1 Exercício 2 parte 2

Seja o estado térmico

$$\rho = Ne^{-\beta H_{AB}} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1.1}$$

e o hamiltoniano

$$H_{AB} = J\vec{\sigma}_A \cdot \vec{\sigma}_B \tag{1.2}$$

Calcule

$$\langle CHSH \rangle = tr(\rho CHSH) \tag{1.3}$$

Onde CHSH é o operador de Bell,

$$CHSH = AB + A'B + AB' - A'B' \tag{1.4}$$

em que $[A_i, B_j] = 0$, $[A_i, A_j] = [B_i, B_j] \neq 0$, $A^2 = 1$, $B^2 = 1$.

1.1 Solução:

1.1.1 Autoestados do Hamiltoniano

Partindo da equação de Schrödinger,

$$H_{AB} |\psi\rangle_{AB} = E_n |\psi\rangle_{AB} \tag{1.5}$$

e considerando um estado genérico no espaço de Hilbert $\mathcal{H}_{AB}$

$$|\psi\rangle_{AB} = \sum_i \sum_j C_{ij} |i\rangle |j\rangle ; \qquad i, j \in \{0, 1\} \tag{1.6}$$

Vamos, a partir da equação de Schrödinger, encontrar os autovalores e autovetores.

$$\begin{aligned} J(\sigma_x^A \sigma_x^B + \sigma_y^A \sigma_y^B + \sigma_z^A \sigma_z^B)(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) \\ = E_n(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) \end{aligned} \tag{1.7}$$

Temos portanto,

$$\begin{aligned} J\sigma_x^A \sigma_x^B(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) + \\ J\sigma_y^A \sigma_y^B(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) + \\ J\sigma_z^A \sigma_z^B(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) = \\ E_n(C_{00} |0\rangle_A |0\rangle_B + C_{10} |1\rangle_A |0\rangle_B + C_{01} |0\rangle_A |1\rangle_B + C_{11} |1\rangle_A |1\rangle_B) \end{aligned} \tag{1.8}$$

Ou seja, considerando que os estados da base $\{|0\rangle, |1\rangle\}$ são autoestados de σ_z,

$$\begin{aligned}
&J(C_{00}|1\rangle_A|1\rangle_B + C_{10}|0\rangle_A|1\rangle_B + C_{01}|1\rangle_A|0\rangle_B + C_{11}|0\rangle_A|0\rangle_B)+\\
&\quad J(C_{00}i^2|1\rangle_A|1\rangle_B - C_{10}i^2|0\rangle_A|1\rangle_B - C_{01}i^2|1\rangle_A|0\rangle_B + C_{11}(-i)^2|0\rangle_A|0\rangle_B)+\\
&\quad\quad J(C_{00}|0\rangle_A|0\rangle_B - C_{10}|1\rangle_A|0\rangle_B - C_{01}|0\rangle_A|1\rangle_B + C_{11}|1\rangle_A|1\rangle_B) =\\
&\quad\quad\quad E_n(C_{00}|0\rangle_A|0\rangle_B + C_{10}|1\rangle_A|0\rangle_B + C_{01}|0\rangle_A|1\rangle_B + C_{11}|1\rangle_A|1\rangle_B)
\end{aligned} \tag{1.9}$$

Agrupando os termos

$$\begin{aligned}
&JC_{11}|1\rangle_A|1\rangle_B + (2JC_{10} - JC_{01})|0\rangle_A|1\rangle_B + (2JC_{01} - JC_{10})|1\rangle_A|0\rangle_B + JC_0|0\rangle_A|0\rangle_B\\
&\quad = E_n(C_{00}|0\rangle_A|0\rangle_B + C_{10}|1\rangle_A|0\rangle_B + C_{01}|0\rangle_A|1\rangle_B + C_{11}|1\rangle_A|1\rangle_B)
\end{aligned} \tag{1.10}$$

Ou seja, temos o seguinte sistema de equações

$$\begin{cases} JC_{11} = E_n C_{11} \\ C_{10} = \frac{(J+E_n)}{2J} C_{01} \\ C_{01} = \frac{(J+E_n)}{2J} C_{10} \\ JC_{00} = E_N C_{00} \end{cases} \tag{1.11}$$

Ou seja,

$$C_{10} = \frac{(J+E_n)^2}{4J^2} C_{10} \tag{1.12}$$

Ou seja,

$$C_{10} = \pm C_{01} \; e \; (J+E_n)^2 = 4J^2 \rightarrow -3J^2 + 2JE_n + E_n^2 = 0 \tag{1.13}$$

Resolvendo a equação de segundo grau para E_n

$$E_n = -J \pm 2J \Rightarrow E_1 = -3J \quad e \quad E_2 = J. \tag{1.14}$$

Se $E_n = -3J$, temos

$$\begin{cases} JC_{11} = -3JC_{11} \\ JC_{10} = -JC_{01} \\ JC_{01} = -JC_{10} \\ JC_{00} = -3JC_{00} \end{cases} \tag{1.15}$$

Ou seja, $C_{11} = C_{00} = 0$ e $C_{01} = -C_{10} = \frac{1}{\sqrt{2}}$, e portanto temos o seguinte autoestado:

$$|\psi_n\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle) \tag{1.16}$$

Se $E_n = J$, temos

$$\begin{cases} C_{11} = C_{11} \\ C_{10} = C_{01} \\ C_{01} = C_{10} \\ C_{00} = C_{00} \end{cases} \tag{1.17}$$

Ou seja,

$$|\psi_n\rangle_{AB} = C_{00}\,|0\rangle\,|0\rangle + C_{11}\,|1\rangle\,|1\rangle) + C_{10}(|1\rangle\,|0\rangle + |0\rangle\,|1\rangle) \tag{1.18}$$

Quaisquer estados que respeitem esta forma são autoestados do hamiltoniano. Em particular podemos escrever $C_{10} = 0$ e $C_{00} = C_{11} = 1/\sqrt{2}$, e portanto,

$$|\psi_2\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle\,|1\rangle + |0\rangle\,|0\rangle) \tag{1.19}$$

Escrevendo $C_{00} = -C_{11} = 1/\sqrt{2}$

$$|\psi_3\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle\,|1\rangle - |0\rangle\,|0\rangle) \tag{1.20}$$

Escrevendo $C_{00} = C_{11} = 0$, $C_{10} = 1/\sqrt{2}$

$$|\psi_4\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle\,|0\rangle + |0\rangle\,|0\rangle) \tag{1.21}$$

Como estes são os autovalores do hamiltoniano, temos que esta é uma base para o espaço de Hilbert $\mathcal{H}_{AB}$. Em particular, estes estados são precisamente os estados de Bell, compostos pelo singleto $|\psi_1\rangle$ e o tripleto $|\psi_2\rangle$, $|\psi_3\rangle$, $|\psi_4\rangle$.

1.1.2 Matriz densidade

Voltando para a matriz densidade do estado térmico, $\rho_\beta = e^{-\beta H_{AB}}$ e lembrando que $\mathbb{I} = \sum_n |\psi_n\rangle\,\langle\psi_n|$

$$\rho_\beta = \sum_n e^{-\beta H_{AB}}\,|\psi_n\rangle\,\langle\psi_n| \tag{1.22}$$

Usando que $H_{AB}\,|\psi\rangle_{AB} = E_n\,|\psi\rangle_{AB}$

$$\rho_\beta = \sum_n e^{-\beta E_n}\,|\psi_n\rangle\,\langle\psi_n|\,. \tag{1.23}$$

Como $tr\rho = 1$, temos que

$$\rho_\beta = \frac{\sum_n e^{-\beta E_n}\,|\psi_n\rangle\,\langle\psi_n|}{\sum_j e^{-\beta E_j}}. \tag{1.24}$$

Ou seja,

$$\rho_\beta = \frac{e^{3\beta J}\,|\psi_1\rangle\,\langle\psi_1| + e^{-\beta J}\,|\psi_2\rangle\,\langle\psi_2| + e^{-\beta J}\,|\psi_3\rangle\,\langle\psi_3| + e^{-\beta J}\,|\psi_4\rangle\,\langle\psi_4|}{e^{3\beta J} + 3e^{-\beta J}}. \tag{1.25}$$

1.1.3 Cálculo da desigualdade de Bell

Queremos Calcular o valor esperado do operador de Bell $C = AB + A'B + AB' - A'B'$. Vamos utilizar que

$$\langle C\rangle = tr(\rho C) = \sum_n \langle\psi_n|\,\rho C\,|\psi_n\rangle \tag{1.26}$$

$$\langle C\rangle = \sum_n \langle\psi_n| \left(\frac{e^{3\beta J}|\psi_1\rangle\langle\psi_1| + e^{-\beta J}|\psi_2\rangle\langle\psi_2| + e^{-\beta J}|\psi_3\rangle\langle\psi_3| + e^{-\beta J}|\psi_4\rangle\langle\psi_4|}{e^{3\beta J}+3e^{-\beta J}} \right)|\psi_n\rangle \tag{1.27}$$

Ou seja,

$$\begin{aligned}
&\langle\psi_1| \left(\frac{e^{3\beta J}|\psi_1\rangle\langle\psi_1| + e^{-\beta J}|\psi_2\rangle\langle\psi_2| + e^{-\beta J}|\psi_3\rangle\langle\psi_3| + e^{-\beta J}|\psi_4\rangle\langle\psi_4|}{e^{3\beta J}+3e^{-\beta J}} \right) C|\psi_1\rangle + \\
&\langle\psi_2| \left(\frac{e^{3\beta J}|\psi_1\rangle\langle\psi_1| + e^{-\beta J}|\psi_2\rangle\langle\psi_2| + e^{-\beta J}|\psi_3\rangle\langle\psi_3| + e^{-\beta J}|\psi_4\rangle\langle\psi_4|}{e^{3\beta J}+3e^{-\beta J}} \right) C|\psi_2\rangle + \\
&\langle\psi_3| \left(\frac{e^{3\beta J}|\psi_1\rangle\langle\psi_1| + e^{-\beta J}|\psi_2\rangle\langle\psi_2| + e^{-\beta J}|\psi_3\rangle\langle\psi_3| + e^{-\beta J}|\psi_4\rangle\langle\psi_4|}{e^{3\beta J}+3e^{-\beta J}} \right) C|\psi_3\rangle + \\
&\langle\psi_4| \left(\frac{e^{3\beta J}|\psi_1\rangle\langle\psi_1| + e^{-\beta J}|\psi_2\rangle\langle\psi_2| + e^{-\beta J}|\psi_3\rangle\langle\psi_3| + e^{-\beta J}|\psi_4\rangle\langle\psi_4|}{e^{3\beta J}+3e^{-\beta J}} \right) C|\psi_4\rangle
\end{aligned} \tag{1.28}$$

Como os estados Bell são ortonormais, temos

$$\langle C\rangle = \frac{e^{3\beta J}\langle\psi_1|C|\psi_1\rangle + e^{-\beta J}\langle\psi_2|C|\psi_2\rangle + e^{-\beta J}\langle\psi_3|C|\psi_3\rangle + e^{-\beta J}\langle\psi_4|C|\psi_4\rangle}{e^{3\beta J}+3e^{-\beta J}} \tag{1.29}$$

Considerando os seguintes operadores de Bell

$$A|0\rangle = e^{i\alpha}|1\rangle; \qquad A|1\rangle = e^{-i\alpha}|0\rangle \tag{1.30}$$

$$B|0\rangle = e^{i\gamma}|1\rangle; \qquad B|1\rangle = e^{-i\gamma}|0\rangle \tag{1.31}$$

Calculando $\langle\psi_1|AB|\psi_1\rangle$

$$\langle\psi_1|AB|\psi_1\rangle = \frac{1}{2}(\langle 1|\langle 0| - \langle 0|\langle 1|)AB(|1\rangle|0\rangle - |0\rangle|1\rangle) \tag{1.32}$$

$$= \frac{1}{2}(\langle 1|\langle 0|AB|1\rangle|0\rangle - \langle 1|\langle 0|AB|0\rangle|1\rangle - \langle 0|\langle 1|AB|1\rangle|0\rangle + \langle 0|\langle 1|AB|0\rangle|1\rangle) \tag{1.33}$$

$$= -\frac{1}{2}(e^{i(\alpha-\gamma)} + e^{-i(\alpha-\gamma)}) = -cos(\alpha-\gamma) \tag{1.34}$$

Ou seja,

$$\langle\psi_1|C|\psi_1\rangle = -cos(\alpha-\gamma) - cos(\alpha'-\gamma) - cos(\alpha-\gamma') + cos(\alpha'-\gamma') \tag{1.35}$$

Calculando $\langle\psi_2|AB|\psi_2\rangle$

$$\langle\psi_2|AB|\psi_2\rangle = \frac{1}{2}(\langle 0|\langle 0| + \langle 1|\langle 1|)AB(|0\rangle|0\rangle + |1\rangle|1\rangle) \tag{1.36}$$

$$= \frac{1}{2}(\langle 0|\langle 0|AB|0\rangle|0\rangle + \langle 0|\langle 0|AB|1\rangle|1\rangle + \langle 1|\langle 1|AB|0\rangle|0\rangle + \langle 1|\langle 1|AB|1\rangle|1\rangle) \tag{1.37}$$

$$= \frac{1}{2}(e^{-i(\alpha+\gamma)} + e^{i(\alpha+\gamma)}) = cos(\alpha+\gamma) \tag{1.38}$$

Ou seja,

$$\langle\psi_2|C|\psi_2\rangle = cos(\alpha+\gamma) + cos(\alpha'+\gamma) + cos(\alpha+\gamma') - cos(\alpha'+\gamma') \tag{1.39}$$

Calculando $\langle\psi_3|\, AB\, |\psi_3\rangle$

$$\langle\psi_3|\, AB\, |\psi_3\rangle = \frac{1}{2}(\langle 0|\langle 0| - \langle 1|\langle 1|)AB(|0\rangle|0\rangle - |1\rangle|1\rangle) \tag{1.40}$$

$$= \frac{1}{2}(\langle 0|\langle 0|\, AB\, |0\rangle|0\rangle - \langle 0|\langle 0|\, AB\, |1\rangle|1\rangle - \langle 1|\langle 1|\, AB\, |0\rangle|0\rangle + \langle 1|\langle 1|\, AB\, |1\rangle|1\rangle) \tag{1.41}$$

$$= -\frac{1}{2}(e^{-i(\alpha+\gamma)} + e^{i(\alpha+\gamma)}) = -cos(\alpha+\gamma) \tag{1.42}$$

Ou seja,

$$\langle\psi_2|\, C\, |\psi_2\rangle = -cos(\alpha+\gamma) - cos(\alpha'+\gamma) - cos(\alpha+\gamma') + cos(\alpha'+\gamma') \tag{1.43}$$

Calculando $\langle\psi_4|\, AB\, |\psi_4\rangle$

$$\langle\psi_4|\, AB\, |\psi_4\rangle = \frac{1}{2}(\langle 1|\langle 0| + \langle 0|\langle 1|)AB(|1\rangle|0\rangle + |0\rangle|1\rangle) \tag{1.44}$$

$$= \frac{1}{2}(\langle 1|\langle 0|\, AB\, |1\rangle|0\rangle + \langle 1|\langle 0|\, AB\, |0\rangle|1\rangle + \langle 0|\langle 1|\, AB\, |1\rangle|0\rangle + \langle 0|\langle 1|\, AB\, |0\rangle|1\rangle) \tag{1.45}$$

$$= \frac{1}{2}(e^{i(\alpha-\gamma)} + e^{-i(\alpha-\gamma)}) = cos(\alpha-\gamma) \tag{1.46}$$

Ou seja,

$$\langle\psi_4|\, C\, |\psi_4\rangle = cos(\alpha-\gamma) + cos(\alpha'-\gamma) + cos(\alpha-\gamma') - cos(\alpha'-\gamma') \tag{1.47}$$

Portanto,

$$\begin{aligned}\langle C\rangle = {} & \frac{-e^{3\beta J}(cos(\alpha-\gamma) + cos(\alpha'-\gamma) + cos(\alpha-\gamma') - cos(\alpha'-\gamma'))}{e^{3\beta J} + 3e^{-\beta J}} \\ & + \frac{e^{-\beta J}(cos(\alpha+\gamma) + cos(\alpha'+\gamma) + cos(\alpha+\gamma') - cos(\alpha'+\gamma'))}{e^{3\beta J} + 3e^{-\beta J}} \\ & + \frac{-e^{-\beta J}(cos(\alpha+\gamma) + cos(\alpha'+\gamma) + cos(\alpha+\gamma') - cos(\alpha'+\gamma'))}{e^{3\beta J} + 3e^{-\beta J}} \\ & + \frac{e^{-\beta J}(cos(\alpha-\gamma) + cos(\alpha'-\gamma) + cos(\alpha-\gamma') - cos(\alpha'-\gamma'))}{e^{3\beta J} + 3e^{-\beta J}}\end{aligned} \tag{1.48}$$

Ou seja,

$$\langle C\rangle = (cos(\alpha-\gamma) + cos(\alpha'-\gamma) + cos(\alpha-\gamma') - cos(\alpha'-\gamma'))\frac{(e^{-\beta J} - e^{3\beta J})}{e^{3\beta J} + 3e^{-\beta J}} \tag{1.49}$$

Considerando $\alpha = 0; \alpha = \frac{\pi}{2}; \gamma = -\frac{\pi}{4}; \gamma = \frac{\pi}{4};$

$$\langle C\rangle = 2\sqrt{2}\frac{(e^{-\beta J} - e^{3\beta J})}{e^{3\beta J} + 3e^{-\beta J}} \tag{1.50}$$

Podemos também escrever

$$\langle C\rangle = -2\sqrt{2}\frac{2sinh(2\beta J)}{e^{2\beta J} + 3e^{-2\beta J}} \tag{1.51}$$

1.1.3.1 Violações da desigualdade

Quando $\beta \to \infty$, $\langle C \rangle \to 2\sqrt{2}$, ou seja, a desigualdade de Bell tende a violação máxima à medida que a temperatura do estado térmico tende a zero. Em contrapartida, ou seja, quando a tende ao infinito a desigualdade tende a zero, ou seja, para $\beta \to 0$, $\langle C \rangle \to 0$. Portanto, o aumento de temperatura destrói o emaranhamento do estado.

Vamos investigar a partir de qual valor da temperatura ocorre a violação, ou seja, $|\langle C \rangle| > 2$

$$2\sqrt{2}\frac{(-e^{-\beta J} + e^{3\beta J})}{e^{3\beta J} + 3e^{-\beta J}} > 2 \tag{1.52}$$

Ou seja,

$$\sqrt{2}(-e^{-\beta J} + e^{3\beta J}) > (e^{3\beta J} + 3e^{-\beta J}) \tag{1.53}$$

Reescrevendo temos,

$$e^{4\beta J} > 4\sqrt{2} + 2 \tag{1.54}$$

ou ainda

$$\beta J > \frac{\ln\left(4\sqrt{2} + 2\right)}{4} \tag{1.55}$$