

5.2. **The orthogeodesic foliation.** Let $\Sigma_0 \in \mathcal{T}(S_0)$ be a hyperbolic metric on S_0, and let λ_0 be a geodesic lamination.

The complement of λ_0 in Σ_0 is a (disconnected) hyperbolic surface whose metric completion has totally geodesic (non-compact) boundary. For each such complementary component Y, away from a piecewise geodesic 1-complex Sp called the *spine*, there is a nearest point in ∂Y. The fibers of the projection map $Y \setminus \mathsf{Sp} \to \partial Y$ form a foliation of $Y \setminus \mathsf{Sp}$ whose leaves extend continuously across Sp to a piecewise geodesic singular foliation $\mathcal{O}(Y)$ called the *orthogeodesic foliation* with n-prong singularities at the vertices of the spine of valence n. Every endpoint of every leaf of $\mathcal{O}(Y)$ meets ∂Y orthogonally; see Figure 6.

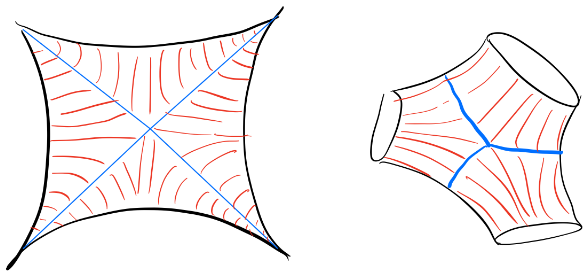


FIGURE 6. Leaves of the orthogeodesic foliation in red with critical graph in blue. The pictured orthogeodesic foliation on the pair of pants is not orientable, though it has orientable representatives in its whitehead equivalence class.

The orthogeodesic foliation $\mathcal{O}(Y)$ is equipped with a transverse measure; the measure assigned to a small enough transversal is Lebesgue after projection to ∂Y, and this assignment is invariant by isotopy transverse to $\mathcal{O}(Y)$. This construction produces a foliation on $\Sigma_0 \backslash \lambda_0$, which extends continuously across the leaves of λ_0 and defines a singular measured foliation $\mathcal{O}_{\lambda_0}(\Sigma_0)$ on S_0; see [CF21, Section 5] for more details.

To a geodesic lamination λ_0, we have produced a map

$$\mathcal{O}_{\lambda_0} : \mathcal{T}(S_0) \to \mathcal{MF}(S_0),$$

corresponding to the Whitehead equivalence class of $\mathcal{O}_{\lambda_0}(\Sigma_0)$.

The following theorem was proved by Calderon-Farre [CF21] building on work of Mirzakhani [Mir08], Thurston [Thu98b], and Bonahon [Bon96].

Theorem 5.1. *The map*

$$\mathcal{O} : \mathcal{T}(S_0) \times \mathcal{ML}(S_0) \to \mathcal{QT}(S_0)$$

that takes a pair (Σ_0, λ_0) to the quadratic differential $\mathcal{O}(\Sigma_0, \lambda_0)$ with real foliation isotopic to $\mathcal{O}_{\lambda_0}(\Sigma_0)$ and imaginary foliation equivalent to λ_0 is a mapping class group equivariant bijection.

The quadratic differential $\mathcal{O}(\Sigma_0, \lambda_0)$ is the square of a holomorphic 1-form if and only if both λ_0 and $\mathcal{O}_{\lambda_0}(\Sigma_0)$ are orientable and co-oriented, i.e., at each point of λ_0, the orientations of λ_0 and $\mathcal{O}_{\lambda_0}(\Sigma_0)$ induce the underlying orientation of S_0.

The following corollary is one of the main ingredients in the proof of Theorem 1.7.

Corollary 5.2. *Let $\lambda_0 \in \mathcal{ML}(S_0)$ be oriented and suppose α is an oriented multi-curve with positive $c\mathbb{Z}$-weights such that every intersection of λ_0 with α is positive and such that λ_0 and α satisfies $S_0 \setminus (\lambda_0 \cup \alpha)$ consists of compact disks. Then there is a unique hyperbolic metric $\Sigma_0 \in \mathcal{T}(S_0)$ such that $\mathcal{O}_{\lambda_0}(\Sigma_0)$ is equivalent to α in $\mathcal{ML}(S_0)$. Moreover, $\mathcal{O}_{\lambda_0}(\Sigma_0)$ is oriented and every intersection with λ_0 is positive.*

Proof. We will prove that the measured foliation equivalent to α lies in $\mathcal{MF}(\lambda_0)$. Suppose not. Then we can find a sequence of simple closed curves γ_n such that

$$\int_{\gamma_n} |d\alpha| + \int_{\gamma_n} |d\lambda_0| \to 0.$$

Since $|d\alpha|$ consists of atomic measures supported on its components, γ_n is eventually disjoint from α. Let μ_0 denote any weak-* accumulation point of $\gamma_n/\ell_{\Sigma_0}(\gamma_n) \in \mathcal{ML}(S_0)$. Continuity of the intersection number gives that the intersection of μ_0 with λ_0 is zero. We conclude that μ_0 either has support contained in λ_0 or is disjoint from λ_0. Both possibilities are impossible, since then μ_0 would cross α essentially. This proves that λ_0 and α bind in the sense of eq. (5.1).

We can apply Theorem 5.1 to deduce that there is a unique hyperbolic metric Σ_0 on S_0 so that $\mathcal{O}_{\lambda_0}(\Sigma_0) = \alpha$. It follows that every non-singular leaf of $\mathcal{O}_{\lambda_0}(\Sigma_0)$ is closed and isotopic to a component of α. The orientation of α is compatible with one of the orientations of $\mathcal{O}_{\lambda_0}(\Sigma_0)$, which then meet λ_0 positively. □

While the map $\mathcal{O}$ from Theorem 5.1 is not continuous, it is established in [CF] that it admits a continuous extension in the following sense.

Theorem 5.3. *Suppose we have $\lambda_n \in \mathcal{ML}(S_0)$ converging in measure to μ_0 and converging to λ_0 in the Hausdorff topology. If $\mathcal{O}(\Sigma_n, \lambda_n) \to q \in \mathcal{QT}(S_0)$, then $\{\Sigma_n\}$ converges to a hyperbolic metric $\Sigma_0 \in \mathcal{T}(S_0)$, and the vertical foliation of q is $\mathcal{O}_{\lambda_0}(\Sigma_0)$.*

We will use this result later on in this section when we construct $\mathbb{Z}$-covers with arbitrarily close metrics but whose bi-minimizing loci are dynamically very different.

Remark 5.4. It is possible to use Theorem 5.3 to prove an analogue of Corollary 5.2 for pairs λ_0 and α, where λ_0 is allowed to be an arbitrary (oriented) chain recurrent geodesic lamination if also the maximal measured part λ_0^m of λ_0 still binds together with α. The two hyperbolic structures Σ_0 and Σ_0^m with $\mathcal{O}_{\lambda_0}(\Sigma_0) = \alpha = \mathcal{O}_{\lambda_0^m}(\Sigma_0^m)$ are different.

See also [CF, Appendix] for more details about extending the results of [CF21] to chain recurrent geodesic laminations.

5.3. **Tight maps with prescribed stretch locus.** In this section, we will prove Theorem 1.7 from the introduction, which we restate here in a slightly more general form.

Theorem 5.5. *Let $\varphi \in H^1(S_0, c\mathbb{Z})$ and let λ_0 be the support of an oriented measured geodesic lamination on S_0. Suppose that φ is Poincaré dual to a homology class represented by an oriented multi-curve α with positive $c\mathbb{Z}$ weights that meets λ_0 transversely and positively and such that $\alpha \cup \lambda_0$ binds S_0.*

Then there is a hyperbolic metric $\Sigma_0 \in \mathcal{T}(S_0)$ *and a* 1*-Lipschitz tight map* $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ *inducing* φ *on homology such that the locus of points whose local Lipschitz constant is* 1 *is equal to* λ_0 *.*

We remark that in Section 3, we proved that the locus of points in $T^1\Sigma_0$ realizing the maximum *directional local Lipschitz constant* of τ_0 is tangent to a geodesic lamination. In Theorem 5.5, we prove the stronger result that the locus of points on Σ_0 realizing the maximum local Lipschitz constant of τ_0 on Σ_0 is a geodesic lamination. C.f. [DU20] for a different construction of tight maps with this property.

The main idea of the proof is to use the dictionary between hyperbolic surfaces equipped with a (measured) geodesic lamination and singular flat surfaces provided by Theorem 5.1 to mimic the following construction. Equipping λ_0 with a transverse measure, there is a unique singular flat structure q with oriented horizontal and vertical foliations equivalent to λ_0 and α, respectively. Integration of cycles on S_0 against α defines a map $q \to \mathbb{R}/c\mathbb{Z}$. This map factors through an oriented metric graph obtained by collapsing the leaves of α, and the leaves of λ_0 are thus mapped locally isometrically to $\mathbb{R}/c\mathbb{Z}$; this is the harmonic map in its homotopy class with respect to the conformal structure on S_0 coming from q.

Proof. We apply Corollary 5.2 to find a unique hyperbolic structure $\Sigma_0 \in \mathcal{T}(S_0)$ such that $\mathcal{O}_{\lambda_0}(\Sigma_0) = \alpha$ and such that $\mathcal{O}_{\lambda_0}(\Sigma_0)$ and λ_0 are positively oriented. The leaf space $\mathcal{G}$ of $\mathcal{O}_{\lambda_0}(\Sigma_0)$ is a directed graph with a metric d induced by integrating the transverse measure; there is an oriented edge for every component of α.

The leaf space $\mathcal{G}$ is obtained by collapsing the leaves of $\mathcal{O}_{\lambda_0}(\Sigma_0)$ in each complementary component Y of $\Sigma_0 \setminus \lambda_0$. In other words, a point $p \in Y$ is identified with its nearest point on λ_0. Let $\pi : \Sigma_0 \to \mathcal{G}$ denote the quotient projection. Then π maps the leaves of λ_0 locally isometrically preserving orientation to $\mathcal{G}$ by construction of the transverse measure on $\mathcal{O}_{\lambda_0}(\Sigma_0)$.

The nearest point projection map onto a geodesic in hyperbolic space is a strict contraction away from the geodesic, so the quotient map $\pi : \Sigma_0 \to \mathcal{G}$ is 1-Lipschitz, and the 1-Lipschitz locus is exactly λ_0. There is a canonical map $\mathcal{G} \to \mathbb{R}/c\mathbb{Z}$ that is orientating preserving, locally isometric along the edges of $\mathcal{G}$ and such that the composition with π induces φ on π_1. Let $\tau_0 : \Sigma_0 \to \mathcal{G} \to \mathbb{R}/c\mathbb{Z}$ denote this composition.

Tightness of τ_0 is an immediate consequence of the construction: any sequence of geodesic curves $\gamma_n \subset \Sigma_0$ that converge in the Hausdorff topology to λ_0 will satisfy

$$\lim_{n\to\infty} \frac{|\varphi(\gamma_n)|}{\ell(\gamma)} = 1,$$

giving a lower bound for the supremum of this ratio over all curves γ, while the Lipschitz constant of τ_0 gives an upper bound. This completes the proof of the theorem. □

5.4. **Minimizing laminations in $\mathbb{Z}$-covers with prescribed dynamics.** Now we use Theorem 5.5 to construction tight 1-Lipschitz maps to the circle whose maximum stretch locus has prescribed dynamical properties. More precisely, given a 1-Lipschitz tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ obtained by collapsing leaves of $\mathcal{O}_{\lambda_0}(\Sigma_0)$ as in the previous section, let γ be the preimage of a point, which is generically a simple multi-curve. We consider the first return map

$$P : \gamma \cap \lambda_0 \to \gamma \cap \lambda_0$$

obtained via the flow tangent to λ_0 in the positive direction for time c.

In this section, we prove Theorem 5.7, in which we assume that γ is connected and show that the dynamics of P can be prescribed by the data of an interval exchange transformation $T : I \to I$.

Interval Exchange transformations. An *interval exchange transformation* (or *IET*) $T : I \to I$ is a peicewise isometric orientation preserving bijection of an interval I to itself determined by a finite partition of I into sub-intervals $I_1, ..., I_p$ and a permutation π that reorders them. Thus T clearly preserves the Lebesgue measure.

An IET is *minimal* if every orbit is dense. A sufficient condition for minimality is that the lengths $\{|I_j|\}_{j=1}^p$ and $|I|$ are rationally independent and π is *irreducible*, i.e., π does not map $\{1, ..., j\}$ to itself for $j < p$. Thus, almost every IET with a given permutation is minimal with respect to the Lebesgue measure on the p-dimensional simplex parameterizing the lengths of the subintervals.

Recall the following definitions:

Definition 5.6.

(1) A transformation $T : I \to I$ preserving a probability measure ν is called *weakly mixing* if $T \times T : I \times I \to I \times I$ is ergodic with respect to $\nu \times \nu$.

(2) Given two probability measure preserving systems (Y, ν_Y, T) and (Z, ν_Z, S), we say that Z is a *factor* of Y if there exist co-null subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ with $TY' \subseteq Y'$ and $SZ' \subseteq Z'$, and there exists a measurable map $\psi : Y' \to Z'$ satisfying

$$\nu_Z = \psi_* \nu_Y \text{ and } \quad \psi \circ T = S \circ \psi \quad \text{on } Y'.$$

We say the systems are *isomorphic* if ψ is additionally assumed to be invertible.

(3) A continuous transformation $T : I \to I$ is called *topologically weakly mixing* if $T \times T$ is topologically transitive in $I \times I$, or equivalently if all continuous and T-invariant functions $f : I \times I \to \mathbb{R}$ are constant.

It is known that almost every IET with irreducible permutation π is weakly topologically mixing [NR97] and weakly mixing [AF07b].

We show the following:

Theorem 5.7. *Given an irreducible IET $T : I \to I$ not equivalent to a circle rotation, there is a closed oriented surface S_0 such that the following holds. For any primitive cohomology class $\varphi \in H^1(S_0; \mathbb{Z})$ and any positive $c > 0$, there is a marked hyperbolic structure $\Sigma_0 \in \mathcal{T}(S_0)$ and a tight Lipschitz map*

$$\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$$

inducing $c \cdot \varphi$ on cohomology; the minimizing lamination λ_0 is equipped with a transverse measure μ_0 and the first return system $(\tau_0^{-1}(x) \cap \lambda_0, \mu_0, P)$ is a factor of (I, Leb, T).

If μ_0 has no atoms (equivalently, T has no periodic orbits or λ_0 has no closed leaves), then the two systems are isomorphic.

Proof. Our plan is to build a (half) translation structure on a closed surface S_0 as a suitable suspension of $T : I \to I$ with constant roof function $c > 0$. The leaf space of the vertical measured foliation is the circle $\mathbb{R}/c\mathbb{Z}$ and the quotient map represents φ. Once we have built this flat structure, we use Theorem 5.5 to hyperbolize this example and conclude the theorem.

We build the flat structure as follows. Endow the rectangle $[0, c] \times I \subset \mathbb{C}$ with its Euclidean structure, and identify its horizontal sides isometrically by translation and its vertical sides by $(c, p) \sim (0, T(p))$. See Figure 7.

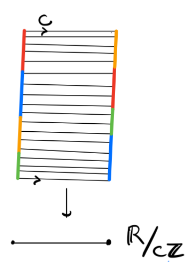


Figure 7. The suspension of an IET with constant roof function. Horizontal black lines each have length c, wrap once around the circle, and becoming the maximal stretch lamination. Vertical lines are pairwise isotopic simple closed curves, except for the critical graph of vertical saddle connections.

The resulting object lies in the moduli space of (half) translation structures of the underlying closed orientable surface S_0. Choose a marking homeomorphism to determine a point $q \in \mathcal{QT}(S_0)$ corresponding to this translation structure. Since the action of the mapping class group of S_0 on $H^1(S_0, c\mathbb{Z})$ acts transitively on primitive $c\mathbb{Z}$-valued cohomology classes, we may (re-)choose the marking so that the quotient map $q \to \mathbb{R}/c\mathbb{Z}$ collapsing the vertical foliation represents the class of $c \cdot \varphi$.

We observe that the non-singular leaves of the vertical foliation are isotopic simple closed curves. Let α denote this isotopy class, oriented suitably.

Then $c\cdot\alpha$ represents the Poincaré dual of $c\cdot\varphi$, and α meets positively the horizontal oriented measured foliation of q. Call the corresponding lamination λ_0 and transverse measure μ_0.

We apply Theorem 5.5 which gives us a hyperbolic metric $\Sigma_0 \in \mathcal{T}(S_0)$ such that $\mathcal{O}_{\lambda_0}(\Sigma_0) = \alpha$ and a 1-Lipschitz tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ inducing φ on π_1, with maximal stretch locus λ_0. Moreover, $\mathcal{O}_{\lambda_0}(\Sigma_0)$ is isotopic to the vertical foliation of q (c.f. [CF21, Proposition 5.10]).

After postcomposing τ_0 with a rotation of the circle, we may assume that the critical graph of $\mathcal{O}_{\lambda_0}(\Sigma_0)$ lies entirely in the fiber over $c\mathbb{Z}$. Then the following follow directly from the construction:

- For any $x, y \in \mathbb{R}/c\mathbb{Z} \setminus \{c\mathbb{Z}\}$, the pre-images $\tau_0^{-1}(x)$ and $\tau_0^{-1}(y)$ are isotopic simple closed curves.
- The Poincaré first return map $P : \tau_0^{-1}(x) \cap \lambda_0 \to \tau_0^{-1}(x) \cap \lambda_0$ is induced by the geodesic flow tangent to λ_0 at time c in the forward direction.

We can define a map $\tau_0^{-1}(x) \cap \lambda_0 \to I$ by choosing a point $p \in \tau_0^{-1}(x)$ and integrating

$$\pi : q \mapsto \int_p^q d\mu_0 \in I,$$

where we use the positively oriented arc from p to q. For a suitable choice of p, this map is dynamics preserving, i.e., $P^n(q) = T^n(\pi(q))$ for all q and n; see [CF21, Proposition 5.10]. If μ_0 has no atoms, then π is also continuous, measure preserving factor map and injective outside of a countable set, hence invertible Lebesgue almost everywhere. When μ_0 has atoms, our factor map goes in the other direction, namely, we can collapse the holes made by the finitely many atoms in the image of π to define a measure preserving map $I \setminus \{\text{singular orbits}\} \to \tau_0^{-1}(x) \cap \lambda_0$ that inverts π. There are only countably many singular orbits, so this mapping is defined almost everywhere and gives the desired factorization. $\square$

5.5. **Geometric convergence.** In this section we construct examples geometrically close $\mathbb{Z}$-covers where the dynamics of the corresponding biminimizing loci are very different. We will use this example to prove Theorem 1.13 that the topology of N-orbit closures can change in the limit along a sequence of regular $\mathbb{Z}$-covers which are arbitrarily close, geometrically.

Theorem 5.8. *Let S_0 be a closed orientable surface of genus ≥ 2 and $S \to S_0$ be a $\mathbb{Z}$-cover. Let $S \to \Sigma$ be a marked hyperbolic metric constructed from a weak-mixing IET as in Theorem 5.7 with minimizing lamination $\lambda \subset \Sigma$. There exist marked hyperbolic metrics $S \to \Sigma_n$ satisfying:*

(1) The minimizing lamination $\lambda_n \subset \Sigma_n$ consists of finitely many (uniformly) isolated geodesics and $\lambda_n \to \lambda$ in the Hausdorff topology.
(2) There are $(1 + \varepsilon_n)$-bi-Lipschitz maps $f_n : \Sigma \to \Sigma_n$ in the homotopy classes of the markings, where $\varepsilon_n \to 0$.

(3) Denote by d the metric on Σ and d_n the metric on Σ_n. There is a constant C such that

$$|d(p,q) - d_n(f_n(p), f_n(q))| < C$$

for all $p, q \in \Sigma$ and n.

Proof. Let $T : I \to I$ be a weak-mixing IET. We can approximate T by periodic IETs $T_n : I \to I$ with the same defining permutation by taking subinvtervals $I_{n,1}, ..., I_{n,p}$ whose lengths are rational multiples of each other and which converge to the subintervals I_1, ..., I_p defining T.

For $c > 0$ fixed, we form the suspended flat structures on the underlying closed surface S_0 as in the proof of Theorem 5.7. After choosing marking appropriately, these correspond to points $q_n \in \mathcal{QT}(S_0)$ which converge to the (marked) suspension q constructed from T and c where the maps to $\mathbb{R}/c\mathbb{Z}$ are homotopic, hence represent the same cohomology class $\varphi \in H^1(S_0, c\mathbb{Z})$.

The horizontal measured foliation of q_n is equivalent to a measured geodesic lamination $\lambda_{0,n}$ which is a weighted multi-curve.

Before we continue, we need the following

Lemma 5.9. *In the Hausdorff topology on geodesic laminations, the supports of $\lambda_{0,n}$ converge to the support to the geodesic lamination λ_0 equivalent to the horizontal foliation of q.*

Proof. There is a natural oriented train track σ with one switch and p branches constructed from the permutation π defining T. The horizontal foliations of q_n are each carried by this train track with weights corresponding to the lengths of the intervals $I_{n,j}$.

There is a splitting sequence $\sigma = \sigma_1 \succ \sigma_2 \succ ...$ such that the intersection of the positive cones of measures carried by σ_i is the set of measures with support equal to λ_0. Since the $\lambda_{0,n}$ are converging to the horizontal foliation of q in measure, for every K, there is an N such that for $n \geq N$, σ_K carries $\lambda_{0,n}$. It follows, using e.g. [ZB04], that $\lambda_{0,n} \to \lambda_0$ in the Hausdorff topology. □

Using Theorem 5.1, let $\Sigma_{0,n} \in \mathcal{T}(S_0)$ be such that $\mathcal{O}(\Sigma_{0,n}, \lambda_{0,n}) = q_n$. Together with Theorem 5.3, Lemma 5.9 implies that $\Sigma_{0,n} \to \Sigma_0 \in \mathcal{T}(S_0)$. Convergence $\Sigma_{0,n} \to \Sigma_0 \in \mathcal{T}(S_0)$ implies that there are $(1+\varepsilon_n)$-bi-Lipschitz maps $f_{0,n} : \Sigma_{0,n} \to \Sigma_0$ in the homotopy class compatible with the markings, where $\varepsilon_n \to 0$.

Let $\tau_{0,n} : \Sigma_{0,n} \to \mathbb{R}/c\mathbb{Z}$ (also $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$) be the 1-Lipschitz tight maps with minimizing laminations $\lambda_{0,n}$ (or λ_0) obtained from either Theorem 5.5 or Theorem 5.7. Lift $f_{0,n}$ to $\mathbb{Z}$-covers $f_n : \Sigma \to \Sigma_n$ and let $\tau_n : \Sigma_n \to \mathbb{R}$ and $\tau : \Sigma \to \mathbb{R}$ be lifts. Then $f_n : \Sigma \to \Sigma_n$ are $(1+\varepsilon_n)$-bi-Lipschitz. We still need to show that the metrics d and $f_n^* d_n$ only differ by an additive error, which follows from the following claim together with Lemma 3.2.

Claim. *For all $p, q \in \Sigma$, there is a C' such that*
$$||\tau(p) - \tau(q)| - |\tau_n(f_n(p)) - \tau_n(f_n(q))|| \leq C'$$
for all n.

Proof of the Claim. Let $k \in \mathbb{Z}$ be such that $|\tau(p) - \tau(k.q)| \leq c$ (there are at most two possibilities). Then $\mathbb{Z}$-equivariance of τ gives
$$c(k-2) \leq |\tau(p) - \tau(q)| \leq c(k+2).$$

Let D be the diameter of Σ_0. Then
$$d(p, k.q) \leq (c + D).$$
Since f_n is $(1+\varepsilon_n)$-bilipschitz, we have
$$d_n(f_n(p), f_n(k.q)) \leq (1+\varepsilon_n)(c+D).$$
Since τ_n is 1-Lipschitz, we obtain
$$|\tau_n(f_n(p)) - \tau_n(f_n(q))| \leq (1+\varepsilon_n)(c+D).$$
Now $\mathbb{Z}$-equivariance of both f_n and u_n give us
$$ck - (1+\varepsilon_n)(c+D) \leq |\tau_n(f_n(p)) - \tau_n(f_n(q))| \leq ck + (1+\varepsilon_n)(c+D).$$

Taking $C' = \max_n\{2c + (1+\varepsilon_n)(c+D)\}$ proves the claim. $\square$

This concludes the proof of the theorem. $\square$

5.6. **An example with a single bi-minimizing ray.** Here, we construct a $\mathbb{Z}$-cover of a closed surface (of any genus) with a single bi-minimizing ray. This construction is elementary and ad hoc, as opposed to being an application of either Theorem 5.5 or Theorem 5.7.

Lemma 5.10. *For any closed oriented surface S_0 of genus at least 2 and primitive class $\varphi \in H^1(S_0, c\mathbb{Z})$, there is a hyperbolic structure $\Sigma_0 \in \mathcal{T}(S_0)$ such that the $\mathbb{Z}$-cover Σ corresponding to φ has a single bi-minimizing geodesic so that $\mathcal{Q}_\omega$ consists of a single geodesic line.*

The construction occupies the rest of the section. The idea is to build a metric structure on a torus with one short boundary component, which we can then attach to any higher genus surface with one boundary component mapping trivially to the circle $\mathbb{R}/c\mathbb{Z}$. The long skinny neck of the torus acts as a barrier: no bi-minimizing geodesic wants to take an excursion into this region.

Proof. Let $\Sigma_0(s, a)$ be a (finite area) hyperbolic metric on a one holed torus with totally geodesic boundary curve γ of length s such that there is a pair of simple closed geodesics α and β intersecting once orthogonally where the length of α is a. The length of β is c. These conditions determine $\Sigma_0(s, a)$ uniquely (up to marking). We allow a to be any positive number (allowing c to be any positive number), and take s such that the collar neighborhood about γ has length larger than the length of β plus the length of α. For example $s = e^{-10}$ and $a = 1$ would work.

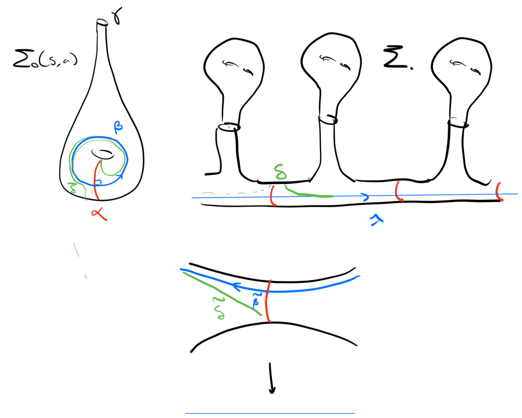


FIGURE 8.

Let Σ_0 be a hyperbolic metric on a surface of genus at least 2 where $\Sigma_0(s,a)$ embeds isometrically. We form a regular $\mathbb{Z}$-cover Σ of Σ_0 by cutting open Σ_0 along α and placing $\mathbb{Z}$ many copies of this building block together, one after the next. Note that this is the cover corresponding to the Poincaré dual of α equipped with the appropriate orientation. Then the preimage λ of β consists of a single bi-infinite geodesic exiting both ends of Σ. We claim that λ is the only bi-minimizing geodesic in Σ.

Suppose $\delta \subset \Sigma$ is bi-minimizing. Then δ exits the ends of Σ and must pass through every component of the preimage of α. The distance in Σ between any pair of points in two neighboring translates of α is at most the length of β plus half that of α. We chose γ with a long collar neighborhood, so δ will never cross γ. We also know that δ will never cross λ, as the shortest path between two points in λ is in λ. This comes from the fact that λ is the concatenation of the unique orthogeodesic segments joining any two components in the preimage of α.

Thus δ is asymptotic to λ in its forward and backward direction, does not cross λ, and its projection to Σ_0 lives in the 1-holed torus $\Sigma_0(s,a)$. There is only one bi-infinite geodesic $\bar{\delta} \subset \Sigma_0(s,a)$ with this property that is not equal to β; it is obtained by cutting open α at its intersection with β and spinning one endpoint of this arc along β in the forward direction and one endpoint in the backward direction; see Figure 8.

Let $\tilde{\delta}$ be the preimage of $\bar{\delta}$ that meets α in the annular covering space of Σ_0 associated to α. In this covering space, there is a 1-Lipschitz retraction onto the preimage $\tilde{\beta}$ of β; the local Lipschitz constant is strictly less than 1 away from the preimage of β. Points on $\tilde{\delta}$ very far from α are very close to $\tilde{\beta}$, so their distance is arbitrarily close to the distance between the closest points in $\tilde{\beta}$. However, the distance along $\tilde{\delta}$ is some definite amount larger, for example, because it spends a definite amount of time in the locus where the Lipschitz constant of this retraction is (definitely) smaller than 1. Since a neighborhood of the union of $\tilde{\delta}$ and $\tilde{\beta}$ embeds isometrically into Σ, this contradicts the assumption that δ was bi-minimizing. We conclude that λ was the only bi-minimizing geodesic.

Using again transitivity of the mapping class group of S_0 on primitive classes in $H^1(S_0, c\mathbb{Z})$ allows us to embed this example in any cohomology class. □

Remark 5.11. Note that whenever Σ contains a single bi-minimizing line, then the set Λ_{nh} of all endpoints in S^1 of quasi-minimizing rays contains exactly two Γ-orbits and is in particular countable.

6. A uniform Busemann function

Fix a 1-Lipschitz tight function $\tau : \Sigma \to \mathbb{R}$ as discussed in section 3. Let $B \subset \Sigma$ be a sufficiently large ball satisfying $p_{\mathbb{Z}}(B) = \Sigma_0$. Notice that the complement of B in Σ has two connected components, one corresponding to unbounded positive values of τ and the other to unbounded negative values. We shall respectively refer to these two components as the positive and negative ends of Σ.

Abusing notation, we shall refer to τ as a function defined over the frame bundle via $\tau \circ p_K$. Consider the function $\beta_+ : G_d/\Gamma \to [-\infty, \infty)$ defined by

$$\beta_+(x) = \lim_{t\to\infty} \tau(a_t x) - t.$$

Notice that since τ is 1-Lipschitz the function $\tau(a_t x) - t$ is monotonically decreasing in t. That is,

$$\beta_+(x) = \inf_{t\geq 0}(\tau(a_t x) - t)$$

and in particular $\beta_+ \leq \tau$.

Lemma 6.1. *The function $\beta_+ : G_d/\Gamma \to [-\infty, \infty)$ is*

1. *upper semicontinuous;*
2. *N-invariant; and*
3. *satisfies for any $x \in G_d/\Gamma$ and $s \in \mathbb{R}$*

$$\beta_+(a_s x) = \beta_+(x) + s.$$

Proof. The function β_+ is defined as a limit of a decreasing sequence of continuous functions, thus implying (1). For (2), note that $d(a_t n x, a_t x) \to 0$ as $t \to \infty$ for any $x \in G_d/\Gamma$ and $n \in N$, and recall that τ is 1-Lipschitz.

Changing variables gives

$$\begin{aligned}\beta_+(a_s x) &= \lim_{t\to\infty}[\tau(a_{t+s}x) - t] = s + \lim_{t\to\infty}[\tau(a_{t+s}x) - (t+s)] = \\ &= s + \beta_+(x),\end{aligned}$$

implying (3). □

Recall the notations from section 3.1. Define

$$\mathcal{Q}_\pm = \{x \in \mathcal{Q} : \lim_{t\to\infty} \tau(a_t x) = \pm\infty\},$$

and respectively $\mathcal{L}_\pm = \mathcal{L} \cap \mathcal{Q}_\pm$. Let x be in $\mathcal{L}_+$, then by definition the geodesic line Ax is in the maximal stretch locus for τ (and is hence bi-minimizing) and facing the positive end. In particular, $\tau(a_t x) = \tau(x) + t$ for all $t \in \mathbb{R}$ and thus

$$\beta_+(x) = \lim_{t\to\infty} [\tau(x) + t - t] = \tau(x).$$

More generally, we have the following characterization of $\mathcal{Q}_+$:

Lemma 6.2. $\mathcal{Q}_+ = \beta_+^{-1}(\mathbb{R})$.

Proof. Recall that $x \in \mathcal{Q}$ if and only if there exists $C > 0$ for which

$$d(x, a_t x) \geq t - C \quad \text{for all } t.$$

On the other hand, $\beta_+(x) \in \mathbb{R}$ if and only if there exists $c > 0$ for which

$$\tau(a_t x) \geq t - c \quad \text{for all } t.$$

By lemma 3.2 we have

$$d(x, a_t x) - C_\tau \leq \tau(a_t x) - \tau(x) \leq d(x, a_t x),$$

and the claim follows.

□

The function β_+ as described above serves as a sort of "uniform Busemann function" for all the non-horospherical limit points of Γ associated to the positive end of Σ. Denote by

$$\mathcal{H}_+(x) := \beta_+^{-1}([\beta_+(x), \infty)) \tag{6.1}$$

the β_+*-horoball through* x. An immediate consequence of the above lemmata is the following:

Corollary 6.3. *For any* $x \in \mathcal{Q}_+$

$$\overline{Nx} \subseteq \mathcal{H}_+(x).$$

Proof. Since β_+ is N-invariant we have $Nx \subseteq \beta_+^{-1}(\beta_+(x))$. Upper semicontinuity ensures the set $\beta_+^{-1}([\beta_+(x), \infty))$ is closed, implying the claim. □

One can similarly define the function $\beta_- : G_d/\Gamma \to (-\infty, \infty]$ by

$$\beta_-(x) = \lim_{t\to\infty} \tau(a_t x) + t = \sup_{t\geq 0}(\tau(a_t x) + t) \geq \tau(x),$$

which would serve as a uniform Busemann function for the negative end of Σ. The function β_- is lower semicontinuous with $\mathcal{Q}_- = \beta_-^{-1}(\mathbb{R})$. Similarly

$$\overline{Nx} \subseteq \mathcal{H}_-(x)$$

for all $x \in \mathcal{Q}_-$, where $\mathcal{H}_-(x) := \beta_-^{-1}((-\infty, \beta_-(x)])$.

7. The recurrence semigroup

This section is devoted to studying the sub-invariance properties of non-maximal horospherical orbit closures, the main result of which is:

Theorem 7.1. *Let $\mathbb{H}^d/\Gamma$ be any $\mathbb{Z}$-cover of a compact hyperbolic d-manifold and let $x \in G_d/\Gamma$ be any quasi-minimizing point. Then there exists a non-compact, non-discrete closed subsemigroup Δ_x of $MA_{\geq 0}$ under which $\overline{Nx}$ is strictly sub-invariant, that is,*

$$\ell\overline{Nx} \subsetneq \overline{Nx} \quad \textit{for all } \ell \in \Delta_x \smallsetminus \{e\}.$$

A special case of this theorem, in dimension $d = 2$, was presented in the introduction as Theorem 1.9(i).

While our focus is on $\mathbb{Z}$-covers, much of the tools developed in this section are applicable in greater generality and we encourage the reader to consider arbitrary discrete subgroups as they read through the text.

Given $x \in G_d/\Gamma$ consider the following subset of MA:

$$\Delta_x = \{\ell \in MA : \ell x \in \overline{Nx}\}.$$

Note that if $\ell_1, \ell_2 \in \Delta_x$ then

$$\ell_2\overline{Nx} = \overline{N\ell_2 x} \subseteq \overline{Nx},$$

where the equality follows from the fact that ℓ_2 normalizes N and the inclusion follows from $\overline{Nx}$ being closed and N-invariant. Therefore

$$\ell_2\ell_1 x \in \ell_2\overline{Nx} \subseteq \overline{Nx},$$

implying:

Lemma 7.2. *Δ_x is a closed subsemigroup of MA.*

We call Δ_x the *recurrence semigroup* of the N-orbit to the "line" MAx (in dimension 2 it is the line Ax). Clearly when x is not quasi-minimizing then $\Delta_x = MA$, as $\overline{Nx} = \mathcal{E}_\Gamma$. In contrast we have:

Lemma 7.3. *If $x \in G_d/\Gamma$ is quasi-minimizing then*

$$\Delta_x \subseteq MA_{\geq 0} \tag{7.1}$$

where $A_{\geq 0} = \{a_t : t \geq 0\}$.

Roughly speaking, the lemma above shows that horospherical orbits at quasi-minimizing points can only accumulate 'forward', deeper into their associated end.

Proof. Let $x = g\Gamma$ and assume in contradiction that $ma_{-t}x \in \overline{Nx}$ for some $t > 0$. Since Δ_x is a semigroup we have $m^k a_{-kt}x \in \overline{Nx}$ for all $k \geq 1$. This implies in particular that for any k and any $\varepsilon > 0$ there exists $n \in N$ and $\gamma \in \Gamma$ satisfying

$$h_\varepsilon m^k a_{-kt} g\gamma = ng$$

for some $h_\varepsilon \in G_d$ with $\|h_\varepsilon\|_{G_d} < \varepsilon$. Having k fixed and choosing ε arbitrarily small we may ensure

$$h'_\varepsilon g\gamma \in NMa_{kt}g$$

where $\|h'_\varepsilon\|_{G_d} < 1$. That is, there exists an element of $g\Gamma$ roughly $kt > 0$ 'deep' inside the horoball bounded by Ng. As $k \geq 1$ was arbitrary this is in contradiction with the assumption that $g\Gamma$ is quasi-minimizing and hence g^+ is a non-horospherical limit point, see Figure 9 and recall Lemma 2.2. $\square$

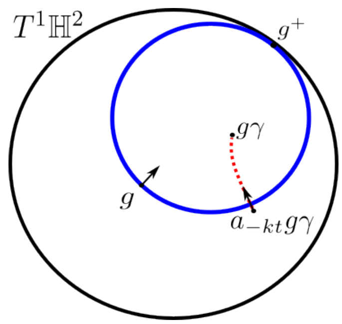


FIGURE 9.

Remark 7.4. In the case of a $\mathbb{Z}$-cover one can deduce Lemma 7.3 directly from Corollary 6.3 and property (3) of Lemma 6.1.

A natural question at this point is — *What is Δ_x for x quasi-minimizing?* In what follows we provide tools to study this question and examples in which a complete answer can be given. See also Bellis [Bel18] which has studied this question in dimension $d = 2$.

A closed subgroup $\Theta \leq G_d$ is called a *geometric limit* of a sequence of closed subgroups $\Gamma_j \leq G_d$ if there exists a subsequence Γ_{j_k} which converges to Θ in the Hasudorff metric when restricted to any compact subset of G_d (this is referred to as convergence in the Chabauty topology on the space of closed subgroups of G_d).

Recalling the Bruhat decomposition in Equation (2.2), we define the following projection $\delta : G_d \to MA$ by

$$\delta(n\ell u) = \ell$$

for all $u \in U, \ell \in MA$ and $n \in N$, and set $\delta(g) = e$ whenever $g \notin NMAU$. We have the following:

Lemma 7.5. *Given $x = g\Gamma$, let Θ be any geometric limit of the family*

$$\{a_t g\Gamma g^{-1} a_{-t}\}_{t\geq 0},$$

and let $e \neq \theta \in \Theta \cap NMAU$. Then $\delta(\theta)x$ is an accumulation point of Nx. In particular, $\delta(\Theta) \subset \Delta_x$.

Note that the conjugate $a_t g\Gamma g^{-1}a_{-t}$ is the stabilizer of the point $a_t x = a_t g\Gamma$ in G/Γ. Therefore, in a sense, geometric limits Θ as above capture the asymptotic geometry "as seen" along the geodesic ray $(a_t x)_{t\geq 0}$. One can consider the following closed subset of G_d associated with the point $x = g\Gamma$:

$$\Xi_x = \bigcap_{k\geq 1}\overline{\bigcup_{t\geq k} a_t g\Gamma g^{-1}a_{-t}}. \tag{7.2}$$

This is the union of all geometric limits Θ as in lemma 7.5, hence as a direct consequence we have:

Lemma 7.6. *For all $x \in G_d/\Gamma$*

$$\delta(\Xi_x) \subseteq \Delta_x.$$

Remark 7.7. See also [LL22, Theorem 1.3] which is somewhat similar in form to the lemma above and deals with horospherical measure rigidity.

Proof of lemma 7.5. Let $\theta_0 = n_0\ell_0u_0$ be an element in $\Theta \cap NMAU$. Hence there exist sequences $\gamma_j \in \Gamma, u_j \in U, \ell_j \in MA$ and $n_j \in N$ satisfying

$$a_{t_j} g\gamma_j g^{-1} a_{-t_j} = n_j\ell_j u_j$$

where $u_j \to u_0, \ell_j \to \ell_0$ and $n_j \to n_0$. The latter convergence relations follow from $NMAU$ being open in G_d and the fact that the multiplication map $N \times MA \times U \to NMAU$ is a diffeomorphism (see Section 2.3). Therefore, for all j

$$g\gamma_j = a_{-t_j} n_j\ell_j u_j a_{t_j} g.$$

This in turn implies

$$\begin{aligned} g\Gamma &= a_{-t_j} n_j\ell_j u_j a_{t_j} g\Gamma = \\ &= (a_{-t_j} n_j a_{t_j})\ell_j(a_{-t_j} u_j a_{t_j}) g\Gamma. \end{aligned}$$

Since $t_j \to +\infty$ we are ensured that $a_{-t_j} u_j a_{t_j} \to e$. Denoting

$$\tilde{n}_j = a_{-t_j} n_j^{-1} a_{t_j} \in N$$

we conclude

$$\tilde{n}_j g\Gamma = \ell_j(a_{t_j} u_j a_{-t_j}) g\Gamma \to \ell_0 g\Gamma.$$

In other words, the N-orbit of x accumulates onto $\ell_0 x$ and in particular $\ell_0 x \in \overline{Nx}$ and $\ell_0 \in \Delta_x$. □

From this simple lemma we draw several useful conclusions:

Proposition 7.8. *Let Θ be a geometric limit as above and assume that Θ is Zariski dense in G_d. Then Nx has an accumulation point on $MAx \setminus Mx$ and in particular Δ_x is non-compact.*

Proof. The set $NMAU \setminus NMU$ is Zariski open. Indeed (see discussion in Section 2.3), since the multiplication map $N \times MA \times U \to NMAU$ is an isomorphism of varieties then NMU is Zariski closed in $NMAU$. Therefore if Θ is Zariski dense there must exist an element $\theta \in \Theta \cap NMAU \setminus NMU$. This directly gives $\delta(\theta) \notin M$, implying the claim. □

More generally, denote by $I(x)$ the limit inferior of the injectivity radius in G_d/Γ along the geodesic ray $(a_t x)_{t\geq 0}$, that is,

$$I(x) = \liminf_{t\to+\infty} \mathrm{Inj}_{G_d/\Gamma}(a_t x).$$

Corollary 7.9. *Assume Γ is torsion-free. Then if $I(x) < \infty$ then Nx has an accumulation point on MAx. Moreover, if $I(x) = 0$ then x itself is an accumulation point of Nx.*

Remark 7.10. See [Bel18, Theorem 1.1] for a different proof of this statement in the case of $d = 2$. See also [CM10].

Proof. The assumption that $I(g\Gamma) < \infty$ implies there exist $R > 0$ and $t_j \to \infty$ for which

$$a_{t_j} g\Gamma g^{-1} a_{-t_j} \cap B_R^{G_d} \neq \{e\} \quad \text{for all } j,$$

where $B_R^{G_d}$ is a ball of radius R around the identity in G_d. In particular, there exists an accumulation point $g_j \to g_0$ where $e \neq g_j \in a_{t_j} g\Gamma g^{-1} a_{-t_j} \cap B_R^{G_d}$. Therefore, we conclude $g_0 \in \Xi_{g\Gamma}$.

If $e \neq g_0 \in NMAU$ then by lemma 7.5 we conclude Nx accumulates onto $\delta(g_0)x$. If $g_j \to e$ then for any $\varepsilon > 0$ we may replace g_j by some power $g_j^m \in a_{t_j} g\Gamma g^{-1} a_{-t_j} \cap B_R^{G_d}$ satisfying $\frac{1}{2}\varepsilon \leq \|g_j^m\|_{G_d} < \varepsilon$ (recall that since Γ is torsion-free, all g_j generate a non-compact subgroup in G_d). Hence $\Xi_{g\Gamma} \cap B_\varepsilon^{G_d} \setminus B_{\frac{1}{2}\varepsilon}^{G_d} \neq \emptyset$ for all small enough ε, implying Nx accumulates onto x. Notice that if $I(g\Gamma) = 0$ then there exist $g_j \to e$ as above.

In the remaining case where $g_0 \in NMA\omega$ we notice that

$$g_0^2 \in NMA\omega NMA\omega = NMAU$$

and so the above argument shows Nx accumulates onto $\delta(g_0^2)x$. □

Corollary 7.11. *If $x \in G_d/\Gamma$ is quasi-minimizing and Θ is a geometric limit of $\{a_t g\Gamma g^{-1} a_{-t}\}_{t\geq 0}$, then*

$$\Theta \cap NMA_{<0}U = \emptyset$$

where $A_{<0} = \{a_t : t < 0\}$.

Proof. By lemma 7.3 we know that $\Delta_x \cap MA_{<0} = \emptyset$. Lemma 7.5 concludes the proof. □

Recall the definition of Ξ_x in (7.2), for $x = g\Gamma$. In some cases the δ projection of Ξ_x completely determines Δ_x. One such immediate case is whenever $\delta(\Xi_x)$ generates, as a closed semigroup, all of $MA_{\geq 0}$. For instance for $d = 2$, whenever e is not an isolated point in $\delta(\Xi_x)$ one can deduce $\Delta_x = A_{\geq 0}$ for x quasi-minimizing.

Another such case is the following:

Lemma 7.12. *Let $x = g\Gamma$ and assume that $g\Gamma g^{-1} \subset \Xi_x$. Then*

$$\overline{\delta(g\Gamma g^{-1})} = \Delta_x$$

where the closure is taken in MA.

Remark 7.13. Note that the hypothesis of this lemma holds for instance whenever the geodesic ray $(a_t g\Gamma)_{t\geq 0}$ in G_d/Γ has $x = g\Gamma$ as an accumulation point. In such a case x is not quasi-minimizing in G_d/Γ and $\overline{Nx} = G_d/\Gamma$ (assuming Γ was Zariski-dense) implying in particular that $\Delta_x = MA$. Hence we conclude that $g\Gamma g^{-1}$ has a dense δ-projection into MA.

Proof. The inclusion $(\subseteq)$ is immediate from lemma 7.6 and the fact that Δ_x is closed in MA. For the other direction, let $\ell_0 \in \Delta_x$, then by definition there exist sequences $\tilde{n}_j \in N$, $\gamma_j \in \Gamma$ and $\varepsilon_j \in G$ with

$$\tilde{n}_j g = \varepsilon_j \ell_0 g \gamma_j$$

and $\varepsilon_j \to e$. Recall that the set $NMAU$ is an open neighborhood of the identity in G_d, therefore for all large j we may write $\varepsilon_j = n_j \ell_j u_j$. Thus

$$\tilde{n}_j g = n_j \ell_j u_j \ell_0 g \gamma_j$$

and

$$g\gamma_j^{-1} g^{-1} = (\tilde{n}_j^{-1} n_j)(\ell_j \ell_0) u_j.$$

The right hand side above is written in $NMAU$ form with δ projection equal $\ell_0 \ell_j$. In other words we get $\ell_0 \ell_j \in \delta(g\Gamma g^{-1})$ and since $\ell_j \to e$, the claim follows. $\square$

Remark 7.14. As mentioned the entire discussion above holds just as well for any discrete group $\Gamma < G_d$, not necessarily a normal subgroup of a uniform lattice. We note that actually much of the discussion extends even further to other ambient groups. Let G be any connected semisimple real algebraic group G (of arbitrary rank) and let $\Gamma < G$ be a discrete subgroup. Let P be a minimal parabolic subgroup with Langlands decomposition $P = MAN$ where A is the Cartan subgroup of $\mathbb{R}$-diagonalizable elements, N a contracting horospherical subgroup with respect to a choice of Weyl chamber $\mathfrak{a}^+$, and M the compact centralizer of A. Given $x \in G/\Gamma$, one may define Δ_x as above and the map δ as a projection from the open Bruhat cell $NMAU$ (where U is the corresponding expanding horospherical subgroup). The proof of lemma 7.5 holds almost verbatim for any geometric limit of $\exp(v_j) g\Gamma g^{-1} \exp(-v_j)$ (i.e. an accumulation point in the Chabauty topology) for a regular sequence $v_j \to \infty$ in $\mathfrak{a}^+$. One can also consider a form of corollary 7.11 with respect to a higher rank notion of horospherical limit point in the Furstenberg boundary $P\backslash G$, see [LO22].

7.1. **Returning to $\mathbb{Z}$-covers.** Let us apply what we have learned to the setting of this paper. Assume from this point on that Γ is a normal subgroup of a uniform lattice $\Gamma_0 < G_d$ with $\Gamma_0/\Gamma \cong \mathbb{Z}$.

Lemma 7.15. *If $a_{t_j} p_{\mathbb{Z}}(x) \to h_0\Gamma_0 \in G_d/\Gamma_0$ for some $x \in G_d/\Gamma$ and $t_j \to \infty$, then $h_0 \Gamma h_0^{-1} \subset \Xi_x$.*

Proof. Denote $x = g\Gamma$ and $p_{\mathbb{Z}}(x) = g\Gamma_0$. Since $a_{t_j} g\Gamma_0 \to h_0\Gamma_0$ in G_d/Γ_0, then there exists a sequence of elements $\eta_j \in \Gamma_0$ satisfying $a_{t_j} g\eta_j \to h_0$. This in turn implies the following geometric convergence of subgroups

$$a_{t_j} g\eta_j \Gamma \eta_j^{-1} g^{-1} a_{-t_j} \to h_0 \Gamma h_0^{-1}.$$

But as $\Gamma \lhd \Gamma_0$ we have $\eta_j \Gamma \eta_j^{-1} = \Gamma$, implying that $h_0\Gamma h_0^{-1}$ is a geometric limit of $\{a_{t_j} g\Gamma g^{-1} a_{-t_j}\}_{j\geq 1}$ and hence contained in Ξ_x. ☐

In certain cases, one geometric limit completely determines the semigroup:

Proposition 7.16. *If $p_{\mathbb{Z}}(x)$ is an accumulation point of the geodesic ray $(a_t p_{\mathbb{Z}}(x))_{t\geq 0}$ in G_d/Γ_0, then $\overline{\delta(g\Gamma g^{-1})} = \Delta_x$.*

Proof. Lemma 7.15 together with Lemma 7.12 imply the claim. ☐

Note that the conditions of this proposition are satisfied for example whenever $p_{\mathbb{Z}}(x)$ lies on a closed geodesic in G_d/Γ_0. Recall the definitions and notations from section 3.1.

Proposition 7.17. *If μ is a minimal component of λ_0 and*

$$\mathcal{L}_\mu = p_{\mathbb{Z}}^{-1}(p_K^{-1}(\mu)) \cap \mathcal{L}$$

is the set of tangent frames to its lift in the $\mathbb{Z}$-cover, then the function $x \mapsto \Delta_x$ is constant on $\mathcal{L}_\mu$.

Proof. Let $g_1\Gamma, g_2\Gamma \in \mathcal{L}_\mu$ be any two points. By definition, since the geodesic flow along μ is minimal, the ray $(a_t g_1 \Gamma_0)_{t\geq 0}$ accumulates onto $g_2\Gamma_0$. Hence $g_2\Gamma g_2^{-1} \subset \Xi_{g_1\Gamma}$ and by lemma 7.5

$$\delta(g_2\Gamma g_2^{-1}) \subset \Delta_{g_1\Gamma}.$$

On the other hand, the ray $(a_t g_2 \Gamma_0)_{t\geq 0}$ also accumulates onto $g_2\Gamma_0$ and hence by Proposition 7.16 we get

$$\Delta_{g_2\Gamma} = \overline{\delta(g_2\Gamma g_2^{-1})}.$$

Since $\Delta_{g_1\Gamma}$ is closed we deduce $\Delta_{g_2\Gamma} \subseteq \Delta_{g_1\Gamma}$. By symmetry, the claim follows. ☐

Let us now establish a few basic properties of these semigroups:

Proposition 7.18. *The semigroup Δ_x is non-compact for all $x \in G_d/\Gamma$.*

Proof. Since G_d/Γ_0 is compact, for any $x \in G_d/\Gamma$ there exists a point $h_0\Gamma_0$ which is an accumulation point of $(a_t p_{\mathbb{Z}}(x))_{t\geq 0}$. This implies that there exists a Zariski dense discrete subgroup $h_0\Gamma h_0^{-1}$ contained in Ξ_x. By proposition 7.8 the claim follows. ☐

Remark 7.19. We deduce in particular that the horospherical flow on a regular cover of a compact hyperbolic d-manifold has no minimal components. Indeed, given $x \in G_d/\Gamma$ the set $F = \overline{Nx}$ is not N-minimal. If $F = G_d/\Gamma$ then the existence of a non-horospherical limit point in a geometrically infinite manifold implies that there exists some $y \in F$ with $\overline{Ny} \neq F$. Otherwise, by lemma 7.3 we know that $\Delta_x \cap MA_{<0} = \emptyset$. On the other hand proposition 7.18 implies that $ma_t \in \Delta_x$ for some $m \in M$ and $t > 0$. Consider $y = ma_t x \in F$. Using lemma 7.3 once more we conclude that $x \notin \overline{Ny}$, implying that F is not minimal.

One can formulate weaker conditions under which the conclusion of the corollary above holds. See [Bel18, Corollary 1.2] for a much general statement in $d = 2$.

Given a set $D \subset MA$ denote by D/M its projection onto $A \cong MA/M$. Since M and A commute, this projection is a group homomorphism.

Proposition 7.20. *The A-component Δ_x/M of the recurrence semigroup is non-discrete for all $x \in G_d/\Gamma$.*

Proof. Denote $x = g\Gamma$ and consider the set of accumulation points in G_d/Γ_0 of $(a_t g\Gamma_0)_{t\geq 0}$. Since G_d/Γ_0 is compact and the above set of accumulation points is closed and A-invariant, it contains an A-minimal subset. In particular, the ray $(a_t g\Gamma_0)_{t\geq 0}$ accumulates onto a point $h_0\Gamma_0$ satisfying that

$$a_{t_j} h_0\Gamma_0 \to h_0\Gamma_0 \quad \text{for some } t_j \to \infty. \tag{7.3}$$

Lemma 7.15 implies that $h_0\Gamma h_0^{-1} \subset \Xi_x$ and hence $\delta(h_0\Gamma h_0^{-1}) \subset \Delta_x$, by Lemma 7.5. We will show that the projected set $\delta(h_0\Gamma h_0^{-1})/M$ is non-discrete.

The group $h_0\Gamma h_0^{-1}$ is Zariski-dense and hence contains an element

$$\gamma_0 \in h_0\Gamma h_0^{-1} \cap NMAU \setminus (NMA \cup MAU \cup NU)\,.$$

In other words, $\gamma_0 = n_0\ell_0u_0$ for some $n_0 \in N$, $u_0 \in U$, and $\ell_0 \in MA$, all not equal to the identity.

By (7.3), we have the following convergence

$$a_{t_j} h_0\Gamma h_0^{-1} a_{-t_j} \to h_0\Gamma h_0^{-1},$$

in the sense of Hausdorff convergence on compact subsets. In particular, there exists a sequence $\gamma_j \in h_0\Gamma h_0^{-1}$ satisfying

$$a_{t_j}\gamma_j a_{-t_j} \to \gamma_0. \tag{7.4}$$

The multiplication map $N \times MA \times U \to NMAU$ is a diffeomorphism implying by (7.4) that there exist $n_j \in N, \ell_j \in MA$, and $u_j \in U$ satisfying

$$a_{t_j}\gamma_j a_{-t_j} = n_j\ell_j u_j \quad \text{for all } j$$

with

$$n_j \to n_0 \quad , \quad \ell_j \to \ell_0 \quad \text{and} \quad u_j \to u_0.$$

In particular

$$\gamma_j = a_{-t_j} n_j \ell_j u_j a_{t_j} = (a_{-t_j} n_j a_{t_j}) \ell_j (a_{-t_j} u_j a_{t_j}) \in h_0 \Gamma h_0^{-1}.$$

Note that $\delta(\gamma_j) = \ell_j$. If $\ell_j/M \neq \ell_0/M$ for some infinite subsequence of j's, this would immediately imply the non-discreteness of $\delta(h_0\Gamma h_0^{-1})/M$, as claimed.

Assume therefore the converse, that $\ell_j/M = \ell_0/M$ for all but finitely many j's. Now consider the sequence of elements $\gamma_j\gamma_0 \in h_0\Gamma h_0^{-1}$. We claim that

Claim. $\delta(\gamma_j\gamma) \to \ell_0^2$ *but* $\delta(\gamma_j\gamma)/M \neq \ell_0^2/M$ *for all large* j.

This would conclude the proof of the proposition.

Consider

$$\gamma_j\gamma = (a_{-t_j} n_j a_{t_j}) \ell_j (a_{-t_j} u_j a_{t_j}) n_0 \ell_0 u_0,$$

and note that by the definition of the map δ we have

$$\delta(\gamma_j\gamma) = \delta(\ell_j (a_{-t_j} u_j a_{t_j}) n_0 \ell_0).$$

As $j \to \infty$ we have $a_{-t_j} u_j a_{t_j} \to e$ and $\ell_j \to \ell_0$, hence

$$\ell_j (a_{-t_j} u_j a_{t_j}) n_0 \ell_0 \to \ell_0 n_0 \ell_0 \in NMA.$$

Since $NMAU$ is an open set containing n_0, we are ensured that the element $(a_{-t_j} u_j a_{t_j}) n_0$ may be written as

$$(a_{-t_j} u_j a_{t_j}) n_0 = n'_j \ell'_j u'_j \quad \text{for all large } j.$$

In particular,

$$\begin{aligned} \ell_j((a_{-t_j} u_j a_{t_j}) n_0) \ell_0 &= \ell_j (n'_j \ell'_j u'_j) \ell_0 = \\ &= (\ell_j n'_j \ell_j^{-1}) \ell_j \ell'_j \ell_0 (\ell_0^{-1} u'_j \ell_0). \end{aligned}$$

Since $\ell_j n'_j \ell_j^{-1} \in N$ and $\ell_0^{-1} u'_j \ell_0 \in U$ we have

$$\delta(\gamma_j\gamma_0) = \delta(\ell_j (a_{-t_j} u_j a_{t_j}) n_0 \ell_0) = \ell_j \ell'_j \ell_0.$$

Since $n'_j \ell'_j u'_j \to n_0$, we know $\ell'_j \to e$ implying that $\delta(\gamma_j\gamma) \to \ell_0^2$.

Showing that $\delta(\gamma_j\gamma)/M \neq \ell_0^2/M$ amounts to showing

$$\ell'_j = \delta((a_{-t_j} u_j a_{t_j}) n_0) \notin M, \quad \text{for all large } j.$$

Since $(a_{-t_j} u_j a_{t_j}) n_0 \in NMAU$ for all large j it suffices to show that

$$(a_{-t_j} u_j a_{t_j}) n_0 \notin NMU, \quad \text{for all large } j.$$

We will do so by using an explicit representation of G_d.

Let us consider the matrix representation of $G_d = \mathrm{SO}^+(d,1)$ as the identity component of the group of $(d+1) \times (d+1)$ matrices preserving the quadratic form $-2x_0x_d + \sum_{i=1}^{d-1} x_i^2$. Under this representation we have the following subgroups

$$MA = \left\{ \begin{bmatrix} e^t & & \\ & m & \\ & & e^{-t} \end{bmatrix} \ : \ m \in \mathrm{SO}(d-1) \quad \text{and} \quad t \in \mathbb{R} \right\},$$

$$N = \left\{ n(s) = \begin{bmatrix} 1 & & \\ s & I_{d-1} & \\ \frac{1}{2}\|s\|^2 & s^t & 1 \end{bmatrix} \ : \ s \in \mathbb{R}^{d-1} \ \text{a column vector} \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{d-1}$, and

$$U = \left\{ u(s) = n(s)^t \ : \ s \in \mathbb{R}^{d-1} \right\}.$$

Returning to our previous notations of $n_0 \in N$ and $u_j \to u_0$ in U, set non-zero vectors s_0, r_0 and r_j in $\mathbb{R}^{d-1}$ satisfying

$$n_0 = n(s_0) \quad , \quad u_0 = u(r_0) \quad \text{and} \quad u_j = u(r_j).$$

Recall that

$$a_{-t_j} u(r_j) a_{t_j} = u(e^{-t_j} r_j) \quad \text{for all } j.$$

Now assume in contradiction that for an unbounded sequence of j's

$$u(e^{-t_j} r_j) n(s_0) = n(v_j) \begin{bmatrix} 1 & & \\ & m_j & \\ & & 1 \end{bmatrix} u(w_j) \in NMU, \tag{7.5}$$

for some sequence of vectors $v_j, w_j \in \mathbb{R}^{d-1}$ and matrices $m_j \in \mathrm{SO}(d-1)$.

The 11-coordinate of the left-hand side of equation (7.5) is

$$\big(u(e^{-t_j} r_j) n(s_0)\big)_{11} = 1 + e^{-t_j} r_j \cdot s_0 + \frac{1}{4}\|e^{-t_j} r_j\|^2 \|s_0\|^2,$$

where $\cdot$ denotes the regular dot product in $\mathbb{R}^{d-1}$. On the other hand, the 11-coordinate of the right-hand side of the equation is 1. Hence we get

$$e^{-t_j} r_j \cdot s_0 = -\frac{1}{4}\|e^{-t_j} r_j\|^2 \|s_0\|^2 \tag{7.6}$$

But since $r_j \to r_0 \neq 0$ we see that

$$e^{-t_j} r_j \cdot s_0 \sim e^{-t_j}$$

while

$$\frac{1}{4}\|e^{-t_j} r_j\|^2 \|s_0\|^2 \sim e^{-2t_j},$$

making the equality in (7.6) impossible for large enough j. □

Remark 7.21. Note that one should not expect a general claim of the sort that the δ projection of any Zariski-dense group in G_d is non-discrete. Indeed, a counter example in $\mathrm{PSL}_2(\mathbb{R})$ is

$$\delta(\mathrm{PSL}_2(\mathbb{Z})) = \left\{ \pm \begin{bmatrix} k & \\ & k^{-1} \end{bmatrix} : k \in \mathbb{Z} \right\}$$

which is discrete.

While Δ_x in a $\mathbb{Z}$-cover is never discrete, under certain geometric conditions the point $e \in \Delta_x$ is isolated. Recall the definition of $\mathcal{L} \subset G_d/\Gamma$ from Section 3.1.

Lemma 7.22. *For any $\rho > 0$ there exists $\varepsilon > 0$ such that if $\tau(x) - \beta_+(x) < \varepsilon$ then $(a_t x)_{t\geq 0} \subset \mathcal{L}^{(\rho)}$, the ρ-neighborhood of $\mathcal{L}$.*

Proof. Consider the function $\varphi_+(y) = \tau(y) - \beta_+(y) \geq 0$. Recall that $\beta_+(y) \leq \tau(y)$ for all $y \in G/\Gamma$ and that $\beta_+(y) = \tau(y)$ if and only if $y \in \mathcal{L}$. In particular, $\varphi_+^{-1}(0) = \mathcal{L}$. The function φ_+ roughly measures how much "spare time" the trajectory $(a_t y)_{t\geq 0}$ has to waste during its voyage into the positive end of G/Γ. Since β_+ is upper semi-continuous, φ_+ is lower semi-continuous. Therefore the set $\varphi_+^{-1}([0,\eta])$ is closed for any $\eta > 0$, and

$$F_\eta = p_{\mathbb{Z}}\left(\varphi_+^{-1}([0,\eta])\right)$$

is compact (the function φ_+ is invariant under the deck transformation because $\beta_+(k.y) = (\tau(k.y) - \tau(y)) + \beta_+(y)$).

Since

$$\bigcap_{\eta>0} F_\eta = p_{\mathbb{Z}}(\varphi_+^{-1}(0)) = p_{\mathbb{Z}}(\mathcal{L}) = \mathcal{L}_0$$

we conclude that for any $\rho > 0$ there exists $\varepsilon > 0$ for which $F_\varepsilon \subset \mathcal{L}_0^{(\rho)}$, implying

$$\varphi_+^{-1}([0,\varepsilon]) \subset \mathcal{L}^{(\rho)}.$$

The function $\varphi_+(a_t x)$ is monotonically decreasing in t, for all $x \in G/\Gamma$ by the 1-Lipschitz of τ and item (2) of lemma 6.1. Therefore if the point x satisfies $\tau(x) - \beta_+(x) < \varepsilon$ then the entire ray admits $(a_t x)_{t\geq 0} \subset F_\varepsilon \subset \mathcal{L}^{(\rho)}$, as claimed. ∎

Corollary 7.23. *If x lies on a uniformly isolated ray in $\mathcal{L}$, that is*

$$(Ax)^{(\rho)} \cap \mathcal{L} = Ax \quad \text{for some } \rho > 0,$$

then $p_K(Nx)$ does not accumulate onto $p_K(x)$ in Σ. In particular, there exists an $\varepsilon > 0$ with $\Delta_x \cap MA_{[0,\varepsilon)} = \{e\}$.

Proof. Assume without loss of generality that ρ is smaller than half the infimal injectivity radius in G_d/Γ and

$$\rho < \frac{1}{2} d_{G_d/\Gamma}(Ax, \mathcal{L} \smallsetminus Ax).$$

Let $\varepsilon > 0$ be the constant corresponding to ρ as per the previous lemma.

Now assume in contradiction that Nx accumulates onto $x' \in G_d/\Gamma$ where $p_K(x') = p_K(x)$. Recall that $\beta_+(x) = \tau(x)$, since $x \in \mathcal{L}$, and that $\beta_+(nx) = \beta_+(x)$ for all $n \in N$. Furthermore, since τ is continuous, if $n_j x \to x'$ then $\tau(n_j x) \to \tau(x')$. Since τ depends only on basepoints we further have $\tau(x') = \tau(x)$. This in turn implies that for all large j we have $\tau(n_j x) - \beta_+(n_j x) < \varepsilon$. By the previous lemma, we conclude that $(a_t n_j x)_{t\geq 0}$ is contained in $(Ax)^{(\rho)}$. But $(a_t n_j x)_{t\geq 0}$ and $(a_t x)_{t\geq 0}$ are asymptotic and ρ is smaller than half the injectivity radius, implying that $n_j x = n' x$ for some small $n' \in N$, with $\|n\| < \rho$. In other words, Γ contains a parabolic element, in contradiction. $\square$

7.2. **The case of a single bi-minimizing line.** We will end this section with a description of the horocycle orbit closures in surfaces such as the ones constructed in Section 5.6.

Theorem 7.24. *Let $\Sigma = G/\Gamma$ be a $\mathbb{Z}$-cover of a compact hyperbolic surface Σ_0, which contains a single bi-minimizing geodesic as its lamination λ and where λ_0 is a closed geodesic in Σ_0. Let $y_0 = h_0\Gamma \in \mathcal{L} = T^1\lambda$ be any bi-minimizing point, then*

$$\overline{Ny_0} = N\Delta y_0$$

where $\Delta = \overline{\delta(h_0\Gamma h_0^{-1})}$, is non-trivial and non-discrete and the horocycle $p_K(Ny_0) \subset \Sigma$ does not accumulate onto $p_K(y_0)$.

Furthermore, all non-maximal horocycle orbit closures in G/Γ are A-translates of $\overline{Ny_0}$ or $\overline{N\omega y_0}$, where ωy_0 is the opposite vector to y_0 in $T^1\lambda$.

Proof. Assume without loss of generality that y_0 is facing the positive end. Since $\mathcal{L}$ contains exactly one geodesic, we conclude that all quasi-minimizing rays in $\mathcal{Q}_+$ are asymptotic to this unique bi-minimizing trajectory. In other words, $\mathcal{Q}_+ = NAy_0$ for the point y_0 as described above. Therefore

$$\overline{Ny_0} = N\Delta_{y_0} y_0.$$

As y_0 is a lift of a point on a closed geodesic in $p_{\mathbb{Z}}(G/\Gamma)$, we have by proposition 7.16 that

$$\Delta_{y_0} = \overline{\delta(h_0\Gamma h_0^{-1})}.$$

In addition, since $\mathcal{L}$ contains a single, clearly isolated, leaf we deduce from corollary 7.23 that $p_K(Ny_0)$ does not accumulate onto $p_K(y_0)$.

Now given any $x \in \mathcal{Q}_+$, we have $Nx \cap Ay_0 = \{a_b y_0\}$ with

$$b = \beta_+(x) - \beta_+(y_0),$$

by Lemma 6.1. Therefore $\overline{Nx} = \overline{Na_b y_0} = a_b \overline{Ny_0}$. An analogous claim holds for $\mathcal{Q}_-$ with respect to ωy_0. $\square$

8. The proximality relation

In the previous section we have studied the intersection of horospherical orbit closures with the geodesic from which they emanate. In this section we study how the horospherical orbit accumulates onto other geodesic rays, showing how this is affected by the way the associated quasi-minimizing rays "fellow travel".

Let $\Gamma' < G_d$ be any Zariski-dense discrete group. The notion of proximality is standard in the theory of topological dynamical systems. We recall its definition in our context:

Definition 8.1. Two points $x, y \in G_d/\Gamma'$ are called *proximal* if

$$\liminf_{t\to\infty} d(a_t x, a_t y) = 0.$$

For example, any two asymptotic points are proximal. On the other hand, for any $x \in \mathcal{Q}$, the points x and $a_s x$ are *not* proximal for any $s \neq 0$.

We begin with the following proposition:

Proposition 8.2. *Let $\Gamma' < G_d$ be any discrete group and let x and y be two quasi-minimizing points in G_d/Γ' satisfying*

$$D = \liminf_{t\to\infty} d_{G_d/\Gamma'}(a_t x, a_t y) < \infty.$$

Then there exists $\ell_0 \in MA$ for which $\ell_0 y \in \overline{Nx}$. Moreover, if $D = 0$, that is if x and y are proximal, then $y \in \overline{Nx}$.

Proof. Let $x, y \in G_d/\Gamma'$ and $D \geq 0$ be as above. By definition, there exist $t_j \to \infty$ and $h_j \in G_d$ satisfying

$$a_{t_j} x = h_j a_{t_j} y,$$

where $\|h_j\|_{G_d} \to D$. As all h_j are bounded we may further assume $h_j \to h_0$ with $\|h_0\|_{G_d} = D$.

We will later show that $h_0 = n_0 \ell_0 u_0 \in NMA\,U$. Under this assumption the proof is quite similar to the proof of lemma 7.5. There exist

$$n_j \to n_0, \quad \ell_j \to \ell_0, \quad \text{and} \quad u_j \to u_0$$

in N, MA and U respectively for which $h_j = n_j \ell_j u_j$. Hence we have

$$a_{t_j} x = n_j \ell_j u_j a_{t_j} y$$

for all j, and consequently

$$(a_{-t_j} n_j^{-1} a_{t_j}) x = \ell_j (a_{-t_j} u_j a_{t_j}) y \to \ell_0 y.$$

Since $a_{-t_j} n_j^{-1} a_{t_j} \in N$ for all j we deduce $\ell_0 y \in \overline{Nx}$.

Whenever $D = 0$ we have $h_0 = \ell_0 = e$ and hence $y \in \overline{Nx}$.

It is left to show $h_0 \in NMAU$. The idea is the following, assume in contradiction that

$$h_0 \in G_d \smallsetminus NMAU = NMA\omega,$$

see (2.2). Recall $a_{-s}\omega = \omega a_s$, which means that $h_0 a_t y$ has the property that its forward endpoint is equal to the backward endpoint of y. Roughly this means that if $a_t x$ is close to $h_0 a_t y$ then, after flowing forward a moderately long time L, $a_{t+L} x$ will be close to $a_{t-L} y$, see Figure 10. On the other hand, the line $\{a_t y\}$ is quasi-minimizing, which means that for all times $s > t + L$, $a_s y$ will be ahead of $a_s x$ by a margin of roughly $2L$. Choosing L large enough relative to D, this contradicts the hypothesis.

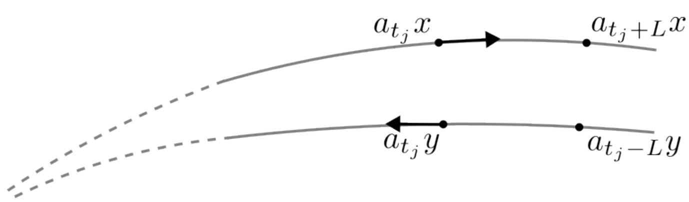


FIGURE 10.

More precisely, denote $h_0 = n_0\ell_0\omega$ with $n_0 \in N, \ell_o \in MA$ and $\varepsilon_j = h_j h_0^{-1} \to e$. Let $L > 0$ be arbitrarily large, then for all sufficiently large j we have $\|a_L\varepsilon_j a_{-L}\| \leq 1$. Compute

$$\begin{aligned} a_{t_j}x &= \varepsilon_j n_0\ell_0\omega a_{t_j}y = \varepsilon_j n_0\ell_0(\omega a_L)a_{t_j-L}y = \\ &= \varepsilon_j n_0\ell_0(a_{-L}\omega)a_{t_j-L}y = \\ &= a_{-L}(a_L\varepsilon_j a_{-L})(a_L n_0 a_{-L})\ell_0\omega a_{t_j-L}y. \end{aligned}$$

Since $n_0 \in N$ we have for L large enough

$$\|(a_L n_0 a_{-L})\ell_0\omega\| \leq \|\ell_0\omega\| + 1 = E,$$

implying for all large j

$$d\left(a_{t_j+L}x, a_{t_j-L}y\right) \leq \|a_L\varepsilon_j a_{-L}\| + \|(a_L n_0 a_{-L})\ell_0\omega\| \leq E + 1. \tag{8.1}$$

Fix j so large so as to satisfy both (8.1) and

$$d(a_{t_{j+1}}x, a_{t_{j+1}}y) \leq D + 1.$$

Now we have

$$\begin{aligned} d(a_{t_j-L}y, a_{t_{j+1}}y) &\leq d(a_{t_j-L}y, a_{t_j+L}x) + d(a_{t_j+L}x, a_{t_{j+1}}x) \\ &\quad + d(a_{t_{j+1}}x, a_{t_{j+1}}y) \\ &\leq (E+1) + (t_{j+1} - (t_j + L)) + (D+1) = \\ &= t_{j+1} - t_j + D + D' + 2 - L. \end{aligned}$$

On the other hand since y is quasi-minimizing, there exists a constant $C > 0$ for which

$$t_{j+1} - (t_j - L) - C \leq d(a_{t_j-L}y, a_{t_{j+1}}y).$$

This in turn implies

$$2L \leq D + E + 2 + C,$$

where all constants C, D, E are independent of L, in contradiction to the fact that L was arbitrary. ☐

Symmetry of the proximality relation immediately implies the following:

Corollary 8.3. *If $x, y \in G_d/\Gamma'$ are proximal then $\overline{Nx} = \overline{Ny}$.*

Let us return to the setting of G_d/Γ a $\mathbb{Z}$-cover of a compact hyperbolic d-manifold. We draw the following conclusion:

Corollary 8.4. *If G_d/Γ is a $\mathbb{Z}$-cover of a compact hyperbolic d-manifold, for any $x, y \in \mathcal{Q}_+$ there exists $\ell \in MA_{\geq 0}$ with $\ell y \in \overline{Nx}$. In other words, $\overline{Nx}$ intersects every geodesic line in $\mathcal{Q}_+$. Similarly for $\mathcal{Q}_-$.*

This proves Theorem 1.9(ii) from the introduction for arbitrary dimension.

Proof. Fix $x, y \in \mathcal{Q}_+$ and denote $b_x = \beta_+(x)$ and $b_y = \beta_+(y)$. For all t we have

$$t + b_x \leq \tau(a_t x) \leq \tau(x) + t \quad \text{and} \quad t + b_y \leq \tau(a_t y) \leq \tau(y) + t,$$

and therefore

$$|\tau(a_t x) - \tau(a_t y)| \leq C = \max\{|\tau(y) - b_x|, |\tau(x) - b_y|\} \quad \text{for all } t.$$

By lemma 3.2 we deduce that

$$d(a_t x, a_t y) \leq C + C_\tau \quad \text{for all } t.$$

The claim now follows from proposition 8.2 and the sub-invariance of $\overline{Nx}$ by the non-compact semigroup $\Delta_x \subseteq MA_{\geq 0}$. □

Recall the notations from Section 3.1 and denote $\mathcal{Q}_\omega^\pm = \mathcal{Q}_\omega \cap \mathcal{Q}_\pm$, the set of points in $\mathcal{Q}_\omega$ facing the positive/negative end of G_d/Γ. The following theorem tells us that it suffices to study the horospherical orbit closures of points in $\mathcal{Q}_\omega^\pm$:

Proposition 8.5. *For any $x \in \mathcal{Q}_+$ there exists $x_0 \in \mathcal{Q}_\omega^+$ such that x and x_0 are proximal in G_d/Γ.*

Proof. Consider the compact topological dynamical system $(G/\Gamma_0, (a_t)_{t\in\mathbb{R}})$ which has $p_\mathbb{Z}(\mathcal{Q}_\omega^+)$ as a closed A-invariant subset. Theorem 3.4 tells us that $(a_t p_\mathbb{Z}(x))_{t\geq 0}$ accumulates onto $p_\mathbb{Z}(\mathcal{Q}_\omega^+)$, that is,

$$\lim_{t\to\infty} d(a_t p_\mathbb{Z}(x), p_\mathbb{Z}(\mathcal{Q}_\omega^+)) = 0.$$

A classical result in topological dynamics [Fur81, Proposition 8.6] implies that there exists $p_\mathbb{Z}(x_0) \in p_\mathbb{Z}(\mathcal{Q}_\omega^+)$ such that $p_\mathbb{Z}(x)$ and $p_\mathbb{Z}(x_0)$ are proximal.

We claim that there exists a lift $x_0 \in \mathcal{Q}_\omega^+ \subset G/\Gamma$ of $p_\mathbb{Z}(x_0)$ for which x and x_0 are proximal. Indeed, assume $t_j \to \infty$ with $d(a_{t_j} p_\mathbb{Z}(x), a_{t_j} p_\mathbb{Z}(x_0)) \to 0$. As $x \in \mathcal{Q}_+ = \beta_+^{-1}(\mathbb{R})$ there exists $T \geq 0$ so large as to ensure

$$|[\tau(a_t x) - \tau(a_s x)] - [t - s]| \leq \frac{c}{4} \quad \text{for all } s, t \geq T,$$

where c is the equivariance constant for the $\mathbb{Z}$-action, i.e. $\tau(k.z) = \tau(z) + kc$ for all $z \in G_d/\Gamma$ and $k \in \mathbb{Z}$. Without loss of generality we may assume that all t_j satisfy $t_j \geq T$ and $d(a_{t_j} p_\mathbb{Z}(x), a_{t_j} p_\mathbb{Z}(x_0)) < \frac{c}{4}$.

The injectivity radius of the $\mathbb{Z}$-action at every point in G_d/Γ is at least c, since if $y_1 = k.y_2$ for some $0 \neq k \in \mathbb{Z}$ then

$$c \leq |k|c = |\tau(y_2) - \tau(y_1)| \leq d(y_1, y_2).$$

Now for any $j \leq 1$ let $x_j \in G_d/\Gamma$ be the unique lift of $p_{\mathbb{Z}}(x_0)$ satisfying

$$d(a_{t_j}x, a_{t_j}x_j) = d(a_{t_j}p_{\mathbb{Z}}(x), a_{t_j}p_{\mathbb{Z}}(x_0)) < \frac{c}{4}. \tag{8.2}$$

Fix some $j \geq 1$, we will show that $x_j = x_1$. By our assumption that $t_j \geq t_1 \geq T$ we know

$$0 \leq \tau(a_{t_j}x) - \tau(a_{t_1}x) \leq t_j - t_1 + \frac{c}{4}.$$

On the other hand, since $x_j \in \mathcal{Q}_\omega^+$ we know

$$\tau(a_{t_j}x_j) - \tau(a_{t_1}x_j) = t_j - t_1.$$

Therefore

$$\begin{aligned}|\tau(a_{t_1}x_j) - \tau(a_{t_1}x_1)| = |[\tau(a_{t_1}x_j) - \tau(a_{t_j}x_j)] + [\tau(a_{t_j}x_j) - \tau(a_{t_j}x)] +\\ + [\tau(a_{t_j}x) - \tau(a_{t_1}x)] + [\tau(a_{t_1}x) - \tau(a_{t_1}x_1)]| =\\ \leq |[t_1 - t_j] + [\tau(a_{t_j}x) - \tau(a_{t_1}x)]| + 2\frac{c}{4} \leq \\ \leq \frac{3c}{4}\end{aligned} \tag{8.3}$$

By definition $a_{t_1}x_j = k.a_{t_1}x_1$ for some $k \in \mathbb{Z}$, the inequality in (8.3) then implies $k = 0$ and $x_j = x_1$ as claimed. Hence, by (8.2) we conclude $x_0 := x_1 \in \mathcal{Q}_\omega^+$ is proximal to x. □

We further draw the following relation between different recurrence semigroups in G_d/Γ:

Proposition 8.6. *For all $x, y \in \mathcal{Q}$ there exists an isometry $f_{x,y}$ of MA satisfying*

$$f_{x,y}(\Delta_y) \subseteq \Delta_x.$$

Furthermore, $f_{x,y}$ acts as a translation on the A-component, that is, there exists an $s \in \mathbb{R}$ for which $f_{x,y}(ma_t) = m'a_{t+s}$ for all $m \in M$ and t.

Proof. Let us first consider $x, y \in \mathcal{Q}_+$. By proposition 8.2 there exists $\ell_1, \ell_2 \in MA$ satisfying $\ell_1 y \in \overline{Nx}$ and $\ell_2 x \in \overline{Ny}$. Therefore

$$\ell_1\overline{Ny} = \overline{N\ell_1 y} \subseteq \overline{Nx},$$

and hence for any $\ell_y \in \Delta_y$

$$\ell_1\ell_y\overline{Ny} = \ell_1\overline{N\ell_y y} \subseteq \ell_1\overline{Ny} \subseteq \overline{Nx}.$$

Now since

$$\ell_2\overline{Nx} \subseteq \overline{Ny},$$

we obtain

$$\ell_1\ell_y\ell_2\overline{Nx} \subseteq \ell_1\ell_y\overline{Ny} \subseteq \overline{Nx},$$

and conclude

$$\ell_1\Delta_y\ell_2 \subseteq \Delta_x.$$

Clearly the map $f_{x,y}(\ell) = \ell_1\ell\ell_2$ satisfies the conditions claimed in the proposition. The same holds when $x, y \in \mathcal{Q}_-$.

Note that the relation on $\mathcal{Q}$ defined by $x \sim y$ if there exists $f_{x,y}$ as in the statement above, is an equivalence relation. We have shown that $\mathcal{Q}_+ \subset [x]_\sim$ whenever $x \in \mathcal{Q}_+$, and similarly for $\mathcal{Q}_-$. To show $[x]_\sim = \mathcal{Q}$ as claimed, it suffices to find two points $x_\pm \in \mathcal{Q}_\pm$ with $x_+ \sim x_-$.

Indeed, let $x_+ \in \mathcal{Q}_\omega^+$ be a point in $\mathcal{Q}_\omega$ with $p_\mathbb{Z}(x_+)$ contained in a minimal A-component of $\mathcal{L}_0$. Therefore, $(a_t p_\mathbb{Z}(x_+))_{t\geq 0}$ accumulates onto $p_\mathbb{Z}(x_+)$ implying that

$$\Delta_{x_+} = \overline{\delta(h_+\Gamma h_+^{-1})}$$

by Lemma 7.12, where $x_+ = h_+\Gamma$.

Let $\omega \in K$ be an element satisfying $\omega a_t \omega^{-1} = a_{-t}$ for all t. Set $x_- = \omega x_+$ and note that $x_- \in \mathcal{Q}_-$ as $x_+ \in \mathcal{Q}_\omega^+$. Since $(a_t x_-)_{t\geq 0}$ also accumulates onto x_- we have

$$\Delta_{x_-} = \overline{\delta(\omega h_+\Gamma h_+^{-1}\omega^{-1})}.$$

Given an element $nma_tu \in NMAU$ we have

$$[\omega(n\ell u)\omega^{-1}]^{-1} = [u'm'a_{-t}n']^{-1} = n'm'a_tu'$$

where $n' \in N, m' \in M, u' \in U$, since conjugation by ω maps $N \leftrightarrow U$ and normalizes M. In particular, for any $\gamma \in h_+\Gamma h_+^{-1}$ we have

$$\delta(\omega\gamma^{-1}\omega^{-1}) = f_\omega(\delta(\gamma))$$

where $f_\omega(ma_t) = (\omega m\omega^{-1})a_t$ is an isometry of MA as claimed. We have thus shown that

$$\Delta_{x_-} = \overline{\delta(\omega h_+\Gamma h_+^{-1}\omega^{-1})} = \overline{f_\omega(\delta(h_+\Gamma h_+^{-1}))} = f_\omega(\Delta_{x_+}),$$

concluding the proof. □

9. Weakly mixing laminations

In this section we study the non-maximal horocycle orbit closures in surfaces with a weakly mixing first return map to a transversal to the maximal stretch lamination.

Let $\Sigma \to \Sigma_0$ be any $\mathbb{Z}$-cover as constructed in Theorem 5.7 together with a 1-Lipschitz tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ and $\tau : \Sigma \to \mathbb{R}$ its lift.

Recall our definition of a "uniform horoball" from Equation (6.1):

$$\mathcal{H}_+(x) = \beta_+^{-1}([\beta_+(x), \infty)) \quad \text{and} \quad \mathcal{H}_-(x) = \beta_-^{-1}((-\infty, \beta_-(x)]).$$

In this section we will prove Theorem 1.8 from the introduction:

Theorem 9.1. *If Σ is a $\mathbb{Z}$-cover surface constructed in Theorem 5.7 from a weakly-mixing and minimal IET, then all non-maximal horocycle orbit closures in Σ are uniform horoballs. That is, for all $x \in \mathcal{Q}_\pm$*

$$\overline{Nx} = \mathcal{H}_\pm(x).$$

Recall the definitions of weakly mixing dynamical systems from Definition 5.6. From the construction that the first return map (Section 5)

$$P_0 : \tau_0^{-1}([0]) \cap \lambda_0 \to \tau_0^{-1}([0]) \cap \lambda_0$$

is minimal and measure theoretically weakly mixing with respect to a measure of full support. This implies in particular that P is topologically weakly mixing.

Lifting this to $\mathcal{L}_+ = \mathcal{L} \cap \mathcal{Q}_+$ in the unit tangent bundle $T^1\Sigma$ gives:

$$P : \tau^{-1}(0) \cap \mathcal{L}_+ \to \tau^{-1}(0) \cap \mathcal{L}_+$$

defined for all $x \in \mathcal{L}_+$ by

$$P(x) = (-1).a_c x$$

where $k.\square$ denotes the isometric action of the element $k \in \mathbb{Z}$ of the deck group, and c is the circumference of $\mathbb{R}/c\mathbb{Z}$, the codomain of τ_0.

The systems

$$(\tau_0^{-1}([0]) \cap \lambda_0, P_0) \quad \text{and} \quad (W_0, P)$$

are isomorphic as topological dynamical systems, implying (W_0, P) is weakly mixing and minimal.

Recall that in a topological dynamical system (X, T) two points $x_1, x_2 \in X$ are called proximal if

$$\liminf_{j\to\infty} d_X(T^j x_1, T^j x_2) = 0.$$

The following can be found in [Gla03, Theorem 1.13]:

Theorem 9.2. *If a continuous dynamical system (X, T) is minimal and topologically weakly mixing then for all $x \in X$ the set*

$$\mathcal{P}_x = \{y \in X : y \text{ is proximal to } x\}$$

is dense in X.

We may now prove the main result of this section:

Proof of Theorem 9.1. Let us first consider a point $x_0 \in W_0 = \tau^{-1}(0) \cap \mathcal{L}_+$. Notice that if $y_0 \in W_0$ is P-proximal to x_0 then it is also A-proximal. That is, if

$$\liminf_{j\to\infty} d(P^j x_0, P^j y_0) = 0$$

then

$$\liminf_{t\to\infty} d(a_t x_0, a_t y_0) = 0.$$

Indeed, since the $\mathbb{Z}$-action and the A-action on G/Γ commute we have

$$P^j x = (-j).a_{jc} x.$$

The metric on W_0 is the one induced from G/Γ, which is $\mathbb{Z}$-invariant hence

$$d(P^j x_0, P^j y_0) < \varepsilon \Longrightarrow d(a_{jc} x_0, a_{jc} y_0) < \varepsilon$$

and the implication follows.

By theorem 9.2, the set of points $y_0 \in W_0$ that are P-proximal, and hence also A-proximal, to x_0 is dense in W_0. Therefore by corollary 8.3 we get that $\overline{Nx_0}$ contains all of W_0 and so for any $y_0 \in W_0$ we have $\overline{Ny_0} \subseteq \overline{Nx_0}$. By symmetry we conclude

$$\overline{Nx_0} = \overline{Ny_0} \quad \text{for all } x_0, y_0 \in W_0. \tag{9.1}$$

Now consider $x \in \mathcal{Q}_+$. Note that

$$\overline{Nx} = \mathcal{H}_+(x) \iff \forall s \in \mathbb{R} \quad \overline{Na_s x} = \mathcal{H}_+(a_s x)$$

by property (3) of lemma 6.1, hence without loss of generality we may replace x with $a_{-\beta_+(x)}x$ and assume $\beta_+(x) = 0$.

If $y \in \beta_+^{-1}(0)$ is any other point, we claim $\overline{Ny} = \overline{Nx}$. By Proposition 8.5, there exist $x_0, y_0 \in \mathcal{L}_+$ such that x and x_0 are proximal, and y and y_0 are proximal. By corollary 8.3 we have

$$\overline{Nx} = \overline{Nx_0} \quad \text{and} \quad \overline{Ny} = \overline{Ny_0}.$$

In particular, we deduce $\beta_+(x_0) = \beta_+(x) = 0$ and similarly for y_0. Recall that on $\mathcal{L}_+$ we have $\beta_+(x_0) = \tau(x_0)$. Therefore $x_0, y_0 \in \tau^{-1}(0)$ and $x_0, y_0 \in W_0$. From (9.1) we conclude

$$\overline{Nx} = \overline{Ny} \quad \text{for all } x, y \in \beta_+^{-1}(0).$$

Showing that $\Delta_{x_0} = A_{\geq 0}$ for some $x_0 \in W_0$ (and hence for all W_0 by proposition 7.17), implies

$$\overline{Nx_0} = \beta_+^{-1}([0, \infty)),$$

concluding the proof. This remaining claim is the content of Theorem 9.3 below. □

9.1. **Full recurrence semigroup.** Using the notations above, we will show:

Theorem 9.3. *There exists $x_0 \in W_0$ for which*

$$\Delta_{x_0} = A_{\geq 0}.$$

As discussed above, this theorem would imply $\Delta_x = A_{\geq 0}$ for all $x \in \mathcal{Q}_+$.

Recall $\mathcal{L}_\pm = \mathcal{L} \cap \mathcal{Q}_\pm$ and let $x, y \in \mathcal{L}_+$ be any two points. Let α be a choice of arc connecting their basepoints, $p_K(x), p_K(y)$ (that is equivalent to choosing lifts of x and y in the universal cover). Let $\mu = \mu(x, y)$ be the geodesic connecting the backward endpoint y^- to the forward endpoint x_+ in the homotopy class of the path

$$(a_t y)_{t \leq 0} * \alpha * (a_t x)_{t \geq 0},$$

going from y^- to y, then along α to x and from there to x^+. See figure 11.

Let $\omega \in G$ be an element in G which reverses orientation, i.e. $a_t\omega = \omega a_{-t}$ for all t.

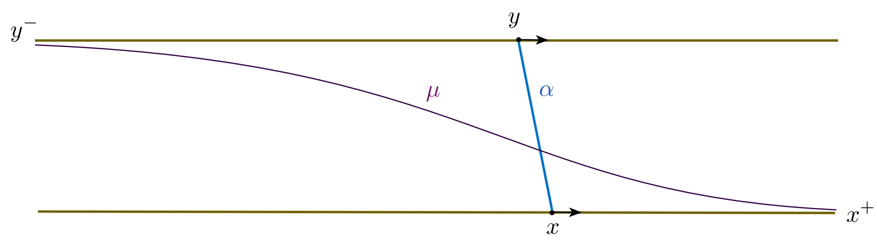


FIGURE 11.

Now if z_0 is any point along μ (with its given orientation), define

$$\Delta(x,y) = \beta_+(z_0) - \beta_-(\omega z_0).$$

Note that z_0 is forward-asymptotic to x and hence is contained in $\mathcal{Q}_+$, and similarly ωz_0 is forward-asymptotic to ωy and contained in $\mathcal{Q}_-$. Therefore, $\Delta(x,y) \in \mathbb{R}$. Moreover, property (3) of lemma 6.1 implies $\Delta(x,y)$ is independent of the choice of z_0. Similarly it doesn't really depend on the points x and y but rather on their endpoints at infinity (and the choice of arc α, which will be implicit in the application).

To see what this is actually measuring consider an N-horocycle segment connecting z_0 to a point $a_t x$ and a U-horocycle segment connecting z_0 to a point $a_s y$.

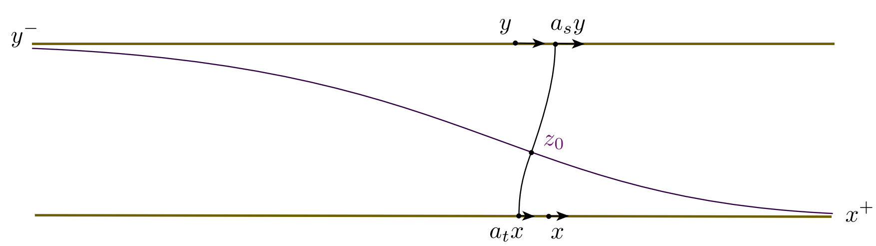


FIGURE 12.

Because $\beta_\pm$ are N-invariant and $\omega N \omega = U$, we have

$$\Delta(x,y) = \beta_+(a_t x) - \beta_-(\omega a_s y).$$

Since x and y are both in $\mathcal{L}_+$ we have

$$\Delta(x,y) = \tau(a_t x) - \tau(a_s y). \tag{9.2}$$

Lemma 9.4. *For all $x, y \in \mathcal{L}_+$ and α we have*

$$\Delta(x,y) \leq 0$$

with equality if and only if the corresponding point z_0 is contained in $\mathcal{L}_+$ and the cross-geodesic μ lies in λ.

Proof. Since

$$\beta_+ \leq \tau \quad \text{and} \quad \beta_- \geq \tau,$$

with equality only on $\mathcal{L}_\pm$ we have

$$\beta_+(z_0) \le \tau(z_0) = \tau(\omega z_0) \le \beta_-(\omega z_0), \tag{9.3}$$

where the equality is due to the value of τ depending only on the basepoint. Hence $\Delta(x, y) \le 0$, with equality only whenever equalities hold in (9.3) and $z_0 \in \mathcal{L}_+$. □

In our application we will consider points x and y that are very close to one another. In particular, whenever $d(x, y)$ is smaller than the minimal injectivity radius in G/Γ, we may choose α to be the unique shortest geodesic segment connecting x to y. Under this assumption and choice of α, equation (9.2) implies $\Delta(x, y)$ varies continuously with the basepoints in λ.

We will need the following estimate:

Lemma 9.5. *Let $x, y \in \mathcal{L}_+$, α, and μ be as above, and let $T > 0$. Rotate $a_{-T}y$ about its basepoint so that it is asymptotic to $\mu^+ = x^+$. Denote the result by y'_T. Then $d_{G/\Gamma}(a_{-T}y, y'_T) = O(e^{-T})$ and*

$$\beta_+(y'_T) = \tau(y'_T) + \Delta(x, y) + O(e^{-T}),$$

where the implicit constant in $O(\cdot)$ depends only on the distance between x and y along α.

The "rotation" in this lemma is determined in the universal cover so there's no ambiguity.

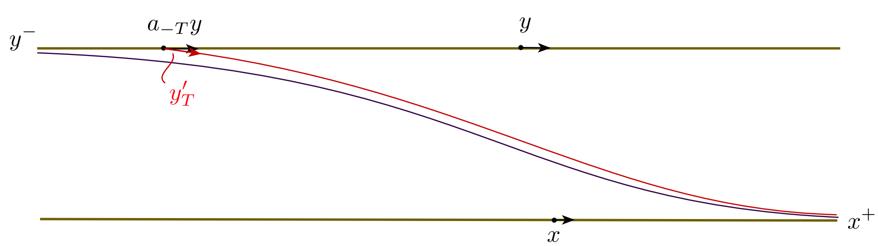


FIGURE 13.

Proof. Because the geodesic μ passes through α, the distance from $\mu \cap \alpha$ to y is bounded by the distance from x to y (this is one spot where two-dimensionality is used in a serious way). This implies that the distance from (the basepoint of) $a_{-T}y$ to μ is bounded by $O(e^{-T})$. The same is true for y'_T which has the same basepoint. Moreover, the rotation moving $a_{-T}y$ to y'_T is also of the order of $O(e^{-T})$.

Now since $a_{-T}y$ is backward asymptotic to μ, one can apply an expanding horocycle u_s to get $z_0 = u_s a_{-T}y$ lying along μ. Similarly since y'_T is forward-asymptotic to μ one can apply a contracting horocycle n_r to obtain $z_1 = n_r y'_T$ lying along μ, see Figure 14.

The distance bound implies that s and r and the distance from z_0 to z_1 are $O(e^{-T})$. In particular we can write $z_0 = a_q z_1$ where $|q| = O(e^{-T})$

Now we can compute $\Delta(x,y)$ using z_0, namely as $\beta_+(z_0)-\beta_-(\omega z_0)$. Since β_+ is N-invariant, the first term is equal to $\beta_+(y'_T)$. Now $\beta_-(\omega z_0)=\beta_-(\omega z_1)\pm q$, and $\beta_-(\omega z_1)=\beta_-(\omega a_{-T}y)$ since β_- is N-invariant.

Putting this together we get

$$\beta_+(y'_T)=\beta_-(\omega a_{-T}y)+\Delta(x,y)+O(e^{-T}).$$

Finally $\beta_-(\omega a_{-T}y)=\tau(a_{-T}y)=\tau(y'_T)$ so we are done.

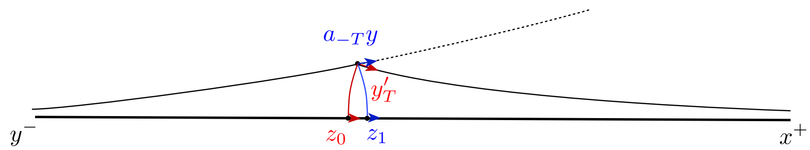


FIGURE 14.

□

Proof of Theorem 9.3. Recall our notation $W_0=\tau^{-1}(0)\cap\mathcal{L}_+$ and let $d\lambda$ denote the transverse measure on W_0 under which $(W_0,P,d\lambda)$ is a weakly mixing system, or equivalently, the system $(W_0\times W_0,P\times P,d\lambda\times d\lambda)$ is ergodic. In particular, almost every $(x,y)\in W_0\times W_0$ has a dense $P\times P$-orbit. By Fubini there exists a point $x_0\in W_0$ for which almost every $y\in W_0$ satisfies that (x_0,y) has a dense $P\times P$-orbit in $W_0\times W_0$. Fix such a point x_0 and let y_j be a sequence of points as above having $y_j\to x_0$. Note that every y_j is also A-proximal to x_0 in G/Γ (because x_0 and y_j are P-proximal).

Now set $C=B_r^{G/\Gamma}(x_0)\cap W_0$, where $r>0$ is smaller than half the injectivity radius at x_0. Since boundary leaves are dense in λ we know that there exist points $x,y\in C$ for which the geodesic μ connecting their corresponding endpoints is not contained in λ. By Lemma 9.4 we are hence ensured $\delta_0=\min_{x,y\in C}(\Delta(x,y))<0$. Pick $\delta\in(\delta_0,0)$.

Using the continuity of Δ in C and the density of the P-orbit of (x_0,y_j), we can find for any j an N arbitrarily large so that $\Delta(P^Nx_0,P^Ny_j)$ lies in $(\delta,\delta/2)$. Since Δ on $\mathcal{L}$ is $\mathbb{Z}$-invariant by (9.2), we deduce that for every j there exist arbitrarily large $T>0$ satisfying $\Delta(a_Tx_0,a_Ty_j)\in(\delta,\delta/2)$. Choose one such T_j for each j, so that $T_j\to\infty$.

Now apply Lemma 9.5, with $a_{T_j}x_0,a_{T_j}y_j$ playing the role of x,y and T_j playing the role of T. We obtain y'_j which is the corresponding rotation of y_j asymptotic to x_0 in forward time. The lemma gives

$$\beta_+(y'_j)=\tau(y'_j)+\Delta(a_{T_j}x_0,a_{T_j}y_j)+O(e^{-T_j}).$$

Since $\tau(y'_j)=\tau(y_j)\to\tau(x_0)=\beta^+(x_0)$ and the Δ term is between $\delta/2$ and δ, what we have, up to taking a subsequence, is

$$s_\delta=\beta_+(x_0)-\lim_{j\to\infty}\beta_+(y'_j)\in[|\delta/2|,|\delta|].$$

Since y'_j is forward-asymptotic to Ax_0 they are related by horocycle flow. More precisely, $y'_j=n_ja_{-s_j}x_0$ for some $n_j\in N$ where $s_j\to s_\delta$ since $s_j=$

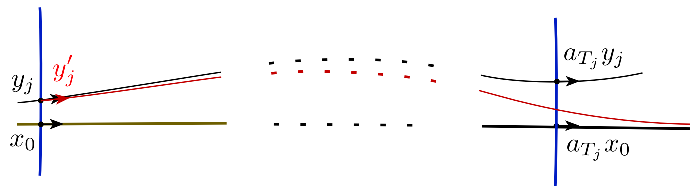


FIGURE 15.

$\beta_+(x_0) - \beta_+(y'_j)$ (by property (3) of lemma 6.1). Therefore we have $a_{s_j}y'_j \in Nx_0$ for all j.

Since $T_j \to \infty$ we have, by lemma 9.5,

$$d_{G/\Gamma}(y'_j, x_0) \leq d_{G/\Gamma}(y'_j, y_j) + d_{G/\Gamma}(y_j, x_0) \to 0,$$

implying

$$a_{s_\delta}x_0 = \lim_{j\to\infty} a_{s_j}y'_j \in \overline{Nx_0}.$$

In other words we have shown $a_{s_\delta} \in \Delta_{x_0}$ for some $s_\delta \in [|\delta/2|, |\delta|]$. Since we can make $|\delta|$ as small as we like, we conclude $\Delta_{x_0} = A_{\geq 0}$ as claimed. □

9.2. **Non-rigidity of horocycle orbit closures.** We may now conclude the non-rigidity result stated in the introduction.

Proof of Theorem 1.13. By Section 5.5, given a $\mathbb{Z}$-cover Σ constructed from a weakly-mixing IET there exist arbitrarily small deformations Σ' of Σ having a bi-minimizing locus λ' consisting of finitely many uniformly isolated geodesics. Since quasi-minimizing points tend to the bi-minimizing locus, we conclude that all quasi-minimizing points in $T^1\Sigma'$ are asymptotic to one of the geodesics in λ'. In particular, such a quasi-minimizing point has the same orbit closure as some point on $T^1\lambda'$. By Corollary 7.23, all horocycles based at $T^1\lambda'$ do not accumulate onto their base.

The above discussion implies in particular, that for any non-maximal N-orbit closure $\overline{Nx}$ in $T^1\Sigma'$, there exists a ball B such that

$$\overline{Nx} \cap B \quad \text{is equal to a single horocyclic segment,} \tag{9.4}$$

this also holds after projecting onto Σ'.

On the other hand, in Σ all N-orbit closures are $NA_{\geq 0}$-sub-invariant implying in particular that no ball B satisfies (9.4). We therefore conclude that no two non-maximal horocycle orbit closures in Σ and in Σ' are homeomorphic. □

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